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IMAGE

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Comments from Thirty Years of Teaching Matrix Algebra to Applied Statisticians

by Shayle R. SEARLE, Cornell University

Abstract. A few simple ideas gleaned from teaching matrix algebra are described. For teaching beginners how to prove theorems a useful mantra is: "Think of something to do: do it and hope."

1. Introduction: Hooked on Matrix Algebra. As a Master's student of mathematics in New Zealand, I had a course on matrices from Jim Campbell who had done his doctorate under A. C. Aitken at Edinburgh. That one could have something like $\mathbf{AB} = 0$ without $\mathbf{A} = 0$ or $\mathbf{B} = 0$ fascinated me—and I was hooked. Eight years later, as a graduate student at Cornell, I gave an informal 1957 summer course on matrix algebra; and after returning to Cornell on faculty that course developed in 1963 into a regular fall semester offering, Matrix Algebra, in the Biometrics Unit. I taught it until 1995, when I retired; and it still continues with 20–30 students a year.

One may well ask "why matrix algebra when linear algebra is widely taught in Mathematics?" Several reasons: I'm no geometer. Dropping a perpendicular from 8-space to 4-space helps my intuition not one bit. I cannot illustrate that perpendicular on a blackboard, and I cannot use my fingers to do so either. I have many kindred spirits in this regard. They come from students minoring in statistics for use in their down-to-earth majors such as agronomy, agricultural economics, plant breeding, pomology and animal breeding. They find no joy in geometry, but they can get the hang of algebra—even to the stage of enjoying it, despite the 8 a.m. hour for the course! That lasted 25 years, whereupon I changed it to 9 a.m., giving the college curriculum committee the reason "After 25 years, I'm tired of 8 a.m.!"

2. Features of the Course. Each lecture began with my immediately launching into the day's topic, saving the end of the hour, when all of a day's attendees would be there, for announcements. And so no repetition was needed.

An 8 a.m. class inevitably generates late arrivals. That never worried me. Mathematical subjects demand using today's new knowledge for tomorrow's work, so that I insisted that late arrival was far better than not coming at all. And with the classroom having a rear door, which I heartily recommend, entry for laggards caused no interruption, except on the day a chronically 30-minute late arrival was greeted with the 8:30 a.m. salutation "Good afternoon".

The course always had homework—every Wednesday, a help session the following Monday and hand it in two days later. By that time, one hoped it was all correct. Each week's work was graded 2, 1 or 0 for satisfactory, mediocre or hopeless, respectively. No student ever argued about this holistic grading. And a whole term's homework counted for no more than 10% of the term's grade; in some years nothing. Yet there was motivation: an early handout explained that the term grade would be F if there was any failure to do homework, no matter how much help had been given. To me, homework is for students to learn to *do* mathematics; it is not inquisitorial for assessing how much they have learned. That is the purpose of exams—and students were told that exam questions would be similar to homework problems—indeed, most exams included at least one of the homework problems.

For a number of years the course was split into two back-to-back 7-week modules, each of two credits and two exams. This came about because agricultural economics graduate students asserted the second part of the course was too theoretical. So with the split they did only the first part and got no further than inverting a matrix (not even to rank and linear independence). So their attendance dwindled almost to zero, coinciding with increasing enrolment of statistics undergraduates. For them I soon learnt that two modules involving four exams was too weighty a presentation, and after 10 years the course returned to a regular 3-lecture, 14-week routine. That was less intensive, more enjoyable for the students, and under less pressure I believe they learnt more, and more easily.

3. Teaching How to do Mathematics. How do we do mathematics? How do we learn how to do mathematics? It seems to me that we don't do a good job of teaching this—of how to do algebra, for example. Many students have difficulty in learning this: they find even simple methods of proof difficult to assimilate. This arises, one must assume, from high school and mandatory college courses in mathematics paying insufficient attention to these matters. Students seem to have little idea as to how mathematicians think, how they go about solving problems, and especially how they *start* trying to solve a problem. So the matrix algebra course attempted to give students ideas about some of the thought processes which are helpful in proving already-established results. One would hope that in learning such techniques students would subsequently find them very useful, both for checking on the validity of results they will find in their own subject-matter research literature, and also for developing their own results *ab initio*. And part of the necessary learning is: how do mathematicians motivate themselves to develop the step-by-step process which leads to final and useful results? The nature of this motivation can be illustrated in the proving of a simple, well known matrix result.

Suppose we want to prove the following little theorem:

If $\mathbf{A}$ *is idempotent, so is* $\mathbf{I} - \mathbf{A}$.

As we all know, the proof is easy:

$$(\mathbf{I}-\mathbf{A})^2 = \mathbf{I}-2\mathbf{A}+\mathbf{A}^2 = \mathbf{I}-2\mathbf{A}+\mathbf{A} = \mathbf{I}-\mathbf{A},$$

and the proof is complete. "But", asks a student, "Where does this come from?" "Why?" asks another; and "So what?" says a third. These reactions, I suggest, stem from unfamiliarity of important steps in the process of proof, steps which are, it appears, seldom emphasized early enough in one's mathematical education. For example, two important steps for algebraic proof are (i) whenever possible convert text statements to algebraic statements and, in doing so, (ii) clearly label the statements as Given, or To Be Proved. In our case this yields

$$\begin{array}{rcl} \text{Given} & : & \mathbf{A}^2 = \mathbf{A}, \text{ the definition of idempotent.} \\ \text{To Be Proven} & : & (\mathbf{I}-\mathbf{A})^2 = \mathbf{I}-\mathbf{A}. \end{array}$$

Now comes the difficult part. We know what has to be proven: we must show that $(\mathbf{I}-\mathbf{A})^2$ can be reduced to $\mathbf{I}-\mathbf{A}$. On thinking about this one soon concludes that it seems (and indeed is) difficult to straightforwardly start at $\mathbf{I}-\mathbf{A}$ and get to $(\mathbf{I}-\mathbf{A})^2$. Actually, one can do this, as follows. Beginning at $\mathbf{I}-\mathbf{A}$ we can observe that $\mathbf{A}(\mathbf{I}-\mathbf{A}) = \mathbf{A}-\mathbf{A}^2 = \mathbf{A}-\mathbf{A} = \mathbf{0}$. Therefore

$$\begin{aligned} \mathbf{I}-\mathbf{A} &= \mathbf{I}-\mathbf{A}-\mathbf{A}(\mathbf{I}-\mathbf{A}) \\ &= \mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{A}^2 = \mathbf{I}-2\mathbf{A}+\mathbf{A}^2 = (\mathbf{I}-\mathbf{A})^2. \end{aligned}$$

This is correct, but it is sort of gimmicky, and not obviously logical. It is much more logical to start from something complicated, in this case $(\mathbf{I}-\mathbf{A})^2$, and try to reduce it to something simpler, $(\mathbf{I}-\mathbf{A})$. This is usually easier than the other way round. So we start with $(\mathbf{I}-\mathbf{A})^2$ and hope to get to $\mathbf{I}-\mathbf{A}$. But now what? $(\mathbf{I}-\mathbf{A})^2 = ?$

The following mantra, to be used iteratively, is what helps.

"Think of something to do: do it and hope."

What comes easily to mind? Expand:

$$(\mathbf{I}-\mathbf{A})^2 = (\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A}).$$

Now what? Use the mantra again, and multiply out:

$$\begin{aligned} (\mathbf{I}-\mathbf{A})^2 &= (\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A}) \\ &= \mathbf{I}(\mathbf{I}-\mathbf{A})-\mathbf{A}(\mathbf{I}-\mathbf{A}) \\ &= \mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{A}^2 = \mathbf{I}-2\mathbf{A}+\mathbf{A}^2. \end{aligned}$$

Now comes an important step in the process of developing results. (iii) Use what is given: in this case $\mathbf{A}^2 = \mathbf{A}$. Thus

$$(\mathbf{I}-\mathbf{A})^2 = \mathbf{I}-2\mathbf{A}+\mathbf{A} = \mathbf{I}-\mathbf{A}.$$

This process, especially the mantra, has helped numerous students to overcome their fear of the proof process and to go on to successfully tackle problems. True, the mantra is not much more than formalizing the trial-and-error process, but after all that is precisely how mathematical progress is often made. It seldom leads to the shortened proof, but who cares about that insofar as learning the process of proof is concerned. Supplementing the mantra there are, of course, a number of technical aids for helping with algebra, some of which are detailed in Searle (1977).

4. Patterns and Relationships. We also need to teach students that mathematicians look for patterns and relationships. For example, in proving that

$$t = (a \;\; b \;\; c)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = p,$$

without calculating the matrix inverse, it is the "pattern" of the row vector also being the first row of the matrix, combined with the thinking about the cofactors in the elements of the matrix inverse, that leads at once to

$$t = (1 \;\; 0 \;\; 0)\begin{pmatrix} p \\ q \\ r \end{pmatrix} = p.$$

5. Two Kinds of Thinking. This "thinking about the cofactors" brings us to what I feel are two kinds of thinking needed for doing mathematics. They can be called "automatic thinking" and "cogitative thinking". Automatic thinking is the kind of straightforward thinking involved in doing algebra, especially in simplifying algebraic expressions; e.g., $\mathbf{I}-2\mathbf{A}+\mathbf{A} = \mathbf{I}-\mathbf{A}$. The second type of thinking is that of cogitating; e.g., consider the statement

$\mathbf{AB}$ is a matrix of columns which
are linear combinations of columns of $\mathbf{A}$.

Automatic (e.g., algebraic) thinking is relatively easy, once one knows the rules of algebra, for example. In contrast, cogitative thinking is not so easy. As soon as we see that statement about $\mathbf{AB}$ we know it is true. But developing the statement requires cogitating (i.e., "thinking hard", according to the dictionary) about the operation of matrix multiplication. Then we hit on the statement about $\mathbf{AB}$. Generally speaking, it is a much harder kind of thinking than that involved in $\mathbf{I}-2\mathbf{A}+\mathbf{A} = \mathbf{I}-\mathbf{A}$. And the real difficulty is to recognize when this kind of cogitating is required. The trick is to know when, in the presence of just the product symbol $\mathbf{AB}$, it will be useful to think of that product as just described. At least a first step in being motivated towards this possibility is to recognize that there are occasions when the "automatic thinking" or "algebraic thinking" has to be replaced by the cogitating (hard thinking). Distinguishing between these two modes of thinking has proven to be one more aid in helping non-mathematics majors to learn a mathematical subject. A simple example involving both kinds of thinking is the following. Suppose

$$\begin{aligned}\mathbf{y}' &= (y_{11}\ y_{12}\cdots y_{1n}\ y_{21}\ y_{22}\cdots y_{2n}\cdots \\ &\qquad \cdots y_{i1}\ y_{i2}\cdots y_{in}\cdots \quad \cdots y_{m1}\ y_{m2}\cdots y_{mn}) \\ &= \left\{_r\ \{_r\ y_{ij}\}_{j=1}^{n}\right\}_{i=1}^{m}.\end{aligned}$$

Then, with $y_{i\cdot} = \sum_{j=1}^{n} y_{ij}$,

$$\begin{aligned}(\mathbf{I}_m \otimes \mathbf{1}_n')(\mathbf{I}_m \otimes \mathbf{J}_n)\mathbf{y} &= (\mathbf{I}_m \otimes n\mathbf{1}_n')\mathbf{y} \\ &= n(\mathbf{I}_m \otimes \mathbf{1}_n')\mathbf{y} = n\{_c\ y_{i\cdot}\}_{i=1}^{m}.\end{aligned}$$

The first two equations here stem from automatic (algebraic) thinking; but the last comes from cogitative thinking.

6. *Doing Mathematics is a Game.* Finally, there is nothing wrong in suggesting to students that doing mathematics is really a game: make the rules and play by them. For example, for scalars

$$ab = 0 \Rightarrow a = 0 \quad \text{and/or} \quad b = 0: \quad \text{and} \quad xy = yx \quad \text{always.}$$

But for matrices the rules are different:

$$\begin{aligned}&\mathbf{AB} = \mathbf{0} \not\Rightarrow \mathbf{A} = \mathbf{0} \quad \text{and/or} \quad \mathbf{B} = \mathbf{0}; \\ &\text{and} \quad \mathbf{XY} = \mathbf{YX} \quad \text{sometimes (not often).}\end{aligned}$$

One must, therefore, always remember which game is being played.

Hopefully some of these ideas may make a contribution to reducing the fear of mathematics that we see in so many students and thence to improving their mathematical literacy!

This paper was presented at the Seventh International Workshop on Matrices and Statistics: Fort Lauderdale, Florida, December 11–14, 1998, and is Paper No. BU-776 in the Department of Biometrics, Cornell University.

Reference

S. R. SEARLE (1977). Proof. *International Journal of Mathematics Education in Science and Technology* 8, 195–202.

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Request for Feedback on Linear Algebra Education

by Frank UHLIG, Auburn University

I would like to ask our members for their ideas on ILAS educational offerings:

Linear Algebra Education Day in Atlanta in 1995 so far was our biggest "production". We had one 1/2 hour educational talk in Chemnitz, and in Madison last summer we again offered several educational activities at the ILAS Conference with one plenary lecture, a double mini-symposium on educational matters and the interactive Linear Algebra textbook presentations.

Our next chance for promoting educational issues is at the ILAS Conferences in Haifa, Israel, June 25–29, 2001, and Auburn, Alabama, in the summer 2002.

For this purpose I would like to ask the ILAS membership for their input: We would like to hear your plans, desires, dreams and ideas now on what you want to see or hear at a future ILAS meeting as regards issues of teaching and learning Linear Algebra.

What can we do to foster educational thoughts and progress in our field? Who could help us? Please specify topics, directions, unmet needs, and so on, to your ILAS Education Committee:

Richard A. BRUALDI: brualdi@math.wisc.edu,
David CARLSON: carlson@math.sdsu.edu,
Jane DAY: day@sjsumcs.sjsu.edu,
Guershon HAREL: harel@math.purdue.edu,
Charles R. JOHNSON: crjohnso@math.wm.edu,
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I will meet many of you in Barcelona. Maybe by then some plans will have crystalized.

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ILAS President/Vice-President's Annual Report: April 1999

1. The following have been elected to ILAS offices with terms that began on March 1, 1999.

Board of Directors: Nicholas J. HIGHAM and Pauline VAN DEN DRIESSCHE (three-year terms ending February 28, 2002).

President: Richard A. BRUALDI.

The following continue in their offices to which they were previously elected:

Vice President: Daniel HERSHKOWITZ (term ends February 28, 2001)

Secretary/Treasurer: James R. WEAVER (term ends February 29, 2000)

Board of Directors: Jane DAY (term ends February 29, 2000), Volker MEHRMANN (term ends February 29, 2000), Jose DIAS DA SILVA (term ends February 28, 2001), Roger A. HORN (term ends February 28, 2001).

The President's Advisory Committee consists of Chi-Kwong Li (chair), Shmuel Friedland, Raphi Loewy, and Frank Uhlig.

2. The Nominating Committee for 1999 consists of: Rajendra Bhatia (chair, appointed by the president), Jim Weaver and Wayne Barrett (appointed by the president's Advisory Committee), and Tom Laffey and Judi McDonald (appointed by the Board of Directors).

3. This year there are elections for two positions on the Board that become vacant on March 1, 2000, and the position of Secretary/Treasurer being vacated by Jim Weaver.

4. The following ILAS conferences are planned for the near future:

a. The 8th ILAS Conference will be held at the Universitat Politecnica de Catalunya, Barcelona, Spain, on July 19–22, 1999. The organizing committee consists of R. Bru, R. A. Brualdi, L. de Alba, I. Garcia-Planas (co-chair), J. M. Gracia, V. Hernandez, N. J. Higham, R. A. Horn, T. J. Laffey (co-chair), G. de Oliveira, F. Puerta (chair), and P. M. van Dooren. The local committee consists of J. Clotet, A. Compta, M. D. Magret, and X. Puerta.

b. The 9th ILAS Conference will be held at the Technion, Haifa, Israel, June 25–29, 2001.

c. The 10th ILAS Conference will be held at Auburn University, Auburn, Alabama, USA, summer 2002.

5. The ILAS Board has agreed that, on an experimental basis, ILAS will consider requests for the sponsorship of an ILAS Lecturer at a conference which is of substantial interest to ILAS members. ILAS will set aside $1,000 per year to support such conferences, with a maximum amount of $500 available for any one conference. The guidelines being used during this experimental period are:

(1) the conference must be of interest to a substantial number of ILAS members;

(2) the same 'organization' is not eligible for support more frequently than once every three years;

(3) the support should be widely distributed geographically.

During this experimental period, the ILAS Board (three executives and six other members) will review proposals and decide on which, if any, will receive support, and how much that support will be. Next year we will review these guidelines and this may result in some changes.

We are now accepting requests for conferences held in 2000. Such requests should be submitted by September 15, 1999. Electronic requests are preferred and should be sent to brualdi@math.wisc.edu. The request should include:

(1) date and place of conference, (2) sponsoring "organization," (3) organizing committee, (4) purpose of conference, (5) invited speakers, to the extent known, (6) expected attendance, (7) proposed ILAS Lecturer, and brief information about the lecturer, (8) amount requested.

ILAS is sponsoring two Lectures in 1999: Hans Schneider (University of Wisconsin–Madison) at the workshop on Applied Linear Algebra honoring Ludwig Elsner (Bielefeld, Germany, January 21–23, 1999) and Gene H. Golub (Stanford University) at the Eighth International Workshop on Matrices and Statistics (Tampere, Finland, August 6–7, 1999).

6. The 1999 ILAS Linear Algebra Prize Committee (Hans Schneider–chair, Angelika Bunse-Gerstner, Biswa Datta, Mirek Fiedler & Russ Merris) has recommended that Ludwig Elsner be awarded the 1999 prize. The prize will be awarded at the ILAS Barcelona meeting in July 1999.

7. ILAS has entered into a cooperative agreement with SIAM for its next Applied Linear Algebra meeting to be held in Raleigh, North Carolina, October 23–25, 2000. The terms of this agreement are:

(1) ILAS will nominate two plenary speakers, and ILAS will support the travel and expenses of these two speakers. (2) The ILAS speakers will be identified in publicity for the meeting and in the program. (3) SIAM will offer the two ILAS speakers free registration. (4) SIAM will offer the two ILAS speakers a free banquet ticket. (5) SIAM will offer ILAS members who are not already a member of the SIAM/SIAG-LA the same reduced registration fee that it offers SIAM/SIAG-LA members.

8. The *Electronic journal of Linear Algebra* (ELA): Volume 1, published in 1996, contained 6 papers. Volume 2, published in 1997, contained 2 papers. Volume 3, the Hans Schneider issue, published in 1998, contained 13 papers. Volume 4, published in 1998 as well, contained 5 papers. Volume 5 is being published now. As of April 1999 it contains 6 papers. ELA's primary site is at the Technion. Mirror sites are located in Temple University (Philadelphia, USA), in the University of Chemnitz (Germany), in the University of Lisbon (Portugal), in The European Mathematical Information Service offered by the European Mathematical Society (EMIS), and in EMIS's 36 Mirror Sites.

We are happy to announce the appointment of five new advisory editors and three new associate editors. The new advisory editors are: Richard A. Brualdi, Ludwig Elsner, Miroslav Fiedler, Shmuel Friedland, and Hans Schneider. The new associate editors are: Ravindra B. Bapat, Stephen J. Kirkland, and Bryan L. Shader. The entire board is shown in ELA's primary homepage:

http://www.math.technion.ac.il/iic/ela/

and in ELA's mirror sites:

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http://hermite.cii.fc.ul.pt/iic/ela/
http://www.tu-chemnitz.de/ftp-home/pub/ela
http://www.emis.de/journals/ELA/

9. George Styan has continued as Editor-in-Chief of IMAGE, with issues no. 20 and 21 published in April and October 1998, respectively.

10. A new Journals Committee has been appointed for a three year term. It consists of Chi-Kwong Li as chair, Hans Schneider, Jim Weaver, Danny Hershkowitz (representing ELA), and George Styan (representing IMAGE). The Committee has been given the immediate charge of exploring the possibility and implications of ILAS publishing an annual (or whatever the quantity of papers suggests) hard copy of ELA, for those who wish to have such in addition to the free electronic availability of ELA.

11. ILAS-NET: As of April 21, 1999, we have circulated 836 ILAS-NET announcements. ILAS-NET currently has 524 subscribers.

12. ILAS INFORMATION CENTER (IIC) has a daily average of 300 information requests (not counting FTP operations). IIC's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz and in the University of Lisbon.

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Dennis Ray Estes: 1941–1999

Dennis Ray Estes was born in Ada, Oklahoma, on June 18, 1941. He received his PhD from Louisiana State University in 1965 under the direction of Gordan Pall. He was a Bateman Research Fellow at the California Institute of Technology from 1966 to 1968. He then moved to the University of Southern California, where he spent 30 years, until his untimely death on February 1, 1999.

He was a leading expert in the arithmetic theory of quadratic forms over rings of algebraic integers and wrote more than 40 papers on quadratic forms, number theory, linear and commutative algebra with over 20 different co-authors including Len Adleman (USC), Hyman Bass (Columbia), John Hsia (Ohio State) and Olga Taussky (Caltech). He was a major speaker at several international conferences in Germany and France, and most recently at the meeting on Quadratic Forms and Lattices in Seoul, Korea, last June.

Estes had 6 PhD students over the years. He was deeply involved in graduate education at USC. He served 12 years as Math Department Vice Chair for Graduate Studies and was a constant source of help and advice to our graduate students. He also served several terms on the University Graduate Advisory Committee.

He is survived by his wife Lee and two daughters, Darcy and Shelly. In recognition of his service to the mathematical community and in particular to graduate students, the University has established the Dennis Ray Estes Graduate Memorial Fund, to benefit graduate students in the Math Deparment. Donations may be sent to the Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA.

The photograph is courtesy Darcy Estes, via John S. Hsia.

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New & Forthcoming Books on Linear Algebra & Related Topics

We present two book reviews, a list of new and forthcoming books on linear algebra and related topics, and some comments on Feliks Ruminovich Gantmacher and his *Theory of Matrices*.

David A. HARVILLE: *Matrix Algebra from a Statistician's Perspective*

Reviewed by Jürgen GROß

Matrix Algebra from a Statistician's Perspective [Springer-Verlag, New York, xvii + 630 pp., 1997, ISBN 0-387-94978-X] is a detailed, essentially self-contained book, giving an abundance of useful results for students, teachers and researchers with an interest in linear statistical models and related topics. Proofs are given for almost all results, and a lot of exercises are provided, whose solutions the author intends to make available on at least a limited basis.

The book contains twenty-two chapters. The first eight chapters are an introduction to matrices and treat 'classical' topics such as submatrices and partitioned matrices, linear dependence and independence (of matrices), row and column spaces, trace of a matrix, geometrical considerations, solution to linear systems, and inverse matrices (including orthogonal and permutation matrices). Although not necessarily required, prior knowledge of the basic concepts in linear algebra makes these chapters easier to grasp. Chapters nine and ten introduce concepts which are of great importance in statistics, namely generalized inverse matrices and idempotent matrices. These are used in chapters eleven and twelve to discuss solutions to linear systems and projections, respectively. Chapter thirteen gives basic properties and useful results on determinants. Chapters fourteen to twenty deal with subjects which are indispensable for the understanding of linear statistical models, such as quadratic forms (as well as linear and bilinear forms), matrix differentiation, Kronecker products, vec and vech operators, constrained and unconstrained minimization of a second-degree polynomial, and the Moore-Penrose inverse. An algorithm for determining the Moore-Penrose inverse (e.g. Greville's algorithm) would have been a nice completion, although a procedure for the computation of a generalized inverse is given in Sec. 9a. Chapter twenty-one treats eigenvalues and eigenvectors, and chapter twenty-two is about linear spaces.

As visible from the title, the book is written from a statistician's perspective. This can of course be confirmed by the choice of subjects in the book, which play a fundamental role in linear statistical models. The author assumes that the reader is aware of the potentiality of applications, or becomes aware of them when attending a course on linear models. A further indication for the statistician's perspective is the exclusion of complex matrices and numbers. Eigenvalues are treated in the next-to-last chapter of the book, and complex numbers are not allowed to be eigenvalues. This approach seems to be motivated from the author's teaching experience. [As a matter of fact, students as well as teachers in statistical sciences sometimes feel it an unnecessary effort to deepen the consequences of the fundamental theorem of algebra, which, among other things, implies that real matrices can have complex eigenvalues which always sum up to a real number, that triangularization can in general only be done by a unitary transformation, or that the absolute value of an eigenvalue is usually understood as the absolute value of a complex number (so that it can happen that no eigenvalue of a real orthogonal matrix is -1 or 1, but all eigenvalues have absolute value 1).] As a disadvantage of this approach one might regard the fact that triangularization, spectral decomposition, or singular value decomposition sometimes give more insight and ease proofs. Certainly, the book is a very good model for proving results without directly using eigenvalues and related canonical forms.

Everyone with an interest in matrix algebra should use this book: as a standard of comparison for his own work, as a source for ideas of proofs, as a supplier of useful results, as a companion for a course in linear models, or as a basis for teaching matrix algebra in statistics.

A. M. MATHAI: *Jacobians of Matrix Transformations and Functions of Matrix Arguments*

Reviewed by Serge B. PROVOST

This excellent book [World Scientific, xxii + 435 pp., 1997, ISBN 981-02-3095-8] makes the evaluation of Jacobians of matrix transformations and of integrals involving scalar functions of matrix argument accessible to a host of researchers and applied scientists. Real multivariable calculus, some properties of matrices and determinants and the basic notions of complex variables are the only prerequisites.

Although Jacobians can be obtained from matrix derivatives which per se are discussed in several books currently available [such as Rogers (1980), Graham (1981), and Magnus and Neudecker (1988)], the monograph being reviewed could very well be the first book dealing directly with Jacobians. The book also distinguishes itself by the inclusion of a detailed coverage of complex Gaussian, complex Wishart and complex matrix-variate statistical distributions.

A rather exhaustive collection of Jacobians of linear (Chapter 1) as well as nonlinear transformations (Chapter 2) are derived for both the real and complex cases. To this reviewer's knowledge, the results obtained for the complex case (Chapters 3 and 4) are new, their derivations usually paralleling their real counterparts. Chapter 5 deals with real-valued scalar functions of matrix argument in the real case; hypergeometric functions are defined in terms the generalized Laplace transform, zonal polynomials and the M-transform and scalar functions of many matrix arguments are introduced, among other topics. Real-valued scalar functions of one or more matrices in the complex case are discussed in Chapter 6.

The material is developed progressively from first principles, lending itself to self-study. The theory is often illustrated with examples taken from distribution theory. Interesting applications and extensions of the results are submitted as exercises at the end of each section. The book also contains a thorough glossary of symbols. Those wishing to further their knowledge of the materials discussed and related topics will find the representative references listed at the end of each section eminently useful. This book could well serve as a primary text for a one-semester graduate course on Jacobians and functions of matrix argument.

In addition to being a valuable reference source in a variety of disciplines such as statistics, engineering, econometrics and physics, this monograph should stimulate research based on multivariable calculus.

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New and Forthcoming Books on Linear Algebra and Related Topics: 1998–1999

Listed here are some new and forthcoming books on linear algebra and related topics that have been published in 1998 or in 1999, or are scheduled for publication in 1999. This list updates and augments our previous list in IMAGE 21 (October 1998), pp. 8–9. The references to CMP (*Current Mathematical Publications*) and MR (*Mathematical Reviews*) were found using MathSciNet and to Zbl (*Zentralblatt für Mathematik/MATH Database*) by visiting the Zbl Web site: http://www.emis.de/cgi-bin/MATH (three free hits to Zbl are allowed for non-subscribers).

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Feliks Ruvimovich Gantmakher and *The Theory of Matrices*

With the republication [E1-IR, E1-IIR] of the English translation by K. A. Hirsch of Gantmacher's well-known work *The Theory of Matrices*, we would like to present here a brief biographical sketch of Gantmacher and some bibliographical information on *The Theory of Matrices*.

Feliks Ruvimovich Gantmakher [Gantmacher] (1908–1964) was a research scientist whose work in both pure and applied mathematics yielded major contributions of wide-ranging theoretical, engineering, and pedagogical importance. [For his research in rocketry, he was awarded the Military Order of the Red Star and for his research in external ballistics, a State Prize of the first degree.]

Gantmacher was born in Odessa on February 23, 1908. After completing gymnasium, his serious education in mathematics began at home under "experienced tutors". At the age of 15, he entered the Odessa Institute of National Education, where he was impressed with the clear, elegant and expressive lectures of S. O. Shatunovski. At the same Institute, he was also impressed with the lectures of the famous engineer G. K. Suslov in applied mathematics and N. G. Chebotarev in pure mathematics: infinite groups and the theory of analytic functions. Suslov helped Gantmacher to obtain a research fellowship in the Applied Mathematics section of the Scientific Research Department of the Odessa Institute and he began teaching there in 1927, the year his research fellowship began. Here began his lifelong project of developing a course of theoretical applied mathematics, the main subject of his distinguished teaching career.

During the summer of 1926 (at the age of 18), Gantmacher wrote his first research paper. This paper "was devoted to a concise exposition of a number of basic relations in affine differential geometry on n-dimensional surfaces in $(n+1)$-dimensional spaces". Beginning in 1929, he published a number of research papers and monographs, co-authored with M. G. Kreĭn, on the theory of matrices and integral equations. These papers introduced and studied an important class of matrices and kernels, called oscillatory. In 1934 Gantmacher entered the Steklov Mathematical Institute, where he presented his doctoral thesis on the theory of semi-simple Lie groups (1938). By 1947, he had become a Professor, and from 1953 he was Head of the Department of Theoretical Applied Mathematics of the Moscow Physical-Technical Institute.

His research in the theory of matrices alone would have earned for him a place of distinction in mathematical history. The fundamental monograph *Teoriya Matrits* (*The Theory of Matrices*), first published in Russian in 1953 [R1], was based on research that took place over many years. This work is widely recognized as a classic in the field and has been translated into English, French, and German. In his detailed review of [R1], Joel L. Brenner [MR 16:438 (1953)] wrote that "The number of subjects which the book treats well is great. Since many of these subjects are in fields commonly classified as applied mathematics, an early translation of the book would appeal to a wide audience". In fact Brenner himself translated the second part of [R-1] in 1959 "adding references, an index and four appendices; also correcting misprints, simplifying a few proofs and making the Second Part of [R-1] a self-contained whole. The translation by K. A. Hirsch is of the complete Russian text with new versions of several paragraphs communicated by the author. The work is an outstanding contribution to matrix theory and contains much material not be found in any other text" [MR 21:6372c (1959), unsigned].

The supplements in the Second Supplemented Edition [R2] comprise, as Brenner noted in his review [MR 34:2585 (1966)], " a chapter on nonsingularity conditions and eigenvalue location theorems, and an appendix by V. B. Lidskiĭ on inequalities concerning singular and proper values. Several other chapters have also been amplified. The appearance of this edition posthumously—Gantmacher died on May 16, 1964—is due to Lidskiĭ's initiative ... The appendix by Lidskiĭ is a beautifully written, self-contained exposition of many important results." The only translation of [R2] appears to be the (now out-of-print) 1986 translation [G4] in German by Helmut Boseck, Dietmar Soyka and Klaus Stengert, with a preface by D. P. Želobenko.

Much of the information above comes from the obituary article by M. A. Aizerman, M. G. Kreĭn, L. A. Lyustenik, M. A. Naimark, L. C. Pontryagin, C. L. Sobolev, C. A. Khristianovich, and Ya B. Shor (in English, translated from the Russian by F. W. Ponting, *Russian Mathematical Surveys* 20, 143–151, 1965). We are also grateful to John Kimmel, Erkki Liski, Heinz Neudecker and Sanjo Zlobec for their help. The photograph is reproduced from the dust-jacket of [R2].

We now present a (possibly) complete listing of all versions of *The Theory of Matrices* by F. R. Gantmacher (only [G4] has the author listed as Felix R. Gantmacher on the title page).

[R1] *The Theory of Matrices.* In Russian. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953, 491 pp. [MR 16:438, Zbl 050.24804].

[R2] *The Theory of Matrices.* In Russian. With an Appendix by V. B. Lidskiĭ. 2nd Supplemented Edition. Izdat. "Nauka", Moscow, 1966, 576 pp. [MR 34 #2585, Zbl 145.03604]. (The "supplements" comprise "a chapter on nonsingularity conditions and eigenvalue location theorems and the Appendix by V. B. Lidskiĭ on inequalities concerning singular and proper values".)

[R3] *The Theory of Matrices.* In Russian. With an Appendix by V. B. Lidskiĭ. 3rd Supplemented Edition. Izdat. "Nauka", Moscow, 1967, 576 pp. (Apparently just a corrected reprint of [R2].)

[R4] *The Theory of Matrices.* In Russian. With an Appendix by V. B. Lidskiĭ. 4th Supplemented Edition. Izdat. "Nauka", Moscow, 1988, 549 pp. ("Basically a reprint of [R2]" [Zbl 666.15002] but with apparently fewer pages.)

[E1-I] *The Theory of Matrices: Volume 1.* In English. Translated from the Russian by K. A. Hirsch. Chelsea, New York, 1959, x + 374 pp., ISBN 0-8284-0133-0 [MR 21:6372c]. (1st part of two-volume set; translation of Chapters 1–10 of [R1].)

[E1-II] *The Theory of Matrices: Volume 2.* In English. Translated from the Russian by K. A. Hirsch. Chelsea, New York, 1959, x + 276 pp., ISBN 0-8284-0131-4 [MR 21:6372c, Zbl 085.01001]. (2nd part of two-volume set; 1st English translation of Chapters 11–14 of [R1].)

[E2-II] *Applications of the Theory of Matrices.* In English. Translated from the Russian by Joel L. Brenner, with the assistance of D. W. Bushaw & S. Evanusa. Interscience, New York, 1959, ix + 317 pp. [MR 21:6372b]. (2nd English translation of Chapters 11–14 of [R1].)

[E1-IR] *The Theory of Matrices: Volume 1.* In English. Reprint Edition. Translated from the Russian by K. A. Hirsch. AMS Chelsea Publishing, Providence, RI, 1998, x + 374 pp., ISBN 0-8218-1376-5 (2-vol. set: ISBN 0-8218-1393-5) [MR 99f:15001, Zbl 990.01407]. (Reprint of [E1-I].)

[E1-IIR] *The Theory of Matrices: Volume 2.* In English. Reprint Edition. Translated from the Russian by K. A. Hirsch. AMS Chelsea Publishing, Providence, RI, 1998, x + 276 pp., ISBN 0-8218-0133-0 (2-vol. set: ISBN 0-8218-1393-5). (Reprint of [E1-II].)

[F1] *Théorie des matrices: Tome 1, Théorie générale.* In French. Translated from the Russian by Ch. Sarthou. Collection Universitaire de Mathématiques, vol. 18, 1966, xiii + 370 pp., Dunod [MR 37:1381a]. (1st part of two-volume set; translation of Chapters 1–10 of [R1].)

[F2] *Théorie des matrices: Tome 2, Questions spéciales et applications.* In French. Translated from the Russian by Ch. Sarthou. Collection Universitaire de Mathématiques, vol. 19, 1966, xii + 268 pp., Dunod [MR 37:1381b, Zbl 136.00410]. (2nd part of two-volume set; 1st translation of Chapters 11–14 of [R1].)

[G1-I] *Matrizenrechnung: Teil I, Allgemeine Theorie* In German. [1st Edition.] Translated from the Russian by Klaus Stengert. Hochschulbücher für Mathematik, vol. 36. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958, xi + 324 pp. [MR 20:3884, Zbl 079.01106/7]. (1st part of two-volume set; translation of the first ten chapters of [R1].)

[G1-II] *Matrizenrechnung: Teil II, Spezielle Fragen und Anwendungen.* In German. [1st Edition.] Translated from the Russian by Klaus Stengert. Hochschulbücher für Mathematik, vol. 37. VEB Deutscher Verlag der Wissenschaften, 1959, vii + 244 pp. [MR 21:6372a, Zbl 085.00904]. (2nd part of two-volume set; translation of Chapters 11–14 of [R1].)

[G2-I] *Matrizenrechnung: Teil I, Allgemeine Theorie* In German. 2nd Corrected Edition. Translated from the Russian by Klaus Stengert. Hochschulbücher für Mathematik, vol. 36. VEB Deutscher Verlag der Wissenschaften, Berlin, 1965, xi + 324 pp. [MR 34:5838].

[G2-II] *Matrizenrechnung: Teil II, Spezielle Fragen und Anwendungen.* In German. 2nd Corrected Edition. Translated from the Russian by Klaus Stengert. Hochschulbücher für Mathematik, vol. 37. VEB Deutscher Verlag der Wissenschaften, Berlin, 1965, vii + 244 pp.

[G3-I] *Matrizenrechnung: Teil I, Allgemeine Theorie* In German. 3rd Edition. Translated from the Russian by Klaus Stengert. [1st Vol. of Two-Volume Set.] Hochschulbücher für Mathematik, vol. 36. VEB Deutscher Verlag der Wissenschaften, 1970, xi + 324 pp. [MR 43:4834, Zbl 223.15001]. (Apparently just a corrected reprint of [G2-I].)

[G3-II] *Matrizenrechnung: Teil II, Spezielle Fragen und Anwendungen.* In German. 3rd Edition. Translated from the Russian by Klaus Stengert and edited by Helmut Boseck. Hochschulbücher für Mathematik, vol. 37. VEB Deutscher Verlag der Wissenschaften, Berlin, 1970, vii + 244 pp. [Zbl 322.15004]. (Apparently just a corrected reprint of [G2-II].)

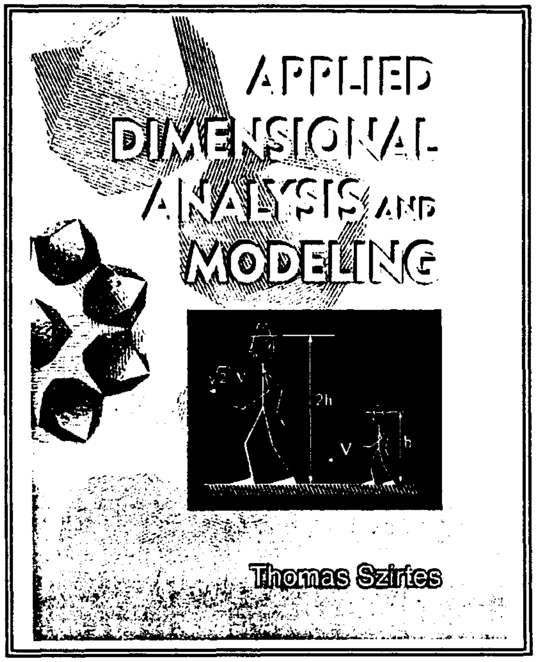
[G4] *Matrizentheorie* by Felix R. Gantmacher. In German. With an Appendix by V. B. Lidskij & a Preface by D. P. Želobenko. [4th Edition.] Translated from the Russian [R2] by Helmut Boseck, Dietmar Soyka & Klaus Stengert. Hochschulbücher für Mathematik, vol. 86, VEB Deutscher Verlag der Wissenschaften, Berlin & Springer-Verlag, Berlin, 654 pp., ISBN 3-326-0001-4 [MR 87i:15001, Zbl 588.15001] & ISBN 3-540-16582-7 [MR 87k:15001]. (Out of print.)

APPLIED DIMENSIONAL ANALYSIS AND MODELING

by Thomas Szirtes

McGraw-Hill, New York, 1998; ISBN: 007-062811-4

815 pages, 200 illustrations, 250 + worked-out examples and case studies; US$99.95

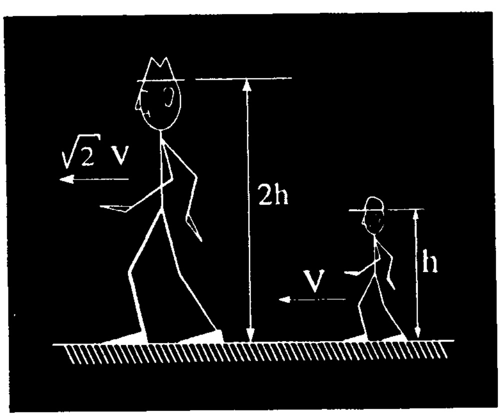


With a most creative use of linear algebra - specifically matrix arithmetic - this book presents the "*dimensional method*", which is an enormously potent tool in design, analyses and tests of engineering and scientific systems and products. The method is also efficient in forming relations among variables by non-analytic means, in verifying analytical formulas, and in the logistics of graphical presentations of multivariable relations.

The most important application of the dimensional method, however, is *model experimentation*, for which the determination of case-specific *Model Laws* and *Scale Factors* are mandatory. These laws and factors are all based on *dimensionless variables*. The book presents a highly elegant and fast method, based on matrix algebra, by which dimensionless variables are generated, and the relevant Model Laws and Scale Factors easily determined.

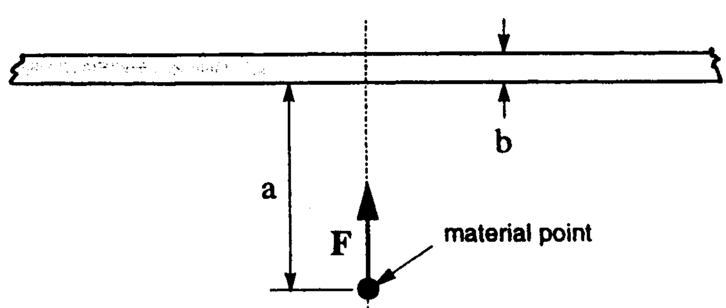
This in-depth landmark reference work also includes over 250 numerical examples and case-studies drawn from the theory and practical utilisation of dimensional method in applied science, engineering, biomechanics, astronomy, geometry and economics.

Did you know that a man *100 %* taller than a child, walks 'most comfortably' a mere *41.4* % faster?



Did you know that the *more* the air in a soap bubble, the *smaller* its internal pressure?

Did you know that the *distance* 'a' of a material point to a large plate has *no effect* on the gravitational force F that the plate imparts on the material point?



These startling facts are all true, as you can easily convince yourself without any analysis, only by some nice matrix arithmetic explained fully in

APPLIED DIMENSIONAL ANALYSIS AND MODELING.

ILAS5.SAM

Seventh International Workshop on Matrices and Statistics, in Celebration of T. W. Anderson's 80th Birthday

Fort Lauderdale, Florida, USA: December 11–14, 1998

This Workshop was organized by N. D'Alessio, R. W. Farebrother, W. D. Hammack, S. Puntanen, D. S. Simon, G. P. H. Styan (chair), H. J. Werner (vice-chair), F. Zhang (local chair), and supported, in part, by the International Linear Algebra Society, Nova Southeastern University, and the Statistical Society of Canada. The photograph on pp. 16–17 is by Simo Puntanen.

The following persons participated:

Yasuo AMEMIYA, Iowa State University, Ames
Dorothy ANDERSON, Stanford, California
Theodore W. ANDERSON, Stanford University, California
Jerry BARTOLOMEO, Nova Southeastern University
Philip V. BERTRAND, Sheffield Hallam University, England
Barbara BONDAR, Ottawa, Canada
James V. BONDAR, Carleton University, Ottawa, Canada
Laura BORYSEWICZ, Nova Southeastern University
N. Rao CHAGANTY, Old Dominion University, Norfolk, VA
Xiao-Wen CHANG, McGill University, Montréal, Canada
Pinyuen CHEN, Syracuse University, New York
John S. CHIPMAN, University of Minnesota, Minneapolis
Roy S. CHOUDHURY, Orlando, Florida
Ka Lok CHU, McGill University, Montréal, Canada
Knut CONRADSEN, Technical University of Denmark, Lyngby
Carles M. CUADRAS, University of Barcelona, Spain
Naomi D'ALESSIO, Nova Southeastern University
Maryse DANSEREAU, Montréal, Canada
Anne DURBIN, London, England
James DURBIN, London School of Economics, London, England
Friedhelm EICKER, Universität Dortmund, Germany
R. William FAREBROTHER, University of Manchester, England
Sheila FAREBROTHER, Shrewsbury, England
Nancy FLOURNOY, American University, Washington, DC
Bernhard D. FLURY, Indiana University, Bloomington
Leah FLURY, Bloomington, Indiana
Steve K. GALITSKY, Spr Inc., Oak Brook, Illinois
Alicia GIOVINAZZO, Nova Southeastern University
Patricie GOMEZ, Cali, Colombia
Jürgen GROß, Universität Dortmund, Germany
Frank J. HALL, Georgia State University, Atlanta
William D. HAMMACK, Nova Southeastern University
Greta HANDING, Phoenix, Arizona
Rebekka HANDING, Phoenix, Arizona
Robert E. HARTWIG, North Carolina State University, Raleigh
David A. HARVILLE, IBM–Watson Res.Center, Yorktown Hghts
Berthold HEILIGERS, Universität Magdeburg, Germany
Harri HIETIKKO, University of Tampere, Finland
Kaisa HUUHTANEN, Tampere, Finland
Pentti HUUHTANEN, University of Tampere, Finland
Shane T. JENSEN, McGill University, Montréal, Canada
André KLEIN, University of Amsterdam, The Netherlands
Tõnu KOLLO, University of Tartu, Estonia
Lynn R. LaMOTTE, Louisiana State University, Baton Rouge
Zhongshan LI, Georgia State University, Atlanta
Erkki P. LISKI, University of Tampere, Finland
Asmaâ MANSOUR, McGill University, Montréal, Canada
Augustyn MARKIEWICZ, Agricultural Univ. of Poznań, Poland
Jyrki MÖTTÖNEN, Tampere Univ. of Technology, Finland
Anne MULDER, Nova Southeastern University
Irja & Tuulia NUMMI, Tampere, Finland
Tapio NUMMI, University of Tampere, Finland
Ingram OLKIN, Stanford University, California
Matjaž OMLADIČ, University of Ljubljana, Slovenia
Vesna OMLADIČ, University of Ljubljana, Slovenia
Yuriko OSHIMA-TAKANE, Montréal, Canada
Christopher C. PAIGE, McGill University, Montréal, Canada
Marianna PENSKY, University of Central Florida, Orlando
Friedrich PUKELSHEIM, Universität Augsburg, Germany
Monika PUKELSHEIM, Augsburg, Germany
Simo PUNTANEN, University of Tampere, Finland
Lianfen QIAN, Florida Atlantic University, Boca Raton
Bhargavi RAO, University Park, Pennsylvania
C. Radhakrishna RAO, Pennsylvania State University
Albert SATORRA, Universitat Pompeu Fabra, Barcelona, Spain
Elke & Siegmar SCHLEIF, Dortmund, Germany
Shayle R. SEARLE, Cornell University, Ithaca, New York
Peter ŠEMRL, University of Ljubljana, Slovenia
Mohammed SHAKIL, Pembroke Plains, Florida
David S. SIMON, Nova Southeastern University
Bimal Kumar SINHA, University of Maryland–Baltimore
T. N. SRIVASTAVA, Concordia University, Montréal, Canada
Michael A. STEPHENS, Simon Fraser Univ., Burnaby, BC, Canada
Evelyn Matheson STYAN, Montréal, Canada / Franklin, Vermont
George P. H. STYAN, McGill University, Montréal, Canada
Gerald E. SUBAK-SHARPE, City College, New York
Yijun SUN, University of Maryland–Baltimore
Yoshio TAKANE, McGill University, Montréal, Canada
Yongge TIAN, Concordia University, Montréal, Canada
Imbi TRAAT, University of Tartu, Estonia
Birgit TRENKLER, Dortmund, Germany
Götz TRENKLER, Universität Dortmund, Germany
Michel VAN DE VELDEN, Univ. Amsterdam, The Netherlands
Hrishikesh D. VINOD, Fordham University, Bronx, New York
Júlia VOLAUFOVÁ, Louisiana State University, Baton Rouge
Heinrich VOSS, Tech.Universität Hamburg–Harburg, Germany
Cheng WANG, Davie, Florida
Tonghui WANG, New Mexico State University, Las Cruces
William E. WATKINS, California State University, Northridge
Hans Joachim WERNER, Universität Bonn, Germany
Haruo YANAI, Center for Univ. Entrance Exam., Tokyo, Japan
Qiang YE, University of Manitoba, Winnipeg, Canada
Fuzhen ZHANG, Nova Southeastern University
Yijun ZUO, Arizona State University, Tempe

Seventh International Workshop on
T. W. Anderson's 80th birthday: Fort L

es and Statistics, in celebration of
ale, Florida, 11–14 December 1998

Selected Forthcoming Linear Algebra Events

We have selected seven forthcoming linear algebra events, scheduled as follows:

- June 7–8, 1999: Regina, Saskatchewan
- July 15–16, 1999: Coimbra, Portugal
- July 19–22, 1999: Barcelona, Spain
- August 6–7, 1999: Tampere, Finland
- June 26–28, 2000: Nafplio, Greece
- July 12–14, 2000: Leuven, Belgium
- December 9–13, 2000: Hyderabad, India.

For information on other linear algebra events please visit the ILAS/IIC Web site:

http://www.math.technion.ac.il/iic/conferences.html

Special Session on the Interaction Between Statistics and Matrix Theory

Regina, Saskatchewan: June 7–8, 1999

Matrix methods in statistics have received some attention in recent years; witness the ongoing series of International Workshops on Matrices and Statistics, which has been running since 1990. At the 1999 Annual Meeting of the Statistical Society of Canada (SSC), which will be held at the Univ. of Regina, June 6–9, 1999, there will be a Special Session on the Interaction Between Statistics and Matrix Theory. This Session will feature six 30-minute lectures on a broad spectrum of topics, as follows:

- S. Ejaz AHMED (Univ. of Regina): "Least squares, preliminary test and Stein-type estimation in general vector $AR(p)$ models" (with A. K. Basu)
- Kjell A. DOKSUM (Univ. of California–Berkeley): "Partial regression curves"
- Carl D. MEYER, Jr. (North Carolina State University): "Aggregation/disaggregation error for Markov chains"
- François PERRON (Univ. de Montréal): "An exponential bound for a $0-1$ function of a reversible Markov chain"
- Serge B. PROVOST (Univ. of Western Ontario): "Hilbert matrices and density estimation" (with Y.-H. Cheong)
- George P. H. STYAN (McGill University): "Some comments and a bibliography on the Craig-Sakamoto Theorem" (with S. W. Drury & Mylène Dumais).

The aims of the session are to highlight the interaction between statistics and matrix theory, and to promote dialogue between researchers in both areas. For further information please contact:

- Ejaz AHMED: ahmed@math.uregina.ca,
- Doug FARENICK: farenick@math.uregina.ca
- Steve KIRKLAND: kirkland@math.uregina.ca.

For the complete SSC Annual Meeting schedule please visit the SSC Conference Web site:

http://www.math.uregina.ca/SSC99/.

Workshop on Geometric & Combinatorial Methods in the Hermitian Sum Spectral Problem

Coimbra, Portugal: July 15–16, 1999

A problem in matrix theory which has interested mathematicians for many years is the following: Given two Hermitian matrices A and B, describe the spectrum of $A+B$ in terms of the spectra of A and B. Recently there were decisive developments in work on this problem, with contributions from algebraic geometry, representation theory, combinatorics and harmonic analysis.

This Workshop on Geometric & Combinatorial Methods in the Hermitian Sum Spectral Problem will gather experts from different fields who have worked on this problem and will take place just before the Barcelona ILAS meeting, July 19–22, 1999.

The provisional list of speakers is as follows:

- Jane DAY, San Jose State University, San Jose, Calif.
- Shmuel FRIEDLAND, Univ. of Illinois, Chicago
- Alexander KLYACHKO, Bilkent Univ., Ankara, Turkey
- Allen KNUTSON, Brandeis University, Waltham, Mass.
- Norman WILDBERGER, Univ. New South Wales, Sydney
- Andrei ZELEVINSKY, Northeastern University, Boston.

Organizing committee: E. Marques de Sá, João F. Queiró, Ana P. Santana. *Sponsors:* CIM, CMUC, Praxis XXI. For more information e-mail: cmuc@mat.uc.pt and/or visit our Web site:

http://www.mat.uc.pt/~jfqueiro/wrkshp2.html

The Eighth ILAS Conference

Barcelona, Spain: July 19–22, 1999

You are kindly invited to attend the 8th Conference of the International Linear Algebra Society (ILAS), which will be held at the *Escola Tècnica Superior d'Enginyers Industrials* in Barcelona (ETSEIB). The subject of the Conference is Linear Algebra in a broad sense, including applications.

Founded in the third century BC, Barcelona today is a large, thriving and important city and port with a population of about 2 million people. In the center of Barcelona is the Plaza de Cataluña, a square—the largest in Spain—rimmed with trees and highlighted by sculptures and fountains. Ramble along *La Rambla*, the tree-lined boulevard that streches for 2 kimometres from the Plaza to the port—it is the very pulse of Barcelona.

The chair of the Organizing Committee is Ferran Puerta of Barcelona: puerta@ma1.upc.es. The co-chair for programs is Tom Laffey and the co-chair for local arrangements is M^a Isabel Garcia-Planas. The other members of the Organizing Committee are: Raphael Bru, Luz DeAlba, J. M. Gracia, V. Hernández, Nick Higham, Roger A. Horn, Paul M. Van Dooren, and Richard A. Brualdi—*ex officio*. The Local Committee consists of: J. Clotet, A. Compta, M. D. Magret, and X. Puerta.

The program will include invited talks of 50 and 30 minutes. The following speakers have agreed to participate: Z. Bai, J. Ferrer, D. Hinrichsen, V. Kaashoek, S. Kirkland, Chi-Kwong Li, N. Mackey, E. Marques de Sá, K. Murota, V. Pták, F. Silva Leite, A. Urbano, and I. Zaballa.

The program will also include several minisymposia. Their titles and organizers are as follows:

- *Linear Systems and Polynomials* (J. M. Gracia)
- *Total Positivity* (T. Ando, J. Garloff)
- *Parallel Asynchronous Methods* (A. Frommer, D. Szyld)
- *Parametrization Problems in Linear Algebra and Systems Theory* (P. Fuhrmann, U. Helmke)
- *Combinatorial Matrix Theory* (H. Schneider, B. Shader)
- *Special Session in Memory of Robert C. Thompson* (J. Day).

A special issue of *Linear Algebra and its Applications* (LAA) will publish the Conference Proceedings. This issue will contain those papers which meet the standards of LAA and are approved through the normal refereeing processes. Submission deadline for papers is November 30, 1999.

The registration form and further information are available in the Conference Web page:

http://www-ma1.upc.es/ilas99/8thILAS99.html

by e-mail from: ilas99@ma1.upc.es. Please send the Registration Form by FAX to 34-93-4011713, by e-mail to: register@ma1.upc.es, or by regular mail to the 8th ILAS'99 Conference Secretariat, Dep. de Matemática Aplicada I, ESTEIB-UPC, Av. Diagonal 647, E-08028 Barcelona, Spain.

The Eighth International Workshop on Matrices and Statistics

Tampere, Finland: August 6–7, 1999

The Eighth International Workshop on Matrices and Statistics will be held at the Univ. of Tampere, Finland, on Friday, August 6 and Saturday, August 7, 1999 (starting with a reception on the Thursday evening, August 5). This Workshop is a Satellite of the 52nd Session of the International Statistical Institute (ISI) to be held in Helsinki from Tuesday, August 10 through Wednesday, August 18, 1999. The sponsors are: Academy of Finland, Finnair—The Official Carrier of the Workshop, Foundation of the Promotion of the Actuarial Profession, International Linear Algebra Society (support awarded for the ILAS Lecturer), Nokia, and the Tampere University Foundation.

The Keynote Speaker for this Workshop is:

★ ANDERSON, T. W., Stanford University, USA: "Canonical analysis and reduced rank regression in autoregressive models".

The Invited Speakers are:

- ANDO, T., Hokusei Gakuen University, Sapporo, Japan: "Arithmetic-geometric mean inequalities for matrices"
- CHRISTENSEN, Ronald, Univ. of New Mexico, Albuquerque, USA: "Checking the independence assumption in linear models by control charting residuals"
- GOLUB, Gene H. (ILAS Lecturer), Stanford University, USA: "Inverting shape from moments"
- MUSTONEN, Seppo, Univ. of Helsinki: "Matrix computations in Survo"
- PUKELSHEIM, Friedrich, Universität Augsburg, Germany: "Kiefer ordering of simplex designs for second-degree mixture models with four or more ingredients" (with Norman R. Draper and Berthold Heiligers)
- SCOTT, Alastair J., University of Auckland, New Zealand: "An extension of Richards's Lemma, with applications to a class of profile likelihood problems"
- SEARLE, Shayle R., Cornell University, Ithaca, NY, USA: "The infusion of matrices into statistics"
- SINHA, Bimal K., Univ. of Maryland–Baltimore County, USA: "Nonnegative estimation of multivariate variance components in unbalanced mixed models"
- STYAN, George P. H., McGill University, Montreal, Canada: "Revisiting Hua's matrix equality and related inequalities, Schur complements, and Sylvester's law of inertia" (with Christopher C. Paige, Bo-Ying Wang & Fuzhen Zhang).

The International Organizing Committee for this Workshop comprises R. William Farebrother (Victoria Univ. of Manchester, England), Simo Puntanen (Univ. of Tampere, Tampere, Finland; chair), George P. H. Styan (McGill University, Montréal, Québec, Canada; vice-chair), and Hans Joachim Werner (Univ. of Bonn, Bonn, Germany). The Local Organizing Committee at the Univ. of Tampere comprises Riitta Järvinen, Erkki Liski, Tapio Nummi, and Simo Puntanen (chair).

Deadlines: Early registration: 1 June 1999; Registration fee FIM 400 (= ca. US$ 80)—after 1 June 1999, add FIM 100; Full-time students: FIM 50. For further information, contact: The Workshop Secretary, Dept. of Mathematics, Statistics & Philosophy, Univ. of Tampere, PO Box 607, FIN-33101 Tampere, Finland; workshop99@uta.fi, FAX +358-3-215-6157, http://www.uta.fi/~sjp/workshop99.html.

A preliminary list of talks by students is as follows:

- Alpargu, Gülhan, McGill U, Montréal, Canada "Matrices in statistical procedures for valid hypothesis testing in regression models with autocorrelated errors"
- Borisov, Mikhail Vladimirovich, Novosibirsk, Russia "The investigation of the stability of two statistical estimators to complex disturbances in the data"
- Dumais, Mylène, McGill U, Montréal, Canada "The Craig-Sakamoto theorem"
- Fugate, Michael, Los Alamos, New Mexico, USA "Adjusted means plots for detecting lack of fit in linear models"
- Grzadziel, Mariusz, Agricultural U Wrocław, Poland "On nonnegative quadratic estimation in linear models"
- Guerra Ones, Valia, I of Math, Cybernetics & Physics, Havana, Cuba "A technique for the numerical calculation of the Moore-Penrose generalized inverse of an ill-conditioned matrix"
- Helle, Sami, U Tampere, Finland "Modeling of the fiber length distribution of mechanical pulp using generalized linear models"
- Jensen, Shane T., McGill U, Montréal, Canada "Some comments and a bibliography on the von Szökefalvi Nagy-Popoviciu and Nair-Thomson inequalities, with extensions and applications to statistics and matrix theory"
- Klein, Thomas, U Augsburg, Germany "Optimal designs for second degree K-model mixture experiments"
- Lu, Chang-Yu, Northeast Normal U, Changchun, Jilin, China "Some matrix results related to admissibility in linear models and generalized matrix versions of Wielandt inequality"
- Luoma, Arto, U Tampere, Finland "Distance criterion in theory of optimal design"
- Nurhonen, Markku, U Minnesota, Minneapolis, USA "Matrices related to diagnostic methods in classification problems"
- Ollikainen, Jyrki, U Tampere, Finland "Comparing traditional cluster methods to neural networks with large medicine data"
- Tian, Yongge, Concordia U, Montréal, Canada "When do matrix equations have block triangular solutions?"
- Wei, Yimin, Fudan U, Shanghai, China "The modification of the Drazin inverse"
- Zhang, Bao-Xue, Northeast Normal U, Changchun, Jilin, China "Some results on estimator of the mean vector and variance in the general linear model and estimating problems for location (scale) models from grouped samples under order restrictions"

A preliminary list of contributed papers is as follows:

- Ahmed, S. E., U Regina, Canada "Improved estimation of characteristic roots in principal analysis"
- Akdeniz, Fikri, U Çukurova, Adana, Turkey "The estimation and analysis of residuals for some biased estimators in linear regression" (Poster)
- Benesch, Thomas, Graz U Technology, Austria "Internet survey by direct mailing"
- Borisov, Mikhail Vladimirovich, Novosibirsk, Russia "The investigation of the stability of two statistical estimators to complex disturbances in the data"
- Chaganty, N. Rao, Old Dominion U, Norfolk, USA "Statistical analysis of some multivariate models using quasi-least squares"
- Chikuse, Yasuko, Kagawa U, Takamatsu City, Japan "Statistical inference on special manifolds"
- Drygas, Hilmar, U Kassel, Germany "Simultaneous SVD: Grillenberger's approach"
- Farebrother, R. W., U Manchester, UK "Optimality results for the method of averages"
- Fellman, Johan, Swedish S of Econ & Bus Admin., Helsinki, Finland "Robust estimators in linear models"
- Gnot, Stanisław, Pedagogical U, Zielona Góra, Poland "Estimation of variance components by maximum likelihood method in certain mixed linear models"
- Groß, Jürgen, U Dortmund, Germany "The Moore-Penrose inverse of a partitioned nonnegative definite matrix"
- Hauke, Jan, Adam Mickiewicz U, Poznań, Poland "On some matrix inequalities and their statistical applications"
- Hovhannissian, Hrant, Northern Dept of NSSP, Gyumri, Armenia "Solving algorithm for the system of linear algebraic equations with five-diagonal, cyclic matrixes"
- Ishak, Maged, U Newcastle, NSW, Australia "Regression methods to model under registration problems in population data"
- John, J. A., U Waikato, Hamilton, New Zealand "Recursive formulae for the average efficiency factor in alpha designs"
- Kasala, Subramanyam, U North Carolina, Wilmington, USA "An exact joint confidence region and test in multivariate calibration"
- Klein, André, U Amsterdam, The Netherlands "A generalization of Whittle's formula"
- Kollo, Tonu, U Tartu, Estonia "Approximation of the distribution of the sample correlation matrix"
- Lee, Alan J., U Auckland, New Zealand "Generating correlated categorical variates via Kronecker products"
- Liu, Shuangzhe, U Basel, Switzerland "On maximum likelihood estimation for the VAR-VARCH model"
- Lomov, Andrei, Sobolev I of Mathematics, Novosibirsk, Russia "Identifiability of a matrix parameter by the stochastically perturbed right null space"
- Markiewicz, Augustyn, Agricultural U Poznań, Poland "Robust linear estimation in an incorrectly specified restricted linear model"
- Maroulas, John, Nat. Tech U, Athens, Greece "A geometric approach to the canonical correlations in two- and three-way layouts"
- Mathew, Thomas, U Maryland, Baltimore County, Catonsville, USA "Tolerance regions and simultaneous tolerance regions in multivariate regression"
- Merikoski, Jorma K., U Tampere, Finland "Simple and good bounds for the Perron root"
- Naumov, Anatoly, Inst. for Medical Statistics and Epidemiology, Munich, Germany "New models of optimal portfolio"

- Nechval, Nicholas A., Aviation U Riga, Latvia "Estimating the true order of a multivariate ARMA model via matrix determinant minimization"
- Nechval, Konstantin N., Aviation U Riga, Latvia "Uniformly undominated estimation of a multivariate normal mean via invariant embedding technique"
- Paramonov, Yuri M., Aviation U Riga, Latvia "Order statistics covariance matrix use for maximum expectation of P-bound for random variable"
- Polasek, Wolfgang, U Basel, Switzerland "Outliers and break points in multivariate time series"
- Puntanen, Simo, U Tampere, Finland "More on partitioned linear models"
- Shakil, Mohammad, Pembroke Pines, Florida, USA "On complexity bounds for some optimization problems"
- Stępniak, Czesław, Agricultural U Lublin, Poland "Information channels and comparison of linear experiments"
- Tiit, Ene-Margit, U Tartu, Estonia "Some extremal properties of correlation matrices"
- Todoran, Dorin, U North Baia Mare, Romania "Mathematical modelling and robot precision"
- Trenkler, Götz, U Dortmund, Germany "Powers and roots of quaternions"
- Troschke, Sven-Oliver, U Dortmund, Germany "Quaternions: A matrix oriented approach"
- Vichi, Maurizio, Societa Italiana di Statistica, Roma, Italy "Least squares approximation of non positive (semi-) definite matrices of correlation coefficients"
- Werner, Hans Joachim, U Bonn, Germany "More on the linear aggregation problem"
- Zaihraiev, Oleksandr, N. Copernicus U, Toruń, Poland "Distance optimality design criterion and stochastic majorization"
- Zmyślony, Roman, Tech U Zielona Góra, Poland "Testing hypotheses for parameters in mixed linear models"
- Zontek, Stefan, Tech U Zielona Góra, Poland "On construction of a basis of Jordan algebra of matrices with applications to a special linear model."

The Fifth Workshop on Numerical Ranges and Numerical Radii

Nafplio, Greece: June 26–28, 2000

The 5th Workshop on "Numerical Ranges and Numerical Radii" will be held in the historical and enjoyable town of Nafplio (Nauplia) in the Peloponnese, Greece, from Monday, June 26 till Wednesday, June 28, 2000, starting with a reception on Sunday evening, June 25. The purpose of the workshop is to stimulate research and to foster the interaction of researchers on the subject. The informal workshop atmosphere will guarantee the exchange of ideas from different research areas and in particular, the researchers may be better informed on the latest developments and the newest techniques.

There will be no registration fees and it is not possible to provide financial support to the participants. All participiants though will have the chance, if they wish, to attend the social events during the workshop without charge. Information about accomodation will be available by December 1999. If you plan to attend the workshop or if you want to receive more information, please contact the organizer : John MAROULAS, Dept. of Mathematics, National Technical University, Zografou Campus, Athens 15781, Greece; maroulas@math.ntua.gr.

Deadlines: To confirm your participation at the workshop : 1 January 2000. To reserve your accomodation : 1 March 2000. To submit the title and absract of your talk : 1 April 2000.

The Third International Conference on Matrix-Analytic Methods in Stochastic Models

Leuven, Belgium: July 12–14, 2000

This conference will provide an international forum for the presentation of recent results on matrix-analytic methods in stochastic models. Its scope includes development of the methodology as well as the related algorithmic implementations and applications in communications, production and manufacturing engineering; it also includes computer experiments in the investigation of specific probability models.

The topics of interest include but are not limited to: methodology, general theory, computational methods, computer experimentation, queueing models, telecommunications modeling, spatial processes, reliability problems, risk analysis, and production and inventory models.

The organizers wish to encourage students to attend the conference. To that effect, financial assistance will be made available on a limited basis and a streamlined submission procedure will be implemented. Details will be published on the conference Web page: http://www.econ.kuleuven.ac.be/mam3. Queries should be addressed to MAM3@econ.kuleuven.ac.be.

The Ninth International Workshop on Matrices and Statistics, in Celebration of C. R. Rao's 80th Birthday

Hyderabad, India: December 9–13, 2000

The Ninth International Workshop on Matrices and Statistics, in Celebration of C. R. Rao's 80th Birthday, will be held in the historic walled city of Hyderabad, in Andra Pradesh, India, on December 9–13, 2000. The program will start with a two-day course on recent advances in Matrix Theory, with Special Reference to Applications in Statistics, on Saturday, December 9, and Sunday, December 10, 2000. This will be followed by presentation of research papers, which will be published in a professional journal after refereeing.

The International Organizing Committee for this Workshop comprises R. W. Farebrother (Manchester), S. Puntanen (Tampere; vice-chair), G. P. H. Styan (McGill), and H. J. Werner (Bonn; chair). For further information e-mail the Local Organizing Committee in India: K. VISWANATH, kvsm@uohyd.ernet.in R. J. R. SWAMY, nhasan@ouastr.ernet.in.

ILAS Treasurer's Report: March 1, 1998–February 28, 1999

by James R. Weaver, *University of West Florida, Pensacola*

Balance on hand March 1, 1998			
Certificate of Deposits (CD)	19,000.00		
Vanguard	8,882.15		
Checking Account	27,702.71		55,584.86
**			
Checking Account Balance on March 1, 1998			27,702.71
March 1998			
Income:			
Dues	844.00		
Interest (Gen)	134.79		
Contributions			
General Fund	40.00		
Hans Schneider Prize	50.00		
Service Charge Refund	10.79	1,079.58	
Expenses:			
Sec. of State	70.00		
Judy K. Weaver(ILAS Files)	127.75	197.75	881.83
April 1998			
Income:			
7th ILAS Conference	4,060.00		
Interest (Gen)	17.94	4,077.94	
Expenses:			
Service Charge	5.05	5.05	4,072.89
May 1998			
Income:			
Dues	640.00		
7th ILAS Conference	4,304.00		
Interest (Gen)	19.87		
Interest on CD (HS)	74.56		
Interest on CD (OT/JT)	29.00		
Interest on CD (FU)	23.12		
ILAS/LAA Lecturer	1,000.00		
Contributions			
General Fund	25.00		
Hans Schneider Prize	25.00	6,140.55	
Expenses:			
Judy K. Weaver (ILAS Files)	35.00		
Producing & Mailing "Image"			
George P.H. Styan	1,315.10		
7th ILAS Conference	6,450.00		
Service Charge	5.35	7,805.45	(1,664.90)
June 1998			
Income:			
Dues	570.00		
Interest (Gen)	20.46		
7th ILAS Conference-Cash	340.06		
7th ILAS Conference-Checks	1,407.00		
Contributions			
General Fund	12.00	2,349.52	
Expences:			
7th ILAS Conf.- Supplies	2,100.00		
7th ILAS Conference-Food/Refreshments	1,189.00		
7th ILAS Conference-Refund	438.00		
Judy K. Weaver (ILAS Files)	84.00		
Service Charge	8.80	3,819.80	(1,470.28)

July 1998			
Income:			
Dues	60.00		
Interest (Gen)	18.14		
Interest on CD (HS)	75.38		
Interest on CD (OT/JT)	29.32		
Interest on CD (FU)	23.63		
7th ILAS Conference-Supplies	25.00		
7th ILAS Conference-Reg.	80.00	311.47	
Expenses:			
Postmaster (Stamps & Mailing)	226.96		
Judy K. Weaver (Dues Notice)	84.00		
Service Charge	7.85	318.81	(7.34)
August 1998			
Income:			
Dues	2,036.51		
7th ILAS Conference	1,921.00		
Interest (Gen)	16.16		
Contributions			
General Fund	191.00		
Hans Schneider Prize	191.00		
OT/JT Lecture Fund	181.00		
F. Uhlig Fund	71.00		
Conference Fund	51.00	4,658.67	
Expenses:			
Judy K. Weaver (ILAS Files)	59.50		
Service Fee	2.59	62.09	4,596.58
September 1998			
Income:			
Dues	280.00		
Interest (Gen)	12.86		
Contributions			
General Fund	45.00		
Hans Schneider Prize	80.00		
OT/JT Lecture Fund	30.00		
F. Uhlig Fund	5.00		
Conference Fund	25.00	477.86	
Expenses:			
Office Depot (Supplies)	61.54		
Postmaster	291.04		
Service Charge	16.50	369.08	108.78
October 1997			
Income:			
Dues	460.00		
Interest (Gen)	12.95		
Interest on CD(HS)	76.23		
Interest on CD(OT/JT)	29.64		
Interest on CD(FU)	21.55		
Collection Item	20.00		
Contributions			
General Fund	20.00		
Hans Schneider Prize	50.00		
Conference Fund	10.00	700.37	
Expenses:			
Judy K.Weaver (ILAS Files & Supplies)	170.94		
Service Charge	2.31	173.25	527.12
November 1998			
Income:			
Dues	640.00		
Interest (Gen)	13.24		
Interest on CD (HS)	25.68		
Interest on CD (OT/JT)	9.99		
Contributions			

General Fund	31.00		
Hans Schneider Prize	40.00		
OT/JT Lecture Fund	40.00		
F. Uhlig Fund	30.00		
Conference Fund	30.00	859.91	
Expenses:			
Producing & Mailing "IMAGE"			
George P.H.Styan	1,325.30		
Service Charge	2.96	1,328.26	(468.35)
December 1998			
Income:			
Dues	940.00		
Interest (Gen)	13.38		
Interest on CD(FU)	21.32		
Contributions			
General Fund	180.00		
Hans Schneider Prize	220.00		
OT/JT Lecture Fund	200.00		
Conference Fund	35.00	1,609.70	
Expenses:			
Postmaster	172.80		
Judy K. Weaver (Dues Notice & ILAS Files)	112.00		
Service Charge	2.53	287.33	1,322.37
January 1999			
Income:			
Interest (Gen)	12.96	12.96	
Expenses:			
ILAS Lecturer	500.00		
Service Charge	6.26	506.26	(493.30)
February 1999			
Income:			
Dues	1,360.00		
Interest (Gen)	11.17		
Interest on CD(HS)	70.78		
Interest on CD(OT/JT)	27.53		
Contributions			
General Fund	152.00		
Hans Schneider Prize	60.00		
OT/JT Lecture Fund	2.00		
F. Uhlig Fund	12.00		
Conference Fund	25.00	1,720.48	
Expenses:	00.00	00.00	1,720.48

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February 28, 1999		36,828.59
Account Balance		
Checking Account	36,828.59	
Certificate of Deposit (FU)	1,500.00	
Certificate of Deposit (Gen.)	10,000.00	
Certificate of Deposit (72% HS & 28% OT/JT)	7,500.00	
Vanguard (72% HS & 28% OT/JT)	9,523.65	65,352.24
General Fund	26,378.70	
Frank Uhlig Educational Fund	2,906.62	
Hans Schneider Prize	16,235.87	
Olga Taussky Todd/John Todd Fund	8,533.11	
Conference Fund	9,707.94	
ILAS/LAA Fund	1,590.00	65,352.24

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James R. Weaver 3/17/99

IMAGE Problem Corner: Problems and Solutions

We look forward to receiving solutions to Problems 18-1, 19-3b, 19-4 & 21-2, which we repeat below; we also present some further comments concerning our solutions to Problem 19-5 in IMAGE 20 (April 1998), pp. 28–31, and the comments on these solutions on page 22 in IMAGE 21 (October 1998). We present solutions to Problems 21-1, 21-3, 21-4 & 21-5, which appeared in IMAGE 21 (October 1998), p. 28. In addition, we introduce seven new problems (p. 32) and invite readers to submit solutions, as well as new problems, for publication in IMAGE. Please send all material both in LaTeX embedded as text in an e-mail to styan@total.net and by p-mail to G. P. H. Styan, Dept. of Mathematics & Statistics, McGill University, 805 Sherbrooke Street West, Montréal, Québec, Canada H3A 2K6.

Problem 18-1: 5×5 **Complex Hadamard Matrices**

Proposed by S. W. DRURY, *McGill University, Montréal, Québec, Canada.*

Show that every 5×5 matrix U with complex entries $u_{j,k}$ of constant absolute value one that satisfies $U^*U = 5I$ can be realized as the matrix $(\omega^{jk})_{j,k}$ where ω is a complex primitive fifth root of unity by applying some sequence of the following: (1) A rearrangement of the rows, (2) A rearrangement of the columns, (3) Multiplication of a row by a complex number of absolute value one, (4) Multiplication of a column by a complex number of absolute value one.

The Editor has not yet received a solution to this problem—indeed even the Proposer has not yet found a solution!

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG, *North Carolina State University, Raleigh, North Carolina, USA,* Peter ŠEMRL, *University of Maribor, Maribor, Slovenia* & Hans Joachim WERNER, *Universität Bonn, Bonn, Germany.*

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \operatorname{Span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

Solution 19-3a.1 by the Proposers: Robert E. HARTWIG, Peter ŠEMRL & Hans Joachim WERNER.

Clearly, $AB = pA + qB$ if and only if $A(B - pI) = qB$ or, equivalently, $qpI = (A - qI)(B - pI)$. Hence $AB = pA + qB$ if and only if $qB(B - pI)^-(B - pI) = qB$ and, for some matrix Z, $A = qB(B - pI)^- + Z\,[I - (B - pI)(B - pI)^-]$. Here $(\cdot)^-$ refers to an arbitrary generalized inverse. In particular, whenever $pq \neq 0$, then $AB = pA + qB$ if and only if $(B - pI)^{-1}$ exists and $A = qB(B - pI)^{-1}$. □

A solution to Part (a), *when* $pq \neq 0$, *was also received from* Christopher C. PAIGE, McGill University, Montréal, QC, Canada. *We look forward to receiving a solution to Part* (b) *of this problem.*

Problem 19-4: Eigenvalues of Non-negative Definite Matrices

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Show that there are constants $\gamma \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, \frac{1}{2})$ such that, if $A_1, \ldots, A_n$ are non-negative definite $r \times r$ matrices of rank one satisfying

(a) $A_1 + \cdots + A_n = I_r$

(b) $\operatorname{trace}(A_i) < \gamma$ for each $i = 1, \ldots, n$,

then there is a subset σ of $\{1, 2, \ldots, n\}$ such that the eigenvalues of $A_\sigma = \sum_{i\in\sigma} A_i$ all lie in the interval $(\varepsilon, 1 - \varepsilon)$.

We look forward to receiving a solution to this problem!

Problem 19-5: Symmetrized Product Definiteness?

Proposed by Ingram OLKIN, *Stanford University, Stanford, California, USA.*

Let A and B be real symmetric matrices. Then prove or disprove:

(a) If A and B are both positive definite (non-negative definite), then $AB + BA$ is positive definite (non-negative definite).

(b) If A and $AB + BA$ are both positive definite (non-negative definite), then B is positive definite (non-negative definite).

Solutions 19-5.1–19-5.5 respectively by David CALLAN, Shuangzhe LIU, David LONDON, Hans Joachim WERNER and Henry WOLKOWICZ, appeared in IMAGE 20 (April 1998), pp. 28–31; for comments on these solutions by Karl E. GUSTAFSON see IMAGE 21 (October 1998), p. 22.

Additional Comment by Karl E. GUSTAFSON, *University of Colorado, Boulder, Colorado, USA.*

In my earlier comments I pointed out that the counter-examples to (a) could all be seen in terms of the sufficient condition: $\sin B < \cos A$, cf. Gustafson (1968). Without going on *ad infinitum*, I would like to add two observations:

(1) The page numbers in my cited references [1, 2, 3] in IMAGE 21, p. 22, should be p. 126, p. 334, p. 62, respectively.

(2) As observed in IMAGE 20 (April 1998), p. 28, this problem is discussed on pp. 120–121 in Lax (1997). Lax (cf. pp. 120–121 and 137–138) gave two proofs of (b) in order to expose nice uses of homotopy and square root arguments, respectively. Solutions 19-5.1–19.5.5 to (b) were (naturally, since the question was stated for real matrices) of the matrix variety, whereas Lax was treating arbitrary selfadjoint operators A and B. Here I would like to provide an alternate proof to (b) which, moreover, bears on (a). This alternate approach is based upon the useful result of Williams (1967): If zero is not in the closure of the numerical range of an operator A in $\mathcal{B}(X)$, i.e., $0 \notin \overline{W(A)}$, then for any operator B in $\mathcal{B}(X)$ one has

$$\sigma(A^{-1}B) \subset \overline{W(B)} \Big/ \overline{W(A)}.$$

Williams's result has the additional merit of needing (in Hilbert space) only a few-lines proof depending essentially only on the fact that the spectrum of a bounded operator is contained within the closure of its numerical range. With more work, the result holds as well in Banach space. Wielandt had some earlier results for matrices although it is unclear whether Williams was aware of those. See Gustafson & Rao (1997, pp. 34–36) for more details.

The proof of (b), for the postive definite case, now goes as follows. Since $AB + BA = 2\,\mathrm{Re}\,BA = 2\,\mathrm{Re}\,AB > 0$ and $A > 0$, we write $B = A^{-1}(AB)$, from which

$$\sigma(B) \subset \overline{W(AB)} \Big/ \overline{W(A)}.$$

The result (b) follows immediately because $\overline{W(B)}$ = convex hull$\{\sigma(B)\}$ = a real interval when B is selfadjoint.

Moreover we may also immediately extract the affirmative spectral content of (a). Given A and B positive definite, we write $A = (A^{-1})^{-1}$, from which

$$\sigma(AB) \subset \overline{W(B)} \Big/ \overline{W(A^{-1})}.$$

Thus, even though $W(AB)$ need not be positive, $\sigma(AB)$ always is. To get $W(AB)$ positive, I introduced my operator trigonometric conditions (cf. Gustafson, 1968).

References

[1] Karl E. GUSTAFSON (1968). The angle of an operator and positive operator products. *Bulletin of the American Mathematical Society* 74, 488–492.

[2] Karl E. GUSTAFSON & Duggirala K. M. RAO (1997). *Numerical Range: The Field of Values of Linear Operators and Matrices.* Springer-Verlag, New York.

[3] Peter LAX (1997). *Linear Algebra.* Wiley, New York.

[4] J. P. WILLIAMS (1967). Spectra of products and numerical ranges. *Journal of Mathematical Analysis and Applications* 17, 214–220.

Problem 21-1: Square Complex Matrix of Odd Order

Proposed by Ludwig ELSNER, *Universität Bielefeld, Bielefeld, Germany.*

Let A denote an $n \times n$ complex matrix, where n is odd. Show that there exist numbers μ_i, $i = 1, 2$ on the unit circle and nonzero vectors $x_1, x_2 \in \mathbb{C}^n$ satisfying

$$A^H x_1 = \mu_1 A x_1, \quad \bar{A} x_2 = \mu_2 A x_2.$$

Solution 21-1.1 by Roger A. HORN, *University of Utah, Salt Lake City, Utah, USA.*

First suppose that A is nonsingular, so we are to show that $B \equiv A^{-1}A^H$ and $C \equiv A^{-1}\bar{A}$ each has at least one eigenvalue with unit modulus. Then $C^{-1} = \bar{C}$ and $B^{-1} = \left(A^{-1}\right)^H A$ is similar to $A\left(A^{-1}\right)^H = B^H$, which is similar to $\bar{B}$. Consider the Jordan structure of a nonsingular matrix D whose inverse is equal to or similar to its conjugate: D^{-1} and $\bar{D}$ have the same Jordan Canonical Form. If λ is an eigenvalue of D that does not have unit modulus, then $\lambda^{-1} \neq \bar{\lambda}$ and every Jordan block of D with eigenvalue λ is paired with another block of the same size with eigenvalue $\bar{\lambda}$. Thus, the sum of the sizes of all the Jordan blocks of D with non-unit eigenvalues is even. If D has odd order, it must therefore have at least one eigenvalue of unit modulus.

If A is singular, then $A_\varepsilon \equiv A + \varepsilon I$ is nonsingular for all sufficiently small positive ε. Let λ_ε and x_ε be a unit eigenvalue and unit eigenvector, respectively, of $B_\varepsilon \equiv A_\varepsilon^{-1}A_\varepsilon^H$, so $B_\varepsilon x_\varepsilon = \lambda_\varepsilon x_\varepsilon$ and $A_\varepsilon^H x_\varepsilon = \lambda_\varepsilon A_\varepsilon x_\varepsilon$. By compactness of the unit circle and the unit ball of $\mathbb{C}^n$ there is a sequence $\varepsilon_k \to 0$, a unit scalar λ, and a unit vector x such that $\lambda_{\varepsilon_k} \to \lambda$ and $x_{\varepsilon_k} \to x$, so $A^H x = \lim_{k\to\infty} A_{\varepsilon_k}^H x_{\varepsilon_k} = \lim_{k\to\infty} \lambda_{\varepsilon_k} A_{\varepsilon_k} x_{\varepsilon_k} = \lambda A x$. The same argument gives the assertion about A and $\bar{A}$. □

Solution 21-1.2 by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

We first establish a lemma, which is of interest in its own right.

Lemma. Let X and Y be $n \times n$ Hermitian and skew-Hermitian matrices, respectively. Then for any eigenvalue λ of XY, either $\lambda \in i\mathbb{R}$, or λ and $-\bar{\lambda}$ occur in pairs as the eigenvalues of XY with the same algebraic multiplicity. The assertion also holds for matrices X and Y satisfying $\bar{X} = X$ and $\bar{Y} = -Y$.

Proof. For any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} p(\lambda) &:= \det(i\lambda I - XY) \\ &= \overline{\det(-i\lambda I - Y^H X^H)} = \overline{\det(-i\lambda I + YX)} = \overline{\det(-i\lambda I + XY)} = (-1)^n \overline{\det(i\lambda I - XY)} = (-1)^n \overline{p(\lambda)}. \end{aligned}$$

Thus for all real λ, $p(\lambda) = \overline{p(\lambda)}$ if n is even, and $p(\lambda) = -\overline{p(\lambda)}$ if n is odd. Accordingly, we may assume that $p(\lambda) = f(\lambda)$ if n is even and $p(\lambda) = if(\lambda)$ if n is odd, where $f(\lambda)$ is some real coefficient polynomial. Further if $C(\lambda)$ is the characteristic polynomial of XY, then $C(\lambda) = p(-i\lambda) = f(-i\lambda)$ or $if(-i\lambda)$. Now if λ is an eigenvalue of XY, then $C(\lambda) = 0$ or $f(-i\lambda) = 0$. Thus $-i\lambda \in \mathbb{R}$, i.e. $\lambda \in i\mathbb{R}$, or $-i\lambda$ and $\overline{-i\lambda} = i\bar{\lambda}$ occur in conjugate pairs (as zeros of $f(\lambda)$ with the same multiplicity). Thus $C(-\bar{\lambda}) = f(-i(-\bar{\lambda})) = f(i\bar{\lambda}) = 0$, and $-\bar{\lambda}$ is an eigenvalue of XY. The lemma is proved. □

It follows from the lemma that if X and Y are further nonsingular, then XY has at least one eigenvalue in the form ir for some real r, since $\det(XY) = \det(X)\det(Y) \in i\mathbb{R}$.

Now we are ready to solve Problem 21-1. Notice that the assertation is equivalent to saying that $A^H - \mu_1 A$ and $\bar{A} - \mu_2 A$ are singular for some μ_1 and μ_2 on the unit circle. We show the case for $A^H - \mu_1 A$. The other one is similar. If $A^H + A$ or $A^H - A$ is singular, then the assertion is true. Suppose otherwise they are both nonsingular. Then by the above argument, the matrix $(A^H + A)^{-1}(A^H - A)$ has at least one eigenvalue in $i\mathbb{R}$, i.e., $\det\left[itI + (A^H + A)^{-1}(A^H - A)\right] = 0$ for some $t \in \mathbb{R}$. Thus

$$\det\left(A^H - \frac{1 - it}{1 + it}A\right) = \frac{1}{(1 + it)^n}\det(A^H + A)\det[itI + (A^H + A)^{-1}(A^H - A)] = 0.$$

Therefore $A^H - \mu_1 A$ is singular, where $\mu_1 = (1 - it)/(1 + it)$ is on the unit circle. □

We note that the conclusion does not hold in general when n is even.

Solution 21-1.3 by the Proposer: Ludwig ELSNER, *Universität Bielefeld, Bielefeld, Germany.*

We assume first that A is nonsingular. In this case we have to show that the matrices $B_1 = A^{-1}A^H$ and $B_2 = A^{-1}\bar{A}$ have eigenvalues μ_1, μ_2 on the unit circle. From $B_1^{-H} = A^H A^{-1}$ we have that B_1 is similar to B_1^{-H}, and hence for any λ in the spectrum of B_1 also $(\bar{\lambda})^{-1}$ is in this spectrum. As there is an odd number of eigenvalues, in this matching one eigenvalue μ_1 is matched with itself, hence it is on the unit circle. In the second case we set $A = T + iS$ with S, T real. We assume also that T is nonsingular. $C = T^{-1}S$ has a real eigenvalue σ. It follows from

$$B_2 = (T + iS)^{-1}(T - iS) = (I + iC)^{-1}(I - iC)$$

that B_2 has an eigenvalue $\mu_2 = (1-i\sigma)/(1+i\sigma)$, hence $|\mu_2| = 1$. The general case follows by the usual continuity argument. □

Problem 21-2: The Diagonal of an Inverse

Proposed by Beresford PARLETT, *University of California, Berkeley, California, USA,*
via Roy MATHIAS, *College of William and Mary, Williamsburg, Virginia, USA.*

Let J be an invertible tridiagonal $n \times n$ matrix that permits triangular factoriziation in both increasing and decreasing order of rows:

$$J = L_+D_+U_+ \quad \text{and} \quad J = U_-D_-L_-.$$

(Here the L's are lower triangular, the U's are upper triangular, and the D's are diagonal.). Show that

$$(J^{-1})_{kk} = [(D_+)_{kk} + (D_-)_{kk} - J_{kk}]^{-1}.$$

We look forward to receiving a solution to this problem!

Problem 21-3: Eigenvalues and Eigenvectors of Two Symmetric Matrices

Proposed by William F. TRENCH, *Trinity University (Emeritus), San Antonio, Texas & Divide, Colorado, USA.*

Find the eigenvalues and eigenvectors of the $n \times n$ symmetric matrices

$$A = \{\min(i,j)\}_{i,j=1}^n \quad \text{and} \quad B = \{\min(2i-1, 2j-1)\}_{i,j=1}^n.$$

Solution 21-3.1 by the Proposer: William F. TRENCH.

For the moment consider the matrix

$$C = \{\min(ai - b, aj - b)\}_{i,j=1}^n$$

where $a > 0$ and $a \neq b$. It is straightforward to verify that

$$C^{-1} = \frac{1}{a}\begin{bmatrix} \frac{2a-b}{a-b} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix}.$$

If $x_0, x_1, \ldots, x_{n+1}$ satisfy the difference equation

$$x_{r-1} - (2 - a\alpha)x_r + x_{r+1} = 0, \quad 1 \leq r \leq n, \tag{1}$$

and the boundary conditions

$$(a - b)x_0 + bx_1 = 0 \quad \text{and} \quad x_n - x_{n+1} = 0, \tag{2}$$

then $x = [x_1\, x_2 \cdots x_n]^T$ satisfies $C^{-1}x = \alpha x$; therefore, α is an eigenvalue of C^{-1} if and only if (1) has a nontrivial solution satisfying (2), in which case x is an α-eigenvector of C^{-1}. The solutions of (1) are given by

$$x_r = c_1\zeta^r + c_2\zeta^{-r}, \tag{3}$$

where ζ is a zero of the reciprocal polynomial

$$p(z) = z^2 - (2 - a\alpha) + 1.$$

Since $p(z) = (z-\zeta)(z - 1/\zeta) = z^2 - (\zeta + 1/\zeta)z + 1$, it follows that if (3) determines an eigenvector of C^{-1} then the corresponding eigenvalue of A^{-1} is

$$\alpha = \frac{1}{a}\left(2 - \zeta - \frac{1}{\zeta}\right). \tag{4}$$

The boundary conditions (2) yield the system

$$\begin{bmatrix} a - b + b\zeta & a - b + b/\zeta \\ \zeta^n - \zeta^{n+1} & 1/\zeta^n - 1/\zeta^{n+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with determinant $\zeta^{-n-1}(\zeta - 1)d(\zeta)$, where

$$d(\zeta) = (a-b)(\zeta^{2n+1} + 1) + b(\zeta^{2n} + \zeta).$$

Therefore, (3) determines an eigenvector of C^{-1} (and of C) if and only if $d(\zeta) = 0$.

If $a = 1$ and $b = 0$ then $C = A$, so (3) determines an eigenvector of A^{-1} if and only if $\zeta^{2n+1} = -1$. Therefore

$$\zeta = \exp\frac{i(2k+1)\pi}{2n+1}$$

for some integer k. Since the boundary conditions (2) require that $x_0 = 0$ and $x_n = x_{n+1}$, (3) implies that

$$x_r = \sin\frac{(2k+1)r\pi}{2n+1}, \quad r = 1, 2, \ldots, n,$$

are the components of an eigenvector corresponding to ζ. From (4), the corresponding eigenvalue of A^{-1} is

$$\alpha = \frac{2}{a}\left(1 - \cos\frac{(2k+1)\pi}{2n+1}\right),$$

and $\lambda = 1/\alpha$ is an eigenvalue of A. Letting $k = 0, 1, \ldots, n-1$ yields a complete set of eigenvalues and eigenvectors of A.

If $a = 2$ and $b = 1$ then $C = B$, so (3) determines an eigenvector of B^{-1} if and only if $\zeta^{2n} = -1$. Therefore

$$\zeta = \exp\frac{i(2k+1)\pi}{2n}$$

conditions (2) require that $x_0 = -x_1$ and $x_n = x_{n+1}$, (3) implies that

$$x_r = \sin\frac{(2k+1)(2r-1)\pi}{4n}, \quad r = 1, 2, \ldots, n,$$

are the components of an eigenvector by corresponding to ζ. From (4), the corresponding eigenvalue of A^{-1} is

$$\alpha = \frac{2}{a}\left(1 - \cos\frac{(2k+1)\pi}{2n+1}\right),$$

and $\lambda = \frac{1}{\alpha}$ is an eigenvalue of B. Letting $k = 0, 1, \ldots, n-1$ yields a complete set of eigenvalues and eigenvectors of B. □

Problem 21-4: Square Complex Matrix, its Moore-Penrose Inverse and the Löwner Ordering
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany.*

(a) Let A be a square matrix with complex entries and Moore-Penrose inverse $B = A^+$. Show that $AA^* \le A^*A$ if and only if $BB^* \le B^*B$, where $\le$ denotes the Löwner-ordering of matrices.

(b) Does this result generalize to operators on a Hilbert space?

Solution 21-4.1 by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Part (a) follows immediately from the observation that if X is a finite complex matrix, then it follows that $XX^* \le X^*X \Leftrightarrow XX^* = X^*X$ (i.e., X is normal), since trace$(X^*X - XX^*) = 0$ for all X. [Note: This is one of the 30 important equivalent conditions for matrix normality in Chapter 8 of Zhang (1999).]

For Part (b) the answer is no: The assertion in Part (a) does not generalize to operators on Hilbert spaces. Take, e.g., cf. Wang & Zhang (1995), the Shifting Operator $S : (c_1, c_2, c_3, \ldots) \mapsto (0, c_1, c_2, c_3, \ldots)$ for the l^2 space. Then $S^* = S^+ : (c_1, c_2, c_3, \ldots) \mapsto (c_2, c_3, \ldots)$. It is easy to verify that $S^+S = S^*S = I$ and $SS^* \le S^*S$. But $S^*S \not\le SS^*$. □

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Solution 21-4a.1 by ROGER A. HORN, *University of Utah, Salt Lake City, Utah, USA.*

To answer Part (a), let $A = V\Sigma W^*$ be a singular value decomposition and compute $A^\dagger = W\Sigma^\dagger V^*$, $\{(A^\dagger)^*A^\dagger\}^\dagger = V\Sigma^2V^* = AA^*$, and $A^\dagger(A^\dagger)^* = W(\Sigma^\dagger)^2W^* = (A^*A)^\dagger$. We use the fact that $X \ge Y \ge 0$ if and only if $\rho(X^\dagger Y) \le 1$, where $\rho(\cdot)$ is the spectral radius function. Then $A^*A \ge AA^*$ if and only if

$$1 \ge \rho\{(A^*A)^\dagger(AA^*)\} = \rho\{(AA^*)(A^*A)^\dagger\} = \rho\{[(A^\dagger)^*A^\dagger]^\dagger[A^\dagger(A^\dagger)^*]\}$$

if and only if $(A^\dagger)^*A^\dagger \ge A^\dagger(A^\dagger)^*$. □

Solution 21-4a.2 by the Proposer: Götz TRENKLER, *Universität Dortmund, Dortmund, Germany.*

We present two solutions to Part (a):

Solution 1: Since $AA^* \le A^*A$ we have $A^*A = AA^* + CC^*$ for some matrix C. Taking traces we obtain $\mathrm{tr}(A^*A) = \mathrm{tr}(AA^*) + \mathrm{tr}(CC^*)$. The identity $\mathrm{tr}(A^*A) = \mathrm{tr}(AA^*)$ then entails $\mathrm{tr}(CC^*) = 0$, i. e. $C = 0$. Hence $AA^* = A^*A$ showing A to be normal (cf. Halmos, 1967, Problem 160). But it is well known that a matrix is normal if and only if its Moore-Penrose inverse is normal. □

Solution 2: By Baksalary, Pukelsheim and Styan (1989, Theorem 4.3) we have $(A^*A)^+ \le (AA^*)^+$, since $\mathrm{rank}(AA^*) = \mathrm{rank}(A^*A)$ (see also Milliken and Akdeniz, 1977). It follows that $A^+A^{*+} \le A^{*+}A^+$, and by $A^{*+} = A^{+*}$ we get $BB^* = B^*B$. □

Comment: It was not possible for the poser of the problem to generalize this result to operators on a Hilbert space (but see the second part of Solution 21-4.1 above). Note, however, that an invertible operator T on a Hilbert space is hypernormal (i. e. $TT^* \le T^*T$) if and only if T^{-1} is hypernormal (cf. Berberian, 1961, p. 161).

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Problem 21-5: Determinants and g-Inverses

Proposed by Simo PUNTANEN, *University of Tampere, Tampere, Finland,* George P. H. STYAN, *McGill University, Montréal, Québec, Canada* & Hans Joachim WERNER, *Universität Bonn, Bonn, Germany.*

For a real matrix B, let B^- denote an arbitrary g-inverse of B satisfying $BB^-B = B$, and let $\{B^-\}$ denote briefly the set of all g-inverses of B. Now let $A \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times k}$ be two given matrices such that A is non-negative definite and symmetric, $X'X = I_k$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$; here $\mathcal{R}(\cdot)$ denotes the range (column space) of $(\cdot)$. Define the matrix function $M(\cdot)$ from $\{A^-\}$ into $\mathbb{R}^{n\times n}$ according to

$$M(A^-) := I_n - XX' + A^-XX'A.$$

(a) Does $M(A^-)$ depend in general on the choice of $A^- \in \{A^-\}$? And if it is so, characterize the restrictive case of invariance.

(b) Does $|M(A^-)|$, with $|\cdot|$ indicating determinant, depend on the choice of $A^- \in \{A^-\}$?

(c) For each $A^- \in \{A^-\}$, determine explicitly $\{M(A^-)^-\}$ in terms of A, X, A^-.

Solution 21-5.1 by the Proposers: Simo PUNTANEN, George P. H. STYAN & Hans Joachim WERNER.

There exists an $n \times (n-k)$ matrix Y so that the matrix $P = (X, Y)$ is orthogonal, and so $X'Y = 0$, $Y'X = 0$, and $Y'Y = I_{n-k}$. Put $W(A^-) := P'M(A^-)P$. Then clearly $|W(A^-)| = |M(A^-)|$ and

$$W(A^-) = \begin{pmatrix} X'A^-XX'AX & X'A^-XX'AY \\ Y'A^-XX'AX & I_{n-k} + Y'A^-XX'AY \end{pmatrix}.$$

Since $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, we know from, e. g., Theorem 2.3 in Werner and Yapar (1996) that $X'A^-X$ is invariant for any choice of A^- and $\mathcal{R}(X'A^-X) = \mathcal{R}(X')$. In view of $X'X = I_k$, clearly $\operatorname{rank}(X) = k$ and so $X'A^-X$ is nonsingular. From

$$W(A^-) = \begin{pmatrix} I_k & 0 \\ Y'A^-X(X'A^-X)^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} X'A^-XX'AX & X'A^-XX'AY \\ 0 & I_{n-k} \end{pmatrix} \tag{1}$$

we then get

$$|M(A^-)| = |W(A^-)| = |X'A^-X| \cdot |X'AX| \tag{2}$$

which is invariant for any choice of A^-.

Observe that the nonsingularity of $X'A^-X$ is equivalent to $|X'A^-X| \neq 0$. Since A is non-negative definite and symmetric, $A\mathcal{R}(X) \cap \mathcal{N}(X') = \{0\}$. This in turn implies that $X'AX$ is nonsingular and so $|X'AX| \neq 0$. From our determinantal formula (2) it now follows that $M(A^-)$ is, irrespective of the choice of A^-, nonsingular. We leave it to the reader to check that the matrix expression on the right hand side of the equality sign in

$$M(A^-)^{-1} = I_n + 2X(X'AX)^{-1}(X'A^-X)^{-1}X' - X(X'AX)^{-1}X'A - A^-X(X'A^-X)^{-1}X',$$

say G, satisfies with $M(A^-)G = I_n$ the defining equation for the nonsingular inverse of $M(A^-)$. Needless to say, $M(A^-)^{-1}$ is then the only g-inverse of $M(A^-)$. In passing, we further mention that $M(A^-)$ can also be derived rather easily, in an alternative constructive manner, from $M(A^-) = PW(A^-)P'$ by exploiting the special lower and upper block triangular structure in the factorization (1) of $W(A^-)$.

Contrary to its determinant, the matrix $M(A^-)$ is generally not invariant for the choice of A^-. This is seen as follows. Trivially, $\mathcal{R}(X) = \mathcal{R}(X(X'AX)^{-1})$. So $M(A^-)X(X'AX)^{-1} = A^-X$. Since $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, it now follows from Theorem 2.1 (ii) & (iii) in Werner and Yapar (1996) that A^-X and so $M(A^-)$ is invariant for the choice of A^- if and only if A is of full column rank. Since A is square, this happens if and only if A is nonsingular and our solution is complete. □

Reference

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IMAGE Problem Corner: New Problems

Problem 22-1: Powers, g-Inverses and Core Matrices

Proposed by Gülhan ALPARGU and George P. H. STYAN, *McGill University, Montréal, Québec, Canada*
& Hans Joachim WERNER, *Universität Bonn, Bonn, Germany.*

For a real matrix A, let A^- denote an arbitrary g-inverse of A satisfying $AA^-A = A$, and let $\{A^-\}$ denote the set of all g-inverses of A. Show that for a real square matrix A the following conditions are equivalent:

(a) $\sum_{i=0}^{h} A^i \in \{(I - A)^-\}$ for some nonnegative integer h.

(b) $A^s = A^{s+1}$ for some nonnegative integer s.

(c) The core part $C(A)$ of A is idempotent. [We recall that a square matrix A has index $\operatorname{ind}(A) = k$ iff there exist a nilpotent matrix N_A of degree k (i.e., $N_A^k = 0$ while $N_A^{k-1} \neq 0$) and a core matrix C_A (i.e., $\operatorname{ind}(C_A) = 1$) such that $A = C_A + N_A, C_A N_A = N_A C_A = 0$; cf. e.g., pp. 175–177 in Adi BEN-ISRAEL & Thomas N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, Wiley, New York, 1974; corrected reprint edition: Krieger, Huntington, NY, 1980.]

Problem 22-2: Square Complex Matrices, Linear Maps and Eigenvalues

Proposed by Ludwig ELSNER, *Universität Bielefeld, Bielefeld, Germany.*

(a) Let M_n denote the set of $n \times n$ complex matrices. Find a *linear* map $S : M_n \longrightarrow M_{n^2}$ with the following property: If $A \in M_n$ has the (not necessarily real) eigenvalues $\lambda_i, i = 1, \ldots, n$ then $S(A)$ has the eigenvalues $\lambda_i, i = 1, \ldots, n$, as well as $\sqrt{\lambda_i \lambda_j}$ and $-\sqrt{\lambda_i \lambda_j}$ for all pairs $i < j$.

(b) Show that there is no linear map $S : M_2 \longrightarrow M_2$ satisfying the following condition: If $A \in M_2$ has the eigenvalues λ, μ then $S(A)$ has the eigenvalues $\pm\sqrt{\lambda\mu}$.

Problem 22-3: The Rank of a Matrix Difference

Proposed by Yongge TIAN, *Concordia University, Montréal, Québec, Canada.*

Let the $n \times n$ matrices A and B be idempotent. Show that $\operatorname{rank}(A - B) = \operatorname{rank}(A - AB) + \operatorname{rank}(B - AB)$.

Problem 22-4: Determinant of a Certain Patterned Matrix

Proposed by William F. TRENCH, *Divide, Colorado, USA.*

Let a, b, c and d be complex numbers. Find the determinant of $G_n = \{\, a(i - j) + b\min(i, j) + c\max(i, j) + d \,\}_{i,j=1}^{n}$.

Problem 22-5: Eigenvalues and Eigenvectors of a Certain Toeplitz Matrix

Proposed by William F. TRENCH, *Divide, Colorado, USA.*

Find the eigenvalues and eigenvectors of the Toeplitz matrix $T = \{\alpha\cos(r - s)t + \beta\sin(r - s)t\}_{r,s=1}^{n}$, with $\alpha^2 + \beta^2 \neq 0$, $n \geq 3$, and $t \neq k\pi$ with k an integer.

Problem 22-6: Idempotency of a Certain Matrix Quadratic Form

Proposed by Michel VAN DE VELDEN, *University of Amsterdam, Amsterdam, The Netherlands,*
Shuangzhe LIU, *Universität Basel, Basel, Switzerland* & Heinz NEUDECKER, *Cesaro, Schagen, The Netherlands.*

Let A be an $n \times n$ symmetric idempotent matrix and let U be an $n \times k$ $(k < n)$ matrix such that $U'U = I_k$. Characterize those matrices U for which $U'AU$ is idempotent.

Problem 22-7: Characterization of a Square Matrix in an Inner-product Inequality

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let A be an $n \times n$ complex matrix. If, for some real number α, $|(Ay, y)| \leq (y, y)^\alpha$ for all n-column complex vectors y, what can be said about A?

Please submit solutions <u>both</u> in LaTeX embedded in an e-mail to styan@total.net <u>and</u> by p-mail (nicely printed copy, please!) to G. P. H. Styan, Dept. of Mathematics & Statistics, McGill University, 805 Sherbrooke Street West, Montréal, QC, Canada H3A 2K6.