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IMAGE

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ILAS President/Vice-President's Annual Report: April 2000

1. The following have been elected to ILAS offices with terms that began on March 1, 2000:

Secretary/Treasurer: Jeff Stuart
(three-year term ending February 28, 2003).

Board of Directors: Harm Bart and Steve Kirkland
(three-year terms ending February 28, 2003).

The following continue in their offices to which they were previously elected:

Vice President: Daniel Hershkowitz
(term ends February 28, 2001)

Board of Directors:
Jose Dias da Silva (term ends February 28, 2001),
Roger Horn (term ends February 28, 2001),
Nicholas Higham (term ends February 28, 2002),
Pauline van den Driessche (term ends February 28, 2002).

2. The President's Advisory Committee consists of Chi-Kwong Li (chair), Shmuel Friedland, Raphi Loewy, and Frank Uhlig.

3. This fall there will be elections for Vice-President (the term of Danny Hershkowitz ends on February 28, 2001) and two members of the Board of Directors (the terms of Jose Dias da Silva and Roger Horn end on February 28, 2001). The ILAS 2000 Nominating Committee has been appointed and it consists of Wayne Barrett, Jane Day (chair), Nick Higham, Chi-Kwong Li, and Michael Tsatsomeros.

4. An ad hoc committee has been appointed and charged to consider changes in the ILAS Bylaws and to possibly make recommendations for change to the full ILAS membership for its approval. The committee consists of Jose Dias da Silva, Raphi Loewy, Hans Schneider (chair), and Frank Uhlig. Any ILAS member who has concerns about our Bylaws should convey them to a member of this committee.

5. The appointment of George P. H. Styan as an editor-in-chief of IMAGE has been extended to May 31, 2003. In addition, Hans Joachim "Jochen" Werner has been appointed as an editor-in-chief of IMAGE for a six year term beginning June 1, 2000 (and so ending on May 31, 2006). Thus beginning June 1, 2000, George and Jochen will be co-editors-in-chief of IMAGE.

6. The 8th ILAS Conference was held at the Universitat Politènica de Catelunya in Barcelona, Spain, on July 19–22, 1999. A report on this conference can be found in IMAGE 23 (October 1999, p. 13).

7. The following ILAS conferences are planned for the near future:

- The 9th ILAS Conference, Technion, Haifa, Israel, June 25–29, 2001.
- The 10th ILAS Conference, "Challenges in Matrix Theory," Auburn University, Auburn, USA, June 10–13, 2002.
- The 11th ILAS Conference, Lisbon, Portugal, Summer 2004.

8. The seventh SIAM Conference on Applied Linear Algebra will take place October 23–26, 2000, at North Carolina State University in Raleigh, North Carolina, USA. This conference is being held in cooperation with ILAS. There will be two ILAS sponsored invited speakers at the conference: Eduardo Marques de Sa' and Hugo J. Woerdeman. The 1999 ILAS Board acted as the selection committee. Under the terms of the agreement with SIAM, ILAS members who are not already a member of the SIAM Activity Group on Linear Algebra (SIAG/LA) will have the same reduced registration fee that it offers SIAG/LA members.

9. Steve Kirkland has been selected as the Olga Tauskky Todd/John Todd Lecturer at the 9th ILAS conference in Haifa in 2001. The selection committee consisted of Roger Horn (chair), Miki Neumann, Andre Ran, and Bit-Shun Tam.

10. Nick Trefethen has been selected as the LAA Lecturer at the 9th ILAS conference in Haifa in 2001. This lecture is sponsored by Elsevier Science Inc., publishers of the journal "Linear Algebra and its Applications." The selection committee consisted of Moshe Goldberg, Volker Mehrmann (chair), George Styan, and Hugo Woerdeman.

11. The next Hans Schneider Prize in Linear Algebra will be awarded at the 10th ILAS conference in Auburn in 2002.

12. ILAS is continuing to consider requests for the sponsorship of an ILAS Lecturer at a conference which is of substantial interest to ILAS members. Each year, US$1,000 is set aside to support such conferences, with a maximum amount of $500 available for any one conference. The guidelines are: (i) the conference must be of interest to a substantial number of ILAS members; (ii) the same "organization" is not eligible for support more than once every three years; (iii) geographically, the support should be distributed widely.

The full ILAS Board (three executives and six other members) reviews proposals and decides on which, if any, will receive support, and how much that support will be. Next year we will review these guidelines and this may result in some changes.

We are now accepting requests for conferences to be held in 2001. Such requests should be submitted by September 1, 2000. Electronic requests are preferred and should be sent to brualdi@math.wisc.edu. The request should include the following:

(1) date and place of conference

(2) sponsoring "organization"

(3) organizing committee

(4) purpose of conference

(5) invited speakers, to the extent known

(6) expected attendance

(7) proposed ILAS Lecturer, with some information about the lecturer

(8) amount requested.

ILAS is sponsoring one lecture in 2000: Chandler Davis at the 5th Workshop on Numerical Ranges and Numerical Radii, Nafplio, Greece, on June 26–28, 2000 [cf. IMAGE 23, p. 16].

13. ILAS has endorsed two conferences to be held in 2000. They are: (i) the Second Conference on Numerical Analysis and Applications, June 11–15, 2000, University of Rousse, Rousse, Bulgaria; (ii) the International Workshop on Parallel matrix algorithms and applications August 18–20, 2000, Neuchâtel, Switzerland—keynote speakers are Ahmed Sameh and Anna Nagurney. ILAS has also endorsed the Rocky Mountain Mathematics Consortium's Summer School on Matrix Theory to be held at the University of Wyoming in 2001—the principal speaker at the summer school will be Charles R. Johnson.

14. ELA—*Electronic journal of Linear Algebra*

- Volume 1, published in 1996, contained 6 papers.
- Volume 2, published in 1997, contained 2 papers.
- Volume 3—the Hans Schneider issue, published in 1998, contained 13 papers.
- Volume 4, published in 1998 as well, contained 5 papers.
- Volume 5, published in 1999, contained 8 papers.
- Volume 6—Proceedings of the Eleventh Haifa Matrix Theory Conference, is being published now. As of April 2000, it contains 6 papers.
- Volume 7, is being published now. As of April 2000, it contains 3 papers. ELA's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz, in the University of Lisbon, in EMIS (European Mathematical Information Service) offered by the European Mathematical Society, and in EMIS's 36 Mirror Sites.

Volumes 1–4 of ELA are now available in book form. The list price is US$20 with a discounted price of US$16 for ILAS members (these prices include postage and handling).

15. ILAS-NET: As of April 16, 2000, we have circulated 953 ILAS-NET announcements. ILAS-NET currently has 533 subscribers.

16. The ILAS INFORMATION CENTER (IIC) has a daily average of 300 information requests (not counting FTP operations). IIC's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz, and in the University of Lisbon.

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Tribute to Jim Weaver

Tuesday, 29 February 2000, was Jim Weaver's last day as ILAS Secretary/Treasurer. As you may recall, Jim was appointed to the position of Treasurer in 1989 by then president Hans Schneider and was elected for two 2-year terms and one 4-year term (03/92–02/94, 03/94–02/96, and 03/96–03/00). So for 11 years he has served as ILAS Treasurer, which was changed to Secretary/Treasurer as a result of changes in our Bylaws.

ILAS owes Jim a big debt of gratitude for his dedicated and professional service for 11 years. As ILAS has grown and assumed a more visible and prominent place in our professional lives, the position of Secretary/Treasurer has become more important and more complicated and has required additional expertise. As ILAS members have witnessed, Jim has responded wonderfully to these new challenges. That ILAS is on a sound financial footing, with detailed documentation of our financial and other records, is testimony to Jim's talents, hard work, and commitment to ILAS.

I hope that you will join me in conveying our deep appreciation to Jim for his service to ILAS. I am happy to know that, according to our Bylaws, Jim remains a member of the ILAS Board for one year after leaving office (until 28 February 2001).

On 1 March 2000, Jeff Stuart began a 4-year term as ILAS Secretary/Treasurer. I look forward to working with Jeff for the next two years. I very much appreciate Jeff's willingness to assume this important position, and I am very confident that he will be a big asset to ILAS and the ILAS Community. Welcome aboard, Jeff! I also want to take this opportunity to thank two retiring members (as of 29 February 2000) of the Board, Jane Day and Volker Mehrmann, for their important service to ILAS, and to welcome two new members of the Board, Harm Bart and Steve Kirkland.

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IMAGE Problem Corner: Problems and Solutions

We are still hoping to receive solutions to Problems 18-1, 19-3b, 21-2 & 23-1, which are repeated below; we present solutions to Problems 23-2 through 23-7, which appeared in IMAGE 23 (October 1999), pp. 28 & 27. In addition, we introduce 9 new problems (pp. 16–17) and invite readers to submit solutions as well as new problems for publication in IMAGE. Please send all material in LaTeX – (a) embedded as text in an e-mail to styan@total.net and (b) 2 copies (nicely printed please) by p-mail to George P. H. Styan, PO Box 270, Franklin, VT 05457-0270, USA. Please make sure that your name as well as your e-mail and p-mail addresses (in full) are included!

Problem 18-1: 5×5 Complex Hadamard Matrices

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Show that every 5×5 matrix U with complex entries $u_{j,k}$ of constant absolute value one that satisfies $U^*U = 5I$ can be realized as the matrix $(\omega^{jk})_{j,k}$ where ω is a complex primitive fifth root of unity by applying some sequence of the following: (1) A rearrangement of the rows, (2) A rearrangement of the columns, (3) Multiplication of a row by a complex number of absolute value one, (4) Multiplication of a column by a complex number of absolute value one.

The Editor has still not received a solution—indeed even the Proposer has not yet found a solution!

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG: hartwig@math.ncsu.edu; *North Carolina State University, Raleigh, North Carolina, USA,*
Peter ŠEMRL: peter.semrl@fmf.uni-lj.si; *University of Maribor, Maribor, Slovenia,*
and Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \mathrm{span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

The Editor has still not received a solution to Problem 19-3b. The solution by the Proposers to Problem 19-3a appeared in IMAGE 22 (April 1999), p. 25. *We look forward to receiving a solution to Problem 19-3b.*

Problem 21-2: The Diagonal of an Inverse

Proposed by Beresford PARLETT: parlett@math.berkeley.edu; *University of California, Berkeley, California, USA,*
via Roy MATHIAS: mathias@math.wm.edu; *College of William and Mary, Williamsburg, Virginia, USA.*

Let J be an invertible tridiagonal $n \times n$ matrix that permits triangular factorization in both increasing and decreasing order of rows:

$$J = L_+D_+U_+ \quad \text{and} \quad J = U_-D_-L_-.$$

(Here the L's are lower triangular, the U's are upper triangular, and the D's are diagonal.). Show that

$$(J^{-1})_{kk} = [(D_+)_{kk} + (D_-)_{kk} - J_{kk}]^{-1}.$$

The Editor has still not received a solution!

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG: chung@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let the $m \times n$ random matrix X be such that $\mathrm{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where 'vec' indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find $\mathrm{E}\det(X'CX)$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

We look forward to receiving a solution to this problem!

Problem 23-2: The Equality of Two 4×4 Determinants

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_2 & b_1 & b_4 & b_3 \\ a_1b_2 & a_2b_1 & a_3b_4 & a_4b_3 \end{vmatrix}$$

The solution by the proposer, who has no imagination, is to expand each determinant and to match the resulting expansions term for term. The proposer has "first dibs" on this solution. Respondents are therefore asked to provide a more elegant solution.

Solution 23-2.1 by Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Let

$$A = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ a_3 & a_4 \end{pmatrix}, \quad C = \begin{pmatrix} b_1 & b_2 \\ a_1b_1 & a_2b_2 \end{pmatrix}, \quad D = \begin{pmatrix} b_3 & b_4 \\ a_3b_3 & a_4b_4 \end{pmatrix},$$

$$\hat{C} = \begin{pmatrix} b_2 & b_1 \\ a_1b_2 & a_2b_1 \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} b_4 & b_3 \\ a_3b_4 & a_4b_3 \end{pmatrix}.$$

If $a_2 \neq a_1$ and $a_4 \neq a_3$, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B) \quad \text{and} \quad \det \begin{pmatrix} A & B \\ \hat{C} & \hat{D} \end{pmatrix} = \det(A)\det(\hat{D} - \hat{C}A^{-1}B).$$

Let

$$F = \hat{D}B^{-1} - \hat{C}A^{-1} = BE_1B^{-1} - AE_2A^{-1},$$

where

$$E_1 = \begin{pmatrix} b_4 & 0 \\ 0 & b_3 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} b_2 & 0 \\ 0 & b_1 \end{pmatrix}.$$

Suppose

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One readily checks that

$$RF^tR^{-1} = B(RE_1R^{-1})B^{-1} - A(RE_2R^{-1})A^{-1} = DB^{-1} - CA^{-1},$$

and hence

$$\det(\hat{D} - \hat{C}A^{-1}B) = \det(F)\det(B) = \det(RF^tR^{-1})\det(B) = \det(D - CA^{-1}B).$$

Thus

$$\det(A)\det(D - CA^{-1}B) = \det(A)\det(\hat{D} - \hat{C}A^{-1}B)$$

under the assumption that $a_1 \neq a_2$ and $a_3 \neq a_4$. By continuity, we see that the two determinants, as polynomials of the variables $a_1, \ldots, a_4, b_1, \ldots, b_4$, are identical. Hence, the result is true for matrices over any commuting ring. □

lution 23-2.2 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

Our elementary solution to this nice problem will show that the determinant in question is even invariant to several further *pairwise permutations* in the underlying matrix. In what follows, let a_i and b_i $(i = 1, 2, 3, 4)$ be given real numbers. For each pair (i, j), with $i, j \in \{1, 2, 3, 4\}$, we define

$$A_{ij} := \begin{pmatrix} 1 & 1 \\ a_i & a_j \end{pmatrix} \quad \text{and} \quad E_{ij} := \begin{pmatrix} b_i & 0 \\ 0 & b_j \end{pmatrix}.$$

In addition, we put

$$M := \begin{pmatrix} A_{12} & A_{34} \\ A_{12}E_{12} & A_{34}E_{34} \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} A_{12} & A_{34} \\ A_{12}E_{21} & A_{34}E_{43} \end{pmatrix},$$

We notice that $\det(M) = \det(N)$ is the claim of Problem 23-2. In order to prove this identity, we begin by deriving an explicit closed form expression for the determinant of M.

THEOREM 1. *Let M be defined as before. Then*

$$\det(M) = (a_2 - a_1)(a_4 - a_3)[b_3b_4 + b_1b_2] + (a_4 - a_1)(a_3 - a_2)[b_1b_4 + b_2b_3] - (a_3 - a_1)(a_4 - a_2)[b_1b_3 + b_2b_4]. \tag{1}$$

Proof: According to the well-known Laplace expansion theorem (see, e.g., §14.2 in [HJK, p. 92]), we obtain

$$\begin{aligned} \det(M) &= \det(A_{12})\det(A_{34})\det(E_{34}) - \det(A_{13})\det(A_{24})\det(E_{24}) + \det(A_{14})\det(A_{23})\det(E_{23}) \\ &\quad + \det(A_{23})\det(A_{14})\det(E_{14}) - \det(A_{24})\det(A_{13})\det(E_{13}) + \det(A_{34})\det(A_{12})\det(E_{12}) \\ &= \det(A_{12})\det(A_{34})[\det(E_{34}) + \det(E_{12})] + \det(A_{14})\det(A_{23})[\det(E_{14}) + \det(E_{23})] \\ &\quad - \det(A_{13})\det(A_{24})[\det(E_{13}) + \det(E_{24})] \end{aligned}$$

by expanding $\det(M)$ along the first two rows of M. Since, in view of $\det(A_{ij}) = a_j - a_i$ and $\det(E_{ij}) = b_ib_j$, this is identical to (1), the proof of Theorem 1 is complete. □

The following corollary is an immediate consequence of the 'symmetry' in our expression (1) in Theorem 1.

COROLLARY 1.1: *Let M be as before. Consider the following six groups of replacements*

(R1): $b_1 \leftrightarrow b_2$, $b_3 \leftrightarrow b_4$ (R2): $b_1 \leftrightarrow b_3$, $b_2 \leftrightarrow b_4$

(R3): $b_1 \leftrightarrow b_4$, $b_2 \leftrightarrow b_3$ (R4): $a_1 \leftrightarrow a_2$, $a_3 \leftrightarrow a_4$

(R5): $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$ (R6): $a_1 \leftrightarrow a_4$, $a_2 \leftrightarrow a_3$,

and, for each $i \in \{1, 2, \cdots, 6\}$, let M_i denote the matrix which is obtained from M according to (Ri). *We then have*

$$\det(M_i) = \det(M),$$

irrespective of the choice of i.

Since modifying M according to (R1) results in N, our corollary, in particular, shows that $\det(M) = \det(N)$ is, as claimed, indeed correct. We continue with considering some special cases of the matrix M.

COROLLARY 1.2: *Let M be as before. In addition, let i, j, r, s be such that $i < j$, $r < s$, and $\{i, j, r, s\} = \{1, 2, 3, 4\}$.*

(i) *If $a_i = a_j$, then* (1) *reduces to*

$$\det(M) = (-1)^{i+j}(a_i - a_r)(a_i - a_s)(b_j - b_i)(b_s - b_r).$$

(ii) *If $b_i = b_j$, then* (1) *reduces to*

$$\det(M) = (-1)^{i+j+1}(b_i - b_r)(b_i - b_s)(a_j - a_i)(a_s - a_r).$$

Corollary 1.2 follows directly from Theorem 1. We conclude our discussion of Problem 23-2 by extending our results to a slightly more general class of matrices. The respective proofs follow along similar lines and are hence omitted.

THEOREM 2. *For $i = 1, 2, 3, 4$: let a_i, b_i, c_i be given real numbers. If*

$$W := \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 b_1 & c_2 b_2 & c_3 b_3 & c_4 b_4 \\ a_1 b_1 & a_2 b_2 & a_3 b_3 & a_4 b_4 \end{pmatrix}$$

then $\det(W) =$ (2)

$$(c_1a_2 - c_2a_1)(c_3a_4 - c_4a_3)(b_3b_4 + b_1b_2) + (c_1a_4 - c_4a_1)(c_2a_3 - c_3a_2)(b_1b_4 + b_2b_3) - (c_1a_3 - c_3a_1)(c_2a_4 - c_4a_2)(b_1b_3 + b_2b_4).$$

COROLLARY 2.1: *Let W be defined as in Theorem 2. Consider the following six groups of replacements*

(R1): $b_1 \leftrightarrow b_2$, $b_3 \leftrightarrow b_4$

(R2): $b_1 \leftrightarrow b_3$, $b_2 \leftrightarrow b_4$

(R3): $b_1 \leftrightarrow b_4$, $b_2 \leftrightarrow b_3$

(R4): $a_1 \leftrightarrow a_2$, $a_3 \leftrightarrow a_4$, $c_1 \leftrightarrow c_2$, $c_3 \leftrightarrow c_4$

(R5): $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$, $c_1 \leftrightarrow c_3$, $c_2 \leftrightarrow c_4$

(R6): $a_1 \leftrightarrow a_4$, $a_2 \leftrightarrow a_3$, $c_1 \leftrightarrow c_4$, $c_2 \leftrightarrow c_3$,

and, for $i \in \{1, 2, \cdots, 6\}$, let W_i denote the matrix which is obtained from W according to (Ri). *Then*

$$\det(W_i) = \det(W),$$

irrespective of the choice of i.

COROLLARY 2.2: *Let W be as above. In addition, let i, j, r, s be such that $i < j$, $r < s$, and $\{i, j, r, s\} = \{1, 2, 3, 4\}$.*

(i) *If $a_i = a_j$ and $c_i = c_j$, then* (2) *reduces to* $\det(W) = (-1)^{i+j}(c_r a_i - c_i a_r)(c_s a_i - c_i a_s)(b_j - b_i)(b_s - b_r)$.

(ii) *If $b_i = b_j$, then* (2) *reduces to* $\det(W) = (-1)^{i+j+1}(b_i - b_r)(b_i - b_s)(c_i a_j - c_j a_i)(c_r a_s - c_s a_r)$.

Reference

[HJK] H.-J. Kowalsky (1967). *Lineare Algebra.* Walter de Gruyter, Berlin.

Problem 23-3: An Inequality Involving a Special Hadamard Product

Proposed by Shuangzhe LIU: Shuangzhe.Liu@maths.anu.edu.au; *Australian National University, Canberra, Australia.*

Let $A > 0$ be an $n \times n$ positive definite Hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, A^T be the transpose of A, A^{-T} be the inverse of A^T, I_n be an $n \times n$ identity matrix and $\odot$ denote the Hadamard product. Show that then, in the Löwner ordering,

$$A \odot A^{-T} \leq \frac{\lambda_1^2 + \lambda_n^2}{2\lambda_1 \lambda_n} I_n.$$

Solution 23-3.1 by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let $H = A \odot A^{-T}$ and let the spectral decomposition of A be $A = U^\star \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U$ with U a unitary matrix. Then, $a_{jk} = \sum_\ell \overline{u_{lj}} \lambda_\ell u_{\ell k}$. Also, $A^{-1} = U^\star \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) U$ and if we denote $B = A^{-T} = U^T \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) \overline{U}$, we have $b_{jk} = \sum_m u_{mj} \lambda_m^{-1} \overline{u_{mk}}$. We find that $h_{jk} = a_{jk} b_{jk} = \sum_{\ell,m} \overline{u_{\ell j}} \lambda_\ell u_{\ell k} u_{mj} \lambda_m^{-1} \overline{u_{mk}}$. We need to bound the largest eigenvalue μ of the hermitian matrix H by $\frac{1}{2}\left(\frac{\lambda_1}{\lambda_n} + \frac{\lambda_n}{\lambda_1}\right)$. Now μ is the maximum value of $z^\star H z$ as z runs over all unit vectors. We find that

$$z^\star H z = \sum_{j,k} \overline{z_j} h_{jk} z_k = \sum_{j,k,\ell,m} (u_{\ell k} z_k \overline{u_{mk}})(\overline{u_{\ell j}}\, \overline{z_j} u_{mj}) \lambda_\ell \lambda_m^{-1} = \sum_{\ell,m} |n_{\ell m}|^2 \lambda_\ell \lambda_m^{-1}$$

where N is the normal matrix given by $n_{\ell m} = \sum_k u_{\ell k} z_k \overline{u_{mk}}$. By comparing the diagonal entries of $NN^\star$ and $N^\star N$ we find that for each ℓ

$$\sum_m |n_{\ell m}|^2 = \sum_m |n_{m\ell}|^2$$

Furthermore, we have

$$\sum_{\ell,m} |n_{\ell m}|^2 = \sum_{j,k,\ell,m} \overline{u_{\ell j}} u_{\ell k} \overline{u_{mk}} u_{mj} \overline{z_j} z_k = \sum_j |z_j|^2 = 1.$$

It remains to apply the following proposition to $q_{\ell,m} = |n_{\ell,m}|^2$.

PROPOSITION. *Let $Q = (q_{\ell m})$ be an $n \times n$ nonnegative matrix with the sum of all entries equal to 1. If further the ℓth row sum equals the ℓth column sum for all ℓ, then for any $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$,*

$$\sum_{\ell,m} q_{\ell,m} \frac{\lambda_\ell}{\lambda_m} \leq \frac{1}{2}\left(\frac{\lambda_1}{\lambda_n} + \frac{\lambda_n}{\lambda_1}\right).$$

Proof. Let $\rho_\ell = \log(\lambda_\ell)$ and let $\rho = \rho_1 - \rho_n$, so that $-\rho \leq \rho_\ell - \rho_m \leq \rho$ for all ℓ and m. Then we must show that $\sum_{\ell,m} q_{\ell m} e^{\rho_\ell - \rho_m} \leq \cosh \rho$. Towards this, we define the function $\varphi(x) = e^x - \frac{\sinh \rho}{\rho} x$. It is clear that $\varphi(\pm\rho) = \cosh(\rho)$ and, since φ is convex, that $\varphi(x) \leq \cosh \rho$ for $-\rho \leq x \leq \rho$. Therefore, since the sum of all entries of Q is unity,

$$\sum_{\ell,m} q_{\ell m} \varphi(\rho_\ell - \rho_m) \leq \cosh \rho$$

which is the same as the desired conclusion since $\sum_{\ell,m} q_{\ell m}(\rho_\ell - \rho_m) = 0$, by the condition on the row and column sums. □

Problem 23-4: Trace and A Partitioned Matrix

Proposed by Heinz NEUDECKER: heinz@fee.uva.nl; *Cesaro, Schagen, The Netherlands.*

Consider the $p \times (p+1)$ matrix $X' = (x \ : \ Y')$, where Y is $p \times p$ nonsingular and $x \neq (1/p)Y'e_p$ with e_p the $p \times 1$ vector with each element equal to 1. Let $M_p := I_p - (1/p)e_p e_p'$ denote the $p \times p$ centering matrix. Prove then that the trace

$$\mathrm{tr}\left((X'M_{p+1}X)^{-1}Y'M_pY\right) = p - 1. \tag{3}$$

Solution 23-4.1 by Gülhan ALPARGU: alpargu@math.mcgill.ca; *McGill University, Montréal, Québec, Canada,* and Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

We note first that there are situations under which the equality (3) does ***not*** hold! We derive, therefore, a condition which is *both* necessary *and* sufficient for (3) to hold. Let the $p \times p$ matrix Y be nonsingular, let $x \neq (1/p)Y'e_p$, and let $X' = (\,x, \quad Y'\,)$. Then, by some easy algebraic transformations, we obtain

$$X'M_{p+1}X = \frac{p}{p+1}\left(x - \frac{1}{p}Y'e_p\right)\left(x - \frac{1}{p}Y'e_p\right)' + Y'M_pY. \tag{4}$$

This shows that $X'M_{p+1}X$ can be written as the sum of two nonnegative definite matrices. Since $x \neq (1/p)Y'e_p$, the first summand has rank 1. Since Y is nonsingular and M_p is an orthogonal projector with rank $p-1$, it is also clear that the second summand has rank $p-1$. So one could be tempted to believe that $X'M_{p+1}X$ is always nonsingular. But this, however, is not always the case. More precisely, we see from our decomposition (4) that $X'M_{p+1}X$ is nonsingular if and only if x is such that

$$x - \frac{1}{p}Y'e_p \notin \mathcal{R}(Y'M_p) \tag{5}$$

[with $\mathcal{R}(\cdot)$ denoting the range (column space) of the matrix $(\cdot)$] or, equivalently, if and only if

$$x'Y^{-1}e_p \neq 1. \tag{6}$$

In other words, when going from $Y'M_pY$ to $X'M_{p+1}X$, then the rank does not always go up by 1. The rank goes up if and only if condition (5) is satisfied. This shows that (3) can hold only if this condition (5) is satisfied.

Let us, therefore, now assume that (5), or equivalently, that (6) holds. Then it is clear that the row spaces and the column spaces of the two summand matrices in decomposition (4) do have only the respective origin in common. Such matrices are said to be *weakly bicomplementary* to each other; see [WGI]. (A pair of weakly bicomplementary matrices is also often said to be a pair of disjoint matrices; cf. [MFR].) If A, B is such a pair of weakly bicomplementary matrices, then each g-inverse of $A+B$ is also a g-inverse of A as well as a g-inverse of B; see [JMW, Th. 2.3]. Therefore, in particular,

$$Y'M_pY(X'M_{p+1}X)^{-1}Y'M_pY = Y'M_pY,$$

which in turn implies that the matrix $Y'M_pY(X'M_{p+1}X)^{-1}$ is a projector onto $\mathcal{R}(Y'M_pY)$. The rank of this projector thus coincides with the rank of the matrix $Y'M_pY$ which is $p-1$. Since for each projector its rank coincides with its trace, our proof is complete. $\square$

Remark: We note that the concept of weak bicomplementarity was used in Solution 22-3.4 (IMAGE 23, p. 23).

References

[JMW] S. K. Jain, S. K. Mitra & H. J. Werner (1996). Extensions of $\mathcal{G}$-based matrix partial orders. *SIAM Journal on Matrix Analysis and Applications*, 17, 834–850.

[MFR] S. K. Mitra (1972). Fixed rank solutions of linear matrix equations. *Sankhyā Series A*, 34, 387–392.

[WGI] H. J. Werner (1986). Generalized inversion and weak bi-complementarity. *Linear and Multilinear Algebra*, 19, 357–372.

A solution was also received from Jos M. F. TEN BERGE: j.m.f.ten.berge@ppsw.rug.nl; *Rijksuniversiteit Groningen, The Netherlands.*

Problem 23-5: An Inequality Involving Diagonal Elements and Eigenvalues

Proposed by Alicja SMOKTUNOWICZ: smok@im.pw.edu.pl; *Warsaw University of Technology, Warsaw, Poland.*

Prove that each eigenvalue λ of $A \in \mathcal{C}^{n\times n}$ such that $\lambda \neq a_{i,i}$ for all $i = 1, \ldots, n$, satisfies

$$\sum_{k=1}^{n} \frac{{r_k}^2}{\mid \lambda - a_{k,k} \mid^2} \geq \frac{n}{n-1}, \qquad \text{where} \qquad r_k = \sqrt{\sum_{j=1, j\neq k}^{n} \mid a_{k,j} \mid^2}.$$

Solution 23-5.1 by Gülhan ALPARGU: alpargu@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let λ be an eigenvalue of A satisfying $\lambda \neq a_{kk}$ for all $k = 1, \ldots, n$. By Gerschgorin's Theorem we then have, for each k, that $|\lambda - a_{k,k}| \leq d_k$ or, equivalently, $|\lambda - a_{k,k}|^2 \leq d_k^2$, where $d_k = \sum_{j=1,j\neq k}^{n} |a_{k,j}|$. From the Cauchy-Schwarz Inequality we then obtain $d_k^2 \leq (n-1)r_k^2$. Combining our observations now, for each $k = 1, 2, \ldots, n$, results in

$$\frac{1}{n-1} \leq \frac{r_k^2}{|\lambda - a_{k,k}|^2}.$$

Since this in turn implies

$$\frac{n}{n-1} \leq \sum_{k=1}^{n} \frac{r_k^2}{|\lambda - a_{k,k}|^2},$$

our proof is complete. □

Solution 23-5.2 by R. B. BAPAT: rbb@isid.ac.in; *Indian Statistical Institute–Delhi Centre, New Delhi, India.*

Let λ be an eigenvalue of A such that $\lambda \neq a_{i,i}$ for all $i = 1, \ldots, n$ and let x be an eigenvector corresponding to λ with $\sum_{k=1}^{n} |x_k|^2 = 1$. Then

$$(\lambda - a_{k,k})x_k = \sum_{j=1,j\neq k}^{n} a_{k,j}x_j, \;\; k = 1, \ldots, n.$$

It follows, by the Cauchy-Schwarz Inequality, that

$$|\lambda - a_{k,k}|^2 |x_k|^2 \leq \left(\sum_{j=1,j\neq k}^{n} |a_{k,j}|^2 \right) \left(\sum_{j=1,j\neq k}^{n} |x_j|^2 \right), \;\; k = 1, \ldots, n.$$

Therefore

$$\sum_{k=1}^{n} \frac{r_k^2}{|\lambda - a_{k,k}|^2} \geq \sum_{k=1}^{n} \frac{|x_k|^2}{\sum_{j=1,j\neq k}^{n} |x_j|^2} = \sum_{k=1}^{n} \frac{|x_k|^2}{1 - |x_k|^2}, \tag{7}$$

where $r_k^2 = \sum_{j=1,j\neq k}^{n} |a_{k,j}|^2$. (Since $\lambda \neq a_{ii}$ for $i = 1, \ldots, n$, x has at least two nonzero coordinates and hence $1 - |x_k|^2$ is nonzero, $k = 1, \ldots, n$.) By the weighted arithmetic mean - harmonic mean inequality,

$$\sum_{k=1}^{n} \frac{|x_k|^2}{1 - |x_k|^2} \geq \frac{1}{\sum_{k=1}^{n} |x_k|^2 (1 - |x_k|^2)} = \frac{1}{1 - \sum_{k=1}^{n} |x_k|^4}. \tag{8}$$

Since $\sum_{k=1}^{n} |x_k|^2 = 1$, an application of the Cauchy-Schwarz Inequality gives

$$\sum_{k=1}^{n} |x_k|^4 \geq \frac{1}{n},$$

and hence

$$\frac{1}{1 - \sum_{k=1}^{n} |x_k|^4} \geq \frac{n}{n-1}. \tag{9}$$

The result then follows from (7)–(9). □

Solution 23-5.3 by Lajos LÁSZLÓ: llaszlo@ludens.elte.hu; *Eötvös Lóránd University, Budapest, Hungary.*

Let λ be an eigenvalue of the $n \times n$ matrix A satisfying $\lambda \neq a_{ii}$, $i = 1 \ldots n$, and x be the associated eigenvector. The kth component of $Ax - \lambda x = 0$ gives

$$|(\lambda - a_{kk})x_k|^2 = \left|\sum_{j\neq k}^{n} a_{kj}x_j\right|^2 \leq r_k^2\, q_k^2,$$

with

$$r_k \equiv \left(\sum_{j\neq k}^{n} |a_{kj}|^2\right)^{1/2} \quad \text{and} \quad q_k \equiv \left(\sum_{j\neq k}^{n} |x_j|^2\right)^{1/2}.$$

Here, $q_k \neq 0$, for $q_k = 0$ would imply $x_k = 0$ and hence $x = 0$, which contradicts the definition of eigenvector. Thus we have

$$\frac{r_k^2}{|\lambda - a_{kk}|^2} \geq \frac{|x_k|^2}{q_k^2},$$

and, by summing up all the n terms,

$$\sum_k \frac{r_k^2}{|\lambda - a_{kk}|^2} \geq \sum_k \frac{|x_k|^2}{q_k^2} \equiv f(x) \geq \frac{n}{n-1}.$$

The last inequality follows from a result in Chapter 3, §D.6 of Marshall & Olkin [MOI]. □

Remark. Of several possible generalizations using Hölder's inequality, one is especially interesting because of the same minimum value. To see this write

$$|(\lambda - a_{kk})x_k| \leq \max_{j\neq k}|a_{kj}| \sum_{j\neq k} |x_k| \equiv R_k\, Q_k,$$

where $Q_k \neq 0$ by the same reasoning as above, and obtain

$$\sum_k \frac{R_k}{|\lambda - a_{kk}|} \geq \sum_k \frac{|x_k|}{Q_k} \equiv F(x) \geq \frac{n}{n-1},$$

since the minimum values for the functions f and F obviously coincide.

Reference

[MOI] A. W. Marshall & I. Olkin (1979). *Inequalities: Theory of Marjorization and its Applications.* Academic Press, New York.

Solution 23-5.4 by the Proposer Alicja SMOKTUNOWICZ: smok@im.pw.edu.pl; *Warsaw University of Technology, Warsaw, Poland.*

We prove that this inequality is essentially based on the inequality between the harmonic and arithmetic mean:

$$\frac{c_1 + \ldots + c_n}{n} \geq \frac{n}{\frac{1}{c_1} + \ldots + \frac{1}{c_n}} \tag{10}$$

where $c_i > 0$ for all i.

Let λ be an eigenvalue of A, and suppose $Ax = \lambda x$, $x \neq 0$. This means that

$$a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,n}x_n = \lambda x_k, \quad k = 1, \ldots, n$$

which is equivalent to

$$(\lambda - a_{k,k})x_k = \sum_{j=1, j\neq k}^{n} a_{k,j}x_j, \quad k = 1, \ldots, n.$$

We obtain

$$|\, \lambda - a_{k,k} \,||\, x_k \,| \leq \sum_{j=1, j\neq k}^{n} |\, a_{k,j} \,||\, x_j \,|.$$

for all $k = 1, \ldots, n$. By the Cauchy-Schwarz inequality we have

$$\sum_{j=1, j \neq k}^{n} | a_{k,j} | | x_j | \leq \sqrt{\sum_{j=1, j \neq k}^{q} | a_{k,j} |^2} \sqrt{\sum_{j=1, j \neq k}^{q} | x_j |^2}.$$

Thus

$$| \lambda - a_{k,k} |^2 | x_k |^2 \leq {r_k}^2 \sum_{j=1, j \neq k}^{q} | x_j |^2, \quad k = 1, \ldots, n. \tag{11}$$

To simplify the notation, let

$$b_k = \sum_{j=1, j \neq k}^{n} | x_j |^2.$$

The elements we have just defined satisfy

$$b_k = \| x \|_2{}^2 - | x_k |^2$$

and

$$\sum_{k=1}^{n} b_k = (n-1) \; \| x \|_2{}^2.$$

Thus, we can rewrite (11) as follows $| \lambda - a_{k,k} |^2 | x_k |^2 \leq {r_k}^2 \, b_k, \; k = 1, \ldots, n$. Notice that by our assumptions, all b_k and $| \lambda - a_{k,k} |$ are positive. From this we obtain

$$\sum_{k=1}^{n} \frac{{r_k}^2}{| \lambda - a_{k,k} |^2} \geq \sum_{k=1}^{n} \frac{| x_k |^2}{b_k} = \| x \|_2{}^2 \sum_{k=1}^{n} \frac{1}{b_k} \; - n. \tag{12}$$

By an application of the harmonic–arithmetic mean inequality (10), with $c_k = 1/b_k$, we have

$$\| x \|_2{}^2 \sum_{k=1}^{n} \frac{1}{b_k} \geq \frac{n^2 \| x \|_2{}^2}{\sum_{k=1}^{n} b_k} = \frac{n^2}{n-1}.$$

This result and (12) give the desired inequality. □

Problem 23-6: Linear Combinations and Eigenvalues

Proposed by Jos M. F. TEN BERGE: j.m.f.ten.berge@ppsw.rug.nl; *Rijksuniversiteit Groningen, Groningen, The Netherlands.*

Suppose we have two real matrices of order $p \times p$, with p even, and with all eigenvalues imaginary. Is it possible to find p linear combinations of the matrices that have at least one real-valued eigenvalue?

I expect that this is not always possible and that the set of matrix pairs that does allow real eigenvalues for linear combinations has positive measure. Is there a place in the literature where I can find such things?

Solution 23-6.1 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*
We begin with considering only 2×2 real matrices. From the literature, cf. e.g., [PLT, p. 55], we know that the characteristic polynomial of a 2×2 matrix A, say, can be written in the form

$$c_A(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A); \tag{13}$$

here $\mathrm{tr}(\cdot)$ and $\det(\cdot)$, respectively, denote, as usual, the trace and the determinant of the matrix $(\cdot)$. By $\mathrm{eig}(A)$ we will denote the set of all eigenvalues of A, i. e., the set of solutions to the characteristic equation $c_A(\lambda) = 0$. In view of (13), we have the following characterization.

THEOREM 1: *Let Im denote the set of all those real 2×2 matrices with purely imaginary eigenvalues. Then $A \in Im$ if and only if*

$$\mathrm{tr}(A) = 0 \qquad \text{and} \qquad \det(A) > 0.$$

In what follows, let $A, B \in Im$. Then, by Theorem 1,

$$A = \begin{pmatrix} a_1 & a_3 \\ a_2 & -a_1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_1 & b_3 \\ b_2 & -b_1 \end{pmatrix},$$

where $-a_1^2 - a_2a_3 > 0$ and $-b_1^2 - b_2b_3 > 0$.

We notice that

$$\mathrm{eig}(A) = \left\{\pm i\sqrt{-a_1^2 - a_2a_3}\right\} \quad \text{and} \quad \mathrm{eig}(B) = \left\{\pm i\sqrt{-b_1^2 - b_2b_3}\right\}.$$

For fixed real numbers p and q, what can we say about the eigenvalues of the matrix $C := pA+qB$? Since $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(B) = 0$, clearly $\mathrm{tr}(C) = 0$ and so

$$\mathrm{eig}(C) = \left\{\pm\sqrt{-\det(C)}\right\}.$$

Hence, if $\det(C) > 0$ or, equivalently, if

$$-p^2(a_1^2 + a_2a_3) - q^2(b_1^2 + b_2b_3) > pq(2a_1b_1 + a_2b_3 + a_3b_2),$$

then $C \in Im$. Otherwise, i.e., if $\det(C) \le 0$ or, equivalently, if

$$-p^2(a_1^2 + a_2a_3) - q^2(b_1^2 + b_2b_3) \le pq(2a_1b_1 + a_2b_3 + a_3b_2), \tag{14}$$

then C has, in contrast to A and B, two purely real eigenvalues, and we indicate this by writing $C \in Re$. The following example illustrates that both situations can indeed occur for different values of p and q.

EXAMPLE 1: Consider

$$A := \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 2 & 4 \\ -3 & -2 \end{pmatrix}.$$

Then, $\mathrm{eig}(A) = \{\pm i\sqrt{3}\}$, $\mathrm{eig}(B) = \{\pm i\sqrt{8}\}$, $\mathrm{eig}(A+B) = \{\pm\sqrt{7}\}$, $\mathrm{eig}(A+2B) = \{\pm 1\}$, and $\mathrm{eig}(A+3B) = \{\pm i\sqrt{21}\}$. Moreover, as the reader may verify,

$$A + qB \in Im \qquad \Longleftrightarrow \qquad q^2 > \frac{1}{8}(18q - 3). \qquad \square$$

Our example, in particular, shows that there do exist matrices $A, B \in Im$ with (i) $pA + qB \in Im$ for infinite many pairs (p, q), and, at the same time, with (ii) $pA + qB \in Re$ for infinite many other pairs (p, q). But does this also mean that both cases are possible for any pair $A, B \in Im$? That the answer is negative is illustrated by our next example.

EXAMPLE 2: Consider

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then, for each pair of real numbers p and q

$$\mathrm{eig}(pA + qB) = \left\{+i\sqrt{p^2+q^2}, -i\sqrt{p^2+q^2}, +i\sqrt{p^2+q^2}, -i\sqrt{p^2+q^2}\right\}.$$

We notice, in passing, that the characteristic polynomial of $pA + qB$ is $c_{pA+qB}(\lambda) = \lambda^4 + 2(p^2 + q^2)\lambda^2 + (p^2 + q^2)^2$. Moreover,

$$\text{eig}(A) = \{+i, -i, +i, -i\}, \qquad \text{eig}(B) = \{+i, -i, +i, -i\}.$$

Hence, we do not only have $A \in Im$ and $B \in Im$ but also $pA + qB \in Im$, for each pair $(p, q) \neq 0$. □

We now let $A, B \in Im$ be again given 2×2 matrices and assume that for some given real numbers p and q, the inequality (14) strictly holds. Then, by continuity, (14) also holds for all other matrix pairs lying in the intersection of Im and a *sufficiently small* neighborhood of (A, B). Needless to emphasize once more, $C \in Im$ only if $\text{tr}(C) = 0$.

Reference

[PLT] P. Lancaster (1969). *Theory of Matrices.* Academic Press, New York.

Problem 23-7: An Inequality Involving Rank and Matrix Powers

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Let A be a square matrix of size $n \times n$. Show that there are n vectors $x_0, x_1, \cdots, x_{n-1}$ such that the square matrix $(x_0, Ax_1, A^2x_2, \cdots, A^{n-1}x_{n-1})$ is nonsingular if and only if

$$\min\{1 + \text{rank}\, A, 2 + \text{rank}\, A^2, \cdots, n - 1 + \text{rank}\, A^{n-1}\} \geq n.$$

Solution 23-7.1 by John ASTON: jaston@bic.mni.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Below we make use of the fact that for each nonnegative integer i, we have $\mathcal{R}(A^i) \subseteq \mathcal{R}(A^{i-1})$, with $\mathcal{R}(\cdot)$ indicating the range (column space) of $(\cdot)$. Observe that we put $A^0 := I_n$, where I_n stands for the $n \times n$ identity matrix. For proving necessity, let there exist n vectors $x_0, x_1, \cdots, x_{n-1}$ such that the matrix

$$\begin{pmatrix} x_0 & Ax_1 & \cdots & A^{n-1}x_{n-1} \end{pmatrix}$$

is nonsingular. Observe that in view of our above inclusion this can happen *only if* $\text{rank}(A^i) \geq n - i$ is satisfied for each $i = 1, 2, \cdots, n - 1$. Since the latter is equivalent to

$$\min\{1 + \text{rank}\, A, 2 + \text{rank}\, A^2, \cdots, n - 1 + \text{rank}\, A^{n-1}\} \geq n,$$

the proof of necessity is complete. To prove sufficiency, let $\text{rank}(A^i) \geq n - i$ be satisfied for each $i = 1, 2, \cdots, n - 1$. In virtue of our above-mentioned inclusion, we can then, of course, choose in backward order the vectors x_i $(i = n - 1,\ n - 2,\ \cdots,\ 1,\ 0)$ such that the matrices

$$M_i := \begin{pmatrix} A^i x_i & A^{i+1}x_{i+i} & \cdots & A^{n-1}x_{n-1} \end{pmatrix}, \qquad i = n-1, n-2, \cdots, 1, 0,$$

are all of full column rank. This completes the proof. □

A solution was also received from the Proposer Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario.*

IMAGE Problem Corner: New Problems

Vol I, Prob 26, §4.4, p. 217

Problem 24-1: Rank of a Skew-Symmetric Matrix

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let A be a complex skew-symmetric matrix. Show that the non-zero singular values of A are equal in pairs and deduce that A necessarily has even rank.

Problem 24-2: Nonnegative Matrix with All Entries Summing to One

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let Q be a real $n \times n$ elementwise nonnegative matrix with the sum of all entries equal to 1 and, for each j, the jth row sum equals the jth column sum. Show that Q is a convex combination of matrices $R(X, \sigma)$ described as follows. Let X be a nonempty subset of $I_n = \{1, 2, \dots, n\}$ with k elements and let σ be a cyclic permutation of X. Then the matrix $R = R(X, \sigma)$ is given by

$$r_{ij} = \frac{1}{k} \quad \text{if } i \in X \text{ and } j = \sigma(i) \quad \text{and} \quad r_{ij} = 0 \text{ otherwise.}$$

Remark. This Problem is related to Solution 23-3.1 in this IMAGE, page 9.

Problem 24-3: A Possible Generalization of Drury's Determinantal Equality

Proposed by George P. H. STYAN: styan@total.net; *McGill University, Montréal, Québec, Canada.*

Let A, B, C, D, E, and F be $n \times n$ complex matrices with

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} A & B \\ E & F \end{pmatrix}$$

and $\det(C) = \det(E)$ and $\det(D) = \det(F)$. Find further necessary and sufficient conditions (if any) so that $\det(R) = \det(S)$.

Remark. This problem is inspired by Problem 23-2 by S. W. Drury—see this IMAGE 24, pp. 6–8.

Problem 24-4: The Positive Definiteness of a Matrix

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Suppose that $p(\lambda)$ and $q(\lambda)$ are any two nonzero polynomials with real coefficients and without common roots and that A is any complex square matrix of order m. Show that the Hermitian matrix $p(A)p(A^*) + q(A^*)q(A)$ is always positive definite.

$A = U\Delta U^*$ reduces $A \to \Delta$, $p(\Delta)p(\Delta)^* + q(\Delta)^* q(\Delta) = [p(\Delta) \;\; q(\Delta)^*] \begin{bmatrix} p(\Delta)^* \\ q(\Delta) \end{bmatrix}$

show that this has full rank. If ∃ a zero diagonal entry above, swap that row for corresp. row below, whose diagonal entry must be nonzero.

Problem 24-5: Two Products Involving Idempotent Matrices

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

(a) Suppose that A and B are two complex Hermitian idempotent matrices of the same size. Show that

$$A(A - B)^{\dagger} B = B(A - B)^{\dagger} A = 0,$$

All we need to assume is that if $\lambda \in \sigma(A)$, and then $|p(\lambda)| + |q(\lambda)| \neq 0$

where $(\cdot)^{\dagger}$ denotes Moore-Penrose inverse.

(b) Suppose the complex matrix A is idempotent. Show that $A(A - A^*)^{\dagger} A^* = A^*(A - A^*)^{\dagger} A = 0$.

Problem 24-6: Rank of a Principal Submatrix

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida.*

Let A be a (complex) positive semidefinite matrix and write $[A]_\alpha$ for the principal submatrix of A consisting of rows α and columns α. Show that for any positive integer k

$$\operatorname{rank}([A]_\alpha) = \operatorname{rank}([A^k]_\alpha) = \operatorname{rank}(\{[A]_\alpha\}^k).$$

Does this generalize to real numbers k or to Hermitian A?

Problem 24-7: Partitioned Positive Semidefinite Matrices

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida.*

Let A be a positive semidefinite matrix. With $\dagger$ for Moore-Penrose inverse, $\circ$ for Hadamard product, and $\geq$ for Löwner ordering, show that

$$A \circ A^\dagger \geq A^\dagger A \circ AA^\dagger \quad \text{and} \quad A + A^\dagger \geq A^\dagger A + AA^\dagger.$$

In particular, if A is invertible, then

$$A \circ A^{-1} \geq I \quad \text{and} \quad A + A^{-1} \geq 2I.$$

Problem 24-8: An Inequality Involving Hadamard Products

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida.*

Let A and B be complex positive semidefinite matrices of the same size. If B is nonsingular, with $\circ$ for Hadamard product and $\geq$ for Löwner ordering, show that

$$A^2 \circ B^{-1} \geq (\operatorname{diag}A)^2(\operatorname{diag}B)^{-1}.$$

In particular, taking $B = I$ and $A = I$, respectively, it follows at once that $\operatorname{diag}(A^2) \geq (\operatorname{diag}A)^2$ and $\operatorname{diag}B^{-1} \geq (\operatorname{diag}B)^{-1}$.

Problem 24-9: Decomposition of Symmetric Matrices

Proposed by Roman ZMYŚLONY: r.zmyslony@im.pz.zgora.pl; *Technical University of Zielona Góra, Poland,*
and Götz TRENKLER: trenkler@statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany.*

It is well-known that a real symmetric $n \times n$ matrix A can be uniquely written as $A = A_+ - A_-$, where A_+ and A_- are nonnegative definite matrices such that $A_+A_- = 0$. Find a procedure to calculate the pair $(A_+,\ A_-)$ without using the eigenvalues and eigenvectors of the matrix A. Assume $n \geq 3$.

Remark: This decomposition of a real symmetric matrix plays a crucial role in the estimation and testing of variance components in linear models.

Please submit solutions, as well as new problems, in LaTeX – (a) embedded as text in an e-mail to styan@total.net and (b) 2 copies (nicely printed please) by p-mail to George P. H. Styan, PO Box 270, Franklin, VT 05457-0270, USA. Please make sure that your name, as well as your e-mail and p-mail addresses are included! We look forward particularly to receiving solutions to Problems 18-1, 19-3b, 21-2 & 23-1!

Selected Forthcoming Linear Algebra Events

Western Canada Linear Algebra Meeting

Winnipeg, Manitoba: May 26–27, 2000

The Western Canada Linear Algebra Meeting (W-CLAM) provides an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting. Previous W-CLAMs were held in Regina (1993), Lethbridge (1995), Kananaskis (1996), and Victoria (1998).

The W-CLAM '00 will be held at the University of Manitoba, Winnipeg, May 26–27, 2000, and is supported by the National Program Committee of the Institutes CRM, Fields, PIms and the University of Manitoba.

We will have three distinguished invited speakers:

- Hans Schneider (University of Wisconsin)
- Bryan Shader (University of Wyoming)
- Henry Wolkowicz (University of Waterloo).

The Web site address is

http://www.math.uregina.ca/~tsat/nr/nr.html

You can register for the meeting online or by contacting Steve Kirkland: kirkland@math.uregina.ca. The participation fee is C$20, to be collected at the meeting. This fee will be waived for participating students and post-doctoral fellows. Support of C$300 per person will be available for a limited number of participating students and post-doctoral fellows on a first-come, first-served basis.

The meeting will be held in Room 106 of the Drake Center, which is on the lower level of building #12 on the campus map. Note that this is NOT the math building—the Drake Center is across campus from Machray Hall where the math department is housed—please visit the Web site for a campus map.

Possibilities for accommodation are as follows:

Holiday Inn Fort Richmond (10–15 minute drive down Pembina). To book, call 204-275-7711 and quote confirmation number 65945650. A block of rooms is reserved under the name "Shivakumar" at the rate of C$69 + taxes, single or double occupancy. (FAX: 204-269-0364.) Book by May 15.

Radisson, downtown (1/2 hour drive) at 288 Portage Ave. Rate C$89 single/double. Phone 204-956-0410 or Fax 204-949-1162.

UM dorms (Speechly/Tache). Washrooms central on each floor, lounge facilities with TV's, coffee-shop/snack-bar, cafeteria during weekdays. An information sheet and accommodation reservation form must be used for reservations. E-mail Michael Tsatsomeros (tsat@math.uregina.ca) or Rob Craigen (craigen@server.maths.umanitoba.ca) to obtain these forms. Rates: single C$34.77, double C$25.65 per person, and there is a one-time administration fee of C$5.

The Seventh SIAM Conference on Applied Linear Algebra

Raleigh, North Carolina: October 23–26, 2000

The Seventh SIAM Conference on Applied Linear Algebra will be held, in cooperation with the International Linear Algebra Society, in the McKimmon Conference Center, North Carolina State University, Raleigh, North Carolina, October 23–26, 2000. The Web site address is

http://www.siam.org/meetings/la00/

Plenary Speakers: Gene Golub, Tom Kailath, Eduardo Marques de Sà (ILAS), Gil Strang, Charlie Van Loan, Richard Varga, Hugo Woerdeman (ILAS), Margaret Wright.

Invited Concurrent Speakers: Tony Chan, Jack Dongarra, Roger Horn, John Lewis, Volker Mehrmann, Dianne O'Leary, Lothar Reichel, Siegfried Rump, Paul Van Dooren.

Minisymposium Organizers: Michele Benzi, Mike Berry, Jim Demmel, Charlie Johnson, Jim Nagy, Dan'l Pierce, Henk Van der Vorst.

Second Announcement
The Ninth International Workshop on Matrices and Statistics in Celebration of C. R. Rao's 80th Birthday

Hyderabad, India: December 9–13, 2000

The Ninth International Workshop on Matrices and Statistics, in Celebration of C. R. Rao's 80th Birthday, will be held in the historic walled city of Hyderabad, in Andhra Pradesh, India, on December 9–13, 2000; Hyderabad is approximately midway between Mumbai (Bombay) and Chennai (Madras) and is the fifth largest city in India, with a population of 6 million. Founded by Quli Qutub Shah in 1591, this large metropolis is unique in its rich architectural glory and blend of linguistic, religious, and ethnic groups and is an ideal place to celebrate C. R. Rao's 80th Birthday. Fine wather is expected wuth a midday high of 20° C.

The program will start with a two-day course on recent advances in Matrix Theory with Special Reference to Applications in Statistics on Saturday, December 9, and Sunday, December 10, 2000. This will be followed by the presentation of research papers in the Workshop proper on Monday, December 11–Wednesday, December 13, 2000; it is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and Its Applications.*

The International Organizing Committee for this Workshop comprises R. William Farebrother (Victoria University of Manchester, Manchester, England, UK), Simo Puntanen (University of Tampere, Tampere, Finland), George P. H. Styan (McGill University, Montréal, Québec, Canada; vice-chair), and Hans Joachim Werner (University of Bonn, Bonn, Germany; chair). The Local Organizing Committee in India includes Rajendra Bhatia (Indian Statistical Institute–Delhi Centre), P. Bhimasankaram (Indian Statistical Institute, Hyderabad), V. Narayana (Indian Statistical Institute, Hyderabad), M. S. Rao (Osmania University, Hyderabad), B. Sidharth (Birla Science Centre, Hyderabad), U. SuryaPrakesh (Osmania University, Hyderabad), R. J. R. Swamy (Osmania University, Hyderabad), P. Udayasree (University of Hyderabad), K. Viswanath (University of Hyderabad).

This Workshop is being organized in Hyderabad by the Indian Statistical Institute and the Society for Development of Statistics, in collaboration with the Birla Science Centre, Osmania University and the University of Hyderabad.

The Short Course will cover generalized matrix inverses (including Moore-Penrose, Drazin and group inverses), matrix partial orderings (including minus and sharp orders), matrix inequalities (including inequalities for determinants, traces, eigenvalues and singuar values), together with statistical applications. In addition there will be a session on statistical proofs of matrix results. We expect that the participants will be researchers in Mathematics, Statistics, Computer Science, Engineering, Physical and Earth Sciences and related areas. It may be assumed that all the participants have a knowledge of Linear Algebra at the level of Hoffman & Kunze [*Linear Algebra*, 2nd ed., Prentice Hall 1971] or Noble & Daniel [*Applied Linear Algebra*, 3rd ed., Prentice Hall 1988.] A set of lecture notes will be provided for each participant.

The Short Course will be followed by the Workshop proper, which will include the presentation of both invited and contributed papers on matrices and statistics. There will also be a special session for papers presented by graduate students. We expect that the topics to be covered will be similar to those covered by the several Special Issues on Linear Algebra and Statistics of *Linear Algebra and Its Applications.* The Web site address is

http://eos.ect.uni-bonn.de/HYD2000.htm

To submit a contributed paper for presentation at the Workshop, please use LaTeX and the style file available from the Web site. The abstract should not exceed 25 lines. Send the title and abstract, plus the names, e-mail addresses and affiliations of <u>all</u> authors by e-mail to Hans Joachim Werner at

werner@united.econ.uni-bonn.de

<u>and</u> a copy by regular mail (p-mail) or FAX to <u>both</u>

Hans Joachim Werner, Institut für Ökonometrie & Operations Research, Ökonometrische Abteilung, Adenauerallee 24-42, Rheinische Friedrich-Wilhelms-Universität Bonn, D-53113 Bonn, Germany; FAX (49-228) 73-9189, <u>and to</u>

P. Bhimasankaram, Indian Statistical Institute, Street No. 8, Habsiguda, Hyderabad–500 007, India; FAX (91-40) 717-3602.

The deadline for receipt of abstracts of contributed papers is September 30, 2000. The registration fees will be as follows:

- *For payment received by September 30, 2000:*
  Short Course: US$30/C$45 (Rs. 250/- Indian residents),
  Workshop: US$60/C$90 (Rs. 500/- Indian residents).
- *For payment received after September 30, 2000:*
  Short Course: US$36/C$54 (Rs. 300/- Indian residents),
  Workshop: US$72/C$108 (Rs. 600/- Indian residents).

There will be no registration fees for students for the Short Course. For students and retired persons the registration fee for the Workshop will be US$30/C$45 (Rs. 250/- Indian residents). There will be no registration fees for accompanying persons. The registration fees cover all conference materials and handouts, lunches, and tea and coffee. We expect that lodging will be available at low cost: full details will be given on the Workshop Web site: http://eos.ect.uni-bonn.de/HYD2000.htm, which will be updated regularly, and in the Third Announcement. The Workshop banquet (US$10/C$15; Rs. 150/- Indian residents) will be on Monday, December 11, 2000.

Indian residents should pay the registration fees through Crossed Demand Draft on any nationalized bank, payable at Hyderabad, in favour of R. J. R. Swamy/NIWMS-2000. Participants not resident in India may pay the registration fees in advance by personal check in either US or Canadian dollars drawn on a US or Canadian bank and made payable to George P. H. Styan, and sent to George P. H. Styan, PO Box 270, Franklin, VT 05457-0270, USA.

We regret that we are not able to accept payment of registration fees by credit card. All correspondence regarding registration fees from participants resident in Europe should be directed to H. J. Werner: werner@united.econ.uni-bonn.de; FAX (49-228) 73-9189, and from participants not resident in India or in Europe should be directed to George P. H. Styan: styan@total.net, FAX (1-514) 398-3899 (e-mail preferred).

All correspondence regarding lodging in Hyderabad for this Workshop should be directed to R. J. R. Swamy, Dept. of Statistics, Osmania University, Hyderabad-500 007, India; rjrs@coolmail.com, FAX (91-40) 717-3602. All correspondence concerning the scientific part of this Workshop should be directed to Hans Joachim Werner and/or P. Bhimasankaram, at their respective addresses given above.

ILAS Treasurer's Report: March 1, 1999–February 29, 2000

by James R. Weaver, *University of West Florida, Pensacola*

Balance on hand March 1, 1999			
Certificate of Deposits (CD)	19,000.00		
Vanguard	9,523.65		
Checking Account	36,828.59		65,352.24
**			
Checking Account Balance on March 1, 1999			36,828.59
March 1999			
Income:			
Interest (Gen)	12.97	12.97	
Expenses:			
Office Depot(Supplies)	13.95		
Judy K. Weaver (ILAS Files & Annual Report)	171.50		
Department of State	70.00		
Service Fee (February)	6.76	262.21	(249.24)
April 1999			
Income:			
Dues	260.00		
Interest (Gen)	11.74		
Interest on CD (FU)	21.08		
Contributions			
General Fund	35.00		
OT/JT Lecture Fund	12.00	339.82	
Expenses:			
J. Bernard Gyamfi (Copy Editing for ELA)	182.00		
Service Fee (March)	.30	182.30	157.52
May 1999			
Income:			
Dues	130.00		
Interest (Gen)	11.02		
Interest on CD (HS)	68.47		
Interest on CD (OT/JT)	26.63		
Contributions			
General Fund	20.00	256.12	
Expenses:			
Service Fee (April)	1.91	1.91	254.21
June 1999			
Income:			
Dues	40.00		
Interest (Gen)	12.71		
Interest on CD (FU)	21.55		
Contributions			
Hans Schneider Prize	30.00	104.26	
Expenses:			
George P. Styan (IMAGE)	1,356.51		
Prof. Gene Golub (ILAS Lect.)	500.00		
Benton's Plaque for H.S.Prize	44.50		
Office Depot (Supplies)	44.42		
Judy K. Weaver (Dues Notice & Files)	129.50		
Service Fee (May)	1.28	2,076.21	(1,971.95)
July 1999			
Income:			
Interest (Gen)	11.33	11.33	

Expenses:			
Chi-Kwong Li (Expense on OT/JT Lecture)	1,300.00		
Ludwig Elsner (Expense as H.S. Prize Winner)	800.00		
Jeffrey L. Stuart (Expenses)	600.00		
Service Fee (June)	1.33	2,701.33	(2,690.00)
August 1999			
Income:			
Dues	1,530.92		
Interest (Gen)	11.83		
Interest on CD (HS)	70.76		
Interest onCD (OT/JT)	27.52		
Contributions			
Elsevier Science Inc. (ILAS-LAA)	1,000.00		
ETSEIB-UPC	500.00		
General Fund	138.50		
Hans Schneider Prize	66.00		
OT/JT Lecture Fund	11.00		
F. Uhlig Fund	6.00		
Conference Fund	21.00	3,383.53	
Expenses:			
Postmaster (Stamps)	247.16		
Richard A. Brualdi (Partial support-8th ILAS Conf.)	166.67		
James R. Weaver (Partial support-8th ILAS Conf.)	166.66		
Daniel Hershkowitz (Partial support-8th ILAS Conf.)	166.67		
Judy K. Weaver (ILAS Files & Dues Notice)	108.50		
Service Fee (July)	.30	855.96	2,527.57
September 1999			
Income:			
Dues	660.00		
Interest (Gen)	11.34		
Contributions			
General Fund	50.00		
Hans Schneider Prize	80.00		
OT/JT Lecture Fund	190.00		
F. Uhlig Fund	10.00		
Conference Fund	10.00	1,011.34	
Expenses:			
Mail Boxes Etc. (Copies)	34.40		
Postmaster (Stamps)	244.70		
Postmaster (Stamps)	30.00		
Service Fee (August)	7.84	316.94	694.40
October 1999			
Income:			
Dues	580.00		
Interest (Gen)	11.15		
Interest on CD (FU)	21.55		
Contributions			
General Fund	60.00		
Hans Schneider Prize	205.00		
OT/JT Lecture Fund	40.00		
Conference Fund	5.00	922.70	
Expenses:			
Judy K.Weaver (ILAS Files)	157.50		
Service Fee (September)	4.39	161.89	760.81

November 1999			
Income:			
Dues	800.00		
Interest (Gen)	12.55		
Interest on CD (HS)	70.78		
Interest on CD (OT/JT)	27.53		
Contributions			
SIAM	200.00		
Hans Schneider Prize	30.00		
OT/JT Lecture Fund	55.00		
F. Uhlig Fund	20.00		
Conference Fund	10.00	1,225.86	
Expenses:			
Office Depot	46.74		
Office Max	18.81		
Postmaster (Stamps)	241.70		
Judy K. Weaver (Files, Dues Notice & Postage)	135.40		
Service Fee (October)	3.63	446.28	779.58
December 1999			
Income:			
Dues	680.00		
Interest (Gen)	12.46		
Contributions			
Cambridge Univ. Press	200.00		
General Fund	221.00		
Hans Schneider Prize	160.00		
OT/JT Lecture Fund	150.00		
F. Uhlig Fund	160.00		
Conference Fund	5.00	1,588.46	
Expenses:			
Service Fee (November)	4.47	4.47	1,583.99
January 2000			
Income:			
Dues	960.00		
Interest (Gen)	12.48		
Interest on CD (FU)	21.31		
Contributions			
General Fund	152.00		
Hans Schneider Prize	290.00		
F.Uhlig Fund	5.00		
Book Sales	288.00	1,728.79	
Expenses:			
George Styan (IMAGE)	1,447.29		
Judy K.Weaver (Filing & Postage)	80.00		
Judy K.Weaver (Supplies)	39.23		
Postmaster (Stamps)	26.68		
Odyssey Press (ELA)	1,783.37		
Register of Copyrights	30.00		
Postmaster (Stamps for ELA)	312.78		
Postmaster "	15.95		
Postmaster "	13.59		
Service Fee (December)	3.57	3,752.46	(2,023.67)
February 2000			
Income:			
Dues	340.00		
Interest (Gen)	11.38		
Interest on CD(HS)	70.78		
Interest on CD(OT/JT)	27.53		
Contributions			
OT/JT Lecture Fund	30.00		
Book Sales	352.00		
ILAS (Transfer)	10,000.00	10,831.69	

Expenses:			
Postage (ELA)	16.79		
Postage (ELA)	10.50		
Postage (Pictures)	9.60		
ILAS	10,000.00		
Judy K. Weaver (Filing)	126.00		
Postmaster (Postage)	16.96		
Judy K. Weaver (Purchase of OT/JT Supplies)	23.22		
Service Fee (January)	8.55	10,211.62	620.07

****

February 29,2000		37,271.88
Account Balance		
Checking Account	37,271.88	
Certificate of Deposit (FU)	1,500.00	
Certificate of Deposit (Gen.)	11,078.00	
Certificate of Deposit (72% HS & 28% OT/JT)	7,500.00	
Vanguard (72% HS & 28% OT/JT)	9,721.11	67,070.99
General Fund	26,923.50	
Frank Uhlig Educational Fund	3,193.11	
Hans Schneider Prize	16,719.83	
Olga Taussky Todd/John Todd Fund	7,885.61	
Conference Fund	9,758.94	
ILAS/LAA Fund	2,590.00	67,070.99

****

James R. Weaver 3/12/2000

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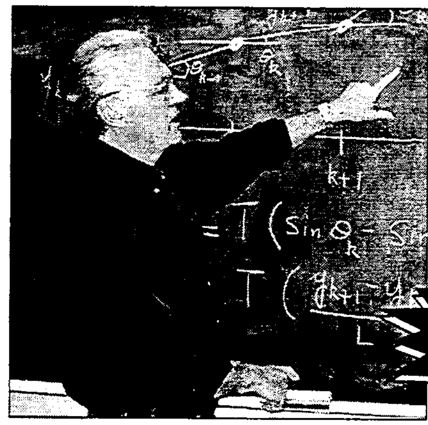

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This book is more rigorous and goes into more depth than some. Other books assume that a preliminary course has already been taken—this one does not. Because of the broad coverage, it has more flexibility than many of those already available. It contains a variety of problems and examples ranging from the elementary to the challenging and from simple applications to discovery problems. The focus on applications is a big difference between this book and the others.

One of the greatest aspects of this book is the inclusion of historical information. These personal insights into some of the greatest mathematicians who developed this subject provide a spark for students and make teaching more interesting for both the teacher and the students.

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The Infusion of Matrices into Statistics

by Shayle R. SEARLE, Cornell University

Key Words and Phrases. Determinant, direct product, Doolittle, experimental design, generalized inverse, multivariate statistics, normal equations, quadratic forms, vec and vech.

1. INTRODUCTION

Matrices were introduced to me in New Zealand in 1949 as a one-year Master's level course taught by an A. C. Aitken doctoral graduate. Two years later at Cambridge University's Statistics Laboratory, the 24-week mathematical statistics course made no use of matrices, not even in the teaching of the bivariate normal distribution or of multiple linear regression. This was a great surprise. But it is totally in line with similar comments from others, reported by Farebrother (1997), who also quotes Grattan-Guinness & Ledermann (1994) as saying "the rise of matrix theory to staple diet has occurred only since the 1950s".

Indeed, not even Aitken himself (much of whose research centered on both statistics and matrices) made a strong pitch for using matrices in statistics. Neither of his two books, *Determinants and Matrices* (Aitken 1939a) or *Statistical Mathematics* (Aitken 1939b), mentions the topic of the other, except for a snippet about quadratic forms in the matrix book. For someone having strong interests in both topics, these are, surely, unexpected omissions. They have been my motivation for trying to trace a little of the infusion of matrices into statistics. It all seems remarkably recent when viewed against the longer history of matrices themselves. Of course, in trying to be an amateur historian one immediately comes face to face with the ocean of literature available and the consequent near-impossibility of assembling every detail and aspect of one's topic. Therefore there are assuredly gaping holes in what follows—and all that can be done is to apologize and ask for help for filling those holes. Circumscribed by such lacework, the paper is arranged under four main headings: origins, the 1930s, special topics, and books. And, generally speaking, these historical notes mostly go no further than the mid 1980s.

2. THE ORIGINS OF MATRICES

In his *Men of Mathematics*, Bell (1937, pp. 400–403; 1953, pp. 440–443), attributes the invention of matrices and their algebra to Cayley (1855, 1858). "Matrices" writes Bell, "grew out of the ... way in which the transformations of the theory of algebraic invariants are combined." However, Farebrother (1997) indicates that others in the eighteenth and nineteenth centuries may well have made greater although indirect contributions than did Cayley, a conclusion, he suggests that is supported by Grattan-Guinness (1994, p. 67).

From that paper, Farebrother (1999, p. 4) gives the following quote as evidence of the slow adaptation of matrices to statistics.

> "Matrix theory was not to emerge until quite late in the 19th century, and became prominent only from the 1920s, ... [The papers by] Farkas (1902), de la Vallée Poussin (1911) and even as late as Haar (1924) are typical examples of the continuing slowness, for they explicitly worked with determinants but wrote out the matrix equations and inequalities longhand."

3. MATRICES ENTER STATISTICS IN THE 1930s

With Grattan-Guinness (1994) having concluded (as previously quoted) that matrices "became prominent only from the 1920s", the year 1930 seems a good starting point for the entry of matrices into statistics. That was the year of volume 1 of the *Annals of Mathematical Statistics*, its very first paper, Wicksell (1930), being "Remarks on regression". Today that would undoubtedly be a welter of matrices; in 1930 the normal equations were solved using determinants. But two years later came the Turnbull & Aitken (1932) book with several applications of matrices to statistics, all of which are still important and widely used today: normal equations $\mathbf{X}'\mathbf{X}\beta = \mathbf{X}'\mathbf{y}$, and $\mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{X}'\mathbf{B}\mathbf{y}$ if errors are assumed to be *not* of "equal importance and uncorrelated"; and $\mathrm{E}(\mathbf{x}'\mathbf{A}\mathbf{x}) = \mathrm{tr}(\mathbf{A}\mathbf{V})$ for $\mathbf{x} \sim (0, \mathbf{V})$ obtained by what today is considered to be a very roundabout derivation.

Then comes the Hotelling (1933) paper on principal components, a landmark both for its content and its matrix usage, although it does make reference to Kowalewski (1909) on determinants. Cochran (1934) is a next important milestone, dealing with the distribution of quadratic forms, and in the same year in Bartlett (1934), we find an early use in Britain of the word *vector* in a title—but there are no matrices in the paper. And the absence of matrices in Kolodziejczyk (1935) and in Welch (1935) is glaringly noticeable by today's standards; this is also true of Cochran (1938). All three of these papers deal with topics which lend themselves so easily to matrix representation, namely linear hypotheses and regression. But even the popular layman's book of that time, *Mathematics for the Million*, Hogben (1936), has no index entry for determinants and only one reference to matrix algebra, as being but one of several names used for "different ways of counting and measuring"! Contrarywise, Frazer, Duncan & Collar (1938) is a substantive book on matrices with virtually nothing in statistics but many applications in engineering, such as the oscillations of a triple pendulum (p. 310) and the disturbed steady motion of an aeroplane (p. 284). So the progress of matrices was slow. Infusion *had* begun, but the steeping was taking a long time. Nevertheless, it was to gain speed.

Craig (1938) was, as a follow-on from Cochran (1934), the first in a long line of papers (by many authors continuing to the present day) dealing with the independence of quadratic forms of normally distributed variables (see Section 4.1). Aitken (1937, 1938) continued publishing matrix results that would be of later value to statistics but with no explicit mention of statistics. And the year 1939 ended with a flourish: the publication of Aitken's two books referred to at the beginning of this paper.

4. SPECIAL TOPICS

We now briefly describe some of the progress that matrices have made into certain statistical topics. The choice of topics here is undoubtedly personal and the references are certainly not complete.

4.1. Quadratic Forms. Turnbull & Aitken (1932) show how to use a canonical form of a (symmetric) matrix to reduce a quadratic form to a sum of squares. Later, as already mentioned, Craig (1938) was an initial paper on the independence of quadratic forms (of "certain estimates of variance", actually), followed by Craig (1943) on "certain quadratic forms". In between, Hsu (1940) and Madow (1938, 1940) touched on the subject, Aitken (1940) dealt with the independence of linear and quadratic forms, and a decade later, Aitken (1950) dealt with the independence of two quadratic forms. Matérn (1949) also contributes. Lancaster (1954) confines his attention to traces and cumulants. The necessity condition for independence is still a hot topic; see, e.g., Driscoll & Krasnicka (1995), Olkin (1997), Driscoll (1999), and the recent MSc thesis by Dumais (2000), which includes 248 references.

4.2. Multivariate Statistics. This branch of statistics has spawned more matrix activity than any other except, perhaps, for linear models.

Wilks (1932) used lots of determinants in his work on the multiple correlation coefficient; but he uses almost no matrices and certainly no matrix algebra. In contrast, Ledermann (1940) deals with a matrix problem arising from factor analysis. And in their discourses on canonical correlations, Bartlett (1941) and Hsu[1] (1941) use matrices aplenty. Bartky (1943) uses matrices sparingly, but gives $\mathbf{I} + \mathbf{M} + \mathbf{M}^2 + \cdots = (\mathbf{I} - \mathbf{M})^{-1}$ without proof or a reference thereto. Bartlett (1947) makes considerable use of matrices in his long "Multivariate Statistics" paper wherein he writes that he has "avoided complicated analytical discussion of theory" but has made use of "matrix and vector algebra". In doing so, he refers to Bartlett (1934) and to a paper by Tukey, "Vector methods in analysis of variance", described as having been on the programme at Princeton, November 1, 1946. Also in the 1940s, Dwyer & MacPhail (1948) have a long paper on matrix derivatives.

From this time on, matrices gather momentum towards becoming standard notation for multivariate statistics, peaking in the classic book by Anderson (1958), and continuing through to the present day in numerous books and papers, including the start of the *Journal of Multivariate Analysis* in 1971.

[1] Indeed Anderson (1983, p. 474) observes that "Hsu promoted the use of matrix theory in statistical theory as well as proving new theorems about matrices".

A matrix operation of particular use in multivariate statistics is the vectorizing of a matrix: writing its columns one under the other in one long vector. Its origin goes back to a Cambridge and London contemporary of Cayley, namely Sylvester (1884), and it is now usually called *vec*. It, and *vech*, an adaptation of vec to symmetric matrices developed in Searle (1978), are considered at length in Henderson & Searle (1979, 1981a). An early use of vech is in Aitken (1949).

Two important features of multivariate analysis are eigenroots and eigenvectors (known originally as latent roots and latent vectors—see Farebrother, 1997, p. 9, for interesting commentary). Whatever their name, they are matrix characteristics which feature frequently in multivariate literature. Early examples are Hsu (1940) on analysis of variance, Anderson (1948), Geary (1948) and Whittle (1953) in dealing with time series, with continuing interest through to Mallows (1961), who is concerned with "latent vectors of random symmetric matrices".

4.3. Solving Normal Equations of Full Rank. Normal equations derived from applying least squares to multiple regression data are usually of full rank. According to Kruskal & Stigler (1997, pp. 91–92), they have had that name since Gaußs (1823). Their derivation and numerical solution have always been a source of great concern. Some early publications presenting normal equations are Wicksell (1930), Aitken & Silverstone (1942) and Bacon (1938) —all of whom use no matrices; but of course Turnbull & Aitken (1932) do. Until the acceptance of matrix inverses, methods for solving normal equations, being, as they are, just simultaneous linear equations, were simply successive elimination with back substitution—or the Cramer's (1750) method using determinants as described previously.

Despite its algebraic clarity, the use of a matrix inverse initially posed considerable arithmetic difficulty. Although today's computers now make light work of that arithmetic, that has come about in only the last forty years; and it is accomplished at speeds that were utterly unimaginable then. During graduate student days in 1959, in a small computing group at Cornell, there was great excitement when we inverted a 10-by-10 matrix in seven minutes. After all, only a year or two earlier a friend had inverted a 40-by-40 matrix by hand, using electric (Marchant or Monroe) calculators. That took six weeks! So it is understandable that statisticians were interested in computational techniques for inverting matrices—and they still are, for that matter, although at a much more sophisticated level than in the pre-computer days. An early beginning to those sophisticated ways was the Doolittle system for doing, and setting out, the individual calculations. The date and location of the publication of this system is perhaps surprising: Doolittle (1851) in the 1878 U.S. Coast and Geodetic Survey Report. Numerous abbreviations, improvements and comments followed, in a variety of publications, including Horst (1932) and Dwyer (1941a, 1941b, 1944) in the statistical literature. In his 1944 paper, Dwyer wrote "The reader should be familiar with elementary matrix theory such as that outlined on pages 1-57 of Aitken's book" (1939a). An alternative approach was Bingham (1941) using the Cayley-

Hamilton theorem and then Newton's equations. Other contributions included Hotelling (1943), Quenouille (1940), Fox (1950) and Fox & Hayes (1951). The arrival of computers soon put an end to these pencil-and-paper based methods, which by then had collected other variants of Doolittle such as abbreviated Doolittle, Crout-Doolittle, and so on.

4.4. Design of Experiments. In the 1950s a widely used-book on experiment design was Cochran & Cox (1950, 2nd ed.); it has but one, brief mention of matrices related to regression. But only a year later Box & Wilson (1951) make substantial use of matrices for normal equations, their solution and resulting sums of squares. Likewise, Tocher (1952) has solid matrix usage. Both papers give, for order n, $(\mathbf{I} + a\mathbf{J})^{-1} = \mathbf{I} - a\mathbf{J}/(1 + an)$; and Tocher extends this to $(\mathbf{I} + \mathbf{A} \otimes \mathbf{J})^{-1} = \mathbf{I} - [(\mathbf{I} + n\mathbf{A})^{-1}\mathbf{A}] \otimes \mathbf{J}$. This is an example of a class of designs involving its own special matrices. There are many other examples, one of the most extensive being factorial designs, for which Cornfield & Tukey (1956) developed far-reaching use of Kronecker (or direct) products of matrices for fixed effects models.

Nelder (1965a, b), Smith & Hocking (1978) and Searle & Henderson (1979) independently extended this to the dispersion matrix for models involving random effects. Another example is Latin squares, for which Yates & Hale (1939) could have benefited from using matrices—but they did not. But Cox (1958) certainly did—in abundance. Design features involving circulant matrices include autocorrelation and spectral density functions (Wise, 1955), circular stationary models (Olkin & Press, 1969) and partial factorial balance (Anderson, 1972); for these Searle (1979) gives methods for inverting circulants of two and three non-zero elements per row.

4.5. Linear Models. H. O. Hartley makes an interesting comment in the published discussion of Tocher (1952, p. 96), suggesting that it was Barnard who introduced the unifying principle of developing "analysis of variance by analogy to regression analysis". This is, of course, the vital connection for what used to be called "fitting constants" and is now usually known as linear models. It is the foundation on which enormous numbers of research papers have been built—and also numerous books, so many in fact that commenting on them is well beyond the scope of this paper. And certainly during the last forty or more years, all of them use matrices as the *lingua franca*. The earliest book to do so seems to be Kempthorne (1952) and a paper of the same decade is Searle (1956). Nevertheless, seven years later, the book by Williams (1959) on regression had only a tiny mention of matrices.

One of the greatest contributions to understanding the apparent quirkiness of normal equations of non-full rank (as is customary with linear models), which have an infinity of solutions, is due to Rao (1962). Using the work of Moore (1920) and Penrose (1955), he showed how a generalized inverse matrix yields a solution to the normal equations and how that solution can be used to establish estimable functions and their estimators—and these results are invariant to whatever generalized inverse is being used. Although the arithmetic of generalized inverses is scarcely any less than that of regular inverses, the use of generalized inverses is of enormous help in understanding estimability and its consequences.

4.6. Other Topics. Of the many other topics and places where matrices have contributed substantially to statistics, only a few are mentioned briefly here.

4.6.1. Probability Theory and Markov Chains. Back-to-back papers using matrices for branching processes and queuing processes are Hammersley & Morton (1954) and Bailey (1954), followed by the books by Kemeny & Snell (1960, 1962).

4.6.2. Age Distribution Vectors. When the proportions (or numbers) in different age groups of an animal population are arrayed in a vector, it is an age distribution vector (Lewis 1942). That vector is seldom constant over time. Leslie (1945, 1948), by using age-specific birth rates as the first row of a matrix, and age-specific survival rates as the sub-diagonal of that matrix (now called a Leslie matrix), shows how pre-multiplying an age-distribution vector for time t by a Leslie matrix yields (deterministically) the age distribution vector at time $t + 1$. In a practical application of this, using monthly Leslie matrices and a killing (diagonal) matrix, Darwin & Williams (1964) adapt this procedure to ascertain optimum months of the year for poisoning rabbits on New Zealand farms. Rabbits there are a seriously overpopulated pest, consuming grass needed for producing meat, wool and milk which sustain the country's economy through exports.

4.6.3. Estimating Genetic Worth. Programs for improving the genetic worth of an animal species through planned matings often include estimating the genetic worth of an untried animal from records of its ancestors. Searle (1963) shows that the correlation between genetic worth and an ancestor-based estimate of it is of the form $\mathbf{x}'\mathbf{A}^{-1}\mathbf{x}$ where $\mathbf{x}'$ is closely related to rows of $\mathbf{A}$. As a result, through knowing how elements of $\mathbf{A}^{-1}$ involve cofactors and the determinant of $\mathbf{A}$, a recurrence relationship for the correlation (based on the number of generations of ancestors) was established. Conclusions are that going beyond grandparents contributes very little to one's estimate of the untried animal. (Horse-racing enthusiasts wanting to buy an untried yearling should take heed!)

4.6.4. Inverting $\mathbf{A} + \mathbf{UBV}$. In statistics, the matrix $\mathbf{A} + \mathbf{UBV}$ arises in various forms. For example, one special case is the dispersion matrix $\mathbf{D}(\mathbf{p}) - \mathbf{pp}'$ for a multinomial random variable, where $\mathbf{p}$ is a vector of probabilities and $\mathbf{D}(\mathbf{p})$ is the diagonal matrix of elements of $\mathbf{p}$. Also, it occurs as an intraclass correlation matrix $(1-\rho)\mathbf{I}+\rho\mathbf{J}$, it arises as discriminant analysis (Bartlett 1951) and it turns up when elements of a matrix are altered (Sherman & Morrison 1950).

What is interesting about the inverse of $\mathbf{A} + \mathbf{UBV}$ is that although

$$(\mathbf{A} + \mathbf{UBV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1},$$

as the standard result is widely known, there are many variations of it, as well as numerous special cases. Henderson & Searle (1981b) give a number of these as well as much of the history.

4.6.5. Partitioned Inverses. The value of generalized inverses in linear models analysis has already been mentioned. They are particularly useful when dealing with partitioned models $E(y) = X_1\beta_1 + X_2\beta_2$. Then the matrix $Q^* =$

$$\begin{pmatrix} A^- & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -A^-B \\ I \end{pmatrix} (D - CA^-B)^-(-CA^- \;\; : \;\; I)$$

is a generalized inverse of

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

provided

$$\mathrm{rank}(\mathbf{Q}) = \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{D} - \mathbf{CA}^-\mathbf{B}),$$

cf. Marsaglia & Styan (1974). For non-singular $\mathbf{Q}$, the generalized inverse Q^- becomes the regular inverse Q^{-1}. In the case of $Q = X'X$ for $X = (X_1 \;\; : \;\; X_2)$, the rank condition always holds, and $Q^* = Q^-$. The use of Q^- in providing $\tilde{\beta} = Q^-X'y$ then leads immediately to very useful results such as $\tilde{\beta}_1$, $\tilde{\beta}_2$ and $R(\beta_1|\beta_2)$, the reduction in sum of squares for fitting β_1 adjusted for β_2.

5. BOOKS

Tracing the intermingling of matrices and statistics in published books could be a long process. Here it will be brief, in terms of a dichotomy: the occurrence of matrices in statistics books, and of statistics in matrix books.

Fisher (1935 *et seq.*) had no matrices in any of its numerous editions. Many sets of numbers are laid out in rectangular arrays, but they are designs, not matrices. Snedecor (1937), Kendall (1943–1952), Mood (1950), and Mood & Graybill (1963) had either no matrices or almost none, but by Kendall & Stuart (1958, Vol. I, 1st ed.), their chapters 15 and 19 had matrix notation for the multi-normal distribution, for quadratic forms, and for least squares. But the Snedecor book took much longer to join the matrix crowd. It is in Snedecor & Cochran (1989, 8th ed.), where the preface heralds the arrival of matrices with the following near-apologia:

> " A significant change in this edition occurs in the notation used to describe the operations of multiple regression. Matrix algebra replaces the original summation operators, and a short appendix on matrix algebra is included."

Cramér (1946) was early in having a whole chapter "Matrices, determinants and quadratic forms", referencing Cochran (1934) for the last of those three topics. But Kempthorne (1952) and Rao (1952) seem to be the first books having substantial sections in matrix notation and making considerable use of matrix algebra. Rao (1952) begins with some thirty pages of matrices, having written in his preface "The problems of multivariate analysis resolve themselves into an analysis of the dispersion matrix and reduction of determinants." Thereafter came Anderson (1958), Graybill (1961), Rao (1965), Searle (1971), Graybill (1976), Seber (1977), and many others too numerous to mention, all using matrices extensively. So do Kotz *et al.* (1985) in the *Encyclopedia of Statistical Sciences* (Vol. 5), and the *Encyclopedia of Biostatistics*, the latter having "Matrix Algebra" as a major entry.

Now to the occurrence of statistics in matrix books. Those by Frazer, Duncan & Collar (1938), Aitken (1939a) and Ferrar (1941) have no statistics. Bellman (1960) has two chapters (10%) on Markov matrices and probability theory. And the Graybill (1969, 1983) and Searle (1966, 1982) books have plenty of statistics, as one would expect from their titles.

Then there are the specialized books on generalized inverses, three in the same year: Pringle & Rayner (1971), Rao & Mitra (1971), both with plenty of statistics, and Boullion & Odell (1971) with a modest tilt to statistics. Shortly thereafter came Ben-Israel & Greville (1974) with virtually none.

Whilst this incursion of statistics in matrix books was taking place, we may note that it was also happening in linear algebra journals. In *Linear Algebra and its Applications* (begun in 1968 and often referred to as LAA) and *Linear and Multilinear Algebra* (started in 1972) the early issues showed little evidence of statistics, whereas nowadays there is quite a good representation of statistics. In particular, LAA periodically now has special issues on linear algebra and statistics; the 1990 special issue (vol. 127) had 656 pages. Many papers in these issues stem from the now almost annual Workshop on Matrices and Statistics. Started in 1990, the only years of this decade in which the workshop has not met are 1991 and 1993 (the next Workshop is scheduled for Hyderabad, India, December 2000). So matrices in statistics are alive and well.

6. OMISSIONS

To those whose work has not been mentioned, my apologies. For a hint at the vastness (in both time and geography) of the literature, they are referred to Puntanen & Styan (1988); it has 1,596 literature references on matrices and statistics and many, many more have been added in supplements during the last eleven years. See also Anderson, Das Gupta & Styan (1977) and Styan (1998) for more references.

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Eigenvalues and Less Horrible Words

Grattan-Guinness & Ledermann (1994, p. 785), as reported by Farebrother (1999, p. 8), are right to object to the term *eigenvalue* as a horrible hybrid of German and English, but Farebrother's etymology of this word is not quite right. It derives from *Eigenwert* and the second part, *Wert*, means *value* and is cognate to the English word *worth* of similar meaning. *Wort* is an archaic term for herb, today found in compounds, as Farebrother points out. My amateur sleuthing in the *Oxford English Dictionary* (OED) has traced *worth* and *wort* back to similar but different Old English (OE) word forms. If the two words have the same root, the OED gives no clue of this.

But here's the curious thing. The dictionary traces the word *root* back to the same OE word as *wort*. The corresponding obsolete German form is *Wurz* (according to *Duden Herkunftswörterbuch*) meaning both root and herb. This appears in modern German as *Wurzel* (root) and *Würze* (spice). Now that's interesting, as eigenvalue was consistently known as *latent root* when I was a student in Scotland in the 1950s. So, in some higher sense, Farebrother is right after all with his translation.

If you wish to retain the sense of the German *eigen* (meaning own) without committing bilingual murder, why not use *proper value* or *proper root*, for the older meaning of *proper* is precisely *one's own*. But to me latent root says it all: a hidden root. Yet I'll make the pessimistic prediction that eigenvalue is now used in so many areas of mathematics and by so many people in other scientific disciplines that it is here to stay.

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Let us Root for *Root*

No matter what the proper translation is for the original German words into *eigenvalue* and *eigenvector*, these are a poor choice. *Eigenroot* is much better than *eigenvalue*; after all, an eigenvalue is a root of an equation and as such the word *root* carries with it the implications that it can be zero, positive or negative, real or complex. Moreover, for lecturing, *root* and *vector* without the word *eigen* bring good clarity to the subject, with no confusion such as is possible between *value* and *vector*; easier, too, for students writing notes. Finally, *root* is in keeping with the early use of latent root and latent vector. So let us root for *root*.

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