The Bulletin of the International Linear Algebra Society

[IL AS] IMAGE [IL AS]

Serving the International Linear Algebra Community

Issue Number 25, pp. 1–32, October 2000

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Printed in Canada by the McGill University Printing Services, Montréal, Québec

IMAGE Problem Corner: Old Problems with Solutions

We are still hoping to receive solutions to Problems 19-3b & 23-1, which are repeated below; we present solutions to IMAGE Problems 18-1, 21-2, and 24-1 through 24-9 and a Comment on Solution 19-4.1 published in IMAGE 23 (October 1999), pp. 19–20. In addition, we introduce eight new problems (pp. 16–17) and invite readers to submit solutions as well as new problems for publication in IMAGE. Please send all material in macro-free LaTeX (a) by e-mail to werner@united.econ.uni-bonn.de and (b) two copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses are included!

Problem 18-1: 5×5 **Complex Hadamard Matrices**

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Show that every 5×5 matrix U with complex entries $u_{j,k}$ of constant absolute value one that satisfies $U^*U = 5I$ can be realized as the matrix $(\omega^{jk})_{j,k}$ where ω is a complex primitive fifth root of unity by applying some sequence of the following: (1) A rearrangement of the rows, (2) A rearrangement of the columns, (3) Multiplication of a row by a complex number of absolute value one, (4) Multiplication of a column by a complex number of absolute value one.

Computer Solution 18-1.1 by the Proposer S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

We do not have a proof to offer, but we have verified the statement numerically using a computer. Without loss of generality, we take the matrix U in the form

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{i\theta_{1,1}} & e^{i\theta_{1,2}} & e^{i\theta_{1,3}} & e^{i\theta_{1,4}} \\ 1 & e^{i\theta_{2,1}} & e^{i\theta_{2,2}} & e^{i\theta_{2,3}} & e^{i\theta_{2,4}} \\ 1 & e^{i\theta_{3,1}} & e^{i\theta_{3,2}} & e^{i\theta_{3,3}} & e^{i\theta_{3,4}} \\ 1 & e^{i\theta_{4,1}} & e^{i\theta_{4,2}} & e^{i\theta_{4,3}} & e^{i\theta_{4,4}} \end{pmatrix},$$

where the $\theta_{j,k}$ are in the interval $[0, 2\pi]$. We may further insist that $e^{i\theta_{1,1}}$ is the closest of the $e^{i\theta_{j,k}}$ to 1, that $0 \leq \theta_{1,1} \leq \pi$ and that $\theta_{1,k}$ is increasing with k and that $\theta_{j,1}$ is increasing with j. The search space is thus a 16-dimensional torus and we proceed by attempting to eliminate 16-dimensional boxes. We eliminate a box $C = \prod_{1 \leq j,k \leq 4} I_{j,k}$ if

- The intervals $I_{j,k}$ are so located that the order relations described above necessarily fail;
- The intervals $I_{j,k}$ are so located that the inner product of two distinct columns is necessarily non-zero;
- The intervals $I_{j,k}$ are so located that the inner product of two distinct rows is necessarliy non-zero;
- The box C is contained in the safe box C_0, to be described shortly.

If we fail to eliminate a box, we divide it into two equal pieces (by dividing a longest $I_{j,k}$) and then check the resulting boxes separately. This process is continued recursively and eventually terminates showing that there are no solutions, except possibly in the safe box C_0. So it remains to show that there are no solutions in the box C_0 aside from the one sought. We use the following lemma:

LEMMA. *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a smooth function which satisfies $f(0) = 0$. Let σ be the smallest singular value of the Jacobian matrix $f'(0)$ and let M be an upper bound for the Frobenius norm of the Hessian tensor $f''(x)$ as x runs over $\mathbb{R}^m$. Then, for any solution of $f(x) = 0$, except $x = 0$ we have $||x|| \geq 2\sigma/M$.*

Proof: Taylor's Theorem gives

$$||f(x) - f(0) - f'(0)x|| \leq \tfrac{1}{2} \sup_{\xi} ||f''(\xi)|| \cdot ||x||^2$$

and since $f(0) = 0$ and $f(x) = 0$, we get $\sigma||x|| \leq ||f'(0)x|| \leq M||x||^2/2$. Since $||x|| \neq 0$, we get the desired result. □

Applying the result to the case in point, with $m = 16$, $n = 20$ (10 complex equations give the orthogonality of distinct rows), we find a rather small ball around the known solution in which there can be no further solutions. It remains then to choose C_0 to be a box entirely contained in this ball.

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG: hartwig@math.ncsu.edu; *North Carolina State University, Raleigh, North Carolina, USA,*
Peter ŠEMRL: peter.semrl@fmf.uni-lj.si; *University of Maribor, Maribor, Slovenia,*
and Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \mathrm{span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

The Editor has still not received a solution to Problem 19-3b. The solution by the Proposers to Problem 19-3a appeared in IMAGE 22, p. 25 (April 1999). *We look forward to receiving a solution to Problem 19-3b.*

Problem 19-4: Eigenvalues of Positive Semidefinite Matrices

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Show that there are constants $\gamma \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, \frac{1}{2})$ such that if $A_1, \ldots, A_n$ are non-negative definite $r \times r$ matrices of rank one satisfying

(a) $A_1 + \cdots + A_n = I_r$

(b) trace$(A_i) < \gamma$ for each $i = 1, \ldots, n$,

then there is a subset σ of $\{1, 2, \ldots, n\}$ such that the eigenvalues of $A_\sigma = \sum_{i \in \sigma} A_i$ all lie in the interval $(\varepsilon, 1-\varepsilon)$.

Comment on Solution 19-4.1 [IMAGE 23 (October 1999), pp. 19–20] by Alexander KOVAČEC: kovacec@gentzen.mat.uc.pt; *Universidade de Coimbra, Coimbra, Portugal.*

In my Solution 19-4.1 [IMAGE 23 (October 1999), pp. 19–20] I used the following theorem:

THEOREM (Bourbaki). *Assume $\{p_i\}_1^m$ are m vectors in the Euclidean unit n-ball: i.e., they sum to a: $p_i \in \mathbb{R}^n$ with $|p_i| \le 1$, $\sum_{i=1}^m p_i = a$. Then there exists a $\sigma \in S_m$ (symmetric group of m elements) such that for all $k \in [m]$*

$$\left| \sum_{i=1}^{k} p_{\sigma(i)} \right| \le (1+a) \cdot 5^{(n-1)/2}.$$

I have since found that there is a complete proof and some history of this theorem in the paper entitled "The remarkable theorem of Lévy and Steinitz" by P. Rosenthal [*The American Mathematical Monthly*, 94, 342–351 (1987)].

Problem 21-2: The Diagonal of an Inverse

Proposed by Beresford PARLETT: parlett@math.berkeley.edu; *University of California, Berkeley, California, USA,*
via Roy MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Let J be an invertible tridiagonal $n \times n$ matrix that permits triangular factorization in both increasing and decreasing order of rows:

$$J = L_+ D_+ U_+ \quad \text{and} \quad J = U_- D_- L_- . \tag{1}$$

(Here the L's are lower triangular, the U's are upper triangular, and the D's are diagonal.). Show that

$$(J^{-1})_{kk} = [(D_+)_{kk} + (D_-)_{kk} - J_{kk}]^{-1}. \tag{2}$$

Solution 21-2.1 by Yongjian HU: yongjian@bnu.edu.cn; *Beijing Normal University, Beijing, China.*

First, we establish some auxiliary propositions which are useful later on. Let J be an arbitrary invertible $n \times n$ matrix and let J, J^{-1} be partitioned into the following 2×2 block matrix forms:

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \tag{3}$$

where the block entries A, E and D, H are square matrices with the same order, respectively. If A is nonsingular, then we denote by J/A the Schur complement of A in J, i.e., $J/A = D - CA^{-1}B$. Similarly, if D is nonsingular, then we write the Schur complement of D in J as J/D, i.e., $J/D = A - BD^{-1}C$.

PROPOSITION 1. *Let J be an arbitrary invertible square matrix of order n and J, J^{-1} be partitioned into the form* (3).

(i) *If A is nonsingular, then*

$$J = \begin{pmatrix} I_r & 0 \\ CA^{-1} & I_{n-r} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & J/A \end{pmatrix} \begin{pmatrix} I_r & A^{-1}B \\ 0 & I_{n-r} \end{pmatrix}, \tag{4}$$

and $H = (J/A)^{-1} = (D - CA^{-1}B)^{-1}$.

(ii) *If D is nonsingular, then*

$$J = \begin{pmatrix} I_r & BD^{-1} \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} J/D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_r & 0 \\ D^{-1}C & I_{n-r} \end{pmatrix}, \tag{5}$$

and $E = (J/D)^{-1} = (A - BD^{-1}C)^{-1}$.

Proof: Eqs. (4) and (5) can be checked directly. By Eqs. (4) and (5), we have

$$J^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} I_r & -A^{-1}B \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (J/A)^{-1} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -CA^{-1} & I_{n-r} \end{pmatrix},$$

and

$$J^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ -D^{-1}C & I_{n-r} \end{pmatrix} \begin{pmatrix} (J/D)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I_r & -BD^{-1} \\ 0 & I_{n-r} \end{pmatrix}.$$

Comparing both sides of the last two equations, we obtain $H = (J/A)^{-1}$ and $E = (J/D)^{-1}$. □

Let us write $J = (J_{ij})_{i,j=1}^n$, $J_p = (J_{ij})_{i,j=1}^p$ and $J_0' = J$, $J_q' = (J_{ij})_{i,j=q+1}^n$ $(1 \le q \le n-1)$. We also may write

$$J = \left[\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & J_{k-1} & & 0 & & 0 & & \\ & & & & J_{k-1,k} & & & & \\ 0 & \cdots & 0 & J_{k,k-1} & J_{kk} & J_{k,k+1} & 0 & \cdots & 0 \\ & & & & J_{k+1,k} & & & & \\ & & 0 & & 0 & & J_k' & & \\ & & & & \vdots & & & & \\ & & & & 0 & & & & \end{array}\right]. \tag{6}$$

PROPOSITION 2. *Let J be an invertible $n \times n$ matrix that permits triangular factorization* (1). *Then all J_p $(1 \le p \le n)$ and J_q' $(0 \le q \le n-1)$ are nonsingular.*

PROPOSITION 3. *Let J be an invertible $n \times n$ matrix that permits triangular factorization* (1). *Then* $(D_+)_{11} = J_{11}$ *and* $(D_+)_{ll} = J_l / J_{l-1}$, $l = 2, \cdots, n$. *Moreover,* $(D_-)_{nn} = J_{nn}$ *and* $(D_-)_{ll} = J'_{l-1}/J'_l$, $l = 1, \cdots, n-1$.

Now let us prove the main result (2). Let J be an invertible tridiagonal $n \times n$ matrix. We write J in the form given by (6) above. By Proposition 3, we have $(D_+)_{kk} = J_k/J_{k-1} = J_{kk} - (0 \ \cdots \ 0 \ \ J_{k,k-1})\, J_{k-1}^{-1}\, (0 \ \cdots \ 0 \ \ J_{k-1,k})^T$ and $(D_-)_{kk} = J'_{k-1}/J'_k = J_{kk} - (J_{k,k+1} \ \ 0 \ \cdots \ 0)\, (J'_k)^{-1}\, (J_{k+1,k} \ \ 0 \ \cdots \ 0)^T$. Let $A = J_{k-1}$ and $D = J'_{k-1}$.
Then, by Proposition 1 we have:

$$
\begin{aligned}
(J^{-1})_{kk} &= H_{11} = [(J/A)^{-1}]_{11} = [(J/J_{k-1})^{-1}]_{11} \\
&= \left\{ \left(\begin{bmatrix} J_{kk} & J_{k,k+1}\ 0 \cdots 0 \\ J_{k+1,k} & \\ 0 & \\ \vdots & J'_k \\ 0 & \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & J_{k,k-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} J_{k-1}^{-1} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ J_{k-1,k} & \cdots & 0 & 0 \end{bmatrix} \right)^{-1} \right\}_{11} \\
&= \left\{ \left(\begin{bmatrix} J_k/J_{k-1} & J_{k,k+1}\ 0 \cdots 0 \\ J_{k+1,k} & \\ 0 & \\ \vdots & J'_k \\ 0 & \end{bmatrix} \right)^{-1} \right\}_{11} = (J_k/J_{k-1} - (J_{k,k+1}\ 0 \cdots 0)(J'_k)^{-1}(J_{k+1,k}\ 0 \cdots 0)^T)^{-1} \\
&= (J_k/J_{k-1} - (J_{kk} - J'_{k-1}/J'_k))^{-1} = (J_k/J_{k-1} + J'_{k-1}/J'_k - J_{kk})^{-1} = ((D_+)_{kk} + (D_-)_{kk} - J_{kk})^{-1}. \qquad (7)
\end{aligned}
$$

Hence, the proof of assertion (2) is completed. □

Solution 21-2.2 by John MURCHLAND: 100326.1223@compuserve.com; *35 Rectory Park, Sanderstead, Surrey CR2 9JR, England.*

The relation can also be written symmetrically as $J_{kk} + (J^{-1})_{kk}^{-1} = (D_+)_{kk} + (D_-)_{kk}$. To ease writing below let $G = J^{-1}$. To help comparisons, factors in terms are kept in d, l, u, g order. For the forward factorization $J = L_+ D_+ U_+$,

$$j_{11} = d_1^+ \ , \tag{8}$$

and for $k = 1$ to $n-1$

$$j_{k,k+1} = d_k^+ u_{k,k+1}^+ \ , \quad j_{k+1,k} = d_k^+ l_{k+1,k}^+ \ , \quad j_{k+1,k+1} = d_k^+ l_{k+1,k}^+ u_{k,k+1}^+ + d_{k+1}^+ \ . \tag{9}$$

For the backward factorization $J = U_- D_- L_-$,

$$j_{nn} = d_n^- \ , \tag{10}$$

and for $k = n-1$ to 1

$$j_{k,k+1} = d_{k+1}^- u_{k,k+1}^- \ , \quad j_{k+1,k} = d_{k+1}^- l_{k+1,k}^- \ , \quad j_{k,k} = d_{k+1}^- l_{k+1,k}^- u_{k,k+1}^- + d_k^- \ . \tag{11}$$

With $G = J^{-1} = U_+^{-1} D_+^{-1} L_+^{-1}$, $D_+ U_+ G L_+ = I_n$, and picking out the k,k element on each side, for $k = 1 \cdots n-1$,

$$d_k^+ g_{kk} + d_k^+ l_{k+1,k}^+ g_{k,k+1} + d_k^+ u_{k,k+1}^+ g_{k+1,k} + d_k^+ l_{k+1,k}^+ u_{k,k+1}^+ g_{k+1,k+1} = 1 \ ,$$

which by using (9) becomes

$$d_k^+ g_{kk} + j_{k+1,k} g_{k,k+1} + j_{k,k+1} g_{k+1,k} + (j_{k+1,k+1} - d_{k+1}^+) g_{k+1,k+1} = 1 \,. \tag{12}$$

Similarly $D_- L_- G U_- = I_n$, and this time picking out the k+1,k+1 element, for $k = 1$ to $n-1$,

$$d_{k+1}^- l_{k+1,k}^- u_{k,k+1}^- g_{kk} + d_{k+1}^- l_{k+1,k}^- g_{k,k+1} + d_{k+1}^- u_{k,k+1}^- g_{k+1,k} + d_{k+1}^- g_{k+1,k+1} = 1 \,,$$

which by (11) becomes

$$(j_{kk} - d_k^-) g_{kk} + j_{k+1,k} g_{k,k+1} + j_{k,k+1} g_{k+1,k} + d_{k+1}^- g_{k+1,k+1} = 1 \,. \tag{13}$$

If we subtract (13) from (12), the two off-diagonal terms cancel and $(d_k^+ + d_k^- - j_{kk}) g_{kk} - (d_{k+1}^+ + d_{k+1}^- - j_{k+1,k+1}) g_{k+1,k+1} = 0$. Thus $(d_1^+ + d_1^- - j_{11}) g_{11} = (d_k^+ + d_k^- - j_{kk}) g_{kk}$ for each $k = 2$ to n. But $(d_1^+ + d_1^- - j_{11}) g_{11} = 1$, since from (8) $d_1^+ = j_{11}$ and from $G = J^{-1} = L_-^{-1} D_-^{-1} U_-^{-1}$, $g_{11} = (L_-^{-1})_{11} (D_-^{-1})_{11} (U_-^{-1})_{11} = 1/d_1^-$, as L_-^{-1} and U_-^{-1} are *unit* (full) triangular matrices. This establishes the relation needed. □

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG: chung@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let the $m \times n$ random matrix X be such that $\mathrm{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where vec indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find E det$(X'CX)$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

We look forward to receiving a solution to this problem!

Problem 24-1: Rank of a Skew-Symmetric Matrix

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let A be a complex skew-symmetric matrix. Show that the non-zero singular values of A are equal in pairs and deduce that A necessarily has even rank.

Solution 24-1.1 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

The result follows readily from the fact (see [HJ, pp.158–159] or see [KL] for an elementary proof) that for any real or complex skew-symmetric matrix A there exists a real orthogonal or complex unitary matrix U such that $U^t A U = A_1 \oplus \cdots \oplus A_k$, where

$$A_j = \begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}$$

with $s_j \geq 0$ for all $j \leq n/2$, and $A_k = [0]$ in case n is odd. □

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[HJ] R. A. Horn & C. R. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press.

[KL] J. Karro & C. K. Li (1997). A unified elementary approach to canonical forms of matrices. *SIAM Review*, 39, 305–309.

Editorial Remarks. We are very grateful to Roger A. Horn (University of Utah) for pointing out that Problem 24-1 is Theorem 6, Chapter XI, Section 4 of Gantmacher (1998); the stronger version in Solution 24-1.1 (special canonical form) is Problem 25 in §4.4 in Horn & Johnson (1990). The more detailed question of the possible Jordan canonical forms for a skew-symmetric complex matrix is also addressed by Gantmacher (1998); for a modern treatment see Horn & Merino (1999).

Moreover, Wedderburn (1934, Th. 13, p. 96) shows that "If A is a skew $n \times n$ matrix of rank r, then (i) r is even; ..., (iii) if $r = n$, [then] the determinant of A is a perfect square ..." and cites Loewy (1910, ch. 2, pp. 79–153). See also Pascal (1900, p. 43): "Eine schiefsymmetrische Determinante von ungeradem Grad ist Null. Eine schiefsymmetrische Determinante von gerader Ordnung is das vollständige Quadrat einer ganzen rationalen Function ihrer Elemente"; Pascal cites Cayley (1847, p. 93). See also Lawford (2000), who asks readers to prove that when n is even then the skew-symmetric matrix A is nonsingular!

References and Further Reading

A. Cayley (1847). Sur les déterminants gauches. *Journal für die reine und angewandte Mathematik*, 38, 93–96. [According to Jeff Miller's Web site http://members.aol.com/jeff570/s.html this paper also has the earliest use of the term "skew-symmetric determinant". For discussion of this paper see Muir (1960, pp. 255–262).]

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R. A. Horn & C. R. Johnson (1990). *Matrix Analysis*. Corrected Reprint Edition. Cambridge University Press.

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J. H. M. Wedderburn (1934). *Lectures on Matrices*. Colloquium Pub. 17. American Mathematical Society. [Dover reprint edition: 1964.]

Problem 24-2: Nonnegative Matrix with All Entries Summing to One

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let Q be a real $n \times n$ elementwise nonnegative matrix with the sum of all entries equal to 1 and, for each j, the jth row sum equals the jth column sum. Show that Q is a convex combination of matrices $R(X, \sigma)$ described as follows. Let X be a nonempty subset of $I_n = \{1, 2, \ldots, n\}$ with k elements and let σ be a cyclic permutation of X. Then the matrix $R = R(X, \sigma)$ is given by

$$r_{ij} = \frac{1}{k} \quad \text{if } i \in X \text{ and } j = \sigma(i) \quad \text{and} \quad r_{ij} = 0 \ \text{ otherwise.}$$

Remark. This Problem is related to Solution 23-3.1 [IMAGE 24 (April 2000), page 9].

Solution 24-2.1 by R. B. BAPAT: rbb@isid.ac.in; *Indian Statistical Institute–Delhi Centre, New Delhi, India.*

See Lemma 3.4.3, page 126, in the book entitled *Nonnegative Matrices and Applications* by R. B. Bapat & T. E. S. Raghavan [Encyclopedia of Mathematics and its Applications, Vol. 64, Cambridge University Press, 1997].

Solution 24-2.2 by Chi-Kwong LI: ckli@math.wm.edu, *The College of William and Mary, Williamsburg, Virginia, USA,* and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Note that the set Q is a polytope in $\mathbb{R}^{n \times n}$ containing elements $A = (a_{ij})$ determined by the inequalities:

$$a_{ij} \geq 0 \tag{14}$$

and the equalities

$$\sum_{i,j} a_{ij} = 1, \qquad \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}, \quad i = 1, \ldots, n. \tag{15}$$

Since a convex polytope is the convex hull of the extreme points, we need to show that the extreme points of Q satisfy the condition described in the problem.

Note that every extreme point of Q must satisfy at least n^2 linear independent equalities from (14) and (15). Now, suppose A is an extreme point of Q with k nonzero rows (and the corresponding columns) with $1 \leq k \leq n$. Applying a permutation similarity to A, we may assume that the first k rows and columns of A are nonzero. Then the entries in the last $n - k$ rows and columns are all nonzero; so we get $n^2 - k^2$ equalities from (14), and we need k^2 additional equalities. Since one can get at most k independent equalities from (15), we need at least $k^2 - k$ equalities from (14) when the entries in the leading $k \times k$ submatrix A_1 is considered. Consequently, A_1 has exactly k nonzero entries. Since A_1 has nonzero row sums and column sums, the k nonzero entries must lie on different rows and different columns of A_1. Thus, the nonzero positions of A_1 correspond to a permutation matrix R.

If R is a length k cycle, then all row and column sums of A_1 are the same implying that $kA_1 = R$. One easily checks that if $A = (B + C)/2$ with $B, C \in Q$, then B and C have the same zero/nonzero structure as A, and hence $B = C = A$. So, A is an extreme point.

If R is not a length k cycle, then there exists a permutation matrix P such that $P^tRP = R_1 \oplus R_2$, where R_1 and R_2 are permutation matrices of lower orders. Accordingly, $P^tAP = A_1 \oplus A_2$. One can find suitable scalar $r, s > 0$ so that $B = (1+r)A_1 \oplus (1-s)A_2$, $C = (1-r)A_1 \oplus (1+s)A_2 \in Q$. So, $A = (B+C)/2$ is not an extreme point of Q. The result follows. □

Solution 24-2.3 by the Proposer S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

The set of all such matrices Q forms a compact polytope which we denote by Π. By the Krein–Millman Theorem, Π is the convex hull of its extreme points. It therefore suffices to show that each extreme point of Π is an $R(X, \sigma)$. Of course, the $R(X, \sigma)$ are extreme in Π, but we do not need to show this. Let us call the pattern of an $R(X, \sigma)$ an $\mathcal{R}$-pattern.

Let Q be an extreme point of Π which is not an $R(X, \sigma)$. Then Q must vanish somewhere on each $\mathcal{R}$-pattern, for otherwise, we could write Q as a non trivial convex combination of an $R(X, \sigma)$ and some other matrix in Π. We claim that Q must have a zero row. To establish the claim, suppose not. Then there is a mapping $\mu : I_n \longrightarrow I_n$ such that $q_{i\mu(i)} > 0$. Consider the successive iterates $\mu^j(I_n)$ $(j = 1, 2, \ldots)$. These are nested nonempty subsets of I_n decreasing as j increases. So, for j large enough, they are all equal to some fixed nonempty subset Y of I_n. Clearly the restriction of μ to Y is surjective and hence a permutation of Y. Choose now σ to be one of the cycles in the cycle decomposition of $\mu|_Y$ and let X be the set of points of Y moved by σ. Together X and σ give an $\mathcal{R}$-pattern on which Q has strictly positive entries. This contradiction establishes the claim. Thus Q has a zero row and it follows from the hypothesis that the corresponding column is also zero. We may now delete this row and column and reduce the size of the problem. Since the result is clear if Q is 1×1, we are done. □

Problem 24-3: A Possible Generalization of Drury's Determinantal Equality

Proposed by George P. H. STYAN: styan@total.net; *McGill University, Montréal, Québec, Canada.*

Let A, B, C, D, E, and F be $n \times n$ complex matrices with

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} A & B \\ E & F \end{pmatrix}$$

and $\det(C) = \det(E)$ and $\det(D) = \det(F)$. Find further necessary and sufficient conditions (if any) so that $\det(R) = \det(S)$.

Remark. This problem is inspired by Problem 23-2 by S. W. Drury—see IMAGE 24 (April 2000), pp. 6–8.

Solution 24-3.1 by Gülhan ALPARGU: alpargu@math.mcgill.ca; *McGill University, Montréal, Québec, Canada,*
Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA,*
and the Proposer George P. H. STYAN: styan@total.net; *McGill University, Montréal, Québec, Canada.*

We assume first that the matrices D and F are nonsingular and so their determinants are both nonzero. Since determinant is multiplicative on the Schur complement, we have

$$\det(R) = \det(D)\det(A - BD^{-1}C) \quad \text{and} \quad \det(S) = \det(F)\det(A - BF^{-1}E).$$

When $\det(D) = \det(F) \neq 0$ then $\det(R) = \det(S)$ if and only if

$$\det(A - BD^{-1}C) = \det(A - BF^{-1}E). \tag{16}$$

When the matrices C and E are also nonsingular, then the condition (16) may be written as

$$\det(C)\det(AC^{-1} - BD^{-1}) = \det(E)\det(AE^{-1} - BF^{-1})$$

and so when $\det(C) = \det(E) \neq 0$ and $\det(D) = \det(F) \neq 0$ then $\det(R) = \det(S)$ if and only if $\det(AC^{-1} - BD^{-1}) = \det(AE^{-1} - BF^{-1})$ or equivalently

$$\det(\delta\, A\, \mathrm{adj}C - \gamma\, B\, \mathrm{adj}D) = \det(\delta\, A\, \mathrm{adj}E - \gamma\, B\, \mathrm{adj}F), \tag{17}$$

where $\mathrm{adj}(\cdot)$ denotes the adjugate matrix of first cofactors transposed and

$$\gamma = \det(C) = \det(E) \neq 0 \quad \text{and} \quad \delta = \det(D) = \det(F) \neq 0.$$

□

The condition (17) involves determinants of order $n \times n$ while $\det(R) = \det(S)$ involves determinants of order $2n \times 2n$. In Problem 23-2 [IMAGE 23 (October 1999), p. 28; see also Solution 23-2.1 by C.-K. Li and Solution 23-2.2 by H. J. Werner, both in IMAGE 24 (April 2000), pp. 6–8], S. W. Drury asked for an elegant proof of $\det R = \det S$, where

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_2 & b_1 & b_4 & b_3 \\ a_1b_2 & a_2b_1 & a_3b_4 & a_4b_3 \end{pmatrix}.$$

With this choice of R and S the condition (17) becomes

$$\det \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \det \begin{pmatrix} z & -x \\ -y & w \end{pmatrix} = wz - xy;$$

here $w = \delta(a_2b_2 - a_1b_1) + \gamma(a_3b_3 - a_4b_4)$, $x = \delta(b_1 - b_2) + \gamma(b_4 - b_3)$, $y = \delta a_1a_2(b_2 - b_1) + \gamma a_3a_4(b_3 - b_4)$, and $z = \delta(a_2b_1 - a_1b_2) + \gamma(a_3b_4 - a_4b_3)$, with $\gamma = (a_4 - a_3)b_3b_4$ and $\delta = (a_2 - a_1)b_1b_2$.

It is tempting to suggest that the condition (17) will still characterize the equality $\det(R) = \det(S)$ when γ and/or δ are 0. But when γ or δ is zero, not much can be said about the condition for $\det(R) = \det(S)$. For example, let $R = D_1P_1$ and $S = D_2P_2$ be 4×4, where D_i is an invertible diagonal matrix and P_i is a permutation matrix ($i = 1, 2$), so that all the 2×2 blocks are singular. Then it is easy to construct totally unrelated D_i, P_i so that $\det(R) = \det(S)$. Similarly, one can create R and S so that the $(2, 1)$ blocks of both of them have determinant 0, and $\det(R) = \det(S)$, but R and S are totally unrelated. For example, only the $(3, 1)$ entry in the first column of R and S are nonzero, and make the $(3, 1)$ minor of each matrix the reciprocal of the $(3, 1)$ entry.

Problem 24-4: The Positive Definiteness of a Matrix

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Suppose that $p(\lambda)$ and $q(\lambda)$ are any two nonzero polynomials with real coefficients and without common roots and that A is any complex square matrix of order m. Show that the Hermitian matrix $p(A)p(A^*) + q(A^*)q(A)$ is always positive definite.

Solution 24-4.1 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Clearly, $B = p(A)p(A^*) + q(A)q(A^*) = p(A)p(A)^* + q(A)q(A)^*$ is positive semidefinite. If B is singular, then there exits a unit eigenvector v such that $0 = v^*Bv = v^*p(A)p(A^*)v + v^*q(A)q(A^*)v$. Hence, there exists an eigenvalue μ of A with eigenvector v such that $0 = p(A^*)v = p(\bar{\mu})v$ and $0 = q(A^*)v = q(\bar{\mu})v$, contradicting the fact that $p(\lambda)$ and $q(\lambda)$ have no common roots. □

Problem 24-5: Two Products Involving Idempotent Matrices

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

(a) Suppose that A and B are two complex Hermitian idempotent matrices of the same size. Show that

$$A(A - B)^{\dagger}B = B(A - B)^{\dagger}A = 0,$$

where $(\cdot)^{\dagger}$ denotes Moore-Penrose inverse.

(b) Suppose the complex matrix A is idempotent. Show that $A(A - A^*)^{\dagger}A^* = A^*(A - A^*)^{\dagger}A = 0$.

Solution 24-5.1 by R. B. BAPAT: rbb@isid.ac.in; *Indian Statistical Institute–Delhi Centre, New Delhi, India.*

We will use the following proposition:

PROPOSITION. *If A and B are complex idempotent matrices of the same size, then for any nonnegative integer k,*

$$A(A-B)^{2k+1}B = 0. \tag{18}$$

Proof. Consider the identities

$$A(A-B)B = 0,\ A(A-B)^3B = 0 \tag{19}$$

$$A(A-B)^2 = A(A-B)A,\ (A-B)^2B = B(B-A)B \tag{20}$$

$$A(A-B)^{2k+1}B = A(A-B)A(A-B)^{2k-3}B(B-A)B,\ k = 1, 2, \ldots. \tag{21}$$

Identities (19) and (20) are easily verified since A and B are idempotent, while (21) follows from (20). Thus (18) holds for $k = 0, 1$ in view of (19) and for general k, the result is proved by induction on k using (21). □

To solve (a) we suppose that A and B are complex Hermitian idempotent matrices of the same size. Recall the well-known representation for the Moore-Penrose inverse (see S. L. Campbell & C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*, Dover, New York: 1991, page 26), which is easily proved using the singular value decomposition: For any complex matrix X,

$$X^\dagger = \int_0^\infty e^{-X^*Xt}X^*\,dt. \tag{22}$$

Using (22), the hypotheses on A and B, and (18) above, we obtain

$$A(A-B)^\dagger B = A\left\{\int_0^\infty e^{-(A-B)^2t}(A-B)dt\right\}B = \sum_{k=0}^\infty \int_0^\infty \frac{(-t)^k}{k!}A(A-B)^{2k+1}Bdt = 0.$$

The proof of $B(A-B)^\dagger A = 0$ and of (b) is similar.

Solution 24-5.2 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

(a) It is known (see [DA, HA]) that if A and B are complex Hermitian idempotent matrices of the same size, then there exists a unitary matrix U such that $U^*AU = A_0 \oplus A_1 \oplus \cdots \oplus A_k$ and $U^*BU = B_0 \oplus B_1 \oplus \cdots \oplus B_k$, where A_0 and B_0 are diagonal matrices, and for $i = 1, \ldots, k$,

$$A_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} \cos^2 t_i & \sin t_i \cos t_i \\ \sin t_i \cos t_i & \sin^2 t_i \end{pmatrix} \qquad \text{for some } t_i \in \mathbb{R}.$$

To prove the given assertion, it suffices to consider the result for each pair (A_i, B_i) for $i = 0, \ldots, k$, which can be readily verified.

(b) It is known (see [DO]) that if A is an idempotent matrix, then there exists a unitary U such that $U^*AU = A_0 \oplus A_1 \oplus \cdots \oplus A_k$, where A_0 is a diagonal matrix, and for each $i = 1, \ldots, k$,

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad a_i > 0.$$

To prove the given assertion, it suffices to consider the result for each A_i for $i = 0, \ldots, k$, which can be readily verified.

For the sake of completeness, we give a proof for the canonical forms used in (a) and (b). First suppose A and B are complex Hermitian idempotent matrices. Without loss of generality, assume that ranks of A and B are k and m with $k \geq m$. Let V be unitary such that $V^*AV = I_k \oplus 0_{n-k}$. Write $V^*BV = YY^*$ so that Y is an $n \times m$ matrix satisfying $Y^*Y = I_m$. Let X_1 be $k \times k$ unitary and X_2 be $m \times m$ unitary so that

$$(X_1 \oplus I_{n-k})YX_2 = \begin{array}{cc} \begin{pmatrix} C \\ 0 \\ R \end{pmatrix} & \begin{array}{l} m \\ k-m \\ n-k \end{array} \end{array}$$

where C is a diagonal matrix with nonnegative entries $c_1 \le \cdots \le c_k$. Suppose the last r diagonal entries of C equal one. Since $(X_1 \oplus I_{n-k})YX_3$ has orthogonal columns, the last r columns of R equal zero, and there exists an $(n-k) \times (n-k)$ unitary matrix X_3 so that

$$X_3 R = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

where S is an $(m-r) \times (m-r)$ upper triangular matrix with nonnegative diagonal entries. Since

$$(X_1 \oplus X_3)YX_2 = \begin{array}{c} \begin{pmatrix} C \\ 0 \\ X_3 R \end{pmatrix} \end{array} \begin{array}{l} m \\ k-m \\ n-k \end{array}$$

has orthogonal columns, we see that S is actually in diagonal form. Now, by applying a suitable permutation similarity to the matrices U^*AU and U^*BU with $U = V(X_1 \oplus X_3)^*$, we get the desired canonical form. Next, we let A be a general complex idempotent matrix. Then A has eigenvalues 1 and 0. Thus there exists a unitary matrix U such that

$$U^*AU = \begin{pmatrix} I_k + A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

so that A_1 and A_3 are upper triangular nilpotent matrices. Since $(U^*AU)^2 = U^*AU$, we see that A_1 and A_3 are zero matrices. Suppose A_2 has singular value decomposition X^*DY. Then by applying a suitable permutation similarity to the matrix $(X \oplus Y)U^*AU(X \oplus Y)^*$, we get the desired canonical form. □

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Solution 24-5.3 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

Our solution to this problem is based on the following known answer to the question: when does $(AB)^\dagger = B^\dagger A^\dagger$ hold?

THEOREM (see Corollary 5.8 in [WG]). *For A and B, let AB be defined. Then the following conditions are equivalent:*

(i) $B^\dagger A^\dagger = (AB)^\dagger$. (ii) $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$; *here* $\mathcal{R}(\cdot)$ *denotes the range (column space) of* $(\cdot)$.

(iii) A^*ABB^* *is EP, i. e.,* $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$.

The conditions (ii) are due to Greville [GN], whereas the condition under (iii) dates back to Arghiriade [AS]. An additional characterization for $(AB)^\dagger = B^\dagger A^\dagger$, in terms of certain less restrictive g-inverses of A and B, is also given in Corollary 5.8 in [GN]; only in passing we mention here that this characterization even allows one to extend the statements of this problem to certain other particular g-inverses.

We are now in the position to prove the desired results. To that extent, let A and B be complex Hermitian idempotent matrices of the same size. Then $A = A^* = A^2 = A^\dagger$ and $B = B^* = B^2 = B^\dagger$. Consequently $B(A-B)A = 0$ or, equivalently, $CA = 0$, where $C := B(A-B)$. Moreover, $C^*CAA^* = 0$ and $AA^*C^*C = 0$, so that, according to our Theorem above, $(CA)^\dagger = A^\dagger C^\dagger$. But $CA = 0$ if and only if $(CA)^\dagger = 0$, and so, in view of $A^\dagger = A$, we now arrive at $(CA)^\dagger = AC^\dagger = 0$. Next check that $B^*B(A-B)(A-B)^* = B - BAB = (A-B)(A-B)^*B^*B$. Therefore, again by our Theorem above, $C^\dagger = (A-B)^\dagger B^\dagger = (A-B)^\dagger B$. Combining our observations now results in $0 = AC^\dagger = A(A-B)^\dagger B$. Conjugate-transposing this equation yields $0 = B(A-B)^\dagger B$, thus completing the proof of part (a) of this problem.

For proving (b), let A be a complex idempotent matrix. Then $A^2 = A$, $(A^*)^2 = A^*$, and $A^*(A-A^*)A = 0$. Let $C := A^*(A-A^*)$ and check that $C^*CAA^* = 0 = AA^*C^*C$. Since in addition $AA^*(A-A^*)(A-A^*)^* = (AA^*)^2 - AA^* = (A-A^*)(A-A^*)^*AA^*$, our Theorem above tells us similarly as before that $A^\dagger(A-A^*)^\dagger(A^*)^\dagger = 0$. In view of $\mathcal{R}(A^{*\dagger}) = \mathcal{R}(A)$ and $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, this turns out to be equivalent to $A^*(A-A^*)^\dagger A = 0$. By $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ we denote the column space (range) and the null space of $(\cdot)$, respectively. On similar lines, one finds $A(A-A^*)^\dagger A^* = 0$ from $A(A-A^*)A^* = 0$. □

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A solution to the first part of the problem was also received from: Néstor Javier THOME: njthome@math.upv.es; *Universidad Politécnica de Valencia, Valencia, Spain.*

Problem 24-6: Rank of a Principal Submatrix

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida.*

Let A be a (complex) positive semidefinite matrix and write $[A]_\alpha$ for the principal submatrix of A consisting of rows α and columns α. Show that for any positive integer k

$$\operatorname{rank}([A]_\alpha) = \operatorname{rank}([A^k]_\alpha) = \operatorname{rank}(\{[A]_\alpha\}^k).$$

Does this generalize to real numbers k or to Hermitian A?

Solution 24-6.1 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Suppose A is Hermitian with nonzero eigenvalues $\lambda_1, \ldots, \lambda_r$, and $A = \sum_{i=1}^r \lambda_i x_i x_i^*$ where x_i is a unit eigenvector of A corresponding to λ_i for $i = 1, \ldots, r$. For any real number k, let

$$A^k = \sum_{i=1}^r \lambda_i^k x_i x_i^*. \tag{23}$$

(Note that $A^0 \neq I$ by our definition, and $A^k = (A^\dagger)^{(-k)}$ for $k < 0$ where $A^\dagger$ is the Moore-Penrose inverse of A.) Then it is clear that

$$\operatorname{rank}([A]_\alpha) = \operatorname{rank}(\{[A]_\alpha\}^k) \tag{24}$$

for any Hermitian A. However, the equality

$$\operatorname{rank}([A^k]_\alpha) = \operatorname{rank}([A]_\alpha) \tag{25}$$

may not be true if A is merely Hermitian. Consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \{1\}, \quad k = 2.$$

Now, suppose A is positive semidefinite with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$, with $A = \sum_{i=1}^r \lambda_i x_i x_i^*$. Let y_i be the vector obtained from x_i by keeping the entries with indices in α. For any real number k, we have $[A]_\alpha = \sum_{j=1}^r \lambda_i y_i y_i^*$ and $[A^k]_\alpha = \sum_{j=1}^r \lambda_i^k y_i y_i^*$. Clearly, there are positive real numbers a and b such that

$$a \sum_{j=1}^r \lambda_i y_i y_i^* < \sum_{j=1}^r \lambda_i^k y_i y_i^* < \sum_{j=1}^r b \lambda_i y_i y_i^*.$$

Thus (25) holds. Note that if we set $H^0 = I$ for any Hermitian matrix H, then for $k = 0$ we have $\operatorname{rank}([A^k]_\alpha) = \operatorname{rank}(\{[A]_\alpha\}^k)$, which may be strictly larger than $\operatorname{rank}([A]_\alpha)$. Also, note that for any function $f : (0, \infty) \to (0, \infty)$ if $f(A) = \sum_{i=1}^r f(\lambda_i) x_i x_i^*$ for any positive semidefinite A with spectral decomposition $A = \sum_{i=1}^r \lambda_i x_i x_i^*$, the above proof shows that

$$\operatorname{rank}([A]_\alpha) = \operatorname{rank}([f(A)]_\alpha) = \operatorname{rank}(f([A]_\alpha)).$$

Furthermore, one may consider replacing $A \mapsto [A]_\alpha$ by a positive linear map $A \mapsto \Phi(A)$. $\square$

Problem 24-7: Partitioned Positive Semidefinite Matrices

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let A be a positive semidefinite matrix. With $\dagger$ for Moore-Penrose inverse, $\circ$ for Hadamard product, and $\geq$ for Löwner ordering, show that

$$A \circ A^\dagger \geq A^\dagger A \circ AA^\dagger \quad \text{and} \quad A + A^\dagger \geq A^\dagger A + AA^\dagger.$$

In particular, if A is invertible, then

$$A \circ A^{-1} \geq I \quad \text{and} \quad A + A^{-1} \geq 2I.$$

Solution 24-7.1 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

We offer an elementary solution to this problem. Let A be a positive semidefinite (possibly complex) $n \times n$ matrix. From the famous *Spectral Theorem* for such a matrix we then know that A can be written as $A = \sum_{i=1}^n \lambda_i u_i u_i^*$, where λ_i, $i = 1, \cdots, n$, are the nonnegative eigenvalues of A and u_i, $i = 1, \cdots, n$, are corresponding eigenvectors; by $(\cdot)^*$ we denote here, as usual, the conjugate transpose of $(\cdot)$. It is also well-known (or can be easily seen by checking the Penrose defining equations) that the Moore-Penrose inverse $A^\dagger$ can be expressed as $A^\dagger = \sum_i \lambda_i^\dagger u_i u_i^*$; we recall that for a scalar c we have

$$c^\dagger = \begin{cases} 0 & \text{if } c = 0 \\ c^{-1} & \text{if } c \neq 0 \end{cases}.$$

Therefore,

$$AA^\dagger = \sum_i \lambda_i \lambda_i^\dagger u_i u_i^* = A^\dagger A, \qquad A \circ A^\dagger = \sum_{i,j} \lambda_i \lambda_j^\dagger (u_i \circ u_j)(u_i \circ u_j)^* \quad \text{and} \quad AA^\dagger \circ A^\dagger A = \sum_{i,j} \lambda_i \lambda_i^\dagger \lambda_j \lambda_j^\dagger (u_i \circ u_j)(u_i \circ u_j)^*.$$

These representations, in particular, directly show that the matrices $AA^\dagger$, $A \circ A^\dagger$ and $AA^\dagger \circ A^\dagger A$ are all positive semidefinite. We next show that $A \circ A^\dagger - AA^\dagger \circ A^\dagger A$ is also true. From the previous representations we obtain

$$\begin{aligned} A \circ A^\dagger - AA^\dagger \circ A^\dagger A &= \sum_{i,j} \lambda_i \lambda_j^\dagger (1 - \lambda_j \lambda_i^\dagger)(u_i \circ u_j)(u_i \circ u_j)^* \\ &= \sum_{i<j} (\lambda_i \lambda_j^\dagger + \lambda_j \lambda_i^\dagger - 2\lambda_i \lambda_i^\dagger \lambda_j \lambda_j^\dagger)(u_i \circ u_j)(u_i \circ u_j)^* \\ &= \sum_{i<j} \lambda_i^\dagger \lambda_j^\dagger (\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j)(u_i \circ u_j)(u_i \circ u_j)^* \;=\; \sum_{i<j} \lambda_i^\dagger \lambda_j^\dagger (\lambda_i - \lambda_j)^2 (u_i \circ u_j)(u_i \circ u_j)^*. \end{aligned}$$

Since all eigenvalues of A are nonnegative, it is now clear that $A \circ A^\dagger \geq AA^\dagger \circ A^\dagger A$ does indeed hold. If A is invertible, then $A^\dagger = A^{-1}$ and so $A \circ A^\dagger \geq A^\dagger A \circ AA^\dagger$ reduces to $A \circ A^{-1} \geq I$. This completes the proof of the first part of this problem.

We proceed with proving $A + A^\dagger \geq A^\dagger A + A^\dagger A$. In view of

$$A + A^\dagger = \sum_i (\lambda_i + \lambda_i^\dagger) u_i u_i^* \qquad \text{and} \qquad A^\dagger A + AA^\dagger = 2AA^\dagger = \sum_i 2\lambda_i \lambda_i^\dagger u_i u_i^*,$$

we obtain

$$A + A^\dagger - A^\dagger A - AA^\dagger = \sum_i (\lambda_i + \lambda_i^\dagger - 2\lambda_i \lambda_i^\dagger) u_i u_i^* = \sum_i \lambda_i^\dagger (\lambda_i^2 + 1 - 2\lambda_i) u_i u_i^* = \sum_i \lambda_i^\dagger (\lambda_i - 1)^2 u_i u_i^* \geq 0,$$

which is the other claimed result. For A being invertible, this trivially reduces to $A + A^{-1} \geq 2I$, and so our offered solution is complete. □

Solution 24-7.2 by the Proposer Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Notice that for X, Y and Z of the same size

$$\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0 \;\Rightarrow\; \begin{pmatrix} Z & Y^* \\ Y & X \end{pmatrix} \geq 0.$$

Thus, with $\star$ denoting either sum $+$ or Hadamard product $\circ$,

$$\begin{pmatrix} X \star Z & Y \star Y^* \\ Y \star Y^* & X \star Z \end{pmatrix} \geq 0,$$

which yields, by pre- and post-multiplying $(I : \pm I)$ and $(I : \pm I)^{\mathsf{T}}$, that $X \star Z \geq \pm(Y^* \star Y)$. The desired inequalities then easily follow, since

$$\begin{pmatrix} A & AA^{\dagger} \\ A^{\dagger}A & A^{\dagger} \end{pmatrix} \geq 0.$$

A solution was also received from Chi-Kwong LI: ckli@math.wm.edu and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

Problem 24-8: An Inequality Involving Hadamard Products

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let A and B be complex positive semidefinite matrices of the same size. If B is nonsingular, with $\circ$ for Hadamard product and $\geq$ for Löwner ordering, show that

$$A^2 \circ B^{-1} \geq (\mathrm{diag}A)^2(\mathrm{diag}B)^{-1}.$$

In particular, taking $B = I$ and $A = I$, respectively, it follows at once that $\mathrm{diag}(A^2) \geq (\mathrm{diag}A)^2$ and $\mathrm{diag}B^{-1} \geq (\mathrm{diag}B)^{-1}$.

Solution 24-8.1 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

The matrices

$$\begin{pmatrix} A^2 & A \\ A & I \end{pmatrix} \text{ and } \begin{pmatrix} B^{-1} & I \\ I & B \end{pmatrix}$$

are positive semidefinite, and hence so is their Hadamard product:

$$\begin{pmatrix} A^2 \circ B^{-1} & A \circ I \\ A \circ I & B \circ I \end{pmatrix} = \begin{pmatrix} A^2 \circ B^{-1} & \mathrm{diag}(A) \\ \mathrm{diag}(A) & \mathrm{diag}(B) \end{pmatrix}.$$

Hence

$$A^2 \circ B^{-1} \geq \mathrm{diag}(A)(\mathrm{diag}(B))^{-1}\mathrm{diag}(A).$$

□

Solution 24-8.2 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *Universität Bonn, Bonn, Germany.*

Our solution to this problem will show that it is easy to generate even more general inequalities of the Löwner type. Our approach is based on two well-known basic observations. First, the cone of positive semidefinite matrices is closed under the Hadamard product; cf. our Solution 24-7.1. And second, a given block-partitioned Hermitian matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

with M_{11} and M_{22} both being square, is positive semidefinite if and only if M_{11} and $M_{22} - M_{21}M_{11}^{-}M_{12}$ are both positive semidefinite. For later use, we mention here that $M_{22} - M_{21}M_{11}^{-}M_{12}$ is invariant to the choice of the g-inverse M_{11}^{-} of M_{11}. These observations enable us now to prove the following theorem.

THEOREM. *Let A and B be two positive semidefinite $n \times n$ matrices. Then*

$$\text{(i)} A^2 \circ B^2 \geq (A \circ B)^2 \geq 0, \quad \text{and} \quad \text{(ii)} A \circ B^{\dagger} \geq (A \circ B^{\dagger}B)(I \circ B)^{-}(A \circ BB^{\dagger}) \geq 0.$$

Proof: For given $A \geq 0$ and $B \geq 0$ we clearly have

$$\begin{pmatrix} I & A \\ A & A^2 \end{pmatrix} = \begin{pmatrix} I \\ A \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix}^* \geq 0, \quad \text{and} \quad \begin{pmatrix} I & B \\ B & B^2 \end{pmatrix} = \begin{pmatrix} I \\ B \end{pmatrix} \begin{pmatrix} I \\ B \end{pmatrix}^* \geq 0.$$

For $C := B^{1/2}$ we further have $C \geq 0$ and $CC^\dagger = BB^\dagger = B^\dagger B = C^\dagger C \geq 0$. Therefore,

$$\begin{pmatrix} B & BB^\dagger \\ B^\dagger B & B^\dagger \end{pmatrix} = \begin{pmatrix} C \\ C^\dagger \end{pmatrix} \begin{pmatrix} C \\ C^\dagger \end{pmatrix}^* \geq 0.$$

Since the cone of positive semidefinite matrices is closed under the Hadamard product, we now easily obtain

$$\begin{pmatrix} I & A \circ B \\ A \circ B & A^2 \circ B^2 \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} I \circ B & A \circ BB^\dagger \\ A \circ B^\dagger B & A^2 \circ B^\dagger \end{pmatrix} \geq 0$$

or, equivalently, $A^2 \circ B^2 \geq (A \circ B)^2 \geq 0$ and $A^2 \circ B^\dagger \geq (A \circ B^\dagger B)(I \circ B)^-(A \circ BB^\dagger)$, respectively. This completes the proof of our Theorem. □

Needless to say, if B is nonsingular, then part (ii) of our previous theorem reduces to the main statement of this problem.

Problem 24-9: Decomposition of Symmetric Matrices

Proposed by Roman ZMYŚLONY: r.zmyslony@im.pz.zgora.pl; *Technical University of Zielona Góra, Zielona Góra, Poland,* and Götz TRENKLER: trenkler@statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany.*

It is well known that a real symmetric $n \times n$ matrix A can be uniquely written as $A = A_+ - A_-$, where A_+ and A_- are nonnegative definite matrices such that $A_+ A_- = 0$. Find a procedure to calculate the pair $(A_+,\ A_-)$ without using the eigenvalues and eigenvectors of the matrix A. Assume $n \geq 3$. [This decomposition of a real symmetric matrix plays a crucial role in the estimation and testing of variance components in linear statistical models.]

Solution 24-9.1 by Chi-Kwong LI: ckli@math.wm.edu, and Roy C. MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA.*

We show that computing the A_+/A_- decomposition is equivalent to computing the eigendecomposition. One way to find A_+ and A_- is to find the polar decomposition of A: $A = PU$. Then $A_+ = (A+P)/2$ and $A_- = (P-A)/2$. On the other hand, given A_+ we can find the positive definite factor in the polar decomposition by $P = 2A_+ - A$. Thus the problems of finding A_+ and of finding P are equivalent.

Now, one may ask, how can we find the polar decomposition? One way to do this without computing eigenvalues and eigenvectors is to compute the sign of A by the iteration $S_0 = A,\ \ S_{i+1} = (S_i + S_i^{-1})/2, i = 0, 1, \ldots$. This iteration is globally convergent, and quadratically convergent provided that A is not singular. Call the limit S, then $P = AS$. However, this is not a finite algorithm. Is there any finite algorithm for the polar decomposition? No, this have been shown by George & Ikramov [*SIAM Journal on Matrix Analysis and Applications,* 18, 264]. In fact, the key idea in their proof is that one can compute the eigendecomposition by computing $n-1$ polar decompositions and $n-1$ QR factorizations. Galois theory tells us that one cannot compute the eigendecomposition in a finite number of steps. In conclusion, there is no finite algorithm for computing A_+, and any algorithm for computing A_+ yields an algorithm for computing the eigendecomposition. □

Solution 24-9.2 by the Proposers Roman ZMYŚLONY: r.zmyslony@im.pz.zgora.pl; *Technical University of Zielona Góra, Zielona Góra, Poland,* and Götz TRENKLER: trenkler@statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany.*

Since A^2 is nonnegative definite (nnd), its unique square root $B = (A^2)^{\frac{1}{2}}$ exists (see Ben-Israel & Greville, *Generalized Inverses: Theory and Applications*, Wiley, 1974, p. 274). The matrix B can be calculated iteratively (see the comment below). Then it is easily seen that $A_+ = \frac{1}{2}(B+A)$ and $A_- = \frac{1}{2}(B-A)$. □

Remark. Let D be nonnegative definite (nnd). Without loss of generality assume that $I - D$ is nnd; otherwise multiply D with a suitably chosen constant, e. g., $1/\mathrm{tr}(D)$. If we set $E = I - D$, then the iteratively defined sequence $(E_j)_j$, where $E_1 = \frac{1}{2}E$, $E_{j+1} = \frac{1}{2}(E + E_j^2)$ converges to $D^{\frac{1}{2}}$.

IMAGE Problem Corner: New Problems

Problem 25-1: Moore–Penrose Inverse of a Skew-Symmetric Matrix

Proposed by Jürgen GROß: gross@amadeus.statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany*,
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and Götz TRENKLER: trenkler@amadeus.statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany.*

Find the Moore–Penrose inverse A^+ of

$$A = \begin{pmatrix} 0 & a_3 & -a_2 & a_5 & -a_4 & a_7 & -a_6 \\ -a_3 & 0 & a_1 & a_6 & -a_7 & -a_4 & a_5 \\ a_2 & -a_1 & 0 & -a_7 & -a_6 & a_5 & a_4 \\ -a_5 & -a_6 & a_7 & 0 & a_1 & a_2 & -a_3 \\ a_4 & a_7 & a_6 & -a_1 & 0 & -a_3 & -a_2 \\ -a_7 & a_4 & -a_5 & -a_2 & a_3 & 0 & a_1 \\ a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & 0 \end{pmatrix}.$$

Problem 25-2: Vandermonde Matrices with Condition Number 1

Proposed by Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA*,
Roy MATHIAS: mathias@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA*,
and Seok-Zun SONG: szsong@cheju.cheju.ac.kr; *Cheju National University, Cheju, Korea.*

An $n \times n$ complex matrix $A = (a_{pq})$ is a Vandermonde matrix if there exist complex numbers $z_1, \ldots, z_n$ such that $a_{pq} = z_q^{p-1}$. Characterize those Vandermonde matrices with condition number equal to one, i.e., Vandermonde matrices which are multiples of unitary matrices.

Problem 25-3: Linear Functions on Similarity Orbits of Complex Square Matrices

Proposed by Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA*,
and Tin-Yau TAM: tamtiny@mail.auburn.edu; *Auburn University, Auburn, Alabama, USA.*

Let $A \in M_n$, the set of $n \times n$ complex matrices, and let $S(A)$ be the similarity orbit of A, i.e., $S(A)$ is the set of matrices in M_n similar to A. Show that $f(S(A))$ is convex for any linear functional f on M_n. What about $g(S(A))$ for some other linear function $g : M_n \to \mathbf{C}^m$ with $m > 1$, is $g(S(A))$ still convex?

Problem 25-4: Two Rank Equalities Associated with Blocks of an Orthogonal Projector

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}; \quad A_{11} \in \mathbf{C}^{m\times n}, \ A_{12} \in \mathbf{C}^{m\times k}, \ A_{21} \in \mathbf{C}^{l\times n}, \ A_{22} \in \mathbf{C}^{l\times k}.$$

(a) Show that the rank of the upper-right $m \times l$ block of $P_A = AA^+$ is

$$\operatorname{rank}((P_A)_{12}) = \operatorname{rank}(A_{11} : A_{12}) + \operatorname{rank}(A_{21} : A_{22}) - \operatorname{rank}(A).$$

Here $(E : F)$ denotes the partitioned block matrix with E placed next to F.

(b) Now suppose that A is Hermitian and idempotent, i.e., an orothogal projector. Show that

$$\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A_{12}).$$

Problem 25-5: Three Inequalities Involving Moore-Penrose Inverses

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Do the three inequalities

$$(A+B)^+ \leq_{\mathsf{L}} A^+ + B^+, \qquad A(A+B)^+B \leq_{\mathsf{L}} A^+ + B^+, \qquad (A:B)(A:B)^+ \leq_{\mathsf{L}} AA^+ + BB^+$$

hold for all nonnegative definite matrices A and B? Have these inequalities been considered before? Here $(A : B)$ denotes the partitioned block matrix with A placed next to B and $\leq_{\mathsf{L}}$ denotes the Löwner ordering.

Problem 25-6: Generalized Inverse of a Matrix Product

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given. It is well known that the reverse order law $(AB)^- = B^-A^-$ does not hold in general for an inner inverse of the matrix product AB. Show, however, that for any A^- and B^-, the product $B^-(A^\dagger ABB^\dagger)^\dagger A^-$ is an inner inverse of AB, that is, the set inclusion

$$\{\, B^-(A^\dagger ABB^\dagger)^\dagger A^- \,\} \subseteq \{\, (AB)^- \}$$

holds, where $(\cdot)^\dagger$ denotes the Moore-Penrose inverse.

Problem 25-7: Hadamard Product of Square Roots of Correlation Matrices

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Show that for any correlation matrices A and B of the same size (nonnnegative definite matrices with diagonal entries 1),

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leq_{\mathsf{L}} I,$$

where $\circ$ stands for the Hadamard product and $\leq_{\mathsf{L}}$ for the Löwner ordering.

Problem 25-8: Several Matrix Orderings Involving Matrix Geometric and Arithmetic Means

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let A, B, and C be n-square complex matrices such that $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq_{\mathsf{L}} 0$ and put

$$M_1 = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}, \quad M_2 = \begin{pmatrix} A & |B^*| \\ |B| & C \end{pmatrix}, \quad M_3 = \begin{pmatrix} A & |B| \\ |B| & C \end{pmatrix}, \quad M_4 = \begin{pmatrix} A & |B^*| \\ |B^*| & C \end{pmatrix}.$$

Here $X \geq_{\mathsf{L}} 0$ denotes the positive semidefiniteness (nonnegative definiteness) of the matrix X and $|X| = (X^*X)^{\frac{1}{2}}$.

Prove or disprove each of the following statements:

(1) $M_1 \geq_{\mathsf{L}} 0$; (2) $M_2 \geq_{\mathsf{L}} 0$; (3) $M_3 \geq_{\mathsf{L}} 0 \Leftrightarrow M_4 \geq_{\mathsf{L}} 0$.

(4) $M_3 \geq_{\mathsf{L}} 0$ if A and B commute; (5) $M_4 \geq_{\mathsf{L}} 0$ if A and B commute;

(6) $M_2 = M_3 = M_4 \geq 0$ when B is Hermitian or normal; (7) $M_2 \geq_{\mathsf{L}} 0 \Rightarrow M_3 \geq_{\mathsf{L}} 0$ and $M_4 \geq_{\mathsf{L}} 0$ but not conversely);

(8) Find necessary and/or sufficient conditions so that $M_5 = \begin{pmatrix} A & (B^*+B)/2 \\ (B^*+B)/2 & C \end{pmatrix} \geq_{\mathsf{L}} 0.$

Please submit solutions, as well as new problems, in macro-free LaTeX (a) by e-mail to werner@united.econ.uni-bonn.de and (b) two copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses are included! We look forward particularly to receiving solutions to Problems 19-3b & 23-1.

A Genealogy of the Spottiswoode Family: 1510–1900

by R. William FAREBROTHER and George P. H. STYAN

William
(d. 1513)

(first) John
1510–1585

(second) John
1565–1639

James
1567–1645

Sir Robert
1596–1646

Alexander
1631–1675

Robert
1637–1688

(third) John
1666–1728

Alexander (governor)
1676–1740

(fourth) John
1711–1793

(seventh) John
1724–1758

(fifth) John
1741–1805

(sixth) John
1780–1866

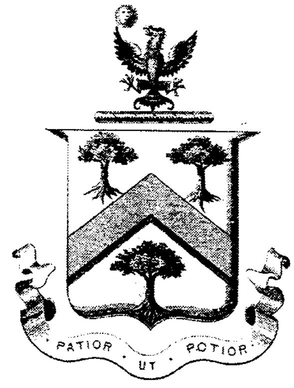
Andrew
1787–1866

Alicia Anne
1810–1900

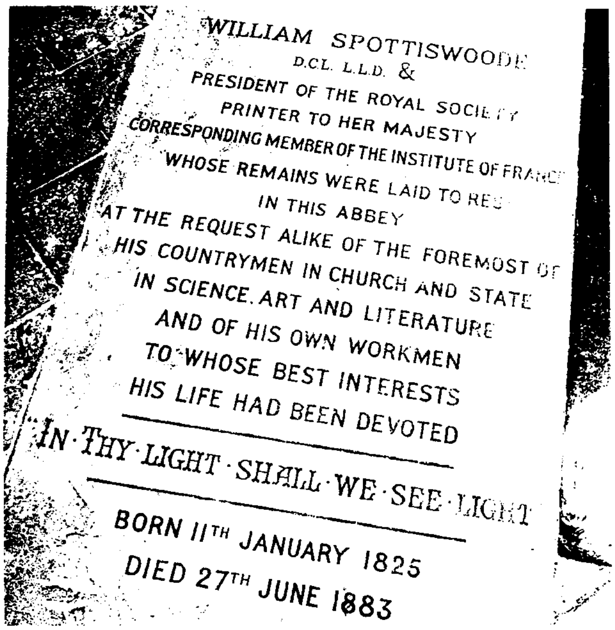
William
1825–1883

In an attempt to identify the great-grandfather of William Spottiswoode (1825–1883), left unresolved in the Spottiswoode genealogy in IMAGE 23 by Farebrother (1999), we outline here some of the history from 1510–1900 of the Spottiswoode, Spottiswood, or Spotswood family as recorded by Barry & Hall (1997), *Burke's Landed Gentry* (1863), Campbell (1868), and the (British) *Dictionary of National Biography* (1995). (Note that, at this period, it is not pertinent to place any weight on the precise spelling of a family's surname.)

The Spottiswoode family traces its ancestry to a Robert Spottiswood, who held the Barony of Spottiswood, Berwickshire, in the reign of King Alexander III (1249–1286) of Scotland. However, we shall begin our account with William Spottiswood of Spottiswood who was killed at the Battle of Flodden Field in 1513. This William's second son, John (1510–1585) was ordained a minister in the Church of England by Archbishop Thomas Cranmer. He was appointed a superintendent of the Church of Scotland with responsibility for the districts of Lothian and Tweedsdale.



This first John had two sons, John (1565–1639) and James (1567–1645), both of whom were also ordained into the Church of England. This second John was successively Archbishop of Glasgow and Archbishop of St Andrews in Scotland whilst James was Bishop of Clogher in Ireland. John compiled an invaluable *History of the Church and State of Scotland*..., originally published in 1655, with a third edition by the Spottiswoode (*sic*) Society in 1847. Both brothers were buried in Westminster Abbey, in St Benedict's Chapel, a small chapel at the western end of the south ambulatory, opening out into the south transept. The large monument to the Earl of Middlesex and his second wife, erected in the middle of St Benedict's chapel in 1645, has removed all trace of the two graves. Nearby in Poet's Corner, however, a slab on the floor, reincised in 1971, covers the grave of William Spottiswoode[1] (1825–1883).



The second John's second son Robert (1596–1646) became president of the Court of Sessions and was knighted. Later, when Secretary of State for Scotland, he was taken prisoner at the battle of Philiphaugh in 1645, tried by the Scottish Parliament for treason and executed in 1646.

[1]His *Elementary Theorems Relating to Determinants* (1851) was the first book ever published on determinants.

This Robert had three sons. The second, Alexander (1631–1675), became an advocate. He was followed in this profession by his son John (1666–1728) who founded 'Spottiswood's College of Law' in Edinburgh in 1703 with himself as Professor. He wrote several legal studies and took an interest in printing. Indeed, he is said to have written the preface to Watson's *History of Printing* published in 1713 and Austen-Leigh (1912) suggests that he may have been "the same Mr John Spottiswoode who sold in 1706 a printing press which he had established for printing law books to Robert Fairbairn of Edinburgh".

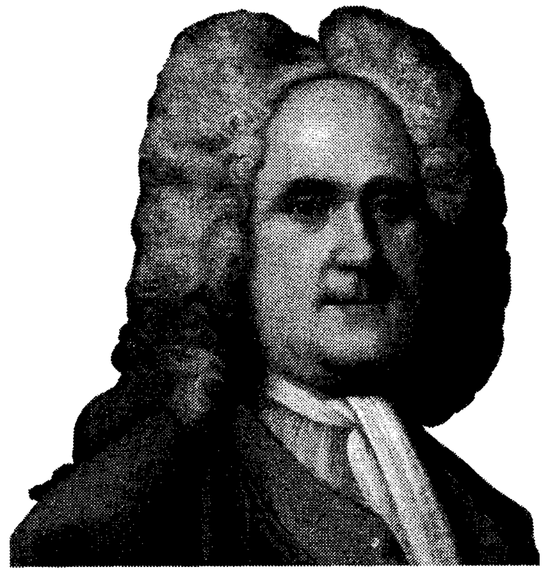
In 1700 this third John recovered the lands of Spottiswood from the heirs of the Bell family to whom it had been sold by his great-grandfather, the Archbishop. In 1710 he married Lady Helen Arbuthnot, daughter of the third Viscount of Arbuthnot, and had a son also named John (1711–1793), identified by Barry & Hall (1997) "as one of the important agricultural improvers of his age ... turning a barren stretch of the Lammermuirs[2] into ... a paradise of forest and productive farmland".

In 1740 this fourth John married Mary Thomson and they had eight children. Their eldest son, our fifth John (1741–1805), married Margaret Strahan in 1779, thus enabling us to identify the fourth John (1711–1793) as the great-grandfather of William Spottiswoode (1825–1883).

This fifth John (1741–1805), like his father and grandfather before him, was Laird of Spottiswoode, and he was succeeded in this dignity by his eldest son, John (1780–1866). As both of his sons had died before him, this sixth John drew up a deed of entail which, in effect, left the lands and barony to his eldest daughter Alicia Anne[3] (1810–1900), and then to her great-nephew John Roderick Charles Herbert Spottiswoode (1882–1946). But for the intervention of this entail, the property and title would presumably have passed to William (1825–1883) and then to his son William Hugh (1864–1915), as the senior male descendents of the fifth John.

Although Campbell (1868) seems to follow the entry in the 1863 edition of *Burke's Landed Gentry* fairly closely, he does not give any account of the descent of the Scottish Spottiswoodes from the third John as he is primarily concerned with the American branch of the family. To place what follows in context we note, as Dodson (1932, p. 3) points out, that "King Charles II of England, Scotland, and Ireland married Catherine of Braganza[4] in 1661 and received as part of her dowry the future British garrison of Tangier[5].

Robert Spotswood or Spottiswood (1637–1688), the third son of Sir Robert Spottiswood (1596–1646), served as physician to the governor of Tangier. Robert married the governor's widow Catherine Elliott and their only son Alexander[6] (1676–1740) was born in Tangier and became Lieutenant-Governor[7] (1710–1722) of the Colony of Virginia; it was said of him "He was the best Governor that Virginia ever had".



Alexander Spotswood rebuilt[8] The College of William and Mary in Williamsburg, Virginia, and captured and executed the notorious pirate Edward Teach, better known as Blackbeard[9]. According to Anderson (1999) "Alexander Spotswood tried to regulate the fur trade with the Indians ... [and] tried to protect the colony from Indian raids. He encouraged settlement along the colony's western frontier and led several expeditions over the Blue Ridge Mountains. ... He acquired an estate of about 85,000 acres in Spotsylvania County ..." The village of Spotsylvania, Virginia, 11 miles south-west of Fredericksburg, was the scene of the Civil War battle of Spotsylvania Court House, May 1864.

The eldest son of Alexander and Anne Brayne Spotswood is our seventh John (1724–1758). According to Shelton (1982) the skeleton of this John was discovered intact in 1931 in the Massaponax Tract of Spotsylvania County, Virginia, which John had inherited from his father who died in 1740. Hieter Atkinson, a crane operator for the Massaponax Sand and Gravel Company,

[2] A range of hills in the Lothian and Borders regions of south-east Scotland.

[3] According to Barry & Hall (1997, p. 74): Alicia Anne, Lady John Scott, is known above all as the creator of the song "Annie Laurie". For a calotype portrait by Hill and Adamson, 1840s, see illustration 14 (after page 72) in Barry & Hall (1997). See also Cant (1984, p. 18), who also discusses the Spottiswoode district of Edinburgh, including Spottiswoode Road, Street and Terrace. Alicia Anne's magnificent Spottiswoode House (designed by William Burn) in Berwickshire was destroyed in 1938; for "before" and "after" photographs see illustrations 16 and 17 (after page 72) in Barry & Hall (1997).

[4] After whom Queens, the largest (in area) of New York City's five boroughs is named. In 1640, John, Duke of Braganza, led a rebellion that drove out the Spaniards and restored Portugal's independence. John became the first king of the House of Braganza, the last Portuguese line of monarchs.

[5] On the African shore of the Strait of Gibraltar and now part of Morocco.

[6] For a full biography of Alexander Spotswood see Dodson (1932).

[7] Governor in all but title and emolument, Alexander was the deputy of George Hamilton, Earl of Orkney, who never set foot in Virginia.

[8] "Since it was burnt down, it has been rebuilt and nicely contrived, altered and adorned by the ingenious Direction of Governor Spotswood" (Kopper 1986, p. 68).

[9] According to Ritchie (1999) "Blackbeard was caught on November 21, 1718, near Ocracoke Inlet, off the North Carolina coast. He fought desperately with sword and pistol until he fell with 25 wounds in his body. Captain Maynard ordered two of his men to cut off Blackbeard's head and hang it on a ship's bow as a trophy and a warning to other pirates, cf. McCarthy (1987, p. 7). His head was taken back to Virginia and displayed on a pole. Blackbeard was born in either Bristol, England, or in Jamaica, and the year is unknown; he is said to have had 14 wives."

was manoeuvering the drag line of his crane when he discovered a vault of solid brick construction and John's skeleton in a walnut coffin inside the vault. The vault now rests in a low hill at a site remote from present mining operations with the area surrounding the vault maintained by employees of the Massaponax Sand and Gravel Company.

ACKNOWLEDGEMENTS

The first author is indebted to Nigel Kimber of the John Rylands University Library of Manchester for reading the whole of a microfilm copy of Campbell's genealogy to him and to Mrs. Elizabeth Roads, Keeper of the Records at the court of the Lord Lyon, Edinburgh, for supplying copies of some of the genealogical material employed in this article. The second author is most grateful to Sue and Dick Lake for their warm hospitality at the Governor's Trace Bed & Breakfast in Williamsburg, Virginia, with its elegantly appointed Alexander Spotswood Room.

The picture of the gravestone of William Spottiswoode was made available to us by Dr Tony Trowles, Librarian of the Westminster Abbey Library. The portrait of Alexander Spotswood comes from the tourist brochure "Virginia's Spotsylvania" and is reproduced with the permission of the Spotsylvania Historical Association. Thanks go also to Sue and Dick Lake, Angelika Van der Linde, and to Joyce and Harold Matheson for drawing our attention to the books by Kopper (1986), Cant (1984) and McCarthy (1987). We first discovered the book by Barry & Hall (1997) in the Cornell University Library (Ithaca, New York).

Very detailed family trees of the Spottiswood family, starting with our first John (1510–1585), are given by Dorothy Spottiswoode Doré in the frontal matter of Barry & Hall (1997, pp. xii–xxiii). See also the *The Dictionary of National Biography* (now available on CD from Oxford University Press), *Burke's Landed Gentry* (1863), the genealogy by Campbell (1868), and the unofficial genealogy (1296–1851) lodged with the court of the Lord Lyon, Edinburgh, Scotland.

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British Acts of Parliament

The October 1999 issue of *Image* contains two articles [1, 2] that are indirectly concerned with the publication of British Acts of Parliament and the medical use of St John's and other worts. Hardly had this issue gone to press, when, on 15 November 1999, the British House of Commons held a debate on whether the prime copies of Acts of Parliament should continue to be printed on goatskin vellum, concluding with a vote in favour of this proposition. And, less than a month later, on 11 December 1999, the *British Medical Journal* published an article by Philipp, Kohnen, and Hiller (1999) on the efficacy of Hypericum obtained from St John's wort in the treatment of mild depression. Clearly, *Image* is a bulletin that deserves to be studied carefully!

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Charles Lutwidge Dodgson: A Biographical and Philatelic Note

by R. William FAREBROTHER, Shane T. JENSEN and George P. H. STYAN

1. INTRODUCTION

Lewis Carroll (1832–1898) is known to millions as the author of *Alice's Adventures in Wonderland* (1865) and *Through the Looking Glass and What Alice Found There* (1870). But few of his readers (even those who are ILAS members) may know that his real name was Charles Lutwidge Dodgson or that he was a mathematician and logician whose book entitled *An Elementary Treatise on Determinants with their Application to Simultaneous Linear Equations and Algebraical Geometry*[1] was published in 1867. He acknowledged, this book was "not one, I am afraid, that can be brought out as by the author of *Alice in Wonderland*".

2. BIOGRAPHICAL SKETCH

Charles Lutwidge Dodgson was born at Daresbury, England, on 27 January 1832, the eldest son and third child of Charles Dodgson, then a Church of England minister in Daresbury, and his wife and first cousin, Frances Jane Lutwidge. Charles Lutwidge Dodgson is, however, far better known by his pseudonym "Lewis Carroll" which he invented by anglicising and reversing a Latin translation "Carolus Ludovicus" of his first two names.

Charles Lutwidge Dodgson was 11 when his father was appointed rector of Croft in North Yorkshire. At the age of 12, Dodgson was sent to school at nearby Richmond, and two years later he entered Rugby. There the shy, industrious boy received much praise from his schoolmasters, but he was not happy, due in part to the treatment he got from his schoolmates. In addition to good grades in classical languages and mathematics Charles earned a reputation for defending himself with his fists.

At the age of 18, Dodgson entered Christ Church, Oxford, and remained there for the rest of his life. He received his bachelor's and master's degrees at Christ Church and taught mathematics to several generations of Oxford students from 1855 until 1881. In 1861 (at the age of 29), Dodgson took deacon's orders of the Church of England but was not ordained a priest. He never married; his friends were mainly fellow faculty members.

As a mathematician, Dodgson was conservative. As a logician, he was more interested in logic as a game than as an instrument for testing reason. He was the author of a number of mathematics books, but, according to Lucas (1901): "His one valuable contribution to mathematics is *Euclid and his Modern Rivals* (London 1879) ... which when stripped of its external eccentricities was a really serious contribution to Euclidean geometry, and went far to vindicate the unique position of Euclid's elements as a first textbook of geometry, by a careful and systematic examination of the various treatises which had been produced by way of substitutes for it."

Dodgson enjoyed the company of children. He gave parties for them and took them to the theatre and on boating trips. On one such expedition his guests were Lorina, Alice, and Edith, the daughters of Dr Henry George Liddell, Dean of Christ Church. On a hot summer's day in 1862 in a meadow along the River Isis (= Thames) he began telling them the stories that later featured in the 'Alice' books.

The ensuing royalties enabled him to teach fewer mathematics classes, and he spent his summers at Eastbourne in Sussex on the south coast of England with his child-friends. He died in his sisters' house in Guildford, Surrey, England, on 14 January 1898, just thirteen days short of his 66th birthday. There is a statue of Lewis Carroll in Guildford.

3. QUEEN VICTORIA WAS NOT AMUSED

According to a popular story, Queen Victoria (1819–1901) was so impressed with *Alice's Adventures in Wonderland* (1865) that she asked the author to send her a copy of his next publication—which apparently was the 1867 book on determinants! Indeed, according to Abbott (1985, p. 45) Dodgson sent Queen Victoria "all of his serious and advanced papers on mathematical subjects, which Queen Victoria was dumbfounded to receive".

On consulting Miss Pamela Clark of the Royal Archive at Windsor Castle, however, we learned that Dodgson had never received an audience with Queen Victoria and had never sent her a copy of any mathematical work! Moreover, there is no excuse for the persistence of this rumour as Dodgson himself explicitly rebutted it in an advertisement attached to the preface to the Third Edition of his *Symbolic Logic*[2], as pointed out in a letter by J. B. Benson to the *Radio Times* of 5 December 1963:

> "... A silly story ... has been going the rounds of the papers about my having presented certain books to Her Majesty the Queen. It is so constantly repeated, and is such absolute fiction, that I think it worthwhile to state, once for all, that it is utterly false in every particular ..."

Apparently, the only presentation copy of a book by Charles Dodgson in the Royal Library is a copy of the first German translation of *Alice's Adventures in Wonderland*, which is inscribed to Princess Beatrice (1857–1944), the youngest of Queen Victoria's nine children and a great-grandmother of King Juan Carlos of Spain.

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[1] We are very grateful grateful to Hans Schneider and Evelyn Matheson Styan for drawing this book to our attention. In fact this was only the second book in English on determinants—the book by William Spottiswoode, *Elementary Theorems Relating to Determinants* (viii + 63 pp., Longman, Brown, Green & Longman, London, 1851) being the first.

[2] Unfortunately we have not been able to locate a copy of the short-lived Third Edition of Lewis Carroll's *Symbolic Logic*.

4. LEWIS CARROLL AND POSTAGE STAMPS

4.1. Mali 1982. We are very grateful to Monty J. Strauss (Professor of Mathematics at Texas Tech University in Lubbock and President of the Mathematical Study Unit of The American Topical Association and The American Philatelic Society: m.strauss@ttu.edu), for making available to us a copy of his extensive checklist of mathematicians on postage stamps as an Excel spreadsheet, which lists many stamps under Lewis Carroll.

As far as we can determine, however, only one of these stamps (Scott #C443, Stanley Gibbons #905) actually contains a (full) portrait of the author. This 110F stamp was issued by Mali on 30 January 1982, to celebrate the "150^e Anniversaire de la naissance de Lewis Carroll". The 130F and 140F stamps show illustrations of scenes from *Alice's Adventures in Wonderland.* The Republic of Mali was formed in 1960; it was previously French Sudan (part of French West Africa). It is a land-locked country in north-east Africa, to the south of Algeria, and is perhaps best known today for one of its major towns: Tombouctou (Timbuktu).

4.2. Tristan da Cunha 1981. Three 1981 stamps (10p, 20p and 30p) and a "souvenir sheet" (Scott #287a, Stanley Gibbons #300–302) featuring these three stamps were issued for the "Centenary of the Rev. Edwin Dodgson's arrival on Tristan da Cunha". Charles, his brother Edwin, and eleven other members (Wilfred, Lucy Lutwidge, Archdeacon Dodgson, Elizabeth, Mary, Louisa, Frances, Skeffington, Henrietta, Caroline, and Margaret) of the Dodgson family are shown. In the photograph of "The Dodgson Family outside Croft Rectory, c. 1860" on the souvenir sheet, Charles is shown seated on the ground at the left in the group but is not shown in any of the stamps.

According to Mackay (1998): "The 10p stamp reproduced a rather stiffly posed portrait of Edwin as a young man, perhaps photographed by his famous brother. The 20p shows him [Edwin] arriving at Tristan as resident missionary in 1881". And, according to the 1996 Scott *Standard Postage Stamp Catalogue* (vol. 1B, p. 785) "The Rev. Edwin H. Dodgson saved [the] population [of Tristan da Cunha] from starvation."

Tristan da Cunha is a group of British volcanic islands in the South Atlantic Ocean to the west of the Cape of Good Hope; in 1938 it was made a dependency of St Helena (where Napoleon Bonaporte was exiled in 1815 and where he died in 1821).

4.3. Kaulbach Island 1976. Strauss's Excel file also lists a 1976 set of four local stamps (and a 1977 overprint) from Kaulbach Island which, as Wood (1983, p. 405) points out, is in the North Atlantic Ocean just off the coast of Nova Scotia: "in Mahone Bay [near Lunenberg] about a mile off the village of Indian Point and about six miles southwest of the town of Chester. Its name is said to be derived from that of an early settler. For several years, the island's owner has created local post stamps used to frank mail to the nearest Canadian post office on the mainland".

Two of the stamps depict Alice: "Alice caught the baby with some difficulty" and "... it was neither more nor less than a pig"; the other two, The Cheshire Cat: "I didn't know that ... cats could grin" and "It vanished very slowly ... ending with the grin". The stamps are "Not valid for the carriage of mail by the Canada Post Office. To be used only in the Kaulbach Island Carriage Service and may be placed only on the back of envelopes".

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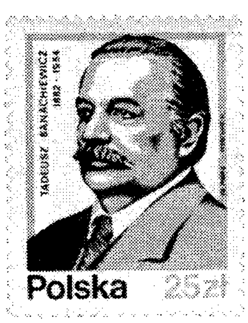
Tadeusz Banachiewicz: 1882–1954

The astronomer and mathematician Tadeusz Banachiewicz was born in Warsaw, Poland, on 13 February 1882. Banachiewicz completed his studies in astronomy at the Warsaw University and obtained also there, in 1904, his degree of the candidate of sciences. As a student he sent a telegram to the Central Astronomical Bureau in Kilonia that on 19 September 1903 the planet Jupiter would cover the changeable star signed as BD 6'6191. After graduating from university he studied in Getynga and worked as an astronomer in Russia (1910–1917) in Kazan and Dorpat. He returned to Poland at the end of 1918. In March 1919 he became professor and director of the Astronomical Observatory at the Jagellonian University in Cracow (Kraków). For the history of the Cracow Observatory visit the Web site http://www.oa.uj.edu.pl/history/history.html.



In 1925 Banachiewicz created the mathematical theory of Cracovians, a special kind of matrix algebra in which columns multiply columns, and which is used, for example, in spherical astronomy (polygonometry), geodesy, celestial mechanics, and the calculation of orbits. Using Cracovians he discovered general formulae of spherical polygonometry, and simplified considerably the algorithm for least squares and the solution of systems of linear equations. Using Cracovians, Banachiewicz was the first to calculate the orbit of Pluto. He published about 240 papers on astronomy, mathematics, mechanics, geodesy and geophysics. He was a vice-president of the International Astronomical Union (1932–1938) and was elected a member of the Polish Academy of Sciences in 1951. He died on 17 September 1954 in Cracow.

According to Ouellette (1981, p. 201) and Henderson & Searle (1981, p. 55), Banachiewicz (1937a, 1937b) is credited for the following formula of the inverse of a partitioned matrix

$$A^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} E^{-1} + E^{-1}F(A/E)^{-1}GE^{-1} & -E^{-1}F(A/E)^{-1} \\ -(A/E)^{-1}GE^{-1} & (A/E)^{-1} \end{pmatrix}$$

where the Schur complement $(A/E) = H - GE^{-1}F$.

When A is symmetric nonnegative definite and singular, then Latour, Puntanen & Styan (1987) identify the reflexive symmetric generalized inverse

$$A^{\#} = \begin{pmatrix} E_{\rm rs}^{-} + E_{\rm rs}^{-}F(A/E)_{\rm rs}^{-}GE_{\rm rs}^{-} & -E_{\rm rs}^{-}F(A/E)_{\rm rs}^{-} \\ -(A/E)_{\rm rs}^{-}GE_{\rm rs}^{-} & (A/E)_{\rm rs}^{-} \end{pmatrix},$$

to be in "Banachiewicz-Schur" form; here the generalized Schur complement $(A/E) = H - GE^{-}F$ with $G = F'$, E^{-} any generalized inverse of E, and $(A/E)_{\rm rs}^{-}$ a relexive symmetric generalized inverse of (A/E).

The stamp is Scott # 2565, Stanley Gibbons #2872 (1983)—see also Jeff Miller's excellent Web site http://jeff560.tripod.com/ which contains many images of mathematical stamps.

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Vlad Ionescu: 1938–2000

Dr Vlad Ionescu, Professor of Systems Theory at the Univ. Polytechnica Bucharest, passed away on May 28, 2000, at the age of sixty-two, in his natal city Bucharest. Vlad Ionescu received the degrees of Engineer (with honours) and Doctor of Science in Electrical Engineering in 1959 and 1970, respectively, both from the Univ. Polytechnica Bucharest. Since 1962 he was a faculty member at the same Univ., where he served in various capacities, including Head of the Automatic Control Department, until his death.

As a teacher, he was consistently judged an excellent, effective and yet charming expositor, explaining many difficult concepts in a simple manner. To his PhD students, he was not only a technical advisor and a coworker but also a confident and close personal friend, exemplary in his generous contribution of time and effort.

In the early days of his career he was mainly involved with Optimization Theory and its applications to power generation. Later on, his interests shifted to the Algebraic and Geometric Theory of Systems, and he wrote four successful monographs devoted to this subject, one of which has been awarded the Prize of The Romanian Academy. For many scientists in the linear algebra community, his name became familiar during the last ten years, particularly through the "Generalized Riccati Theory" and its applications to the robust control of linear dynamical systems. An impressive list of publications, including fifteen monographs (one yet to appear), three textbooks and over one hundred fifty papers in archival journals or proceedings of conferences, has established for him a respectable position in the international community. Vlad Ionescu was also an effective administrator. He served on the Editorial Board of the *International Journal of Control* (London) and the *Revue Roumaine des Sciences Techniques, Série Électrotechnique et Énergétique* (Bucharest), and as a member of the IFAC Committee on Linear Systems & Robust Control.

At the age of sixty-two, when most respected scientists are happy with an advisory role for their graduate students, Professor Vlad Ionescu was conducting a very active and enthusiastic scientific life, thinking about new papers and books, planning to participate in scientific conferences. A Member of the Romanian Academy since 1996, he was nominated in 2000 for the Prize of Excellence in Romanian Culture next to the most renowned personalities of the Romanian cultural elite.

We all mourn with great sorrow the passing of this great scientist, generous educator, and cherished friend.

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Sally Rear: 1945–2000

My LAA secretary, Sally Rear, died unexpectedly on Thursday, October 19, 2000, apparently of an allergic reaction. She was a most conscientious and careful secretary, who served LAA with dedication for over 10 years.

In the course of her service, she must have sent messages to several hundred linear algebraists around the world. I relied on her totally for keeping the LAA operation on track. She will be greatly missed.

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ILAS News

The ballots for this year's ILAS elections have now been counted. In accordance with the election rules in the ILAS By-Laws, on March 1, 2001,

Roger Horn will become ILAS Vice-President (3 year term)
Tom Markham and Daniel Szyld will become members of the ILAS Board (3 year terms).

We look forward to working with Roger, Tom, and Daniel in service to the linear algebra community.

I want to personally thank the members of the nomination committee: Jane Day (Chair), Wayne Barrett, Nick Higham, Chi-Kwong Li, and Michael Tsatsomeros for their efforts on behalf of ILAS. I also want to thank Jane Day and Wasin So for counting the ballots in a timely manner.

The revision to the ILAS By-Laws recently approved by the ILAS Board of Directors allowed for the possibility of the appointment of an Outreach Director. I am very happy to report that, with the consent of the Board of Directors, I have appointed Jim Weaver as ILAS Outreach Director. Jim's appointment extends to the end of my term as ILAS President, that is, February 28, 2002.

Those of you who attended the SIAM-SIAG/LA Conference in Raleigh this month, probably noticed that Jim was already on the job, recruiting new members and selling the print version of ELA, Volumes 1-4.

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Workshops and Conferences in Linear Algebra and Matrix Theory

Western Canada Linear Algebra Meeting

Winnipeg, Manitoba: 26–27 May 2000

The on-going series of Western Canada Linear Algebra Meetings (W-CLAM) provides an opportunity for western Canadian mathematicians working in linear algebra and related areas to meet and exchange accounts of their recent research. The meetings are regionally based, but they also attract participants from beyond Canada's four western provinces. Previous W-CLAMs were held in Regina, Lethbridge, Kannanaskis, and Victoria.

The most recent W-CLAM was held in Winnipeg on May 26–27, 2000. The meeting was attended by more than 30 participants and featured three invited speakers: Hans Schneider, Bryan Shader and Henry Wolkowicz. In addition, contributed talks were given by S. Cokus, R. Craigen, S. Fallat, C.-H. Guo, H. Kharaghani, J. McDonald, J. Molitierno, C. Neal, D. Olesky, S. Pati, P. Psarrakos, R. Reams, M. Tsatsomeros and P. van den Driessche. These lectures covered a variety of aspects of linear algebra, including qualitative and combinatorial matrix theory, spectral theory, matrix analysis, and applied linear algebra. And as at previous W-CLAMs, there was plenty of lively, informative, and informal discussion.

The next W-CLAM is tentatively scheduled to take place in 2002 and will be held in Regina.

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The Second Conference on Numerical Analysis and Applications

Rousse, Bulgaria: 11–15 June 2000

The Second Conference on Numerical Analysis and Applications was held June 11–15, 2000, at the University of Rousse, Bulgaria. There were about a hundred participants with about twenty Bulgarians and more than eighty from thirty-three countries such as Germany, Italy, Ireland, Japan, The Netherlands, Russia, USA, Yugoslavia, and elsewhere.

Professor Plamen Yalamov and others at Rousse University did an excellent job running the conference. Everyone learned a great deal. The social program included an excursion to the Kaliopa Museum in Rousse, the Bassarbovo Monastery, and the Cherven Castle.

The conference was organized into four tracks: (1) Numerical Linear Algebra, (2) Numerical Methods for Differential Equations, (3) Numerical Modeling, and (4) High-Performance Computing. The program covered these topics evenly, which made it easy to find an interesting talk to attend.

Some of the highly successful mini-symposia were: (1) Robust Numerical Methods for Singularly Perturbed and Related Problems, (2) Numerical Methods in Orbit Determination, (3) Numerical Modeling in Mechanics, (4) Recent Advances in Structured Matrices and Applications, and (5) Computational Methods in Economics, Finance and Statistics.

At the Westen Canada Linear Algebra Meeting (W-CLAM) at Winnipeg, Manitoba, Canada, May 2000, are (from left to right): Steve Kirkland, Chun-Hua Guo, Cora Neal, Panos Psarrakos, Dan Scully, Jason Molitierno, Brenda Kroschel, Peter Lancaster, Pauline van den Driessche, Henry Wolkowicz, Sarah Naqvi, Bryan Shader, Judi McDonald, Aiden Roy, Hans Schneider, Rob Craigen, LeRoy Beasley, Hadi Kharaghani, Steve Pierce, Robert Reams, Shaun Fallat, Dale Olesky, P. N. Shivakumar, Michael Tastsomeros, Shaun Cokus, and Sukanta Pati.

Keynote speakers included R. Chan (Hong Kong), R. Freund (USA), A. V. Gulin (Russia), A. Griewank (Germany), P. W. Hemker (The Netherlands), B. Jovanovich (Yugoslavia), M. Kaschiev (Bulgaria), R. D. Lazarov (USA/Bulgaria), J. J. Miller (Ireland), G. I. Shishkin (Russia), and E. E. Tyrtyshnikov (Russia). The Organizing Committee consisted of Lubin Vulkov and Plamen Yalamov (Bulgaria), Marcin Paprzycki (USA).

Here is an exciting story connected with the Rousse conferences. When David Kincaid was at the First Conference on Numerical Analysis and Applications four years ago in Rousse, he was walking back to his hotel after a full day at the conference when he saw one of the secretaries from the conference walking in the same direction. So he said "hello" and she became enthusiastic when she saw from his name tag that he was from Texas. Since they each didn't speak the other's language, they communicated the best they could. It was clear she wanted him to follow her, which he did, not knowing where they were going! It turned out she took him to her apartment to talk to her daughter, Iskra, who was applying to colleges in the USA and one of them was The University of Texas at Austin where he worked. To his surprise he next saw Iskra three months later on campus in Texas as a student. He saw her frequently over the next several years since she majored in astronomy and his office was in the physics-mathematics-astronomy building.

During the Second Rousse conference when he asked her mother about Iskra, there came another surprise. She showed him pictures of Iskra's graduation from UT Austin and said that Iskra was now at Princeton University! This is only one of many success stories connected with the Rousse conferences.

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Fifth Workshop on Numerical Ranges and Numerical Radii

Nafplio, Greece: 26–28 June 2000

The Fifth Workshop on Numerical Ranges and Numerical Radii took place in Nafplio, Greece, 26–28 June 2000. There were over 30 participants whose names are listed below. The program featured Chandler Davis as the *ILAS Lecturer* who "updated" us on numerical ranges. There were 27 talks in total, all very interesting, on numerical ranges, their generalizations and applications.

The participants had the opportunity to attend social events, courtesy of our sponsors, including an outdoor classical concert by a German youth orchestra, an excursion to the Castle of Palamidi, and a guided trip to the ancient theatre of Epidauros and the temple of Aesklepios. The banquet took place at a port-side fish taverna.

In the closing remarks, Chandler Davis paid tribute to the workshop's friends and contributors, those present and those who have contributed with their work and ideas throughout the years to the development of the field. As a newcomer to the world of numerical ranges, I particularly appreciated the high caliber and thoughtful presentations by all speakers. They set the stage for future research with open problems and new ideas. We all look forward to the next workshop.

On behalf of the organizing committee (Maroulas, Tsatsomeros, Psarrakos, Adam) I would like to express our gratitude to our sponsors: The National Technical University of Athens, the City of Nafplio, and especially ILAS.

The participants in the workshop were: Maria Adam, Tsuyoshi Ando, Natália Bebiano, Avi Berman, Mao-Ting

Group photo from the Fifth Workshop on Numerical Ranges and Numerical Radii: Nafplio, Greece, June 2000.

Chien, Thanassis Chryssakis, Chandler Davis, Doug Farenick, Susana Furtado, Moshe Goldberg, Charles Johnson, Peter Lancaster, Heinz Langer, Gorazd Lešnjak, Chi-Kwong Li, Ioannis Maroulas, Judith McDonald, Boris Mirman, Hiroshi Nakazato, Kazuyoshi Okubo, João da Providencia, Psarrakos Panayiotis, Salemi Abbas, Peter Šemrl, Ahmed Sourour, Ilya Spitkovsky, Bit-Shun Tam, Christiane Tretter, Michael Tsatsomeros, Dimitris Valougeorgis, Markus Wagenhofer, Pei Yuan Wu, Takanori Yamamoto, and Xingzhi Zhan.

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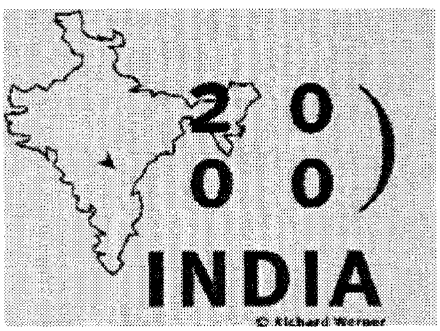


Ninth International Workshop on Matrices and Statistics

Hyderabad, India: 9–13 December 2000

The Ninth International Workshop on Matrices and Statistics, in Celebration of C. R. Rao's 80th Birthday, will be held in the historic walled city of Hyderabad, in Andhra Pradesh, India, on December 9–13, 2000. This Workshop is being organized in Hyderabad by the Indian Statistical Institute (ISI) and the Society for Development of Statistics, in collaboration with the Birla Science Centre, Osmania University and the Univ. of Hyderabad.

Hyderabad is approximately midway between Mumbai (Bombay) and Chennai (Madras) and is one of India's largest cities, with a population of about 6 million. Founded by Quli Qutub Shah in 1589 as a royal capital, Hyderabad was a large and important princely feudatory state in India; the ruler, the Nizam of Hyderabad, was considered to be the world's richest man at the time of the state's annexation to India in 1947. Postage stamps were issued from 1869 to 1949.

The city of Hyderabad became the capital of the state of Hyderabad in 1950 and the capital of Andhra Pradesh in 1956. Hyderabad has many palaces and the Char Minar, built in 1591, with four minarets (towers) and four arches through which the main streets of the city pass. Hyderabad is a major trading center and manufactures textiles, glassware, paper, flour, and railway cars, and is unique in its rich architectural glory and blend of linguistic, religious, and ethnic groups. It is an ideal place to celebrate C. R. Rao's 80th Birthday. Fine weather is expected with a midday high of about 20° C/60° F.

The program will start with a two-day Short Course on recent advances in Matrix Theory with Special Reference to Applications in Statistics on Saturday, December 9, and Sunday, December 10, 2000. This will be followed by the presentation of research papers in the Workshop proper on Monday, December 11–Wednesday, December 13, 2000; it is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications.*

The International Organizing Committee for this Workshop comprises R. William Farebrother (Victoria Univ. of Manchester, Manchester, England, UK), Simo Puntanen (Univ. of Tampere, Tampere, Finland), George P. H. Styan (McGill Univ., Montréal, Québec, Canada; vice-chair), and Hans Joachim Werner (Univ. of Bonn, Bonn, Germany; chair). The Local Organizing Committee in India includes R. Bhatia (ISI–Delhi), P. Bhimasankaram (ISI–Hyderabad), G. S. R. Murthy (ISI–Hyderabad), V. Narayana (ISI–Hyderabad), M. S. Rao (Osmania Univ., Hyderabad), B. Sidharth (Birla Science Centre, Hyderabad), U. SuryaPrakesh (Osmania Univ., Hyderabad), R. J. R. Swamy (Osmania Univ., Hyderabad), P. Udayasree (Univ. of Hyderabad), and K. Viswanath (Univ. of Hyderabad).

For up-to-date information on this Workshop please visit

http://eos.ect.uni-bonn.de/HYD2000.htm

The Short Course will be held in the air-conditioned auditorium at the Prof. G. Rami Reddy Centre for Distance Education on the Osmania Univ. campus with a capacity of 300. This auditorium has a public address system, OHP and facilities for PowerPoint presentations. The Workshop proper and the Workshop banquet will be held in the Auditorium of the Birla Science Centre in downtown Hyderabad.

Accomomdation has been arranged for delegates in reasonably good guestrooms on and around the Osmania Univ. campus. The tariff ranges from US$15–25 per person per day. Rooms have also been reserved in hotels as follows:

	5-star	3-star	motel
Single US$ per day	80-125	40–50	20-30
Double US$ per day	120-160	50-60	25-35

Preliminary Programme of Invited Papers for the Short Course

C. R. Rao (PennState): Statistical proofs of some matrix inequalities

R. B. Bapat (ISI–Delhi): Generalized inverses–I: Existence of generalized inverse: Ten proofs and some remarks

P. Bhimasankaram (ISI–Hyderabad): Generalized inverses–II

G. H. Golub (Stanford): Numerical Linear Algebra–I: Numerical methods for solving least squares problems with constraints

T. Kailath (Stanford): Numerical Linear Algebra–II: Displacement structure–Theory and applications

S. Puntanen (Tampere): Some comments on several matrix inequalities with applications to canonical correlations–I

G. P. H. Styan (McGill): Some comments on several matrix inequalities with applications to canonical correlations–II

Y. Takane (McGill): Matrices with special reference to applications in psychometrics

H. J. Werner (Bonn): On some estimating and predicting techniques in the general Gauss-Markov model

The Short Course will be followed by the Workshop proper, which will include the presentation of both invited and contributed papers on matrices and statistics. There will also be a special session for papers presented by graduate students.

Preliminary Programme of Invited Papers for the Workshop

C. R. Rao (PennState): Anti-eigen and singular values of a matrix and their applications to statistics

F. Akdeniz (Adana): Some comments on the ridge regression and the Liu estimators

T. Ando (Sapporo): From determinant and trace inequalities to majorization of eigenvalues

J. K. Baksalary (Zielona Góra): Revisiting outer inverses in the context of nonlinear matrix equations and partitioned martices

A. Ben-Israel (Rutgers): On some geometric problems in $\mathbf{R}^n$.

K. P. S. Bhaskara Rao (ISI–Bangalore): Title to be announced

K. Conradsen (DTU–Lyngby): The use of the Wishart distribution in edge detection in radar images

S. Das Gupta (ISI–Calcutta): Parametric identifiability and model-preserving constraints

S. J. Kirkland (Regina): Markov Chains whose transition matrix has large exponent

R. Kumar (Dayalbagh Educational Institute–Agra): Characterizations of certain eigenvalue functions and their bounds

K. M. Prasad (Gangtok): Dimension function for modules over commutative rings

J. N. K. Rao (Carleton): On empirical likelihood methods for linear regression models under imputation for missing responses

P. S. S. N. V. P. Rao: Title to be announced.

P. Šemrl (Ljubljana): Locally linearly dependent operators

D. Sengupta (ISI–Calcutta): Some observations and questions on the linear model.

G. Trenkler (Dortmund): Estimation of the cross product of two expectation vectors

J. Volaufová (Baton Rouge): Some estimation and testing problems in multistage linear models

H. Yanai (Tokyo): Some relationships between projectors and rank-addititivity

Preliminary Programme for the Special Session for Graduate Students

O. M. Baksalary (Poznań): Idempotency of linear combinations of idempotent and tripotent matrices

A. Guterman (Moscow): Minus order and its linear preservers

J. Mashreghi (McGill): Complex variable methods in linear algebra

Y. Tian: Moore–Penrose inverses of matrix products

Preliminary Programme for Contributed Papers

G. M. Antony & K Visweswara Rao: Factors responsible for variations in human development index in Indian states: results of factor analysis

Th. Benesch & V. Goyal: Electronic commerce in India

P. Druilhet: Optimality criteria in experimental designs matrices

V. S. Gudi & B. N. Nagnur: A note on the second-order efficiency of the BAN-estimators in the multinomial family

C. P. S. Gungund: Unbiased estimation in a certain class of independent increment processes

V. V. Haragopal & S. N. N. Pandit: Dependent count streams as Markov chains (an exploratory study)

M. Ishak: Statistical techniques to analyse hospitalization pattern of the Australian indigeneous population

D. N. Kashid & S. R. Kulkarni: A more general criterion for subset selection in multiple linear regression

G. Lešnjak: On the algebraic version of Cochran's theorem

A. R. Meenakshi: On products of range-symmetric fuzzy matrices

A. R. Meenakshi & S. Krishnamoorthy: k_{EP} generalized inverses

S. K. Mukhopadhyay: Probabilistic forecasting of monsoon daily rainfall by Markov chain model

A. L. N. Murthy: Complementarity problems and positive definite

H. Naseri: Some examples of linear prediction with infinite variance in electricity distributions

S. N. N. Pandit: Some new matrix operations with applications to clustering, modelling and cryptography

J.-S. Park, D. D. Cox & T. Y. Kim: Fast cross validation of best linear unbiased predictors by downdating Cholesky decomposition

D. Petz: From the Cramér–Rao inequality to the analogue of Fisher information in quantum probability

Th. D. Popescu: New method for change detection with application in structural systems

I. Rama Bhadra Sarma & B. Rami Reddy: Inversion of a matrix by bordering

G. R. Maruthi Sankar: A multiple selection index model for selection of genotypes for dryland conditions

B. Schaffrin: Establishing equivalent systems for universal kriging

J. Seberry: Short amicable sets of matrices with application to orthogonal designs

V. K. Sharma, A. V. Dattatreya Rao & A. Vasideva Rao: Software for reliability computation in life-testing models

Y. Takane: On the Lagrange–(Wedderburn–)Guttman theorem

N. J. Thome: Generalized inverses and a block-rank-equation

N. Trandafilov: On the simultaneous eigenvalue-like decomposition of several matrices with application to multivariate analysis

C. Varghese & A. R. Rao: Robustness of change-over designs

R. P. Venkataraman: Matrix ensembles with different characteristic eigenvalue densities

To submit a contributed paper for presentation at the Workshop, please use LaTeX and the style file available from the Web site. The abstract should not exceed 25 lines. Send the title and abstract, plus the names, e-mail addresses and affiliations of <u>all</u> authors by e-mail to Hans Joachim Werner at

werner@united.econ.uni-bonn.de

and a copy by classical p-mail or FAX to both Hans Joachim Werner, Institut für Ökonometrie & Operations Research, Adenauerallee 24-42, Rheinische Friedrich-Wilhelms-Universität Bonn, D-53113 Bonn, Germany; FAX (49-228) 73-9189, and to P. Bhimasankaram, ISI, Street No. 8, Habsiguda, Hyderabad–500 007, India; FAX (91-40) 717-3602.

The registration fees will be as follows:
Short Course: US$36/C$54/Euro 39.6 (Rs. 300/- Indian res.),
Workshop: US$72/C$108/Euro 79,2 (Rs. 600/- Indian res.).

There will be no registration fees for students for the Short Course. For students and retired persons the registration fee for the Workshop will be US$30/C$45/Euro 33 (Rs. 250/- Indian residents). There will be no registration fees for accompanying persons. The registration fees cover all conference materials and handouts, lunches, and tea and coffee.

Participants not resident in India may pay the registration fees in advance (a) by personal check in either US or Canadian dollars drawn on a US or Canadian bank and made payable to George P. H. Styan, and sent to George P. H. Styan, PO Box 270, Franklin, VT 05457-0270, USA, or (b) through Bank Transfer in Euro: Account number 146005939 with Sparkasse Bonn (BLZ 38050000), Friedensplatz 1-3, D-53111 Bonn, Germany. The Receiver of the Payment should be marked as Hans Joachim Werner and the Subject of the Payment should be marked as NIWMS-2000. All bank fees are at the sender's expense. Please send also a copy of the receipt of your bank payment by airmail to Hans Joachim Werner, Institute for Econometrics and Operations Research, Econometrics Unit, Univ. of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. We regret that we are not able to accept payment by credit card. Indian residents should pay the registration fees through Crossed Demand Draft on any nationalized bank, payable at Hyderabad, in favour of R. J. R. Swamy/NIWMS-2000.

All correspondence regarding lodging in Hyderabad for this Workshop should be directed to R. J. R. Swamy, Dept. of Statistics, Osmania University, Hyderabad-500 007, India swamy_rjr@yahoo.co.uk, FAX (91-40) 717-3602 or P. Bhimasankaram, ISI, Street No.8, Habsiguda, Hyderabad-500 007, India; pbhim@rediffmail.com, FAX (91-40) 717-3602. All correspondence concerning the scientific part of this Workshop should be directed to Hans Joachim Werner or to G. P. H. Styan.

International Linear Algebra Conference

Haifa, Israel: 25–29 June 2001

This conference will be held under the auspices of the Institute of Advanced Studies in Mathematics at the Technion and the International Linear Algebra Society (ILAS). This is the 9th conference organized by ILAS since 1989. It is also the 12th conference in a sequence of matrix theory meetings organized by the Technion since 1984.

Provisional list of speakers: D. Alpay, R. Bapat, A. Bruckstein, J. Bunch, A. Bunse-Gerstner, N. Cohen, H. Dym, K.-H. Foerster, P. Fuhrmann, I. Gohberg, C. R. Johnson, G. Kalai, S. Kirkland (OT/JT Speaker), W. Luxemburg, R. Merris, R. Meshulam, R. Nabben, I. Olkin, V. Olshevsky, S. Pierce, F. Puerta, J. Queiro, A. Ran, L. Rodman, P. Šemrl, B. Shader, J. Shao, Tin-Yau Tam, N. Trefethen (ILAS-LAA Speaker), F. Uhlig, W. Watkins, and H. J. Werner.

Organizing committee: Moshe Goldberg (co-chair), Daniel Hershkowitz (co-chair), Raphael Loewy (co-chair), Abraham Berman, Richard A. Brualdi, David London, Ludwig Elsner, Leonid Lerer, Uriel G. Rothblum and Abraham Zaks.

On Thursday, June 28, 2001, the conference banquet will be held. Estimated cost of the banquet is about US$45.00. On Wednesday, June 27, 2001, there will be a half-day excursion. Estimated cost of the excursion is about $30.00.

The deadline for registration is February 28, 2001. Please make every effort to mail/email your form in time to ensure that it reaches us before that date. The conference will commence on the morning of Monday, June 25, 2000, so participants should reach Haifa by Sunday evening. Lectures will be held from Monday through Friday. There will be a US$50 registration fee. Suggestions for minisymposia are welcome; the deadline for such suggestions is December 31, 2000. If you are interested in presenting a talk, please let us have a title and short abstract no later than February 28, 2001.

Hotel rooms will be available for the conference at the hotels listed below at the following rates:

- *Marom Hotel:*
  Double (2 persons sharing): US$72.00, Single: $55.00
- *Shulamit Hotel:*
  Double (2 persons sharing): $94.00, Single: $68.00
- *Nof Hotel:*
  Double (2 persons sharing): $116.00, Single: $92.00

Prices include breakfast and service charges. These rates also apply to periods directly before and after the conference. In order to benefit from the special conference rates, please contact us no later than February 28, 2001. However, as the number of rooms available in each hotel is limited, we suggest that you reserve your accommodation as soon as possible if you wish to ensure that you have a room at the hotel of your choice.

The weather in Haifa in June is sunny and warm, with temperatures at about 28° C/80° F. Humidity is moderate and there is very little chance of rain. Light clothing and informal dress are recommended.

Further information: Please visit our Web site:

http://www.math.technion.ac.il/institute/linear.htm

or contact Sylvia Schur, Dept. of Mathematics, Technion–Israel Insitutute of Technology, 32000 Haifa, Israel; tel. (972-4) 829-4278, FAX (972-4) 832-4654, iasm@techunix.technion.ac.il

Tenth International Workshop on Matrices and Statistics

Voorburg, The Netherlands: 2–3 August 2001

The Tenth International Workshop on Matrices and Statistics will be held under the auspices of the International Statistical Institute in or near Voorburg, The Netherlands, in the late summer of 2001. Our current target dates are 2–3 August 2001.

The International Organizing Committee for this Workshop comprises R. William Farebrother (Victoria Univ. of Manchester), Simo Puntanen (Univ. of Tampere; vice-chair), George P. H. Styan (McGill University; chair), and Hans Joachim Werner (Univ. of Bonn). The Local Arrangements Committee is chaired by Patrick J. F. Groenen (Univ. of Leiden).

The International Statistical Institute Permanent Office is located in Voorburg not far from an old house (beautifully restored) of Christiaan Huygens (1629–1695), the Dutch physicist, astronomer, and mathematician. According to the *World Book Encyclopedia*, Gottfried Wilhelm von Leibniz (1646–1716) is indebted to Huygens "from whom he learnt much of his mathematics".

International Workshop on Spectral Theory

Cochin (Kerala), India: 6–15 August 2001

It is proposed to organize an International Workshop from August 6–15, 2001 at the Cochin University of Science and Technology (CUSAT), Cochin (Kerala), India. The programme aims at bringing together mathematicians who work in the area of Spectral Theory. A variety of techniques from Linear Algebra, Numerical Functional Analysis and Wavelet Theory are used to address important questions in this area. The main idea of the Workshop is to get a collective picture of the current activities in this area. The programme will also discuss applications of spectral theory to other disciplines such as Digital Signal Processing and Theoretical Physics. There will be several Invited Lectures from eminent researchers. A partial list of expected speakers is as follows: M. Ahues, R. Alam, R. Bhatia, R. G. Douglas, S. Grudsky, R. P. Kulkarni, S. H. Kulkarni, B. V. Limaye, A. Largillier, J. Lemordant, A. M. Mathai, M. G. Nadkarni, M. T. Nair, Radha Ramakrishnan, E. Schock, S. Roch, V. S. Sunder, M. Uchiyama, K. R. Unni, Vittal Rao, and Vijay Kodiyalam.

Invited speakers will be provided with board and lodging at Cochin free of cost during the Workshop in addition to travel expenses within India. Due to shortage of funds, it will not be possible to provide international airfare. The programme will also encourage young researchers in the relevant areas to present their work before the experts.

The International Advisory Committee consists of the following members: K. Babu Joseph (Vice Chancellor, CUSAT–Chairman), R. G. Douglas, R. Bhatia, B. V. Limaye, A. M. Mathai, M. G. Nadkarni, E. Schock, K. R. Unni, The Organizing Committee consists of the following members. R. Alam, R. P. Kulkarni, S. H. Kulkarni, M. T. Nair, M. N. N. Namboodiri (Convener), and T. Thrivikraman (Co-Convener).

Challenges in Matrix Theory: 2001–2002

New *Challenges in Matrix Theory* may be submitted in every January of odd-numbered years. After review, the accepted new-challenge papers will appear in an early issue of *Linear Algebra and its Applictions* (LAA) in the following even year.

This is a *Call for Submission* of new-challenge papers in January 2001. Those new to the project should consult the "Call for Challenges" in LAA (1996), vol. 233, pp. 1–3. A new set of matrix challenges has just been published in LAA (2000), vol. 304, pp. 179–200. The first set appeared in LAA (1998), vol. 278, pp. 285–336. For more information contact Frank Uhlig (Challenges Coordinator): uhligfd@auburn.edu and/or visit http://www.auburn.edu/~uhligfd

Tenth ILAS Conference

Auburn, Alabama, USA: 10–13 June 2002

The Tenth ILAS Conference will be held at Auburn University from Monday, June 10 to Thursday, June 13, 2002. The conference will take place at the Auburn University Hotel and Conference Center, all under one roof. Auburn is a small university town in Alabama, USA, about 100 miles southwest of Atlanta, Georgia. Atlanta has the largest airport in the US, with many nonstop international flight connections.

Details for the conference, e.g., invited speakers, social events, travel, registration and hotel information will be made available in time on a dedicated website and on ILAS-NET.

The organizing committee has been formed. Its task is to select speakers who can bring "Challenges in Matrix Theory" (see above) to the participants. The conference will encompass all branches of Linear Algebra and Matrix Theory, i.e., core, applied, and numerical. For more information, please contact Frank Uhlig (Chair of the Organizing Committee): uhligfd@auburn.edu and/or visit http://www.auburn.edu/~uhligfd

Eleventh International Workshop on Matrices and Statistics

Lyngby, Denmark: Summer 2002

The Eleventh International Workshop on Matrices and Statistics will be held at the Technical University of Denmark in Lyngby, near Copenhagen, in the summer of 2002.

The International Organizing Committee for this Workshop comprises R. William Farebrother (Victoria Univ. of Manchester), Simo Puntanen (Univ. of Tampere), George P. H. Styan (McGill University), and Hans Joachim Werner (Univ. of Bonn). The Local Arrangements Committee is chaired by Knut Conradsen (Technical Univ. of Denmark).

Eigenvectors are Nonzero Vectors Scaled by a Linear Map

Eigenvectors and eigenvalues are among the most important concepts of linear algebra—and at the same time among the most mysterious for students. I have had graduate students in mathematics tell me that they didn't really know what an eigenvector or an eigenvalue was. One even talked about writing a Master's thesis to figure eigenvectors out!

Perhaps one reason for this is that these concepts are traditionally presented (for a square matrix A) as solutions to the algebraic (but nonlinear) system $Ax = \lambda x, x \neq 0$. This defines them, but it does not shed any light on their significance. Giving several applications demonstrates their utility—but not the reason for that utility. What I propose to do here is to discuss a nonstandard way of looking at these ideas, which I think does give a reason for their importance. I will also propose a renaming of these concepts, to emphasize their special nature.

Let L denote a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let's examine the effect of this map on a specific vector x. Since $L(0) = 0$ is always true, we will assume that $x \neq 0$. What is the simplest nontrivial way that the linear map L could act on the vector x? The answer is surely for the map to take the vector into some scalar multiple of itself, in other words, for $L(x) = \lambda x$ to hold for some scalar λ. (The traditional equation is not the focus of the discussion, but a summary of it.) Now in general the linear map L will not do this for all nonzero vectors x. Those vectors $x \neq 0$ for which $L(x) = \lambda x$ does hold are thus in some sense special ones. As we often do in mathematics, we will try to "build up" the action of the linear map on all vectors in $\mathbb{R}^n$ from its action on these special vectors.

How can we rename these concepts so that they embody these ideas? Let us first call the operation of scalar multiplication *scaling*, and then say that a nonzero vector x is *λ-scaled* by L, or, more simply, *scaled* by L if $L(x) = \lambda x$ for some scalar λ , that is, if the action of L on x is to scale x by λ. The value λ we will call a *scaling factor* of L. The vectors scaled by L are organized by the scaling factors λ. We will call the set of vectors λ-scaled by L, together with the zero vector, the *space λ-scaled* by L. We must in practice usually determine the scaling factors before determining the spaces scaled by the linear map.

An $n \times n$ real matrix A of course determines the linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ which takes each vector x into Ax, and the scaled vectors, scaling factors, etc., for A are the corresponding objects for this linear map. (And, if we can "build up" the action of this map on all vectors in $\mathbb{R}^n$ from its action on scaled vectors, we say that the matrix is diagonalizable.)

In $\mathbb{R}^2$, it is easy to show geometrically that orthogonal projection onto a line through the origin and that reflection across such a line are linear, and it is easy to determine the scaled vectors and scaling factors of these linear maps. It is easy to show geometrically that rotation by a fixed angle about the origin is a linear map and to show that this linear map has no scaled vectors and no scaling factors.

The concepts discussed here are geometric. Their definition and illustration do not depend on matrices and determinants, even though in general we employ these objects to actually compute scaling factors and scaled vectors.

David H. CARLSON: carlson@saturn.sdsu.edu
Dept. of Mathematical Sciences, San Diego State Univ.

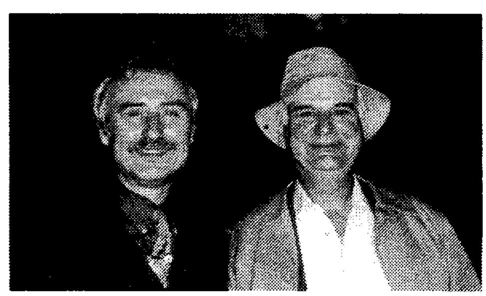
What is a Matrician?

Let us examine some English words derived from the Latin roots *pater* (father) and *mater* (mother). The first of these roots gives us *paternal*, *paternity*, *patriarch*, *patricide*, *patrimony*, *patrician*, and *patron*; whilst the second gives us *maternal*, *maternity*, *matriarch*, *matricide*, *matrimony*, and *matron*. As the word *matrician* has not yet entered the English language, we may either use it to denote a member of the ruling class of a matriarchy or to denote a person who studies matrices in their medical, geological, or mathematical senses.

In this context, I should also mention that the word *matriculate* (to enter in a register) is formed from *matricula* (a register), the Latin diminutive of *matrix* (a womb). Finally, I note that *Matricia* seems more appropriate as a female variant of *Patrick* than the more conventional *Patricia*, and that *fraternity* and *sorority* derive from the roots *frater* (brother) and *soror* (sister).

R. William FAREBROTHER: R.W.Farebrother@man.ac.uk
Dept. of Economic Studies, Victoria Univ. of Manchester

Hans Joachim Werner: IMAGE Editor-in-Chief



As announced in IMAGE 24 (April 2000, page 2), the appointment of George P. H. Styan as Editor-in-Chief of IMAGE has been extended to 31 May 2003. In addition, Hans Joachim "Jochen" Werner (Universität Bonn) has been appointed Editor-in-Chief of IMAGE for a six-year term beginning June 1, 2000 (and so ending on 31 May 2006). Thus since 1 June 2000, George and Jochen are joint Editors-in-Chief of IMAGE.

Many thanks go to Richard Werner, who took this photograph of Jochen (on the left) and George after dinner at the excellent Restaurant *Zum Fässchen* in Meckenheim (Rhein) near Bonn, Germany, on 3 June 2000.