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IMAGE Problem Corner: Problems 25-1 to 25-7 With Solutions*

We present solutions to IMAGE Problems 25-1 through 25-8 published in IMAGE 25 (October 2000), pp. 16–17; solutions to Problem 25-8 are on pp. 21–23 below. We are still hoping to receive solutions to Problems 19-3b and 23-1 [see IMAGE 25 (October 2000), pp. 3 & 6]. In addition, we introduce five new problems (pp. 24–25 below) and invite readers to submit solutions as well as new problems for publication in IMAGE. Please submit all material (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 25-1: Moore–Penrose Inverse of a Skew-Symmetric Matrix

Proposed by Jürgen GROß: gross@statistik.uni-dortmund.de; *Universität Dortmund, Dortmund, Germany*,
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Find the Moore–Penrose inverse A^+ of

$$A = \begin{pmatrix} 0 & a_3 & -a_2 & a_5 & -a_4 & a_7 & -a_6 \\ -a_3 & 0 & a_1 & a_6 & -a_7 & -a_4 & a_5 \\ a_2 & -a_1 & 0 & -a_7 & -a_6 & a_5 & a_4 \\ -a_5 & -a_6 & a_7 & 0 & a_1 & a_2 & -a_3 \\ a_4 & a_7 & a_6 & -a_1 & 0 & -a_3 & -a_2 \\ -a_7 & a_4 & -a_5 & -a_2 & a_3 & 0 & a_1 \\ a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & 0 \end{pmatrix}.$$

Solution 25-1.1 by Jerzy K. BAKSALARY: jkbaks@lord.wsp.zgora.pl; *Tadeusz Kotarbiński Pedagogical University, Zielona Góra*, and Oskar Maria BAKSALARY: baxx@main.amu.edu.pl; *Adam Mickiewicz University, Poznań, Poland.*

Let a denote the 7×1 vector whose successive components are a_i, $i = 1, \ldots, 7$. Since the transpose of A satisfies $A' = -A$, the well-known formula $A^+ = A'(A'A)^+$ takes the form

$$A^+ = A(A^2)^+.$$

The structure of A ensures that the entries of A^2 are

$$(A^2)_{ii} = -\alpha + a_i^2,\ i = 1, \ldots, 7 \quad \text{and} \quad (A^2)_{ij} = a_i a_j,\ i, j = 1, \ldots, 7;\ i \neq j,$$

where $\alpha = a'a = \sum_{i=1}^{7} a_i^2$. This means that

$$A^2 = -\alpha I_7 + aa' = -\alpha(I_7 - \alpha^+ aa').$$

The matrix $I_7 - \alpha^+ aa'$ is idempotent and Hermitian (i.e., an orthogonal projector), and therefore has the Moore–Penrose inverse equal to itself. Consequently, $A^+ = A(A^2)^+$ takes the form

$$A^+ = -\alpha^+ A(I_7 - \alpha^+ aa').$$

But the structure of A ensures also that $Aa = 0$, and thus the solution is

$$A^+ = -\alpha^+ A. \qquad \square$$

Problem 25-8 with solutions are on pp. 21–23 below; new problems on pp. 24–25.

Solution 25-1.2 by R. B. BAPAT: rbb@isid.ac.in; *Indian Statistical Institute–Delhi Centre, New Delhi, India.*

Clearly, if each a_i is zero then A^+ is the zero matrix. Below let $a_i \neq 0$ for some i. Then we show that

$$A^+ = -(aa^T)^{-1}A,$$

where $a = [a_1, a_2, a_3, a_4, a_5, a_6, a_7]$. It is easily verified that $Aa^T = 0$ and that $AA^T = (aa^T)I - a^T a$. Also, $AA^T = -A^2 = A^T A$ since A is skew-symmetric. Put $X := -(aa^T)^{-1}A$. Then it follows from these observations that $AXA = A$, $XAX = X$ and that AX (which equals XA) is symmetric. Hence the result is proved. □

Solution 25-1.3 by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

We consider the matrix A over the field of real numbers. Clearly $A = -A^T$. Let $b = (a_1 : a_2 : \cdots : a_7)^T$; then $Ab = 0$. Let

$$M = \begin{pmatrix} a_0 & -b^T \\ -b & A - a_0 I \end{pmatrix};$$

then $MM^T = sI$, where $s = a_0^2 + a_1^2 + \cdots + a_7^2$. If $s \neq 0$, then $M^{-1} = s^{-1}M^T$. When $a_0 = 0$ we write $M = M_0$ with

$$M_0 = \begin{pmatrix} 0 & -b^T \\ -b & A \end{pmatrix} \quad \text{and} \quad M_0^{-1} = \frac{1}{s}\begin{pmatrix} 0 & -b^T \\ -b & A^T \end{pmatrix};$$

here $s = a_1^2 + \cdots + a_7^2 \neq 0$. Further to the orthogonality of the rows and columns of M, we note that

$$M_0^{-1} = \begin{pmatrix} 0 & -b^+ \\ -(b^T)^+ & A^+ \end{pmatrix}$$

and so $A^+ = (1/s)A^T$. □

COMMENT 1. We may write $M = PNQ$, where $P = \mathrm{diag}(1, -1, \cdots, -1, 1)$, $Q = \mathrm{diag}(1, \cdots, 1, -1)$ and

$$N = \begin{pmatrix} +a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & +a_7 \\ +a_1 & +a_0 & -a_3 & +a_2 & -a_5 & +a_4 & -a_7 & -a_6 \\ +a_2 & +a_3 & +a_0 & -a_1 & -a_6 & +a_7 & +a_4 & +a_5 \\ +a_3 & -a_2 & +a_1 & +a_0 & +a_7 & +a_6 & -a_5 & +a_4 \\ +a_4 & +a_5 & +a_6 & -a_7 & +a_0 & -a_1 & -a_2 & -a_3 \\ +a_5 & -a_4 & -a_7 & -a_6 & +a_1 & +a_0 & +a_3 & -a_2 \\ +a_6 & +a_7 & -a_4 & +a_5 & +a_2 & -a_3 & +a_0 & +a_1 \\ -a_7 & +a_6 & -a_5 & -a_4 & +a_3 & +a_2 & -a_1 & +a_0 \end{pmatrix}.$$

The matrix N is a real matrix representation of an octonion (for more details on octonions see, e.g., [1, 3]). The upper-left 4×4 block of N is a real matrix representation of a quaternion. From the theory of quaternionic and octonionic algebras, we may derive many interesting properties of the matrix N. The corresponding results can be used in matrix analysis over octonionic algebras. Some introductory work was presented in [2].

COMMENT 2. If the matrix A is over the field of complex numbers, then A^+ is not necessarily equal to $(1/s)A^T$. The condition $A^T = -A$ does not help us find the Moore–Penrose inverse A^+ and, in fact, there seems to be no "nice" expression for A^+. If, however, we replace all a_i in the lower triangular part of A by their complex conjugates then $A^* = -A$, and $A^+ = (1/t)A^*$ with $t = |a_1|^2 + \cdots + |a_7|^2$.

References

[1] R. D. Schafer (1966). *An Introduction to Non-associative Algebras.* Academic Press, New York.

[2] Y. Tian (2000). Matrix representations of octonions and their applications. *Advances in Applied Clifford Algebras*, 10, 61–90.

[3] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov & A. I. Shirshov (1982). *Rings That Are Nearly Associative.* Translated from the Russian by Harry F. Smith. Academic Press, New York.

Solution 25-1.4 by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let $A = U^*DU$ be a spectral decomposition of the Hermitian (real symmetric) matrix A. It suffices to characterize the diagonal matrix D. We compute $A^2 = -\alpha I_7 + aa'$, where $a = (a_1, a_2, \ldots, a_7)$ and $\alpha = a'a$. By considering the eigenvalues of A^2, we have $D^2 = \operatorname{diag}(-\alpha, \ldots, -\alpha, 0)$ and $D^+ = -\alpha^+ D$. It follows that $A^+ = -\alpha A$. $\square$

A solution was also received from the Proposers Jürgen GROß, Sven-Oliver TROSCHKE *and* Götz TRENKLER, *who observed that:*

For a fixed vector $a = (a_1, \ldots, a_7)^\top$ and arbitrary $x \in \mathbb{R}^7$ the linear mapping $a \times x = Ax$ defines the vector cross product in $\mathbb{R}^7$.

Problem 25-2: Vandermonde Matrices with Condition Number 1

Proposed by Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA,*
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and Seok-Zun SONG: szsong@cheju.cheju.ac.kr; *Cheju National University, Cheju, Korea.*

An $n \times n$ complex matrix $A = (a_{pq})$ is a Vandermonde matrix if there exist complex numbers $z_1, \cdots, z_n$ such that $a_{pq} = z_q^{p-1}$. Characterize those Vandermonde matrices with condition number equal to one, i. e., Vandermonde matrices which are multiples of unitary matrices.

Solution 25-2.1 by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

The length of the first row of A is $\sqrt{n}$, so this is also the length of the qth column. So $\sum_{p=1}^{n} |z_q|^{2(p-1)} = n$ and it follows that $|z_q| = 1$ for $q = 1, 2, \ldots, n$. We now use the fact that qth and rth columns are orthogonal for $q \neq r$. This gives $\sum_{p=1}^{n} (\overline{z_q} z_r)^{p-1} = 0$, and it follows that $\overline{z_q} z_r = z_q^{-1} z_r$ is an nth root of unity and that $z_q \neq z_r$. It follows that $z_q = z_1 \omega^{k_q}$, where ω is a primitive nth root of unity and where $k_1, k_2, \ldots, k_n$ are distinct elements of $\{0, 1, 2, \ldots, n-1\}$. So, after rearranging the columns, A is the matrix of the DFT on $\mathbb{Z}(n)$.

Another approach to this problem uses the orthogonality of the first row and the remaining rows and Newton's identities to show that $e_k(z_1, \ldots, z_n) = 0$ for $k = 1, \ldots, n-1$. One may deduce that after rearranging the z_q, we have $z_q = z\omega^q$ for some fixed constant z. One can then show that $|z| = 1$ as in the first step of the proof given above. We leave the details to the reader.

A solution was also received from the Proposers Chi-Kwong LI, Roy MATHIAS and Seok-Zun SONG.

Problem 25-3: Linear Functions on Similarity Orbits of Complex Square Matrices

Proposed by Chi-Kwong LI: ckli@math.wm.edu; *The College of William and Mary, Williamsburg, Virginia, USA,*
and Tin-Yau TAM: tamtiny@mail.auburn.edu; *Auburn University, Auburn, Alabama, USA.*

Let $A \in M_n$, the set of $n \times n$ complex matrices, and let $S(A)$ be the similarity orbit of A, i.e., $S(A)$ is the set of matrices in M_n similar to A. Show that $f(S(A))$ is convex for any linear functional f on M_n. What about $g(S(A))$ for some other linear function $g : M_n \to \mathbb{C}^m$ with $m > 1$, is $g(S(A))$ still convex?

Solution 25-3.1 by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

The following lemma shows that $f(S(A))$ is either a singleton or $\mathbb{C}$.

LEMMA. *Let A and B be $n \times n$ complex matrices and suppose that* $\{\operatorname{tr}(BP^{-1}AP); P\ n \times n,$ *invertible*$\} \neq \mathbb{C}$. *Then either A or B is a scalar multiple of the identity.*

PROOF. Let ξ, η be orthogonal vectors. Then $\eta^\star \xi = 0$. Let $N = \xi \otimes \eta^\star$ so that $N^2 = 0$. Now define for $z \in \mathbb{C}$, $P = I + zN$. Then $P^{-1} = I - zN$. We find

$$\operatorname{tr}(BP^{-1}AP) = \operatorname{tr}(BA) + z\eta^\star[B, A]\xi - z^2(\eta^\star A\xi)(\eta^\star B\xi)$$

Now a complex quadratic polynomial maps onto $\mathbb{C}$ unless it is constant, so we deduce that $\eta^\star[B, A]\xi = 0$ and $(\eta^\star A\xi)(\eta^\star B\xi) = 0$

for all orthogonal ξ and η. Here $[B, A]$ means the commutator $BA - AB$. We make no use of the first of these identities. Now consider the closed subset $X = \{(\xi,\eta); \xi,\eta \in \mathbb{C}^n,\ \eta^\star\xi = 0\}$. Then X is a closed subset of $\mathbb{C}^n \times \mathbb{C}^n$ and therefore is complete as a metric space. We have

$$X = \{(\xi,\eta) \in X; \eta^\star A\xi = 0\} \cup \{(\xi,\eta) \in X; \eta^\star B\xi = 0\}.$$

By the Baire Category Theorem one or other of the subsets on the right has non-empty interior in X. Let us suppose without loss of generality that it is the first of these. Then, extending by linearity (first with ξ fixed and η varying in $\xi^\perp$ and then again with η fixed and ξ varying in $\eta^\perp$), we find that

$$\xi,\eta \in \mathbb{C}^n,\ \eta^\star\xi = 0 \qquad \Longrightarrow \qquad \eta^\star A\xi = 0.$$

It now follows first that $A\xi$ is a scalar multiple of ξ for every vector $\xi \in \mathbb{C}^n$ and then that $A = \lambda I$ for a suitable $\lambda \in \mathbb{C}$.

On the other hand, $g(S(A))$ need not be convex in case $m = 2,\ n = 2$. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A, -A \in S(A)$. So, if $g(S(A))$ is convex, then $0 \in g(S(A))$. Let g be the linear functional that picks out the off-diagonal elements. Then $S(A)$ must contain a diagonal matrix, which is impossible. □

Solution 25-3.2 by the Proposers Chi-Kwong LI and Tin-Yau TAM.

Note that for every linear functional on M_n there is a complex matrix C such that the linear functional can be represented as $X \mapsto \mathrm{tr}CX$. Thus,

$$f(S(A)) = \{\mathrm{tr}(CS^{-1}AS) : S \in M_n \text{ is invertible }\}.$$

We claim that either

(i) C or A is a scalar matrix, and $f(S(A))$ is a singleton, or

(ii) neither C nor A is a scalar matrix and $f(S(A)) = \mathbb{C}$.

PROOF. If C or A is a scalar matrix, then clearly $f(S(A))$ is a singleton. If it is not the case, then C has a Jordan block of size larger than one, or C is diagonalizable with at least two distinct eigenvalues. In either case, C is similar to a matrix C_1 with diagonal entries $c_1, \ldots, c_n$, $(1,2)$ entry equal to 1, and (i,j) entry equal to 0 if $j - i \notin \{0,1\}$. Similarly, A is similar to a matrix A_1 with diagonal entries $a_1, \ldots, a_n$, $(1,2)$ entry equal to 1, and (i,j) entry equal to 0 if $j - i \notin \{0,1\}$. Then $\mathrm{tr}(C_1A_1) = \sum_{j=1}^n a_jc_j \in f(S(A))$. Furthermore, for any $z \in \mathbb{C}$, there exists a 2×2 invertible matrix R such that

$$R\begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix}R^{-1} = \begin{pmatrix} a_1 & 0 \\ z & a_2 \end{pmatrix}.$$

So, $\mathrm{tr}(C_1(R \oplus I_{n-2})A_1(R^{-1} \oplus I_{n-2})) = z + \sum_{j=1}^n a_jc_j \in f(S(A))$. Thus, $f(S(A)) = \mathbb{C}$, i.e., (ii) holds. The proof of our claim is complete.

In general, suppose $g : M_n \to \mathbb{C}^k$ is linear. Then $g(S(A))$ may not be convex even if $k = 2$ as shown by the example in Solution 25-3.1 by S. W. Drury. Nevertheless, there are examples that $g(S(A))$ is convex for $k = n^2 - n + 1$. For instance, suppose $A \in M_n$ has distinct eigenvalues $a_1, \ldots, a_n$, and $g(X) = (\mathrm{tr}X, X_0)$, where X_0 is obtained from X by setting all the diagonal entries of X to 0. By a result of Friedland (1972), for any $Y_0 \in M_n$ with zero diagonal, one can find a diagonal matrix D such that $D + Y_0$ has eigenvalues $a_1, \ldots, a_n$. Thus, $D + Y_0 \in S(A)$ and $g(D + Y_0) = (\mathrm{tr}A, Y_0)$. Hence,

$$g(S(A)) = \{(\mathrm{tr}A, Y_0) : Y_0 \in M_n \text{ has zero diagonal }\}$$

and is clearly convex. In general, it would be interesting to determine the conditions on A and g so that $g(S(A))$ is convex.

Reference

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Problem 25-4: Two Rank Equalities Associated with Blocks of an Orthogonal Projector

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}; \quad A_{11} \in \mathbb{C}^{m\times n},\ A_{12} \in \mathbb{C}^{m\times k},\ A_{21} \in \mathbb{C}^{l\times n},\ A_{22} \in \mathbb{C}^{l\times k}.$$

(a) Show that the rank of the upper-right $m \times l$ block of $P_A = AA^+$ is

$$\mathrm{rank}\big((P_A)_{12}\big) = \mathrm{rank}(A_{11} : A_{12}) + \mathrm{rank}(A_{21} : A_{22}) - \mathrm{rank}(A).$$

Here $(E : F)$ denotes the partitioned block matrix with E placed next to F.

(b) Now suppose that A is Hermitian and idempotent, i.e., an orthogonal projector. Show that

$$\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{12}).$$

Solution 25-4.1 by Jerzy K. BAKSALARY: jkbaks@lord.wsp.zgora.pl; *Tadeusz Kotarbiński Pedagogical University, Zielona Góra,* and Oskar Maria BAKSALARY: baxx@main.amu.edu.pl; *Adam Mickiewicz University, Poznań, Poland.*

We solve the parts (a) and (b) of the problem in the reverse order. First observe that without the additional assumption that $n = m$, $k = l$ the result in (b) is invalid. This is seen from the example in which $m = 2$, $n = 1$, $k = 2$, $l = 1$, and

$$A_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A_{21} = (\,0\,) \quad \text{and} \quad A_{22} = (\,\tfrac{1}{2} \;\; \tfrac{1}{2}\,).$$

Then $\mathrm{rank}(A) = 2$, whereas $\mathrm{rank}(A_{11}) + \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{12}) = 1$. Since every orthogonal projector is a nonnegative definite matrix, it follows from Theorem 1 in [AL] that if $n = m$ and $k = l$, then the blocks A_{ij} $(i, j = 1, 2)$ satisfy

$$A_{11}A_{11}^{+}A_{12} = A_{12}, \quad A_{12}A_{22}^{+}A_{22} = A_{12}, \quad A_{21}A_{11}^{+}A_{11} = A_{21} \quad \text{and} \quad A_{22}A_{22}^{+}A_{21} = A_{21}. \tag{1}$$

Corollary 19.1 in [MS] assures that the conditions (1) are sufficient for the rank of A to be additive on the Schur complement, i.e.,

$$\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \mathrm{rank}(A_{22} - A_{21}A_{11}^{+}A_{12}). \tag{2}$$

An immediate consequence of (1) is that

$$A_{22}A_{22}^{+}(A_{21}A_{11}^{+}A_{12}) = A_{21}A_{11}^{+}A_{12} \quad \text{and} \quad (A_{21}A_{11}^{+}A_{12})A_{22}^{+}A_{22} = A_{21}A_{11}^{+}A_{12}. \tag{3}$$

On the other hand, from the idempotency of A it follows that $A_{21} = A_{21}A_{11} + A_{22}A_{21}$, $A_{12}A_{21} = (I - A_{11})A_{11}$, and $A_{21}A_{12} = A_{22}(I - A_{22})$. Hence, in view of (1),

$$\begin{aligned}
(A_{21}A_{11}^{+}A_{12})A_{22}^{+}(A_{21}A_{11}^{+}A_{12}) &= A_{21}A_{11}^{+}A_{12}A_{22}^{+}(A_{21}A_{11} + A_{22}A_{21})A_{11}^{+}A_{12} \\
&= A_{21}A_{11}^{+}(A_{12}A_{22}^{+}A_{21}A_{12} + A_{12}A_{21}A_{11}^{+}A_{12}) \\
&= A_{21}A_{11}^{+}[A_{12}(I - A_{22}) + (I - A_{11})A_{12}] = A_{21}A_{11}^{+}(2A_{12} - A_{21}^{*}) = A_{21}A_{11}^{+}A_{12}.
\end{aligned} \tag{4}$$

According to Theorem 17 in [MS], the conditions (3) and (4) are necessary and sufficient for rank subtractivity of A_{22} and $A_{21}A_{11}^{+}A_{12}$, i.e.,

$$\mathrm{rank}(A_{22} - A_{21}A_{11}^{+}A_{12}) = \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{21}A_{11}^{+}A_{12}). \tag{5}$$

Moreover, since A_{11} is a nonnegative definite matrix, it follows that

$$\mathrm{rank}(A_{21}A_{11}^{+}A_{12}) = \mathrm{rank}(A_{12}^{*}A_{11}^{+}A_{11}A_{11}^{+}A_{12}) = \mathrm{rank}(A_{11}A_{11}^{+}A_{12}) = \mathrm{rank}(A_{12}). \tag{6}$$

Substituting (5) and (6) into (2) yields the required equality

$$\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{12}). \tag{7}$$

The result in part (a) can be obtained as a corollary to (7). Denoting $(A_{11} : A_{12})$ and $(A_{21} : A_{22})$ by A_1 and A_2, respectively, and assuming that $A^+ = (G_1 : G_2)$, the equality (7) applied to

$$P_A = AA^+ = \begin{pmatrix} A_1G_1 & A_1G_2 \\ A_2G_1 & A_2G_2 \end{pmatrix}$$

takes the form

$$\text{rank}(AA^+) = \text{rank}(A_1G_1) + \text{rank}(A_2G_2) - \text{rank}(A_1G_2). \tag{8}$$

It is clear that $\text{rank}(AA^+) = \text{rank}(A)$. Moreover, since the nonnegative definiteness of AA^+ implies the range inclusions $\mathcal{R}(A_1G_2) \subseteq \mathcal{R}(A_1G_1)$ and $\mathcal{R}(A_2G_1) \subseteq \mathcal{R}(A_2G_2)$, it follows that

$$\text{rank}(A_i) \geq \text{rank}(A_iG_i) = \text{rank}(A_i(G_1 : G_2)) \geq \text{rank}(A_iA^+A) = \text{rank}(A_i), \quad i = 1, 2.$$

Consequently, the equality (8) can be expressed in the form $\text{rank}(A) = \text{rank}(A_1) + \text{rank}(A_2) - \text{rank}((P_A)_{12})$, as required. □

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[AL] A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics*, 17, 434–440.

[MS] G. Marsaglia & G. P. H. Styan (1974). Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra*, 2, 269–292.

Solution 25-4.2 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *University of Bonn, Bonn, Germany.*

First, we give an elementary proof of (a). Then we show that the (claimed) statement (b) is correct, irrespective of the orthogonal projector A, if and only if $n = m$ and $k = l$.

PROOF OF (a): Let A be an arbitrary but fixed matrix of rank r, and let $U \in \mathbb{C}^{(m+l)\times r}$ be a matrix such that $U^*U = I_r$ and $\mathcal{R}(U) = \mathcal{R}(A)$. In other words, let the column unitary matrix U be such that its columns form a basis for the range (column space) $\mathcal{R}(A)$ of A. Then there uniquely exists a matrix $V^* \in \mathbb{C}^{r\times(n+k)}$ such that

$$A = UV^* = \begin{pmatrix} U_1V_1^* & U_1V_2^* \\ U_2V_1^* & U_2V_2^* \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{and} \quad V^* = (\,V_1^* \quad V_2^*\,)$$

are partitioned in accordance with A. So, in particular, $U_1 \in \mathbb{C}^{m\times r}$ and $U_2 \in \mathbb{C}^{l\times r}$. Since $U^*U = U_1^*U_1 + U_2^*U_2 = I_r$, it is easy to see that

$$UU^* = \begin{pmatrix} U_1U_1^* & U_1U_2^* \\ U_2U_1^* & U_2U_2^* \end{pmatrix} = P_A.$$

Clearly $(P_A)_{12} = U_1U_2^* \in \mathbb{C}^{m\times l}$. Consequently,

$$\text{rank}((P_A)_{12}) = \dim\, \mathcal{R}((P_A)_{12}) = \dim\, \mathcal{R}(U_2^*) - \dim\, \mathcal{R}(U_2^*) \cap \mathcal{N}(U_1),$$

where $\mathcal{N}(\cdot)$ denotes the null space of $(\cdot)$. Since

$$\text{rank}(U_2) = \text{rank}(U_2^*) = \dim\, \mathcal{R}(U_2^*), \text{rank}(A_{11} \; : \; A_{12}) = \text{rank}(U_1V_1^* \; : \; U_1V_2^*) = \text{rank}(U_1V^*) = \text{rank}(U_1) = \dim\, \mathcal{R}(U_1)$$

and

$$\text{rank}(A_{21} \; : \; A_{22}) = \text{rank}(U_2V_1^* \; : U_2V_2^*) = \text{rank}(U_2V^*) = \text{rank}(U_2) = \dim\, \mathcal{R}(U_2)$$

it is now clear that the claimed rank equation under (a) is correct if and only if

$$r = \dim \mathcal{R}(U_1) + \dim\, \mathcal{R}(U_2^*) \cap \mathcal{N}(U_1).$$

In view of $I = U_1^*U_1 + U_2^*U_2$ we get $U_2^*U_2x = x$ for all $x \in \mathcal{N}(U_1)$, which implies that $\mathcal{N}(U_1) \subseteq \mathcal{R}(U_2^*)$ and hence, $\mathcal{R}(U_2^*) \cap \mathcal{N}(U_1) = \mathcal{N}(U_1)$. Since by the well-known *Dimension Theorem* $\dim \mathcal{R}(U_1) + \dim\, \mathcal{N}(U_1) = r$, our proof of (a) is complete. □

DISCUSSION OF (b): Now suppose that A is Hermitian and idempotent, i. e., an orthogonal projector. Then $A = P_A$. When $n = m$ and $l = k$, $V^* = U^*$,

$$\text{rank}(A_{11} \; : \; A_{12}) = \text{rank}(U_1) = \text{rank}(U_1U_1^*) = \text{rank}(A_{11})$$

and

$$\text{rank}(A_{21} \; : \; A_{22}) = \text{rank}(U_2U^*) = \text{rank}(U_2) = \text{rank}(U_2U_2^*) = \text{rank}(A_{22})$$

so that the desired result is a direct consequence of (a). That the result, however, does not necessarily hold when $n \neq m$ or $l \neq k$ can be seen by means of the 3×3 matrix $A := I_3$. Let A_{11} be the upper left 2×1 subblock of this identity matrix, and let the matrices A_{21}, A_{12} and A_{22} be chosen accordingly. Then $\text{rank}(A) = 3$, whereas $\text{rank}(A_{11}) + \text{rank}(A_{22}) - \text{rank}(A_{12}) = 1$. □

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The rank equality (a) with A_{11} and A_{22} possibly rectangular, and the rank equality (b) with both A_{11} and A_{22} square, follow at once from the following theorem.

THEOREM. *Let $A \in \mathbb{C}^{h\times h}$, $X \in \mathbb{C}^{h\times m}$ and $Y \in \mathbb{C}^{h\times l}$ be such that $(X : Y)$ has full row rank h and $X^*Y = 0$. Then*

$$\operatorname{rank}(X^* P_A Y) = \operatorname{rank}(X^* A) + \operatorname{rank}(Y^* A) - \operatorname{rank}(A). \tag{9}$$

To prove (a), we let $h = m + l$ and put $X^* = (I_m : 0)$ and $Y^* = (0 : I_l)$. When A_{11} and A_{22} are square then the result (b) follows at once from (a) since when A is Hermitian and idempotent then $P_A = A$, $\operatorname{rank}(A_{11} : A_{12}) = \operatorname{rank}(A_{11})$ and $\operatorname{rank}(A_{21} : A_{22}) = \operatorname{rank}(A_{22})$. These last two equalities hold for all nonnegative definite A, not necessarily Hermitian idempotent, and so then $\operatorname{rank}((P_A)_{12}) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A)$. When A is nonnegative definite then A is "block rank-additive", i.e., $\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22})$, if and only if $((P_A)_{12}) = 0$; for other characterizations of block rank-additivity see Drury *et al.* (2000, Th. 3, p. 16).

When A_{11} and A_{22} are not square and A is nonnegative definite, not necessarily idempotent, then possibly $\operatorname{rank}(A_{11} : A_{12}) > \operatorname{rank}(A_{11})$ and so $\operatorname{rank}((P_A)_{12}) > \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A)$. For example, let $A = \operatorname{diag}(1, 2, 3)$ and $A_{11} = (1, 0)'$, the upper left 2×1 submatrix of A. Then $P_A = I_3$, $\operatorname{rank}(A_{11}) = 1 = \operatorname{rank}(A_{22})$ and $\operatorname{rank}(A) = 3$; thus $\operatorname{rank}((P_A)_{12}) = 0 > 1 + 1 - 3 = -1$. See also the discussion of this point in Solutions 25-4.1 and 25-4.2 above.

The rank equality (9) was proved by Drury *et al.* (2000, p. 13) for A nonnegative definite (and thus square); but their proof also works for A not necessarily nonnegative definite, indeed not even square! We use the following lemma due to Puntanen (1985, p. 12; 1987, Th. 3.4.1, p. 34); see also Baksalary & Styan (1993), Drury *et al.* (2000, Lemma 2, p. 13) and Tian & Styan (2000):

LEMMA. *Let the complex matrices F and G be $h \times m$ and $h \times l$, respectively, and let $P_F = FF^+$ and $P_G = GG^+$ denote the corresponding orthogonal projectors, with $Q_F = I_h - P_F$ and $Q_G = I_h - P_G$. Then*

$$\begin{aligned}
\operatorname{rank}(P_F P_G Q_F) &= \operatorname{rank}(P_F P_G) + \operatorname{rank}(P_F : P_G) - \operatorname{rank}(F) - \operatorname{rank}(G) && (10)\\
&= \operatorname{rank}(P_F P_G) + \operatorname{rank}(Q_F P_G) - \operatorname{rank}(G) && (11)\\
&= \operatorname{rank}(P_G P_F) + \operatorname{rank}(Q_G P_F) - \operatorname{rank}(F) = \operatorname{rank}(P_G P_F Q_G). && (12)
\end{aligned}$$

If we put $F = X$ and $G = A$ then the left-hand side of (10) becomes the left-hand side of (9), i.e., $\operatorname{rank}(X^* P_A Y)$, since with $(X : Y)$ of full row rank we have $Q_X = P_Y$, and (11) becomes the right-hand side of (9). From (10) and (12) we see that $\operatorname{rank}(P_F P_G Q_F) = \operatorname{rank}(P_G P_F Q_G)$ and so we can interchange F and G.

PROOF. We will make repeated use of the formula

$$\operatorname{rank}(K : L) = \operatorname{rank}(K) + \operatorname{rank}(Q_K L) = \operatorname{rank}(L : K), \tag{13}$$

see Marsaglia & Styan (1974, Th. 5, p. 274). We have

$$\begin{aligned}
\operatorname{rank}(P_F P_G Q_F) &= \operatorname{rank}((P_F P_G Q_F)^*) = \operatorname{rank}(Q_F P_G P_F) = \operatorname{rank}(P_F : P_G P_F) - \operatorname{rank}(F) && (14)\\
&= \operatorname{rank}(Q_G P_F : P_G P_F) - \operatorname{rank}(F) && (15)\\
&= \operatorname{rank}(Q_G P_F) + \operatorname{rank}(P_G P_F) - \operatorname{rank}(F) && (16)\\
&= \operatorname{rank}(P_F : P_G) - \operatorname{rank}(G) + \operatorname{rank}(P_G P_F) - \operatorname{rank}(F), && (17)
\end{aligned}$$

which proves (10). To go from (14) to (15), we use the equalities

$$\operatorname{rank}(P_F : P_G P_F) = \operatorname{rank}(P_F - P_G P_F : P_G P_F) = \operatorname{rank}(Q_G P_F : P_G P_F),$$

while to go from (15) to (16), we use the virtual disjointness of the column spaces of $Q_G P_F$ and $P_G P_F$. Otherwise we have used only (13), which we may also use to deduce (11) from the right-hand side of (10); since the right-hand side of (10) is "symmetric" in F and G, both the equalities in (12) follow at once, using $\operatorname{rank}(P_F P_G) = \operatorname{rank}((P_F P_G)^*) = \operatorname{rank}(P_G P_F)$. □

References

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Problem 25-5: Three Inequalities Involving Moore–Penrose Inverses

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Do the three inequalities

$$(A+B)^+ \leq_{\mathrm{L}} A^+ + B^+, \qquad A(A+B)^+B \leq_{\mathrm{L}} A^+ + B^+, \qquad (A:B)(A:B)^+ \leq_{\mathrm{L}} AA^+ + BB^+$$

hold for all nonnegative definite matrices A and B? Have these inequalities been considered before? Here $(A:B)$ denotes the partitioned block matrix with A placed next to B and $\leq_{\mathrm{L}}$ denotes the Löwner ordering.

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None of the three proposed inequalities holds for all nonnegative definite matrices A and B. Matrices constituting counter-examples for the first and third inequalities are

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

for which

$$A^+ + B^+ - (A+B)^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{5}{4} & -\frac{7}{4} \end{pmatrix}$$

and

$$AA^+ + BB^+ - (A:B)(A:B)^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

are not nonnegative definite. In these two cases there is no possibility to construct counter-examples for nonzero matrices of order one. This is possible, however, in case of the second proposed inequality, even under the additional condition that both matrices involved are equal. If $AB = (\alpha)$, with a real α satisfying $\alpha > 2$, then

$$A^+ + B^+ - A(A+B)^+B = \left(\tfrac{1}{\alpha}\right) + \left(\tfrac{1}{\alpha}\right) - \left(\tfrac{\alpha^2}{2\alpha}\right) = \left(\tfrac{4-\alpha^2}{2\alpha}\right),$$

which is not nonnegative.

In the second part of our solution we establish a theorem related to the third inequality. Since for any complex matrix K, the product $P_K = KK^+$ represents the orthogonal projector on the column space $\mathcal{C}(K)$, the proposed inequality can be re-expressed in the form

$$P_{(A:B)} \leq_{\mathrm{L}} P_A + P_B \tag{18}$$

irrespective of whether or not the matrices A and B are nonnegative definite. A necessary and sufficient condition for validity of (18) appears to be the commutativity of P_A and P_B.

THEOREM. *For any $m \times n$ and $m \times p$ complex matrices A and B, the orthogonal projector $P_{(A:B)}$ is a predecessor of the sum of orthogonal projectors $P_A + P_B$ in the sense of Löwner ordering if and only if $P_A P_B = P_B P_A$.*

PROOF. It is known (and easy to verify) that the column space (range)

$$\mathcal{C}(A : B) = \mathcal{C}(A) \overset{\perp}{\oplus} \mathcal{C}((I - P_A)B), \tag{19}$$

where I denotes the identity matrix of order m and the symbol $\overset{\perp}{\oplus}$ indicates that the subspaces constituting a direct sum are orthogonal. Consequently, $P_{(A:B)} = P_A + P_{(I-P_A)B}$ and (19) can be reexpressed as

$$P_{(I-P_A)B} \leq_{\mathrm{L}} P_B. \tag{20}$$

It is well known that the Löwner ordering between orthogonal projectors is equivalent to the inclusion of the subspaces on which they project, and therefore (20) holds if and only if $\mathcal{C}((I - P_A)B) \subseteq \mathcal{C}(B)$. But this means that

$$P_B(I - P_A)P_B = (I - P_A)P_B, \tag{21}$$

and since (21) is clearly equivalent to $P_A P_B = P_B P_A$, the proof is complete. □

References

[BA] J. K. Baksalary (1987). Algebraic characterizations and statistical implications of the commutativity of orthogonal projectors. In *Proceedings of the Second International Tampere Conference on Statistics* (T. Pukkila & S. Puntanen, eds.), Dept. of Mathematical Sciences, University of Tampere, Tampere, pp. 113–142. [See Theorem 1.]

[HS] R. E. Hartwig & G. P. H. Styan (1987). Partially ordered idempotent matrices. In *Proceedings of the Second International Tampere Conference on Statistics* (T. Pukkila & S. Puntanen, eds.), Dept. of Mathematical Sciences, University of Tampere, Tampere, Finland, pp. 361–383. [See Theorem 5.8.]

Problem 25-6: Generalized Inverse of a Matrix Product

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Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given. It is well known that the reverse order law $(AB)^- = B^- A^-$ does not hold in general for an inner inverse of the matrix product AB. Show, however, that for any A^- and B^-, the product $B^-(A^\dagger ABB^\dagger)^\dagger A^-$ is an inner inverse of AB, that is, the set inclusion $\{\, B^-(A^\dagger ABB^\dagger)^\dagger A^- \,\} \subseteq \{\, (AB)^- \,\}$ holds, where $(\cdot)^\dagger$ denotes the Moore–Penrose inverse.

Solution 25-6.1 by Jerzy K. BAKSALARY: jkbaks@lord.wsp.zgora.pl; *Tadeusz Kotarbiński Pedagogical University, Zielona Góra,* and Oskar Maria BAKSALARY: baxx@main.amu.edu.pl; *Adam Mickiewicz University, Poznań, Poland.*

Let P_1 and P_2 be orthogonal projectors, i.e., $P_i^2 = P_i = P_i^*$, $i = 1, 2$. Then

$$(P_1 P_2)^+ = P_2(P_1 P_2)^+ P_1, \tag{22}$$

which follows straightforwardly from the definition of the Moore–Penrose inverse by verifying that

$$P_1 P_2^2 (P_1 P_2)^+ P_1^2 P_2 P_1 P_2 (P_1 P_2)^+ P_1 P_2 = P_1 P_2,$$

$$P_2 (P_1 P_2)^+ P_1^2 P_2^2 (P_1 P_2)^+ P_1 = P_2 (P_1 P_2)^+ P_1 P_2 (P_1 P_2)^+ P_1 = P_2 (P_1 P_2)^+ P_1$$

and noting that the products

$$P_1 P_2^2 (P_1 P_2)^+ P_1 = (P_1 P_2 (P_1 P_2)^+)^* P_1 = (P_1^2 P_2 (P_1 P_2)^+)^* = P_1 P_2 (P_1 P_2)^+$$

$$\text{and} \qquad P_2 (P_1 P_2)^+ P_1^2 P_2 = P_2 ((P_1 P_2)^+ P_1 P_2)^* = ((P_1 P_2)^+ P_1 P_2^2)^* = (P_1 P_2)^+ P_1 P_2$$

are both Hermitian.

Since $A^+ A$ and BB^+ are the orthogonal projectors on the column spaces $\mathcal{C}(A^*)$ and $\mathcal{C}(B)$, respectively, we adopt the notation $A^+ A = P_{A^*}$ and $BB^+ = P_B$. Then on account of (22), it follows that

$$
\begin{aligned}
ABB^-(P_{A^*}P_B)^+A^-AB &= ABB^-P_B(P_{A^*}P_B)^+P_{A^*}A^-AB = AP_B(P_{A^*}P_B)^+P_{A^*}B \\
&= AP_{A^*}P_B(P_{A^*}P_B)^+P_{A^*}P_BB = AP_{A^*}P_BB = AB,
\end{aligned}
$$

and so $B^-(P_{A^*}P_B)^+A^-$ is a generalized inverse of AB irrespective of the choice of generalized inverses A^- and B^-. □

Solution 25-6.2 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *University of Bonn, Bonn, Germany.*

It is well known that for the Moore–Penrose $M^\dagger$ of a complex matrix M we have

$$\mathcal{R}(M^\dagger) = \mathcal{R}(M^*), \quad \mathcal{N}(M^\dagger) = \mathcal{N}(M^*), \quad P_M = MM^\dagger, \quad \text{and} \quad P_{M^*} = M^\dagger M,$$

where $\mathcal{R}(M)$, $\mathcal{N}(M)$, M^*, P_M, and P_{M^*} denote the range (column space) of M, the null space of M, the conjugate transpose of M, the orthogonal projector onto $\mathcal{R}(M)$ (along $\mathcal{N}(M^*)$), and the orthogonal projector onto $\mathcal{R}(M^*)$ (along $\mathcal{N}(M)$), respectively. Therefore $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger A$ and hence,

$$A^\dagger ABB^-(A^\dagger ABB^\dagger)^\dagger A^-ABB^\dagger = A^\dagger ABB^-BB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger AA^-ABB^\dagger = A^\dagger ABB^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger ABB^\dagger = A^\dagger ABB^\dagger,$$

which in turn directly implies that $ABB^-(A^\dagger ABB^\dagger)^\dagger A^-AB = AB$. This completes our proof. □

A solution to this problem was also received from the Proposer Yongge TIAN.

Problem 25-7: Hadamard Product of Square Roots of Correlation Matrices

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Show that for any correlation matrices A and B of the same size (nonnnegative definite matrices with diagonal entries 1),

$$A^{1/2} \circ B^{1/2} \leq_{\mathsf{L}} I,$$

where $\circ$ stands for the Hadamard product and $\leq_{\mathsf{L}}$ for the Löwner ordering.

Solution 25-7.1 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *University of Bonn, Bonn, Germany.*

Since the square root of a nonnegative definite matrix is again nonnegative definite and the cone of nonnegative definite matrices is closed under the Hadamard product, the claim is equivalent to $\lambda_{\max}(A^{1/2} \circ B^{1/2}) \leq 1$, where $\lambda_{\max}(\cdot)$ stands for the largest eigenvalue of the matrix $(\cdot)$. In order to prove this, observe first that $A^{1/2} \circ A^{1/2}$ and $B^{1/2} \circ B^{1/2}$ are both doubly stochastic matrices because A and B are correlation matrices. Hence we know that the Euclidean length of each row in $A^{1/2}$ and of each row in $B^{1/2}$ is 1. According to Theorem 5.5.3 in [HJ], $\sigma_{\max}(A^{1/2} \circ A^{1/2}) \leq r_{\max}(A^{1/2})c_{\max}(B^{1/2})$, with $r_{\max}(\cdot)$, $c_{\max}(\cdot)$ and $\sigma_{\max}(\cdot)$, respectively, denoting the largest Euclidean row length, the largest Euclidean column length and the largest singular value of $(\cdot)$. Since for a nonnegative definite matrix its singular values coincide with its eigenvalues, we now obtain, as claimed, $\lambda_{\max}(A^{1/2} \circ B^{1/2}) \leq 1$. □

We proceed by considering two special cases. First, we note that for each correlation matrix A the largest eigenvalue $\lambda_{\max}(A^{1/2} \circ A^{1/2}) = 1$, since $A^{1/2} \circ A^{1/2}$ is a stochastic matrix. Second, we collect a few results concerning 2×2 correlation matrices

$$R_x := \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}.$$

Clearly, R_x is a correlation matrix if and only if $-1 \leq x \leq 1$. Then

$$\operatorname{spec}(R_x) = \{1 + x,\ 1 - x\},$$

where $\operatorname{spec}(\cdot)$ indicates the spectrum of $(\cdot)$. Putting $r_x := \sqrt{1+x}$ and $s_x := \sqrt{1-x}$, we have

$$R_x^{1/2} = \frac{1}{2}\begin{pmatrix} r_x + s_x & r_x - s_x \\ r_x - s_x & r_x + s_x \end{pmatrix}, \qquad \operatorname{spec}(R_x^{1/2}) = \{r_x,\ s_x\},$$

$$R_x^{1/2} \circ R_y^{1/2} = \frac{1}{4} \begin{pmatrix} r_x r_y + r_x s_y + s_x r_y + s_x s_y & r_x r_y - r_x s_y - s_x r_y + s_x s_y \\ r_x r_y - r_x s_y - s_x r_y + s_x s_y & r_x r_y + r_x s_y + s_x r_y + s_x s_y \end{pmatrix}$$

and

$$\operatorname{spec}(R_x^{1/2} \circ R_y^{1/2}) = \left\{ \tfrac{1}{2}\left(r_x r_y + s_x s_y\right), \ \tfrac{1}{2}\left(r_x s_y + s_x r_y\right) \right\}.$$

Next, consider

$$\frac{1}{\sqrt{2}} \begin{pmatrix} r_x \\ s_x \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} r_y \\ s_y \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} s_y \\ r_y \end{pmatrix},$$

and observe that the Euclidean length of each of these vectors is unity. By applying the Cauchy–Schwarz inequality to suitably chosen pairwise combinations of these three vectors, it is now possible to obtain again, but in an alternative manner, our former observation that both eigenvalues of the 2×2 matrix $R_x^{1/2} \circ R_y^{1/2}$ are less than or equal to 1. For most of these 2×2 matrices, both eigenvalues are even strictly less than 1. More precisely, at least one of these eigenvalues is 1 if and only if $x = y$ in which case the spectrum of the Hadamard product reduces to

$$\operatorname{spec}(R_x^{1/2} \circ R_x^{1/2}) = \{1, \ \sqrt{1 - x^2}\}.$$

We invite the interested reader to find corresponding closed-form expressions for the eigenvalues when the correlation matrices are of order $n \times n$ with $n \geq 3$.

Reference

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Solution 25-7.2 by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada,*
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The matrices

$$X = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I & B^{1/2} \\ B^{1/2} & B \end{pmatrix}$$

are both nonnegative definite, and hence so is their Hadamard product

$$X \circ Y = \begin{pmatrix} I & A^{1/2} \circ B^{1/2} \\ A^{1/2} \circ B^{1/2} & I \end{pmatrix}.$$

We note that $A \circ I = I = I \circ B$ since A and B are correlation matrices. The Schur complement $I - (A^{1/2} \circ B^{1/2})^2$ is nonnegative definite and so all the eigenvalues of $(A^{1/2} \circ B^{1/2})^2$ are at most 1. Hence the eigenvalues of $A^{1/2} \circ B^{1/2}$ are at most 1 and so $A^{1/2} \circ B^{1/2} \leq_{\mathsf{L}} I$, as required. □

Another proof follows from the following observation, see [1]: For any nonnegative definite A, B we have

$$A \circ B \leq_{\mathsf{L}} \tfrac{1}{2}(A^2 + B^2) \circ I.$$

Reference

[1] F. Zhang (2000). Matrix inequalities by means of block matrices. Dept. of Math, Science and Technology, Nova Southeastern University, Fort Lauderdale, Florida, Preprint 11 pp. To be published in *Mathematical Inequalities & Applications.*

Problem 25-8 with solutions are on pp. 21–23 below; new problems on pp. 24–25 ⇒

Tenth International Workshop on Matrices and Statistics

Voorburg, The Netherlands: 2–3 August 2001

The Tenth International Workshop on Matrices and Statistics will be held under the auspices of the International Statistical Institute in Voorburg, The Netherlands, on Thursday, August 2, and Friday, August 3, 2001. Voorburg is near Den Haag (The Hague) and the Workshop will be held in the Statistics Netherlands (Central Bureau of Statistics) building which houses the Permanent Office of the International Statistical Institute (ISI). Invited speakers include T. W. Anderson (USA), J. K. Baksalary (Poland), H. Bozdogan (USA), N. R. Chaganty (USA), S. Ghosh (USA), J. C. Gower (UK), J. Groß (Germany), W. J. Heiser (Leiden), E. J. Kontoghiorges (Switzerland), and F. Zhang (USA).

This Workshop is being organized by the Data Theory Group from Leiden University and will be a Satellite Meeting of the 53rd Session of the International Statistical Institute (Seoul, Korea: 22–29 August 2001); this Workshop is endorsed by the International Linear Algebra Society and supported, in part, by CANdiensten, distributors of S-PLUS. Further information can be found at our Web site: http://matrix.fsw.leidenuniv.nl

This Workshop will be an opportunity for all those working in Matrices and Statistics to meet and exchange ideas. As usual there will be sessions of invited papers and sessions of contributed papers; it is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications.*

Accommodation will be in the Mövenpick Hotel (Stationsplein 8; fax (31-70) 337-3700) directly opposite the Voorburg railway station and a short walk from the Hofwijck Manor and the Huygensmuseum. According to the *World Book Encyclopedia*, Gottfried Wilhelm von Leibniz (1646–1716) is indebted to Christiaan Huygens (1629–1695), the Dutch physicist, astronomer, and mathematician "from whom he learnt much of his mathematics"; Huygens invented the pendulum clock, and a replica is housed in the museum. The Workshop dinner will be held in the imposing Kurhaus (1885) in Scheveningen.

The International Organizing Committee for this Workshop comprises R. W. Farebrother (UK), S. Puntanen (Finland, vice-chair), G. P. H. Styan (Canada, chair), and H. J. Werner (Germany). The National Program Committee includes J. ten Berge (Groningen), D. Berze (ISI), P. J. F. Groenen (Leiden, chair), W. J. Heiser (Leiden), H. A. L. Kiers (Groningen), J. R. Magnus (Tilburg), J. J. Meulman (Leiden), and W. R. van Zwet (Leiden).

The following talks are expected:

Y. I. Abramovich, N. K. Spencer & A. Y. Gorokhov: Maximum likelihood completion of partially specified covariance matrix estimates
T. W. Anderson: Title to be announced.
J. K. Baksalary: Projectors—a bridge between linear algebra and mathematical statistics
O. M. Baksalary: Idempotency of linear combinations of three idempotent matrices, two of which are disjoint
H. Bensmail & H. Bozdogan: Supervised and unsupervised clustering for mixed data
H. Bozdogan & J. R. Magnus: Misspecification resistant model selection using information complexity
C. Carroll: The need for robust network analysis techniques for field studies of multiplex business interactions
N. R. Chaganty & A. Vaish: Wishartness and independence of quadratic forms in correlated singular normal vectors
K. L. Chu & G. P. H. Styan: Equalities and inequalities for canonical efficiency factors
H. Drygas: Autoregressive error processes and cubic splines
R. W. Farebrother: On rotation, reflection, and retroflection matrices
J. Fortiana & A. Esteve: Estimable functions in distance-based ANOVA
P. Foschi & E. J. Kontoghiorghes: Estimating SUR models with orthogonal regressors—computational aspects
S. Ghosh & C. Burns: Comparison of four new general classes of search designs for factor screening experiments with factors at three levels
J. C. Gower: Applications of the modified Leverrier–Fadeev algorithm
J. Hauke, R. Bru & J. Mas: On nonsingularity of some matrices used in an iterative method for the solution of linear systems
W. J. Heiser: Christiaan Huygens—physicist, astronomer and mathematician
I. Ibraghimov: Improving stability of PARDEC algorithm for three- and multi-way decompositions
H. Kelderman: Conditions on graphical models with interchangeable measurements
T. Klein: Analysis of a quadratic subspace of invariant symmetric matrices as a tool for experimental design
P. Knottnerus: An antique Pythagorean view of today's statistics
R. Kumar: Bounds for singular values
B. Lausen: Prognostic modelling by DNA array data
N. F. W. Lehmann: Random spectrum of (generalized) inverse Wishart matrices
A. Markiewicz: Robust linear estimation with a misspecified covariance matrix
E. Meijer: Matrix algebra for higher order moments
S. Mukherjee: Box–Jenkins analysis of synodic time series
D. S. G. Pollock: Circulant matrices and time-series analysis
S. Puntanen & J. Groß: Extensions of the Frisch–Waugh theorem
T. A. G. M. Steerneman: The Moore–Penrose inverse of $Q\Sigma^{-1}Q$
J. Szulc & F. Uhlig: Real eigenvalues of a real matrix
N. T. Trendafilov: Principal components extraction based on their optimality
W. de Winter: Matrices and statistics with S-PLUS
H. Yanai, H. J. Werner & Y. Takane: More on g-inverses, projectors and Cochran's theorem
F. Zhang: Matrix inequalities by means of block matrices.

Fifth China Conference on Matrix Theory and Its Applications

Shanghai, China: 14–18 August 2002

The Fifth China Conference on Matrix Theory and Its Applications will be held in Shanghai University, Shanghai, from 14 to 18 August 2002. This Conference will be a satellite meeting of the 2002 International Congress of Mathematicians (ICM) to be held in Beijing from 20 to 28 August 2002. The conference registration fee is US$100, which also covers meals for four-and-a-half days and a sightseeing tour. For more information, contact Erxiong Jiang: ejiang@fudan.edu.cn or Chuanqing Gu: guchqing@guomai.sh.cn

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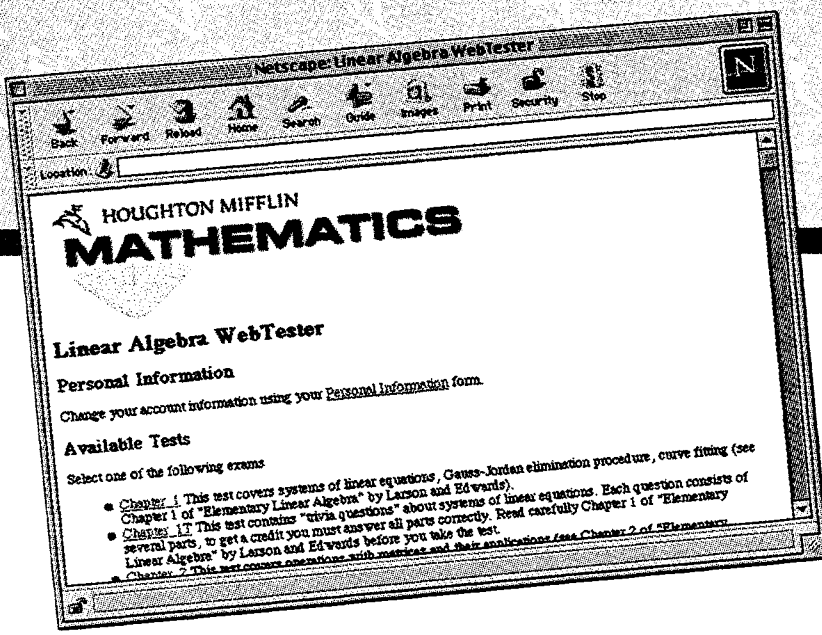

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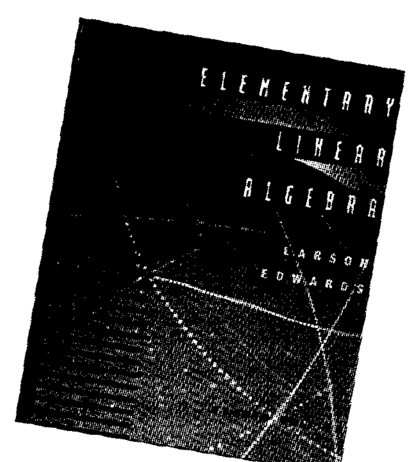

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WebTester accompanies the Fourth Edition of *Elementary Linear Algebra,* by Larson/Edwards, with practice tests that correlate chapter-by-chapter to the material in the text. WebTester is available shrinkwrapped with the *Elementary Linear Algebra* text. It can also be purchased separately for use with any linear algebra text.

For more information about WebTester, go to **college.hmco.com** or contact your Houghton Mifflin sales representative. To try WebTester yourself, email **college_math@hmco.com** to request a passkey.

tal difference and illustrated it by recent numerical experiments carried out by her group at CERFACS. These experiments show that Krylov-type methods need an accurate inner process only at the beginning of the convergence. This led Amina Bouras and Valérie Frayssé (CERFACS) to propose and study a relaxation strategy for Krylov-type methods which allows the inner accuracy to deteriorate as convergence proceeds. This counter-intuitive strategy can be shown to provide a substantial reduction of the overall computer time required to obtain the solution with a prescribed backward error.

Many important industrial domains can be successfully tackled by this approach. As an example, Valérie Frayssé presented promising results on domain decomposition methods for heterogeneous and anisotropic problems, obtained jointly with Amina Bouras and Luc Giraud at CERFACS. Returning to inexact Newton methods, the last speaker Andreas Frommer from the University of Wuppertal discussed their implementation and performance on parallel computers.

The workshop ended with a lively round-table discussion. Everyone acknowledged that the industrial needs presented during the workshop were particularly challenging and the engineers were strongly urged to provide the community with detailed examples and data. The workshop gathered 6 engineers and 24 researchers from 5 countries (although this would increase to 13 if the multi-ethnic distribution of CERFACS researchers is included). Its moderate size and the nice and informal atmosphere really favoured fruitful discussions and indeed fostered the birth of new collaborations. Abstracts, slides and pictures are available from the Web site: http://www.cerfacs.fr/algor/iter2000.html

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Oberwolfach Meeting on Nonnegative Matrices, M-matrices and Applications

Oberwolfach: 26 November–1 December 2000

It is not very often that a meeting on some aspect of Linear Algebra takes place at the famous Mathematics Institute in the Black Forest in south-western Germany. Therefore, extra credit should be given to the three organizers Daniel Hershkowitz, Volker Mehrmann and Hans Schneider, for putting together such a meeting. As is customary at Oberwolfach, the meeting is limited to fifty participants (group photo below) who arrive on Sunday for a Monday–Friday meeting. There were some twenty talks and the rest of the time was devoted to discussions and informal sessions. Until late at night, one could often see groups of two or more people around the table or in front of a blackboard. Some participants called this "the spirit of Oberwolfach." This "spirit" appears to be responsible for the several theorems that reportedly were proved during the meeting.

As the week progressed some of these new results were presented at the talks. In an informal session Monday after dinner, Hans Schneider gave a short talk in which he posed a challenge problem. It is interesting to note that this talk was a follow-up to the talk he presented in Oberwolfach in 1982. A few people took up the challenge and thanks to "the spirit of Oberwolfach" and to several discussions and collaborations, the appropriate theorems were proved by Thursday. Luckily, the community at large will have the opportunity to read about many of the talks presented, and the new theorems proved, since there will be a special issue of *Linear Algebra and its Applications* devoted to papers presented at the meeting. The special editors are Daniel Hershkowitz, Judith McDonald and Reinhard Nabben. Abstracts of the talks are available from the Web site: http://www.mfo.de

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Participants of the Meeting on Nonnegative Matrices, M-matrices and Applications—Oberwolfach, Germany: 30 November 2000.

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TO CELEBRATE THE
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ORGANIZERS: ISI


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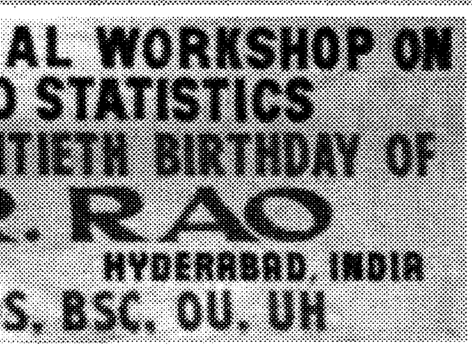
AL WORKSHOP ON
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INDIA

Ninth International Workshop on Matrices and Statistics

Hyderabad, India: 9–13 December 2000

The Ninth International Workshop on Matrices and Statistics, in Celebration of C. R. Rao's 80th Birthday, was held in Hyderabad (Andhra Pradesh), India, from December 9–13, 2000. This Workshop was organized in Hyderabad by the Indian Statistical Institute (ISI) and the Society for Development of Statistics, in collaboration with the Birla Science Centre, Osmania University and the University of Hyderabad. The International Organizing Committee (IOC) comprised R. W. Farebrother (England, UK), S. Puntanen (Finland), G. P. H. Styan (Canada; vice-chair), and H. J. Werner (Germany; chair). The Local Organizing Committee in India included R. Bhatia (ISI-Delhi), P. Bhimasankaram (ISI-Hyderabad), G. S. R. Murthy (ISI–Hyderabad), V. Narayana (ISI-Hyderabad), M. S. Rao (Osmania Univ., Hyderabad), B. Sidharth (Birla Science Centre, Hyderabad), U. SuryaPrakesh (Osmania University, Hyderabad), R. J. R. Swamy (Osmania Univ., Hyderabad), P. Udayasree (Univ. of Hyderabad), and K. Viswanath (Univ. of Hyderabad).

The purpose of this Workshop was to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. The Workshop, with more than 100 participants (group photo on pp. 16–17 above) from 17 countries, began with the Inauguration Ceremony on Saturday, December 9. Inaugural addresses were given by D. C. Reddy (Vice-Chancellor, Osmania University), H. J. Werner (Univ. of Bonn, IOC Chair), and M. Sudhakara Rao (President of the Society for Development of Statistics). This was followed, on the Saturday and Sunday, by a Short Course comprising the ten invited one-hour talks on recent advances in Matrix Theory, with Special Reference to Applications in Statistics.This Short Course was followed by the presentation of about forty further research papers on Monday, December 11 – Wednesday, December 13. The Workshop proper included sessions of invited talks (40 minutes), contributed talks (15 minutes) and talks by students (20 minutes). Most Workshop abstracts are included in the *Workshop Programme with Abstracts as of 2000-11-29* which can be downloaded from the Workshop Web site: http://eos.ect.uni-bonn.de/HYD2000.htm. It is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and Its Applications*; the Eighth Special Issue on Linear Algebra and Statistics is vol. 321 (December 2000).

The Short Course was held in the Auditorium at the Prof. G. Rami Reddy Centre of Distance Education on the Osmania University campus and the Workshop proper in the Birla Science Centre in downtown Hyderabad. On Tuesday, December 12, there was a special ceremony in the Bhaskara Auditorium in the Birla Science Centre celebrating the 80th Birthday of C. R. Rao (newspaper article below from the *Indian Express*, 9 December 2000). His Excellency Dr. C. Rangarajan, Governor of Andhra Pradesh, was the Chief Guest and Speaker; further Addresses, Remarks and Felicitations to C. R. Rao were presented by Malakonda Reddy (Executive Secretary, MediCiti Hospitals, Hyderabad), M. Sudhakara Rao (President: Society for Development of Statistics), S. B. Rao (Former Director, ISI) and H. J. Werner (Univ. of Bonn, IOC Chair). A ballet performance by the children of the MNR Educational Institutions was followed by the Workshop Banquet.

On Monday evening, December 10, the cultural programme was highlighted by the play "Journey through Mathematics: The Crest of the Peacock" directed by Professor Shewalker and staged by students of the Sarojini Naidu School of Performing Arts. The play began in the days of the Indus Valley Civilisation in Mohenjodaro (3000–1750 BC) and ended in the 20th century. The marble of the Naubat Pahad, Birla Mandir, was a marvelous backdrop for this "unique theatre for Mathematics". *–Eds.*

The Sultan of Stats is 80 today

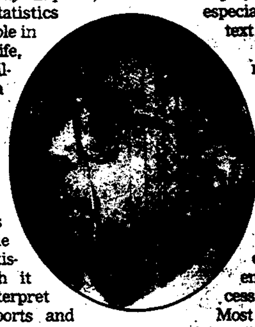
BY B KRISHNA MOHAN
December 8

PROF. Calyampudi Radhakrishna Rao, or CR Rao as he is more widely known, is reckoned among the top statisticians in the world. The Indian born US academic's contributions to the field are both profound and prolific. He has published over 400 research papers.

To celebrate Prof. CR Rao's 80th birthday, an international workshop on 'Matrices and Statistics' is being jointly organised by the Indian Statistical Institute (ISI), the Society for Development of Statistics, Birla Science Centre, Osmania University and the University of Hyderabad from December 9 to 13.

The workshop, the ninth in the series, would be held at the Centre for Distance Education at the Osmania University on the first two days and at the Bhaskara auditorium of the BM Birla Science Centre the next three.

Talking to *City Express*, Prof. Rao said statistics played a vital rôle in all spheres of life, rendering viability to the data collected by helping draw out inferences.



The prediction of demographic trends was possible only with statistics. In health it was used to interpret diagnostic reports and take decisions on medication. It was used in grading samples to carry out bulk transactions. Industry, meteorology and psychology were just a few of the fields in which statistics was used to arrive at conclusions.

He said that in coming years the "SIGM" quartet of technologies — space, informatics, genome and management — would played a vital role, especially in the context of globalisation.

As for the use of matrices, he said they enabled statisticians to arrive at more coherent conclusions as the error percentages were quantified. The use of computers too had enhanced the success rate.

Most of Rao's fundamental contributions to statistical theory were offshoots of the practical problems he addressed himself to. His 'orthogonal arrays' are used in controlling and improving productivity and quality in manufacturing processes.

To date, Rao has received honours from universities in 15 countries and has authored 14 books. The Cramer-Rao inequality law, the Rao-Blackwell theorem, Rao's score test are some of his well-known contributions to the discipline.

Two of his books, *Linear Statistical Inference* and *Statistics and Truth*, have been translated world-wide.

Earlier, ISI director SB Rao addressing mediapersons, said the workshop was intended as a platform for interaction between statisticians. About 140 statisticians, including 40 from abroad, will be participating in the workshop.

International organising committee members HJ Werner form Germany, vice-chairman S Puntanen, RW Fairbrother, GPH Styan, national committee members P Bhimasankaran, RJR Swamy, K Vishwanath, M Sudhakar Rao and several other experts attended the meeting.

Workshops and Conferences in Linear Algebra and Matrix Theory

International Workshop on Parallel Matrix Algorithms and Applications

Neuchâtel, Switzerland: 17–19 August 2000

This International Workshop on Parallel Matrix Algorithms and Applications (PMAA) was organised by Erricos Kontoghiorghes, Oliver Besson and Paolo Foschi (University of Neuchâtel), Peter Arbenz (ETH Zürich, Switzerland) and M. Paprzycki (University of Southern Mississippi, Hattiesburg) in co-operation with the Society for Industrial and Applied Mathematics Activity Group on Linear Algebra. It was endorsed by the International Linear Algebra Society (ILAS) and took place in the University of Neuchâtel from 17–19 August 2000. The principal aim of the workshop was to provide a forum for the exchange of ideas and competence for specialists in those areas of parallel computing that are based on matrix algorithms. Four tutorial sessions were delivered on the first day of the workshop, the underlying theme being Parallel Numerical Applications. During the second and third days, there were two invited talks, forty selected talks, and one tutorial session. The workshop was attended by approximately one hundred delegates from more than twenty countries.

In the first tutorial session, Marcin Paprzycki (University of Southern Mississippi, Hattiesburg) provided a very interesting overview of the recent developments and the state-of-the-art in the areas of high performance hardware, tools, environments, and libraries as related to matrix algorithms. Some current research projects that were deemed to be worth watching were also summarised. During the second session Maurice Clint (Queens University of Belfast, Northern Ireland) addressed some aspects of iterative and direct methods for the partial eigensolution of large, sparse, real symmetric matrices. In particular, several important features of the Lanczos method were discussed and the performances of some recently developed variants of the method were analysed in the context of their efficient implementation on a number of machines with different architectures.

In the first part of the third tutorial session, Yousef Saad (University of Minnesota, Minneapolis) presented a very informative overview of the state-of-the-art in general iterative solvers for the parallel solution of large irregularly structured sparse linear systems. The emphasis was placed on pre-conditioning methods with a bias towards those algorithms with good inherent parallelism. In the second part, methods specifically designed for message-passing parallel platforms were considered and their implementation was illustrated by means of PSPARSLIB, a portable library for solving sparse linear systems on distributed memory parallel platforms. In the fourth and final tutorial session of the first day, Marcus Grote (ETH Zürich, Switzerland) presented the SPAI pre-conditioner as an alternative to the ILU pre-conditioner, which is difficult to parallelize, and the Polynomial and Block-Jacobi pre-conditioners, which are easy to parallelize but are not effective for many difficult ill-conditioned problems. The SPAI algorithm computes explicitly a sparse approximate inverse for use as a pre-conditioner, is very general in its applicability, and was shown to be effective even when applied to difficult problems.

The first invited talk was presented in a very stimulating manner by Ahmed Sameh (Purdue University, West Lafayette, Indiana) and focused on the topic of Parallel Algorithms for Solving Indefinite Systems. The systems referred to are linear, nonsymmetric, indefinite and sparse and arise in the simulation of incompressible fluid flows involving thousands of rigid particles. Two approaches to the handling of these linear systems, each of which is amenable to efficient implementation on a parallel computer, were outlined. In the first, the linear systems are solved by using a novel multilevel algorithm. In the second, effective ordering of the irregular grid nodes was shown to yield an indefinite subsystem which, after suitable pre-conditioning, may be solved using a novel hybrid (direct-iterative) scheme. The effectiveness of the two approaches was demonstrated by the use of numerical experiments.

In the second invited talk entitled Parallel Computation of Financial Equilibria as Variational Inequalities, Anna Nagurney (University of Massachusetts, Amherst) reviewed some large scale applications, notably traffic network equilibrium problems and spatial price problems, in which parallel computing has been successfully applied. She also illustrated clearly how parallel computing may be used to solve a variety of general financial equilibrium problems. The methodologies discussed included variational inequality theory and projected dynamical systems.

The selected talks were grouped into one session that was concerned with SVD and its Applications and five parallel sessions. The SVD session provided four papers which dealt with block two-sided Jacobi SVD algorithms, a new bi-diagonalisation method, and algorithms for Information Retrieval. Eight papers were delivered in two Applications sessions on a variety of topics, including Air Pollution Modelling, Earth Science Calculations, Geomechanics, and Structural Mechanics. Two Linear Systems sessions also provided eight papers. Among the issues addressed were pre-conditioning, high performance parallel direct solvers for sparse positive definite systems, scalable parallel sparse LU factorisation, partitioning of sparse matrices for multiprocessor systems and inversion of symmetric matrices.

Typical of the twenty papers presented in the remaining sessions on (i) Sparse Approximate Inverses, (ii) Augmented Systems, (iii) Iterative Methods, (iv) Eigenvalues, (v) Parallel Models, and (vi) Metacomputing were, respectively: (i) Practical Use and Implementation of Sparse Approximate Inverse Preconditioners, by Edmond Chow (Lawrence Livermore National Laboratory, Livermore, California), (ii) On Computing Some of the Principal Eigenfrequencies of Cavity Resonators, by Roman Geus and Peter Arbenz (ETH Zürich, Switzerland), (iii) Trigonometric Implications of Domain Decomposition Methods, by Karl Gustafson (University of Colorado, Boulder), (iv) Parallel Pseudospectrum Calculations for Large Matrices, by Dany Mezher and Bernard Philippe (INRIA-IRISA, Rennes, France), (v) A Parallel Triangular Inversion Using Elimination, by Ayse Kiper (Middle East Technical University, Ankara, Turkey), and (vi) Matrix Computation on the Internet, by M. Gonzalez and S. Petiton (Laboratoire ASCI, France).

The final tutorial session entitled Parallel Metaheuristics for Combinatorial Optimization Problems was presented by Panos Pardalos (University of Florida, Gainesville). Search techniques for approximating the global optimal solution of combinatorial optimisation problems were reviewed and recent developments on parallel implementation of genetic algorithms, simulated annealing, tabu search, variable neighbourhood search, and greedy randomised adaptive procedures were also discussed. The tutorial was run in parallel with two sessions of selected talks.

As illustrated above, the organisers had prepared an excellent programme that addressed a wide range of current research issues associated with those areas of parallel computing based on matrix algorithms. There was a good mixture of both theoretical and applications talks and tutorial sessions and, in general, these provided a stimulating basis for many fruitful discussions and exchanges of ideas. Selected peer-reviewed papers from the workshop will be published in a special issue of *Parallel Computing*.

The ancient city of Neuchâtel provided an ideal setting for the workshop. As described in the brochures, it has everything: a lake straight from a picture postcard, a stunning choice of bistros and restaurants, good hotels and leisure facilities, a lively nightlife, museums, chateaux, ancient buildings, and easy access by air, rail and road. The workshop dinner was held on the last day in the beautiful 18th century Hôtel du Peyrou and contributed significantly to the overall enjoyment of the workshop.

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Two Industrial Days on Inner-Outer Iterations

Toulouse, France: 11–12 September 2000

On 11–12 September 2000, CERFACS (Centre Européen de Recherche et de Formation Avancée en Calcul Scientifique, Toulouse, France) hosted two Industrial Days on Inner-Outer Iterations, a workshop focusing on numerical quality and software coupling. This workshop was organized at the request of three out of the four industrial shareholders of CERFACS, namely AEDS (the new European consortium for Defence and Aerospace), CNES (the French NASA) and EDF (French Electricity). More and more often in their simulations, these industries are making heavy use of either embedded iterative processes (at two or more levels) or coupled software that exchanges information successively between the subunits. Preliminary discussions led to the acknowledgement that the state-of-the-art in the understanding of the numerical behaviour of embedded or coupled processes was not advanced enough to fulfill the needs of the many users.

A workshop was therefore planned to bring together engineers and researchers to allow the former to better specify their needs and the latter to present up-to-date tools and existing research results on the topic. This resulted in a two-day workshop supported by SMAI and organised as follows. On the first morning of the workshop, there were two tutorials. After the welcome address given by Iain Duff (CERFACS & RAL), the first speaker, Sven Hammarling (Nag Ltd), gave a lively and clear introduction to error analysis particularly concentrating on backward error analysis and discussed its implementation in modern software such as LAPACK. Then Andreas Griewank (Technical University, Dresden) presented the basics of automatic differentiation, a powerful tool to perform forward error analysis and a sensitivity analysis on complex codes. He also described its use for embedded processes arising in optimal design. In the afternoon, industrial users were invited to report on their experience in using embedded processes. Jean-Louis Vaudescal (EDF), described various applications in mechanics and neutronics particularly focusing on a generalised eigenvalue problem arising in the modelling of a nuclear core. Such a problem is solved by successive iterations with Chebyshev acceleration but each step requires the solution of a linear system which has to be computed with an iterative solver due to the characteristics of the problem. Jean-Louis Vaudescal insisted on the extreme sensitivity of the convergence of the outer process with respect to the accuracy of the inner one. Philippe Homsi (EADS) concluded the first day with a brief description of the major issues encountered in software coupling for multidisciplinary physics such as the coupling of acoustics and structure for aircraft design.

On the second day of the workshop, expert researchers had the opportunity to present the state-of-the-art in the control of embedded iterative processes. Even when there are only two levels of iterations corresponding to solving a linear problem, many questions concerning the optimal tuning of the inner process remain open. A pioneer in the field of the control of inner-outer iterations and inexact methods for Linear Algebra, Gene Golub (Stanford University) opened this session with an overview of his contributions (many with his young colleagues) to the topic, including inexact variants of the Chebyshev method, an inexact Uzawa method, inexact conjugate gradient methods or the Lanczos process with inner-outer iterations. When the outer process is the Chebyshev method, Golub has shown that it is possible in some cases to define an optimal strategy for monitoring the inner process while minimising the global cost.

Whereas it is known that an outer Newton-like method requires inner accuracy increasingly as convergence proceeds, the picture is completely different when the outer scheme is a Krylov-type method. In her talk, Françoise Chaitin-Chatelin (CERFACS & Université Toulouse 1) focused on this fundamen-

IMAGE Problem Corner: Problem 25-8 With Solutions*

Problem 25-8: Several Matrix Orderings Involving Matrix Geometric and Arithmetic Means

Proposed by Fuzhen ZHANG: zhang@nova.edu; *Nova Southeastern University, Fort Lauderdale, Florida, USA.*

Let A, B, and C be n-square complex matrices such that $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq_{\mathsf{L}} 0$, and put

$$M_1 = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}, \quad M_2 = \begin{pmatrix} A & |B^*| \\ |B| & C \end{pmatrix}, \quad M_3 = \begin{pmatrix} A & |B| \\ |B| & C \end{pmatrix}, \quad M_4 = \begin{pmatrix} A & |B^*| \\ |B^*| & C \end{pmatrix}.$$

Here $X \geq_{\mathsf{L}} 0$ denotes the positive semidefiniteness (nonnegative definiteness) of the matrix X, and $|X| = (X^*X)^{\frac{1}{2}}$.

Prove or disprove each of the following statements:

(1) $M_1 \geq_{\mathsf{L}} 0$; (2) $M_2 \geq_{\mathsf{L}} 0$; (3) $M_3 \geq_{\mathsf{L}} 0 \Leftrightarrow M_4 \geq_{\mathsf{L}} 0$.

(4) $M_3 \geq_{\mathsf{L}} 0$ if A and B commute; (5) $M_4 \geq_{\mathsf{L}} 0$ if A and B commute;

(6) $M_2 = M_3 = M_4 \geq 0$ when B is Hermitian or normal; (7) $M_2 \geq_{\mathsf{L}} 0 \Rightarrow M_3 \geq_{\mathsf{L}} 0$ and $M_4 \geq_{\mathsf{L}} 0$ but not conversely;

(8) Find necessary and/or sufficient conditions so that $M_5 = \begin{pmatrix} A & (B^* + B)/2 \\ (B^* + B)/2 & C \end{pmatrix} \geq_{\mathsf{L}} 0$.

Solution 25-8.1 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *University of Bonn, Bonn, Germany.*

We prove (4) and (7), disprove (1), (2), (3), (5) and (6), and give a necessary and sufficient condition for (8). We note that an n-square complex matrix W is nonnegative definite if and only if it is Hermitian and $x^*Wx \geq 0$ for all $x \in \mathbb{C}^n$. In what follows, we often make use of the following result, due to Albert (1969), characterizing $W \geq_{\mathsf{L}} 0$ in terms of the subblocks in a square partitioning of W.

THEOREM 1. *Let $W \in \mathbb{C}^{m\times m}$ and let $W = \begin{pmatrix} K & M \\ N & L \end{pmatrix}$, where K is a square subblock of W. Then $W \geq_{\mathsf{L}} 0$ if and only if the conditions $K \geq_{\mathsf{L}} 0$, $N = M^*$, $\mathcal{R}(M) \subseteq \mathcal{R}(K)$ and $L - NK^\dagger M \geq_{\mathsf{L}} 0$ are all satisfied; here $\mathcal{R}(K)$ and $K^\dagger$, respectively, denote the range (column space) of K and the Moore–Penrose inverse of K, respectively. An equivalent set of conditions for $W \geq_{\mathsf{L}} 0$ to hold is obtained by interchanging L and K as well as N and M. So, in particular, $W \geq_{\mathsf{L}} 0$ only if $\mathcal{R}(N) \subseteq \mathcal{R}(L)$ and $L \geq_{\mathsf{L}} 0$.*

EXAMPLE 1. Take

$$M := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Needless to say, $M \geq_{\mathsf{L}} 0$. Partition $M =: \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, where A, B, and C are all of order 2×2. Then, in virtue of Theorem 1, also $A \geq_{\mathsf{L}} 0$ and $C \geq_{\mathsf{L}} 0$. Check that

Problems 25-1 to 25-7 with solutions are on pp. 2–12 above; new problems on pp. 24–25 below.

$$|B^*| = \sqrt{\frac{2}{5}}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad |B| = \sqrt{\frac{5}{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad A^\dagger = A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Since $\mathcal{R}(|B^*|) \not\subseteq \mathcal{R}(C)$, it follows from Theorem 1 that $M_1 = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \not\geq_{\mathsf{L}} 0$, and so this example shows that (1) does not necessarily hold. Since $|B| \neq |B^*|$, clearly $M_2 = \begin{pmatrix} A & |B^*| \\ |B| & C \end{pmatrix} \not\geq_{\mathsf{L}} 0$ and so (2) is also disproved. Moreover, since $C - |B|A^\dagger|B| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq_{\mathsf{L}} 0, \mathcal{R}(|B|) \subseteq \mathcal{R}(A)$ [note that A is nonsingular] and $A \geq_{\mathsf{L}} 0$, we have $M_3 = \begin{pmatrix} A & |B| \\ |B| & C \end{pmatrix} \geq_{\mathsf{L}} 0$. However, $C - |B^*|A^\dagger|B^*| = \frac{1}{5}\begin{pmatrix} 11 & 7 \\ 7 & -1 \end{pmatrix} \not\geq_{\mathsf{L}} 0$ and so $M_4 = \begin{pmatrix} A & |B^*| \\ |B^*| & C \end{pmatrix} \not\geq_{\mathsf{L}} 0$. In other words, M also serves as an example for which (3) does not hold. □

We proceed by proving (7). Clearly, if $M_2 \geq 0$, then necessarily $M_2 = M_2^*$ and so, in particular, $|B| = |B^*|$ or, equivalently, $B^*B = BB^*$. Since in this case we have $M_2 = M_3 = M_4$, the first part of (7) is established, irrespective of whether M is nonnegative definite or not. That the converse implication, however, is incorrect is seen by our next example.

EXAMPLE 2. Let $M := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, with $A = I_2$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $C = I_2$. From Theorem 1, it is easy to check that $M \geq_{\mathsf{L}} 0$. Next observe that $|B| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $|B^*| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $|B| \neq |B^*|$, clearly

$$M_2 = \begin{pmatrix} A & |B^*| \\ |B| & C \end{pmatrix} \not\geq_{\mathsf{L}} 0.$$

Nevertheless, applying Theorem 1 to

$$M_3 = \begin{pmatrix} A & |B| \\ |B| & C \end{pmatrix} \qquad \text{and} \qquad M_4 = \begin{pmatrix} A & |B^*| \\ |B^*| & C \end{pmatrix},$$

respectively, yields $M_3 \geq_{\mathsf{L}} 0$ and $M_4 \geq_{\mathsf{L}} 0$; for observe that

$$A = I_2 = A^{-1}, \quad A \geq_{\mathsf{L}} 0, \quad \mathcal{R}(A) = \mathbb{C}^2, \quad C - |B|A^\dagger|B| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \geq 0, \quad \text{and} \quad C - |B^*|A^\dagger|B^*| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq_{\mathsf{L}} 0.$$

This example thus proves that $M_2 \geq_{\mathsf{L}} 0$ is not a necessary condition for $M_3 \geq_{\mathsf{L}} 0$, $M_4 \geq_{\mathsf{L}} 0$, and $M \geq_{\mathsf{L}} 0$ to hold simultaneously. This completes the proof of the second part of (7). □

We next prove (4), that is, we show that if the n-square matrices A and B commute, then $M_3 \geq_{\mathsf{L}} 0$ is always implied by $M \geq_{\mathsf{L}} 0$. To do so, let

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0, \qquad \text{and let} \qquad AB = BA.$$

Then, in view of Theorem 1, $A \geq_{\mathsf{L}} 0$, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $C - B^*A^\dagger B \geq_{\mathsf{L}} 0$. Clearly, $A \geq_{\mathsf{L}} 0$ implies that $AA^\dagger = A^\dagger A$. Since $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ is equivalent to $AA^\dagger B$, it is now not difficult to see that $AB = BA \iff AB^* = B^*A \iff A^\dagger B = BA^\dagger$. Moreover, $AB^*B = B^*BA$ and $A^\dagger B^*B = B^*BA^\dagger$. Since A and B^*B are normal matrices, it is well known that they commute if and only if they have a common set of orthonormal eigenvectors. Let P be the $n \times n$ unitary matrix whose column vectors are these eigenvectors. Then $A = PDP^*$ and $B^*B = PEP^*$ for some nonnegative diagonal matrices D and E. Since $(B^*B)^{1/2}$ can be written as $(B^*B)^{1/2} = PE^{1/2}P^*$, it is clear that A and $(B^*B)^{1/2}$ and so $A^\dagger$ and $(B^*B)^{1/2}$ do also commute. Therefore $C - |B|A^\dagger|B| = C - |B|^2A^\dagger = C - B^*BA^\dagger C - B^*A^\dagger B \geq_{\mathsf{L}} 0$. We already know that $A \geq_{\mathsf{L}} 0$. According to Theorem 1, we thus have $M_3 \geq_{\mathsf{L}} 0$ if (and only if) $\mathcal{R}(|B|) \subseteq \mathcal{R}(A)$ can be shown. Since $A \geq_{\mathsf{L}} 0$, clearly $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$. The equation $AB = BA$ therefore implies that $\mathcal{R}(AB) = \mathcal{R}(B)$ and $\operatorname{rank}(AB) = \operatorname{rank}(B)$. Since $\operatorname{rank}(B) = \operatorname{rank}(B^*)$ and $\operatorname{rank}(AB) = \operatorname{rank}(B^*A)$, and since $B^*A = AB^*$ implies that $\operatorname{rank}(B^*A) = \operatorname{rank}(AB^*)$, we arrive at $\operatorname{rank}(AB^*) = \operatorname{rank}(B^*)$. Since $B^*A = AB^*$ also implies that $\mathcal{R}(AB^*) \subseteq \mathcal{R}(B^*)$, we indeed have $\mathcal{R}(AB^*) = \mathcal{R}(B^*)$. Therefore, $\mathcal{R}(B^*) \subseteq \mathcal{R}(A)$. Since $\mathcal{R}(B^*) = \mathcal{R}(B^*B) = \mathcal{R}((B^*B)^{1/2})$, we obtain, as desired, $\mathcal{R}(|B|) \subseteq \mathcal{R}(A)$.

That the implication (5) is *not* always true follows from our next counter-example.

EXAMPLE 3. Let $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, with $A = I_2$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$. Then $A \geq_{\mathsf{L}} 0, \mathcal{R}(B) \subseteq \mathcal{R}(A) = \mathbb{C}^2$ and $C - B^*A^\dagger B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq_{\mathsf{L}} 0$. So, according to Theorem 1, $M \geq_{\mathsf{L}} 0$. Since $A = I_2$, trivially $AB = BA$. Moreover,

$$|B^*| = \sqrt{\frac{2}{5}}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \Rightarrow \quad M_4 = \begin{pmatrix} A & |B^*| \\ |B^*| & C \end{pmatrix} \not\geq_{\mathsf{L}} 0;$$

and so $\mathcal{R}(|B^*|) \not\subseteq \mathcal{R}(C)$ and hence, according to Theorem 1, indeed $M_4 \not\geq_{\mathsf{L}} 0$. □

We now disprove the implication (6).

EXAMPLE 4. Let $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, with $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Then $A \geq_{\mathsf{L}} 0, \mathcal{R}(B) \subseteq \mathcal{R}(A) = \mathbb{C}^2$, and $C - B^*A^\dagger B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \geq_{\mathsf{L}} 0$. According to Theorem 1, therefore $M \geq_{\mathsf{L}} 0$. Since $B^*B = BB^* = I_2$, the matrix B is normal and so $M_2 = M_3 = M_4$. We note that $|B| = |B^*| = I_2$. Since $C - |B|A^\dagger|B| = C - A^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \not\geq_{\mathsf{L}} 0$, Theorem 1 tells us that $M_2 \not\geq_{\mathsf{L}} 0$. □

We conclude our discussion of this problem by characterizing

$$M_5 = \begin{pmatrix} A & (B^* + B)/2 \\ (B^* + B)/2 & C \end{pmatrix} \geq_{\mathsf{L}} 0$$

in terms of the subblocks of M_5. Theorem 1, of course, already tells us that $M_5 \geq_{\mathsf{L}} 0$ holds true if and only if $A \geq_{\mathsf{L}} 0, \mathcal{R}(B^* + B) \subseteq \mathcal{R}(A)$ and $C - \frac{1}{4}(B^* + B)A^\dagger(B^* + B) \geq_{\mathsf{L}} 0$ are all satisfied. Of course, if A, B and C are such that

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq_{\mathsf{L}} 0,$$

then it can be expected that this characterization might be simplified. And, by means of Theorem 1, it is indeed easy to see that in such a case $M_5 \geq_{\mathsf{L}} 0$ holds if and only if both

$$\mathcal{R}(B^*) \subseteq \mathcal{R}(A) \qquad \text{and} \qquad C - \tfrac{1}{4}(B^* - B)A^\dagger(B^* - B) \geq_{\mathsf{L}} 0$$

hold. □

Reference

A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics,* 17, 434–440.

A solution to this problem was also received from the Proposer Fuzhen ZHANG.

IMAGE Problem Corner: New Problems

Problem 26-1: Degenerate Complex Quadratic Forms on Real Vector Spaces

Proposed by S. W. DRURY: drury@math.mcgill.ca; *McGill University, Montréal, Québec, Canada.*

Let C be a complex symmetric $n \times n$ matrix. Suppose that A and B are the real and imaginary parts of C respectively (i.e. $a_{jk} = \Re c_{jk}$, $b_{jk} = \Im c_{jk}$). Suppose that $\det(B - tA) = 0$ for all real t. Prove or find a counterexample to the following statement. There necessarily exist nonnegative integers p, q and r such that $p + q + r = n$ and $q > r$ and a real invertible $n \times n$ matrix P, such that

$$P'CP = \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & Y \\ 0 & Y' & 0 \end{pmatrix},$$

where X is a $p \times p$ complex matrix and Y is a $q \times r$ complex matrix. The cases $p = 0$ and $r = 0$ are allowed. For instance, $p = n-1$, $q = 1$, $r = 0$ is the case where there is a non-zero real vector in the null space of C.

Problem 26-2: Almost Orthogonality of a Skew-Symmetric Matrix

Proposed by L. LÁSZLÓ: llaszlo@ludens.elte.hu; *Eötvös Loránd University, Budapest, Hungary.*

By adding an extra first row and column to the matrix of Problem 25-1 (see page 2 above), we obtain the 8×8 matrix

$$A_8 = \begin{pmatrix} +a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ +a_1 & +a_0 & +a_3 & -a_2 & +a_5 & -a_4 & +a_7 & -a_6 \\ +a_2 & -a_3 & +a_0 & +a_1 & +a_6 & -a_7 & -a_4 & +a_5 \\ +a_3 & +a_2 & -a_1 & +a_0 & -a_7 & -a_6 & +a_5 & +a_4 \\ +a_4 & -a_5 & -a_6 & +a_7 & +a_0 & +a_1 & +a_2 & -a_3 \\ +a_5 & +a_4 & +a_7 & +a_6 & -a_1 & +a_0 & -a_3 & -a_2 \\ +a_6 & -a_7 & +a_4 & -a_5 & -a_2 & +a_3 & +a_0 & +a_1 \\ +a_7 & +a_6 & -a_5 & -a_4 & +a_3 & +a_2 & -a_1 & +a_0 \end{pmatrix}$$

(see also Solution 25-1.3 on pp. 3–4 above). Denote by A_k its principal submatrix of order k, and observe that A_2, A_4 and A_8 are skew symmetric and (apart from a scalar multiple) orthogonal for any real a_i with $a_0 = 0$. Can this construction be continued? Note that only the sign pattern of A_{16} is to be properly chosen, for the distribution of the a_i is easy to follow: if the sign-free matrix A_k is denoted by M_k, then

$$M_{2k} = \begin{pmatrix} M_k & N_k \\ N_k & M_k \end{pmatrix},$$

where N_k comes from M_k by increasing the subscripts of the a_i by k, e.g.,

$$M_2 = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \Longrightarrow N_2 = \begin{pmatrix} a_2 & a_3 \\ a_3 & a_2 \end{pmatrix}.$$

Problem 26-3: Factorization of Generalized Inverses of Matrix Polynomials

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Queen's University, Kingston, Ontario, Canada.*

(a) Consider an $n \times n$ matrix A and two scalars λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Prove that if

$$\text{range}\,(\lambda_1 I_n - A) \cap \text{range}\,(\lambda_2 I_n - A) \neq \{0\}$$

and

$$\text{range}\,(\lambda_1 I_n - A^T) \cap \text{range}\,(\lambda_2 I_n - A^T) \neq \{0\}$$

then the set equality

$$\{\,[(\lambda_1 I_n - A)(\lambda_2 I_n - A)]^-\,\} = \frac{1}{\lambda_2 - \lambda_1}\{\,(\lambda_1 I_n - A)^- - (\lambda_2 I_n - A)^-\,\}$$

holds, where $(\cdot)^-$ denotes a generalized (or inner) inverse of a matrix.

(b) Consider an $n \times n$ matrix A and k scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$. If

$$\text{range}\,(\lambda_1 I_n - A) \cap \text{range}\,(\lambda_2 I_n - A) \cap \cdots \cap \text{range}\,(\lambda_k I_n - A) \neq \{0\},$$

and

$$\text{range}\,(\lambda_1 I_n - A^T) \cap \text{range}\,(\lambda_2 I_n - A^T) \cap \cdots \cap \text{range}\,(\lambda_k I_n - A^T) \neq \{0\},$$

then prove or disprove that

$$\{\,[(\lambda_1 I_n - A)(\lambda_2 I_n - A)\ldots(\lambda_k I_n - A)]^-\,\} = \left\{\frac{1}{p_1}(\lambda_1 I_n - A)^- + \frac{1}{p_2}(\lambda_2 I_n - A)^- + \ldots + \frac{1}{p_k}(\lambda_k I_n - A)^-\right\},$$

where

$$p_i = (\lambda_1 - \lambda_i)\ldots(\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i)\ldots(\lambda_k - \lambda_i), \quad i = 1, 2, \cdots, k.$$

Problem 26-4: Commutativity of EP Matrices

Proposed by Yongge TIAN: ytian@mast.queensu.ca; *Quuen's University, Kingston, Ontario, Canada.*

The square complex matrix A is said to be EP whenever the column space (range) $\mathcal{R}(A) = \mathcal{R}(A^*)$. Let A and B be $n \times n$ complex matrices and let A^+ denote the Moore–Penrose inverse of A.

(a) Supppose that $AB = BA$. Show that A is EP if and only if $A^+B = BA^+$.

(b) Suppoose that A and B are both EP. Show that $AB = BA$ if and only if $A^+B^+ = B^+A^+$.

Problem 26-5: Convex Matrix Inequalities

Proposed by Bao-Xue ZHANG: baoxuezhang@sohu.com; *Beijing Institute of Technology, Beijing, China.*

Show that for any Hermitian nonnegative definite matrices A and B of the same size and for any real number λ satisfying $0 \leq \lambda \leq 1$,

(a) $\{P\,[\lambda A + (1-\lambda)B]\,P\}^+ \leq_{\mathrm{L}} \lambda A^+ + (1-\lambda)B^+$,

(b) $[\lambda A + (1-\lambda)B]^+ \leq_{\mathrm{L}} \lambda A^+ + (1-\lambda)B^+ + [\lambda A + (1-\lambda)B]^+ \left\{M\,[\lambda A + (1-\lambda)B]^+\,M\right\}^+ [\lambda A + (1-\lambda)B]^+$,

where P stands for the orthogonal projector onto $\mathcal{R}(A) \cap \mathcal{R}(B)$, $\mathcal{R}(\cdot)$ denotes the range (column space) of $(\cdot)$, $(\cdot)^+$ indicates the Moore-Penrose inverse of $(\cdot)$, $M := I - P$, and $\leq_{\mathrm{L}}$ stands for the Löwner ordering.

Please submit solutions and new problems (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

ILAS Treasurer's Report: March 1, 2000–February 28, 2001

by Jeffrey L. Stuart, *University of Southern Mississippi, Hattiesburg*

Balance on hand March 1, 2000			
Certificates of Deposit (CD)		20,078.00	
Vanguard (ST Fed. Bond Fund 991.346 Shares)		9,721.11	
Checking account		37,271.88	67,070.99

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Checking Account Balance on March 1, 2000			37,271.88
March 2000			
Income:			
Interest (General Fund)	17.01		
Interest (Uhlig CD)	21.32	38.33	
Expenses:			
FNB Checks and Bank Fees	47.89	(47.89)	(9.56)
April 2000			
Income:			
Interest (General Fund)	34.57		
Dues	520.00		
Schneider Fund	10.00		
Book Sales (7)	132.00		
Uncredited income in FL	805.29	1501.86	
Expenses:			
FL Department of State	70.00		
ELA Copy Editing	153.00		
FNB New Account Charges	47.50		
Chandler Davis (ILAS Lecture)	500.00		
Supplies and Postage	87.94		
Wages (ILAS Files)	56.00	(914.44)	687.42
May 2000			
Income:			
Interest (General Fund)	35.71		
Interest (Schneider/Todd CD)	96.15		
Federal Tax Filing Fee Refund	15.04		
Dues	80.00		
Institutional Dues	200.00		
Elsevier Sciences Inc.LAA Fund	1000.00		
Book Sales (4)	72.00	1498.90	
Expenses:			
		(0.00)	1498.90
June 2000			
Income:			
Interest (General Fund)	118.02		
Interest (Uhlig CD)	21.55	139.57	
Expenses:			
		(0.00)	139.57
Transferred to checking: Uhlig CD	1500.00		
July 2000			
Income:			
Interest (General Fund)	104.27		
Book Sales (2)	36.00	140.27	
Expenses:			
IMAGE #24 (Print and Mail)	1355.34		

Postage (Book Sales)	111.97		
Postage (1st Dues Mailing)	473.26		
Supplies	114.51		
Wages (Dues, Ballot Mailing)	117.50	(2172.58)	(2032.31)
August 2000			
Income:			
Interest (General Fund)	115.39		
Interest (Schneider/Todd CD)	98.31		
Dues	1140.00		
Schneider Fund	315.00		
Uhlig Fund	100.00		
Todd Fund	240.00		
Conference Fund	10.00		
General Fund	380.00		
Book Sales (1)	20.00	2418.70	
Expenses:			
ELA Copy Editing	170.00		
Supplies and Postage	20.62	(190.62)	2228.08
September 2000			
Income:			
Interest (General Fund)	119.02		
Dues	720.00		
Schneider Fund	25.00		
Conference Fund	15.00		
General Fund	156.00		
Book Sales (8)	140.00	1175.02	
Expenses:			
Postage (Ballots)	150.55		
Wages (Ballot Mailing)	14.00	(164.55)	1010.47
October 2000			
Income:			
Interest (General Fund)	127.15		
Dues	120.00		
Book Sales (1)	20.00	267.15	
Expenses:			
Hugo Woerdeman (ILAS Lecture)	400.00		
Postage and Photocopies	62.56		
Returned Foreign Check	20.00		
Returned Checks (2) and Fees	42.00	(524.56)	(257.41)
Transferred to checking:			
Schneider (72%)/Todd (28%) CD	7500.00		
November 2000			
Income:			
Interest (General Fund)	139.89	139.89	
Expenses:			
ELA Copy Editing	391.00		
Foreign Check Fee	12.00		
Returned Check and Fee	17.00	(420.00)	(280.11)
December 2000			
Income:			
Interest (General Fund)	145.16		

	Dues	1160.00		
	Schneider Fund	100.00		
	Todd Fund	120.00		
	Conference Fund	200.00		
	General Fund	85.00		
	Book Sales (4)	80.00	1890.16	
Expenses:				
	Postage (2nd Dues Mailing)	148.00		
	Wages (Dues Mailing)	28.00		
	Returned Foreign Checks (2)	40.00		
	Supplies and Photocopying	35.00	(251.00)	1639.16
January 2001				
Income:				
	Interest (General Fund)	129.41		
	Dues	1060.00		
	Schneider Fund	180.00		
	Uhlig Fund	10.00		
	Conference Fund	30.00		
	General Fund	320.00		
	Book Sales (1)	16.00	1745.41	
Expenses:				
			(0.00)	1745.41
February 2001				
Income:				
	Interest (General Fund)	104.10		
	Interest on CD (General Fund)	844.78		
	Interest on CD (General Fund)	67.32		
	Dues	820.00		
	Institutional Dues	600.00		
	Schneider Fund	35.00		
	Todd Fund	25.00		
	Conference Fund	5.00		
	General Fund	5.00		
	Book Sales (3)	48.00	2554.20	
Expenses:				
	IMAGE #25 (Print and Mail)	1398.05		
	Returned Foreign Checks (2)	40.00	(1438.05)	1116.15
Transferred to checking:				
	General Fund CD's	11,078.00		

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February 28, 2001 Checking Account Balance ($135.00 checks outstanding) **64,870.65**

Net Account Balances on February 28, 2001		
Vanguard (ST Fed. Bond Fund 1053.184 Shares)		
(72% Schneider Fund and 28% Todd Fund)	10,879.39	
Checking account	64,735.65	**75,615.04**

General Fund	31,001.62	
Conference Fund	10,018.94	
ILAS/LAA Fund	3,590.00	
Olga Taussky Todd/John Todd Fund	8,831.47	
Frank Uhlig Education Fund	3,345.98	
Hans Schneider Prize Fund	18,827.03	75,615.04

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ILAS President/Vice Presidents' Annual Report: April 2001

1. The following persons have been elected to ILAS offices with terms that began on March 1, 2001.
Vice President: Roger Horn
(three-year term ending February 29, 2004).
Board of Directors: Tom Markham and Daniel Szyld
(three-year terms ending February 29, 2004).
The following continue in their offices to which they were previously elected:
Secretary/Treasurer: Jeff Stuart (term ends February 28, 2003)
Board of Directors:
Nicholas Higham (term ends February 28, 2002),
Pauline van den Driessche (term ends February 28, 2002),
Harm Bart (term ends February 28, 2003),
Steve Kirkland (term ends February 28, 2003).

2. The President's Advisory Committee continues as last year: Chi-Kwong Li (chair), Shmuel Friedland, Raphi Loewy, and Frank Uhlig.

3. This fall there will be elections for President (Richard Brualdi's term ends on February 28, 2002) and two members of the Board of Directors (the terms of Nicholas Higham and Pauline van den Driessche end on February 28, 2002). The ILAS 2000 Nominating Committee has been appointed in accordance with the ILAS By-Laws and consists of Avi Berman, Roger Horn (chair), Tom Laffey, Hans Schneider, and Bryan Shader.

4. There was no ILAS conference in the year 2000. ILAS did cooperate with the SIAM Activity Group on Linear Algebra (SIAG-LA) in their seventh triennial conference on Applied Linear Algebra, held in Raleigh, North Carolina on October 23–25, 2000. There were two invited speakers who were chosen by the ILAS Board and sponsored by ILAS at this conference. They were Eduardo Marques de Sà and Hugo J. Woerdeman. Due to a medical problem, Sà was unable to participate in the conference. We are grateful to Hans Schneider who, with short notice, replaced Sà in the program. Under the terms of the agreement with SIAM, ILAS members, who were not already members of the SIAG-LA, had the same reduced registration fee of that of SIAG-LA members. SIAM is reciprocating at the ILAS meeting in Auburn in 2002 (see below).

5. The following ILAS conferences are planned:

a. The 9th ILAS Conference, Technion, Haifa, Israel, June 25–29, 2001. At this conference Nick Trefethen will deliver the second LAA Lecture.

b. The 10th ILAS Conference, "Challenges in Matrix Theory," Auburn University, Auburn, USA, June 10–13, 2002. At this conference Steve Kirkland will deliver the Fourth Olga Tauskky Todd/John Todd Lecture. The SIAG-LA is cooperating with ILAS in this conference. The SIAG-LA has chosen and will support two invited speakers, Michele Benzi of Emory University and Misha Kilmer of Tufts University. Members of the SIAG-LA will be offered the same reduced registration fee as ILAS members.

c. There is no ILAS conference being planned for 2003, in which year the SIAG-LA will hold its triennial conference (July 23–26, 2003, Williamsburg, VA). The 11th ILAS Conference will be held in Lisbon, Portugal, Summer, 2004.

d. It is expected that the ILAS Board will approve a 2005 conference site at its meeting in Haifa in June. One proposal has been received. Any additional proposals or inquiries should be sent to the ILAS president no later than May 2001.

6. The next Hans Schneider Prize in Linear Algebra, for research, contributions, and achievements at the highest level of Linear Algebra, will be awarded at the 10th ILAS conference in Auburn in 2002. A five member Prize committee has been appointed by the ILAS president upon the advice of the ILAS Executive Board: Jesse Barlow, Harm Bart (Chair), Ludwig Elsner, Roger Horn, and Frank Uhlig. The ILAS president serves as ex-officio member. Nominations, to include a brief biographical sketch and a supporting statement, should be sent to the Chair, Harm Bart, by November 15, 2001. The prize guidelines can be found at http://www.math.technion.ac.il/iic/ILASPRIZE.html

7. For the past few years, ILAS has considered requests for the sponsorship of an ILAS Lecturer at a conference which is of substantial interest to ILAS members. Each year, $1,000 has been set aside to support such lecturers, with a maximum amount of $500 available for any one conference. At its June meeting in Haifa, the ILAS Board is expected to review this program. ILAS is sponsoring one Lecturer in 2001: Chi-Kwong Li at the Matrix Analysis Session organized by Judi McDonald at the Canadian Mathematical Society Meeting, June 2–4, 2001 in Saskatoon, Saskatchewan (Canada). One goal of this session is to increase the profile of Matrix Analysis in Canada.

8. ILAS has endorsed the Rocky Mountain Mathematics Consortium's Summer School on Combinatorics and Matrix Theory to be held at the University of Wyoming from July 23 to August 3, 2001. The principal speaker at the summer school will be Charles R. Johnson. The other speakers are Sean Fallat and Bryan Shader.

9. Somewhat modifying recommendations of an ad hoc By-Laws Committee, the ILAS Board approved in August 2000 some changes in its By-Laws. The approved changes are:

(I) Article 7. Journals

Section 1 (A) (ELA) and Section 2 (A) (IMAGE) are reworded to read: The President shall appoint at the direction of the Board of Directors and after consultation with the Journals Committee not more than two (2) Editors-in Chief who will serve three (3) year terms, with the possibility of reappointment.

Sections 1, (B), (8) and 2, (B), (7) are added to to read:

Six months before the expiration of a term of any Editor-in-Chief, the Editors-in-Chief shall present a report to the Board of Directors (to be discussed with the Journals Committee) on the operation of the journal since the last such report.

(II) A new ARTICLE 9. Outreach Director is added:

The President, with the consent of the Board of Directors, may appoint an Outreach Director. The duties of the Outreach Director shall be as provided for in (1)–(4) of this section.

(1) Overseeing the publication of a print version of the Electronic Journal of Linear Algebra, and similar publications, including their marketing and distribution.

(2) Development of relations with other professional and learned societies.

(3) Recruitment of individual and corporate members of ILAS.

(4) Other activities of an outreach nature as directed by the ILAS Executive Board.

(5) The Outreach Director shall be an ex-officio non-voting member of the Board of Directors and reports to and is responsible to it and the Executive Board.

(6) The term of the Outreach Director shall end with the term of the ILAS President who makes the appointment. The term may be extended according to the rules of appointment stated above.

(III) Article 10. ELECTIONS now reads:

1. At an appropriate time, the President shall inform the membership of the officers to be elected in the next elections and encourage members to suggest candidates to the Nominating Committee. The Nominating Committee shall consider such suggestions. A suggested candidate who is not nominated by the Nominating Committee may still be nominated under the provisions of Article 6, Section 2(B).

2. The President shall circulate over ILAS Net and/or another appropriate electronic net the nominations for officers by the Nominating Committee made under Article 6 Section 2(B) and shall inform the membership of the possibility for additional nominations as provided by the said article. At least two weeks shall be allowed for the additional nominations.

3. Elections for the Board of Directors (including the Executive Officers) shall be held between September 1 and December 31 for terms beginning March 1 of the following year.

4. A member of the Executive Board will be deemed elected upon receiving an absolute majority of votes cast. If no candidate receives such a majority, there shall be a new election to decide between the two candidates who receive the most votes in the first election.

5. In elections of other members of the Board of Directors, each voter may cast one vote for each of as many candidates as there are open positions. Candidates receiving the most votes sufficient to fill the number of vacancies will be deemed elected provided they receive a vote on at least one-third of the ballots received. If there remain vacancies, then those candidates, numbering twice the number of remaining vacancies, who have received the most votes among those not already declared a winner will, with their consent, compete in a new election.

6. In the case of ties in an election, a decision is determined by lot.

7. Voting shall be done by hard mail, with proper precautions for the security of the ballots and the anonymity of voters.

8. The votes will be counted by the Chair of the Nominating Committee and another person, normally an ILAS member who holds no office in ILAS.

9. The results of the elections shall be transmitted by the Chair of the Nominating Committee to the President who informs the ILAS membership of the winners.

(IV) As a result of the new Article 9, previous Articles 9, 10, 11, 12, 13, 14 are renumbered as Articles 11, 12, 13, 14, 15, 16 respectively.

10. The revision to the ILAS By-Laws recently approved by the ILAS Board of Directors allowed for the possibility of the appointment of an Outreach Director. With the consent of the Board of Directors, the president has appointed Jim Weaver as ILAS Outreach Director. In accordance with the By-Laws, Jim's appointment extends to the end of the current term of the ILAS President, that is, February 28, 2002.

11. As of this writing, 310 people have renewed their membership in ILAS for 2001. If you have not yet done so, please renew with the Secretary/Treasurer Jeff Stuart as soon as possible. ILAS also has five corporate sponsors: Cambridge University Press, Elsevier Science Inc., Houghton Mifflin, Jones & Bartlett Publishers, and the Society for Industrial and Applied Mathematics (SIAM).

12. *The Electronic Journal of Linear Algebra* (ELA):

Volume 1, published in 1996, contained 6 papers.

Volume 2, published in 1997, contained 2 papers.

Volume 3 (the Hans Schneider issue), published in 1998, contained 13 papers.

Volume 4, published also in 1998, contained 5 papers.

Volume 5, published in 1999, contained 8 papers.

Volume 6 (Proceedings of the Eleventh Haifa Matrix Theory Conference), published in 1999 and 2000, contained 8 papers.

Volume 7, published in 2000, contained 14 papers.

Volume 8, is being published now; as of April 2001 it contains 3 papers.

Volumes 1–4 of ELA are now available in book form. The list price is $20 with a discounted price of $16 for ILAS members. The Outreach Director is currently working on the publication of a book containing Volumes 5–7 of ELA. It is expected that samples will be available at the Haifa meeting this June. The anticipated cost is $20 for ILAS members and $25 for others (including shipping).

ELA's primary Web site is at the Technion (Haifa, Israel). Mirror sites are at Temple University (Philadelphia), at the University of Chemnitz, at the University of Lisbon, in the the European Mathematical Information Service (EMIS) of the European Mathematical Society, and in the 36 EMIS mirror sites.

13. ILAS-NET: As of April 13, 2001, we have circulated 1044 ILAS-NET announcements. ILAS-NET currently has 547 subscribers.

14. ILAS INFORMATION CENTER (IIC) has a daily average of 300 information requests (not counting FTP operations). IIC's primary Web site is at the Technion. Mirror sites are located in Temple University (Philadelphia), in the University of Chemnitz and in the University of Lisbon.

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Tribute to Danny Hershkowitz

Wednesday, 28 February 2001, was Danny Hershkowitz's last day as ILAS Vice President. As you may recall, Danny was the founding secretary of ILAS when it was called the International Matrix Group (IMG) and served three 2-year terms (1989–1991, 1991–1993, and 1993–1995), first as an appointed position and later as an elected position. [Due to a change in ILAS by-laws, with the term beginning on March 1, 1995, the position of Secretary was combined with that of Treasurer.] Danny began an elected 3-year term as Vice President on March 1, 1995 and was re-elected for another 3-year term, now ending.

The ILAS community is very indebted to Danny for his dedicated, professional, and multifacted service to ILAS, in particular for his service these last six years as Vice President. Besides serving as Secretary and Vice President, Danny has been, and continues to be, the Manager of ILAS-Net and Editor-in-Chief of *The Electronic Journal of Linear Algebra* (ELA). According to our Bylaws, Danny remains a voting member of the ILAS Board for one year after leaving office (until 28 February 2002). I am sure that you will join me in this expression of deep appreciation to Danny for his work these last six years as ILAS Vice President and his continued work for ILAS.

On 1 March 2001, Roger Horn begins a 3-year term as ILAS Vice President. I look forward to working with Roger during my last year as ILAS President. We are indebted to Roger for agreeing to take on this responsibility and for standing for election. I also want to take this opportunity to thank two retiring members (as of 28 February 2001) of the Board, José Dias da Silva and Roger Horn (mentioned above) for their important service to ILAS. At the same time I want to welcome two new members of the Board, Tom Markham and Daniel Szyld, and I look forward to collaborating with them. Continuing members of the Board are Harm Bart, Nicholas Higham, Steve Kirkland, and Pauline van den Driessche. As Outreach Director, Jim Weaver continues as a non-voting member of the Board.

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Some New Books

R. B. Bapat (2000). *Linear Algebra and Linear Models*, Second edition. Springer-Verlag, x + 138 pp., ISBN 0-387-98871-8/hbk.

Hal Caswell (2001). *Matrix Population Models: Construction, Analysis, and Interpretation*, Second edition. Sinauer Associates, xxii + 722 pp., ISBN 0-87893-096-5/hbk.

Phoebus J. Dhrymes (2000). *Mathematics for Econometrics*, Third edition. Springer-Verlag, xiv + 240 pp., ISBN 0-387-98995-1/pbk.

Douglas R. Farenick (2001). *Algebras of Linear Transformations*. Springer-Verlag, xi + 238 pp., ISBN 0-387-95062-1/hbk.

M. J. R. Healy (2000). *Matrices for Statistics*, Second edition. Oxford University Press, ix + 147 pp., ISBN 0-19-850703-8/hbk, 0-19-850702-X/pbk.

S. K. Jain & A. D. Gunawardena, with the cooperation of P. B. Bhattacharya (2001). *Linear Algebra with MATLAB Drills*. Key College Publishers, ISBN 1-9301-9016-6/CD.

Carl Dean Meyer, Jr. (2000). *Matrix Analysis and Applied Linear Algebra*, SIAM, xii + 718 pp. + CD, ISBN 0-89871-454-0/hbk + *Solutions Manual* (iv + 171 pp./pbk).

Jakob Riishede Møller (2000). *On Matrix-Analytic Methods and on Collective Risk in Life Insurance*, Doctoral dissertation, Lund University, vi + 125 pp., ISBN 91-628-4384-2.

Hannes Stoppel & Birgit Griese (2001). *Übungsbuch zur linearen Algebra: Aufgaben und Lösungen*, Third revised edition, in German. Vieweg, ix + 286 pp., ISBN 3-528-27288-0/pbk.

F. Szidarovszky & S. Molnar (2001). *Introduction to Matrix Theory: With Applications to Business and Economics*. World Scientific, 420 pp., ISBN 981-02-4504-1/hbk, 981-02-4513-0/pbk.

Richard S. Varga (2000). *Matrix Iterative Analysis*, Second edition. Springer-Verlag, x + 358 pp., ISBN 3-540-66321-5/hbk.

Gareth Williams (2001). *Linear Algebra with Applications*. Fourth edition. Jones & Bartlett, xvii + 647 pp., ISBN 0-7637-1451-8/hbk.

Robin Wilson (2001). *Stamping Through Mathematics*. Springer-Verlag, 128 pp., ISBN 0-387-98949-8/hbk.

The Sinister Ex-Rector

Simo Puntanen's use of the term 'ex-rector' in his report of the 1999 Tampere meeting [IMAGE 23 (October 1999), pp. 12–13] prompts me to enter a caveat against this modern usage of the prefix ex- as it may lead to misunderstandings as I will attempt to ex-plain to a certain ex-tent in a brief ex-tract from a much longer and more ex-act ex-position of the subject that ex-amines many further ex-amples including ex-change, ex-cite, ex-claim, ex-eat, ex-it, ex-on, ex-pert, ex-pose, ex-pound, ex-port, ex-press, and ex-tort. I should also mention the possible confusion between the original and modern meanings of the prefix when it is combined with words of Latin origin as when an ex-director becomes ex- directory or an ex-rector becomes ex-rectory!

I am indebted to my nephew Phillip Carr for drawing this intriguing problem to his attention some years ago by misunderstanding the traditional meaning of the word 'exeat' in a game of charades. Hopefully, this was not the source of his decision not to apply for a place at the University of Ex-eter!

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