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Editor-in-Chief: George P. H. STYAN
gphs@videotron.ca, styan@total.net
Dept. of Mathematics & Statistics, McGill University
805 ouest, rue Sherbrooke Street West
Montréal (Québec), Canada H3A 2K6

Hans Joachim WERNER: *Editor-in-Chief*
werner@united.econ.uni-bonn.de
Institute for Econometrics and Operations Research
Econometrics Unit, University of Bonn
Adenauerallee 24-42, D-53113 Bonn, Germany

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Results of ILAS Elections

The ballots for this year's ILAS elections have now been counted. In accordance with the election rules in the ILAS By-Laws, on March 1, 2002,

Danny HERSHKOWITZ becomes ILAS President (3-year term),
Ravi BAPAT and Miki NEUMANN
become members of the ILAS Board (3-year terms).

We congratulate these winners but we also give our sincere appreciation to all of those who showed their dedication to ILAS by agreeing to stand for election.

We all look forward to working with Danny, Ravi, and Miki in service to our linear algebra community.

I want to personally thank the members of the nomination committee [Roger Horn (Chair), Avi Berman, Tom Laffey, Hans Schneider, and Bryan Shader] for their efforts on behalf of ILAS. I also want to thank Roger Horn, Raphi Loewy, and Judi MacDonald for counting the ballots in a timely manner.

Richard A. BRUALDI, *ILAS President:* brualdi@math.wisc.edu
Dept. of Mathematics, University of Wisconsin–Madison, WI 53706-1388, USA

Some Observations on "A Genealogy of the Spottiswoode Family"

We are indebted to Mrs Dorothy Spottiswoode Doré of "The Spottiswoode Family History Society" (Nottingham, England), for several observations on our recent articles on the genealogy of William Spottiswoode [2] and the Spottiswoode family [3].

First, we should note that our genealogical table [3, p. 19] is not intended to represent a full genealogy of the Spottiswoode family, but only a table that is sufficient to trace the links between three principal members of the family: John (1565–1639), the Archbishop of St Andrews; Alexander (1676–1740), the Lieutenant-Governor of the Colony of Virginia; and William (1825–1883), the author of the first book ever published on determinants (1851) and President of the Royal Society (1878–1883). For a far more detailed, if still incomplete, genealogy of the Spottiswoode family, we again refer to the tables in the frontal matter of the book by Barry & Hall [1, pp. xii–xxiii]. We understand that these tables may be considerably extended using information in the records Mrs Doré has on file.

Second, we should note that the seven Johns in our genealogy [3, p. 19] were numbered for the convenience of our readers, and that these numbers have no other significance. In particular, they do not indicate the passage of the lairdship of Spottiswoode.

Third, the ancestry of our first John (1510–1585), superintendent of the Church of Scotland, is not known. However, it seems certain that he was not a son of the William Spottiswood of Spottiswoode (actually Archibald) who died at Flodden Field in 1513. This William should therefore be deleted from the head of the genealogical table and from the opening paragraphs of the article [3, p. 19].

Fourth, the entail of 1863 was drawn up by our sixth John (1780–1866) to protect his wife Helen Wauchope and his widowed daughter Alicia Ann, see Barry & Hall [1, pp. xxii–xxiii], with the added proviso that those following were required to take the name of 'Spottiswoode'.

Fifth, the undifferenced arms of the Spottiswoodes of Spottiswoode illustrated in [2, p. 19] are those of the lairds of Spottiswoode including our fifth John (1741–1805) and our sixth John (1780–1866). As the son of a younger brother of our sixth John, William (1825–1883) was obliged to matriculate a distinct set of arms with the Office of the Lord Lyon in Edinburgh. These arms are illustrated in a special form in an armorial panel in the Speech Room of Harrow School. In this panel, the arms of William Spottiswoode (1825–1883) are *impaled* (set side-by-side) with those of the Royal Society of London. The arms of the Royal Society appear in the more prestigious position on the left of the shield as seen (or on the right as worn), and William's own on the right (or on the left as worn).

Finally, we note, with Barry & Hall [1, pp. 3–4] that, however spelt, the name of Spottiswoode should be pronounced 'Spotswood' as the letters 'is' represent the old Scottish genitive case usually represented by 'his' in English. Mackenzie [4, pp. 108, 312] notes two contemporary documents referring to 'Godis holy word' and to 'Jesus Christ his birth, passion, resurrection'.

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[1] T. Barry & D. Hall (1997). *Spottiswoode: Life and Labour on a Berwickshire Estate, 1753–1793.* Introduction by Douglas Hall and Commentary by Tom Barry. Tuckwell Press, East Linton, East Lothian, Scotland.

[2] R. W. Farebrother (1999). A genealogy of William Spottiswoode 1825–1883. *Image*, 23, 3–4.

[3] R. W. Farebrother & G. P. H. Styan (2000). A genealogy of the Spottiswoode Family: 1510–1900. *Image*, 25, 19–21.

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R. William FAREBROTHER: Msrbsrf@fs1.ec.man.ac.uk
11 Castle Road, Bayston Hill, Shrewsbury, England SY3 0NF

George P. H. STYAN: gphs@videotron.ca
Dept. of Mathematics and Statistics, McGill University
Montréal (Québec), Canada H3A 2K6

Magic Squares, Melancholy and The Moore–Penrose Inverse

Dietrich Trenkler and Götz Trenkler

1. Introduction

A couple of years ago, the second author came across the article by Booth & Booth (1955), where it is shown that the inverse of the matrix for a semi-magic square, i.e., a matrix whose row and column sums are all identical, is also semi-magic. Having been interested in this topic for a long time, he asked himself if a similar statement was possible for the magic square which can be found in the famous engraving by Albrecht Dürer (1471–1528) entitled "Melencolia" (Fig. 1). The date of the work, 1514, which is also the year of Dürer's mother's death, appears in the middle of the fourth row.

From the classic book by William Symes Andrews (1960, p. 146) we quote the following wonderful description:

> The symbolism of this engraving has interested to a marked degree almost every observer. The figure of the brooding genius sitting listless and dejected amid her uncompleted labors, the scattered tools, the swaying balance, the flowing sands of the glass, and the magic square of 16 beneath the bell — these and other details reveal an attitude of mind and a connection of thought, which the great artist never expressed in words, but left for every beholder to interpret for himself.

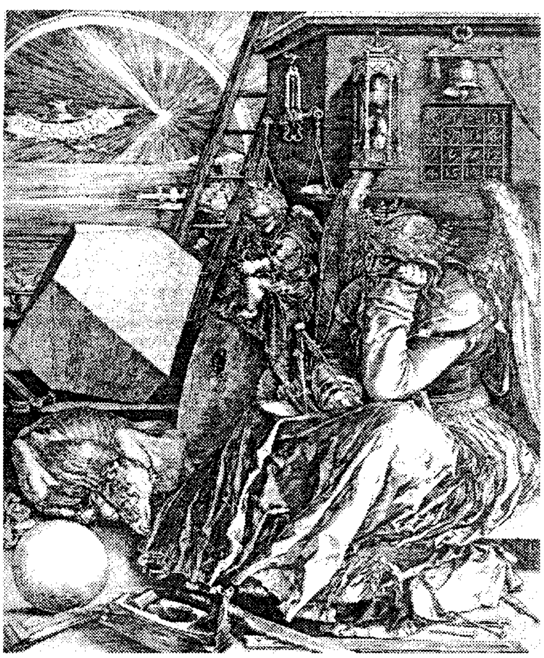


FIG. 1: Dürer's "Melencolia".

Indeed, the square beneath the bell (see Fig. 2) has some remarkable properties. Column, row and diagonal sums are all equal to 34 coinciding with the sum of elements of the four 2×2 corner subsquares. We will call 34 the "magic constant."

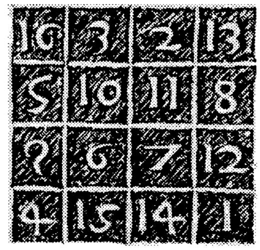


FIG. 2: Dürer's magic square.

There are still 72 more patterns of numbers with sum 34 detectable in Dürer's magic square. All patterns with sum 34 are given in Fig. 3 (below).

The matrix for Dürer's magic square,

$$D = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix},$$

is singular with rank 3. The result by Booth & Booth (1955) is, therefore, not applicable here. Nevertheless, one might ask if the Moore–Penrose inverse (MP-inverse) of D also has magical properties. This was answered in the affirmative by Trenkler & Knautz (1991), who identified D^+ as

$$D^+ = \frac{1}{34 \cdot 80} \begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}.$$

The row, column and diagonal sums of D^+ are $1/34$, but there are more magical patterns in D^+, as we will see in Fig. 4 below.

It is easy to verify that $DD^+ \neq D^+D$. Hence the matrix D for Dürer's magic square is not EP, and consequently the Drazin inverse $D^\#$ and MP-inverse D^+ differ. Recall that a matrix is EP if and only if its column space (range) coincides with the column space of its transpose (Campbell & Meyer, 1979, ch. 4 & 7). Nevertheless $D^\#$ turns out to be magic.

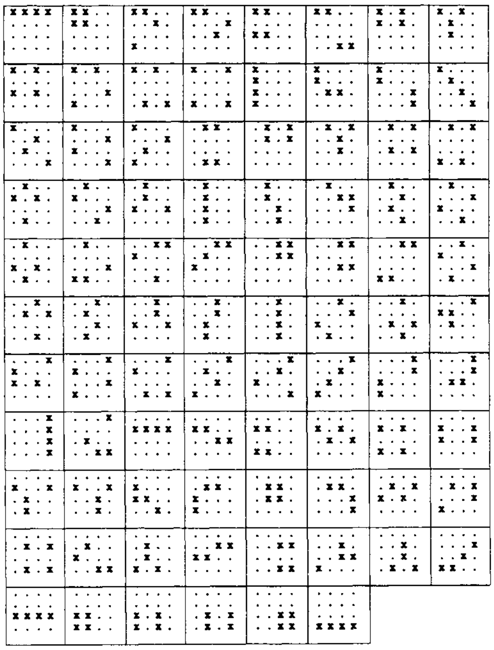


FIG. 3: Patterns in D with sum 34.

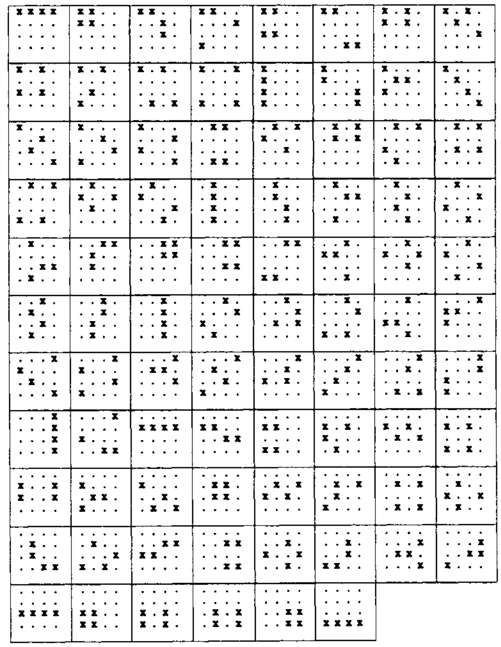


FIG. 4: Patterns in D^+ with sum 1/34.

We find that

$$D^\# = \frac{1}{34 \cdot 64}\begin{pmatrix} 271 & -171 & -205 & 169 \\ -103 & 67 & 101 & -1 \\ 33 & -69 & -35 & 135 \\ -137 & 237 & 203 & -239 \end{pmatrix}.$$

Surprisingly enough, the patterns of numbers in $D^\#$ adding to $1/34$ are the same as those of D (Fig. 3). The matrix D for Dürer's magic square and its Moore–Penrose inverse D^+ have 52 patterns in common, and there are 34 patterns occurring only in D and 34 patterns to be seen only in D^+.

The magic square of type $n \times n$ in its classical form contains the integers $1, 2, \ldots, n^2$ such that every row, column and the two main diagonals add to $n(n^2+1)/2$. In early history such squares were devised in Arabia, China, Greece, India and Japan (Hirayama & Abe 1983), where they also served as charms against all kinds of ailments. One of these squares, also known as "Loh-Shu" or "scroll of the river Loh", is attributed[1] to the Great Emperor Yu, c. 2000 B.C. According to a Chinese legend, the Great Emperor Yu once was standing by the Yellow River when a tortoise rose up from the water, delighting the Emperor with the following array on its back:

4	9	2
3	5	7
8	1	6

According to Ball & Coxeter (1987), this array is sometimes to be seen on the deck of large passenger ships, for scoring in such games as shuffleboard.

The introduction of magic squares into Europe may be attributed to the Greek mathematician Emanuel Moschopulus, who lived in Constantinople in the fifteenth century (Ball & Coxeter, 1987).

Methods for constructing magic squares are described extensively in the book by Moran (1982). The easiest method, also described there, is applicable to any square of odd order n. This so-called pyramid method starts with the desired, but empty, square which is then augmented by additional boxes so that one obtains the "pyramid pattern", which is considered here for $n = 3$ and $n = 5$:

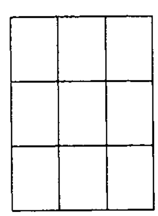


FIG. 5: Empty box.

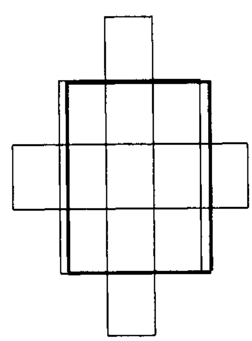
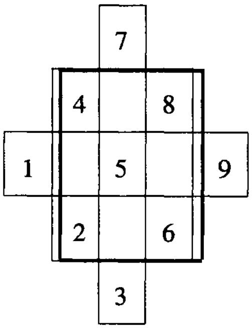


FIG. 6: Pyramid pattern.

We subsequently enter the numbers $1, 2, \ldots, 9$ in the pyramid of Fig. 6 to get the following arrangement of Fig. 7:

[1] Also attributed to Fuh-Hi, the mythical founder of Chinese civilisation, c. 2800 B.C.



4	3	8
9	5	1
2	7	6

FIG. 7: Filled-up pyramid. FIG. 8: 3×3 magic square.

The final magic square is obtained by reflecting the outside numbers 1, 3, 7 and 9 to the empty positions opposite the original location. How this method works for $n = 5$ is illustrated in Figs. 9 and 10.

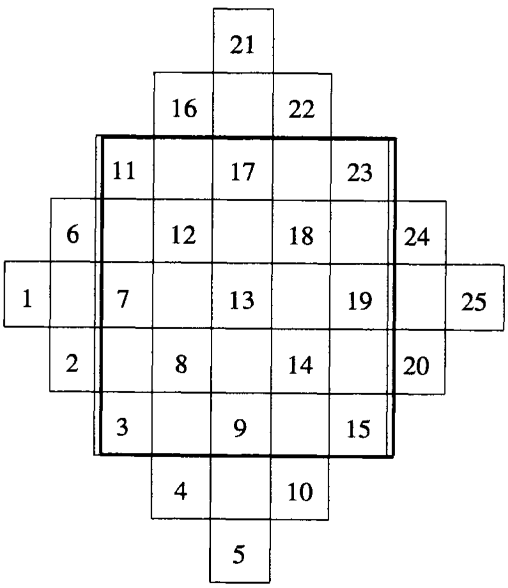


FIG. 9: Filled-up pyramid.

More than 10 additional methods for constructing odd-order magic squares are discussed in the book by Moran (1982), including the famous method of the French writer Simon de La Loubère (1642–1729) presented in his *Relation du Royaume de Siam* published in 1693.

The formation of even-order magic squares is more challenging. It appears that there is no universal method in sight which is suitable for designing even-order magic squares of any size. Each even number has its own peculiarities, as beautifully illustrated in Moran's book.

One of the most interesting 8×8 semi-magic squares is due to a Mr. Beverly and was published in 1848 (Fig. 11). Its horizontal and vertical sums are 260. The four 4×4 corner squares are also semi-magic with magic constant 130. Furthermore, in each 4×4 corner square the four 2×2 subsquares also add to 130. Beginning in the upper left-hand corner, it is possible to make knight moves as in chess and follow the integers 1 through 64 in numerical order (Fig. 12). By its symmetry, this knight's path is remarkable, indeed!

11	4	17	10	23
24	12	5	18	6
7	25	13	1	19
20	8	21	14	2
3	16	9	22	15

FIG. 10: 5×5 magic square.

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

FIG. 11: Beverly's semi-magic square.

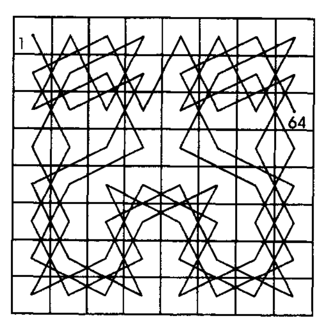


FIG. 12: Knight's tour.

Another kind of symmetry can be found in the reversible magic square, which to the best of our knowledge first appeared in Lanners (1977, p. 75) and was rediscovered by James & James (2000), see our Fig. 13. If we turn it upside down we detect another magic square!

96	11	89	68
88	69	91	16
61	86	18	99
19	98	66	81

FIG. 13: Reversible magic square.

Is it possible to construct magic squares containing prime numbers only? Yes! The famous British puzzle expert Henry Ernest Dudeney (1857–1930) found the following array in 1900 where 1 is regarded as a prime number (Devendran, 1990, p. 25):

7	61	43
73	37	1
31	13	67

FIG. 14: Magic square with prime numbers.

There also exists a 3×3 magic square made with consecutive primes, albeit pretty big ones! In 1987 Martin Gardner (see Gardner 1996) offered \$100 to anyone who found such a square. Harry Nelson of Lawrence Livermore Laboratories was the successful winner who used a Cray computer to produce the following "simple" square:

1,480,028,201	1,480,028,129	1,480,028,183
1,480,028,153	1,480,028,171	1,480,028,189
1,480,028,159	1,480,028,213	1,480,028,141

FIG. 15: Magic square with consecutive primes.

According to Domenicano & Hargittai (2000) there is in Italy, close to the small town of Capestrano (in the heart of Abruzzo 40 km from L'Aquila), at the entrance of the Church of San Pietro and Oratorium, a well-preserved 5×5 alphabetical magic square, consisting of the following five words: ROTAS, OPERA, TENET, AREPO and SATOR (Fig. 16).

R	O	T	A	S
O	P	E	R	A
T	E	N	E	T
A	R	E	P	O
S	A	T	O	R

FIG. 16: 5×5 alphabetical magic square of Capestrano.

Four of the five words are Latin. AREPO is the exception, though it recalls the word ARATRO, the ablative of the Latin word for plough. The symmetry of the array becomes evident when reading ROTAS and OPERA backwards to get AREPO and SATOR. TENET reads the same from either end, i.e., it is palindromic.

2. Semi-Magic Squares

The matrix A denoting a semi-magic square may be characterized by the properties

$$Ae^{(n)} = se^{(n)} = A'e^{(n)},$$

where A' is the transpose of A, $e^{(n)}$ denotes the vector of n ones and s is the magic constant. The entries of A are real numbers, not necessarily integers. If the sums of the two diagonals are also equal to s, the matrix A represents a magic square. Hence the class of magic squares is contained in the class of semi-magic squares. Semi-magic squares have been continually discussed over the years, see, e.g., Weiner (1955), Khan (1957), Watson (1966) and Mayoral (1996). Other subclasses are the double-centered matrices where $s = 0$, see Sharpe & Styan (1965) and Styan & Subak-Sharpe (1997), and the doubly-stochastic matrices with $s = 1$ and all entries of A being nonnegative, see Montague & Plemmons (1973) and Minc (1988).

If $s(A)$ denotes the magic constant s of A, we can state the following properties

$$\begin{aligned} s(A+B) &= s(A) + s(B) \\ s(\alpha A) &= \alpha s(A) \\ s(A') &= s(A) \\ s(AB) &= s(A) \cdot s(B), \end{aligned}$$

where A and B are matrices for semi-magic squares of type $n \times n$ and α is a real number. Hence the set of semi-magic squares of the same type $n \times n$ forms a subalgebra R_n of the set of all real $n \times n$ matrices. As indicated in our introduction, Booth & Booth (1955) showed that the inverse of the matrix for a nonsingular semi-magic square A is semi-magic, again with the magic constant $1/s(A)$. In Schmidt & Trenkler (2001) this result is extended to singular semi-magic squares. Indeed, they showed that if A is semi-magic, then A^+ is semi-magic with magic constant $s(A)^+$. Here A^+ denotes the M-inverse of A, and $s(A)^+ = 0$ if $s(A) = 0$ and $s(A)^+ = 1/s(A)$, otherwise.

Consider now

$$M_n = \{A | A = \alpha e^{(n)} e^{(n)'}, \alpha \in \mathbb{R}\}$$

and

$$N_n = \{A | A \in R_n \text{ and } s(A) = 0\}.$$

It is clear that M_n and N_n are both ideals of R_n. As stated in Weiner (1955), the algebra R_n is the direct sum of these two ideals. Furthermore, the matrices $A_{ij} = (e_1 - e_i)(e_1 - e_j)'; i, j = 2, \dots, n$, where the e_i are the members of the canonical basis of $\mathbb{R}^n$, generate N_n. Since the A_{ij} are linearly independent, the algebra N_n has dimension $(n-1)^2$.

$$N = \begin{pmatrix} k+l+m+n & -l-m & -k-n \\ -k-l & l & k \\ -m-n & m & n \end{pmatrix}. \tag{2}$$

We note that the magic constant $s = s(A)$ since $Ne^{(3)} = 0 = N'e^{(3)}$. From this we get $((s/3)e^{(3)}e^{(3)'})N = N((s/3)e^{(3)}e^{(3)'}) = 0$, and hence $A^+ = (s^+/3)e^{(3)}e^{(3)'} - N^+$.

If $\rho = ln - km \neq 0$, then the MP-inverse of N is $N^+ = \frac{1}{9\rho} \times$

$$\begin{pmatrix} -k+l-m+n & -k+l+2m-2n & 2k-2l-m+n \\ 2k+l-m-2n & 2k+l+2m+4n & -4k-2l-m-2n \\ -k-2l+2m+n & -k-2l-4m-2n & 2k+4l+2m+n \end{pmatrix}.$$

For $\rho = 0$ the MP-inverse of N is also calculated in Groß et al. (1999), along with rank, eigenvalues and eigenvectors of A (for arbitrary ρ).

The matrix B for Beverly's 8×8 semi-magic square B (Fig. 11) has rank 5 and MP-inverse

$$B^+ = \frac{1}{6928663040} \begin{pmatrix} -42857041 & -8977065 & 22454127 & 42704331 & -31517297 & 5342929 & 11114383 & 28384337 \\ 22454127 & 42663095 & -42857041 & -8935829 & 12954143 & 29894209 & -33357057 & 3833057 \\ 4085361 & 16299017 & 29641905 & -36701931 & 8743313 & -22838705 & 24983953 & 2435791 \\ 29641905 & -36660695 & 4085361 & 16257781 & 23144193 & 925919 & 10583073 & -21328833 \\ 2576815 & 43322871 & -22979729 & -9595605 & -16482017 & 5736257 & -3920897 & 27991009 \\ -22979729 & -9636841 & 2576815 & 43364107 & -2081137 & 29500881 & -18321777 & 4226385 \\ 49519217 & -36000919 & -15791951 & 15598005 & -6291967 & -23232033 & 40019233 & 2829119 \\ -15791951 & 15639241 & 49519217 & -36042155 & 38179473 & 1319247 & -4452207 & -21722161 \end{pmatrix}.$$

We observe that for $A \in R_n$ and $\alpha \neq 0$ we have

$$\left(A + \alpha e^{(n)} e^{(n)'}\right)^+ = A^+ + \beta e^{(n)} e^{(n)'},$$

where $\beta =$

$$\begin{cases} \frac{1}{\alpha n^2} & \text{if} \quad s(A) = 0 \\ -\frac{1}{ns(A)} & \text{if} \quad s(A) \neq 0 \text{ and } s(A) + \alpha n = 0 \\ -\frac{1}{s(A)(s(A)+\alpha n)} & \text{if} \quad s(A) \neq 0 \text{ and } s(A) + \alpha n \neq 0 \end{cases}.$$

As Dürer's magic square D is also semi-magic, then by this result we obtain

$$(D + \delta e^{(n)} e^{(n)'})^+ = D^+ + \gamma e^{(n)} e^{(n)'}$$

for given $\delta \in \mathbb{R}$ and for some γ. Thus the rich pattern of D^+ is inherited by $(D + \delta e^{(n)} e^{(n)'})^+$.

Groß et al. (1999) studied semi-magic squares of type 3×3 and showed that the associated matrices have the form

$$A = (s/3)e^{(3)} e^{(3)'} - N, \tag{1}$$

where

All rows and columns of B^+ add to $1/260$. Its four 4×4 corner squares are semi-magic again with magic constant $1/520$.

The knight-moves structure has disappeared, with two exceptions. The largest entry $49519217/6928663040$ occurs twice, and the distance between the positions of these two entries is a knight's move. The same is true for true for the smallest entry $-42857041/6928663040$ which again occurs twice and the distance between the positions of these two entries is again a knight's move.

The eigenvalues of B are $0, 0, 0, 260$ plus the four roots of the polynomial

$$x^4 - 22x^3 - 3784x^2 + 113792x + 172032.$$

These four roots are all real, two of them negative.

Finally we should mention Benjamin Franklin (1706–1790), the U.S. statesman, scientist and writer. Well-known for the invention of the lightning conductor, he is less renowned for his semi-magic squares, but see Marder (1940) and Murphy & Walz (2001). The matrix F for Franklin's 16×16 square (Andrews, 1960, p. 97) has rank 3 and magic constant 2056; the row and column sums of the MP-inverse F^+ are, of course, $1/2056$. A comprehensive report on Franklin's semi-magic squares can be found in the recent article by Pasles (2001).

3. Magic Squares

The matrix M for a magic square of type $n \times n$ with magic constant s is characterized by the following four equations

$$\begin{array}{lrcl} \text{(i)} & Me^{(n)} & = & se^{(n)} \\ \text{(ii)} & M'e^{(n)} & = & se^{(n)} \\ \text{(iii)} & \operatorname{tr}(M) & = & s \\ \text{(iv)} & \operatorname{tr}(RM) & = & s, \end{array}$$

where $\operatorname{tr}(\cdot)$ denotes trace and $R = (r_{ij})$ is the $n \times n$ reflection matrix with entries $r_{i,n-i+1} = 1$ and $r_{ij} = 0$ if $j \neq n-i+1$; $i,j = 1,\ldots,n$. If M is a magic square of type 3×3, it can be written as

$$M = (s/3)e^{(3)}e^{(3)'} + N,$$

where

$$N = \begin{pmatrix} u & v-u & -v \\ -u-v & 0 & u+v \\ v & u-v & -u \end{pmatrix},$$

see Wardlaw (1992) and Trenkler (1994), and (1) and (2) above.

We observe that $\det(M) = \rho \cdot s$, where $\rho = 3(u^2 - v^2)$. Hence M is singular if and only if $\rho = 0$ or $s = 0$. In the nonsingular case we have

$$M^{-1} = \frac{1}{3s}e^{(3)}e^{(3)\prime} + \frac{1}{\rho}N.$$

We now consider the singular case.

(a) $\rho \neq 0,\ s = 0$. Then $M = N$ and $M^+ = N^+ = (1/\rho)N$, where we used the identities $NN = \rho(I_3 - (1/3)e^{(3)}e^{(3)\prime})$ and $NNN = \rho N$.

(b) $\rho = 0,\ v = u$. Then $M^+ = (s^+/3)e^{(3)}e^{(3)'} + (u^+/12)ba'$, where $a = (1,-2,1)'$ and $b = (1,0,-1)'$.

(c) $\rho = 0,\ v = -u$. Then we have $M^+ = (s^+/3)e^{(3)}e^{(3)'} + (u^+/12)ab'$, with a and b as in (b).

From these formulas it becomes clear that for the matrix M of 3×3 magic squares the inverse M^{-1} or MP-inverse M^+ is also magic.

Magic squares of type 4×4 behave differently. The matrix

$$G = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -3 & 1 & 1 \end{pmatrix}.$$

for the 4×4 magic square given by Groß (2000) has rank 3. The MP-inverse is

$$G^+ = \frac{1}{16}\begin{pmatrix} -3 & -3 & 5 & 1 \\ 1 & 1 & 1 & -3 \\ 5 & -3 & -3 & 1 \\ -3 & 5 & -3 & 1 \end{pmatrix},$$

which obviously is not magic.

In general, the matrix for 4×4 magic squares has the form

$$M = \begin{pmatrix} a+b & f+h & e+g-h & c+d \\ c+e & d+g & a+f & b \\ d+f & b+e & c & a+g \\ g & a+c-h & b+d+h & e+f \end{pmatrix}.$$

If $h = 0$, these magic squares have the additional property that the entries in the four 2×2 corner squares also add to the magic constant $s = s(M) = a+b+c+d+e+f+g$. For instance, the matrix G is of this type. We call such magic squares "super-magic squares". There exist super-magic squares whose MP-inverse is also super-magic. Consider the class of matrices

$$S = \begin{pmatrix} -c-f-g & f & g & c \\ c & g & f & -c-f-g \\ f & -c-f-g & c & g \\ g & c & -c-f-g & f \end{pmatrix}.$$

Clearly, S is the special case of M with $a = d = e = h = 0$ and $b = -c-f-g$. The characteristic polynomial of S is

$$p(\lambda) = [8(c+f)(c+g)(f+g) + \lambda^3]\lambda,$$

and so S is singular.

Let us now assume that $\alpha = (c+f)(c+g)(f+g) \neq 0$. Then

$$S^+ = \frac{1}{8\alpha}\begin{pmatrix} a_1+b_1 & f_1 & e_1+g_1 & c_1+d_1 \\ c_1+e_1 & d_1+g_1 & a_1+f_1 & b_1 \\ d_1+f_1 & b_1+e_1 & c_1 & a_1+g_1 \\ g_1 & a_1+c_1 & b_1+d_1 & e_1+f_1 \end{pmatrix},$$

where $a_1 = -4cf - 4cg - 4fg - 2f^2 - 2g^2$, $d_1 = 2c^2 - 2g^2$, $e_1 = 2c^2 - 2f^2$ and $b_1 = c_1 = f_1 = g_1 = -c^2 + f^2 + fg + g^2 + c(f+g)$. Hence S^+ is also super-magic.

It might be interesting to identify all cases of magic squares of order 4 such that their MP-inverse is magic again. This is postponed for future research.

4. Most Perfect Pandiagonal Magic Squares

Let us consider the 4×4 matrix

$$P_0 = \begin{pmatrix} 1 & 15 & 4 & 14 \\ 8 & 10 & 5 & 11 \\ 13 & 3 & 16 & 2 \\ 12 & 6 & 9 & 7 \end{pmatrix}.$$

Obviously, P_0 represents a magic square. But P_0 has other properties. All its 6 "broken" diagonals

$$(1,6,16,11) \quad (8,15,9,2) \quad (13,10,4,7)$$

$$(14,8,3,9) \quad (4,11,13,6) \quad (15,5,2,12)$$

also have magic constant $s(P_0) = 34$ as their sum. Magic squares in which all the broken diagonals add to the magic constant are called "pandiagonal". But our P_0 here has still more to offer. As with the matrix D for Dürer's magic square, the sum of the four 2×2 corner squares of P_0 is 34. Thus P_0 is super-magic.

Following Ollerenshaw & Brée (1998), we call a pandiagonal super-magic square a most-perfect pandiagonal magic square, or an MPPM-square.

We will now consider only 4×4 MPPM-squares. It is readily seen that they are of the form

$$P = \begin{pmatrix} a+b+e & c+d+e & a+c & b+d \\ a+c+d & b & a+b+d+e & c+e \\ b+d+e & a+c+e & c+d & a+b \\ c & a+b+d & b+e & a+c+d+e \end{pmatrix}$$

with magic constant $s = s(P) = 2(a+b+c+d+e)$. The eigenvalues of P are given by

$$\lambda_1 = 0,\ \lambda_2 = s,$$

$$\lambda_3 = -2\sqrt{ae+cd-bd},\ \lambda_4 = -2\sqrt{ae+cd-bd} = -\lambda_3.$$

Thus P here is always singular; when $ae + cd < bd$, then two eigenvalues are always purely imaginary.

Chater & Chater (1949) proved that MPPM-squares of even order are singular in general, in accordance with our result. As seen above in our Section 3, the MP-inverse of a magic square need not be magic. MPPM-squares of order 4 behave differently! Indeed, the MP-inverse of an MPPM-square is an MPPM-square again.

Let $a+b+c+d+e \neq 0,\ (b-c)^2+e^2 \neq 0$ and $a^2+d^2 \neq 0$, then

$P^+ =$

$$\frac{1}{8}\begin{pmatrix} a_1+b_1+e_1 & c_1+d_1+e_1 & a_1+c_1 & b_1+d_1 \\ a_1+c_1+d_1 & b_1 & a_1+b_1+d_1+e_1 & c_1+e_1 \\ b_1+d_1+e_1 & a_1+c_1+e_1 & c_1+d_1 & a_1+b_1 \\ c_1 & a_1+b_1+d_1 & b_1+e_1 & a_1+c_1+d_1+e_1 \end{pmatrix},$$

where $a_1 = 2\beta e,\ b_1 = -(a+d)\gamma+\alpha-\delta\beta,\ c_1 = (d-a)\gamma+\alpha-\delta\beta,\ d_1 = 2\beta(c-b)$ and $e_1 = 2a\gamma$ with the additional constants $\alpha = (a+b+c+d+e)^{-1},\ \beta = [(b-c)^2+e^2]^{-1},\ \gamma = (a^2+d^2)^{-1}$ and $\delta = c+e-b$. It can be shown that in the other cases $a+b+c+d+e = 0$ or $(b-c)^2+e^2 = 0$ or $a^2+d^2 = 0$ the MP-inverse of P is MPPM again. This will be reported later.

More details about MPPM-squares can be found in the marvellous book by Ollerenshaw & Brée (1998), which is highly recommended to anyone who is interested in pandiagonal magic squares.

Acknowledgements

We are indebted to Jürgen Groß and Sven-Oliver Troschke for conversations, suggestions and computational help. Heinrich Hemme helped us trace the origin of the reversible magic square of Fig. 13. Furthermore, we offer thanks to Nina Buschmeyer, Irina Czogiel and Johanna Puscher for their technical help, and to Eva Brune for her conscientious typing of the manuscript. Special thanks go to Jürgen Kühl for allowing us to reproduce Figures 1 and 2 from the Web site http://www.informatik.uni-hamburg.de/bib/ausstellung/Tafel01.html and to George Styan for inviting us to write this paper.

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Dietrich TRENKLER: dtrenkler@nts6.oec.uni-osnabrueck.de
Dept. of Economics, University of Osnabrück, D-49060 Osnabrück, Germany

Götz TRENKLER: trenkler@statistik.uni-dortmund.de
Dept. of Statistics, University of Dortmund, D-44227 Dortmund, Germany

Paul Daniels and Spaces of Magic Squares

I did not know that the MP-inverse of a magic square may not be a magic square. In my undergraduate lectures on magic squares I sometimes mention Paul Daniels's magic square—Paul Daniels (b. 1938) is a well-known British magician and TV personality who I often saw on Canadian TV in the 1970s. I copied the matrix for Daniels's magic square once from TV. It is:

$$\begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix}.$$

The MP-inverse of this magic square IS a magic square!

One of my favourite exercises for linear algebra undergraduates is to replace a given set of, say, three 3×3 magic squares, by an orthogonal set of magic squares using Gram–Schmidt and the usual inner product $\mathrm{tr}(A'B)$. So, magic squares have their place in linear algebra courses if only for illustrations.

I wonder what is known about spaces of magic squares and their dimensions. For example, the dimension of 4×4 magic squares is 8, but I wonder if the dimension is known for 5×5 magic squares, etc. I think that this might still be an open question, see T. J. Fletcher (1972), *Linear Algebra: Through its Applications*, Van Nostrand Reinhold, London.

Sanjo ZLOBEC: zlobec@math.mcgill.ca
Dept. of Mathematics and Statistics, McGill University
Montréal (Québec), Canada H3A 2K6

Magic Squares on the Web and the Determinants of 4 × 4 Magic Squares

Possibly my favourite Web site for bibliographic research is AMICUS Web from the National Library of Canada:

http://amicus.nlc-bnc.ca:80/aaweb/amilogine.htm?

For "Title Keyword" = Magic Squares, I found 39 records.

For Andrews (1960) I discovered that the "other writers" are Paul Carus, L. S. Frierson and C. A. Browne, Jr., and that the book includes an introduction by Paul Carus. The 1960 Dover reprint edition is an "Unabridged and unaltered publication of the second edition published by Open Court, Chicago, in 1917". The first edition was published in 1908.

Marder (1940) is "The first of a series of four papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes, under the general title 'The intrinsic harmony of number' ". Also included is "The simple method of raising any 8×8 square to the 16×16 size with the aid of an auxiliary square preserving all the qualities of the original" and "Three entirely new methods of producing the bent diagonals of Benjamin Franklin".

Harvey Heinz of Surrey, B. C., Canada, maintains a wonderful 25-page Web site on magic squares at

http://www.geocities.com/CapeCanaveral/Launchpad/4057/

© Harvey D. Heinz. This Web site links directly to an extensive annotated bibliography of more than 160 items on magic squares (and magic stars), Franklin squares, and order-5 pandiagonal magic squares. Here I found that in Andrews (1960) "The essays which comprise this volume appeared first in an American journal called *The Monist* between 1905 and 1916". Ball & Coxeter (1987) was "Originally published in 1892, H. S. M. Coxeter brought it up to date with the 1938 publication of the 11th edition, the 12th edition in 1974 and edition 13 in 1987. Chapter 7 is on magic squares".

Furthermore Heinz & Hendricks (2000) has 228 pp., 171 captioned illustrations and tables, 239 terms defined, 2 appendices of bibliographies, ISBN 0-9687985-0-0. It is available postage paid (in North America) for C\$32.00 or US\$25.00 per copy from H. D. Heinz, 15450 92A Avenue, Surrey, B. C., Canada V3R 9B1; harveyheinz@home.com

From the Heinz Web site I was led to the Japanese Web site

http://www.pse.che.tohoku.ac.jp/~msuzuki/MagicSquare.html

run by Matsumi Suzuki. Here I found a photograph taken at Gaudi's cathedral in Barcelona of a magic square with magic constant 33 "which is popularly acknowledged as the age at which Jesus Christ died".

The most helpful Web site © Glenn Fleishman

http://isbn.nu/cgi-bin/design-books

allows one to search for books and compare prices. If the ISBN (International Standard Book Number) is known then one can go directly to the price comparisons. For Pickover (2001), visiting isbn.nu/0-691-07041-5 showed prices ranging from US\$17.00–\$23.88 and availability from 1–5 days to 6–15 days.; visit isbn.nu/toc/0-691-07041-5 for the table of contents. The ISBN may be entered with or without hyphens. For "magic squares" in the title I found 13 books including

Shen, C. T. (1989). *General Solutions for Even Order Magic Squares.* Victory Press, Monterey, CA.

The "Zentralblatt MATH" Web site © European Mathematical Society, FIZ Karlsruhe & Springer-Verlag

http://www.emis.de/ZMATH/

has a "Demo mode with a limit of max. 3 allowed answers for unregistered users". I found 9 books on "magic squares" including

Yu. V. Chebrakov (1995). *Magic Squares: Number Theory, Algebra, Combinatorial Analysis* (in Russian). Populyarnye Lektsii po Matematike. Sankt-Peterburgskij Gosudarstvennyj Tekhnicheskij Universitet, ISBN 5-7422-0015-3, Zbl 0879.05013.

I also found 9 books on "magic squares" in "MathSciNet: *Mathematical Reviews* on the Web" © American Mathematical Society

http://www.ams.org/mathscinet/search

including

J. Sesiano, (1996). *Un Traité Médiéval sur les Carrés Magiques* (in French). Annotated edition and translation of an anonymous Arabic text describing various modes of construction. Presses Polytechniques et Universitaires Romandes, Lausanne, ISBN 2-88074-310-9, MR 98e:01007.

The MathPages Web site

http://mathpages.com/

has several interesting articles on magic squares. In "The Determinants of 4×4 Magic Squares" it is observed that "Up to sign, there are only 12 distinct determinants for 4×4 magic squares (using the elements 0 to 15)". The numbers of squares having each of these determinants are listed there.

Moreover, the author then wonders "if all the magic squares (of a given order) with the same determinant constitute a group under some set of 'simple' transformations. It's certainly true for the sets of 32, since these can be shown to constitute a group under rotation, reflection, etc. But what about the larger sets?

Interestingly, the unique (up to sign and 32-group transformations) square with determinant 6720 is the only square whose determinant is divisible by any prime other than 2, 3, or 5.

It's also interesting to experiment with various matrix operations on two magic squares A, B. For example, the squares given by ABA^{-1} have the same row and column sums as A and B. The commutator $AB - BA$ also gives some interesting results, as does replacing each element of a square A with its cofactor."

George P. H. STYAN: gphs@videotron.ca
Dept. of Mathematics and Statistics, McGill University
Montréal (Québec), Canada H3A 2K6

The Naming of Parts

Some recent research suggests that the following terminology may be useful:

If the real matrix $B = -A$ is idempotent, i.e., $B^2 = B$, then A is "skew idempotent": $A^2 = -A$.

If the imaginary matrix $C = iA$ is tripotent, i.e., $C^3 = C$, then A is "skew tripotent": $A^3 = -A$.

If the imaginary matrix $C = iA$ is involutory i.e., $C^2 = I$, then A is "skew involutory": $A^2 = -I$.

Moreover "involutory" not "involutary".

Readers may have other suggestions.

R. William FAREBROTHER: Msrbsrf@fs1.ec.man.ac.uk
11 Castle Road, Bayston Hill, Shrewsbury, England SY3 0NF

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New & Forthcoming Books on Linear Algebra & Related Topics

Listed here are some new and forthcoming books on linear algebra and related topics that have been published in 1999–2001 or are scheduled for publication in 2002. This list updates and augments our previous lists in IMAGE 23 (October 1999), pp. 10–11, and 26 (April 2001), page 32.

In identifying these books we have visited the Web sites for various publishers, as well as AMICUS Web, isbn.nu, MathSciNet and Zentralblatt MATH, which also helped us find books on magic squares, see pp. 10–11 above (where Web addresses are given). "Telegraphic Reviews" from *The American Mathematical Monthly* were also helpful. Pagination and ISBNs are given when known. Please send all additions and corrections to George Styan: styan@total.net

S. N. Afriat (2000). *Linear Dependence: Theory and Computation.* Kluwer/Plenum, New York: xv + 175 pp., ISBN 0-306-46428-4.

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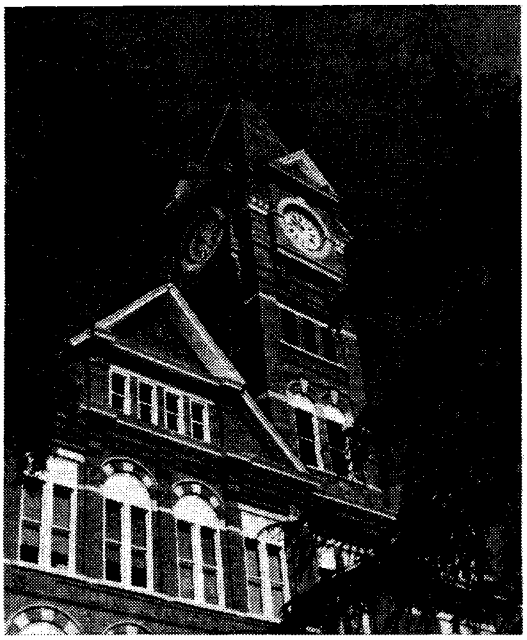
CHALLENGES IN MATRIX THEORY



10th ILAS Conference

Department of Mathematics

Auburn University

Auburn, AL 36849–5310, USA

June 10 - 13, 2002

Call for papers: **Abstracts** for *contributed and invited papers* on any aspect of Linear Algebra and Matrix Theory are due

March 15, 2002

electronically at ilas2002@auburn.edu .

Program: The task of this conference is to bring together theoreticians and practitioners of Linear Algebra and Matrix Theory to describe and exchange ideas about current research and to develop future challenges in the field.

Invited Speakers: T. Ando, Japan; H. Bart, Netherlands; M. Benzi, Emory; R. Brualdi, Wisconsin; D. Calvetti, Case Western; G. Chartier, France; M. Choi, Canada; M. Chu, NC State; G. Dahl, Norway; P. v. d. Driessche, Canada; S. Fallat, Canada; M. Goldberg, Israel; J. Groß, Germany; M. Kilmer, Tufts; L. Lerer, Israel; M. Neubauer, CalState; O. Nevanlinna, Finland; M. Overton, Courant Institute; L. Qui, China; C. Rao, Penn State; F. Roush, Alabama State; H. Schneider, Wisconsin; A. Sierpinska, Canada; J. d. Silva, Portugal; W. So, San Jose State; M. Tsatsomeros, Washington State; E. Tyrtshnikov, Russia; H. v. d. Vorst, Netherlands

Invited Mini-Symposia: Six invited Mini-Symposia:

Matrix Extensions and Interpolation Problems: L. Rodman: lxrodm@math.wm.edu and H. Woerdeman: hugo@math.wm.edu

Matrices in Max Algebras: S. Gaubert: Stephane.Gaubert@inria.fr and G. J. Olsder: G.J.Olsder@its.tudelft.nl

Nonlinear Matrix Equations: A. Ran: ran@cs.vu.nl

Matrices in Control Problems: A. C. Antoulas: aca@rice.edu

Complexity in Numerical Linear Algebra: V. Olshevsky: volshevsky@cs.gsu.edu

Linear Algebra Education: M. Trigueros: trigue@itam.mx and K. Weller: wellerk@bethel-in.edu

Anyone wanting to contribute to a mini-symposium is invited to contact the mini-symposium organizers directly.

Conference Site: Auburn University Hotel and Dixon Conference Center on the campus of Auburn University.

Travel Information: Auburn lies about 100 miles southwest of the Atlanta Airport along I–85. Auburn is in the Central Time Zone, while Atlanta is on Eastern time. There is frequent limousine service between Auburn and the Atlanta airport. However, departures before noon from Atlanta require a previous night stop-over at some Atlanta Airport hotel since the earliest (7 am) limousine arrives at 9:45 am in Atlanta due to the time zone change.

Partial Funding by: Auburn University, Oak Ridge Associated Universities, National Security Agency, others pending.

With the cooperation of the SIAM Special Interest Group on Linear Algebra.

Support for Graduate Students etc.: Support for graduate students and participants in need has been requested. It is available upon request to the organizers.

Excursion: We are planning an excursion for the late afternoon of Tuesday, June 11, 2002. We will visit Tuskegee University and the Carver Museum there. After dinner we travel on to the Alabama Shakespeare Festial in Montgomery for a choice of two plays.

Details for Registration etc.: The LaTeX template for the abstracts is available at our website www.auburn.edu/ilas2002. Further details, such as registration forms and hotel and travel information will be made available early in 2002 at the website and over ILASNET.

Contact: For the conference : ilas2002@auburn.edu or Frank Uhlig, conference chair: uhligfd@auburn.edu

The Ninth ILAS Conference and Twelfth Haifa M

heory Conference: Haifa, Israel: 25–29 June 2001.

Conferences and Workshops in Linear Algebra and Matrix Theory

The Ninth ILAS Conference and the Twelfth Haifa Matrix Theory Conference

Haifa, Israel: 25–29 June 2001

The Ninth Conference of the International Linear Algebra Society (ILAS) was held at the Technion on June 25–29, 2001 (a group photo is on pp. 18–19). This conference also constituted the Twelfth Haifa Matrix Theory Conference. It was organized by the Institute of Advanced Studies in Mathematics at the Technion, which was the main sponsor of the conference.

The number of people who attended the meeting was 109, representing twenty-five countries. The conference was characterized by many interesting talks that covered a wide range of topics in linear algebra. There were 25 invited talks and over 50 contributed talks. In addition, there were minisymposia on the following topics: (i) Linear Algebra in Statistics; (ii) Matrix functions, and (iii) Education in Linear Algebra.

The LAA Lecturer was Lloyd N. Trefethen, who talked on Stability of Gaussian elemination. The Taussky-Todd Lecturer was Stephen Kirkland, who talked on Extremizing algebraic properties of tournament matrices.

An ILAS Business Meeting was held during the afternoon of June 26. A reception was held at the end of the first day of the conference and an excursion, on the afternoon of Wednesday, June 27. It consisted of a tour of Haifa, including the magnificent Bahai Gardens which had opened just a few weeks earlier. The conference banquet was held on the evening of Thursday, June 28, at the Forcheimer Faculty Center and was attended by over 90 participants. The banquet speaker was Mr. Carl Alpert who is considered to be the Historian of the Technion. He talked about various events in the history of the Technion.

Proceedings of the conference will appear as special issues of *Linear Algebra and its Applications* (LAA) and *The Electronic Journal of Linear Algebra* (ELA); each paper can be submitted either to LAA or to ELA.

Daniel HERSHKOWITZ: hershkow@techunix.technion.ac.il
Raphael LOEWY: loewy@techunix.technion.ac.il
Dept. of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel

Bill Watkins, Jochen Werner, Jerzy Baksalary, Ingram Olkin and Adi Ben-Israel at the Haifa Conference: June 2001.

The Tenth International Workshop on Matrices and Statistics

Voorburg, The Netherlands: 2–3 August 2001

The 10th International Workshop on Matrices and Statistics was held at Statistics Netherlands (CBS) in Voorburg, The Netherlands, on August 2–3, 2001. This Workshop was organized by the Data Theory Group from Leiden University and was a Satellite Meeting of the 53rd Biennial Session of the International Statistical Institute; this Workshop was also endorsed by the International Linear Algebra Society (ILAS). The Workshop was financially supported by CANdiensten. The Permanent Office of the International Statistical Institute was very kind to support the Workshop and to mediate in hosting the workshop at the CBS.

The International Organizing Committee for the workshop consisted of R. William Farebrother, Simo Puntanen, George P. H. Styan (chair), and Hans Joachim Werner. The National Program Committee included Jos ten Berge, Daniel Berze, Patrick J. F. Groenen (chair), Henk A. L. Kiers, Jan R. Magnus, Jacqueline J. Meulman, and Willem R. van Zwet. A group photo of most the participants taken in front of the CBS appears below.

The Tenth International Workshop on Matrices and Statistics: Voorburg, The Netherlands: 2–3 August 2001.

The purpose of the workshop was to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. A total of 54 participants from 14 countries joined the workshop.

The Workshop was opened by Marcel P. R. Van den Broecke, Director of the Permanent Office of the International Statistical Institute. This was followed by sessions of invited and contributed papers. Participants can submit their papers for publication, after a refereeing process, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications* (LAA). The keynote speakers were E. Kontoghiorghes and T. W. Anderson. There were nine invited speakers: J. K. Baksalary, H. Bozdogan, N. R. Chaganty, S. Ghosh, J. C. Gower, J. Groß, T. Mueller, A. C. Vandal, and F. Zhang. Another 28 contributed papers were presented in 9 sessions.

The lunch buffets at the CBS provided a good atmosphere to stimulate contacts and exchange ideas. The social program included a welcome reception on August 2, 2001, at the Howijck mansion in Voorburg, where Christiaan Huygens (1629–1695) had his residence for 16 years. The workshop dinner was held on August 3 in the Kurhaus Hotel in Scheveningen. An excellent Indonesian rice table was served. The exceptionally good weather made it possible to enjoy great beach views on the Kurhaus terrace. On August 3, the gourmet dinner was enjoyed by thirteen participants at the one Michelin star Savelberg Restaurant in Voorburg. About 16 participants joined the Delft excursion on Saturday, August 4. Without any doubt, the visit to the working windmill Molen de Roos and the explanations by the miller were the highlights of this excursion. Please vist http://matrix.fsw.leidenuniv.nl for several photographs taken during this Workshop.

Patrick J. F. GROENEN: groenen@fsw.leidenuniv.nl
Data Theory Group, Leiden University
P.O. Box 9555, 2330 RB Leiden, The Netherlands

International Conference on Combinatorial Matrix Theory

Pohang, Korea: 14–17 January 2002

This International Conference on Combinatorial Matrix Theory will be held at Pohang University of Science and Technology (Postech) in Pohang, Korea, 14–17 January 2002. The conference will be on Combinatorics, Combinatorial Matrix Theory and related areas of Computational and Core Matrix Theory.

The Combinatorial and Computational Mathematics Center at Pohang University of Science and Technology, a science research center funded by the Korean Science and Engineering Foundation (KOSEF), is organizing the conference to bring together researchers with interests in the interactions between combinatorics and matrix theory. It is anticipated that the conference will provide a forum for interaction between a broad group of scientists and engineers with diverse interests in combinatorics and matrix theory.

The conference chairs are: Richard A. Brualdi and Suk-Geun Hwang. The international advisory committee comprises Arnold Kräuter, Bryan Shader, and Jia-Yu Shao. The local coordinator at Postech is Hyun Kwang Kim: hkkim@postech.ac.kr

Invited plenary speakers are Steve Kirkland: "An Approach to Algebraic Connectivity Via Nonnegative Matrices" and Jian Shen: "Some problems on digraphs". Other invited speakers include: Kazuyoshi Okubo, Jason Zhongshan Li, Miroslav Fiedler, Bolian Liu, Qiao Li, Willem Haemers, Ian Wanless, Richard Brualdi, Arnold Kräuter, Bryan Shader, Charles Johnson, Jennifer Seberry, and Jia-yu Shao.

Participants presenting papers are invited to submit them for possible publication in a special issue of the international journal, *Linear Algebra and Its Applications* (LAA), to one of the special editors: Suk-Geun Hwang, Arnold Kräuter, Bryan Shader, and Jia-yu Shao. Papers will be refereed in accordance with the normal high standards of LAA. April 30, 2002 is the deadline for papers to be submitted to one of the special editors.

An excursion to Kyongju will take place on Wednesday afternoon. Kyungju was the capital of the Shilla Kingdom (57 BC - 935 AD) and is Korea's tourist highlight. The cost for the excursion including transportation, entrance fees, and dinner is US$50.

The Web site of the conference contains up-to-date information including electronic forms for submission of individual presentations. Please visit

http://matrix.skku.ac.kr/sglee/postech/postech.htm

Iterative Solvers for Large Linear Systems: Latsis-Symposium 2002

Zürich, Switzerland: 18–21 February 2002

This Conference, to be held at the Swiss Federal Institute of Technology /Eidgenoössische Technische Hochschule (ETH) in Zürich, Switzerland, from 18 through 21 February 2002 will commemorate 50 years of conjugate gradients and will also host a celebration of the 70th birthday of Gene Howard Golub (b. 29 February 1932).

In 1952, M. Hestenes and E. Stiefel published their seminal paper "Methods of conjugate gradients for solving linear systems" [*J. Research Nat. Bur. Standards* 49 (1952), 409–436]. This conference has the purpose to commemorate this event, review the early and later developments, survey the tremendous impact of this paper, and discuss current research in the area of Krylov space methods and their preconditioning. There will also be a series of talks on large-scale applications that make heavy usage of iterative methods.

The following invited speakers are expected: Bernd Beckermann, Howard Elman, Roland Freund, Anne Greenbaum, Tim

Kelley, Gerard Meurant, Dianne O'Leary, Chris Paige, Youcef Saad, Gene Wachspress, Henk van der Vorst, Charbel Farhat, Michael Griebel, Van Henson, Wei-Pai Tang, Ray Tuminaro, Gabriel Wittum, Fritz L. Bauer, Urs Hochstrasser, John Todd.

For more information please visit the Web site

http://www.cg50.ethz.ch

Western Canada Linear Algebra Meeting

Regina, Saskatchewan: 10–11 May 2002

The Western Canada Linear Algebra Meeting (W-CLAM) provides an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting. Previous W-CLAMs were held in Regina (1993), Lethbridge (1995), Kananaskis (1996), Victoria (1998), and Winnipeg (2000).

The W-CLAM '02 will be held at the University of Regina in Regina on May 10–11, 2002. The participation fee is C$20, to be collected at the meeting. This fee will be waived for participating students and postdoctoral fellows. We anticipate being able to offer support of up to C$300 per person for a limited number of participating students and postdoctoral fellows to be distributed on a first-come, first-served basis.

The deadline for the submission of titles and abstracts is April 1, 2002. We expect to have online registration available by January 2002; the department's website http://www.math.uregina.ca will then have a link to the W-CLAM Web site. For further information contact either Steve Kirkland: kirkland@math.uregina.ca or Shaun Fallat: sfallat@math.uregina.ca

The Sixth Workshop on Numerical Ranges and Numerical Radii

Auburn University, Alabama: 7–8 June 2002

The Sixth Workshop on "Numerical Ranges and Numerical Radii" (6-WONRA) will be held at Auburn University, Alabama, from June 7 to June 8, 2002, in conjunction with the ILAS conference on "Challenges in Matrix Theory" (10–13 June 2002, see page 17).

Purpose. The purpose of the workshop is to stimulate research and foster interaction of researchers interested in the subject. The informal workshop atmosphere will guarantee the exchange of ideas from different research areas and, hopefully, the participants will leave informed of the latest developments and newest ideas. For some background about the subject and previous meetings please visit the WONRA Web site:

http://www.math.wm.edu/ ckli/wonra.html

Confirmed Particiapants. T. Ando, Natália Bebiano, Chandler Davis, Charles Dolberry, Charlie Johnson, Tom Morley, Hiroshi Nakazato, Peter Nylen, Yiu-Tung Poon, João da Providencia, Leiba Rodman, Ilya Spitkovsky, Bit-Shun Tam, Michael Tsatsomeros, Frank Uhlig, and Wen Yan.

There will be no registration fees for this Workshop. Limited financial support to participants based on need may be requested. To confirm your participation and reserve your accommodation : April 1, 2002. To submit the title and abstract of your talk : April 15, 2002.

Computational Linear Algebra with Applications

Milovy, Czech Republic: 4–10 August 2002

Following the successful meeting on Iterative Methods and Parallel Computing in 1997 we will organize the second Milovy meeting (Milovy 2002) from August 4th to 10th, 2002. The meeting will be held at the Devet Skal Hotel (literally "Nine Rocks") in Milovy, a small village located in the Bohemian-Moravian Highlands about 100 km (c. 62.5 miles) from Prague.

The meeting will concentrate on computational methods, namely on methods of numerical linear algebra, their (parallel) implementations and applications. Based on the contributions, we plan to structure the program into subthemes such as matrix theory and its applications, sparse direct solvers, iterative methods for linear systems, preconditioning techniques, large-scale eigen-problems, inverse problems and regularization. Special attention will be paid to application areas not restricted to PDEs, control, optimization or image processing. The scientific program is planned for five full days (Monday through Friday), with a free Wednesday afternoon reserved for an excursion.

The Program Committee and the Local Organization: Mario Arioli, Michele Benzi, Iain Duff, Michael Eiermann, Roland Freund, Anne Greenbaum, Martin Gutknecht, Ivo Marek, Volker Mehrmann, Gerard Meurant, Jim Nagy, Zdenek Strakos, Daniel Szyld, Henk van der Vorst, Olof Widlund, Mirek Tuma and Miro Rozloznik will be in charge of the local organization.

Registration and Submission of Abstracts: The registration form can be found on the conference Web site

http://www.cs.cas.cz/~milovy/main97.htm

The meeting facility holds up to 150 people. Therefore, early registration is recommended. The deadline for submitting an abstract is January 15, 2002. All participants will be notified about the acceptance of their contributions by February 15, 2002. Conference proceedings will be published as a special issue of the research journal *BIT*. All submitted papers will be subject to standard refereeing procedures. The submission deadline will be November 30, 2002. The conference fees include registration fee, full board in Milovy, bus transportation from and to Prague and excursions on Wednesday afternoon. The early registration fee (paid before April 30, 2002) is US$450 (double room), US$530 (single room).

Before and after the conference the participants will have an opportunity to visit the historical parts of Prague. We have made several group-reservations in a variety of hotels, with relevant information available on the Web site. All participants will be responsible for making their own hotel reservations. We would appreciate it if you send us information about your reservation (for the purpose of transport to and from Milovy). Those participants who wish to stay in a student dormitory should indicate this interest on the registration form. conference organizers.

Fifth China Conference on Matrix Theory and Its Applications

Shanghai, China: 14–18 August 2002

The Fifth China Conference on Matrix Theory and Its Applications will be held at Shanghai University, Shanghai, from 14 to 18 August 2002. This Conference will be a satellite meeting of the 2002 International Congress of Mathematicians (ICM) to be held in Beijing, 20–28 August 2002: http://www.icm2002.org.cn/

The themes of the conference will cover all aspects of Matrix Theory and Linear Algebra, such as traditional Matrix Theory, Combinatorial Matrix Theory and Numerical Linear Algebra, Matrix Computations etc. The following experts are interested in attending the conference: T. Ando, Zhaojun Bai, A. Berman, Frank Hall, Bixin Tan, and Huisheng Zheng.

The conference registration fee is US$100, which also covers meals for four-and-a-half days and a sightseeing tour. For more information, contact Erxiong Jiang: Department of Mathematics, Fudan University, 200433 Shanghai, China: ejiang@fudan.edu.cn or Chuanqing Gu: Dept. of Mathematics, Shanghai University, 200136, Shanghai, China: guchqing@guomai.sh.cn Please send your suggestions and comment to: icmsec@beijing.icm2002.org.cn

The Eleventh International Workshop on Matrices and Statistics

Lyngby, Denmark: late August 2002

The Eleventh International Workshop on Matrices and Statistics will be held at the Technical University of Denmark in Lyngby, near Copenhagen, in late August 2002. This Workshop will be hosted by the Division of Image Analysis and Computer Graphics in the Department of Informatics and Mathematical Modelling at the Technical University of Denmark and has been endorsed by the International Linear Algebra Society (ILAS).

The International Organizing Committee R. William Farebrother, Simo Puntanen, George P. H. Styan, and Hans Joachim Werner (chair): werner@united.econ.uni-bonn.de Knut Conradsen: kc@imm.dtu.dk, hw@imm.dtu.dk is in charge of local arrangements.

The purpose of this Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. This Workshop will provide a forum through which statisticians working in the field of linear algebra and matrix theory may be better informed of the latest developments and newest techniques and may exchange ideas with researchers from a wide variety of countries.

This Workshop is the eleventh in a series. The previous ten Workshops were held as follows: (1) Tampere, Finland: August 1990, (2) Auckland, New Zealand: December 1992, (3) Tartu, Estonia: May 1994, (4) Montréal (Québec), Canada: July 1995, (5) Shrewsbury, England: July 1996, (6) Istanbul, Turkey, August 1997, (7) Fort Lauderdale, Florida, USA, December 1998: in Celebration of T. W. Anderson's 80th Birthday, (8) Tampere, Finland, August 1999, (9) Hyderabad, India, December 2000: in Celebration of C. R. Rao's 80th Birthday, (10) Voorburg, The Netherlands, August 2001.

Topics in Linear Algebra Conference

Ames, Iowa, USA: 13–14 September 2002

The Topics in Linear Algebra Conference will be held at Iowa State University on Friday, September 13 and Saturday, September 14, 2002. This meeting is sponsored by the Institute for Mathematics and Its Applications and Iowa State University and has been endorsed by the International Linear Algebra Society (ILAS). Invited Speakers include Sam Hedayat, David P. Jacobs, Charles R. Johnson, C. K. Li, and Hans Schneider.

The conference is organized around the following topics: Matrix Completion Problems, Numerical Ranges, Matrix Stability and Convergence, Applications of Linear Algebra to Non-Associative Algebra, and Statistical Applications of Linear Algebra. For each topic, there will be a presentation by an invited speaker, a session for contributed papers, and a work session. There will also be a contributed paper session for areas of linear algebra within the focus of the conference but not specifically within one of the topics. To register, please do the following by May 1, 2002:

- Send an e-mail containing your name, e-mail address, telephone number, and classical p-mail address to Huaiqing Wu: isuhwu@iastate.edu
- Send a check or money order for the US$40 registration fee made out to Iowa State University Mathematics Department to Jan Nyhus, Mathematics Department, 400 Carver Hall, Iowa State University, Ames, IA 50011. (The registration fee is US$20 for students.)
- If you wish to present a contributed paper (20 minutes), e-mail Luz DeAlba: luz.dealba@drake.edu. The e-mail should include your name, the contributed paper session (one of the 5 topics above or "related"), and should have an abstract in LaTeX or MS-Word attached.

The conference is organized by Leslie Hogben, Bryan Cain, Luz DeAlba, Irvin Hentzel, Mark Mills, Yiu-Tung Poon, and Huaiqing Wu. For more information please visit our Web site:

http://www.math.iastate.edu/lhogben/TLA/homepage.html

IMAGE Problem Corner: Old Problems, Many With Solutions

We present solutions to IMAGE Problems 26-2 through 26-5 published in IMAGE 26 (April 2001), pp. 24–25. We are still hoping to receive solutions to Problems 19-3b, 23-1 and to Problem 26-1, which are repeated below. In addition, we introduce six new problems (page 32 below) and invite readers to submit solutions as well as new problems for publication in IMAGE. Please submit all material (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenaueralle 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG, *North Carolina State University, Raleigh, North Carolina, USA:* hartwig@math.ncsu.edu
Peter ŠEMRL, *University of Maribor, Maribor, Slovenia:* peter.semrl@fmf.uni-lj.si
and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \mathrm{span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

The Editor has still not received a solution to Problem 19-3b. The solution by the Proposers to Problem 19-3a appeared in IMAGE 22, p. 25 (April 1999). *We look forward to receiving a solution to Problem 19-3b.*

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG, *University of Wisconsin–Madison, Madison, Wisconsin, USA:* mchung@stat.wisc.edu

Let the $m \times n$ random matrix X be such that $\mathrm{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where vec indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find E $\det(X'CX)$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

We look forward to receiving a solution to this problem!

Problem 26-1: Degenerate Complex Quadratic Forms on Real Vector Spaces

Proposed by S. W. DRURY, *McGill University, Montréal, Québec, Canada:* drury@math.mcgill.ca

Let C be a complex symmetric $n \times n$ matrix. Suppose that A and B are the real and imaginary parts of C respectively (i.e., $a_{jk} = \Re c_{jk}$, $b_{jk} = \Im c_{jk}$). Suppose that $\det(B - tA) = 0$ for all real t. Prove or find a counterexample to the following statement. There necessarily exist nonnegative integers p, q and r such that $p + q + r = n$ and $q > r$ and a real invertible $n \times n$ matrix P, such that

$$P'CP = \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & Y \\ 0 & Y' & 0 \end{pmatrix},$$

where X is a $p \times p$ complex matrix and Y is a $q \times r$ complex matrix. The cases $p = 0$ and $r = 0$ are allowed. For instance, $p = n - 1$, $q = 1$, $r = 0$ is the case where there is a non-zero real vector in the null space of C.

The Editors have not yet received a solution to this problem—indeed even the Proposer has not yet found a solution!

Problem 26-2: Almost Orthogonality of a Skew-Symmetric Matrix

Proposed by L. LÁSZLÓ, *Eötvös Loránd University, Budapest, Hungary:* llaszlo@ludens.elte.hu

By adding an extra first row and column to the matrix of Problem 25-1 (see IMAGE 26 (April 2001), p. 2), we obtain the 8×8 matrix

$$A_8 = \begin{pmatrix} +a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ +a_1 & +a_0 & +a_3 & -a_2 & +a_5 & -a_4 & +a_7 & -a_6 \\ +a_2 & -a_3 & +a_0 & +a_1 & +a_6 & -a_7 & -a_4 & +a_5 \\ +a_3 & +a_2 & -a_1 & +a_0 & -a_7 & -a_6 & +a_5 & +a_4 \\ +a_4 & -a_5 & -a_6 & +a_7 & +a_0 & +a_1 & +a_2 & -a_3 \\ +a_5 & +a_4 & +a_7 & +a_6 & -a_1 & +a_0 & -a_3 & -a_2 \\ +a_6 & -a_7 & +a_4 & -a_5 & -a_2 & +a_3 & +a_0 & +a_1 \\ +a_7 & +a_6 & -a_5 & -a_4 & +a_3 & +a_2 & -a_1 & +a_0 \end{pmatrix}$$

(see also Solution 25-1.3 on pp. 3–4 above). Denote by A_k its principal submatrix of order k, and observe that A_2, A_4 and A_8 are skew symmetric and (apart from a scalar multiple) orthogonal for any real a_i with $a_0 = 0$. Can this construction be continued? Note that only the sign pattern of A_{16} is to be properly chosen, for the distribution of the a_i is easy to follow: if the sign-free matrix A_k is denoted by M_k, then

$$M_{2k} = \begin{pmatrix} M_k & N_k \\ N_k & M_k \end{pmatrix},$$

where N_k comes from M_k by increasing the subscripts of the a_i by k, e.g., $M_2 = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \Longrightarrow N_2 = \begin{pmatrix} a_2 & a_3 \\ a_3 & a_2 \end{pmatrix}$.

Solution 26-2.1 by Fátima SILVA LEITE, *Departamento de Matemática, Universidade de Coimbra, Coimbra, Portugal:* fleite@mat.uc.pt

The answer is no! Indeed, that construction cannot be continued, due to the following result in algebraic topology, which has been an open problem for many years and was completely solved in the early 1960s; see Adams (1962):

the only parallelizable spheres are S^1, S^3 and S^7.

A k-dimensional differentiable manifold M is said to be parallelizable if there exists a set of k C^∞-vector fields on $M, X_1, \cdots, X_k$, satisfying the following: $\forall a \in M,\ span\,\{X_1(a), \cdots, X_k(a)\} = T_a M$, where $T_a M$ denotes the tangent space to M at the point a. In this case, we say that $\{X_1, \cdots, X_k\}$ is a C^∞*-frame field* for M.

The structure of the matrices A_2, A_4 and A_8 exhibits a frame field for the paralellizable spheres. Indeed, the first column of A_2, A_4 and A_8 can be identified with a generic point on the surface of S^1, S^3 and S^7, respectively, or equivalently with the radius vector from the origin to a. The other columns are mutually orthogonal and also orthogonal to the radius vector. So, for every a, those columns span the tangent space to the sphere at the point a.

If a matrix A_n, with $n \neq 2, 4, 8$, having the properties of the matrices A_2, A_4 and A_8 could be constructed, it would exhibit a frame field for S^{n-1}. But such a field doesn't exist.

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[1] J. Adams (1962). Vector fields on spheres. *Annals Mathematics: Second Series,* 75, 603–632.

[2] I. Eganova & M. Shirokov (1984). Orthomatrices and octonions. *Reports on Mathematical Physics,* 20, 1–11.

[3] F. Silva Leite (1993). The geometry of hypercomplex matrices. *Linear and Multilinear Algebra,* 34, 123–132.

Solution 26-2.2 by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

The matrix A_k proposed in the problem is closely linked to real representations of complex numbers, quaternions and octonions. For A_2, it can be regarded as a real matrix representation of the complex number $a = a_0 + a_1 i$ through the mapping

$$\phi_2 : a = a_0 + a_1 i \in \boldsymbol{C} \longrightarrow \phi_2(a) = A_2 = \begin{bmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{bmatrix}. \tag{1}$$

This mapping satisfies

(a) $a = b \Longleftrightarrow \phi_2(a) = \phi_2(b)$,
(b) $\phi_2(a+b) = \phi_2(a) + \phi_2(b)$,
(c) $\phi_2(ab) = \phi_2(a)\phi_2(b)$; $\phi_2(\lambda a) = \lambda\phi_2(a)$ with λ real,
(d) $\phi_2(\overline{a}) = \phi_2^T(a)$,
(e) $\phi_2(a^{-1}) = \phi_2^{-1}(a)$, if $a \neq 0$,
(f) $\det[\phi_2(a)] = |a|^2$.

For the matrix A_4, it is well known as the real matrix representation of the quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$, where $i^2 = j^2 = k^2 = -1$, $ijk = -1$, through the mapping

$$\phi_4 : a = a_0 + a_1 i + a_2 j + a_3 k \longrightarrow \phi_4(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}. \tag{2}$$

This mapping satisfies

(a) $a = b \Longleftrightarrow \phi_4(a) = \phi_4(b)$.
(b) $\phi_4(a+b) = \phi_4(a) + \phi_4(b)$.
(c) $\phi_4(ab) = \phi_4(a)\phi_4(b)$, $\phi_4(\lambda a) = \lambda\phi_4(a)$.
(d) $\phi_4(\overline{a}) = \phi_4^T(a)$.
(e) $\phi_4(a^{-1}) = \phi_4^{-1}(a)$, if $a \neq 0$.
(f) $\det[\phi_4(a)] = |a|^4$.

Moreover the diagonal matrix aI_4 is universally similar to $\phi_4(a)$, see Tian (1999).

Although the matrix A_8 given in the problem is skew-symmetric and orthogonal (apart from a scalar multiple), it cannot be regarded as a matrix representation of an element in the well-known eight-dimensional Clifford or Cayley–Dickson algebra.

An alternative expression of A_8 is

$$\phi_8(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix},$$

which is skew-symmetric and orthogonal (apart from a scalar multiple). This matrix is a matrix representation of the octonion $a = a_0 + a_1 e_1 + \cdots + a_7 e_7$ (i.e., an element in the eight-dimensional Cayley–Dickson algebra). The multiplication rules for the basis are listed in the following matrix

$$E_8^T E_8 = \begin{bmatrix} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_3 & -e_2 & e_5 & -e_4 & -e_7 & e_6 \\ e_2 & -e_3 & -1 & e_1 & e_6 & e_7 & -e_4 & -e_5 \\ e_3 & e_2 & -e_1 & -1 & e_7 & -e_6 & e_5 & -e_4 \\ e_4 & -e_5 & -e_6 & -e_7 & -1 & e_1 & e_2 & e_3 \\ e_5 & e_4 & -e_7 & e_6 & -e_1 & -1 & -e_3 & e_2 \\ e_6 & e_7 & e_4 & -e_5 & -e_2 & e_3 & -1 & -e_1 \\ e_7 & -e_6 & e_5 & e_4 & -e_3 & -e_2 & e_1 & -1 \end{bmatrix},$$

where $E_8 = [\,1,\ e_1,\ \cdots,\ e_7\,]$, see also Zhevlakov et al. (1982). It is well known that octonionic algebra is non-associative. The mapping ϕ_8 satisfies

(a) $a = b \Longleftrightarrow \phi_8(a) = \phi_8(b)$.
(b) $\phi_8(a+b) = \phi_8(a) + \phi_8(b)$.
(c) $\phi_8(aba) = \phi_8(a)\phi_8(b)\phi_8(a)$, but $\phi_8(ab) \neq \phi_8(a)\phi_8(b)$.
(d) $\phi_8(\overline{a}) = \phi_8^T(a)$.
(e) $\phi_8(a^{-1}) = \phi_8^{-1}(a)$, if $a \neq 0$.
(f) $\det[\phi_8(a)] = |a|^8$.

For more details on real matrix representations of octonions and applications for octonionic matrices, see Tian (2000).

Consider a sedenion $a = a' + a''e$, where a' and a'' are two octonions. According to the Cayley-Dickson process, see Schafer (1966) and Zhevlakov et al. (1982), the addition and multiplication for sedenions are defined by

$$a + b = (\,a' + a''e\,) + (\,b' + b''e\,) = (\,a' + b'\,) + (\,a'' + b''\,)e$$

and

$$ab = (\,a' + a''e)(b' + b''e) = (a'b' - \overline{b''}a'') + (b''a' + a''\overline{b'}\,)e,$$

where $\overline{b'}$, $\overline{b''}$ denote the conjugates of the octonions b' and b''. Through the matrix representation of the octonion presented above, one can easily get a 16×16 real matrix representation $\phi_{16}(a)$ for any sedenion $a = a' + a''e$ which is skew-symmetric and $\phi_8(a')$ is its principal submatrix. Various interesting properties for $\phi_{16}(a)$ can be derived through the sedenion algebra. It is well known, however, that the sedenion algebra is not a division algebra, i.e., there exist two nonzero sedenions a and b such that $ab = 0$. In that case, it is easy to prove that the corresponding real matrix representations $\phi_{16}(a)$ and $\phi_{16}(b)$ are singular. Thus the matrix $\phi_{16}(a)$ and $\phi_{16}(b)$ cannot be orthogonal. This fact tells us that one cannot construct a skew-symmetric and orthogonal matrix A_{2^k} $(k \geq 4)$ through matrix representations of Clifford and Cayley–Dickson algebras.

Our final conclusion is that there does not exist A_{2^k} $(k \geq 4)$ which is skew-symmetric and orthogonal (apart from a scalar multiple). If so, then one can construct a 2^k-dimensional algebra $\mathcal{A}$ from the matrix A_{2^k} such that $\mathcal{A}$ is a division algebra. But it is well known that there do not exist 2^k-dimensional division algebras over the reals when $k \geq 4$; see, e.g., Schafer (1966) and Zhevlakov et al. (1982). □

References

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[4] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov & A. I. Shirshov (1982). *Rings that are Nearly Associative.* Translated from the Russian by Harry F. Smith. Pure and Applied Mathematics 104. Academic Press, New York.

Problem 26-3: Factorization of Generalized Inverses of Matrix Polynomials
Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

(a) Consider an $n \times n$ matrix A and two scalars λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Prove that if

$$\text{range}\,(\lambda_1 I_n - A) \cap \text{range}\,(\lambda_2 I_n - A) \neq \{0\}$$

and

$$\text{range}\,(\lambda_1 I_n - A^T) \cap \text{range}\,(\lambda_2 I_n - A^T) \neq \{0\}$$

then the set equality

$$\{\,[\,(\lambda_1 I_n - A)(\lambda_2 I_n - A)\,]^-\,\} = \frac{1}{\lambda_2 - \lambda_1}\{\,(\lambda_1 I_n - A)^- - (\lambda_2 I_n - A)^-\,\}$$

holds, where $(\cdot)^-$ denotes a generalized (or inner) inverse of a matrix.

(b) Consider an $n \times n$ matrix A and k scalars $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$. If

$$\text{range}\,(\lambda_1 I_n - A) \cap \text{range}\,(\lambda_2 I_n - A) \cap \cdots \cap \text{range}\,(\lambda_k I_n - A) \neq \{0\},$$

and

$$\text{range}\,(\lambda_1 I_n - A^T) \cap \text{range}\,(\lambda_2 I_n - A^T) \cap \cdots \cap \text{range}\,(\lambda_k I_n - A^T) \neq \{0\},$$

then prove or disprove that

$$\{\,[\,(\lambda_1 I_n - A)(\lambda_2 I_n - A)\ldots(\lambda_k I_n - A)\,]^-\,\} = \left\{\frac{1}{p_1}(\lambda_1 I_n - A)^- + \frac{1}{p_2}(\lambda_2 I_n - A)^- + \ldots + \frac{1}{p_k}(\lambda_k I_n - A)^-\right\},$$

where

$$p_i = (\lambda_1 - \lambda_i)\ldots(\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i)\ldots(\lambda_k - \lambda_i), \quad i = 1, 2, \cdots, k.$$

Solution 26-3.1 by Hans Joachim WERNER: *University of Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

We prove the following more general result:

THEOREM 1. *Let A be a real or complex $n \times n$ matrix, and let λ_i $(i = 1, 2, \cdots, k)$ be scalars such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. For $A_{\lambda_i} := \lambda_i I - A$, $i = 1, 2, \cdots, k$, we then have*

$$\left\{\left(\prod_{1=1}^{k} A_{\lambda_i}\right)^-\right\} = \left\{\sum_{i=1}^{k} \frac{1}{p_i} A_{\lambda_i}^-\right\}, \tag{3}$$

where $p_i := \prod_{j \neq i}(\lambda_j - \lambda_i)$.

PROOF: For convenience we put

$$p_{k-1}(\lambda) := \sum_{i=1}^{k} \frac{\prod_{j\neq i}(\lambda_j - \lambda)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}.$$

It is obvious that the degree of this polynomial cannot exceed $k - 1$. For all $i = 1, 2, \cdots, k$, clearly $p_{k-1}(\lambda_i) = 1$. This in turn implies that $p_{k-1}(\cdot) \equiv 1$. So the degree of the polynomial $p_{k-1}(\lambda)$ is 0 and we do have $p_{k-1}(A) = I$ for every square matrix A. Since the matrices A_{λ_i} $(i = 1, 2, \cdots, k)$ commute with each other, we now obtain

$$\left(\prod_{i=1}^{k} A_{\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{1}{p_i} A_{\lambda_i}^-\right)\left(\prod_{i=1}^{k} A_{\lambda_i}\right) = \sum_{i=1}^{k} \frac{1}{p_i} A_{\lambda_i}\left(\prod_{j \neq i} A_{\lambda_j}^2\right) = \left(\prod_{i=1}^{k} A_{\lambda_i}\right) p_{k-1}(A) = \prod_{i=1}^{k} A_{\lambda_i},$$

thus completing the proof of inclusion $\supseteq$ in (3). To prove the converse inclusion, observe first that according to Theorem 2.4.1 in Rao & Mitra (1971) we now know that a general solution to the matrix equation

$$\left(\prod_{i=1}^{k} A_{\lambda_i}\right) G \left(\prod_{i=1}^{k} A_{\lambda_i}\right) = \left(\prod_{i=1}^{k} A_{\lambda_i}\right)$$

is

$$G=\left(\sum_{i=1}^{k}\frac{1}{p_i}A_{\lambda_i}^{-}\right)+V(I-CC^{-})+(I-C^{-}C)W, \tag{4}$$

where $C:=\prod_{i=1}^{k}A_{\lambda_i}$, where $A_{\lambda_i}^{-}$ $(i=1,2,\cdots,k)$ and C^{-} are particular g-inverses of A_{λ_i} $(i=1,2,\cdots,k)$ and C, respectively, and where V and W are arbitrary matrices. Evidently, whenever $i\neq j$, $\mathcal{N}(A_{\lambda_i})\subseteq\mathcal{R}(A_{\lambda_j})$ and $\mathcal{N}(A_{\lambda_i})\cap\mathcal{N}(A_{\lambda_j})=\{0\}$; here $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the range (column space) and the null space, respectively, of $(\cdot)$. Therefore,

$$\mathcal{N}\left(\prod_{i=1}^{k}A_{\lambda_i}\right)=\bigoplus_{i=1}^{k}\mathcal{N}(A_{\lambda_i})\qquad\text{and}\qquad\mathcal{N}\left(\prod_{i=1}^{k}A_{\lambda_i}^{*}\right)=\bigoplus_{i=1}^{k}\mathcal{N}(A_{\lambda_i}^{*}), \tag{5}$$

with $\oplus$ indicating a direct sum. With this in mind, it is clear that (4) can be rewritten as

$$\left(\sum_{i=1}^{k}\frac{1}{p_i}A_{\lambda_i}^{-}\right)+V(I-CC^{-})+(I-C^{-}C)W=\sum_{i=1}^{k}\frac{1}{p_i}\left[A_{\lambda_i}^{-}+V_i(I-A_{\lambda_i}A_{\lambda_i}^{-})+(I-A_{\lambda_i}^{-}A_{\lambda_i})W_i\right] \tag{6}$$

for suitable choices of V_i and W_i $(i=1,2,\cdots,k)$. Since the expressions in the brackets are, again according to Theorem 2.4.1 in Rao & Mitra (1971), g-inverses of the corresponding matrices A_{λ_i} $(i=1,2,\cdots,k)$, the proof of our theorem is complete. □

Although our Theorem 1 contains, of course, as a special case also an answer to part (a) of Problem 26-3, we offer below an alternative proof for that particular case. To this end, let λ_1 and λ_2 be two scalars with $\lambda_1\neq\lambda_2$. Moreover, let A be a given square matrix. Put $A_{\lambda_1}:=\lambda_1 I-A$ and $A_{\lambda_2}:=\lambda_2 I-A$. Then $A_{\lambda_1}-A_{\lambda_2}=(\lambda_1-\lambda_2)I$, and it is clear that A_{λ_1} and $-A_{\lambda_2}$ is a pair of parallel summable matrices; for a detailed discussion of parallel summable matrices we refer the interested reader to Rao & Mitra (1971, p. 188 ff). We note that the parallel sum of A_{λ_1} and $-A_{\lambda_2}$, which is defined as

$$A_{\lambda_1}\pm(-A_{\lambda_2}):=A_{\lambda_1}\left[A_{\lambda_1}+(-A_{\lambda_2})\right]^{-}(-A_{\lambda_2}),$$

reduces to $A_{\lambda_1}\pm(-A_{\lambda_2})=-(\lambda_1-\lambda_2)^{-1}A_{\lambda_1}A_{\lambda_2}$. Theorem 10.1.8(d) in Rao & Mitra (1971) tells us that $A_{\lambda_1}^{-}+(-A_{\lambda_2})^{-}$ is one choice of $[A_{\lambda_1}\pm(-A_{\lambda_2})]^{-}$ and that conversely every g-inverse of $[A_{\lambda_1}\pm(-A_{\lambda_2})]$ is expressible as $A_{\lambda_1}^{-}+(-A_{\lambda_2})^{-}$ for suitable choices of these g-inverses. Clearly, $\{(-A_{\lambda_2})^{-}\}=\{-A_{\lambda_2}^{-}\}$ and

$$\left\{\left[-(\lambda_1-\lambda_2)^{-1}A_{\lambda_1}A_{\lambda_2}\right]^{-}\right\}=\left\{(\lambda_2-\lambda_1)\left(A_{\lambda_1}A_{\lambda_2}\right)^{-}\right\}.$$

Combining these observations now results in

$$\left\{(A_{\lambda_1}A_{\lambda_2})^{-}\right\}=\left\{(\lambda_2-\lambda_1)^{-1}\left(A_{\lambda_1}^{-}-A_{\lambda_2}^{-}\right)\right\},$$

thus completing our alternative proof for claim (a).

Reference

[1] C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and Its Applications.* Wiley, New York

Problem 26-4: Commutativity of EP Matrices

Proposed by Yongge TIAN *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

The square complex matrix A is said to be EP whenever the column space (range) $\mathcal{R}(A)=\mathcal{R}(A^{*})$. Let A and B be $n\times n$ complex matrices and let A^{+} denote the Moore–Penrose inverse of A.

(a) Suppose that $AB=BA$. Show that A is EP if and only if $A^{+}B=BA^{+}$.

(b) Suppose that A and B are both EP. Show that $AB=BA$ if and only if $A^{+}B^{+}=B^{+}A^{+}$.

Solution 26-4.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* jbaks@lord.wsp.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

Since an orthogonal projector (a Hermitian idempotent matrix) is uniquely determined by a subspace onto which it projects, and since MM^+ and M^+M are the orthogonal projectors onto $\mathcal{R}(M)$ and $\mathcal{R}(M^*)$, respectively, it is clear that $M \in \mathcal{C}_{n,n}$ is an EP matrix if and only if $MM^+ = M^+M$.

The first part of our solution corrects the assertion in part (a) of the problem. The correction consists in deleting the implication that if $AB = BA$ and $A^+B = BA^+$, then A is an EP matrix. Two simple counterexamples are obtained by taking B as the zero or identity matrix, in which cases the two equalities above are fulfilled by any $A \in \mathcal{C}_{n,n}$.

LEMMA. *If $M \in \mathcal{C}_{n,n}$ is an* EP *matrix, then any $N \in \mathcal{C}_{n,n}$ such that $MN = NM$ satisfies $M^+N = NM^+$.*

PROOF. In view of $MM^+ = M^+M$, $MN = NM$, and the properties $MM^+M = M$ and $M^+MM^+ = M^+$ of the Moore–Penrose inverse M^+, it is clear that

$$\mathcal{R}(NM^+) = \mathcal{R}(NM^+M) = \mathcal{R}(NMM^+) = \mathcal{R}(NM) = \mathcal{R}(MN) \subseteq \mathcal{R}(M),$$

$$\mathcal{R}[(M^+N)^*] = \mathcal{R}[(MM^+N)^*] = \mathcal{R}[(M^+MN)^*] = \mathcal{R}[(MN)^*] = \mathcal{R}[(NM)^*] \subseteq \mathcal{R}(M^*).$$

Hence $MM^+NM^+ = NM^+$, $M^+NM^+M = M^+N$, and then

$$M^+N = M^+NM^+M = M^+NMM^+ = M^+MNM^+ = MM^+NM^+ = NM^+,$$

which concludes the proof. □

The second part of our solution extends the assertion in part (b) of the problem.

THEOREM. *If $A, B \in \mathcal{C}_{n,n}$ are* EP *matrices, then the following four commutativity conditions are mutually equivalent:*

$$\text{(i) } AB = BA, \quad \text{(ii) } AB^+ = B^+A, \quad \text{(iii) } A^+B^+ = B^+A^+, \quad \text{(iv) } A^+B = BA^+.$$

PROOF. It is clear that $M \in \mathcal{C}_{n,n}$ is an EP matrix if and only if M^+ has the same property. In view of this observation, the chain of implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) follows immediately from Lemma by substituting $M = B$ and $N = A$ to show that (i) ⇒ (ii), $M = A$ and $N = B^+$ to show that (ii) ⇒ (iii), $M = B^+$ and $N = A^+$ to show that (iii) ⇒ (iv), and, finally, by substituting $M = A^+$ and $N = B$ to show that (iv) ⇒ (i). □

Solution 26-4.2 by J. BELL, J. William HELTON & Anthony SHAHEEN, *Univ. of California at San Diego, La Jolla, Calif., USA:*
jbell@math.ucsd.edu, helton@ucsd.edu, ashaheen@math.ucsd.edu

DEFINITION. A matrix $M \in M_n(C)$ is said to be EP if M has the same column space as M^*. Equivalently, M is EP if $M^* = MG$ for some invertible matrix G.

A slight modification of the original Problem 26-4 is

THEOREM. *For matrices $A,\ B \in M_n(C)$ we have:*
(a) If A and B commute, then A is EP only if $A^\dagger B - BA^\dagger = 0$.
(b) If A and B are EP, then A and B commute if and only if $A^\dagger$ and $B^\dagger$ commute.

SOLUTION. Our goal is to give a proof based upon fairly automatic use of computer algebra. We use the *NCGB* package of *NCAlgebra* a vailable from www.math.ucsd.edu/~ncalg

(a) Problem 26-4 asked whether, under the condition that A and B commute, A is EP if and only if $A^\dagger B = BA^\dagger$. Observe that only one implication can always be true, for if B is the identity matrix, then B commutes with both A and its pseudo-inverse regardless of whether A is EP or not.

In the other direction, if A and B commute and A is EP, then we show that $A^\dagger$ and B commute using a noncommutative (NC) Gröbner basis algorithm. Such algorithms in the commutative case are the main tool for solving collections of polynomial equations or at least for reducing them to a "nice" form. For noncommutative polynomials GB's are very experimental. Recall that the input to a Gröbner basis algorithm has two parts. The first part is a list of polynomials, denoted by S, which in our solution of this problem is:

1. The polynomials

$$\left\{(AA^{\dagger})^{*} - AA^{\dagger}, (A^{\dagger}A)^{*} - A^{\dagger}A, AA^{\dagger}A - A, A^{\dagger}AA^{\dagger} - A^{\dagger}\right\} \tag{7}$$

which when set to zero define the pseudo-inverse of A.

2. The polynomial

$$\left\{AB - BA\right\}. \tag{8}$$

3. The fact that A is EP gives

$$\left\{AG - A^{*}, GG^{-1} - 1, G^{-1}G - 1\right\}.$$

4. One way to show that $A^{\dagger}$ and B commute is to create an additional variable Q and we adjoin $Q - A^{\dagger}B + BA^{\dagger}$, since we achieve our goal if Q is 0.

5. We adjoin the adjoints of the polynomials above. (NCAlgebra has a convenient command that accomplishes this.)

The second part of the input is an order on monomials which typically is induced by an order on the variables in the problem. In our case the order is degree lexicographic order induced by the following order on variables:

$$(A^{\dagger}) < (A^{\dagger})^{*} < A < A^{*} < B < B^{*} < G < G^{-1} < G^{*} < (G^{*})^{-1} < Q < Q^{*}. \tag{9}$$

Recall that in a Gröbner basis algorithm variables which are high in the order tend to be eliminated before variables which are low in the order.

We use the command NCProcess, with options set so that it makes and sorts an NC Gröbner basis by running Mora's algorithm for four iterations and removes some redundant polynomials. This makes a long list of polynomial equations, one of which is $Q = 0$. This shows that part (a) is true. The run took 49 minutes on a 1 GigaHertz Pentium III machine with one gigabyte of RAM.

(b) Suppose A and B commute. Again we use the Gröbner basis package applied to the starting relations:

1. The hypothesis that both A and B are EP is equivalent to the existence of G and H satisfying

$$\left\{AG - A^{*}, GG^{-1} - 1, G^{-1}G - 1, BH - B^{*}, HH^{-1} - 1, H^{-1}H - 1\right\}.$$

2. Applying part (a) to the pairs A and B which commute, B and A which commute, A and itself, and B and itself allows us to adjoin the relations

$$\left\{AB^{\dagger} - B^{\dagger}A, BA^{\dagger} - A^{\dagger}B, AA^{\dagger} - A^{\dagger}A, BB^{\dagger} - B^{\dagger}B\right\}.$$

3. The polynomial

$$\left\{AB - BA\right\}. \tag{10}$$

4. We adjoin the defining polynomials for the pseudo-inverse of A given in equation (7) and the corresponding polynomials for B.

5. Once again, our objective is achieved if the additional variable Q satisfying the equation

$$Q - A^{\dagger}B^{\dagger} + B^{\dagger}A^{\dagger}$$

is zero.

6. We adjoin the adjoints of the above equations.

We choose a degree lexicographic order with Q at the top. (We found, when Q is placed at the top of the order, the result to be insensitive to the rest of the order in 4 experiments.) We ran NCProcess with options set to 2 iterations and among the output was $Q = 0$. It follows that $A^{\dagger}$ and $B^{\dagger}$ commute, thus establishing the "if" part of (b). These runs took less than a minute.

The converse is obtained similarly. In the input polynomials we merely replace

$$\left\{AB - BA\right\}$$

by

$$\left\{A^{\dagger}B^{\dagger} - B^{\dagger}A^{\dagger}\right\}$$

and redefine Q by adjoining the polynomial $Q - AB + BA$ instead of $Q - A^{\dagger}B^{\dagger} + B^{\dagger}A^{\dagger}$; we do this since our goal is now to show that A and B commute. Again we choose a degree lexicographic order with Q at the top and find after 3 iterations and about 15 minutes that $Q = 0$. Again we tried some variations on the orders and in each case found that $Q = 0$.

This solves problem Problem 26-4 by fairly automatic use of NCAlgebra's noncommutative Gröbner basis package and suggests the following problem to us:

> Now that some noncommutative computer algebra is available, which of the purely algebraic problems (with a modest number of variables and equations) that have been posed over the past years in IMAGE succumb in an automatic fashion to computer algebra?

Solution 26-4.3 by Hans Joachim WERNER: werner@united.econ.uni-bonn.de; *University of Bonn, Bonn, Germany.*

We begin by recalling some well-known facts. First, $AA^+ = P_{\mathcal{R}(A)}$ and $A^+A = P_{\mathcal{R}(A^*)}$, that is, AA^+ and A^+A are orthogonal projectors onto the range of A (along the null space of A^*) and onto the range of A^* (along the null space of A), respectively. Second, A is EP iff $\mathcal{R}(A) = \mathcal{R}(A^*) = \mathcal{R}(A^+)$ or, equivalently, iff $\mathcal{N}(A) = \mathcal{N}(A^*) = \mathcal{N}(A^+)$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote, as usual, the range (column space) and the null space of $(\cdot)$, respectively. Consequently, A is EP iff $AA^+ = A^+A$. By means of these facts, we are now able to prove the following two theorems.

THEOREM 1. *Let A be EP. Then $AB = BA$ iff $A^+B = BA^+$.*

PROOF: Let A be EP. Moreover, let $AB = BA$. Then $\mathcal{R}(BA) \subseteq \mathcal{R}(A)$ and $B\mathcal{N}(A) \subseteq \mathcal{N}(A)$. Consequently,

$$BA^+ = A^+ABA^+ = A^+BAA^+ = A^+B,$$

that is, $BA^+ = A^+B$. The converse implication follows on similar lines. □

Since a matrix M is EP iff M^+ is EP, and since $(M^+)^+ = M$, the following theorem is a direct consequence of Theorem 1.

THEOREM 2. *Let A and B be a pair of EP matrices and let AB be defined. The following conditions are then equivalent:*

(a) $AB = BA$, (b) $A^+B = BA^+$, (c) $AB^+ = B^+A$, (d) $A^+B^+ = B^+A^+$.

We conclude with mentioning that if $B = 0$, then for each matrix A of the same size as B, clearly $AB = BA = 0$ and $A^+B = BA^+ = 0$, even when A is not EP. This shows that claim (a) of Problem 26-4 is incorrect.

Problem 26-5: Convex Matrix Inequalities

Proposed by Bao-Xue ZHANG: baoxuezhang@sohu.com; *Beijing Institute of Technology, Beijing, China.*

Show that for any Hermitian nonnegative definite matrices A and B of the same size and for any real number λ satisfying $0 \leq \lambda \leq 1$,

(a) $\{P\,[\lambda A + (1-\lambda)B]\,P\}^+ \leq_{\mathsf{L}} \lambda A^+ + (1-\lambda)B^+$,

(b) $[\lambda A + (1-\lambda)B]^+ \leq_{\mathsf{L}} \lambda A^+ + (1-\lambda)B^+ + [\lambda A + (1-\lambda)B]^+ \left\{M\,[\lambda A + (1-\lambda)B]^+\,M\right\}^+ [\lambda A + (1-\lambda)B]^+$,

where P stands for the orthogonal projector onto $\mathcal{R}(A) \cap \mathcal{R}(B)$, $\mathcal{R}(\cdot)$ denotes the range (column space) of $(\cdot)$, $(\cdot)^+$ indicates the Moore-Penrose inverse of $(\cdot)$, $M := I - P$, and $\leq_{\mathsf{L}}$ stands for the Löwner ordering.

Solution 26-5.1 by Jerzy K. BAKSALARY: jbaks@lord.wsp.zgora.pl; *Zielona Góra University, Zielona Góra, Poland* and Oskar Maria BAKSALARY: baxx@main.amu.edu.pl; *Adam Mickiewicz University, Poznań, Poland.*

Theorem 1 in Albert (1969) asserts that, for $U \in \mathcal{C}_{m,n}$ and Hermitian $V \in \mathcal{C}_{m,m}$ and $W \in \mathcal{C}_{n,n}$ (i.e., $V = V^*$ and $W = W^*$), the inequality

$$0 \leq_{\mathrm{L}} \begin{pmatrix} V & U \\ U^* & W \end{pmatrix}$$

holds if and only if

$$0 \leq_{\mathrm{L}} V, \quad \mathcal{R}(U) \subseteq \mathcal{R}(V), \quad \text{and} \quad U^*V^+U \leq_{\mathrm{L}} W, \tag{11}$$

or, equivalently,

$$0 \leq_{\mathrm{L}} W, \quad \mathcal{R}(U^*) \subseteq \mathcal{R}(W), \quad \text{and} \quad UW^+U^* \leq_{\mathrm{L}} V. \tag{12}$$

Consider the matrices

$$A_{\#} = \begin{pmatrix} A^+ & P \\ P & PAP \end{pmatrix} \quad \text{and} \quad B_{\#} = \begin{pmatrix} B^+ & P \\ P & PBP \end{pmatrix}.$$

Since $0 \leq_{\mathrm{L}} A$, it is clear that $0 \leq_{\mathrm{L}} A^+$ and $\mathcal{R}(A^+) = \mathcal{R}(A)$, thus implying $\mathcal{R}(P) \subseteq \mathcal{R}(A^+)$. Moreover, $P(A^+)^+P = PAP \leq_{\mathrm{L}} PAP$, which completes the set of conditions (11) concerning $A_{\#}$. Hence it follows that $0 \leq_{\mathrm{L}} A_{\#}$ and, by analogous arguments, $0 \leq_{\mathrm{L}} B_{\#}$. Consequently,

$$0 \leq_{\mathrm{L}} \lambda A_{\#} + (1-\lambda)B_{\#} = \begin{pmatrix} \lambda A^+ + (1-\lambda)B^+ & P \\ P & P[\lambda A + (1-\lambda)B]P \end{pmatrix},$$

and therefore, on account of the third condition in (12),

$$P\{P[\lambda A + (1-\lambda)B]P\}^+P \leq_{\mathrm{L}} \lambda A^+ + (1-\lambda)B^+. \tag{13}$$

Applying the equality

$$Q(QKQ)^+Q = (QKQ)^+, \tag{14}$$

which holds for any quadratic matrix K and any Hermitian idempotent matrix Q of the same order, transforms (13) to the desired form.

A solution to part (b) of the problem is established in a substantially strengthened form.

THEOREM. *Let A and B be Hermitian nonnegative definite matrices of the same order, let λ be a real number satisfying* $0 \leq \lambda \leq 1$, *and let $C = \lambda A + (1-\lambda)B$. Moreover, let P denote the orthogonal projector onto $\mathcal{R}(A) \cap \mathcal{R}(B)$ and let $M = I - P$. Then*

$$0 \leq_{\mathrm{L}} C^+ - C^+(MC^+M)^+C^+ = (PCP)^+ \leq_{\mathrm{L}} \lambda A^+ + (1-\lambda)B^+. \tag{15}$$

PROOF. The latter inequality in (15) is just part (a) of the problem, which has been proved above, while the former one follows when we apply (11) and (12) with (14) to the matrix

$$C_{\#} = \begin{pmatrix} C^+ & C^+M \\ MC^+ & MC^+M \end{pmatrix}.$$

Consequently, the proof reduces to establishing that

$$C^+ = C^+M(MC^+M)^+MC^+ + P(PCP)^+P, \tag{16}$$

where the matrices on the right-hand side are modified according to (14). Let $C_0 = C^{1/2}$. Then $\mathcal{R}(C_0) = \mathcal{R}(C)$, and since the column spaces of all matrices appearing in (16), as well as the column spaces of their conjugate transposes, are contained in $\mathcal{R}(C) = \mathcal{R}(C^*)$, it follows that (16) is equivalent to

$$C_0C^+C_0 = C_0C^+M(MC^+M)^+MC^+C_0 + C_0P(PCP)^+PC_0. \tag{17}$$

Further, since for any matrix L the product $P_L = L(L^*L)^+L^*$ represents the orthogonal projector onto $\mathcal{R}(L)$, and since $C_0C^+ = C_0^+ = C^+C_0$, equality (17) may be interpreted as

$$P_{C_0} = P_{C_0^+M} + P_{C_0P}.$$

In view of $\mathcal{R}(P) \subseteq \mathcal{R}(C_0)$, it follows that

$$C_0^+C_0P = P_{C_0}P = P. \tag{18}$$

Combining (18) with $MP = 0$ we get $P_{C_0^+M}P_{C_0P} = 0$, which according to the Theorem in §42 of the Halmos (1958) means that $P_{C_0^+M} + P_{C_0P}$ is the orthogonal projector onto $\mathcal{R}(C_0^+M : C_0P)$. Since an orthogonal projector is uniquely determined by a subspace onto which it projects, and since obviously $\mathcal{R}(C_0^+M : C_0P) \subseteq \mathcal{R}(C_0)$, the proof reduces to showing that

$$\mathrm{r}(C_0P : C_0^+M) = \mathrm{r}(C), \tag{19}$$

where $\mathrm{r}(\cdot)$ denotes the rank of a matrix argument. On account of (18), it follows that

$$\mathrm{r}(C_0P : C_0^+M) = \mathrm{r}(C_0^+C_0P : C_0^+C_0^+M) = \mathrm{r}(P : C^+M). \tag{20}$$

But according to Theorem 5 in Marsaglia & Styan (1974),

$$\mathrm{r}(P : C^+M) = \mathrm{r}(P) + \mathrm{r}(MC^+M). \tag{21}$$

Moreover,

$$\mathrm{r}(MC^+M) = \mathrm{r}(MC^+) = \mathrm{r}(MP_C) = \mathrm{r}(P_C - PP_C) = \mathrm{r}(P_C - P).$$

Since $P_CP = P$, it follows from the Theorem in §42 of the Halmos (1958) that $P_C - P$ is a projector, and therefore

$$\mathrm{r}(MC^+M) = \mathrm{tr}(P_C - P) = \mathrm{tr}(P_C) - \mathrm{tr}(P) = \mathrm{r}(C) - \mathrm{r}(P). \tag{22}$$

Consequently, combining (20), (21), and (22) yields the desired equality (19).

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Solution 26-5.2 by Hans Joachim WERNER, *University of Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de;
and Bao-Xue ZHANG, *Beijing Institute of Technology, Beijing, China:* baoxuezhang@sohu.com

Our proposed solution to this problem is based on the following two theorems.

THEOREM 1. *Let V be an $n \times n$ Hermitian nonnegative definite matrix. Moreover, let P be an $n \times n$ orthogonal projector such that $\mathcal{R}(P) \subseteq \mathcal{R}(V)$, and let $\mathcal{P}(P|V)$ denote the set of all those (generally oblique) projectors Q which satisfy simultaneously $\mathcal{R}(Q) = \mathcal{R}(P)$ and $V\mathcal{N}(P) \subseteq \mathcal{N}(Q)$; here $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the range (column space) and the null space, respectively, of $(\cdot)$. Then we have:*

(a) *$\mathcal{R}(V) = \mathcal{R}(P) \oplus V\mathcal{N}(P)$; here $\oplus$ indicates a direct sum.*

(b) *QV and $(I-Q)V$ are invariant for any choice of $Q \in \mathcal{P}(P|V)$.*

(c) *For each $Q \in \mathcal{P}(P|V)$, $\mathcal{R}(QV) = \mathcal{R}(P)$, $\mathcal{R}((I-Q)V) = V\mathcal{N}(P)$ and $QV(I-Q)^* = 0$.*

(d) *For each $Q \in \mathcal{P}(P|V)$, $V = QV + (I-Q)V$, and QV and $(I-Q)V$ are Hermitian nonnegative definite.*

(e) *$P(PV^+P)^+PV^+ \in \mathcal{P}(P|V)$, and so for each $Q \in \mathcal{P}(P|V)$, $QV = P(PV^+P)^+P = (PV^+P)^+$.*

(f) *For each $Q \in \mathcal{P}(P|V)$, $(I-Q)V = VM(MVM)^+MV$ where $M := I-P$.*

(g) *$V = (PV^+P)^+ + VM(MVM)^+MV \geq_{\mathrm{L}} (PV^+P)^+$ where $M := I-P$.*

(h) *$V^+ = (PVP)^+ + V^+M(MV^+M)^+MV^+ \geq_{\mathrm{L}} (PVP)^+$ where $M := I-P$.*

PROOF: Since $\mathcal{R}(P) \subseteq \mathcal{R}(V)$, (a) follows immediately from (2.6) in Werner & Yapar (1996), and so it is evident that $\mathcal{P}(P|V)$ is not empty. From Theorem 2.5 in Werner & Yapar (1996) we even know that $P(PV^+P)^+PV^+$ is in $\mathcal{P}(P|V)$; this proves the first part of (e). Theorem 2.7(i) & Theorem 2.7(iii) in Werner & Yapar (1996) give us (b) and (c). Equation $QV(I-Q)^* = 0$ from (c) implies $QV = QVQ^* = VQ^*$ and $(I-Q)V = (I-Q)V(I-Q)^* = V(I-Q)^*$. Since in view of $I = Q + (I-Q)$ we trivially have $V = QV + (I-Q)V$, the proof of (d) is now complete. Combining the first part of (b) with the first part of (e) gives $QV = P(PV^+P)^+P$ since $PV^+V = P$. By checking the defining equations of the Moore–Penrose inverse of PV^+P it is seen that $P(PV^+P)^+P = (PV^+P)^+$, and so we also have the second part (e). Since $M = I - P$, $\mathcal{R}(M) = \mathcal{N}(P)$. We note that $VM(MVM)^+M$ is a projector onto $\mathcal{R}(VM) = V\mathcal{N}(P)$ along a subspace that includes $\mathcal{N}(M) = \mathcal{R}(P)$. Therefore, in view of (a), (b) & (c), $(I-Q)V = VM(MVM)^+MV$ is invariant for any choice of $Q \in \mathcal{P}(P|V)$. This is (f). From (d), (e) & (f) we obtain (g). Since $\mathcal{R}(V^+) = \mathcal{R}(V)$, $\mathcal{R}(P) \subseteq \mathcal{R}(V^+)$, and so, in view of (g) and $(V^+)^+ = V$, it is clear that (h) also holds true. □

THEOREM 2. *Let A and B be Hermitian nonnegative definite matrices of the same size. Then we have:*

(a) *$A + B$ is a Hermitian nonnegative definite matrix with $\mathcal{R}(A+B) = \mathcal{R}(A) + \mathcal{R}(B)$ and $\mathcal{N}(A+B) = \mathcal{N}(A) \cap \mathcal{N}(B)$.*

(b) *$A(A+B)^-B$ is invariant under the choice of g-inverse $(A+B)^-$.*

(c) *$A(A+B)^+B = B(A+B)^+A$ is Hermitian nonnegative definite with $\mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(A) + \mathcal{N}(B)$ as its range (column space) and null space, respectively.*

(d) *If P denotes the orthogonal projector onto $\mathcal{R}(A) \cap \mathcal{R}(B)$ along $\mathcal{N}(A) + \mathcal{N}(B)$, then $(A(A+B)^+B)^+ = P(A^- + B^-)P = P(A^+ + B^+)P$ irrespective of the choice of A^- and B^-.*

The proof of this theorem is left as an exercise to the interested reader. In this context, we mention that Theorem 10.1.8 in Rao & Mitra (1971) deals with a less restrictive situation; we also note that $A(A+B)^-B$ is often called the parallel sum of A and B.

We are now in a position to prove the claimed inequalities. First, let λ be such that $0 < \lambda < 1$. We recall that for each matrix C and each scalar $c \neq 0$, and we have $(C^+)^+ = C$ and $(cC)^+ = c^+C^+$ with $c^+ = c^{-1}$. Hence, in view of Theorem 2(d),

$$\begin{aligned} [P(\lambda A + (1-\lambda)B)P]^+ &= \lambda^+A^+\left[\lambda^+A^+ + (1-\lambda)^+B^+\right]^+(1-\lambda)^+B^+ \\ &= A^+\left[(1-\lambda)A^+ + \lambda B^+\right]^+B^+. \end{aligned}$$

For convenience, we now put $W := (1-\lambda)A^+ + \lambda B^+$. Clearly, $W = A^+ + \lambda(B^+ - A^+) = B^+ + (1-\lambda)(A^+ - B^+)$. The matrices A^+ and B^+ are Hermitian nonnegative definite with $\mathcal{R}(A^+) = \mathcal{R}(A)$ and $\mathcal{R}(B^+) = \mathcal{R}(B)$. Therefore, since $0 < \lambda < 1$, according to Theorem 2(a), W is Hermitian nonnegative definite with $\mathcal{R}(W) = \mathcal{R}(A) + \mathcal{R}(B)$. Consequently, $WW^+(B^+ - A^+) = B^+ - A^+$ and $(A^+ - B^+)W^+W = A^+ - B^+$. With this in mind, we get

$$\begin{aligned} A^+W^+B^+ &= \left[\lambda A^+ + (1-\lambda)A^+ + \lambda B^+ - \lambda B^+\right]W^+\left[\lambda B^+ + (1-\lambda)B^+ + (1-\lambda)A^+ - (1-\lambda)A^+\right] \\ &= \left[W + \lambda(A^+ - B^+)\right]W^+\left[W + (1-\lambda)(B^+ - A^+)\right] \\ &= W + \lambda(A^+ - B^+) + (1-\lambda)(B^+ - A^+) + \lambda(1-\lambda)(A^+ - B^+)W^+(B^+ - A^+) \\ &= A^+ + (1-\lambda)(B^+ - A^+) - \lambda(1-\lambda)(A^+ - B^+)W^+(A^+ - B^+) \\ &= \lambda A^+ + (1-\lambda)B^+ - \lambda(1-\lambda)(A^+ - B^+)W^+(A^+ - B^+) \end{aligned}$$

and thus, since W^+ is Hermitian nonnegative definite, also

$$\lambda A^+ + (1-\lambda)B^+ - A^+W^+B^+ = \lambda(1-\lambda)(A^+ - B^+)W^+(A^+ - B^+) \geq_{\mathrm{L}} 0.$$

This completes the proof of inequality (a) in case of $0 < \lambda < 1$. For $\lambda = 0$, inequality (a) reduces to $(PBP)^+ \leq_{\mathrm{L}} B^+$, which is the result of Theorem 1(h). For $\lambda = 1$, inequality (a) becomes $(PAP)^+ \leq_{\mathrm{L}} A^+$, which again holds according to Theorem 1(h).

So it remains to show that inequality (b) also holds true. To this end, let $0 \leq \lambda \leq 1$ be arbitrary but fixed. For convenience, put $H := \lambda A + (1-\lambda)B$. Then, according to Theorem 1(h), $H^+ = (PHP)^+ + H^+M(MH^+M)^+MH^+$. From inequality (a), we already know that $(PHP)^+ \leq_{\mathrm{L}} \lambda A^+ + (1-\lambda)B^+$. By combining these two observations we obtain the claimed inequality (b).

References

[1] C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and Its Applications.* Wiley, New York.

[2] H. J. Werner & C. Yapar (1996). On inequality constrained generalized least squares selections in the general possibly singular Gauss–Markov model: a projector theoretical approach. *Linear Algebra and its Applications*, 237/238, 359–393.

IMAGE Problem Corner: New Problems

Problem 27-1: A Class of Square Roots of Involutory Matrices

Proposed by Richard William FAREBROTHER, *11 Castle Road, Bayston Hill, Shrewsbury, England:* Msrbsrf@fs1.ec.man.ac.uk

Let A and B be $n \times n$ nonsingular real matrices satisfying $ABA = B$ and $BAB = A$. Show that $M = AA = BB$ is involutory (i.e., $M^2 = I$), and identify a possible application for matrices of this type (see also "The Naming of Parts" on page 11 above).

Problem 27-2: Specific Generalized Inverses

Proposed by Jürgen GROß & Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:*
gross@statistik.uni-dortmund.de, trenkler@statistik.uni-dortmund.de

Characterize the class $\mathcal{A}$ of $m \times n$ matrices A with complex entries such that $(A^*A)^kA^*$ is a generalized inverse of A for an arbitrary positive integer k.

Problem 27-3: A Word Problem

Proposed by Charles R. JOHNSON, *The College of William and Mary, Williamsburg, Virginia:* crjohnso@m24.math.wm.edu

Let $V(A,B)$ and $W(A,B)$ be "words" in the two letters A and B, e.g., W could be $W(A,B) = AABABBBA$. Show that any given W is identically equal to V if and only if $W(L,U) = V(L,U)$ for the particular matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 27-4: An Inequality for Hadamard Products Involving a Correlation Matrix

Proposed by Shuangzhe LIU, *Australian National University, Canberra, Australia:* lius@maths.anu.edu.au

Show that for a $k \times k$ positive definite correlation matrix A

$$A \circ A^{-1} + I \leq_{\mathsf{L}} \frac{(c+d)^2}{2cd}(A \circ A)^{-1},$$

where $\circ$ stands for the Hadamard product, $\leq$ stands for the Löwner ordering, and c and d are the largest and smallest eigenvalues of A, respectively. This inequality is a counterpart to an inequality established by G. P. H. Styan (1973): Hadamard products and multivariate statistical analysis, *Linear Algebra and Its Applications*, 6, 217–240.

Problem 27-5: Orthogonal Projectors and the Löwner Ordering

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

How are the complex orthogonal projectors P and Q (i.e., Hermitian idempotent matrices with complex entries) related if they satisfy the following inequality in the Löwner ordering

$$\sqrt{2}\,(P+Q)^{\frac{1}{2}} \leq_{\mathsf{L}} P + Q,$$

where $(P+Q)^{\frac{1}{2}}$ denotes the unique nonnegative definite square root of $P+Q$?

Problem 27-6: Inequalities of Hadamard Products of Nonnegative Definite Matrices

Proposed by Xingzhi ZHAN, *Tohoku University, Aoba-ku, Sendai 980-8579, Japan:* zhan@math.is.tohoku.ac.jp

Let A and B be nonnegative definite matrices of the same size and let s and t be positive real numbers such that $s + t = 1$. Prove or disprove that

$$A^s \circ B^t \leq_{\mathsf{L}} (A \circ I)^s (B \circ I)^t \leq_{\mathsf{L}} (sA + tB) \circ I,$$

where $\circ$ stands for the Hadamard product and $\leq_{\mathsf{L}}$ for the Löwner ordering.