The Bulletin of the International Linear Algebra Society

IMAGE

Serving the International Linear Algebra Community

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ILAS Presidents' and Vice President's Annual Report: April 2002

1. The following persons have been elected to ILAS offices with terms that began on March 1, 2002.

President: Daniel Hershkowitz
(three-year term ending February 28, 2005).

Board of Directors: Ravi Bapat and Michael Neumann
(three-year terms ending February 28, 2005).

The following continue in their offices to which they were previously elected:

Vice-President: Roger Horn (term ends February 29, 2004)

Secretary/Treasurer: Jeff Stuart (term ends February 28, 2003)

Board of Directors:
Harm Bart (term ends February 28, 2003),
Steve Kirkland (term ends February 28, 2003),
Tom Markham (term ends February 29, 2004),
Daniel Szyld (term ends February 29, 2004).

2. Ludwig Elsner (Bielefeld, Germany) has been appointed to a three-year term (2002–2004) as an Editor-in-chief of *The Electronic Journal of Linear Algebra* (ELA) and joins Danny Hershkowitz in that position. Shaun Fallat (Regina, Canada) has been appointed as ILAS-Net/IIC Manager. Jim Weaver (Pensacola, USA) has been reappointed as ILAS Outreach Director. Michael Tsatsomeros (Pullman, USA) has been appointed as Assistant Managing Editor of ELA.

3. The president's Advisory Committee this year consists of Paul Van Dooren, Pauline van den Driessche, Moshe Goldberg, Leiba Rodman (chair).

4. This fall there will be elections for Secretary/Treasurer (the term of Jeff Stuart ends on February 28, 2003) and two members of the Board of Directors (to replace retiring members Harm Bart and Steve Kirkland). We have started the procedure of appointing the ILAS 2002 Nominating Committee. The President has appointed Peter Lancaster to chair this Committee. The Board of Directors has appointed Rien Kaashoek and Volker Mehrmann to the Committee. Two additional members will be selected by the ILAS Advisory Committee.

5. The 9th ILAS Conference took place on June 25–29, 2001 at the Technion in Haifa, Israel. The co-chairs of the organizing committee were Moshe Goldberg, Daniel Hershkowitz, and Raphael Loewy. There were 109 registered participants from 25 countries. Lloyd Trefethen (Oxford, UK) gave the triennial LAA Lecture on "Stability of Gaussian elimination". Steve Kirkland (Regina, Canada) gave the triennial Taussky-Todd lecture on "Extremizing algebraic properties of tournament matrices". The conference organizers offered an excursion consisting of a tour of the beautiful port city of Haifa, including the Bahai Gardens. ILAS expresses its gratitude to the Institute of Advanced Studies (Shimon Reich, Director) at the Technion for its financial and administrative support. A banquet was held on the last night of the conference with Moshe Goldberg as the emcee. A delightful after-dinner talk was given by Mr. Carl Alpert, the historian of the Technion.

6. Tsuyoshi Ando (Sapporo, Japan) and Peter Lancaster (Calgary, Canada) have been selected to receive the 2002 Hans Schneider Prize in Linear Algebra for outstanding lifetime contributions to the field of linear algebra. Professor Ando will be awarded his prize at the 10th ILAS Conference in Auburn. Professor Lancaster will be awared his prize at the 11th ILAS Conference in Coimbra. The selection committee consisted of Jesse Barlow, Harm Bart (Chair), Ludwig Elsner, Roger Horn, Frank Uhlig, Richard Brualdi (ex officio).

7. The following ILAS conferences are planned for the near future:

a. The 10th ILAS Conference, "Challenges in Matrix Theory", Auburn University, Auburn, Alabama, USA, June 10–13, 2002: www.auburn.edu/ilas2002

At this conference T. Ando (Sapporo, Japan) will be awarded the ILAS Hans Schneider Prize in Linear Algebra and will deliver his Prize Lecture.

b. The 11th ILAS Conference, Coimbra, Portugal, Summer, 2004. At this conference Peter Lancaster (Calgary, Canada)) will be awarded the ILAS Hans Schneider Prize in Linear Algebra and will deliver his Prize Lecture.

c. There is no ILAS conference planned for 2003, in which year the SIAG-LA will hold its triennial conference (July 23–26, 2003, Williamsburg, Virginia).

d. The 12th ILAS Conference, Regina, Canada, Summer 2005.

8. ILAS has recently endorsed several conferences of particular interest to ILAS members. They are:

(a) Multilinear Algebra and Matroid Theory Workshop at the Complexo Interdisciplinar da Universidade de Lisboa (Portugal), March 24–26, 2002; contact Jose Dias da Silva.

(b) 6th-Western Canada Linear Algebra Meeting (WCLAM), University of Regina (Canada), May 10–11, 2002; contact Shaun Fallat.

(c) Computational Linear Algebra with Applications, Milovy (Czech Republic), August 4–10, 2002; contact Daniel Szyld.

(d) Topics in Linear Algebra Conference, Iowa State University (USA), September 13–14, 2002; contact Leslie Hogben.

9. ILAS has continued to consider requests for the sponsorship of an ILAS Lecturer at a conference which is of substantial interest to ILAS members. Currently a maximum of US$1500 is budgeted for this program, with a maximum of US$750 for any one ILAS speaker. ILAS is sponsoring three Lecturers in 2002:

(a) Volker Mehrmann at the Householder XV Meeting, Peebles (Scotland), June 17–21, 2002; contact Nick Higham.

(b) Avi Berman (Technion, Israel) at the 5th China Matrix Conference, Shanghai (China), August 14–18, 2002; contact Erxiong Jiang.

(c) Sam Hedayat at the Topics in Linear Algebra Conference, Iowa State University (USA), September 13–14, 2002.; contact Leslie Hogben.

10. *The Electronic Journal of Linear Algebra* (ELA):

Volume 1, published in 1996, contained 6 papers.

Volume 2, published in 1997, contained 2 papers.

Volume 3 (the Hans Schneider issue), published in 1998, contained 13 papers.

Volume 4, published also in 1998, contained 5 papers.

Volume 5, published in 1999, contained 8 papers.

Volume 6 (Proceedings of the Eleventh Haifa Matrix Theory Conference), published in 1999 and 2000, contained 8 papers.

Volume 7, published in 2000, contained 14 papers.

Volume 8, published in 2001, contained 12 papers.

Volume 9, is being published now. As of May 12, 2002, it contains 8 papers.

ELA's primary site is at the Technion: www.math.technion.ac.il/iic/ela/ Mirror sites are located in Temple University, in the University of Chemnitz, in the University of Lisbon, in EMIS—The European Mathematical Information Service offered by the European Mathematical Society, and in 36 EMIS Mirror Sites.

Volumes 1–7 (1996–2000) of ELA are in print, bound as two separate books: vol. 1–4 and 5–7. Copies can be ordered from Jim Weaver: jweaver@uwf.edu

11. IMAGE: *The Bulletin of the International Linear Algebra Society* continues with Editors-in-Chief George P. H. Styan and Hans-Joachim Werner, and Associate Editors: Jerzy K. Baksalary, Oskar Maria Baksalary, S. W. Drury, Stephen J. Kirkland, Steven J. Leon, Chi-Kwong Li, Simo Puntanen, Peter Šemrl, and Fuzhen Zhang. The number of pages has been increased to 36 effective with IMAGE 27 (October 2001).

12. ILAS-NET: As of May 5, 2002, we have circulated 1178 ILAS-NET announcements. ILAS-NET currently has 493 subscribers.

13. ILAS INFORMATION CENTER (IIC) has a daily average of 300 information requests (not counting FTP operations). IIC's primary site is at Regina. Mirror sites are located in the Technion, in Temple University, in the University of Chemnitz and in the University of Lisbon.

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2002 ILAS Business Meeting

12 June 2002: Auburn, Alabama, USA

The 2002 ILAS Business Meeting will be held on Wednesday, June 12, 2002, starting at 17:30 hours (5:30 pm) in The Auditorium of the Auburn University Hotel and Dixon Conference Center Auburn, Alabama, USA. This Business Meeting will be held during the 10th ILAS Conference, "Challenges in Matrix Theory", Auburn University, Auburn, Alabama, USA, June 10–13, 2002: www.auburn.edu/ilas2002

1. Welcome, and thanks to Richard A. Brualdi for his 6 years as ILAS President.

2. Reading of Notice of Meeting.

3. Reading of the Minutes of Previous Meeting in Haifa, Israel, 2001.

4. Report of President/Vice-President.

5. Report of Secretary/Treasurer.

6. Reports of Committees.

A. Report on ELA (Ludwig Elsner/Danny Hershkowitz).

B. Report on IMAGE (George Styan/Hans Joachim Werner).

C. Report on IIC and ILAS-Net (Shaun Fallat).

D. Outreach Director's Report (Jim Weaver).

E. Report of 2002 Nominating Committee (Peter Lancaster, chair).

F. Report of the retiring Education Committee: Frank Uhlig, (chair); David Carlson, Luz DeAlba, Jane Day, Guershon Harel, Charles Johnson, Jeff Stuart.

G. Report of the retiring Journals Committee (Chi-Kwong Li, chair) [Committee consists of Danny Hershkowitz, Chi-Kwong Li, Hans Schneider, George Styan, Jim Weaver].

7. Future ILAS Conferences.

(a) Coimbra, summer 2004. (b) Regina, summer 2005.

8. Other business. **9.** Adjournment.

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Another Web Site on Magic Squares

Further to the articles on magic squares in IMAGE [27 (October 2001), 3–11], I would like to draw readers to my Web site: www.magic-squares.de, which includes the results of about 75 algorithms to create magic squares; recently added are algorithms to construct inlaid magic squares, as well as pandiagonal and most-perfect magic squares, and the 25×25 bimagic square—the first bimagic cube in the world—discovered by John R. Hendricks in June 2000.

There is also a choice of using flash-animations instead of simple html-tables. The opening page features the oldest known magic square of the world: from Loh-Shu in 2800 BC.

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ILAS Treasurer's Annual Report: March 1, 2001–February 28, 2002

Net Account Balances on March 1, 2001			
Vanguard (ST Fed. Bond Fund 1053.184 Shares)			
(72% Schneider Fund and 28% Todd Fund)		10,879.39	
Checking account		64,735.65	**75,615.04**
General Fund		31,001.62	
Conference Fund		10,018.94	
ILAS/LAA Fund		3,590.00	
Olga Taussky Todd/John Todd Fund		8,831.47	
Frank Uhlig Education Fund		3,345.98	
Hans Schneider Prize Fund		18,827.03	75,615.04

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March 1, 2001 through February 28, 2002

Income:			
Interest	792.82		
Dues	6880.00		
Book Sales	815.00		
General Fund	515.00		
Conference Fund	20.00		
ILAS\LAA Fund	2000.00		
Taussky-Todd Fund	635.67		
Uhlig Education Fund	130.00		
Schneider Prize Fund	1008.15	12,796.64	
Expenses:			
IMAGE (1 issue)	1395.33		
Speakers (3)	2450.00		
ELA Print II Production	1956.61		
Fees	99.48		
Labor - Mailing & Conference	334.00		
Postage	881.04		
Supplies and Copying	217.73		
Bad Checks	140.00	7474.19	

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February 28, 2002 Checking Account Balance		**69,286.21**	
Net Account Balances on February 28, 2002			
Vanguard (ST Fed. Bond Fund 1111.153 Shares)			
(72% Schneider Fund and 28% Todd Fund)		11,578.21	
Checking account		69,281.21	**$80,859.42**
General Fund		34,402.18	
Conference Fund		10,038.94	
ILAS/LAA Fund		4,840.00	
Olga Taussky Todd/John Todd Fund		8,267.14	
Frank Uhlig Education Fund		3,475.98	
Hans Schneider Prize Fund		19,835.18	$80,859.42

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George Salmon and the Theory of Determinants in 1876

The 39-year-old Karl Pearson (1857–1936) seems to have thought that readers of his 1896 article on "regression, heredity and panmixia" would find a thirty-year-old edition of George Salmon's *Lessons Introductory to the Modern Higher Algebra* (1866) a suitable guide to the theory of determinants. This unexpected discovery prompted me to investigate the life and works of this previously little known (at least to me) author. I found George Salmon listed in the *Dictionary of Scientific Biography* (1975) and cited in the *Theory of Determinants in the Historical Order of Development* by Muir (1911, 1920). A biography and a portrait of George Salmon appear on the "The MacTutor History of Mathematics Archive" Web site[1]: www-groups.dcs.st-and.ac.uk/~history/BiogIndex.html

From the article by McConnell (1975) we learn that George Salmon (1819–1904), who was a lifelong member (student, Fellow, and Provost) of Trinity College Dublin, and a personal friend of Cayley and Sylvester, played an important role in the application of the theory of invariants and covariants of algebraic forms to the geometry of curves and surfaces. He wrote a series of well-respected textbooks including *A Treatise on Conic Sections* (1848), *A Treatise on Higher Plane Curves* (1852), and *A Treatise on the Analytic Geometry of Three Dimensions* (1862), as well as the work cited above. According to McConnell (1975),

> These four treatises on conic sections, higher plane curves, modern higher algebra, and the geometry of three dimensions not only gave a comprehensive treatment of their respective fields but also were written with a clarity of expression and an elegance of style that made them models of what a textbook should be. They were translated into every western European language and ran into many editions (each incorporating the latest developments); they remained for many years the standard advanced textbooks in their respective subjects.

Although Muir (1911, p. 106; 1920, pp. 12, 52, 55, 66) certainly mentions three editions of Salmon's *Lessons Introductory to Modern Higher Algebra* (1859, 1866, 1876), together with two German translations (1863, 1877), one French translation (1868), a Polish quasi-translation (1874), and a Belgian derivative (1874) which mentions Salmon's name in its title, he seems to do so without any great enthusiasm.

By contrast, the descriptions by Muir (1920) of the numerous alternative texts in several European languages are very much more fulsome. In particular, we note his descriptions (*op. cit.*, pp. 55, 58, 66) of the historical sections of the works by Günther (1875, 1877), Studnička (1876) and Mellberg (1876). In this connection, it is interesting to observe that these three volumes were published in Germany, Bohemia, and Finland, respectively, and that they feature prominently in the short list of works cited by Knobloch (1994) in his article on the history of determinants in the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences.*

[1] Created and maintained by John J. O'Connor and Edmund F. Robertson in the School of Mathematics and Statistics, University of St Andrews, Scotland.

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K. Pearson (1896). Mathematical contributions to the theory of evolution, III: Regression, heredity and panmixia. *Philosophical Transactions of the Royal Society of London, Series A: Mathematical and Physical Sciences*, 187, 253–318. [Reprinted in *Karl Pearson's Early Statistical Papers* (E. S. Pearson, ed.), Cambridge University Press, pp. 113–178 (1948).]

G. Salmon (1848). *A Treatise on Conic Sections: Containing an Account of Some of the Most Important Modern Algebraic and Geometric Methods.* Hodges & Smith, Dublin.

G. Salmon (1852). *A Treatise on Higher Plane Curves: Intended as a Sequel to "A Treatise on Conic Sections".* Hodges & Smith, Dublin.

G. Salmon (1859). *Lessons Introductory to the Modern Higher Algebra.* Hodges & Smith, Dublin. [2nd edition: Hodges & Smith, Dublin, 1866; 3rd edition: Hodges, Figgis & Co., Dublin, 1876; 4th edition: Hodges, Figgis & Co., Dublin, 1885; 5th edition: Chelsea, New York, 1964.]

G. Salmon (1862). *A Treatise on the Analytic Geometry of Three Dimensions.* Hodges & Smith, Dublin.

F. J. Studnička (1876). A. L. Cauchy als formaler Begründer der Determinanten-Theorie. Eine literarisch-historische Studie. *Abhandlungen der Königlichen Böhmischen Gesellschaft der Wissenschaften,* Series 6, 8 (für 1875–1876), 46 pp.

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Sir Thomas Muir and Nineteenth-Century Books on Determinants

R. William Farebrother, Shane T. Jensen & George P. H. Styan

As pointed out by Turnbull (1935, p. 79) "For prolonged singleness of purpose amid official duties Muir's five-volume *History of Determinants* is a monument unparalleled in the scientific world. It is a classic: if there exists anywhere a more detailed and comprehensive history of theoretical knowledge, one would be interested to hear of it".

This *magnum opus* by Sir Thomas Muir (1844–1934) was published in five volumes between 1890 and 1930, with a reprint edition of the first four volumes published by Dover in 1960:

Thomas Muir (1890). *The Theory of Determinants in the Historical Order of Development, Part I: Determinants in General, Leibniz (1693) to Cayley (1841).* Macmillan, London, xi + 278 pp. [This is the original version of "Volume I" of the 5-volume set but the phrase "Volume I" does not appear on the title page.]

Thomas Muir (1906). *The Theory of Determinants in the Historical Order of Development, Part I: General Determinants up to 1841, Part II: Special Determinants up to 1841.* Second Edition. Macmillan, London, xi + 491 pp. [This is the "Second Edition" of Volume 1 of the 5-volume set but (again) the phrase "Volume I" does not appear on the title page; there is no mention in the Preface (dated 19 July 1905) of an earlier edition, which would be the 1890 edition cited above. Reprinted by Dover in 1960.]

Thomas Muir (1911). *The Theory of Determinants in the Historical Order of Development, Volume II: The Period 1841 to 1860.* Macmillan, London, xvi + 475 pp. [Volume 2 of the 5-volume set—only this edition was published. Reprinted by Dover in 1960.]

Thomas Muir (1920). *The Theory of Determinants in the Historical Order of Development, Volume III: The Period 1861 to 1880.* Macmillan, London, xxvi + 503 pp. [This is Volume 3 of the 5-volume set—only this edition was published. Reprinted by Dover in 1960.]

Thomas Muir (1923). *The Theory of Determinants in the Historical Order of Development, Volume IV: The Period 1880 to 1900.* Macmillan, London, xxxi + 508 pp. [Volume 4 of the 5-volume set—only this edition was published. Reprinted by Dover in 1960.]

Sir Thomas Muir (1930). *Contributions to the History of Determinants: 1900–1920.* Blackie, London, xxiv + 408 pp. [Volume 5 of the 5-volume set but the phrase "Volume V" does not appear on the title page; also referred to as the "Supplement" to the previous four volumes; author is now "Sir" Thomas Muir. Only this edition was published— not reprinted by Dover.]

Thomas Muir (1960). *The Theory of Determinants in the Historical Order of Development—Four Volumes Bound as Two, Volume One: General and Special Determinants up to 1841, Volume Two: The Period 1841 to 1860; Volume Three: The Period 1861–1880, Volume Four: The Period 1880 to 1900.* Reprint Edition. Dover, New York. [Reprint vol. 1: xi + 491 pp. & xvi + 475 pp. Reprint vol. 2: xxvi + 503 pp. & xxxi + 508 pp. "Unabridged and Unaltered Republication" of the four volumes I–IV above (but not "Volume V"), bound as a two-volume set (hardcover).]

"And until within a few days of his death [in 1934] a sixth volume was in preparation, with the date 1940 [when Muir would have been 96] as the goal" [Turnbull (1935, p. 79)].

In 1882, Muir first[1] published *A Treatise on the Theory of Determinants*, which ultimately grew into the first volume of his *magnum opus*. This was revised and enlarged (from 240 to 766 pages) by William Henry Metzler (b. 1863) and published first privately in 1930 and then by Longmans, Green and Company, London, in 1933 (with a Dover reprint in 1960).

Thomas Muir (1882). *A Treatise on the Theory of Determinants with Graduated Sets of Exercises for Use in Colleges and Schools.* Macmillan: London, vi + 240 pp.

Thomas Muir (1930). *A Treatise on the Theory of Determinants.* Revised and Enlarged by William H. Metzler. "Privately published": Albany, New York, iv + 766 pp. [Not really a "Revised Edition" of Muir (1882) but a very different book.]

Thomas Muir (1933). *A Treatise on the Theory of Determinants.* Revised and Enlarged by William H. Metzler. Longmans, Green and Company, London, iv + 766 pp. [Apparently identical to Muir (1930).]

Thomas Muir (1960). *A Treatise on the Theory of Determinants.* Revised and Enlarged by William H. Metzler. Reprint Edition. Dover Books on Advanced Mathematics. Dover, New York, iv + 766 pp. ["Unabridged and corrected (paperback) republication of Muir (1933).]

Thomas Muir was born on 25 August 1844 in Stonebyres, Falls of Clyde, Lanarkshire, Scotland, and died on 21 March 1934 in Rondebosch (near Cape Town), South Africa.

From the two obituaries by Turnbull[2] (1934, 1935), we learn that Muir was brought up at Biggar, a neighbouring small country town [to Stonebyres], where his father was a souter [shoemaker]. Muir was educated at the University of Glasgow and showed great ability in Greek, his favourite subject. Sir William Thomson[3] (1824–1907) recognized Muir's "uncommon mathematical power" and persuaded him to study mathematics.

Muir became a tutor at St Andrews University in Scotland, then travelled in Europe, meeting many of the top mathematicians and acquiring a taste for collecting books. "In 1871 he

[1]The National Library of Canada AMICUS entry no. 6222555 states that the Dover Reprint Edition Muir (1960) was "Originally published. Paris: Rothschild, 1874". We have not found any corroboration of this early edition and there seems to be no mention of it in Muir (1960).

[2]Herbert Westren Turnbull (1885–1961).

[3]Knighted in 1866 for his work on laying a submarine cable between Ireland and Newfoundland, Thomson was raised to the peerage in 1892 (as Baron Kelvin of Largs) in recognition of his work in engineering and physics. For more information and a nice portrait, visit http://www.bingo-ev.de/~kg666/verschie/physiker/kelvin.htm

was appointed an assistant to the professor of mathematics at the University of Glasgow, a post which he held for three years. It was then that he established his fame as a teacher of mathematics, communicating his first paper to the Royal Society of Edinburgh and becoming a Fellow. From 1874 to 1892 he was chief mathematical master in the Glasgow High School, being responsible also for the teaching of science" [Turnbull (1934, p. 179)].

"In 1892 the post of Superintendent-General of Education in the Cape Colony [South Africa] ... became vacant, and Mr. Cecil Rhodes, Prime Minister in the Cape Colony ... selected Muir", who "was then considering the offer of the Chair of Mathematics at the newly opened Leyland Stanford University in California". Moreover, Muir "occupied the post of Superintendent-General of Education in the Cape Colony until the Union in 1910, continuing as Superintendent-General [of Education] of the Cape Province [of the Union of South Africa] until his retirement in 1915 at the age of seventy-five years" [Turnbull (1934, p. 180)].

In 1882, Muir received the honorary LL.D. degree from the University of Glasgow; in 1900 he was elected a Fellow of the Royal Society and received the CMG [Companion of (the Order of) St Michael and St George] in 1901; Muir was knighted in 1915.

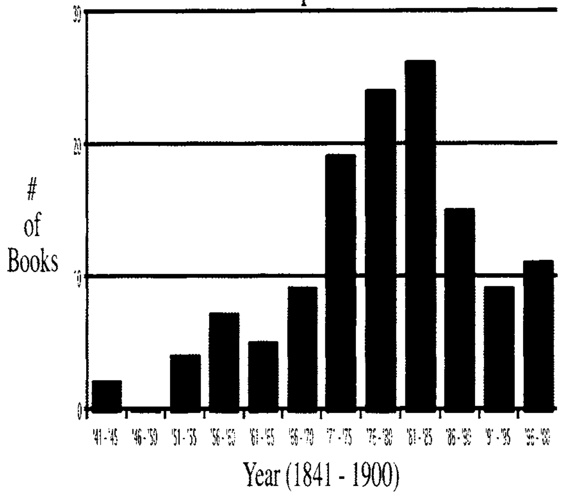
"Muir had wonderful health: until he was 80 he had occasion to consult a doctor only four times and a dentist but twice" [Turnbull (1934, p. 182)]. In 1931, at the age of 87, he showed surprising abilities to keep with the modern flavour of his subject when he wrote that he "welcomed the light matrix proofs in contrast to the heavy footed method of thirty-five years ago" [Turnbull (1935, p. 79)].

Mainly using Muir's five-volume *magnum opus*, we have identified 131 books on determinants published in the 19th century. These books are identified on pages 8–15. The life span of the author is given in column 4. Information about the edition of the book and other comments, often about translation, appear in column 6. Quotations from the five-volume *magnum opus* by Muir appear in the third column from the right, with reference to the volume and page number in the last 2 columns. Pages 8–15 are printed from an Excel spreadsheet, which is available by e-mail from the third author.

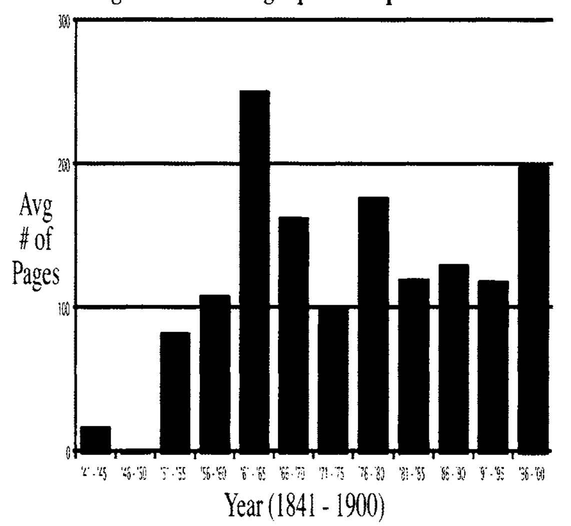
The first books on determinants seem to be Jacobi's "De determinantibus functionalibus" (1841), the 16-page booklet by Cayley (1844) and the 63-page book by Spottiswoode (1851) which was revised and republished in the *Journal der reine und angewandte Mathematik*, vol. 51, pp. 209-271 & 328–381 (1856). For more about William Spottiswoode (1825–1883) see IMAGE 23 (October 1999), pp. 3–5, 25 (October 2000), pp. 19–21, and 27 (October 2001), page 2.

Almost half of these 131 books are in German, with 58 entries. Next come books in French (18), English (17) and Italian (12). There are 4 books in each of Polish, Russian and Spanish; 3 in Dutch; 2 in each of Czech, Latin, Portuguese and Swedish and 1 in each of Croatian, Norwegian and Serbian.

Number of Books per Five-Year Period



Average Number of Pages per Book per Five-Year Period



References

H. W. T[urnbull] (1934). Sir Thomas Muir 1844–1934. *Obituary Notices of Fellows of the Royal Society*, 1, 178–184. [On page 178 is an excellent portrait of Muir "taken from a photograph by Elliott and Fry, Ltd".]

H. W. Turnbull (1935). Thomas Muir. *The Journal of The London Mathematical Society*, 10, 76–80.

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#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
1	1884	Albuquerque, J. A.		Primeiros principios da theoria dos determinantes para usos dos lyceus		Portuguese	55	Guillard, Aillaud \& Co., Paris	"Elementary textbook with suitable exercises, somewhat resembling Mansion's 'Eléments'. "	4	84
2	1888	Amorétti, F. / Morales, C. M.		Teoría elemental de las determinantes, y sus principales aplicaciones al álgebra y la geometria		Spanish	180	Buenos-Ayres [Buenos Aires, Argentina]	"In extent it is similar to that of Bacas/Escandon (1883), but in other respects it differs significantly for the better."	4	89
3	1883	Bacas, D. / Escandon, R.		Teoría elemental de las determinantes y sus aplicaciones al álgebra y á la trigonometria		Spanish	196	Madrid	"It takes over from Dostor two pages on 'determinantes multiples' and some of its 'applications to trigonometry' are less appropriate"	4	82
4	1857	Baltzer, Richard	1818-1887	Theorie und Anwendung der Determinanten, mit Beziehung auf die Originalquellen		German	129	Verlag von S. Hirzel, Leipzig	"The good qualities . . . belonging to Brioschi (1854) are still more conspicuous . . . three years later, and the historical footnotes . . . give it a special additional value. "	2	97
5	1861	Baltzer, Richard	1818-1887	Théorie et applications des déterminants: avec l'indication des sources originales		French	235	Mallet-Bachelier, Paris	Translation of Baltzer 1857 by Guillaume Jules Hoüel 1823-1886 from German into French .	2	97
6	1864	Baltzer, Richard	1818-1887	Theorie und Anwendungen der Determinanten	2te vermehrte Auflage	German	224	Verlag von S. Hirzel, Leipzig	"The enlargement referred to in the title would have accounted to about forty pages if the former size of page had been retained . . . "	3	13
7	1870	Baltzer, Richard	1818-1887	Theorie und Anwendung der Determinanten	3te verbesserte Auflage	German	241	Verlag von S. Hirzel, Leipzig	"In this edition the 'Theorie' is enlarged to the extent of four or five pages"	3	30
8	1875	Baltzer, Richard	1818-1887	Theorie und Anwendung der Determinanten	4te verbesserte Auflage	German	247	Verlag von S. Hirzel, Leipzig	"The only fresh matter in this edition is the statement that in the case of the evanescent determinant ..."	3	379
9	1881	Baltzer, Richard	1818-1887	Theorie und Anwendung der Determinanten	5te verbesserte und vermehrte Auflage	German	279	Verlag von S. Hirzel, Leipzig	"The new edition ... has received only slight enlargement, but ... careful revision, and amendment ..."	4	79
10	1878	Baraniecki, M		Teorya wyznaczników (determinantów): Kurs uniwersytecki		Polish	600	Paris	"In bulk ... greatly surpasses all its predecessors"	3	75
11	1878	Bartl, E.		Einleitung in die Theorie der Determinanten. Zum Gebrauch an Mittelschulen sowie zum Selbstunterricht		German	96	Prag [now Prague, Cezch Republic]	"... additional explanations and illustrations ..."	3	72
12	1854	Brioschi, Francesco	1824-1897	La theorica dei determinanti, e le sue principali applicazioni		Italian	116	Pavia	"Second separately-published text-book on determinants ... mainly on the same lines as the first"	2	87
13	1856	Brioschi, Francesco	1824-1897	Théorie des déterminants et leurs principales applications		French	216	Mallet-Bachelier, Paris	Translation of Brioschi 1854 from Italian into French by Edouard Combescure b. 1824	2	87
14	1856	Brioschi, Francesco	1824-1897	Theorie der Determinanten und ihre hauptsächlichen Anwendungen		German	102	Duncker \& Humblot, Berlin	Translation of Brioschi 1854 from Italian into German, and with a "Vorwort" by Karl Friedrich Schellbach 1805-1892	2	87
15	1898	Budisavljevic, E.		Grundzüge der Determinanten-Theorie und der projektivischen Geometrie		German	492	W. Braumuller, Wien [Vienna, Austria]	"not seen"	4	95
16	1883	Bunkofer, W.		Die ersten Elemente der Determinantentheorie		German	27	Tauberbischoffs-heim	"Equally elementary to Krober (1883) but less workmanlike"	4	83
17	1844	Cayley, Arthur	1821-1895	On the theory of determinants		English	16	Pitt Press, Cambridge		2	17

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
18	1853	Chio, F.		Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants		French	32	Turin	"Consideration of determinants whose elements are binomial"	2	79
19	1890	de Prado, G. Ferninadez		Elementos de la teoriá de los determinantes, y sus applicaciones á la eliminacion y á la teoría de las formas		Spanish	302?	Eduardo Areas, Madrid	"A clear but stiffly formal exposition...with no unfamiliar material, and with no exercises for practice"	4	91
20	1876	Diekmann, Joseph	1848-1905	Einleitung in die Lehre von den Determinanten, und ihre Anwendung auf dem Gebiete der niederen Mathematik. Zum Gebrauch an Gymnasien, Realschulen u. andern höhern Lehranstalten, sowie zum Selbstunterricht		German	88	Essen	"... pressed on teachers his views as to the part that ought to be taken by determinants in a school course"	3	59
21	1889	Diekmann, Joseph	1848-1905	Anwendung der Determinanten und Elemente der neueren Algebra auf dem Gebiete der niederen Mathematik		German	111	B. G. Teubner, Leipzig	"... what effective applications can be made of even a very elementary knowledge of the theory"	4	90
22	1867	Dodgson, Charles L[utwidge]	1832-1898	An Elementary Treatise on Determinants, with their application to simultaneous linear equations and algebraic geometry		English	143	Macmillan, London	"This is a text-book quite unlike all its predecessors. Professedly its main aim is logical exactitute" / Author = Lewis Carroll = Carolus Lodovicus	3	24
23	1873	Dölp, Heinrich		Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben		German	94	Ludwig Brill, Darmstadt		3	49
24	1877	Dölp, Heinrich		Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben	2te Auflage, bearbeitet von W. Soldau	German	94	Ludwig Brill, Darmstadt	"... is an improvement on the first [edition] ..."	3	67
25	1887	Dölp, Heinrich		Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben	3te Auflage	German	94	Ludwig Brill, Darmstadt	"... practically [a] reprint ..."	4	86
26	1893	Dölp, Heinrich		Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben	4te Auflage	German	95	Ludwig Brill, Darmstadt	"... almost a complete reprint"	4	93
27	1899	Dölp, Heinrich		Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben	5te Auflage	German	95	Ludwig Brill, Darmstadt	"have not seen"	4	95
28	1877	Dostor, Georges	b. 1820	Éléments de la théorie des déterminants, avec application à l'algèbre, la trigonométrie et la géométrie analytique dans le plan et dans l'espace, à l'usage des classes de mathématiques spéciales		French	352	Gauthier-Villars, Paris	"... lengthiest text-book as yet published ..."	3	67
29	1883	Dostor, Georges	b. 1820	Éléments de la théorie des déterminants, avec application à l'algèbre, la trigonométrie et la géométrie analytique dans le plan et dans l'espace, à l'usage des classes de mathématiques spéciales	2. éd.	French	361	Gauthier-Villars, Paris	"Dostors book is ... slightly larger"	4	82
30	1883	Egidi, G.		Trattato elementare dei determinanti e loro applicazioni		Italian	196	Roma	"Well-intentioned contribution but shows no advance on previous Italian textbooks"	4	83
31	1876	Falk, M.		Lärobok i determinant-teoriens första grunder, för högre läroanstalter och till sjelfstudium		Swedish	96	Upsala [Uppsala, Sweden]	"... establishment and illustration of twelve carefully formulated theorems, including three on Jacobians ..."	3	59
32	1872	Fontebasso, D.		I primi elementi della teoria dei Determinanti, e loro applicazioni all' algebra ed alla geometria, proposti agli alunni degli istituti tecnici		Italian	134	Treviso	"Fontebasso's aim was a text-book simpler still in style than Trudi's ... there is nothing very praiseworthy in the result"	3	43

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
33	1858	Gallenkamp, W.		Die einfachsten Eigenschaften und Anwendungen der Determinanten		German	12	Duisburg	"A workmanlike twelve-page exposition"	2	101
34	1874	Garbieri, G[iovanni]		I Determinanti, con numerosi applicazioni, Parte prima: Utlie agli studiosi di matematica nei primi corsi universitari		Italian	267	Bologna	"In bulk this exceeds all previous Italian text-books except Trudi's (1862)"	3	53
35	1893	Garbieri, G[iovanni]		Teoria ed applicazioni dei determinanti		Italian	100	Reggio d'Emilia	"... somewhat incorrectly viewed as a second edition of Garbieri (1874) ..."	4	93
36	1899	Gavrilovitch, B.		Teoriya determinanata		Serbian	278	Beograd	"A workmanlike text-book ... with a wide and accurate knowledge of others"	4	95
37	1881	Groll, R.		Die Determinanten für den Schulgebrauch		German	20	Quackenbruck	"...teaches by means of the simplest particular instances, and is not sparing in explanation"	4	80
38	1876	Guldberg, A. S.	1866-	Determinanternes Teori [Udgivet med bidrag af videnskabernes seskab i Trondhjem]		Norwegian	88	Kristiania [Oslo, Norway]	"It makes no pretension to freshness of manner or matter."	3	59
39	1875	Günther, Siegmund	1848-1923	Lehrbuch der Determinantentheorie, für Studirende		German	236	E. Besold, Erlangen	"Günther's aim was ... to produce a fairly comprehensive text-book less condensed than Baltzer's and in general more suitable for learners"	3	54
40	1877	Günther, Siegmund	1848-1923	Lehrbuch der Determinantentheorie, für Studirende	2te … vermehrte … Auflage	German	209	E. Besold, Erlangen	"... all that the title-page proclaims it to be" / 2te durchaus umgearbeitet vermehrte und durch eine Aufgaben-sammlung bereicherte Auflage	3	66
41	1886	Hanus, P. H.	1855-1941	An elementary treatise on the theory of determinants		English	218	Ginn and Company, Boston	"the result is a fresh,whole,more homogenous than the works of some authors who are less frank"	4	87
42	1872	Hattendorff, Karl	1834-1882	Einleitung in die Lehre von den Determinanten		German	60	Schmorl & von Seefeld, Hannover	"... aims at keeping the wants of the young student constantly in view ..."	3	35
43	1887	Hattendorff, Karl	1834-1882	Einleitung in die Lehre von den Determinanten	2te Auflage	German	60	Hannover	"practically [a] reprint"	4	86
44	1871	Hesse, Ludwig Otto	1811-1874	Die Determinanten, elementar behandelt		German	46	B. G. Teubner, Leipzig	"As might have been expected ... excellent ..."	3	35
45	1871	Hesse, Ludwig Otto	1811-1874	De determinanten, elementair behandeld	Translated from German into Dutch by F. van Wageningen	Dutch	48	Gebr. Belinfante, s Gravenhage [The Hague, The Netherlands]		3	35
46	1872	Hesse, Ludwig Otto	1811-1874	Die Determinanten, elementar behandelt	2te Auflage	German	48	B. G. Teubner, Leipzig		3	35
47	1880	Hesse, Ludwig Otto	1811-1874	Die Determinanten, elementar behandelt	Translated into Polish by Zdziarski	Polish		Warsaw		4	79
48	1881	Heun, Karl	b. 1859	Die Kugelfunktionen und Lamé'schen Funktionen als Determinanten		German	32	Druck der Dieterichschen Univ.-Buchdrückerei, Göttingen			

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
49	1889	Horta, F. P.		Estudo elementar dos determinantes, sequido de uma parte complementar relativa, principalmente, aos determinntes funcionaes		Portuguese	176	Lisboa	"Supplies Portuguese students with a reliable textbook of a more advanced type than Albuquerque (1884)"	4	90
50	1871	Hoüel, Guillaume Jules	1823-1886	Notions élémentaires sur les déterminants		French		Paris	"... accurately describes itself"	3	31
51	1841	Jacobi, C[arl] G[ustav] J[acob]	1804-1851	De determinantibus functionalibus		Latin			translated into German by P. Stäckel (1896)	4	257
52	1858	Janni, G.		Saggio di una teorica elementare de determinanti		Italian	40	Napoli	"There is nothing sufficiently fresh in the treatment"	2	99
53	1882	Jenrich, P.		Beiträge zur Methodik des mathematischen Unterrichts mit Berücksichtigung der Determinantenfrage		German	41	Magdeburg		4	77
54	1886	Jürgens, E.		Zur Aulfösung linearer Gleichungssyteme und numerischen Berechnung von Determinanten		German	20	Aachen	"... solution of any set of equations in which the known quantities are numbers in the ordinary decimal notation ..."	4	101
55	1882	Kaiser, H.		Die Anfangsgründe der Determinantes, in Theorie und Anwendung		German	41	Bergmann, Wiesbaden	"...[teaches] by means if general propostions accompanied by the minimum of illustration"	4	80
56	1885	Kaiser, H.		Die Determinanten für den ersten Unterricht in der Algebra		German	23	Bergmann, Wiesbaden	"resembles school-programs of the most elementary type"	4	84
57	1885	Kerschen-steiner, Georg	1854-1932	Dr. Paul Gordan's Vorlesungen Über Invariantentheorie, Erster Band: Determinanten	P[aul Albert] Gordan (1837 - 1912)	German	202	B. G. Teubner, Leipzig	"The first half, though freshly written, shows no improvement on previous expositions...the second half is considerably more interesting"	4	85
58	1889	Killing, W.	b. 1847	De determinante quodam disquistiones mathematicae		Latin?	16	Braunsberg		4	477
59	1892	Korczynski, J.		Elementary theory of determinants		Polish	42	Kraków [Cracow, Poland]	"...'program' of the classroom-type"	4	93
60	1883	Krober		Die Determinanten für den Unterricht bearbeitet		German	30	Bischweiler	"Fair specimen of the school 'program' of the purely classroom type"	4	83
61	1880	Leboulleux, L.		Traite élémentaire des determinants		French	175	Paris	"This I have only seen referred to"	4	83
62	1884	Leboulleux, L.		Traite élémentaire des déterminants, contenant 299 exercises		French	184	Genève	"The author is a willing and profuse guide, but not always safe"	4	83
63	1874	Legge, A. di		Teoria dei determinanti		Italian		Roma		3	57
64	1879	Lit, R. R.		Beginselen van de leer der determinanten, met voorbeelden en toepassingen		Dutch	38	Amsterdam	"... nothing ... more than [the title] would lead one to expect"	3	78
65	1875	Mansion, Paul	1844-1919	Éléments de la théorie des déterminants, d'après Baltzer et Salmon		French	44	Mons [Belgium]	"... singularly clear though short exposition"	3	55

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
66	1876	Mansion, Paul	1844-1919	Introduction à la théorie des déterminants. A l'usage des établissements d'instruction moyenne		French	24	Gand [Ghent, Belgium]	"... still more elementary [than Mansion (1875)]"	3	55
67	1878	Mansion, Paul	1844-1919	Elemente der Theorie der Determinanten. Mit vielen Übungsaufgaben		German	50	Leipzig	"... an improvement on what we have already seen to be good ..."	3	73
68	1880	Mansion, Paul	1844-1919	Éléments de la théorie des déterminants avec de nombreux exercices	3. éd.	French	63	H. Manceaux, Mons [Belgium]	"Mansion very considerably enlarges and improves his ... edition of 1878"	4	55
69	1882	Mansion, Paul	1844-1919	Introduction à la théorie des déterminants	2. éd.	French	32	Gand [Ghent, Belgium]	"In these new editions [1882-1886] Mansion enlarges and improves both the *Introduction* and the *Éléments* ..."	4	82
70	1883	Mansion, Paul	1844-1919	Éléments de la théorie des déterminants avec de nombreux exercices	4. éd.	French	80	Gauthier-Villars, Paris		4	82
71	1886	Mansion, Paul	1844-1919	Elemente der Theorie der Determinanten mit vielen Übungsaufgaben	2te vermehrte Auflage	German	56	B. G. Teubner, Leipzig	"... the German edition ... does not differ from the French edition of 1883."	4	82
72	1899	Mansion, Paul	1844-1919	Elemente der Theorie der Determinanten mit vielen Übungsaufgaben	3te vermehrte Auflage	German	103	B. G. Teubner, Leipzig	"... the German edition is now closely assimilated to the French."	4	96
73	1891	Martone, M.		Introduzione alla teoria delle serie, Parte prima: I determinanti Wronskiani e la legga suprema		Italian	43	Maccarone, Catanzaro [Calabria, Italy]	"not seen"	4	247
74	1872	Maurer, F.		Grundzüge der Determinantenlehre		German		Budweis [Germany, now \v{C}eské Bud\v{e}jovice, Czech Republic]	"School-program of the simplest type."	3	31
75	1876	Mellberg, E. J.		Teorin för determinant-kalkylen		Swedish	121	Helsingfors [Helsinki, Finland]	"... dissertation ... devotes about seventy pages to an exposition of the theory and applications ..."	3	58
76	1884	Méray, [Huges] Charles [Robert]	1835-1911	Exposition nouvelle de la théorie des formes linéaires et des déterminants		French	104	Gauthier-Villars, successeur de Mallet-Bachelier, Paris		4	23
77	1863	Mikelli, A.		Dei determinanti		Italian		Mantua [Mantova, Italy]		3	12
78	1892	Miller, G[eorge] A[bram]	1863-1951	An introduction to the study of determinants, with examples and applications		English	110	D. Van Nostrand, New York	"belongs to a series of pocket booklets on technical science where simplicity and clearness of exposition were held to be the most important requisites."	4	92
79	1878	Mollame, V.		I Determinanti, e loro applicazioni all' algebra ed alla geometria analitica. Nuova esposizione elementare		Italian	125	Napoli	"Notwithstanding its bulk it does not go deeply into the theory ..."	3	73
80	1874	Muir, Thomas	1844-1934	A treatise on determinants with graduated sets of exercises for use in colleges and schools		English		J. Rothschild, Paris	unconfirmed, but AMICUS entry no. 6222555 says that the 1960 Dover Reprint Edition of the revised version of Muir (1882) was originally published by Rothschild, Paris (1874)		
81	1882	Muir, Thomas	1844-1934	A treatise on determinants with graduated sets of exercises for use in colleges and schools		English	240	Macmillan, London	"... intended to make readers interested in the theory for itself, and not merely as a pleasant auxiliary to other branches of elementary mathematics"	4	80

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
82	1890	Muir, Thomas	1844-1934	The theory of determinants in the Historical Order of its Development, Part I: Determinants in general; Leibniz (1693) to Cayley (1841)		English	278	Macmillan, London	"A reprint, separately published, of six papers communicated to the Royal Society of Edinburgh during the years 1886-1889 ... It takes up to, but does not include, Cayley."	4	45
83	1876	Müller, H.		Kurze und schulgemässe Behandlung der Determinanten		German	19	Metz [now in France]	"... very elementary"	3	60
84	1891	Niemöller, F.		Anwendung der linearen Ausdehnungslehre von Grassmann auf die Theorie der Determinanten		German	22	Gymn. Osnabrück	"[has] the object of including as many so-called 'applications' as possible"	4	48
85	1893	Ott, A.		Über Determinanten		German	26	Weimar	"...'program' of the classroom-type"	4	93
86	1897	Pascal, Ernesto	b. 1865	I determinanti, teoria et applicazioni; con tutte le più recenti ricerche	Manuali Hoepli. Serie scientifica. No. 223 e 224	Italian	330	U. Hoepli, Milano	"A very attractive pocket text-book ... For 3 lire (about half-a-crown) Italian students receive better value than those of any other nationality. Even the German translation costs about four times as much (10 marks)"	4	94
87	1900	Pascal, Ernesto	b. 1865	Die Determinanten: Eine Darstellung ihrer Theorie und Anwendungen mit Rücksicht auf die neuen Forschungen	Berichtigte deutsche Ausgabe, von Dr. Hermann Leitzmann	German	266	B. G. Teubner, Leipzig	"A carefully produced translation. Addition to the text are noticeably few"	5	102
88	1887	Peck, William G[uy]	1820-1892	Elementary treatise on determinants		English	47	A. S. Barnes and Company, New York	"...a quite elementary exposition and, though filling a gap in America, is of merely local interest"	4	87
89	1900	Prang, C.		Einführung in die Theorie und den Gebrauch der Determinanten		German	53	Mayer & Müller, Berlin	"Useful school textbook of a type already common, and having no outstanding feature or merit"	5	103
90	1874	Reidt, Friedrich	1834-1894	Vorschule der Theorie der Determinanten für Gymnasien und Realschulen		German	66	B. G. Teubner, Leipzig	"... workmanlike elementary class-book"	3	49
91	1867	Reiss, Max		Beiträge zur Theorie der Determinanten		German	113	B. G. Teubner, Leipzig	"It is only the first section of this long and important memoir that comes up here for consideration."	3	21
92	1884	Reuschle, C.		Determinantentheorie, Erstes Kapital: Einführung in die Determinantentheorie		German	16	Stuttgart	"resembles school-programs of the most elementary type"	4	84
93	1898	Roe, Edward Drake	1859-1929	Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen ...	Dissertation	German	52	B. G. Teubner, Leipzig	"Simple study of the final expansion of a bigradient"	4	347
94	1859	Salmon, George	1819-1904	Lessons introductory to the modern higher algebra		English	147	Dublin	"... this historically interesting text-book ..."	2	106
95	1863	Salmon, George	1819-1904	Vorlesungen zur Einführung in die Algebra der linearen Transformationen	Deutsch bearbeitet von Dr. Wilhelm Fiedler	German	271	Leipzig	"Fielder improves the original ..."	3	12
96	1866	Salmon, George	1819-1904	Lessons introductory to the modern higher algebra	2nd edition	English	296	Dublin	"[nothing] really fresh appear[s] in regard to general theory"	3	12
97	1868	Salmon, George	1819-1904	Leçons d'algèbre supérieure.	Traduit de l'anglais par M. Bazin. Augmenté de notes par M. Hermite	French	247	Paris	"Bazin ... makes considerable curtailments."	3	12

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
98	1876	Salmon, George	1819-1904	Lessons introductory to the modern higher algebra	Third edition	English	318	Dublin	"The first six chapters are increased, but ... the added matter had already appeared elsewhere."	3	57
99	1877	Salmon, George	1819-1904	Vorlesungen zur Einführung in die Algebra der linearen Transformationen.	Deutsch bearbeitet von Dr. Wilhelm Fiedler. Zweite Auflage	German	477	Leipzig	"... follows closely the third English edition"	3	66
100	1884	Schrader, W.	1817-1907	Die Determinanten im Schulunterricht, und ihre Anwendung auf Gegenstände der analytischen Geometrie.		German	23	H. W. Schmidt, Halle a. S.	"a school program with only six pages expository of determinants"	4	84
101	1887	Schrader, W.	1817-1907	Beiträge zur Theorie der Determinanten: Neue Sätze und eine neue Bezeichnug		German	156	H. W. Schmidt, Halle a. S.	"Notwithstanding the title, this is in form and reality a textbook"	4	88
102	1880	Scott, Robert Forsyth	1849-1933	A treatise on the theory of determinants and their applications to analysis and geometry		English	251	"Cambridge at the University Press"	"This is noteworthy as giving for the first time to the Englishspeaking-student a reasonably full account of the subject. No branch of any note is neglected ..."	4	78
103	1878	Sersawy, V.		Die Fundamente der Determinantentheorie		German	42	Wien [Vienna, Austria]	"...gives much less matter [than Mollamee (1878)] and is less concerned with explanation and illustration ..."	3	73
104	1882	Shaposhnikoff		Elements of the theory of determinants		Russian	35	Moscow	"not seen"	4	82
105	1885	Sickenberger, A.		Die Determinanten in genetischer Behandlung. Eine Einführung in die Lehre von den Determinanten		German	80	Ackermann, München	"This school-program of the expository type is better and much more extensive than any of its immediate predecessors."	4	85
106	1887	Sickenberger, A.		Die Determinanten in genetischer Behandlung	2te Abdruck	German	80	Ackermann, München	"practically a reprint"	4	86
107	1858	Sperling, I.		The theory of determinants and its most important applications		Russian		St. Petersburg	"This dissertation I have failed to see"	2	102
108	1851	Spottiswoode, William	1825-1883	Elementary theorems relating to determinants		English	63	Longman, Brown, Green, and Longman, and G. Bell, London	"First separately-published elementary work on the subject"	2	54
109	1853	Spottiswoode, William	1825-1883	Elementary theorems relating to determinants	2nd Edition, rewritten and much enlarged by the author	English	117	*Journal für die reine und angewandte Mathematik*	"Rearranged, partly rewritten, and much enlarged" [Not a separate book but published in the *Journal für die reine und angewandte Mathematik*, vol. 51, pp. 209-271, 328-381]	2	81
110	1896	Stäckel, P[aul Gustav], ed.	1862-1919	Über die Functionaldeterminanten [by Carl Gustav Jacob Jacobi (1804-1851)]	translation of Jacobi (1841) from Latin into German	German	72	W. Engelmann, Leipzig	"At the close of this translation of Jacobi (1841), the editor appends 13 pages of useful notes." / Ostwalds Klassiker der exakten Wissenschaften 78		
111	1897	Stoffäes, Eugene	1854-?	Théorie des déterminants		French		Lille		4	94
112	1869	Studnicka, Frantisek Josef	1836-1903	O determinantech		Czech	64	V Praze		3	29
113	1870	Studnicka, Frantisek Josef	1836-1903	The elements of the theory of determinants		Russian		Prag [now Prague, Cezch Republic]	"not seen"	3	29

#	Year pub.	Author	Life span	Title	Edition / Comment	Language	# pp.	Publisher	Quotation from Muir / Comment	Muir vol	Muir page
114	1871	Studnicka, Frantisek Josef	1836-1903	Einleitung in die Theorie der Determinanten		German	66	Prag [now Prague, Cezch Republic]		3	29
115	1900	Studnicka, Frantisek Josef	1836-1903	Einführung in die Theorie der Determinanten		German	230	Verlag des Vereins böhmischer Mathematiker, Prag	not in Muir?		
116	1882	Suarez A. / Gasco, L. G.		Lecciones de coordinatoria con las determinantes y sus principales aplicaciones		Spanish	451	Valencia	"...an unusually large portion being occupied with the auxiliary subject of 'coordinatoria'."	4	82
117	1885	Tartinville, A.		Première leçon sur les déterminants		French		Paris	"not seen"	4	85
118	1886	Tartinville, A.		Seconde leçon sur les déterminants		French		Paris	"not seen"	4	85
119	1882	Thomson, William [Lord Kelvin]	1824-1907	An introduction to determinants, with numerous examples: For use in schools and colleges		English	101	J. Thin, Edinburgh	"An elementary exposition accompanied by a variety and sufficiency of easy exercises..."	4	80
120	1862	Trudi, Nicola	1811-1884	Teoria de' determinanti, e loro applicazioni		Italian	268	B. Pellerano, Napoli	". . . the exposition, which is not a little influenced by Brioschi and Baltzer, being full and simple"	2	288
121	1870	Unterhüber, A.		Einleitung in die Theorie der Determinanten		German		Breslau [now Wroclaw, Poland]	"School program of the simples type"	3	31
122	1872	Unterhüber, A.		Über Determinanten		German	14	Leoben [Austria]	"School-program of the simplest type."	3	31
123	1883	Valenta, J.		The theory of determinants		Czech		Prag [now Prague, Cezch Republic]	"not seen"	4	83
124	1878	Vascenko-Zacharenko, M.		Theory of determinants and theory of forms		Russian	501	Kieff [Kiev, Ukraine]	"... much of the space is occupied with algebraical matter that is not closely relevant"	3	76
125	1870	von Zelewski, A.		Ein Beitrag zur Theorie der Determinanten		German	31	Breslau [now Wroclaw, Poland]	"Very simple but very formally arranged ..."	3	29
126	1877	von Zelewski, A.		Nauka owyznacznikach z zastosowaniami [Theory of Determinants with Applications]		Polish	191	Kraków [Cracow, Poland]	"I have not seen"	3	67
127	1893	Weichold, G.		Lehrbuch der Determinanten und deren Anwendungen, I. Teil: Nach System Kleyer bearbeitet		German	380	J. Maier, Stuttgart	"Instruction is conveyed in it by question-and-answer with the aid of numerous solved and unsolved exercises and a wealth of explanations."	4	92
128	1893	Weld, Laenas Gifford	b. 1862	A short course in the theory of determinants		English	238	Macmillan, New York & London	"textbook not unlike others already in English...but aiming at still further simplification of exposition and exercises."	4	92
129	1873	Wisselink, D. B.		Iets over Determinanten		Dutch	38	Deventer	"It is the first instance-not a translation-of the subject being dealt with in the language of Holland"	3	49
130	1875	Wright, Rev. W[illiam] J[ames]	b. 1831	Tracts relating to modern higher mathematics, Tract No. 1: Determinants		English	72	C. F. Hodgson, London	"Everywhere except on the title-page ... is styled 'Elementary' "	3	60
131	1878	Zahradnik, K[arl]		O determinantih drugoga i trecega stupnja: Za porabu visih srednjih ucilista		Croatian	39	Zagreb	"This little introduction confines itself in the main to determinants of the second and third order"	3	73

Did Karl Pearson Use Matrix Algebra in 1896?

In her article on Karl Pearson (1857–1936) published in *Statisticians of the Centuries,* Eileen Magnello (2001, p. 252) asserts that:

> In this seminal paper on "Regression, Heredity and Panmixia" in 1896, Pearson introduced matrix algebra into statistical theory.

And again, Magnello (*op. cit.*, p. 253):

> Pearson set up the Drapers' Biometric Laboratory in 1903 following a grant from the Worshipful Drapers' Company ... The methodology incorporated in the Drapers' Biometric Laboratory was twofold: the first was mathematical, and included the use of Pearson's statistical methods, matrix algebra and analytical solid geometry.

See also Magnello (1998, pp. 3311–3312). If Magnello's statements regarding the use of matrix algebra by Pearson and his colleagues are at all justified, then there must have been a break of some thirty years in the use of matrix algebra in statistical work between the late 1890s and the alternative mid-1930s starting date maintained by Farebrother (1996, 1999) and Searle (2000).

Although I have not checked the original source material, the books by Stigler (1986) and Hald (1998) do not seem to contain any evidence in support of Magnello's thesis. However, Hald (1998, p. 621) observes that:

> Edgeworth (1892) indicates the solution for $m = 4$, and in a later paper (1893) he gives the general solution in determinantal form. Pearson (1896, §10b) named the multivariate normal ... Edgeworth's theorem for $m = 3$.

This passage suggests that Magnello (1998, 2001) may have used the phrase "matrix algebra" to refer to the determinantal analysis performed by Edgeworth (1893) and Pearson (1896) in their description of the multivariate normal distribution.

References

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S. M. Stigler (1986). *The History of Statistics: The Measurement of Uncertainty before 1900.* Belknap Press of Harvard University Press.

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Accurate Solution of Eigenvalue Problems III

In the last 15 years, there have been several advances in the accurate solution of eigenvalue problems. Well-known advances include fast and more accurate methods for solving the symmetric tridiagonal eigenproblem, more accurate methods for computing the singular value decomposition, and further understanding of the conditioning theory of the nonsymmetric eigenvalue problem.

To recognize these advances and to encourage further advances, we are proposing to have a special issue of *Linear Algebra and its Applications* on Accurate Solution of Eigenvalue Problems. So far, we have completed one special issue (vol. 309) on this problem area and a second one is pending. Since both of these issues received many strong submissions, we expect the same for the third.

This special issue is in coordination with the International Workshop on Accurate Solution of Eigenvalue Problems IV to be held in Split, Croatia, on June 24–27, 2002. The participants in the workshop will be heartily encouraged to submit papers to the special issue. Submissions, consistent with the themes of the workshop, are also welcome from non-participants.

The editors for this special issue will be Jesse L. Barlow, Department of Computer Science and Engineering, The Pennsylvania State University, University Park, PA 16802–6106, USA; Beresford N. Parlett, Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA; and Kresimir Veselić, FernUniversität Hagen, Lehrgebiet Mathematische Physik, Postfach 940, D-58084 Hagen, Germany. Manuscripts submitted to this issue will be refereed according to standard procedures for *Linear Algebra and its Applications* (LAA). The deadline for submissions is January 15, 2003.

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A Note on Hamilton and Quaternions

When Sir William Rowan Hamilton (1805–1865) invented a four-dimensional generalisation of complex numbers in 1843, he named the elements of this algebra *quaternions* for the unit of four soldiers in the Roman army. Jesus Christ was presumably taken to execution by such a unit, see *The Gospel of Saint John* (ch. 19, v. 24), and Saint Peter was guarded in prison by four quaternions, see *The Acts of the Apostles* (ch. 12, v. 4).

Modern newlyweds may be intrigued to learn that in 1864 Isaac Todhunter (1820–1884), the historian of geodesy, probability, and the calculus of variations, took some of Hamilton's work on quaternions to read on his honeymoon, as observed in the 1963 *Biometrika* article [1] by Maurice George Kendall (1907–1983).

The following quotation appears at the end of the 1976 paper [2] entitled "Hamilton's Discovery of Quaternions" by Bartel Leendert van der Waerden (1903–1996):

> The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs; and we doubt not that in the great mathematical birth of 1843, the Quaternions of Hamilton, there is as much real promise of benefit to mankind as in any event of Victoria's reign.

This passage is ascribed to a Thomas Hill, presumably the inventor of the single transferable voting system who was the brother of Sir Rowland Hill (1795–1879, the inventor of the uniform postal rate), and a direct ancestor of Sir Austin Bradford Hill (1897–1991) the medical statistician and his son Ian David Hill sometime Algorithms Editor of *Applied Statistics.*

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[1] M. G. Kendall (1963). Isaac Todhunter's history of the mathematical theory of probability. *Biometrika*, 50, 204–205.

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Biographies, Portraits and Stamps on the Web

Biographies and portraits of Sir William Rowan Hamilton, Sir Maurice Kendall, Isaac Todhunter, and Bartel Leendert van der Waerden appear on the "The MacTutor History of Mathematics Archive" Web site, created and maintained by John J. O'Connor and Edmund F. Robertson in the School of Mathematics and Statistics, University of St Andrews, Scotland: www-groups.dcs.st-and.ac.uk/~history/BiogIndex.html The Web site: www.maths.tcd.ie/pub/HistMath/People/Hamilton/ by David R. Wilkins (School of Mathematics, Trinity College Dublin) is devoted to Sir William Rowan Hamilton.

From the Glassine Surfer's Web site "Sir Rowland Hill and The Penny Black" www.glassinesurfer.com/f/gsrowlandhill.shtml we learn that "The man behind the first stamps was an Englishman named Rowland Hill, a reformer who followed the ideas of utilitarian reformer Jeremy Bentham (1748–1832), who argued that the purpose of government was to provide the greatest happiness of the greater number of people and that the more democractic government was the better off people would be".

Three postage stamps have been issued that are associated with Sir William Rowan Hamilton and quaternions. Issued by Ireland (Éire, Irish Free State) on 15 November 1943 to commemorate the "Centenary of the Announcement of the Discovery of the Mathematical Formula of Quaternions by William Rowan Hamilton" are two stamps with a portrait of "Rowan Hamilton": $\frac{1}{2}$d, deep green (Scott #126, Stanley Gibbons #131) and $2\frac{1}{2}$d, red brown (Scott #127, Stanley Gibbons #132).

Issued by Ireland (Éire, Republic of Ireland) on 4 May 1983 featuring "Sir William Rowan Hamilton's formulae for the multiplication of quaternions": 29p, black, brown & yellow (Scott #562, Stanley Gibbons #557).

The $\frac{1}{2}$d and the 29p are on page 73 of *Stamping Through Mathematics* by Robin J. Wilson (Springer, 2001), where these and many other mathematical stamps are illustrated in full colour. All three stamps, plus many other mathematical stamps, are included in the Web site "Images of Mathematicians on Postage Stamps": http://jeff560.tripod.com/ maintained by Jeff Miller, a mathematics teacher at Gulf High School in New Port Richey, Florida. I am very grateful to Jeff Miller for allowing us to reproduce here his jpeg file of the 29p Hamilton stamp.

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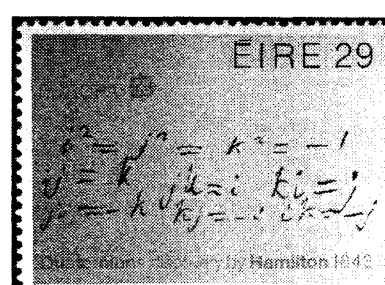

Conferences and Workshops in Linear Algebra and Matrix Theory

Linear Algebra & Matrix Theory Special Sessions 2002 AMS Southeastern Regional Conference

Atlanta, Georgia: 8–10 March 2002

The 2002 American Mathematical Society Southeastern Regional Conference was held at Georgia Institute of Technology, Atlanta, March 8–10, 2002. The Linear Algebra and Matrix Theory Special Sessions, organized by Frank Hall and Zhongshan Li, had more than twenty invited talks, which covered a wide range of topics, such as various problems on eigenvalues and singular values, combinatorial matrix theory, and operator theory. Each talk ran from 20 to 25 minutes. The speakers and titles of their talks are listed below.

Session I: Charles R Johnson, the number of unitary similarity classes among a congruence class with fixed spectrum; Ilya M Spitkovsky, Positive eigenvalues and two-letter generalized words; Tin-Yau Tam, Generalization of Ky Fan–Amir-Moéz–Horn–Mirsky's result on the eigenvalues and real singular values of a matrix; Sivaram K Narayan, Spectrally stable matrices; Mihaly Bakonyi, Extensions of positive definite functions on ordered groups; Hongyu He, Some inequalities on $A \times B$ for classical groups.

Session II: Chi-Kwong Li, Off-diagonal block of a two by two block Hermitian matrix; Tom Morley, Schur complements and M-matrices; George D Poole, On the maximal number of odd submatrices of 0-1 matrices; Frank Uhlig, The role of proof in comprehending and teaching elementary linear algebra.

Session III: James R Weaver, Properties of the Brualdi-Li tournament matrix; Rohan Hemasinha, Some graph and matrix theoretic properties of divisor tournaments; Luz M DeAlba, The P- and P_0-matrix completion problems; Judith J McDonald, Ray-patterns of matrices; Frank J Hall, On ranks of matrices associated with trees; Zhongshan Li, Sign patterns that require all distinct eigenvalues.

Session IV: Mei-Qin Chen, An eigenvalue problem arising in the design of 2-D filters with periodic coefficients; Leiba Rodman, Robust solutions of matrix equations; Michael J Tsatsomeros, Principal pivot transforms; Jun Wang (a thesis advisee of Vadim Olshevsky), A unified approach to the eight versions of fast discrete cosine and sine transforms; Vladimir Bolotnikov, On finite dimensional backward shift invariant subspaces of the Arveson space; Shmuel Friedland, On spaces of matrices containing a nonzero matrix of bounded rank.

On the evening of March 9, participants of the Linear Algebra and Matrix Theory special sessions had a wonderful time at a dinner party at Vadim and Rosa Olshevsky's house.

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Summer Graduate Workshop in Matrix Theory Topics in Combinatorial Matrix Theory

Regina, Saskatchewan: 28 July – 17 August 2002

The Department of Mathematics and Statistics at the University of Regina is pleased to offer a summer graduate workshop on Matrix Theory. Regular daily lectures will be given by Stephen Kirkland. The title of this workshop is "Topics in Combinatorial Matrix Theory". The workshop will also feature special lectures by Richard A. Brualdi, University of Wisconsin–Madison.

This workshop can be taken for graduate credit by standard arrangements and through contact with our department. Specific questions concerning the workshop can be directed by e-mail to gradapps@math.uregina.ca and electronic registration is available through the online registration form at www.math.uregina.ca/~workshop/ Please visit this Web page for detailed information about the Workshop. The organizers of the Workshop can be reached by mail: Summer Workshops, c/o Dept. of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2.

Fifth International Conference on China Matrix Theory and its Applications

Shanghai, China: 14–18 August 2002

The Fifth International Conference on China Matrix Theory and its Applications (ICMTA) will be held at Shanghai University, Shanghai, China, from 14 to 18 August 2002. This Conference, supported by the National Natural Science Foundation of China, Science and Technology Committee of Shanghai, and the Chinese Mathematical Society, will be a satellite meeting of the 2002 International Congress of Mathematicians (ICM) to be held in Beijing, 20–28 August 2002: www.icm2002.org.cn/

The themes of the conference will cover all aspects of Matrix Theory and Linear Algebra, such as traditional Matrix Theory, Combinatorial Matrix Theory and Numerical Linear Algebra, Matrix Computations, etc. The conference registration fee is US$100, which also covers meals for four-and-a-half days and a sightseeing tour.

Invited speakers include: T. Ando, Zhaojun Bai, Raymond Chan, B. Datta, C. R. Johnson, Chi-Kwong Li, Tin-Yan Tam, Frank Uhlig, Zhongci Shi, and Zhexian Wan.

The Local Organizing Committee includes Erxiong Jiang (chair) and Zhongzhi Bai (secretary-general). For more information, contact ICMTA, Dept. of Mathematics, Shanghai University, Box 28, 99 Shang Da Road, Shanghai 200436, China: FAX 86-21-3603-3287, ejiang@fudan.edu.cn or ejiang@amil.shu.edu.cn

11th International Workshop on Matrices and Statistics

Lyngby, Denmark: 29–31 August 2002

The Eleventh International Workshop on Matrices and Statistics (EIWMS-2002), in Celebration of George P. H. Styan's 65th Birthday, will be held at the Technical University of Denmark (DTU) in Lyngby, near Copenhagen, on August 29–31, 2002. This Workshop will be hosted by the Section of Image Analysis and Computer Graphics in the Department of Informatics and Mathematical Modelling (IMM) at the Technical University of Denmark and has been endorsed by the International Linear Algebra Society (ILAS).

The purpose of this Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. The Workshop will provide a forum through which statisticians working in the field of linear algebra and matrix theory may be better informed of the latest developments and newest techniques and may exchange ideas with researchers from a wide variety of countries.

The International Organizing Committee for this Workshop comprises R. William Farebrother (England), Simo Puntanen (Finland), and Hans Joachim Werner (Germany; chair). The Local Organizing Committee at the Danish Technical University in Lyngby comprises Knut Conradsen (chair), Bjarne K. Ersbøll, Per Christian Hansen, and Allan Aasbjerg Nielsen. Our Workshop Secretary is Ms. Helle Welling. This Workshop will include the presentation of both invited and contributed papers on matrices and statistics. We also plan to have a special session for papers presented by graduate students. It is expected that many of these papers will be published, after refereeing, in the 10th Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications*; the 9th Special Issue is at the printer with publication expected in early summer 2002. Contributed papers are welcome! Details for submission of a paper are given on the Workshop Web site: www.imm.dtu.dk/matrix02/ Abstracts should arrive by June 15, 2002.

The registration fees will be as follows: For payment received by July 31, 2002: Euro 240; after July 31, 2002: Euro 280. The registration fee includes: Admission to all scientific sessions, abstract booklet, coffee/tea/cake in the mornings and in the afternoons, lunch three days, reception Thursday, August 29th at Lyngby Town Hall. A cancellation fee after August 1, 2002 amounts to Euro 40. On Friday afternoon, August 30, 2002, there is a joint excursion. The outing first goes to Roskilde Dome where the Danish Kings and Queens are buried, then coffee and on to the Viking Museum, followed by the Workshop Dinner at Peter Lieps House, Deer Park (close to the university). There is an additional fee of Euro 100 per person for the joint excursion (including the Workshop Dinner).

Delegates are kindly requested to make their own hotel reservations. The Local Organizing Committee has reserved a block of rooms in 5 different Hotels, all at reduced rates. For details please visit the Workshop Web site: www.imm.dtu.dk/matrix02/

SIAM Conference on Applied Linear Algebra

Williamsburg, Virginia: 15–19 July 2003

The SIAM Conference on Applied Linear Algebra (LA03) will be held at The College of William and Mary, Williamsburg, Virginia, July 15–19, 2003. This conference is sponsored by the SIAM Activity Group on Linear Algebra (SIAG/LA). In cooperation with the International Linear Algebra Society (ILAS).

The Program Committee Chairs are Roy Mathias and Hugo Woerdeman, both of The College of William and Mary. Other members of this Committee are Raymond Chan (Univ. of Hong Kong), John Gilbert (Xerox Co.), Per Christian Hansen (Technical University of Denmark), Nicholas Higham (Univ. of Manchester), Ilse Ipsen (North Carolina State University), Horst Simon (National Energy Research Scientific Computing Center), and Paul Van Dooren (Université Catholique de Louvain).

Invited Plenary Speakers will include George Cybenko (Dartmouth College), Heike Fassbender (TU Munich), Andreas Frommer (Bergische Universität-Gesamthochschule Wuppertal), Rich Lehoucq (Sandia National Laboratories), James G. Nagy (Emory University) Michael Overton (New York University), G. W. (Pete) Stewart (University of Maryland), and Gilles Villard (CNRS/École Normale Supérieure de Lyon).

The Web site is www.siam.org/meetings/la03/index.htm

12th International Workshop on Matrices and Statistics

Dortmund, Germany: 5–8 August 2003

The 12th International Workshop on Matrices and Statistics will be held at the University of Dortmund (Dortmund, Germany) on August 5–8, 2003, during the week immediately before the 54th Biennial Session of the International Statistical Institute (ISI) in Berlin. This Workshop, which will be an ISI satellite meeting, will be hosted by the Department of Statistics at the University of Dortmund and will be cosponsored by the Bernoulli Society.

The International Organizing Committee comprises R. William Farebrother (England), Simo Puntanen (Finland), George P. H. Styan (Canada; vice-chair), and Hans Joachim Werner (Germany; chair): werner@united.econ.uni-bonn.de The Local Organizing Committee at the University of Dortmund comprises Jürgen Groß, Götz Trenkler (chair): trenkler@statistik.uni-dortmund.de, and Claus Weihs.

This Workshop in Germany will be the 12th in a series. The 11th International Workshop on Matrices and Statistics will be held at the Danish Technical University in Lyngby, Denmark: 29–31 August 2002.

New & Forthcoming Books on Linear Algebra & Related Topics

Douglas R. Farenick: Algebras of Linear Transformations

Reviewed by S. W. Drury

In the undergraduate curriculum, this book [*Algebras of Linear Transformations* by Douglas R. Farenick, Springer: xi + 238 pp., ISBN 0-387-95062-X (2001)] is suitable for a third course in linear algebra—a course that is hardly ever given. It is also the book that will allow the first-year graduate student to walk securely before running into fields like Banach Algebras, Operator Algebras, Rings and Modules, or Representation Theory. In short, it develops finite-dimensional linear algebra in the algebraic direction from stuff that is in the grasp of any serious mathematics undergraduate into the realm of the graduate curriculum. This is material usually considered too advanced for undergraduates and yet, on its own, too mundane for a graduate course.

But in the hands of Professor Farenick, this material is anything but mundane! The presentation has been carefully crafted and it maintains a keen interest on the part of the reader as the material unfolds. Above all, this is a book from which one can actually learn the subject. There are numerous examples that bring the ideas alive and a wealth of explanatory remarks that make for clarity and facilitate comprehension. The notes at the end of each chapter detail its historical context and taken as a whole, they chronicle the development of algebraic linear algebra out of the nineteenth century well into the twentieth. Another important feature of the notes is that they include suggestions for further reading for the reader who wishes to follow more recent developments.

It is very clear that the author spent a considerable amount of time devising the exercises that one finds at the end of each chapter. They are well thought out and instructive and the student who does them all will develop a substantially deeper knowledge of the material presented.

The first chapter on "Linear Algebra" brings the reader up to snuff on the basics. Despite the remedial intent, there is nothing rushed about the presentation. The undergraduate who has allowed a topic such as field extensions or bases in infinitely dimensional spaces to slip between the cracks will find it here beautifully presented.

The second chapter on "Algebras" presents many examples including the quaternions and group algebras. It goes on to discuss the spectral properties of elements, to prove classification theorems for finite-dimensional division algebras and to study matrix algebras over such algebras. Nonassociative algebras are not considered in this book.

The third chapter entitled "Invariant Subspaces" also deals with idempotents and projections, triangularizations, functional calculus and the polar decomposition. It also sows the seeds for topics that will be covered in more detail later in the book such as representations and von Neumann algebras.

Chapter 4 covers the "Wedderburn Theory", starting with nilpotency, progressing to both the nil radical and the Jacobson radical and culminating in the structure theorems for simple and semisimple algebras. Modules make a brief appearance here as a tool for determining the structure of simple algebras.

Chapter 5 deals with "Operator Algebras" at least in the finite-dimensional case. Here we find discussed von Neumann algebras, involutions, representability, the Wedderburn theory (again) and C^*-algebras. Von Neumann algebras and C^* algebras are, of course, essentially the same in the finite-dimensional case. The author takes the view that von Neumann algebras are closed unital involutive subalgebras of $\mathfrak{B}(\mathfrak{H})$ and C^* algebras are defined abstractly. This chapter is very useful groundwork for graduate students who will eventually get involved in the infinite-dimensional analogues of these concepts. One significant feature of this chapter is that many of the concepts are discussed over both $\mathbb{C}$ and $\mathbb{R}$, something which is usually not attempted in most texts.

The final chapter is on "Tensor Products". A long and gentle section develops the idea for vector spaces starting from the universal property. This is used to motivate the usual definition as a quotient of a free vector space. The author continues by discussing connections with spaces of linear mappings. Later in the chapter, the tensor product of algebras and finally operator algebras are considered and used as a tool to reframe Wedderburn's Theorem.

There are few misprints and typos, but §3.4 contains an unfortunate error. A corrected version of this section, together with a list of other misprints, can be found on the author's Web site www.math.uregina.ca/~farenick/fixups.pdf

To sum up, this is a remarkable book, with an unusual selection of material, usually accessed from a more advanced viewpoint. By restricting himself to the finite-dimensional case, Professor Farenick has put this material within the reach of the advanced mathematics undergraduate, who, in accepting the challenge, cannot help but be motivated by it.

Innovations in Multivariate Statistical Analysis: A Festschrift for Heinz Neduecker

Reviewed by Hans Joachim Werner

The last 30 years have witnessed a steadily increasing influence of matrix analysis in many scientific disciplines such as econometrics, psychometrics and multivariate analysis. This Festschrift for Heinz Neudecker [*Innovations in Multivariate Analysis: A Festschrift for Heinz Neduecker* edited by R. D. H. Heijmans, D. S. G. Pollock & A. Satorra, Advanced Studies in Theoretical and Applied Econometrics, Volume 36, Kluwer: xiii + 295 pp., ISBN 0-7923-8636-1 (2000)] is a collection of 20 essays aiming "to illustrate how powerful the tools of matrix theory have become as weapons in the statistician's armory".

The first three chapters of the book are concerned with the theory of matrix inequalities which, for example, can be very powerful tools in distribution theory and in the theory of statistical estimation. In Chapter 1, G. Alpargu & G. P. H. Styan show, from a historical perspective, how the well-known Kantorovich inequality (which the authors prefer to call the Frucht–Kantorovich Inequality) is related to several other inequalities; the chapter ends with an extensive bibliography. In Chapter 2, by S. Liu, a variety of matrix trace versions of the Kantorovich-type inequalities as well as some relevant matrix and determinant inequalities are collected. Mathematical and statistical applications are studied, and some new inequalities involving the Hadamard product are derived or conjectured. In Chapter 3, P. Bekker & T. Wansbeek present some results on matrix inequalities relevant in econometrics. The range of topics treated includes the multinomial probit model, regression analysis, the MINQUE approach and measurement error models.

The following two chapters deal with covariance matrices. Chapter 4, by G. Trenkler, is concerned with the covariance matrix of the multinomial distribution which may be rank-deficient. In Chapter 5, inspired by A. D. Chesher's "Testing for neglected heterogeneity" [*Econometrica*, 52, 865–872 (1984)], B. P. M. McCabe & S. J. Leybourne consider White's so-called information-matrix test as a general method of testing for random parameter variation in statistical models. There they allow, in particular, for quite arbitrary dependence and nonstationarity in the data generating mechanism.

Chapters 6 and 7 are devoted to correspondence analysis. M. van de Velden, in Chapter 6, clarifies the connections and differences between dual scaling and correspondence analysis of rank order data. Chapter 7, by C. M Cuadras, J. Fortiana & M. J. Greenacre, extends the correspondence analysis to a continuous bivariate distribution. The results are applied to a generalization of the Farlie–Gumbel–Morgenstern family of bivariate distributions.

The concern of Chapter 8 by R. H. Koning & G. Ridder is utility maximization. Most microeconometric demand studies are based on the assumption of utility maximization over a linear budget constraint which implies certain restrictions on the parameters of the demand function, the so-called integrability conditions. The analogue of these integrability conditions for discrete choice models is developed in this chapter and is used to assess the degree of rationality in choices of the mode of payment.

Chapters 9 and 10 deal with autoregressive models. By assuming a hierarchical prior distribution, which imposes constraints in the form of tightness on the lag distribution, and by proposing a Gibbs sampling approach to find the posterior distributions of the hyperparameters, W. Polasek analyzes, in Chapter 9, different types of autoregressive and vector autoregressive models with a latent variables or errors-in-variables structure from a Bayesian perspective. In Chapter 10, J. R. Magnus & Th. J. Rothenberg contribute to least-squares autoregression with near-unit root. In such autoregressive models, assumptions about the initial conditions can matter a lot when one of the roots of the characteristic polynomial is near unity. This chapter provides further evidence and offers some simple explanations for some well-known unusual features of least-squares estimates, test statistics, and forecasts in the leading case of the first-order autoregressive model.

The topic of Chapter 11 by F. Windmeijer is the generalized method of moments for dynamic panel data models that combines the differenced equation with the level equation. The inverse of the moment matrix of the instruments is used as the initial weight matrix in the criterion function of the estimation. This chapter assesses the potential efficiency loss resulting from the use of this weight matrix.

Chapters 12 and 13 are concerned with the stability of linear dynamic systems. In Chapter 12, F. J. H. Don considers a system of first order linear differential equations with a singular transition matrix. He shows that a rank and an order condition are jointly sufficient to ensure the existence of a non-explosive solution. Chapter 13, written by R. W. Farebrother, deals with a class of reflection matrices which are related to the Samuelson–Wise conditions for stability of a linear difference equation. This chapter also discusses a class of permutation matrices associated with functions defining chaotic and subchaotic pseudo-random numbers.

Conditioning in multivariate time series models is the topic of Chapter 14 by H. P. Boswijk. The concept of S-ancillarity is contrasted with the concept of strong exogeneity. The next four chapters are devoted to asymptotic theory in statistics. Sometimes, such as in principal component analysis, one may wish to test whether the columns of a given matrix lie in the subspace spanned by the eigenvectors associated with eigenvalues of a sample covariance matrix. In Chapter 15, T. Kollo develops a test statistic based on the asymptotic properties of eigenprojections of covariance matrices. Chapter 16 by D. G. Nel & P. C. N. Groenewald contains a detailed discussion of a Fisher–Cornish type expansion of Whishart matrices. Chapter 17 by A. Satorra is on scaled and adjusted tests in multi-sample analysis of moment structures. Moment structure analysis is widely used in behavioral, social and economic studies to analyze structural relations between variables, some of which may be latent. The main purpose of this chapter is to make corrections to score, difference and Wald test statistics which can be used in testing a specific set of restrictions on the parameters of the model when the alternative hypothesis is that the restrictions do not hold. In Chapter 18, R. D. Heijmans studies the asymptotic properties of sums of powers of residuals in the classical linear model.

Nonlinear dynamic optimization models are widely used in theoretical and empirical economic modelling, particularly in the field of optimal growth and intertemporal macroeconomic modelling. In Chapter 19, H. M. Amman & D. A. Kendrick describe an algorithm for solving a broad class of nonlinear optimization problems in discrete time. By making successive quadratic-linear approximations of the dynamic optimization model and computing the resulting algebraic Riccati matrix iteratively, a direct method for calculating the steady state solution is presented.

The final Chapter 20 by D. S. G. Pollock concerns some puzzles which arose in the context of computer graphics and on the interface between multilinear algebra and multivariate statistics. Here Pollock takes a fresh look at the multilinear algebra which lies at the heart of the techniques of matrix differential calculus to which the seminal paper by Heinz Neudecker "Some theorems on matrix differentiation with special reference to Kronecker products" [*Journal of the American Statistical Association* 64, 953–963 (1969)] has made substantial contributions.

On the whole, this book provides us with a cross section of some innovative work in the field of matrix methods and multivariate statistical analysis. Although most chapters of this book are unrelated, it will be useful for all students and practitioners in subjects which rely on modern methods of statistical analysis. It can serve not only as a reference book but also as a source of inspiration for future research. Needless to say, since this book is a Festschrift for Heinz Neudecker, most of the chapters are related to, or seem to be inspired by, his extensive scientific work.

New and Forthcoming Books

Listed here are some new and forthcoming books on linear algebra and related topics that have been published in 2000–2002 or are scheduled for publication in 2002 or 2003. This list updates and augments our previous lists in IMAGE 26 (April 2001), page 32, and 27 (October 2001), pp. 13–15. Please send all additions and corrections to George Styan: gphs@videotron.ca

A. Abragin & S. Dontsov (2000). *Linear Algebra, It's Easy: Training Aid* (in English). Voronezh State University: 249 pp., ISBN 5-9273-0052-9/hbk.

O. Axelsson & V. A. Barker (2001). *Finite Element Solution of Boundary Value Problems: Theory and Computation*, Reprint Edition. Classics in Applied Mathematics 35. SIAM: xxiii + 432 pp., ISBN 0-89871-499-0/pbk. [Originally published: Academic, 1984.]

T. H Barr (2001). *Vector Calculus*, Second Edition. Prentice Hall: 429 pp., ISBN 0-13-088005-1/hbk.

H. Benker (2000). *Mathematik mit MATLAB: Eine Einführung für Ingenieure und Naturwissenschaftler* (in German). Springer: xii + 496 pp., ISBN 3-540-67372-5.

A. Beutelspacher & M.-A. Zschiegner (2001). *Lineare Algebra interaktiv: Eine CD-ROM mit Tausenden von Übungsaufgaben* (in German). Vieweg, Wiesbaden, CD-ROM with textbook, ISBN 3-528-06890-6.

S. Bosch (2001). *Lineare Algebra* (in German). Springer-Lehrbuch. Springer: x + 284 pp., ISBN 3-540-41853-9/pbk.

D. Busneag & D. Piciu (2001). *Linear algebra* (in Romanian). Editura Universitaria, Craiova: 158 pp., ISBN 973-8043-62-2/pbk.

V. F. Butusov, N. Ch. Krutitskaya & A. A. Shishkin (2001). *Linear Algebra in Questions and Problems* (in Russian). FIZMATLIT, Moscow: 247 pp., ISBN 5-9221-0022-X.

D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, eds. (2002). *Linear Algebra Gems: Assets for Undergraduate Mathematics*. MAA Notes 59. Mathematical Association of America: xvi + 328 pp., ISBN 0-88385-170-9/pbk.

R. C. Dalang, & A. Chaabouni (2001). *Algèbre linéaire: Aide-mémoire, exercices et applications* (in French). Lausanne: Presses Polytechniques et Universitaires Romandes, Lausanne: xii + 322 pp., ISBN 2-88074-483-0/pbk.

J. Farlow, J. E. Hall, J. M. McDill & B. H. West (2002). *Differential Equations & Linear Algebra*. Prentice-Hall International: xxiv + 641 pp. & CD-ROM, ISBN 0-13-086250-9/hbk.

T. Furuta (2001). *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space*. Taylor & Francis: 272 pp., ISBN 0-415-26799-4.

Z. Gajic (2003). *Linear Dynamic Systems and Signals*. In press. Prentice Hall: 700 pp., ISBN 0-201-61854-0/hbk.

V. L. Girko (2001). *Theory of Stochastic Canonical Equations, Volume I*. Mathematics and its Applications 535. Kluwer: xxiv + 497 pp., ISBN 1-4020-0073-1.

V. L. Girko (2001). *Theory of Stochastic Canonical Equations, Volume II*. Mathematics and its Applications 535. Kluwer: xxvi + 463 pp., ISBN 1-4020-0074-X.

A. Givental (2001). *Linear Algebra and Differential Equations*. Berkeley Mathematics Lecture Notes 11. American Mathematical Society & Berkeley Center for Pure and Applied Mathematics: viii + 132 pp., ISBN 0-8218-2850-9.

F. Hansen (2000). *Operator Inequalities Associated with Jensen's Inequality*. Institute of Economics, Univ. Copenhagen: v + 43 pp.

D. J. Hartfiel (2002). *Nonhomogeneous Matrix Products*. World Scientific: x + 224 pp., ISBN 981-02-4628-5.

J. H. Hubbard & B. B. Hubbard (2002). *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach*, Second Edition. Prentice Hall: 816 pp., ISBN 0-13-041408-5/hbk.

G. Jäger (2001). *Algorithmen zur Berechnung der Smith-Normalform und deren Implementation auf Parallelrechnern* (in German). Dissertation. Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen 29, Fachbereich Mathematik, Universität Essen, vi + 84 pp.

S. Keith (2001). *Visualizing Linear Algebra Using Maple*. Prentice Hall: 250 pp., ISBN 0-13-041816-1/pbk.

A. Mukherjea (2000). *Topics in Products of Random Matrices*. Lectures on Mathematics and Physics: Mathematics. Tata Institute of Fundamental Research 87. Narosa Publishing House, New Delhi & Tata Institute of Fundamental Research, Mumbai: vi + 121 pp., ISBN 81-7319-297-9.

W. Müller (2001). *Lineare Algebra* (in German), 3rd Edition. Bayreuther Mathematische Schriften 62. Mathematisches Institut, Univ. Bayreuth: v + 227 pp., ISSN 0172-1062.

S. Nakamura (2002). *Numerical Analysis and Graphic Visualization with MATLAB*. Prentice Hall: 534 pp., ISBN 0-13-065489-2/hbk.

V. Olshevsky, ed. (2001). *Structured Matrices in Mathematics, Computer Science, and Engineering, II.: Proceedings of an AMS-IMS-SIAM Joint Summer Research Conference, University of Colorado, Boulder, CO, USA, June 27–July 1, 1999*. Contemporary Mathematics 281. American Mathematical Society: xiii + 344 pp., ISBN 0-8218-2092-3/pbk; ISSN 0271-4132.

G. W. Recktenwald (2001). *Introduction to Numerical Methods and MATLAB: Implementations and Applications*. Prentice Hall: 786 pp., ISBN 0-201-30860-6/hbk.

K. Sigmon & T. A. Davis (2002). *MATLAB Primer*, Sixth Edition. Chapman & Hall/CRC: xii + 163 pp., 1-58488294-8/pbk.

IMAGE Problem Corner: Old Problems, Many With Solutions

We present solutions to IMAGE Problems 26-1 and 27-1 through 27-6 published in IMAGE 26 (April 2001), p. 24, and IMAGE 27 (October 2001), p. 36, respectively. We are still hoping to receive solutions to Problems 19-3b and 23-1, which are repeated below. In addition, we introduce 10 new problems (4 on page 36 and 6 more on page 35) and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG, *North Carolina State University, Raleigh, North Carolina, USA:* hartwig@math.ncsu.edu
Peter ŠEMRL, *University of Maribor, Maribor, Slovenia:* peter.semrl@fmf.uni-lj.si
and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \operatorname{span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

The Editor has still not received a solution to Problem 19-3b. The solution by the Proposers to Problem 19-3a appeared in IMAGE 22 (April 1999), p. 25. *We look forward to receiving a solution to Problem 19-3b.*

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG, *University of Wisconsin–Madison, Madison, Wisconsin, USA:* mchung@stat.wisc.edu

Let the $m \times n$ random matrix X be such that $\operatorname{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where vec indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find E det$(X'CX)$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

We look forward to receiving a solution to this problem!

Problem 26-1: Degenerate Complex Quadratic Forms on Real Vector Spaces

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Let C be a complex symmetric $n \times n$ matrix. Suppose that A and B are, respectively, the real and imaginary parts of C (i.e., $a_{jk} = \Re c_{jk}$, $b_{jk} = \Im c_{jk}$). Suppose that $\det(B - tA) = 0$ for all real t. Prove or find a counterexample to the following statement. There necessarily exist nonnegative integers p, q and r such that $p + q + r = n$ and $q > r$ and a real invertible $n \times n$ matrix P, such that

$$P'CP = \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & Y \\ 0 & Y' & 0 \end{pmatrix},$$

where X is a $p \times p$ complex matrix and Y is a $q \times r$ complex matrix. The cases $p = 0$ and $r = 0$ are allowed. For instance, $p = n-1$, $q = 1$, $r = 0$ is the case where there is a non-zero real vector in the null space of C.

Solution 26-1.1 by Vladimir V. SERGEĬCHUK, *Institute of Mathematics, Kiev, Ukraine:* sergeich@ukrpack.net

A stronger statement holds over an arbitrary field $\mathcal{F}$ of characteristic $\neq 2$ with involution $a \mapsto \bar{a}$ (possibly $a = \bar{a}$ for all $a \in \mathcal{F}$); this follows from Theorem 4 in Sergeĭchuk (1988).

Let (A, B) be a pair of singular $n \times n$ matrices over $\mathcal{F}$ such that $\bar{A}^T = \varepsilon A$ and $\bar{B}^T = \delta B$, where $\varepsilon, \delta \in \{1, -1\}$. Assume that the pair is direct-sum-indecomposable; that is, there is no nonsingular matrix $S \in \mathcal{F}^{n \times n}$ such that

$$(\bar{S}^T AS, \bar{S}^T BS) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right),$$

where A_1 and B_1 are square matrices of the same size. Then there exists a nonsingular matrix $S \in \mathcal{F}^{n \times n}$ such that

$$(\bar{S}^T AS, \bar{S}^T BS) = \left(\begin{pmatrix} 0 & \varepsilon G^T \\ G & 0 \end{pmatrix}, \begin{pmatrix} 0 & \delta H^T \\ H & 0 \end{pmatrix} \right),$$

where

$$G = \begin{pmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}.$$

□

Reference

V. V. Sergeĭchuk (1988). Classification problems for systems of forms and linear mappings. *Mathematics of the USSR Izvestiya,* 31 (3), 481–501. [English translation from the Russian original: *Izvestiya Akademii Nauk SSSR, Seriya Mathematicheskaya*, 51 (6), 1170–1190 (1987).]

Problem 27-1: A Class of Square Roots of Involutory Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* Msrbsrf@fs1.ec.man.ac.uk

Let A and B be $n \times n$ nonsingular real matrices satisfying $ABA = B$ and $BAB = A$. Show that $M = AA = BB$ is involutory (i.e., $M^2 = I$), and identify a possible application for matrices of this type [see also "The Naming of Parts" in IMAGE 27 (October 2001), p. 11].

Solution 27-1.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* Msrbsrf@fs1.ec.man.ac.uk

A solution is established to a generalized version of the problem, in which the set of real nonsingular matrices is extended to the set of nonzero quadratic complex matrices with index not exceeding one. Let us recall that $\text{Ind}(A) = 0$ if A is invertible, and $\text{Ind}(A) = 1$ whenever $\text{rank}(A) = \text{rank}(A^2)$; see Campbell & Meyer (1979, p. 121). It is known that the inequality $\text{Ind}(A) \leq 1$ is a necessary and sufficient condition for the existence of the group inverse of A, which is the unique solution $A^{\#}$ to the equations

$$AA^{\#}A = A,\ A^{\#}AA^{\#} = A^{\#},\ AA^{\#} = A^{\#}A; \tag{1}$$

see Campbell & Meyer (1979, p. 124). Clearly, $AA^{\#}$ represents a projector onto the column space of A, henceforth denoted by $\mathcal{C}(A)$. In passing, we note that $A^{\#}$ is identical with the Moore-Penrose inverse A^+ if and only if A is an EP matrix, i.e., $\mathcal{C}(A) = \mathcal{C}(A^*)$; see Campbell & Meyer (1979, p. 129).

THEOREM 1. *Let A and B be nonzero quadratic complex matrices of the same order such that* $\text{Ind}(A) \leq 1$ *and* $\text{Ind}(B) \leq 1$. *Then any two of the following equalities:*

$$ABA = B, \tag{2}$$

$$BAB = A, \tag{3}$$

$$A^2 = B^2, \tag{4}$$

imply the third.

PROOF. From (2) and (3) it is immediately seen that

$$A^2 = A(BAB) = (ABA)B = B^2,$$

which is (4). Further, (2) shows that the columns of B lie in $\mathcal{C}(A)$, so that $AA^{\#}B = B$. Consequently, it follows that if (2) and (4) hold along with $AA^{\#} = A^{\#}A$, then

$$BAB = (AA^{\#}B)AB = A^{\#}(ABA)B = A^{\#}B^2 = A^{\#}A^2 = AA^{\#}A = A,$$

which is (3). The proof that (3) and (4) imply (2) is analogous. □

Introducing into consideration the matrix $C = AB$ in addition to A and B affords the possibility of formulating several cyclic type properties.

THEOREM 2. *Let A and B be nonzero quadratic complex matrices of the same order such that* $\mathrm{Ind}(A) \leq 1$ *and* $\mathrm{Ind}(B) \leq 1$ *and let $C = AB$. If A and B satisfy any two (and therefore all three) of the equations*

$$ABA = B, \ BAB = A, \ A^2 = B^2, \tag{5}$$

then

$$AB = C, \ BC = A, \ CA = B, \tag{6}$$

$$ABC = A^2 = BCA = B^2 = CAB = C^2, \tag{7}$$

$$ACB = A^4 = CBA = C^4 = BAC = B^4 = P, \tag{8}$$

where

$$P = AA^{\#} = BB^{\#} = CC^{\#} \tag{9}$$

is a projector onto $\mathcal{C}(A) = \mathcal{C}(B) = \mathcal{C}(C)$. Moreover,

$$A^{\#} = B^{\#}AB = CAC^{\#}, \ B^{\#} = C^{\#}BC = ABA^{\#}, \ C^{\#} = A^{\#}CA = BCB^{\#}. \tag{10}$$

PROOF. The equalities in (6) are trivial consequences of the definition of C and the conditions (2) and (3). Further, an immediate consequence of (6) is that $ABC = C^2 = A^2$, $BCA = A^2 = B^2$, and $CAB = B^2 = C^2$, which leads to (7). Now, let M denote $A^2 = B^2 = C^2$. Then $ACB = A^2B^2 = A^4 = M^2$, $CBA = AB^2A = A^4 = M^2$, and $BAC = BA^2B = B^4 = M^2$, which is (8). From (6) it is seen that $\mathcal{C}(C) \subseteq \mathcal{C}(A)$, $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, and $\mathcal{C}(B) \subseteq \mathcal{C}(C)$, and hence $\mathcal{C}(A) = \mathcal{C}(B) = \mathcal{C}(C)$. Consequently,

$$M^2 = B^2C^2 = AA^{\#}B^2C^2 = A^{\#}(AB)(BC)C = A^{\#}CAC = A^{\#}BC = A^{\#}A,$$

$$M^2 = C^2A^2 = BB^{\#}C^2A^2 = B^{\#}(BC)(CA)A = B^{\#}ABA = B^{\#}B,$$

$$M^2 = A^2B^2 = CC^{\#}A^2B^2 = C^{\#}(CA)(AB)B = C^{\#}BCB = C^{\#}AB = C^{\#}C,$$

which shows that P in (8) can be expressed as in (9).

From (8), (9), and the definition of the group inverse, it follows that

$$A^2 = (AA^{\#}A)^2 = A^4(A^{\#})^2 = P(A^{\#})^2 = A^{\#}A(A^{\#})^2 = (A^{\#})^2,$$

so that

$$A^{\#} = A^{\#}AA^{\#} = A(A^{\#})^2 = A^3.$$

Similarly, $B^{\#} = B^3$, $C^{\#} = C^3$, and then, in view of (6) and (7),

$$A^{\#} = A^2A = C^2A = CB, \ B^{\#} = B^2B = A^2B = AC, \ C^{\#} = C^2C = B^2C = BA. \tag{11}$$

The relationships in (11) lead to those in (10) quite straightforwardly. □

Notice that in the particular case where A and B are nonsingular, Theorem 1 remains unchanged, while part (8) of Theorem 2 yields the following.

COROLLARY. *Let A and B be complex nonsingular matrices of the same order. Then any two of the equalities* (2), (3), (4) *imply that $A^4 = B^4 = C^4 = I$, which means that A^2, B^2, and C^2 are all involutions.*

The results given above are familiar in the special case of $A^2 = B^2 = C^2 = -I$, when they may be identified with the Hamiltonian conditions associated with the matrix representation of quaternions:

$$AB = C = -BA, \; BC = A = -CB, \; CA = B = -AC.$$

Reference

S.L. Campbell & C.D. Meyer, Jr. (1979). *Generalized Inverses of Linear Transformations.* Pitman, London. [Unabridged corrected reprint edition: Dover, New York (1991).]

Solution 27-1.2 by Chi-Kwong LI, Edward POON & David STANFORD, *The College of William and Mary, Williamsburg, Virginia:* ckli@math.wm.edu, poon@math.wm.edu, stanford@math.wm.edu

This problem is really an exercise in group theory and will be stated as such.

PROPOSITION. *Suppose a, b are elements in a group such that*

$$(i)\ aba = b, \qquad (ii)\ bab = a.$$

Then $m = aa = bb$ satisfies $m^2 = 1$, where 1 denotes the group identity.

PROOF. Equation (i) gives $ba = a^{-1}b$ while (ii) gives $ba = ab^{-1}$. Thus $a^{-1}b = ab^{-1}$, i.e., $b^2 = a^2$. Writing $m = a^2$ gives $m^2 = aa^2a = ab^2a = b^{-1}babba = b^{-1}(bab)ba = b^{-1}aba = b^{-1}b = 1$. □

Solution 27-1.3 by Hans Joachim WERNER, *University of Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

We offer the following result, which is more general.

THEOREM 1. *Let A and B be real or complex matrices with $ABA = B$ and $BAB = A$. Then $A^2 = B^2$, $A^5 = A$, $B^5 = B$, and $A^\#$ and $B^\#$, the unique group inverses of A and B, respectively, both exist. Precisely, $A^\# = A^3$ and $B^\# = B^3$.*

PROOF. Clearly, $ABA = B \Rightarrow (AB)^2 = B^2$ and $A^2BA^2 = B$. Likewise, $BAB = A \Rightarrow (AB)^2 = A^2$ and $B^2AB^2 = A$. Consequently, $A^2 = B^2$. Therefore, $A^5 = A$ and $B^5 = B$. In view of the equality $A^5 = A$, trivially $AA^3A = A$, $A^3AA^3 = A^3$, and $A^3A = AA^3$, thus showing that the matrix A^3 satisfies the equations defining the unique group inverse $A^\#$ of A; see Definition 7.2.4 in Campbell & Meyer (1979, p. 124). Hence $A^\#$ exists and $A^\# = A^3$. Along similar lines, $B^5 = B$ implies that $B^\#$ exists and $B^\# = B^3$. □

When A and B are nonsingular, the group inverses of A and B are obviously equal to A^{-1} and B^{-1}, respectively. The equations $A^\# = A^3$ and $B^\# = B^3$ in Theorem 1 are then equivalent to $A^4 = I$ and $B^4 = I$. Since the matrices A^2 and B^2 are thus involutory, our solution is complete.

Reference

S.L. Campbell & C.D. Meyer, Jr. (1979). *Generalized Inverses of Linear Transformations.* Pitman, London. [Unabridged corrected reprint edition: Dover, New York (1991).]

Solutions to this problem were also received from Alain BOURGET & Gülhan ALPARGU, Heinz NEUDECKER, Shayle R. SEARLE, *and from* Fuzhen ZHANG.

Problem 27-2: Specific Generalized Inverses

Proposed by Jürgen GROß & Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* gross@statistik.uni-dortmund.de, trenkler@statistik.uni-dortmund.de

Characterize the class $\mathcal{A}$ of $m \times n$ matrices A with complex entries such that $(A^*A)^kA^*$ is a generalized inverse of A for an arbitrary positive integer k.

Solution 27-2.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

We solve the problem in a generalized version. Let $\mathcal{C}_{m,n}$ denote the set of $m \times n$ complex matrices and let $\mathcal{C}_n^{\geq}$ be a subset of $\mathcal{C}_{n,n}$ consisting of Hermitian nonnegative definite matrices. The solution refers to the following concept.

DEFINITION. Let $N \in \mathcal{C}_n^{\geq}$ of rank $a > 0$ have a spectral decomposition of the form $N = U\Lambda U^*$, where $U \in \mathcal{C}_{n,a}$ satisfies $U^*U = I_a$ and $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_a)$ is a diagonal matrix comprising positive eigenvalues of N. Then, for any $\alpha \in \mathcal{R}$, N^α is defined by the formula $N^\alpha = U\Lambda^\alpha U^*$, where $\Lambda^\alpha = \mathrm{diag}(\lambda_1^\alpha, ..., \lambda_a^\alpha)$.

According to this definition, $N^{-1} = U\Lambda^{-1}U^*$ and $N^0 = U\Lambda^0 U^* = UU^*$ signify the Moore-Penrose inverse of N (customarily denoted by N^+) and the orthogonal (with respect to the standard inner product) projector onto the range (column space) of N, respectively. Moreover, it is clear that if $\alpha = -1$, then the equation

$$NN^\alpha N = N \tag{12}$$

is fulfilled by every $N \in \mathcal{C}_n^{\geq}$. A solution for other values of α is given in Theorem 1 below, and an answer to the original problem is included in a corollary to it.

THEOREM 1. *For any nonzero $N \in \mathcal{C}_n^{\geq}$, the statements:*
(a) *N^α is a generalized inverse of N for some $\alpha \in \mathcal{R} \setminus \{-1\}$,*
(b) *N^α is a generalized inverse of N for every $\alpha \in \mathcal{R} \setminus \{-1\}$,*
are equivalent and hold if and only if N is a projector, i.e., $N = N^2$.

PROOF. On account of the well known definition, N^α is a generalized inverse (inner inverse) of N if and only if it satisfies (12). For N decomposed as in Definition, (12) reduces to

$$\lambda_i^{\alpha+2} = \lambda_i, \quad i = 1, ..., a. \tag{13}$$

It is clear that (13) holds for some $\alpha \in \mathcal{R} \setminus \{-1\}$ if and only if $\lambda_1 = ... = \lambda_a = 1$, which further ensures that (13) holds for every $\alpha \in \mathcal{R} \setminus \{-1\}$. Since in particular, for $\alpha = 0$, equation (12) takes the form $N^2 = N$, the proof is complete. □

COROLLARY. *For any nonzero $A \in \mathcal{C}_{m,n}$, the requirements that $(A^*A)^\alpha A^*$ be a generalized inverse of A for some $\alpha \in \mathcal{R} \setminus \{-1\}$ and for every $\alpha \in \mathcal{R} \setminus \{-1\}$ are equivalent and hold if and only if A is a partial isometry, i.e., $A = AA^*A$.*

PROOF. A matrix $(A^*A)^\alpha A^*$ is a generalized inverse of A if and only if $A(A^*A)^\alpha A^*A = A$. But this is equivalent to

$$A^*A(A^*A)^\alpha A^*A = A^*A,$$

i.e., to equation (12) with $N = A^*A$. Consequently, the result follows from Theorem 1 since the premultiplier A^* can be cancelled in $A^*A = A^*AA^*A$. □

Finally notice that if N^α in Theorem 1 is a generalized inverse of N, then it actually is its Moore-Penrose inverse N^+. A similar observation is valid for a matrix of the form $(A^*A)^\alpha A$, and therefore the well known result, which asserts that $A^+ = (A^*A)^+A$ is true for any $A \in \mathcal{C}_{m,n}$ (see Theorem 1.2.1(P6) in Campbell & Meyer (1979)), can be supplemented by the statement that $A^+ = (A^*A)^\alpha A^*$ holds for any $\alpha \in \mathcal{R} \setminus \{-1\}$ whenever A is a partial isometry.

Reference

S.L. Campbell & C.D. Meyer, Jr. (1979). *Generalized Inverses of Linear Transformations.* Pitman, London. [Unabridged corrected reprint edition: Dover, New York (1991).]

Solution 27-2.2 by Hans Joachim WERNER: *University of Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Let $\mathcal{A}$ denote the set of all those complex matrices A such that, for an arbitrary positive integer k, $(A^*A)^kA^*$ is a generalized inverse of A, i. e., $A(A^*A)^{k+1} = A$. We offer the following characterizations for such matrices.

THEOREM 1. *The following conditions are all equivalent:*

(a) $A \in \mathcal{A}$;

(b) A^*A *is idempotent;*

(c) A^*A *is the orthogonal projector onto the range (column space) of* A^* *along the null space of* A;

(d) A^* *is a generalized inverse of* A, *i. e.,* $AA^*A = A$;

(e) A *is a generalized inverse of* A^*, *i. e.,* $A^*AA^* = A^*$;

(f) $A^* \in \mathcal{A}$;

(g) AA^* *is idempotent;*

(h) AA^* *is the orthogonal projector onto the range of* A *along the null space of* A^*;

(i) *The singular values of* A *are all equal to* 0 *and/or* 1;

(j) $A = UV^*$ *for some column-unitary matrices* U *and* V, *i. e.,* $U^*U = I = V^*V$;

(k) A *is a partial isometry, i. e.,* $A^+ = A^*$, *with* A^+ *denoting the Moore-Penrose inverse of* A.

PROOF. In the singular value decomposition of the complex $m \times n$ matrix A of rank r, say, A is decomposed into $A = UDV^*$, where U and V are column-unitary matrices of order $m \times r$ and $n \times r$, respectively, and where D is a diagonal matrix of order $r \times r$ with positive diagonal elements; see Theorem 3.5 in Zhang (1999, p. 66). Then, for an arbitrary $k \in \mathbb{N}$, clearly $(A^*A)^k = VD^{2k}V^*$, thus showing that the diagonal elements of D^{2k} are the nonzero eigenvalues of the nonnegative definite Hermitian matrix $(A^*A)^k$. Now, let $A \in \mathcal{A}$. Then $A(A^*A)^kA^*A = A$, and so $(A^*A)^{k+2} = A^*A(A^*A)^kA^*A = A^*A$. But this can happen if and only if $D^{2(k+2)} = D^2$ or, equivalently, if and only if $D = I$. Since then, irrespective of $k \in \mathbb{N}$, $(A^*A)^k = A^*A$ is idempotent, the proof of (a) $\Rightarrow$ (b) is complete.

Next notice that A^*A is nonnegative definite and Hermitian, $\mathcal{R}(A^*A) = \mathcal{R}(A^*)$ and $\mathcal{N}(A^*A) = \mathcal{N}(A)$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote, as usual, the range (column space) and the null space of $(\cdot)$, respectively. If (b) holds, it is hence clear that A^*A is the orthogonal projector onto $\mathcal{R}(A^*)$ [along its (orthogonal) complement, $\mathcal{N}(A)$, with respect to the usual inner product]. So we have (b) $\Rightarrow$ (c). Evidently, (c) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a). Since in the spectral decomposition of A, the elements along the main diagonal of D are the nonzero singular values of A, it is clear that (b) $\iff$ (i) $\iff$ (j). Trivially, (d) $\iff$ (e), and so, in view of our already established results, also (e) $\iff$ (f) $\iff$ (g) $\iff$ (h). Since $AA^*A = A \iff A^*AA^* = A^*$, and since AA^* and A^*A are both Hermitian, it is also clear that (d) is equivalent to (k). This completes the proof. □

Reference

F. Zhang (1999). *Matrix Theory: Basic Results and Techniques.* Springer-Verlag, New York.

Solution 27-2.3 by the Proposers Jürgen GROß & Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:*
gross@statistik.uni-dortmund.de, trenkler@statistik.uni-dortmund.de

An $m \times n$ matrix A belongs to the class $\mathcal{A}$ if and only if A is a partial isometry, i.e., $A^\dagger = A^*$, where $A^\dagger$ denotes the Moore-Penrose inverse of A. To see this, let $(A^*A)^kA^*$ be a generalized inverse of A. Then $(A^*A)^kA^*A = (A^*A)^{k+1}$ is Hermitian idempotent. Consequently, also A^*A is Hermitian idempotent, which can be seen for example from a spectral decomposition of A^*A and the fact that any Hermitian idempotent matrix has only 0 and 1 as eigenvalues. From Ben-Israel and Greville (1974, p. 252), A^*A is Hermitian idempotent if and only if A is a partial isometry. Conversely, let A be a partial isometry so that $A^\dagger = A^*$. Then

$$(A^*A)^kA^* = (A^\dagger A)^kA^\dagger = A^\dagger AA^\dagger = A^\dagger$$

is a generalized inverse of A. □

Reference

A. Ben-Israel & T. N. E. Greville (1974). *Generalized Inverses. Theory and Applications.* John Wiley, New York. [Corrected reprint edition, Krieger (1980).]

Solutions to this problem were also received from Chi-Kwong LI, Edward POON & David STANFORD, *and* Fuzhen ZHANG.

Problem 27-3: A Word Problem

Proposed by Charles R. JOHNSON, *The College of William and Mary, Williamsburg, Virginia:* crjohnso@m24.math.wm.edu

Let $V(A, B)$ and $W(A, B)$ be "words" in the two letters A and B, e.g., W could be $W(A, B) = AABABBBA$. Show that any given W is identically equal to V if and only if $W(L, U) = V(L, U)$ for the particular matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Solution 27-3.1 by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

There is nothing to show for the necessity. To show the sufficiency, we prove that if $W(L, U) = V(L, U)$, then W and V begin with the same letter L or U on the left (or right). That is, we show that $LP \neq UQ$ for any words P and Q in L and U. If, otherwise, $LP = UQ$, then $P = L^{-1}UQ$. Upon computation,

$$L^{-1}U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since Q is a 2×2 nonnegative nonsingular matrix, $L^{-1}UQ$ will have at least a negative entry in the second row. On the other hand, P is a nonnegative nonsingular matrix, it cannot equal $L^{-1}UQ$. A contradiction. This process can be repeated after a cancellation of the same letter-nonsigular matrix L (or U). □

Solutions to this problem were also received from Steven J. LEON *and from* Chi-Kwong LI, Edward POON & David STANFORD.

Problem 27-4: An Inequality for Hadamard Products Involving a Correlation Matrix

Proposed by Shuangzhe LIU, *Australian National University, Canberra, Australia:* lius@maths.anu.edu.au

Show that for a $k \times k$ positive definite correlation matrix A

$$A \circ A^{-1} + I \leq_{\mathsf{L}} \frac{(c+d)^2}{2cd}(A \circ A)^{-1},$$

where $\circ$ stands for the Hadamard product, $\leq_{\mathsf{L}}$ stands for the Löwner ordering, and c and d are the largest and smallest eigenvalues of A, respectively. This inequality is a counterpart to an inequality established by G. P. H. Styan (1973): Hadamard products and multivariate statistical analysis, *Linear Algebra and Its Applications*, 6, 217–240.

Solution 27-4.1 by Chi-Kwong LI, Edward POON & David STANFORD, *The College of William and Mary, Williamsburg, Virginia:*

ckli@math.wm.edu, poon@math.wm.edu, stanford@math.wm.edu

We use two facts, see Bhatia & Davis (2000):

FACT 1. *For any positive definite matrix,* $I \leq_{\mathsf{L}} A \circ A^{-1}$.

FACT 2. *For any linear positive unital map* ϕ *on the matrix algebra* M_n, *and for any positive definite matrix* A *with largest eigenvalue* d *and smallest eigenvalue* c,

$$\phi(A^{-1}) \leq_{\mathsf{L}} \frac{(c+d)^2}{4cd}\phi(A)^{-1}.$$

Note that for a correlation matrix A, the map $\phi(B) = A \circ B$ is a positive unital map. Thus

$$A \circ A^{-1} + I \leq_{\mathsf{L}} A \circ A^{-1} + A \circ A^{-1} = 2\phi(A^{-1}) \leq_{\mathsf{L}} 2\frac{(c+d)^2}{4cd}\phi(A)^{-1} = \frac{(c+d)^2}{2cd}(A \circ A)^{-1}.$$

□

Reference

R. Bhatia & C. Davis (2000). A better bound on the variance. *The American Mathematical Monthly*, 107, 353–357.

Solution 27-4.2 by George VISICK, *Belgravia, London, England, UK:* gv94511@gsk.com

Evidently, $cI - A \geq_{\mathsf{L}} 0$ and $A - dI \geq_{\mathsf{L}} 0$. Multiplying these commuting matrices results in $-A^2 + (c+d)A - cdI \geq_{\mathsf{L}} 0$. Pre- or postmultiplying this inequality by A^{-1} gives, after some reordering,

$$(c+d)I - A \geq_{\mathsf{L}} cdA^{-1}.$$

We may now validly form the Hadamard product with matrix A and, since $A \circ I = I$ for a correlation matrix, we get

$$(c+d)I - A \circ A \geq_{\mathsf{L}} cdA \circ A^{-1}. \tag{14}$$

Completing the square on the left results in $\frac{1}{4}(c+d)^2(A \circ A)^{-1} - \left(\frac{1}{2}(c+d)(A \circ A)^{-1/2} - (A \circ A)^{1/2}\right)^2 \geq_{\mathsf{L}} cdA \circ A^{-1}$ from which, neglecting the subtracted squared term, we arrive at

$$\frac{(c+d)^2}{2cd}(A \circ A)^{-1} \geq_{\mathsf{L}} 2A \circ A^{-1} \geq_{\mathsf{L}} A \circ A^{-1} + I\,. \tag{15}$$

The final part follows since $A \circ A^{-1} \geq_{\mathsf{L}} I$. The inequality (14) is stronger than the required inequality(15). It depends essentially on both c and d, while (15) depends only on the ratio c/d. □

Solution 27-4.3 by the Proposer Shuangzhe LIU, *Australian National University, Canberra, Australia:* lius@maths.anu.edu.au

We first prove that for $k \times k$ positive definite correlation matrices A and B

$$A \circ B^{-1} + A^{-1} \circ B + 2I \leq_{\mathsf{L}} \frac{(c+d)^2}{cd}(A \circ B)^{-1}, \tag{16}$$

where $\circ$ stands for the Hadamard product, $\leq$ stands for the Löwner ordering, and c and d are positive numbers such that the eigenvalues of A and B are contained in the interval $[d, c]$.

We have the following matrix Kantorovich inequality (see, e.g., Liu and Neudecker, 1997)

$$Y'NMNY \leq_{\mathsf{L}} \frac{(p+q)^2}{4pq} Y'NY(Y'M^{-1}Y)^{-1}Y'NY,$$

where M and N are positive definite matrices of the same order, and $p > 0$ and $q > 0$ are the largest and smallest eigenvalues of MN, respectively. We use

$$M = A^{-1} \otimes B^{-1}, N = A \otimes I + I \otimes B \quad \text{and} \quad Y = J,$$

where $\otimes$ stands for the Kronecker product, J is the $k^2 \times k$ selection matrix (see, e.g., Rao and Rao (1998, §6.3), and Zhang (1999, §6.5) such that $Y'M^{-1}Y = A \circ B$, $Y'NY = A \circ I + I \circ B = 2I$ and $Y'NMNY = A \circ B^{-1} + A^{-1} \circ B + 2I$ for $k \times k$ correlation matrices A, B and I. Then

$$A \circ B^{-1} + A^{-1} \circ B + 2I \leq_{\mathsf{L}} \frac{(p+q)^2}{pq}(A \circ B)^{-1}.$$

Note that $MN = A^{-1} \otimes I + I \otimes B^{-1}$. The eigenvalues of A^{-1} and B^{-1} are contained in $[1/c, 1/d]$, and therefore the largest eigenvalue p and smallest eigenvalue q of MN are bounded: $p \leq 2/d$ and $q \geq 2/c$, which can be derived from, e.g., Zhang (1999, Theorem 6.19). We can then replace p and q by $2/d$ and $2/c$, respectively, and get (16). Letting $A = B$ establishes

$$A \circ A^{-1} + I \leq_{\mathsf{L}} \frac{(c+d)^2}{2cd}(A \circ A)^{-1}.$$

□

References

S. Liu & H. Neudecker (1997). Kantorovich inequalities and efficiency comparisons for several classes of estimators in linear models. *Statistica Neerlandica,* 51, 345–355.

C. R. Rao & M. B. Rao (1998). *Matrix Algebra and Its Applications to Statistics and Econometrics.* World Scientific, Singapore.

F. Zhang (1999). *Matrix Theory: Basic Results and Techniques.* Springer-Verlag, New York.

Problem 27-5: Orthogonal Projectors and the Löwner Ordering

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

How are the complex orthogonal projectors P and Q (i.e., Hermitian idempotent matrices with complex entries) related if they satisfy the following inequality in the Löwner ordering

$$\sqrt{2}\,(P+Q)^{1/2} \leq_{\mathsf{L}} P+Q,$$

where $(P+Q)^{1/2}$ denotes the unique nonnegative definite square root of $P+Q$?

Solution 27-5.1 by Chi-Kwong LI, Edward POON & David STANFORD, *The College of William and Mary, Williamsburg, Virginia:* ckli@math.wm.edu, poon@math.wm.edu, stanford@math.wm.edu

Note that if A, B are commuting nonnegative definite matrices, $A \leq_{\mathsf{L}} B \iff A^2 \leq_{\mathsf{L}} B^2$. As $\sqrt{2}(P+Q)^{1/2}$ and $P+Q$ commute, it follows that $\sqrt{2}(P+Q)^{1/2} \leq_{\mathsf{L}} P+Q \iff 2(P+Q) \leq_{\mathsf{L}} (P+Q)^2 \iff (P-Q)^2 \leq_{\mathsf{L}} 0 \iff P=Q.$ □

Solution 27-5.2 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

The projectors P and Q are identical. Let $R = \frac{1}{2}(P+Q)$. Then the displayed inequality becomes $R^{1/2} \leq_{\mathsf{L}} R$. On the other hand we have $R \leq_{\mathsf{L}} R^{1/2}$; see Bellman (1968, Theorem 1). Hence $R^{1/2} = R$, which implies that $(P-Q)^*(P-Q) = 0$. □

Reference

R. Bellman (1968). Some inequalities for the square root of a positive definite matrix. *Linear Algebra and its Applications*, 1, 321–324.

Solutions to this problem were also received from Jerzy K. BAKSALARY & Oskar Maria BAKSALARY; George VISICK *and from* Fuzhen ZHANG.

Problem 27-6: Inequalities of Hadamard Products of Nonnegative Definite Matrices

Proposed by Xingzhi ZHAN, *Tohoku University, Sendai, Japan:* zhan@math.is.tohoku.ac.jp

Let A and B be nonnegative definite matrices of the same size and let s and t be positive real numbers such that $s+t=1$. Prove or disprove that

$$A^s \circ B^t \leq_{\mathsf{L}} (A \circ I)^s (B \circ I)^t \leq_{\mathsf{L}} (sA+tB) \circ I,$$

where $\circ$ stands for the Hadamard product and $\leq_{\mathsf{L}}$ for the Löwner ordering.

Solution 27-6.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl; and Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

The first inequality is just the content of Theorem 7.2 of Ando (1998), while the second (involving two diagonal matrices) is an immediate consequence of Young's inequality; see, e.g., Ando (1998, p. 75). See also Ando (1979). □

References

T. Ando (1979). Concavity of certain maps on positive definite matrices and applications to Hadamard products. *Linear Algebra and its Applications*, 26, 203–241.

T. Ando (1998). *Operator-Theoretic Methods for Matrix Inequalities.* Hokusei Gakuen University, Sapporo, Japan.

Solution 27-6.2 by S. W. DRURY, *McGill University, Montréal, Québec, Canada:* drury@math.mcgill.ca

The right-hand inequality follows easily from the geometric-arithmetic mean inequality. The left-hand inequality is more interesting.

LEMMA 1. *Let P and R be diagonal $m \times m$ matrices with positive entries and suppose that Q is a Hermitian $m \times m$ matrix such that $\begin{pmatrix} P & Q \\ Q & R \end{pmatrix}$ is nonnegative definite. Then $Q \leq_{\mathsf{L}} (PR)^{1/2}$.*

PROOF. It follows easily from the hypotheses that $P^{-1/2}QR^{-1/2}$ is a contraction and therefore that all of its eigenvalues have absolute value less than or equal to unity. On the other hand, the similar matrix $H = R^{-1/4}P^{-1/4}QP^{-1/4}R^{-1/4}$ is Hermitian and therefore $H \leq_{\mathsf{L}} I$. The desired conclusion now follows. □

LEMMA 2. *Let $n = 2^k$ with k a nonnegative integer. Let p and q be nonnegative integers with $n = p + q$. Let A and B be positive definite matrices of the same size. Then it follows that*

$$A^p \circ B^q \leq_{\mathsf{L}} (A^n \circ I)^{p/n}(B^n \circ I)^{q/n}.$$

PROOF. The proof is by induction on k. If $k = 0$, then $n = 1$ and either $p = 1$ and $q = 0$ or $p = 0$ and $q = 1$. In either case, the result is trivial. So the induction starts. Suppose now that $k \geq 1$ and that the result is proved for $k - 1$. We will establish it for k. If p (and therefore also q) is even, then the result follows from the induction hypothesis applied to $k_1 = k - 1, p_1 = p/2, q_1 = q/2$, $A_1 = A^2$ and $B_1 = B^2$. Therefore we may assume that p and q are both odd. Now, the Hadamard product of nonnegative definite matrices is nonnegative definite, so

$$\begin{pmatrix} A^{p-1} \circ B^{q+1} & A^p \circ B^q \\ A^p \circ B^q & A^{p+1} \circ B^{q-1} \end{pmatrix} = \begin{pmatrix} A^{p-1} & A^p \\ A^p & A^{p+1} \end{pmatrix} \circ \begin{pmatrix} B^{q+1} & B^q \\ B^q & B^{q-1} \end{pmatrix}$$

is nonnegative definite. But now, $p - 1$ and $q + 1$ are both even, so applying the inductive hypothesis, we have $A^{p-1} \circ B^{q+1} \leq_{\mathsf{L}} (A^n \circ I)^{(p-1)/n} (B^n \circ I)^{(q+1)/n}$. So, using this and another similar adjustment, we have that

$$\begin{pmatrix} (A^n \circ I)^{(p-1)/n} (B^n \circ I)^{(q+1)/n} & A^p \circ B^q \\ A^p \circ B^q & (A^n \circ I)^{(p+1)/n} (B^n \circ I)^{(q-1)/n} \end{pmatrix}$$

is also nonnegative definite. The induction step is completed by Lemma 1. □

Replacing A^n by A and B^n by B in Lemma 2 yields the left-hand inequality of the problem with $s = p/n$ and $t = q/n$ nonnegative dyadic rationals adding to unity, at least in the positive definite case. Straightforward limiting arguments now give the result for s and t nonnegative reals summing to unity and A and B nonnegative definite.

Solution 27-6.3 by George VISICK, *Belgravia, London, England, UK:* gv94511@gsk.com

We shall prove that the Löwner inequalities

$$A^s \circ B^t \ \leq_{\mathrm{L}} \ (A \circ I)^s (B \circ I)^t \ \leq_{\mathrm{L}} \ (sA + tB) \circ I \tag{17}$$

with $0 \leq s, t$ and $s + t = 1$, do indeed hold for nonnegative definite matrices. Ando (1979) derives the inequality

$$A \circ B \ \leq_{\mathrm{L}} \ (A^p \circ I)^{1/p} (B^q \circ I)^{1/q} \ ,$$

where $0 \leq p, q$, and $1/p + 1/q = 1$, with a non-trivial proof. The left hand inequality of (17) follows from this by setting $s = p^{-1}, t = q^{-1}$ and replacing $A^{1/s}$ by A and $B^{1/s}$ by B. The right hand inequality of (17) is just the arithmetic-geometric mean inequality applied to the diagonal elements of A and B, that is

$$a_{ii}^s \, b_{ii}^t \leq s a_{ii} + t b_{ii} \ .$$

□

Reference

T. Ando (1979). Concavity of certain maps on positive definite matrices and applications to Hadamard products. *Linear Algebra and its Applications,* 26, 203–241.

A solution was also received from the Proposer Xingzhi ZHAN.

More New Problems[1]

Problem 28-5: A Range Equality for Moore-Penrose Inverses

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario:* ytian@mast.queensu.ca

Suppose A and B are complex $m \times n$ and $m \times k$ matrices, respectively. Show that if $\mathrm{range}(A) \cap \mathrm{range}(B) = \{0\}$, then

$$\mathrm{range}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = \mathrm{range}\begin{pmatrix} A^* \\ B^* \end{pmatrix},$$

where $(\cdot)^\dagger$ and $(\cdot)^*$ denote the Moore-Penrose inverse and the conjugate transpose of $(\cdot)$, respectively.

Problem 28-6: Square Roots and Additivity

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Let A and B be nonnegative definite matrices of the same type and let $\leq_{\mathsf{L}}$ denote the Löwner-ordering. Is it true that

$$(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2} \ ?$$

Problem 28-7: Partial Isometry and Idempotent Matrices

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $\mathbf{A}$ be an idempotent complex matrix. Show that $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is a partial isometry, i.e. $\mathbf{A}\mathbf{A}^*\mathbf{A} = \mathbf{A}$, where $\mathbf{A}^*$ denotes the conjugate transpose of $\mathbf{A}$.

Problem 28-8: Another Inequality for Hadamard Products

Proposed by George VISICK, *Belgravia, London, England, UK:* gv94511@gsk.com

Show that for any positive definite matrix A

$$I + A \circ A^{-1} \leq_{\mathsf{L}} \frac{c+d}{c^{1/2}d^{1/2}} A^{1/2} \circ A^{-1/2},$$

where c is the largest and d the smallest eigenvalue of A, $\circ$ denotes the Hadamard product, and $\leq_{\mathsf{L}}$ denotes the Löwner ordering (so that $H \leq_{\mathsf{L}} G$ means that $G - H$ is nonnegative definite). This is akin to IMAGE Problem 27-4, and is also a rough converse of $2(A^{1/2} \circ A^{-1/2})^2 \leq_{\mathsf{L}} I + A \circ A^{-1}$, which is (16) in the paper by G. Visick (2000): A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product, *Linear Algebra and its Applications*, 304, 45–68.

Problem 28-9: A Relative Perturbation Bound

Proposed by Yimin WEI, *Fudan University, Shanghai, China:* ymwei@fudan.edu.cn
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Let A be a nonsingular matrix and let $\rho(\cdot)$ denote spectral radius. If $B = A + E$ and $\mathrm{rank}(AE - EA) \leq 1$ for some matrix E, show that for any eigenvalue $\tilde{\lambda}$ of B, there exists an eigenvalue λ of A such that $|\tilde{\lambda} - \lambda|/|\lambda| \leq \rho(A^{-1}E)$.

Problem 28-10: Inequalities Involving Square Roots

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

(a) Let H and K be Hermitian matrices of the same order, let $\geq_{\mathsf{L}}$ denote the Löwner ordering and let the nonnegative definite Hermitian "square root" $|K| = (K^*K)^{1/2}$. Are $H \geq_{\mathsf{L}} \pm K$ and $H \geq_{\mathsf{L}} |K|$ equivalent?

(b) Let A, B and C be square complex matrices of the same order such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is nonnegative definite. Show that $A^{1/2}CA^{1/2} \geq_{\mathsf{L}} B^*B$ if (i) A and B commute or (ii) A and C commute and B is Hermitian. Must B in (b) be Hermitian?

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[1] *Problems 28-1 through 28-4 are on page 36.*

IMAGE Problem Corner: New Problems[2]

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Institute for Econometrics and Operations Research, Econometrics Unit, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 28-1: Regular and Reflected Rotation Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

An $n \times n$ Jacobi, Givens, or elementary rotation matrix is an $n \times n$ matrix with unit elements on its diagonal and zeroes elsewhere except for its iith, ijth, jith, and jjth positions ($i \neq j$) which contain the real values $c, s, -s$, and c, satisfying $c^2 + s^2 = 1$. Show that any real orthogonal matrix with a determinant of plus one may be expressed as the product of at most $n^2/2$ elementary rotation matrices, and deduce that any real orthogonal matrix with a determinant of minus one may be expressed as the product of at most $(n^2 - 1)/2$ elementary rotation matrices and a single $n \times n$ diagonal matrix with $n - 1$ elements of plus one and one element of minus one on its diagonal. Such matrices may be named *reflected rotation matrices* by contrast with the (proper) *rotation matrices* of the main result.

Problem 28-2: Linear Combinations of Imaginary Units

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

Let $i,\ j,\ k$ denote the imaginary units of the algebra of quaternions. Then, it is well known that these units satisfy the conditions $i^2 = j^2 = k^2 = ijk = -1$. Let v denote the 3×1 matrix of imaginary units $v = [i\ j\ k]'$, and let $p,\ q,\ r$ be arbitrary 3×1 real matrices. Find conditions such that the linear combinations $i_o = p'v,\ j_o = q'v,\ k_o = r'v$ satisfy the conditions $i_o^2 = j_o^2 = k_o^2 = i_o j_o k_o = -1$.

Problem 28-3: Ranks of Nonzero Linear Combinations of Certain Matrices.

Proposed by Shmuel FRIEDLAND, *University of Illinois at Chicago, Chicago, Illinois, USA:* friedlan@uic.edu
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Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Show that any nonzero real linear combination of these four matrices has rank at least 3. [The proposers prefer a solution which does not depend on the use of a computer package such as Maple.]

Problem 28-4: A Rank Identity for Block Circulant Matrix

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario:* ytian@mast.queensu.ca

Let $A_1,\ A_2,\ \ldots,\ A_k$ be $m \times n$ matrices, and denote by A the $km \times kn$ block circulant matrix $A = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}$.

Show that $\operatorname{rank}(A) = \operatorname{rank}(\,A - P'NQ\,) + \operatorname{rank}(P'NQ)$, where $P = \frac{1}{\sqrt{k}}[\,I_m, \ldots, I_m\,],\ Q = \frac{1}{\sqrt{k}}[\,I_n, \ldots, I_n\,]$ and $N = A_1 + \cdots + A_k$.

[2] *Please find 6 more new problems on page 35.*