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Serving the International Linear Algebra Community

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Conferences and Workshops in Linear Algebra and Matrix Theory

Western Canada Linear Algebra Meeting: Regina, Saskatchewan, 10–11 May 2002

Report by Stephen J. Kirkland

The most recent Western Canada Linear Algebra Meeting (WCLAM) was held at the University of Regina on May 10 and 11, 2002. This was the sixth in the ongoing series of WCLAMs, which have been held roughly every two years since 1993.

WCLAM 2002 featured 18 talks by speakers from Canada, the United States, and Germany. The lectures covered a range of research areas associated with linear algebra, including matrix theory, operator theory, graph theory, applied mathematics, numerical analysis and combinatorics. In addition to the contributed talks, the meeting featured lectures given by three invited speakers: Professor Jane Day (San Jose State University), Professor Ludwig Elsner (Universität Bielefeld), and Professor Chris Godsil (University of Waterloo).

The WCLAM Organizing Committee comprised S. Fallat (University of Regina), H. Kharaghani (University of Lethbridge), S. Kirkland (University of Regina), P. Lancaster (University of Calgary), D. Olesky (University of Victoria), M. Tsatsomeros (Washington State University), and P. van den Driessche (University of Victoria).

As with the previous meetings in this series, WCLAM 2002 was not only a forum for disseminating research results, but also a venue for establishing new research contacts in an informal atmosphere. This year's meeting received financial support from the National Programme Committee of the Institutes, the University of Regina Conference Fund, and the University of Regina Faculty of Science; the WCLAM organizing committee extends its thanks to each of these organizations.

From left to right: Sarah Carnochan-Naqvi, Sivaram Narayan, David Barnes, Manon Mireault, Dale Olesky, Jeff Stuart, Karen Johannson, Yury Ionin, Michelle Davidson, Hadi Kharaghani, Pauline van den Driessche, Volker Runde, Will Gibson, Chris Godsil, Rob Craigen, Peter Zizler, Colin Carbno, Shaun Fallat, Jason Molitierno, Mahmoud Manjegani, Yongge Tian, Michael Doob, Francesco Barioli, Doug Farenick, Nathan Krislock, Chris Leurer, Peter Lancaster, Barb Pidkowich, Paul Binding, Karim Naqvi, Ludwig Elsner, Patrick Browne, Jane Day, Chun-Hua Guo, Lorraine Dame, Xiaoping Liu, and Jeremy Hinks.

Sixth Workshop on "Numerical Ranges and Numerical Radii": Auburn, Alabama, 7–8 June 2002

Report by Chi-Kwong Li

The Sixth workshop on "Numerical Ranges and Numerical Radii," sponsored by the Auburn University, was held on June 7–8, 2002, in conjunction with the Tenth International Linear Algebra Conference, June 10–13, 2002. The Workshop took place in Parker Hall, where the mathematics department of Auburn University is located.

The two-day meeting began with opening remarks by Dr. Stew Schneller (Dean of the College of Sciences and Mathematics, Auburn University) and Dr. Michel Smith (Chair of the Department of Mathematics, Auburn University). Eighteen talks on different aspects of numerical ranges and radii were given; see program and abstracts in http://www.math.wm.edu/~ckli/pgm.pdf for details. The Workshop photo below was taken in front of Parker Hall, and the Workshop dinner took place in the evening of June 7 at the China Palace restaurant.

There were about three dozen participants for the workshop from more than ten different countries: Tsuyoshi Ando (Hokkaido University), Ari Aluthge (Marshall University), Natália Bebiano (University of Coimbra), Raymond Chan (University of Hong Kong), Mao-Ting Chien (Soochow University), Man-Duen Choi (University of Toronto), Gianni Codevico (Katholieke Universiteit Leuven), Charles Dolberry (Auburn University), John Drew (College of William and Mary), Jan Hauke (Adam Mickiewicz University), Masatoshi Ito (Tokyo University of Sciences), Sang-Gu Lee (Sungkyunkwan University), Tom Milligan (College of William and Mary), Tom Morley (Georgia Tech.), Hiroshi Nakazato (Hirosaki University), Marek Niezgoda (Agricultural University of Lublin), Peter Nylen (Auburn University), Kazuyoshi Okubo (Hokkaido University of Education), Ed Poon (College of William and Mary), Yiu-Tung Poon (Iowa State University), João da Providência (U. of Coimbra), Martine Reurings (Free University Amsterdam), Leiba Rodman (College of William and Mary), Lemos Rute (Aveiro University), Abbas Salemi (University of Toronto), Jose Dias da Silva (University of Lisbon), Graca Soares (Universidade de Trás-os-Montes e Alto Douro), Ilya Spitkovsky (College of William and Mary), Nung-Sing Sze (University of Hong Kong), Bit-Shun Tam (Tamkang University), Michael Tsatsomeros (Washington State University), Frank Uhlig (Auburn University), Takeaki Yamazaki (Kanagawa University), Wen Yan (Auburn University), Fuzhen Zhang (Nova Southeastern University).

Similar to the previous meetings, participants exchanged many ideas, results and problems on the subject in a very friendly atmosphere. Furthermore, it was announced that the Seventh workshop on "Numerical Ranges and Numerical Radii" will take place in Coimbra, Portugal, in the summer of 2004, organized by Natália Bebiano.

Tenth ILAS Conference, "Challenges in Matrix Theory": Auburn, Alabama, 10–13 June 2002

Report by Frank Uhlig

This report details the organizational, financial, and logistic aspects of the Conference. The scientific results are the topic of a special issue of *Linear Algebra and its Applications* where the conference proceedings will appear in late 2003 or early 2004. I write about the technical aspects of the conference here in order to share my experiences with the participants, sponsors, future conference organizers, and the general ILAS membership. The group photo appears in the centerfold of this issue of IMAGE, pp. 18–19. The conference program is still accessible on our Web site http://www.auburn.edu/ilas2002

One of the main tasks and activities of the *International Linear Algebra Society* is to hold meetings to exchange research results and ideas in our chosen field of Matrix Theory and its applications. This Conference, with its theme of *Challenges in Matrix Theory*, was originally conceived during Yik-Hoi Au-Yeung's retirement conference in Hong Kong in the summer of 1995. There the author proposed to start a collection of challenging matrix problems in LAA and to hold a subsequent conference at Auburn in 2002. Three sets of *Challenges* have been published in *Linear Algebra and its Applications* (LAA) vol. 278 (1998), pp. 285–336; vol. 304 (2000), pp. 179–200; and vol. 345 (2002), pp. 261–267. As far as we could find out, none of the 9 published challenge problems has found a solution. Therefore, rather than highlighting challenges solutions, each invited speaker was asked to share some of his/her matrix challenges with the audience and to publish these in the proceedings.

People: There were 135 abstracts submitted to the conference from 28 countries on 5 continents and 164 persons registered from 24 countries on 3 continents. In all, 125 talks were given. Among them were 10 plenary 50-minute lectures by researchers from 6 different countries, 15 invited 25-minute lectures from 7 countries, 34 presentations in 6 different invited mini-symposia, and 66 contributed talks in 13 subject specific sections. There were 20 graduate students from 11 countries in attendance, as well as 12 accompanying family members. So far, 31 papers and several mini-symposia reports have been submitted to the proceedings.

Finances: With a Conference budget of $32,000 (all figures approximate) and the average travel, food, and lodging expenses estimated at about $1,000 per participant, the total economic impact of this ILAS conference amounted to around $180,000.

Outlays			*Income*		
Speaker fees and participant support:	$	11,100	Registration fees	$	13,000
Materials:	$	1,800	National Security Agency	$	8,000
Flyers, program, photo, Sunday wages, supplies, Excursion:	$	4,400	Oak Ridge Associated Universities	$	3,000
Theater tickets, bus, food and water, Conference Center:	$	14,180	Auburn University	$	8,000
Lecture halls, internet, snacks, dinner, wine, BBQ, kosher, and ILAS reimbursement	$	520			
	$	32,000		$	32,000

Needs: Special consideration was given to the needs for funds for travel by the invited speakers and mini-symposia organizers, as well as to active researchers from countries with substantially lower resources. Each invited speaker was asked to state his/her perceived funding need from a proposed target fee for the three different categories of plenary talk, invited talk, or mini-symposium organizer. Due to the release of funds by several (locally well-funded) invitees for which I am very grateful, we were able to help a large number of not-so-well-funded invitees and, moreover, to extend support to 6 needy participants at large and to 3 ILAS officers. We also waived all fees for the 20 graduate students who participated. The general fees of $ 130 for participants and $ 90 for family members covered every expense, from the handouts, dinners, snacks, the excursion to the theater, etc, all in one price. Moreover 5 publishing companies had sent us linear algebra related books for distribution to the participants. All of these well over 200 text and research books were eagerly and gladly taken, many to far away shores.

Conference Design: Our aim in scheduling was to let the subject areas develop in a beneficial flow throughout the duration of the conference. Therefore we combined the contributed talks into 13 subject areas and scheduled these talks after main talks with related contents. We could achieve all of this with no more than 3 parallel sessions at any given time. The schedule was dense and long on each day. But despite this, the very last talk at noon on Thursday still had over 120 listeners.

Highlights: Mathematical highlights for me were T. Ando's Schneider Prize Lecture on Wednesday morning, as well as Hans Schneider's historical survey of Matrix Theory in the 1950s through the works of Jean Alexandre Eugène Dieudonné (1906–1992) and Helmut Wielandt (1910–2001). All of of the invited and contributed talks were excellent. But unfortunately we cannot mention everyone here, except for D. Calvetti who gave us four talks altogether. The real highlights are the proceedings of this Conference, the fifth overall at Auburn University, three of which I lent a hand to, with the first two being the children of Emilie Virginia Haynsworth (1916–1985).

Thanks: To everyone for this very successful Conference! Thanks to the Organizing Committee! And, last but not least, thanks to our generous sponsors!

Computational Linear Algebra with Applications: Milovy, Czech Republic, 4–10 August 2002

Report by Michele Benzi

On August 4–10, 2002 an international conference on "Computational Linear Algebra with Applications" was held in Milovy, a small village located in the Bohemian-Moravian Highlands about 100 kilometers (65 miles) from Prague, Czech Republic. The conference, which was endorsed by ILAS, brought together 87 researchers from 21 countries, including a number of mathematicians from central and eastern European nations. Funding was provided by Germany's GAMM (Gesellschaft für Angewandte Mathematik und Mechanik) and by a grant of the National Science Foundation. The latter was used to support several US-based participants, including a few young researchers and graduate students.

The international program committee was co-chaired by Zdeněk Strakoš of the Institute of Computer Science of the Czech Academy of Sciences and by Ivo Marek of the Faculty of Mathematics and Physics of Charles University. The conference was impeccably organized thanks to the efforts of the local organizing committee (Miro Rozložník, Mirek Tůma, Hana Bílková, Karel Segeth, and Petr Tichý), who worked around the clock to handle even the smallest crisis as well as numerous last-minute requests from the speakers, frequently concerning the use of audio-visual equipment. A special issue of the journal *BIT: Numerical Mathematics* will publish selected papers from the conference (the guest editors are Zdeněk Strakoš and Daniel Szyld).

The scientific program consisted of 27 plenary talks and a total of 40 contibuted talks (in parallel sessions). Virtually every topic in numerical linear algebra was covered: direct and iterative methods for large sparse systems of equations, preconditioning techniques, eigenvalue problems, least-squares, ill-posed problems, Markov chains, matrix functions, parallel algorithms, special matrices, and so forth. A number of applications of numerical linear algebra to engineering and scientific problems were also covered, ranging from image processing for biomedical applications to finite elements modeling in solid and fluid mechanics.

If any conclusion can be drawn from a broad-themed conference like this one, it is that the field of numerical linear algebra continues to thrive and remains a very active area of research. The field displays a healthy balance of research in the theoretical foundations, in the development of new algorithms, and in the applications to real-life problems.

The vitality of the field is guaranteed by new ideas and problems being constantly added to the old ones, and to some of the old ideas being revisited under a new light. There is also much cross-fertilization taking place between previously separated areas. For instance, developers of iterative methods for large linear systems are making use of techniques developed by the direct methods community in order to develop more robust solvers and preconditioners. The end result is that both camps benefit from this interaction.

It is impossible to give a fair account of the many excellent talks that were heard at the meeting in a brief report like this one. Just to mention a couple of highlights, several of the talks dealt with some 'hot' topics currently attracting much attention. These include the theory of flexible and inexact iterative methods for both linear equations and eigenvalue problems, and the development of effective preconditioners for KKT-type (saddle-point) systems. Also, the number of talks devoted to the solution of ill-posed problems is witness to the growing importance of this area, which continues to present researchers with challenging problems stretching the capabilities of current algorithms to the limit.

A wonderful feature of this meeting was that the scientific program, while intense, did not exhaust all the available time and energy of the participants. A two-and-a-half hour break after lunch allowed for much discussion and interaction between participants, and at least a few new research collaborations were hatched during such breaks. Perhaps an even higher number of collaborations would have been started were it not for the staggering amounts of food and superb Czech beer that were available, buffet-style, to the conference-goers!

The social program included a trip to the small historic town of Telč, just a few miles away from the Austrian border. The town, which belongs to the UNESCO world heritage list, boasts a beautiful castle (going back to the fourteenth century) and a large market square flanked with Renaissance-era buildings showing the influence of Italian architects and artists on the culture of this region.

Another major social event was the banquet which, as usual, was preceded and followed by toasts and speeches. One of these was given by Gene Golub, who reminisced about his first visit to Czechoslovakia (1964) on the occasion of a meeting that was attended also by a few other of the Milovy participants, including Miroslav Fiedler, Ivo Marek, and Karel Segeth. Gene took note of the amazing changes that have taken place in the Czech Republic in recent years, and praised the organizers not only for the excellent level of the scientific program, but also for the friendly and stimulating atmosphere of the meeting.

This conference had a precursor in the "Czech-US Workshop on Iterative Methods and Parallel Computing" which was held in the same venue in June of 1997. Many of the participants of Milovy '97 where so impressed with that conference that they returned for Milovy 2002. Although a precise date has not been set, another Milovy conference is being planned, and readers of IMAGE may look forward to it. Further information on the conference, together with many photos and the slides of some of the talks, is available at http://www.cs.cas.cz/~milovy

Fifth International Conference on China Matrix Theory and its Applications
Shanghai, China, 14–18 August 2002

Report by Erxiong Jiang and Fuzhen Zhang

The Fifth International Conference on China Matrix Theory and its Applications was held at Shanghai University, Shanghai, China, August 14-18, 2002. This was a satellite meeting of the 2002 International Congress of Mathematicians (ICM), Beijing, China, August 20–28, 2002.

The conference opened with warm welcomes by the Organizing Committee and by the officers from the University and the Education committee of Shanghai Municipal Government. A sightseeing tour on August 16 to a scenic southern Chinese village Zhou-zhuang, known as China's Venice, in suburb Shanghai, gave everyone a wonderful time to cherish. A banquet full of gourmet Chinese foods highlighted the evening of August 17 in the Lehu (meaning "happiness") Hotel.

As many as 139 people attended the conference, including 25 from countries other than China. About 113 talks including 13 plenary (45-minute) presentations were given. Topics ranged from traditional linear algebra to combinatorial matrix theory and to matrix computation.

The invited plenary speakers and titles of their talks were: F. Uhlig, Quadratic Matrix Equations; X. Zhan, On the Unitary Orbit of Complex Matrices; F. Hall, On Ranks of Matrices ; C. R. Johnson, Product of Matrices; A. Berman (ILAS lecturer), Graphs of Matrices and Matrices of Graphs; F. Zhang, Block Matrix Techniques and Matrix Inequalities; J. W. Demmel, Can we do Numerical Linear Algebra in Polynomial Time?; E. Jiang, Matrix Eigenvalue Perturbation problem; R. Chan, Wavelet Algorithms for High-Resolution Image Reconstruction; Z. Bai, Numerical Linear Algebra Techniques in Reduced-Order Modeling of Dynamical systems; C.-K. Li, Results and Techniques in Matrix Inequalities; B.-S. Tam, Strong Linear Preservers of Symmetric Doubly Stochastic Matrices; C. Gu, Matrix Pade Approximation and Rational Interpolation in Inner Product Space.

The Conference was supported by the National Natural Science Foundation of China, Science and Technology Committee of Shanghai, the Chinese Mathematical Society, and the International Linear Algebra Society. The local organizing committee included E. Jiang, Z. Cao, G. Wang, Z. Bai, M. Pang, Y. Wang, C. Gu, and the academic committee consisted of E. Jiang, Z. Bai, J. Li, S. Shao, Z. Li, and B. Tam.

Eleventh International Workshop on Matrices and Statistics: Lyngby, Denmark, 29–31 August 2002

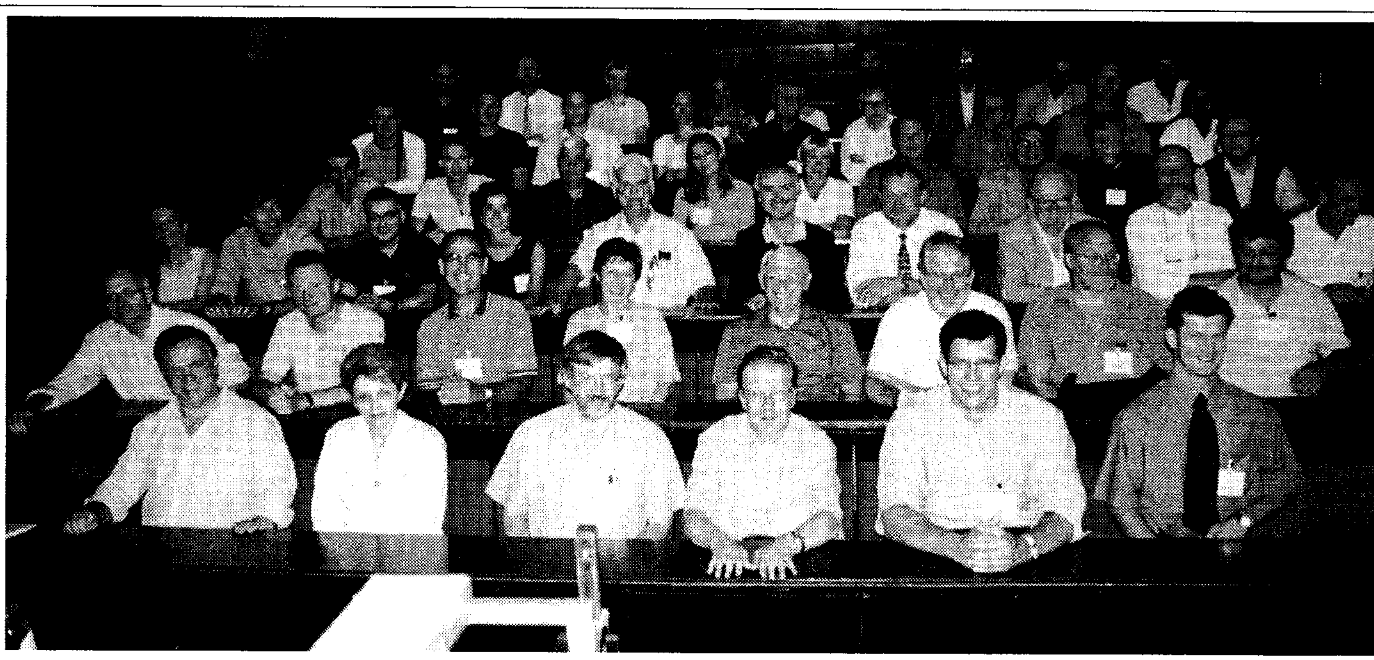
Report by Hans Joachim Werner

The Eleventh International Workshop on Matrices and Statistics (EIWMS-2002), in Celebration of George P. H. Styan's 65th Birthday, was held at the Technical University of Denmark (DTU) in Lyngby, near Copenhagen, on August 29–31, 2002. This Workshop was hosted by the Section of Image Analysis and Computer Graphics in the Department of Informatics and Mathematical Modelling (IMM) at the Technical University of Denmark and was endorsed by the International Linear Algebra Society (ILAS).

The International Organizing Committee (IOC) for this workshop consisted of R. William Farebrother (Victoria University of Manchester, Manchester, England, UK), Simo Puntanen (University of Tampere, Tampere, Finland), and Hans Joachim Werner (University of Bonn, Bonn, Germany; chair). The Local Organizing Committee (LOC) at the Danish Technical University in Lyngby consisted of Knut Conradsen (chair), Bjarne Kjær Ersbøll, Per Christian Hansen, and Allan Aasbjerg Nielsen. The Workshop Secretary was Ms. Helle Welling. The group photograph below is by Simo Puntanen (see also the photograph on page 12).

The purpose of the workshop was to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. More than 65 participants from 22 different countries joined this workshop. The Workshop was opened by Dr. Lars Pallesen, Rector of the Technical University of Denmark. This was followed by sessions of invited and contributed papers. The 14 invited speakers were Fikri Akdeniz, Theodore W. Anderson, Jerzy K. Baksalary, Lars Eldén, Bjarne Kjær Ersbøll, Klaus Bagggesen Hilger, Agnar Høskuldsson, Bent Jørgensen, Harald Martens, Allan Aasbjerg Nielsen, Hans Bruun Nielsen, Simo Puntanen, George P. H. Styan and Jerzy Waśniewski. Another 32 papers were presented in several contributed paper sessions. It is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications* (LAA); the Ninth Special Issue on Linear Algebra and Statistics is vol. 354 (October 2002). The Workshop *Programme with Abstracts* Booklet can still be downloaded from the Workshop Web site: http://www.imm.dtu.dk/matrix02/

The social program included a Reception at Lyngby Rådhus (Town Hall), hosted by the Mayor of Lyngby, on Thursday evening, August 29. On Friday, August 30, there was an Afternoon Outing to the Roskilde Fjord which is about 30 km from Copenhagen. In Roskilde—one of Denmark's oldest cities—we visited the 12th century red brick Dome Church (being on UNESCO's World Heritage List) with its many magnificent sarcophaguses. All the past kings and queens of Denmark are buried in this Cathedral—over 1000 years of Danish monarchy. After a coffee break we then had about two hours to imagine life in the times of the Vikings. The *Vikingeskibshallen* (The Viking Ship Museum) houses the five original Viking ships excavated in Roskilde Fjord; the ships are exhibited and their story is told with models, posters, and a film in the Exhibition Hall. In the evening of the same day a delicious Workshop Dinner was served at Peter Lieps House, near the University in Lyngby. Without any doubt, this Workshop indeed provided a good atmosphere to stimulate contacts and exchange ideas.



Topics in Linear Algebra Conference: Ames, Iowa, 13–14 September 2002

Report by Leslie Hogben

The Topics in Linear Algebra Conference was held at Iowa State University on September 13–14, 2002. The main topics of the conference were matrix completions, numerical ranges, applications of linear algebra to statistics, and applications of linear algebra to non-associative algebra. There were 16 talks and the meeting was attended by more than 40 people. Sam Hedayat of the University of Illinois-Chicago was the ILAS Lecturer and spoke on Extending Saturated D-Optimal Resolution III Two-Level Factorial Designs by Adding More Runs. Other invited speakers and their titles included Charles R. Johnson (College of William and Mary), "Eigenvalues, Eigenvectors, Multiplicities and Graphs," Shaun Fallat (University of Regina), "The Totally Positive Matrix Completion Problem with Few Unspecified Entries," Chi-Kwong Li (College of William and Mary), "Numerical Ranges and Norm Estimation," David P. Jacobs (Clemson University), "Algorithms, Computation, and Non-associative Identities," Murray Bremner (University of Saskatchewan), "Quantization of Lie and Jordan triple systems," Alicia Labra (Universidad de Chile), "Representations of Train Algebras of Rank 3," Hans Schneider (University of Wisconsin–Madison), "Simultaneous and Two-Sided Diagonal Similarity and Diagonal Equivalence of Matrices." Work sessions were held on matrix completions, numerical ranges, and applications to nonassociative algebra.

The conference was held in the Pioneer Room of the ISU Memorial Union, and the works of Iowa artists provided an attractive backdrop that was incorporated into one of the talks. A conference dinner was held on the evening of Friday, September 13, and on Saturday, September 14, participants attended the ISU Department of Mathematics picnic.

The meeting was sponsored by the Institute for Mathematics and Its Applications, International Linear Algebra Society, and Iowa State University. The conference was organized by Leslie Hogben, Bryan Cain, Irvin Hentzel, Y. T. Poon, Amy Wangsness, Huaiqing Wu, all of Iowa State University, Luz DeAlba of Drake University, and Mark Mills of Central College. The program, including abstracts, is available in PDF format on the Web site http://www.math.iastate.edu/lhogben/TLA/homepage.html as are some photographs.

From left to right, front row: Sam Hedayat, Eric Key, Leslie Hogben, Hans Schneider, Miriam Schneider, Bryan Cain; middle row: Luz deAlba, Shaun Fallat, Huaiqing Wu, Jane Day, Alicia Labra, Charles Johnson, Amy Wangsness, Edward Poon, Sivaram Narayan, Michael Prophet, Anna Romanowska; back row: Doug Kilburg, Joseph Keller, Michael Rieck, Irvin Hentzel, Mark Mills, Seamus Riordan, Dean Isaacson, David Jacobs, Jonathan Smith.

Special Session on "Matrices and Statistics"

Halifax, Nova Scotia: 10 June 2003

The 2003 annual meeting of the Statistical Society of Canada will be held at Dalhousie University in Halifax, Nova Scotia, from Sunday, June 8 to Wednesday, June 11, 2003. A special session on "Matrices and Statistics," organized by George P. H. Styan, will be held on Tuesday, June 10, 2003; the invited speakers are Jerzy K. Baksalary (University of Zielona Góra, Poland), Simo Puntanen (University of Tampere, Finland) and Hans Joachim Werner (University of Bonn, Germany).

The conference will take place on the Dalhousie campus which is centrally located in Halifax within walking distance of all the major hotels. Included in your registration fee will be a lobster banquet at Pier 21, the restored immigration shed and museum on the harbour. Hotel space can be quite tight in Halifax in June and we urge you to book early. We are holding a block of rooms at the Lord Nelson Hotel: C$138 per night, www.lordnelsonhotel.com and at the Cambridge Suites; C$149 per night, www.cambridgesuiteshotel.com until May 1, 2003, in addition to rooms in the campus residences.

The Program Committee is chaired by Doug Wiens (University of Alberta). The Local Arrangements Organizing Committee is chaired by Chris Field (Dalhousie University). Local arrangements information will be updated regularly in the conference website: http://www.ssc.ca/main/meetings/halifax_e.html

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SIAM Conference on Applied Linear Algebra

Williamsburg, Virginia: 15–19 July 2003

The SIAM Conference on Applied Linear Algebra (LA03) will be held at The College of William and Mary, Williamsburg, Virginia, July 15–19, 2003. This conference is sponsored by the SIAM Activity Group on Linear Algebra (SIAG/LA). In cooperation with the International Linear Algebra Society (ILAS).

The Program Committee Chairs are Roy Mathias and Hugo Woerdeman, both of The College of William and Mary. Other members of this Committee are Raymond Chan (Univ. of Hong Kong), John Gilbert (Xerox Co.), Per Christian Hansen (Technical University of Denmark), Nicholas Higham (Univ. of Manchester), Ilse Ipsen (North Carolina State University), Horst Simon (National Energy Research Scientific Computing Center), and Paul Van Dooren (Université Catholique de Louvain).

The conference themes are broad and inclusive, and papers in all areas of linear algebra, matrix theory and their applications, will be solicited. The themes include:

Core linear algebra: matrix inequalities, Kronecker products, symbolic computations, graphs and matrices.

Numerical linear algebra: large-scale eigenvalue problems, optimization, polynomial eigenvalue problems, foundations of computational mathematics, lattice QCD calculations.

Applications: information retrieval, computational biomedicine, dynamical systems, quantum information, systems and control, image processing.

The invited plenary speakers include: George Cybenko: "Linear Algebra in Quantum Computation," Heike Fassbender: "Structured Linear Algebra Problems in Control," Andreas Frommer: "Lattice QCD Calculations," Rich Lehoucq: "Large-scale Eigenvalue Problems," Judith McDonald (ILAS speaker), James G. Nagy: "Kronecker Products in Image Restoration," Michael Overton: Optimizing Matrix Stability," Bryan Shader (ILAS speaker): "Nonnegative matrix pairs, 2-D dynamical systems, and road-colorings," G. W. (Pete) Stewart: "Open Problems and Future Directions in Numerical Linear Algebra," Gilles Villard: "Symbolic Computations."

Invited Minisymposia. Rajendra Bhatia & Qiang Ye: "Matrix Inequalities and Applications," Inderjit Dhillon: "Linear Algebra in Data Mining and Information Retrevial," Sabine Van Huffel & Nicola Mastronardi: "Linear Algebra in Computational Bio-medicine," Chi-Kwong Li & Leiba Rodman : "Indefinite Inner Products and Applications," Volker Mehrmann & Françoise Tisseur: "Numerical Solutions of Polynomial Eigenvalue Problems," Esmond G. Ng: "Linear Algebra Algorithms in Science Applications," Stephen Vavasis: "Foundations of Computational Mathematics in Numerical Linear Algebra." The Invited Business Meeting Speaker will be Michael Steuerwalt: "The NSF and Applied Linear Algebra." The Invited Banquet Speaker will be Roger Horn : "Five Fundamental Facts in Matrix Analysis."

We will publish on-line refereed proceedings, which will be available at the conference. The meeting Web site is http://www.siam.org/meetings/la03 The deadline for submission of minisymposium speakers' abstracts: and for submission of contributed abstracts for lecture or poster presentations is 13 January 2003.

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12th International Workshop on Matrices and Statistics

Dortmund, Germany: 5–8 August 2003

The 12th International Workshop on Matrices and Statistics (IWMS-2003) will be held at the University of Dortmund (Dortmund, Germany) on August 5-8, 2003, during the week immediately before the 54th Biennial Session of the International Statistical Institute (ISI) in Berlin. This Workshop, which will be an ISI satellite meeting, will be hosted by the Department of Statistics at the University of Dortmund and will be cosponsored by the Bernoulli Society. It has also been endorsed by the International Linear Algebra Society (ILAS).

The purpose of this Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. This Workshop will provide a forum through which statisticians may be better informed of the latest developments and the newest techniques in matrix theory and may exchange ideas with researchers from a wide variety of countries. This Workshop will include the presentation of both invited and contributed papers on matrices and statistics; it is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications* —the 9th Special Issue is Volume 354 (October 15, 2002). Contributed papers are welcome! Abstracts should arrive by May 15, 2003. Details for submission of a paper are given on the Workshop Web site: http://www.statistik.uni-dortmund.de/IWMS/main.html

The International Organizing Committee comprises Richard William Farebrother (Shrewsbury, England, UK), Simo Puntanen (Univ. of Tampere, Finland), George P. H. Styan (McGill Univ., Montréal, Canada; vice-chair), and Hans Joachim Werner: werner@united.econ.uni-bonn.de (Univ. of Bonn, Germany; chair). The Local Organizing Committee at the University of Dortmund comprises Jürgen Groß, Götz Trenkler: trenkler@statistik.uni-dortmund.de, and Claus Weihs. The Workshop Secretary is Mrs. Eva Brune: iwms2003@statistik.uni-dortmund.de (Department of Statistics, IWMS- 2003, Univ. of Dortmund, Vogelpothsweg 87, D-44221 Dortmund, Germany).

This Workshop in Germany will be the 12th in a series. The previous eleven Workshops were held as follows: (1) Tampere, Finland: August 1990; (2) Auckland, New Zealand: December 1992; (3) Tartu, Estonia: May 1994; (4) Montréal (Québec), Canada: July 1995; (5) Shrewsbury, England, UK: July 1996; (6) İstanbul, Turkey: August 1997, as an ISI Satellite Meeting; (7) Fort Lauderdale, Florida, USA: December 1998; (8) Tampere, Finland: August 1999, as an ISI Satellite Meeting; (9) Hyderabad, India: December 2000; (10) Voorburg, The Netherlands: August 2001, as an ISI Satellite Meeting; and (11) Lyngby, Denmark: August 2002.

Hans Joachim Werner, George P. H. Styan and Simo Puntanen at the reception in the Rådhus in Lyngby, Denmark, 29 August 2002, for the 11th International Workshop on Matrices and Statistics. Photograph by Oskar Maria Baksalary.

It is expected that the 13th International Workshop on Matrices and Statistics will be held near Poznań, Poland, in August 2004, and the 14th in Auckland, New Zealand, in April 2005 (as a satellite to the 55th Biennial Session of the International Statistical Institute to be held in Sydney, Australia, April 5–12, 2005).

For further more detailed information (paper submission, registration fees, accommodation, deadlines, etc.) please visit our Workshop Web site http://www.statistik.uni-dortmund.de/IWMS/main.html or contact either Hans Joachim Werner (IOC-chair) at werner@united.econ.uni-bonn.de or Götz Trenkler (LOC-chair) at trenkler@statistik.uni-dortmund.de

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International Conference on Matrix Analysis and Applications

Fort Lauderdale, Florida, 14–16 December 2003

The international conference on Matrix Analysis and Applications will be held at Nova Southeastern University, Fort Lauderdale, Florida, USA, December 14–16, 2003. The aim of this mathematical meeting is to stimulate research and interaction of researchers interested in all aspects of linear and multilinear algebra, matrix analysis and applications, and to provide an opportunity for researchers to exchange ideas and recent developments on these subjects. The conference is sponsored by the International Linear Algebra Society (ILAS) and Nova Southeastern University.

The organizing committee consists of Tsuyoshi Ando (Hokkaido University, Japan), Chi-Kwong Li (College of William and Mary, USA), George P. H. Styan (McGill University, Canada), Hugo Woerdeman (College of William and Mary, USA and Catholic University-Belgium), and Fuzhen Zhang (Nova Southeastern University, USA)

The invited ILAS Lecturer will be Roger Horn (University of Utah). There will be no registration fee. A reception and a pool party will take place in the evenings of the 13th and 15th, respectively. The conference hotel is Best Western Rolling Hills Resort: www.bestwestern.com/rollinghillsresort which is within walking distance to the conference site: www.nova.edu To register, contact Chi-Kwong Li: ckli@math.wm.edu For local information, contact Fuzhen Zhang: zhang@nova.edu The Web site is http://www.resnet.wm.edu/~cklixx/nova03.html

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A Simple Approach to the Matrix Representation of Quaternions

In [FGT], Farebrother, Groß & Troschke (2002) discussed the representation of quaternions in matrix form and identified 48 valid solutions (together with a complementary set of 48 invalid solutions) by means of a simple algebraic argument.

Let Q be a scalar multiple of a 4×4 orthonormal matrix which has a common element on its principal diagonal and is skew symmetric about this diagonal. Writing a, b, c, d for the elements in the first row of this matrix and their negations in the first column, we have:

$$Q = \begin{pmatrix} a & b & c & d \\ -b & a & . & . \\ -c & . & a & . \\ -d & . & . & a \end{pmatrix}.$$

Now, the second row of this matrix is orthogonal to the first if and only if its third and fourth elements take the values md and $-mc$. Further m must take the value ± 1 if the second row is to have the same squared length as the first. Substituting these values into the second row of the matrix and their negations in the second column, we find that the fourth element in the third row must take the value md. Whence we have the matrix

$$Q = \begin{pmatrix} a & b & c & d \\ -b & a & md & -mc \\ -c & -md & a & mb \\ -d & mc & -mb & a \end{pmatrix}.$$

If we set $m = +1$ in this general form, then the FGT matrix representation of the quaternion $q = a + ch + bj + dk$ is $Q_1 = aI + cH_1 + bJ_1 + dK_1$, where the 4×4 skew symmetric signed permutation matrices H_1, J_1, K_1 (implicitly defined here) play the roles of the imaginary units h, j, k of the algebra of quaternions.

Similarly, if we set $m = -1$, then the FGT matrix representation of the quaternion $q = a + bh + cj + dk$ as $Q_2 = aI + bH_2 + cJ_2 + dK_2$. where the 4×4 matrices H_2, J_2, K_2 again play the roles of the imaginary units of the algebra of quaternions.

Further, if the elements b, c, d are inserted in the first row of the original matrix in any order and are associated with any combination of plus and minus signs, then we have a total of 96 solutions to this problem. Only 48 of these solutions, however, satisfy the true Hamiltonian conditions in which the matrix associated with the first imaginary unit when postmultiplied by that associated with the second imaginary unit is equal to that associated with the third imaginary unit. The other 48 solutions satisfy the anti-Hamiltonian conditions in which this product yields the negation of the third matrix.

Adopting any one of the 48 systems satisfying the true Hamiltonian conditions, we may easily show that any two matrices in the chosen system may be added and multiplied to form sum and product matrices within the same system. In this way, we find that the algebra of quaternions is immediately available to anyone able to master the elements of 4×4 matrix algebra.

Memorandum:

$$Q_1 = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}.$$

For a note on Sir William Rowan Hamilton (1805–1865) and illustrations of three postage stamps assciated with Hamilton, see IMAGE no. 28 (April 2002), page 17.

Reference

[FGT] R. W. Farebrother, J. Groß & S.-O. Troschke (2002). Matrix representation of quaternions. *Linear Algebra and its Applications*, in press.

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Early Statistical Applications of the Theory of Determinants

Readers of the recent note in IMAGE by Farebrother (2002) may be interested to learn that in Farebrother (1996, 1997, 1999) we identified early statistical applications of the theory of determinants in papers by Jacobi (1841), Todhunter (1869) and Glaisher (1879). In particular, we noted that Jacobi (1841) and Glaisher (1879) contain applications of the theory of determinants to the explicit solution of the linear least squares problem. See also Glaisher (1874).

References

R. W. Farebrother (1996). Some early statistical contributions to the theory and practice of linear algebra. *Linear Algebra and Its Applications*, 237, 205–224.

R. W. Farebrother (1997). Notes on the early history of elemental set methods, in *L_1-Statistical Procedures and Related Topics* (Y. Dodge, ed.), Lecture Notes–Monograph Series 31, Institute of Mathematical Statistics, Hayward, California, pp. 161–170.

R. W. Farebrother (1999). *Fitting Linear Relationships: A History of the Calculus of Observations 1750–1900*. Springer-Verlag, New York.

R. W. Farebrother (2002). Did Karl Pearson use matrix algebra in 1896? *Image*, no. 28, page 16.

J. W. L. Glaisher (1874). On the solution of the equations in the method of least squares. *Monthly Notices of the Royal Astronomical Society*, 34, 311–314.

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C. G. J. Jacobi (1841). De formatione et proprietatibus determinantium (in Latin). *Crelle Journal für die reine und angewandte Mathematik*, 22, 285–318. [Reprinted in *C. G. J. Jacobi's Gesammelte Werke* (K. Weierstrass, ed.), Vol. 3, Georg Reimer, Berlin, 1884, pp. 355–392. German translation by P. Stäckel: Über die Bildung und die Eigenschaften der Determinanten, Ostwald's Klassiker der exakten Wissenschaften 77, W. Engelmann, Leipzig, pp. 1–49, 1896.]

I. Todhunter (1869). On the method of least squares. *Transactions of the Cambridge Philosophical Society*, 11, 219–230.

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Denis Serre: *Matrices, Theory and Applications*

Reviewed by S. W. DRURY, *McGill University*

This book [*Matrices: Theory and Applications* by Denis Serre, Graduate Texts in Mathematics 216, New York: Springer, 2002, xv + 202 pp., ISBN 0-387-95460-0] is a graduate text on matrices based on a course given at the École Normale Supérieure de Lyon between 1998 and 2001. In part, it was previously published in French under the title *Les matrices: théorie et pratique* [Paris: Dunod, 2001, vii + 168 pp., ISBN 2-10-005515-1].

The book purveys an intensely personal view of matrix theory. It requires a certain sophistication from the reader, not only in the breadth of knowledge needed to grasp the wide variety of topics presented, but also in the depth of knowledge needed to follow up some of the ideas that are described in the text but not developed in detail. The author has gone out of his way to find proofs that are new, stimulating and different from the ones usually found in the standard treatments. For example, the proof of the Perron–Frobenius Theorem using the Brouwer Fixed Point Theorem is definitely not the one found in Horn and Johnson's *Matrix Analysis*. One can argue that the Brouwer Fixed Point Theorem is being used as a slegehammer to crack a nut, but its application does draw attention to a device that might be employed in a related context.

This is definitely not a textbook, in the sense that it does not pretend to cover all known topics in a given area. Rather it is a book that is intended to whet the reader's appetite in a variety of different facets of matrix theory. Often a theorem or proposition is followed by a note directing the reader to sources for further reading. The very extensive exercises at the end of each chapter are well thought out, instructive, and in many instances allow the reader to discover on his/her own topics that would be treated in detail in other texts.

The first three chapters, entitled "Elementary Theory," "Square Matrices" and "Matrices with Real or Complex Entries," cover the basics that should be known by a graduate student in any mathematical field. Aside from the usual fare, we find a discussion of the Pfaffian in chapter 2. The next four chapters cover theoretical topics. The chapter on norms also presents their relation to the spectrum. We find here the Riesz–Thorin Theorem, a result that has had a powerful influence in analysis for decades, but perhaps one that is not familiar to some practitioners in matrix theory. We do not find a discussion of unitarily invariant norms which are easily learnt elsewhere. The Gerschgorin disks do appear here however. The fifth chapter is on nonnegative matrices, presenting the Perron–Frobenius Theorem, Birkhoff's Theorem on doubly stochastic matrices and its connection with majorization. The sixth chapter is algebraic in flavour. It starts by reviewing some algebraic preliminaries and then presents the similarity theory of matrices valued in a principal ideal domain. It goes on to discuss Jordan reduction in this context.

Chapter 7 is transitional in nature. It lays the groundwork that will be needed in the remaining chapters of the book which cover more applied topics. The matrix exponential, the polar decomposition and the singular value decomposition are discussed here. There are also a few short sections on the classical matrix groups.

The last three chapters give an introduction to, and an overview of, matrix computations, usually with an eye to applications in partial differential equations. While the presentation is both stimulating and inspiring, the serious reader will soon be forced to consult the standard books for this area. Chapter 8 on matrix factorizations does the LU, blockwise LU, Choleski and QR factorizations. It discusses the complexity of matrix multiplication and inversion. The Moore–Penrose inverse is introduced in this chapter. The next chapter deals with iterative methods for solving linear systems. The conjugate gradient method and its implementation are discussed in this chapter. The final chapter entitled "Approximation of Eigenvalues" deals with Householder reduction and Hessenberg matrices, the QR method, the Jacobi method, Givens rotations and power methods. Finally Leverrier's method of computing the characteristic polynomial is given.

The book has few typographical errors. While the use of the English language is everywhere first class, the nomenclature is sometimes very French. For example the author uses the term *bistochastic* where most would write *doubly stochastic*. There is an unfortunate error: Corollary 3.1.1 on page 45 is not true. A list of errata, together with the solutions to the exercises and some additional problems are to be found on the author's Web site: http://www.umpa.ens-lyon.fr/~serre/publi.html In some ways, the book is its own undoing. Any graduate student on whom it has the intended impact will sooner or later be forced to abandon it in favour of more systematically written, duller and more specialized texts. For the experienced matrix practitioner, it is certainly a valuable book to have on the shelf.

Tenth ILAS Conference, "Challenges in Matr

heory": Auburn, Alabama, 10-13 June 2002.

New and Forthcoming Books in Linear Algebra and Related Topics

This list of 61 new or forthcoming books, prepared by George P. H. Styan, updates and augments our lists in IMAGE 27 (October 2001), pp. 13-15 and IMAGE 28 (April 2002), page 23. We have included some dissertations and some journal special issues, which are coded D and J, respectively, in column 6; other codes used in column 6 are B for a regular book, E for an internet resource, F for a Festschrift, and P for a conference proceedings. The ISBN is given in column 8 with ``hbk" denoting hardcover and ``pbk" paperback; ISSNs are given for some books without ISBNs.

In column 9 is given the source of the reference: AMICUS (National Library of Canada), MR (MathSciNet/Mathematical Review), OCLC (WorldCat), and Zbl (Zentralblatt MATH); when the review in MR or Zbl is signed then the name of the reviewer is given in parentheses. Please send all corrections and additions to George Styan at styan@math.mcgill.ca

1	Ablamowicz, Rafal; Fauser, Bertfried (eds.)	2000	Clifford algebras and their applications in mathematical physics. Proceedings of the 5th conference, Ixtapa-Zihuatanejo, Mexico, June 27--July 4, 1999. Volume 1: Algebra and physics.	English	P	Progress in Physics 18. Boston, MA: Birkhäuser. xxv, 461 p.	0-8176-4182-3/hbk	Zbl 0940.00026
2	Anton, Howard; Busby, Robert C.	2003	Contemporary linear algebra.	English	B	New York, NY: Wiley. xviii, 594, [40] p.	0-471-16362-7	OCLC 50782323
3	Baker, Andrew	2002	Matrix groups: an introduction to Lie group theory.	English	B	Springer Undergraduate Mathematics Series. London; New York: Springer, xi, 330 p.	1-85233-470-3	OCLC 47667223
4	Barker, V.A.; Blackford, L.S.; Dongarra, J.; Du Croz, J.; Hammarling, S.; Marinova, M.; Wasniewski, J.; Yalamov, P.	2001	LAPACK95 users' guide.	English	B	Software, Environments, and Tools 13. Philadelphia: Society for Industrial and Applied Mathematics (SIAM). xviii, 258 p.	0-89871-504-0 pbk	Zbl 0992.65013 (Octavian Pastravanu)
5	Barlow, Jesse L.; Parlett, Beresford N.; Veseli\'{c}, Kre\v{s}imir (eds.)	2000	Proceedings of the International Workshop on Accurate Solution of Eigenvalue Problems: held at Pennsylvania State University, University Park, PA, July 20--23, 1998.	English	FJP	*Linear Algebra and its Applications* 316, 1-3. New York: North-Holland. iii, 361 p.	ISSN 0024-3795	MR 2000m65010
6	Bhaskara Rao, K. P. S.	2002	The theory of generalized inverses over commutative rings. (With a foreword by Adi Ben-Israel.)	English	B	Algebra, Logic and Applications. 17. London: Taylor and Francis. xi, 167 p.	0-415-27248-3 hbk	MR 1 919 967; Zbl 0992.15003 (Kim Hang Kim)
7	Bhatia, Rajendra; Li, Chi-Kwong; Okubo, Kazuyoshi; Tsatsomeros, Michael J. (eds.)	2002	Special issue dedicated to Professor T. Ando.	English	FJ	*Linear Algebra and its Applications* 341, 1-3. New York: North-Holland. xii, 420 p.	ISSN 0024-3795	MR 1 873 603
8	Blyth, Thomas Scott; Robertson, Edmund Frederick	2002	Basic linear algebra. (2nd edition.)	English	B	Springer Undergraduate Mathematics Series. London: Springer-Verlag. ix, 232 p.	1-85233-662-5 pbk	Zbl 01774894
9	Böttcher, Albrecht; Karlovich, Yuri I.; Spitkovsky, Ilya M.	2002	Convolution operators and factorization of almost periodic matrix functions.	English	B	Operator Theory: Advances and Applications 131. Basel: Birkhäuser. xii, 462 p.	3-7643-6672-9	MR 1 898 405
10	Calvetti, Daniela; Reichel, Lothar (eds.)	2000	Mathematical journey through analysis, matrix theory and scientific computation: papers from the conference held in honor of Richard S. Varga on the occasion of his 70th birthday at Kent State University, Kent, OH, March 25--27, 1999.	English	FJP	*Numerical Algorithms* 25, 1-4. Dordrecht: Kluwer. xvi, 406 p.	ISSN 1017-1398	MR 2001j:00034
11	Cullum, Jane K.; Willoughby, Ralph A.	2002	Lanczos algorithms for large symmetric eigenvalue computations, Vol. I: Theory. (Reprint edition. Originally published: Boston: Birkhäuser, 1985.)	English	B	Classics in applied mathematics 41. Philadelphia: Society for Industrial and Applied Mathematics (SIAM). x, 273 p.	0-89871-523-7 pbk	AMICUS 27406229
12	Dewilde, Patrick; Olshevsky, Vadim; Sayed, Ali H. (eds.)	2002	Special issue on structured and infinite systems of linear equations.	English	J	*Linear Algebra and its Applications* 343/344. New York: North-Holland. iv, 478 p.	ISSN 0024-3795	MR 1 878 934
13	Diaz Sordo, Cristina	2002	Algebra matricial.	Spanish	E	Internet Resource	http://fresno.pntic.mec.es/%7Ejvaamond	OCLC 48846188

14	Egleston, Patricia D.	2001	Nonnegative matrices with prescribed spectra.	English	D	Thesis (Ph.D.), Dissertation Committee Chair: Sivaram Narayan, Dept. of Mathematics, Central Michigan University, viii, 132 p.	none	OCLC 51081872
15	Feng, Bao Qi	2001	Matrix inequalities.	English	D	KSU Dissertations (Dept. of Mathematics and Computer Science), Thesis (Ph. D.)--Kent State University. x, 103 p.	none	OCLC 49559817
16	Gantmacher, F. P.; Krein, M. G.	2002	Oscillation matrices and kernels and small vibrations of mechanical systems. (Revised edition. Translation based on the 1941 Russian original, Edited and with a preface by Alex Eremenko.)	English	B	Providence, RI: AMS Chelsea Publishing. viii, 310 p.	0-8218-3171-2	MR 1 908 601
17	Ghali, Amin; Neville, Adam.M.; Brown, Tom G.	2003	Structural analysis: a unified classical and matrix approach. (5th edition.)	English	B	New York: Spon Press (Taylor & Francis). c. 904 p.	0-415-28091-5 hbk; 0-415-28092-3 pbk	AMICUS 27598848
18	Gohberg, Israel; Langer, Heinz (eds.)	2002	Linear operators and matrices: the Peter Lancaster anniversary volume.	English	F	Operator Theory: Advances and Applications 130. Basel: Birkhäuser Verlag. viii, 281 p.	3-7643-6655-9	MR 2002m:47002
19	Goode, Stephen W.	2000	Differential equations and linear algebra. (2nd edition.)	English	B	Upper Saddle River, NJ: Prentice Hall. xv, 703 p.	0-13-263757-X	Zbl 0958.34001 (M. C. Anisiu)
20	Hagerty, Gary	2001	Finding a few eigenvalues of large sparse nonsymmetric matrices.	English	D	Thesis (Ph. D.)--Washington State University. xiii, 177 p.	none	OCLC 49749349
21	Hardy, Kenneth	2003	Linear algebra for engineers and scientists.	English	B	Harlow: Pearson Education. 576 p.	0-13-906728-0	OCLC 50782472
22	Hazra, A. K.	1998	Matrix algebra, calculus and generalized inverse: Vol. I, II, III.	Bengali medium	B	Dhaka: Bangla Academy. xvi, 448 p.; xvi, 502 p.; xii, 367 p.	984-07-3800-3; 984-07-3867-4; 984-07-3868-2	http://www.buet.ac.bd/math/Publications/Pub_Hazra.doc
23	Higham, Nicholas J.	2002	Accuracy and stability of numerical algorithms. (2nd edition.)	English	B	Philadelphia: Society for Industrial and Applied Mathematics (SIAM). xxx, 680 pp.	0-89871-521-0 hbk	http://www.maths.man.ac.uk/~higham/asna/
24	Higham, Nicholas J. ; Horn, Roger; Laffey, Thomas J.; Puerta, Ferran (eds.)	2001	Proceedings of the Eighth Conference of the International Linear Algebra Society. Held at the Universitat Politecnica de Catalunya, Barcelona, July 19--22, 1999.	English	JP	*Linear Algebra and its Applications* 332/334. New York: North-Holland. xii, 609 p.	ISSN 0024-3795	MR 1 839 422
25	Hohn, Franz E.	2003	Elementary matrix algebra. (3rd edition. Reprint of 1973 edition, New York: Macmillan.)	English	B	Mineola, NY: Dover; Newton Abbott: David & Charles, 544 p.	0-486-42534-7	AMICUS 27571005
26	Horiguchi, Yuki Fatos	1996	Matrix inequalities as Hermitian completion problems.	English	D	Thesis (Honors)--College of William and Mary. 49 p.	none	OCLC 36161384
27	Hunter, Katherine	2001	Eigenvalues and singular values of circulant and block circulant matrices.	English	D	Thesis (B.S.)--California Polytechnic State University. 1 v. (various pagings)	none	OCLC 48961822
28	Hurley, Donal J.; Vandyck, Michael A.	2000	Geometry, spinors and applications.	English	B	London: Springer-Verlag. Chichester: Praxis Publishing. xvii, 369 p.	1-85233-223-9 hbk	Zbl 0933.53001 (J. D. Zund)
29	Huynh, D. V.; Jain, S. K.; López-Permouth, S. R. (eds.)	2000	Algebra and its applications. Proceedings of the international conference, Athens, OH, USA, March 25--28, 1999.	English	P	Contemporary Mathematics 259. Providence, RI: American Mathematical Society (AMS). xi, 569 p.	0-8218-1950-X/pbk; ISSN 0271-4132	Zbl 0947.00022

30	Jänich, Klaus	2002	Lineare Algebra. (9th edition.)	German	B	Springer-Lehrbuch. Berlin: Springer-Verlag. xii, 271 p.	3-540-43587-5 pbk	Zbl 01764141
31	Jurlewicz, Teresa	2000	Algebra liniowa 2. Kolokwia i egzaminy.	Polish	B	Matematyka dla Studentów Politechnik. Wroclaw: Oficyna Wydawnicza GiS. 93 p.	83-85941-62-2 pbk	Zbl 01745706
32	Jurlewicz, Teresa; Skoczylas, Zbigniew	2000	Algebra liniowa 2. Przyklady i zadania. (4th augmented edition.)	Polish	B	Matematyka dla Studentów Politechnik. Wroclaw: Oficyna Wydawnicza GiS. 144 p.	83-85941-71-1 pbk	Zbl 01745690
33	Jurlewicz, Teresa; Skoczylas, Zbigniew	2001	Algebra liniowa 2. Definicje, twierdzenia, wzory. (6th extended edition.)	Polish	B	Matematyka dla Studentów Politechnik. Wroclaw: Oficyna Wydawnicza GiS. 145 p.	83-85941-89-4 pbk	Zbl 01745689
34	Jurlewicz, Teresa; Skoczylas, Zbigniew	2001	Algebra liniowa 1. Przyklady i zadania. (7th revised edition.)	Polish	B	Matematyka dla Studentów Politechnik. Wroclaw: Oficyna Wydawnicza GiS. 167 p.	83-85941-80-0 pbk	Zbl 01745688
35	Jurlewicz, Teresa; Skoczyslas, Zbigniew	2001	Algebra liniowa 1. Definicje, twierdzenia, wzory. (8th edition, revised and augmented.)	Polish	B	Matematyka dla Studentów Politechnik. Wroclaw: Oficyna Wydawnicza GiS. 163 p.	83-85941-79-7 pbk	Zbl 01745687
36	Kharab, Abdelwahab; Guenther, Ronald B.	2001	An introduction to numerical methods. A Matlab approach.	English	B	Boca Raton, FL: Chapman & Hall/CRC. 431 p. & 1 CD-ROM.	1-58488-281-6	Zbl 0993.65001 (Matti Vuorinen)
37	Krammer, Bettina	2002	Algorithmische lineare Algebra für Polynommatrizen.	German	B	Regensburger Mathematische Schriften 31 & 32. Regensburg: Univ. Regensburg, Fakultät für Mathematik, 202 p. & Online-Resource (1594 k).	ISSN 0179-9746	Zbl 0997.65067
38	Latouche, Guy; Taylor, Peter	2002	Matrix-analytic methods: theory and applications. Proceedings of the 4th International Conference held in Adelaide, July 14--16, 2002.	English	P	Singapore & River Edge, NJ: World Scientific Publishing Co. Inc. xvi, 416 p.	981-238-051-5	MR 1 923 875
39	Lay, David C.	2003	Linear algebra and its applications. (3rd edition.)	English	B	Reading, MA: Addison-Wesley. xvi, 492 p.	0-201-70970-8	AMICUS 27625282
40	Liebler, Robert	2002	Basic matrix algebra with algorithms and applications.	English	B	Boca Raton, FL: Chapman & Hall/CRC	1-58488-333-2	OCLC 50583486
41	Lipschutz, Seymour and Lipson, Marc Lars	2002	Linear algebra: based on Schaum's outline of theory and problems of linear algebra, 3rd edition. (Kimberly S. Kirkpatrick, abridgement editor.)	English	B	Schaum's Easy Outlines. New York : McGraw-Hill. 156 p.	0-07-139880-5	AMICUS 27548462
42	Loss, M.; Ruskai, M. B. (eds.)	2002	Inequalities: Selecta of Elliott H. Lieb. (Edited, with a preface and commentaries by M. Loss and M. B. Ruskai.)	English	B	Berlin: Springer-Verlag. xi, 711 p.	3-540-43021-0	MR 1 922 236; Zbl pre01825514
43	Magnus, J. R.; Neudecker, H.	2002	Matrix differential calculus with applications in statistics and econometrics. (S. A. Ayvazyan scientific editor for the translation from the English revised edition, New York: Wiley, 1999.)	Russian	B	Moscow: Fizmatlit, 496 p.	5-9221-0262-1	photocopy of frontal pages
44	Marques de Sá, E.; Queiró, J. F.; Santana, A.P. (eds.)	2000	Special Issue: Workshop on Geometric and Combinatorial Methods in the Hermitian Sum Spectral Problem. Papers from the workshop held at the University of Coimbra, Coimbra, July 15--16, 1999.	English	JP	*Linear Algebra and its Applications* 319, 1-3. New York: North-Holland. xii, 81 p.	ISSN 0024-3795	MR 2001g:15002
45	Meyberg, Kurt; Vachenauer, Peter	2001	Höhere Mathematik 1. Differential- und Integralrechnung. Vektor- und Matrizenrechnung. (6th corrected edition.)	German	B	Springer-Lehrbuch. Berlin: Springer-Verlag. xv, 529 p.	3-540-41850-4 pbk	Zbl 0982.00001 (J. Appell)

46	Nagy, James; Benzi, Michele; Ferng, William; Sun, Xiaobai (eds.)	2000	Conference celebrating the 60th Birthday of Robert J. Plemmons: Papers from the conference held at Wake Forest University, Winston-Salem, NC, January 1999.	English	FJP	*Linear Algebra and its Applications* 316, 1-3. New York: North-Holland. iv, 287 p.	ISSN 0024-3795	MR 2001d:00055
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50	Poole, George	2003	Linear algebra: a modern introduction.	English	B	Pacific Grove, CA: Brooks/Cole-Thomson Learning, xxvi, 763 p.	0-534-34174-8	AMICUS 25963289
51	Puntanen, Simo; Styan, George P. H.; Werner, Hans Joachim (eds.)	2002	Ninth special issue on linear algebra and statistics.	English	J	*Linear Algebra and its Applications* 354, 1-3. New York: North-Holland.xii, 288 p.	ISSN 0024-3795	MR 1 927 642
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53	Queiroz, Marcelo; Júdice, Joaquim; Humes Júnior, Carlos	2001	The symmetric eigenvalue complementarity problem.	English	B	São Paulo, Brasil: Universidade de São Paulo, Instituto de Matemàtica e Estatística RT-2001-10. 29 p.	none	OCLC 48814738
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55	Sachkov, V. N.; Tarakanov, V. E.	2002	Combinatorics of nonnegative matrices. (Translated from the 2000 Russian original by Valentin F. Kolchin.)	English	B	Translations of Mathematical Monographs 213. Providence, RI: American Mathematical Society. ix, 269 p.	0-8218-2788-X hbk	MR 1 905 938; Zbl 01805713
56	Serre, Denis	2002	Matrices: theory and applications. (Translated from the 2001 French original.)	English	B	New York, NY: Springer-Verlag. xvi, 202 p.	0-387-95460-0	MR 1 923 507
57	Vinson, Jade P.	2001	Closest spacing of eigenvalues.	English	D	Thesis (Ph. D.)--Princeton University, 66 p.	none	OCLC 49366672
58	Vlamou, Elena	2000	Linear systems, determinants & linear transformations.	Greek	B	Algebra (Athens) 2. Athens: ``V" Publications. 144 p.	960-8073-05-7	Zbl 0990.15001 (Efstratios Rappos)
59	Watkins, David S.	2002	Fundamentals of matrix computations. (2nd edition.)	English	B	New York, NY: Wiley. xiv, 618 p.	0-471-21394-2	Zbl 01805736
60	Wildberger, N.J.	2001	Algebraic structures associated to group actions and sums of Hermitian matrices.	English	B	Textos de Mathmática, Série B 28. Coimbra: Universidade de Coimbra, Departamento de Matemática. vi, 32 pp.	972-8564-30-9	MR 2002i:20099 (Mohammed Sifi)
61	Zhan, Xingzhi (not Xinghzi)	2002	Matrix inequalities.	English	B	Lecture Notes in Mathematics 1790. Berlin: Springer-Verlag. vii, 116 p.	3-540-43798-3 pbk	MR 1 927 396; Zbl 01774881

IMAGE Problem Corner: Old Problems, Many With Solutions

We present solutions to IMAGE Problems 28-1, 28-2 and 28-4 through 28-10 published in IMAGE 28 (April 2002), pp. 36 & 35. We are still hoping to receive solutions to Problems 19-3b, 23-1 and 28-3, which are repeated below. In addition, we introduce 13 new problems (8 on page 36 and 5 more on page 35) and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 19-3: Characterizations Associated with a Sum and Product

Proposed by Robert E. HARTWIG, *North Carolina State University, Raleigh, North Carolina, USA:* hartwig@math.ncsu.edu
Peter ŠEMRL, *University of Maribor, Maribor, Slovenia:* peter.semrl@fmf.uni-lj.si
and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

(a) Characterize square matrices A and B satisfying $AB = pA + qB$, where p and q are given scalars.

(b) More generally, characterize linear operators A and B acting on a vector space $\mathcal{X}$ satisfying $ABx \in \operatorname{span}(Ax, Bx)$ for every $x \in \mathcal{X}$.

The Editor has still not received a solution to Problem 19-3b. The solution by the Proposers to Problem 19-3a appeared in IMAGE 22 (April 1999), p. 25. *We look forward to receiving a solution to Problem 19-3b.*

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG, *University of Wisconsin–Madison, Madison, Wisconsin, USA:* mchung@stat.wisc.edu

Let the $m \times n$ random matrix X be such that $\operatorname{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where vec indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find E det$(X'CX)$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

We look forward to receiving a solution to this problem!

Problem 28-1: Regular and Reflected Rotation Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

An $n \times n$ Jacobi, Givens, or elementary rotation matrix is an $n \times n$ matrix with unit elements on its diagonal and zeroes elsewhere except for its iith, ijth, jith, and jjth positions ($i \neq j$) which contain the real values $c, s, -s$, and c, satisfying $c^2 + s^2 = 1$. Show that any real orthogonal matrix with a determinant of plus one may be expressed as the product of at most $n^2/2$ elementary rotation matrices, and deduce that any real orthogonal matrix with a determinant of minus one may be expressed as the product of at most $(n^2 - 1)/2$ elementary rotation matrices and a single $n \times n$ diagonal matrix with $n - 1$ elements of plus one and one element of minus one on its diagonal. Such matrices may be named *reflected rotation matrices* by contrast with the (proper) *rotation matrices* of the main result.

Solution 28-1.1 (with some slight editorial modifications)
by Iwona WRÓBEL, *Warsaw University of Technology, Poland:* aquila_poison_ivy@wp.pl

We offer an elementary solution to the following improved statement: *For $n \geq 2$, every proper rotation matrix is a product of no more than $f(n) = n(n-1)/2$ elementary rotation matrices.* Needless to say, for $n = 1$, the formula in the problem under discussion is not valid, although the result is trivial.

We prove our statement by induction. Let A be an $n \times n$ orthogonal matrix with determinant one. Since, for $n = 2$, the matrix A is itself an elementary rotation matrix, formula $f(2) = 1$ holds. Next, suppose that $n > 2$ and that the result is true for matrices of order $n - 1$. Let $A = (a_1, a_2, \ldots, a_n)$, where $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, n$. By elementary consideration or according to Golub & van Loan (1996, p. 226), there exist rotation matrices $T_1, \ldots, T_{n-1} \in \mathbb{R}^{n \times n}$ such that $T_1 \cdots T_{n-1} a_1 = T a_1 = e_1$. Then

$$TA = (Ta_1, Ta_2, \ldots, Ta_n) = (e_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \begin{pmatrix} 1 & b^T \\ 0 & B \end{pmatrix}.$$

Clearly, as a product of two orthogonal matrices, the matrix TA is orthogonal. So $b = 0$, and hence

$$TA = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}, \tag{1}$$

where B is an orthogonal matrix of order $n-1$. By our induction assumption, B is a product of at most $f(n-1) = (n-1)(n-2)/2$ elementary rotation matrices. So the matrix A is a product of at most $(n-1) + (n-1)(n-2)/2 = n(n-1)/2 = f(n)$ elementary rotation matrices, thus completing the induction proof. (For reflected rotation matrices, one needs only multiply it with a single diagonal matrix with $n - 1$ elements of plus one and one element of minus one along its diagonal, and apply the above result.) □

Reference

Gene H. Golub & Charles F. Van Loan (1996). *Matrix Computations* (Third Edition). Johns Hopkins University Press, Baltimore.

Editorial remarks. Iwona WRÓBEL also offered another proof for the weaker result that at most $n^2/2$ elementary rotation matrices are needed using a similar idea. A sketch of the proof of the best bound using $n(n-1)/2$ elementary rotation matrices using Lie theory was also received from Fátima SILVA LEITE, University of Coimbra, Portugal: fleite@mat.uc.pt; for more details, see Leite (1991) and related references mentioned there.

Reference

F. Silva Leite (1991): Bounds on the order of generation of $SO(n, \mathbf{R})$ by one-parameter subgroups. *Rocky Mountain Journal of Mathematics*, 21, 879–911 & 1183–1188.

Problem 28-2: Linear Combinations of Imaginary Units

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

Let $i,\ j,\ k$ denote the imaginary units of the algebra of quaternions. Then, it is well known that these units satisfy the conditions $i^2 = j^2 = k^2 = ijk = -1$. Let v denote the 3×1 matrix of imaginary units $v = [i\ j\ k]'$, and let $p,\ q,\ r$ be arbitrary 3×1 real matrices. Find conditions such that the linear combinations $i_o = p'v,\ j_o = q'v,\ k_o = r'v$ satisfy the conditions $i_o^2 = j_o^2 = k_o^2 = i_o j_o k_o = -1$.

Solution 28-2.1 by Fátima SILVA LEITE, *University of Coimbra, Coimbra, Portugal:* fleite@mat.uc.pt

We find it convenient to use a slightly different notation. A quaternion $\underline{a}$ is identified with a pair (a_0, a), where $a_0 \in \mathbb{R}$ is called the real part of $\underline{a}$, and $a \in \mathbb{R}^3$ is the imaginary part of $\underline{a}$. A pure quaternion is a quaternion with zero real part. In such a case, a quaternion $\underline{a}$ may be simply identified with its imaginary part. Similarly, if a quaternion has zero imaginary part, it may be identified with a real number. The three imaginary units are usually denoted by i, j, k and, consequently, the imaginary part of a quaternion may be written as $a = a_1 i + a_2 j + a_3 k$, for $a_1, a_2, a_3 \in \mathbb{R}$. With this notation, the problem has the following equivalent formulation.

PROBLEM: Find conditions on three arbitrary pure quaternions a, b and c, in order that they satisfy the following conditions: $a^2 = b^2 = c^2 = abc = -1$.

The crucial identity to prove the theorem below is the following formula for the product of two quaternions $\underline{a} = (a_0, a)$ and $\underline{b} = (b_0, b)$ in terms of the usual inner product $<.,.>$, and the vector cross product $\times$ of their imaginary parts:

$$\underline{ab} = (a_0 b_0 - < a, b >, a_0 b + b_0 a + a \times b). \tag{2}$$

In particular, for the product of two pure quaternions one has

$$ab = (- < a, b >, a \times b) \quad \text{and} \quad a^2 = - \parallel a \parallel^2, \tag{3}$$

and for the product of three pure quaternions a trivial calculation using (2) gives

$$abc = (- < a \times b, c >, - < a, b > c + < c, a > b - < c, b > a). \tag{4}$$

THEOREM 1. *A necessary and sufficient condition for three arbitrary pure quaternions* a, b *and* c *to satisfy* $a^2 = b^2 = c^2 = abc = -1$ *is that they form an orthonormal basis of* $\mathbb{R}^3$.

PROOF OF NECESSITY: We prove first that if $abc = -1$, then a, b and c are linearly independent. Indeed, it follows from (4) that if $abc = -1$, then a and b are linearly independent (otherwise $a \times b = 0$). But c must also be linearly independent of the previous two, otherwise it would be orthogonal to $a \times b$ and consequently the real part of abc would be zero. It now follows from the linear independence of a, b and c and from (4) that if $abc = -1$, then a, b and c are mutually orthogonal. Since by assumption $a^2 = b^2 = c^2 = -1$, due to (3), we also get that $\parallel a \parallel = \parallel b \parallel = \parallel c \parallel = 1$.

PROOF OF SUFFICIENCY: Now assume that a, b and c are orthonormal vectors of $\mathbb{R}^3$. Then it follows immediately from (3) that $a^2 = b^2 = c^2 = -1$ and from (4) that the imaginary part of abc is zero, while the real part is equal to -1. □

Solution 28-2.2 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

The solution is presented in the following form.

PROPOSITION. *Let* i, j, k *denote the imaginary units of the algebra of quaternions. Further, let* $v = (i, j, k)'$, *let* $p = (p_1, p_2, p_3)'$, $q = (q_1, q_2, q_3)'$, $r = (r_1, r_2, r_3)'$ *be* 3×1 *real matrices, and let* $i_0 = p'v$, $j_0 = q'v$, $k_0 = r'v$. *Then*

$$i_0^2 = -1,\ j_0^2 = -1,\ k_0^2 = -1,\ i_0 j_0 k_0 = -1 \tag{5}$$

if and only if the matrix

$$S = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}, \tag{6}$$

having p', q', *and* r' *as its successive rows, is orthogonal, i.e.,* $SS' = I_3$.

PROOF. Since i, j, k satisfy $ij = k = -ji$, $ik = -j = -ki$, $jk = i = -kj$, it follows that the first three conditions in (5) are equivalent to

$$p'p = 1,\ q'q = 1,\ r'r = 1. \tag{7}$$

From (5) it also follows that $i_0 j_0 = k_0$, $i_0 k_0 = -j_0$, $i_0 k_0 = i_0$. It can straightforwardly be verified that

$$i_0 j_0 = -p'q + R_1 i + R_2 j + R_3 k, \tag{8}$$

where R_m denotes the cofactor of r_m in the matrix S of the form (6), $m = 1, 2, 3$. Hence the equality $i_0 j_0 = k_0$ implies $p'q = 0$, and similar arguments lead to $p'r = 0$ and $q'r = 0$. Combining these observations with (7) shows that S must be orthogonal.

Conversely, the orthogonality of S obviously implies conditions (7). Moreover, in view of (8) and $p'q = 0$, it is seen that $i_0 j_0 k_0 = -(r_1 R_1 + r_2 R_2 + r_3 R_3) - (r_2 R_3 - r_3 R_2)i + (r_1 R_3 - r_3 R_1)j - (r_1 R_2 - r_2 R_1)k$. It is clear that $r_1 R_1 + r_2 R_2 + r_3 R_3 = \det(S)$. Since S is an orthogonal matrix, it follows that $-(r_1 R_1 + r_2 R_2 + r_3 R_3) = -1$. Consequently, it remains to show that

$$r_2 R_3 = r_3 R_2,\ r_1 R_3 = r_3 R_1,\ r_1 R_2 = r_2 R_1. \tag{9}$$

These equalities appear to be consequences of the conditions $p'q = 0$, $p'r = 0$, and $q'r = 0$. In fact,

$$r_2 R_3 - r_3 R_2 = r_2(p_1 q_2 - p_2 q_1) + r_3(p_1 q_3 - p_3 q_1) = p_1(q_2 r_2 + q_3 r_3) - q_1(p_2 r_2 + p_3 r_3) = p_1(-q_1 r_1) - q_1(-p_1 r_1) = 0,$$

and the remaining equalities in (9) follow similarly. □

Editorial remarks. A similar solution was also received from
Dennis I. MERINO, Southeastern Louisiana University, Hammond, Louisiana, USA: dmerino@selu.edu

Problem 28-3: Ranks of Nonzero Linear Combinations of Certain Matrices

Proposed by Shmuel FRIEDLAND, *University of Illinois at Chicago, Chicago, Illinois, USA:* friedlan@uic.edu
and Raphael LOEWY, *Technion–Israel Institute of Technology, Haifa, Israel:* loewy@technunix.technion.ac.il

Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Show that any nonzero real linear combination of these four matrices has rank at least 3. [The proposers prefer a solution which does not depend on the use of a computer package such as Maple.]

We look forward to receiving a solution to this problem!

Problem 28-4: A Rank Identity for Block Circulant Matrix

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario:* ytian@mast.queensu.ca

Let $A_1, A_2, \ldots, A_k$ be $m \times n$ matrices, and denote by A the $km \times kn$ block circulant matrix $A = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}$.

Show that $\operatorname{rank}(A) = \operatorname{rank}(A - P'NQ) + \operatorname{rank}(P'NQ)$, where $P = \frac{1}{\sqrt{k}}[I_m, \ldots, I_m]$, $Q = \frac{1}{\sqrt{k}}[I_n, \ldots, I_n]$ and $N = A_1 + \cdots + A_k$.

Solution 28-4.1 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Put $B := A - P'NQ$ and $C := P'NQ$. Then $A = B + C$. According to Theorem 2.3 in Jain, Mitra & Werner (1996), $\operatorname{rank}(B + C) = \operatorname{rank}(B) + \operatorname{rank}(C)$ if and only if B and C is a pair of *weakly bicomplementary* (often also called *disjoint*) matrices, i. e., if and only if $\mathcal{R}(B) \cap \mathcal{R}(C) = \{0\}$ and $\mathcal{R}(B') \cap \mathcal{R}(C') = \{0\}$, where $\mathcal{R}(\cdot)$ denotes the range (column space) of the matrix $(\cdot)$. Therefore it suffices to prove that the matrices B and C satisfy these range conditions. For that purpose, observe first that

$$C = \left(\frac{1}{k}E_k \otimes I_m\right) A \qquad \text{and} \qquad B = \left(I_{mk} - \frac{1}{k}E_k \otimes I_m\right) A, \tag{10}$$

where I_r stands for the $r \times r$ identity matrix, where E_r denotes the $r \times r$ matrix whose elements are all equal to unity, and where $\otimes$ indicates the Kronecker product. Since the matrices $(1/k)E_k \otimes I_m$ and $I_{mk} - (1/k)E_k \otimes I_m$ are idempotent (symmetric) matrices, or equivalently, (orthogonal) projectors, their column spaces have only the origin in common; see, e. g., Lancaster (1969, pp. 82–83) or Werner & Yapar (1996, pp. 362–363). In view of (10), this directly implies that $\mathcal{R}(B) \cap \mathcal{R}(C) = \{0\}$. From

$$C' = \left(\frac{1}{k}E_k \otimes I_n\right) A' \qquad \text{and} \qquad B' = \left(I_{nk} - \frac{1}{k}E_k \otimes I_n\right) A'$$

we analogously obtain $\mathcal{R}(B') \cap \mathcal{R}(C')$. This completes our solution.

References

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P. Lancaster (1969). *The Theory of Matrices.* Academic Press, New York.

H. J. Werner & C. Yapar (1996). On inequality constrained generalized least squares selections in the general possibly singular Gauss–Markov model: a projector theoretical approach. *Linear Algebra and its Applications*, 237/238, 359–393.

Solution
~~Problem~~ **28-4.2** by the Proposer Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

It is easy to see by block Gaussian elimination that

$$\operatorname{rank}\begin{pmatrix} -A & 0 & A \\ 0 & N & NQ \\ A & P'N & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -A & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & A - P'NQ \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}(P'NQ) + \operatorname{rank}(A - P'NQ),$$

since $\operatorname{rank}(N) = \operatorname{rank}(P'NQ)$. Moreover,

$$P'N = AQ', \qquad NQ = PA, \qquad QQ' = I_n, \qquad PP' = I_m.$$

Similarly

$$\operatorname{rank}\begin{pmatrix} -A & 0 & A \\ 0 & N & NQ \\ A & P'N & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & P'N & A \\ 0 & N & NQ \\ A & 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ A & 0 & 0 \end{pmatrix} = 2\operatorname{rank}(A)$$

and so $\operatorname{rank}(A) = \operatorname{rank}(A - P'NQ) + \operatorname{rank}(P'NQ)$, as desired.

Problem 28-5: A Range Equality for Moore–Penrose Inverses

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Suppose A and B are complex $m \times n$ and $m \times k$ matrices, respectively. Show that if $\operatorname{range}(A) \cap \operatorname{range}(B) = \{0\}$, then

$$\operatorname{range}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = \operatorname{range}\begin{pmatrix} A^* \\ B^* \end{pmatrix},$$

where $(\cdot)^\dagger$ and $(\cdot)^*$ denote the Moore–Penrose inverse and the conjugate transpose of $(\cdot)$, respectively.

Solution 28-5.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

We will establish a more general result. The symbols $\mathcal{R}(K)$ and $\mathcal{N}(K)$ will stand for the range and null space of $K \in \mathbb{C}_{m,n}$, respectively. Moreover, $K\{1,3,4\} = \{G_K \in \mathbb{C}_{n,m} : KG_KK = K, KG_K = (KG_K)^*, G_KK = (G_KK)^*\}$ $K\{2,4\} = \{H_K \in \mathbb{C}_{n,m} : H_KKH_K = H_K, H_KK = (H_KK)^*\}$, where the asterisk superscript denotes the conjugate transpose of a matrix.

THEOREM. *For given $A \in \mathbb{C}_{m,n}$ and $B \in \mathbb{C}_{m,p}$ such that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, the inclusions*

$$\mathcal{R}(\begin{pmatrix} H_A \\ H_B \end{pmatrix}) \subseteq \mathcal{R}(\begin{pmatrix} A^* \\ B^* \end{pmatrix}) \subseteq \mathcal{R}(\begin{pmatrix} G_A \\ G_B \end{pmatrix}) \tag{11}$$

hold for any $G_A \in A\{1,3,4\}$, $G_B \in B\{1,3,4\}$ and $H_A \in A\{2,4\}$, $H_B \in B\{2,4\}$.

PROOF. It is clear that, for any complex matrices K and L with the same number of rows, $\mathcal{R}(K) \subseteq \mathcal{R}(L)$ if and only if $\mathcal{N}(L^*) \subseteq \mathcal{N}(K^*)$. Consequently, (11) can be reexpressed as

$$\mathcal{N}((G_A^* : G_B^*)) \subseteq \mathcal{N}((A : B)) \subseteq \mathcal{N}((H_A^* : H_B^*)), \tag{12}$$

where $(.:.)$ denotes a column-wise partitioned matrix. If $x = (x_1^* : x_2^*)^* \in \mathcal{N}((G_A^* : G_B^*))$, i.e., if

$$G_A^* x_1 + G_B^* x_2 = 0, \tag{13}$$

then

$$G_A^* x_1 \in \mathcal{R}(G_A^*) \cap \mathcal{R}(G_B^*). \tag{14}$$

But $\mathcal{R}(G_A^*) = \mathcal{R}(G_A^* A^*) = \mathcal{R}(AG_A) = \mathcal{R}(A)$ and, similarly, $\mathcal{R}(G_B^*) = \mathcal{R}(B)$. Hence, on account of the assumption $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, it follows from (14) that $G_A^* x_1 = 0$ and, in view of (13), $G_B^* x_2 = 0$. Consequently, $Ax_1 = AG_A Ax_1 = AA^* G_A^* x_1 = 0$ and, similarly, $Bx_2 = 0$, which shows that $x \in \mathcal{N}((\, A : B\,))$.

The second part of (12) follows by analogous arguments. If $x \in \mathcal{N}((\, A : B\,))$, i.e., if $Ax_1 + Bx_2 = 0$, then $Ax_1 \in \mathcal{R}(A) \cap \mathcal{R}(B)$ and thus $Ax_1 = 0$ and $Bx_2 = 0$. Consequently, $H_A^* x_1 = (H_A A H_A)^* x_1 = H_A^* H_A A x_1 = 0$ and, similarly, $H_B^* x_2 = 0$, which implies that $x \in \mathcal{N}((\, H_A^* : H_B^*\,))$, thus completing the proof. □

Since $A^+ \in A\{1,3,4\} \cap A\{2,4\}$ and $B^+ \in B\{1,3,4\} \cap B\{2,4\}$, the equality of the ranges stated in Problem 28-5 is an immediate corollary to Theorem.

Solution 28-5.2 by Ravi B. BAPAT, *Indian Statistical Institute, New Delhi, India:* rbb@isid.ac.in

Let S be the space of $(n+k) \times m$ vectors of the form $\begin{pmatrix} u \\ 0 \end{pmatrix}$, where $u \in \text{range}(A^*)$ and let T be the space of $(n+k) \times m$ vectors of the form $\begin{pmatrix} 0 \\ v \end{pmatrix}$, where $v \in \text{range}(B^*)$. Clearly

$$\text{range} \begin{pmatrix} A^* \\ B^* \end{pmatrix} \subseteq S + T. \tag{15}$$

Since $S \cap T = \{0\}$ and $\text{range}(A) \cap \text{range}(B) = \{0\}$,

$$\dim(S+T) = \dim(S) + \dim(T) = \text{rank}(A) + \text{rank}(B) = \text{rank}[A, B]. \tag{16}$$

It follows from (15) and (16) that $\text{range} \begin{pmatrix} A^* \\ B^* \end{pmatrix} = S + T$. Noting that $\text{range}(A^*) = \text{range}(A^\dagger)$, a similar argument shows that $\text{range} \begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = S + T$, and the proof is complete.

Solution 28-5.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Let $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the range (column space) and the null space of $(\cdot)$, respectively. Clearly, $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \iff \mathcal{N}(A^*) + \mathcal{N}(B^*) = \mathbb{C}^m$. Therefore,

$$\mathcal{R}(\begin{pmatrix} A^* \\ B^* \end{pmatrix}) = \begin{pmatrix} A^* \\ B^* \end{pmatrix} \mathbb{C}^m = \begin{pmatrix} A^* \\ B^* \end{pmatrix} (\mathcal{N}(A^*) + \mathcal{N}(B^*)) = A^*\mathcal{N}(B^*) \times B^*\mathcal{N}(A^*) = \mathcal{R}(A^*) \times \mathcal{R}(B^*),$$

where '$\times$' indicates a cartesian product. In view of $\mathcal{N}(A^+) = \mathcal{N}(A^*)$ and $\mathcal{N}(B^+) = \mathcal{N}(B^*)$, likewise

$$\mathcal{R}(\begin{pmatrix} A^+ \\ B^+ \end{pmatrix}) = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} \mathbb{C}^m = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} (\mathcal{N}(A^*) + \mathcal{N}(B^*)) = A^+\mathcal{N}(B^*) \times B^+\mathcal{N}(A^*) = \mathcal{R}(A^+) \times \mathcal{R}(B^+).$$

Since $\mathcal{R}(A^+) = \mathcal{R}(A^*)$ and $\mathcal{R}(B^+) = \mathcal{R}(B^*)$, the claim is plain. □

A solution was also received from the Proposer: Yongge TIAN.

Problem 28-6: Square Roots and Additivity

Proposed by Dietrich TRENKLER, *Universität Osnabrück, Osnabrück, Germany:* dtrenkler@nts6.oec.uni-osnabrueck.de
and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A and B be nonnegative definite matrices of the same type and let $\leq_{\mathsf{L}}$ denote the Löwner-ordering. Is it true that

$$(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2} \ ?$$

Solution 28-6.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl

It is clear that the inequality in Problem 28-6 holds if A and B commute. This condition ensures the existence of a unitary matrix U such that UAU^* and UBU^* are both diagonal, and thus the problem reduces to the set of obvious scalar inequalities $(\lambda_i + \mu_i)^{1/2} \leq \lambda_i^{1/2} + \mu_i^{1/2}$, with λ_i and μ_i being (nonnegative) eigenvalues of A and B, respectively.

In general, the inequality in question is not true. For example, if A and B are orthogonal projectors, i.e., $A = A^2 = A^*$ and $B = B^2 = B^*$, then $A^{1/2} = A$ and $B^{1/2} = B$, and hence $(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2}$ is equivalent to $A + B \leq_{\mathsf{L}} (A+B)^2$, which is for instance not fulfilled by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Editors' Remarks 28-6.2 by Chi-Kwong LI, *The College of William & Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu
and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

We supplement the preceding Solution 28-6.1 by mentioning two observations. First, one could be tempted to believe that the commutativity of A and B is not only sufficient but also necessary for $(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2}$ to hold. That this, however, is not the case is exhibited by the nonnegative definite Hermitian matrices

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}.$$

Although these matrices do not commute, we nevertheless have

$$(A+B)^{1/2} = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{pmatrix} \leq_{\mathsf{L}} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = A^{1/2} + B^{1/2}.$$

Second, we prove that the characterization in question is true on the important subclass of idempotent Hermitian matrices.

THEOREM. *Let A and B be idempotent Hermitian matrices of the same order n. Then $(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2}$ if and only if A and B commute.*

PROOF. In view of Solution 28-6.1, sufficiency is clear. For proving necessity, let $(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2}$. Since A and B are idempotent Hermitian matrices, clearly $A^{1/2} = A = A^*$ and $B^{1/2} = B = B^*$. Therefore, $(A+B)^{1/2} \leq_{\mathsf{L}} A^{1/2} + B^{1/2} \Leftrightarrow (A+B)^{1/2} \leq_{\mathsf{L}} A + B \Leftrightarrow A + B \leq_{\mathsf{L}} (A+B)^2 \Leftrightarrow AB + BA \geq_{\mathsf{L}} 0 \Leftrightarrow \Re\,(x^*ABx) \geq 0$ for all vectors $x \in \mathbb{C}^n$. Hence, in particular, $0 \leq \Re\,((y+\lambda z)^*AB(y+\lambda z)) = \Re\,(y^*By + \lambda y^*Bz) = y^*By + \Re\,(\lambda y^*Bz)$ for all vectors $y \in \mathcal{R}(A)$ and $z \in \mathcal{R}(I-A)$ and for all scalars $\lambda \in \mathbb{C}$. But this holds true if and only if $AB(I-A) = 0$ or, equivalently, $AB = BA$. □

A solution to this problem was also received from the Proposers: Dietrich TRENKLER & Götz TRENKLER.

Problem 28-7: Partial Isometry and Idempotent Matrices

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be an idempotent complex matrix. Show that A is an orthogonal projector if and only if A is a partial isometry, i.e., $AA^*A = A$, where A^* denotes the conjugate transpose of A.

Solution 28-7.1 by Marek ALEKSIEJCZYK, *University of Warmia and Mazury, Olsztyn, Poland:* maralek@matman.uwm.edu.pl

Let A be an idempotent complex matrix (we may assume that $A \neq 0$). We shall prove that

$$A = A^* \Leftrightarrow AA^*A = A.$$

The proof of "$\Rightarrow$" is trivial. To prove the converse, let us assume that $AA^*A = A$. Multiplying this equality by A^* we obtain $A^*AA^*A = A^*A$, so A^*A is an orthogonal projection. Then $||A^*A|| = ||A||^2 = 1$, so $||A|| = 1$, and hence A is an orthogonal projection. □

↑ ? why ?

Solution 28-7.2 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

If A is an orthogonal projector, i.e., $A = A^2 = A^*$, then obviously $AA^*A = A$. Conversely, if $A = A^2$ and $AA^*A = A$, then

$$(A - AA^*)(A - AA^*)^* = AA^* - A^2A^* - A(A^*)^2 + AA^*AA^* = 0,$$

and hence $A = AA^* = A^*$. In the case of real matrices, the characteristic given in Problem 28-7 is a version of part (v) of Theorem 9 in Trenkler (1994). □

Reference

G. Trenkler (1994). Characterizations of oblique and orthogonal projectors. In *Proceedings of the International Conference on Linear Statistical Inference LINSTAT'93* (T. Caliński & R. Kala, eds.), Kluwer Academic Publishers, Dordrecht, pp. 255–270.

Solution 28-7.3 by Vladimir V. SERGEĬCHUK, *Institute of Mathematics, Kiev, Ukraine:* sergeich@ukrpack.net

A canonical matrix of a projector $A = A^2$ on a unitary space is

$$A = I \oplus \begin{pmatrix} 1 & a_1 \\ 0 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 & a_l \\ 0 & 0 \end{pmatrix} \oplus 0, \qquad \begin{matrix} a_1, \ldots, a_l \in \mathbb{R}, \\ a_1 \geq \ldots \geq a_l > 0, \end{matrix}$$

see Djokovic (1991), Ikramov (1996, 2000), or Sergeĭchuk (1998, p. 46). Since

$$\begin{pmatrix} 1 & a_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_i & 0 \end{pmatrix} \begin{pmatrix} 1 & a_i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 + a_i^2 & a_i + a_i^3 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & a_i \\ 0 & 0 \end{pmatrix},$$

$AA^*A = A$ if and only if $A = I \oplus 0$, the last matrix is a canonical matrix of an orthogonal projector.

References

D. Z. Djokovic (1991). Unitary similarity of projectors. *Aequationes Mathematicae*, 42, 220–224.

Kh. D. Ikramov (1996). On the canonical form of projectors with respect to unitary similarity. (In Russian.) *Zhurnal Vychislitel'noĭ Matematiki i Matematicheskoĭ Fiziki*, 36 (3), 3–5. [English translation in *Computational Mathematics and Mathematical Physics*, 36 (3), 279–281 (1996).]

Kh. D. Ikramov (2000). The canonical form as a tool for proving the properties of projectors. (In Russian.) *Zhurnal Vychislitel'noĭ Matematiki i Matematicheskoĭ Fiziki*, 40 (9), 1285–1290. [English translation in *Computational Mathematics and Mathematical Physics*, 40 (9), 1233–1238 (2000).]

V. V. Sergeĭchuk (1998). Unitary and Euclidean representations of a quiver. *Linear Algebra and its Applications*, 278, 37–62.

Solution 28-7.4 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Necessity is trivial, since $A^3 = A$ and $A = A^*$. To show sufficiency we derive the following identity $\mathrm{tr}[(A - A^*)^*(A - A^*)] = 0$. To see this, observe that $(A - A^*)^*(A - A^*) = A^*A - A^*A^* - AA + AA^* = A^+A - A^* - A + AA^+$, where use is made of $A^2 = A$ and $A^* = A^+$, with A^+ the Moore–Penrose inverse of A. Taking traces we obtain $\mathrm{tr}[(A - A^*)^*(A - A^*)] = 2\mathrm{tr}(A^+A) - 2\mathrm{tr}(A)$. Since $\mathrm{tr}(A^+A) = \mathrm{rk}(A) = \mathrm{tr}(A)$, we arrive at the desired equality.

Solution 28-7.5 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

We prove the following slightly more informative characterizations. As usual, let $\mathcal{R}(\cdot)$ denote the range (column space) of the matrix $(\cdot)$.

THEOREM. *Let A be an idempotent complex matrix, i. e., let $A^2 = A$. The following conditions are then equivalent:*

(a) $A = A^*$, *i. e., A is Hermitian.*

(b) $A = AA^*A$, *i. e., A is a partial isometry.*

(c) $\mathcal{R}(A) = \mathcal{R}(A^*)$, *i. e., A is an EP matrix.*

PROOF. If $A = 0$, the results are trivial. For $A \neq 0$, we offer a proof based on a singular value decomposition of A. Let $r = \operatorname{rank}(A)$. Then A can be written as $A = UD_rV^*$, where U and V are (column-unitary matrices) such that $U^*U = V^*V = I_r$ and where D_r is an $r \times r$ diagonal matrix with the r positive singular values of A along its main diagonal. In view of this decomposition, clearly

$$A^2 = A \quad \Longleftrightarrow \quad D_rV^*U = I_r \quad \Longleftrightarrow \quad V^*U = D_r^{-1}.$$

(a) $\Longleftrightarrow$ (b): Trivially, (a) $\Rightarrow$ (b). To prove the converse, let $A = AA^*A$. Then $AA^* = (AA^*)^2$, i. e., $AA^* = UD_r^2U^*$ is idempotent. But this can happen if and only if $D_r^2 = I$ or, equivalently, $D_r = I$. Consequently, $A = UV^*$ and $V^*U = I_r$, and so, in view of $V^*V = U^*U = V^*U = U^*V = I_r$, also $(V-U)^*(V-U) = 0$, or equivalently, $V = U$. As claimed, we thus arrive at $A = UU^* = A^*$.

(a) $\Longleftrightarrow$ (c): Trivially, (a) $\Rightarrow$ (c). To prove the converse, let $\mathcal{R}(A) = \mathcal{R}(A^*)$. Clearly, A is an EP matrix if and only if $\mathcal{R}(V) = \mathcal{R}(U)$. There thus exists a matrix Q such that $V = UQ$. Since $I_r = V^*V = Q^*U^*UQ = Q^*Q$, $D_r^{-1} = V^*U = Q^*U^*U = Q^*$. But $Q^*Q = I_r$, and so $D_r^{-2} = I_r$, or equivalently, $D_r = I = Q$. Hence $A = UU^* = A^*$, and our proof is complete. □

Editorial remarks. Solutions 28-7.1 & 28-7.2 both also work for any C^*-algebra. In fact, from the assumptions we have $A^2 = A$, $(A^*)^2 = A^*$, $AA^*A = A$, $A^*AA^* = A^*$. With this in mind it is easy to check that $(A - A^*)^3 = 0$. Since $A - A^*$ is a normal operator, this implies that $A - A^* = 0$.

A solution to this problem was also received from

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Problem 28-8: Another Inequality for Hadamard Products

Proposed by George VISICK, *Belgravia, London, England, UK:* gv94511@gsk.com

Show that for any positive definite matrix A

$$I + A\circ A^{-1} \leq_{\mathsf{L}} \frac{c+d}{c^{1/2}d^{1/2}} A^{1/2}\circ A^{-1/2},$$

where c is the largest and d the smallest eigenvalue of A, $\circ$ denotes the Hadamard product, and $\leq_{\mathsf{L}}$ denotes the Löwner ordering (so that $H \leq_{\mathsf{L}} G$ means that $G - H$ is nonnegative definite). This is akin to IMAGE Problem 27-4, and is also a rough converse of $2(A^{1/2}\circ A^{-1/2})^2 \leq_{\mathsf{L}} I + A\circ A^{-1}$, which is (16) in the paper by G. Visick (2000): A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product, *Linear Algebra and its Applications*, 304, 45–68.

Solution 28-8.1 by the Proposer George VISICK, *Belgravia, London, England, UK:* gv94511@gsk.com

For convenience, let $H = A^{1/2}$. The eigenvalues λ_i of H are the square roots of those of A. Let κ denote the ratio of the largest λ_i to the smallest one. We need to use

$$\lambda_i/\lambda_j + \lambda_j/\lambda_i \leq \kappa + 1/\kappa$$

(since the left-hand side increases if the larger of λ_i, λ_j is increased, or the smaller is decreased). Let $H = UDU^*$, where $UU^* = I$, and the diagonal elements of the diagonal matrix D are the eigenvalues λ_i of H. Now we have an exercise in Kronecker products.

$$H\otimes H^{-1} + H^{-1}\otimes H = (U\otimes U)\,(D\otimes D^{-1} + D^{-1}\otimes D)\,(U\otimes U)^* \leq_{\mathsf{L}} (U\otimes U)\,(\kappa + 1/\kappa)\,(I\otimes I)\,(U\otimes U)^* = (\kappa + 1/\kappa)\,I\otimes I.$$

Multiplying by the positive definite matrix $H^{-1}\otimes H$ yields

$$I\otimes I + H^{-2}\otimes H^2 \quad \leq_{\mathsf{L}} \quad (\kappa + 1/\kappa)\,H^{-1}\otimes H\ ,$$

and picking out the appropriate principal submatrix

$$I + A\circ A^{-1} \quad \leq_{\mathsf{L}} \quad (\kappa + 1/\kappa)\,A^{1/2}\circ A^{-1/2}\ ,$$

which, since $\kappa = (c/d)^{1/2}$, is the required inequality.

Problem 28-9: A Relative Perturbation Bound

Proposed by Yimin WEI, *Fudan University, Shanghai, China:* ymwei@fudan.edu.cn
and Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

Let A be a nonsingular matrix and let $\rho(\cdot)$ denote spectral radius. If $B = A + E$ and $\operatorname{rank}(AE - EA) \leq 1$ for some matrix E, show that for any eigenvalue $\tilde{\lambda}$ of B, there exists an eigenvalue λ of A such that $|\tilde{\lambda} - \lambda|/|\lambda| \leq \rho(A^{-1}E)$.

Solution 28-9.1 by the Proposers Yimin WEI, *Fudan University, Shanghai, China:* ymwei@fudan.edu.cn
and Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

Since $\operatorname{rank}(AE - EA) \leq 1$, A and E are simultaneously triangularizable; see Prasolov (1994, p. 175). Without loss of generality, we may assume that A, E and B are all upper triangular. With appropriate numbering of the eigenvalues λ_i of A and $\tilde{\lambda}_j$ of B, counting algebraic multiplicities, we have $\lambda_i = (A)_{ii}$ and $\tilde{\lambda}_i = (B)_{ii}$ and since the eigenvalues of $A^{-1}E$ are

$$(A^{-1}E)_{ii} = \frac{(E)_{ii}}{(A)_{ii}} = \frac{(B)_{ii} - (A)_{ii}}{(A)_{ii}} = \frac{\tilde{\lambda}_i - \lambda_i}{\lambda_i},$$

the desired inequality then follows immediately.

NOTE: The statement may be compared to the well-known Bauer–Fike theorem on eigenvalue perturbation.

Reference

V. V. Prasolov (1994). *Problems and Theorems in Linear Algebra.* Translations of Mathematical Monographs, Vol. 134. American Mathematical Society, Providence.

Problem 28-10: Inequalities Involving Square Roots

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

(a) Let H and K be Hermitian matrices of the same order, let $\geq_{\mathsf{L}}$ denote the Löwner ordering and let the nonnegative definite Hermitian "square root" $|K| = (K^*K)^{1/2}$. Are $H \geq_{\mathsf{L}} \pm K$ and $H \geq_{\mathsf{L}} |K|$ equivalent?

(b) Let A, B and C be square complex matrices of the same order such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is nonnegative definite. Show that $A^{1/2}CA^{1/2} \geq_{\mathsf{L}} B^*B$ if (i) A and B commute or (ii) A and C commute and B is Hermitian. Must B in (b) be Hermitian?

false!

Solution 28-10.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

(a) It is well known that any Hermitian matrix K may be represented as

$$K = U_1 D_1 U_1^* - U_2 D_2 U_2^*, \tag{17}$$

where $U_1^* U_1 = I_a$, $U_2^* U_2 = I_b$, $U_1^* U_2 = 0$, and D_1 and D_2 are diagonal matrices comprising on their diagonals all (a say) positive eigenvalues and moduli of all (b say) negative eigenvalues of K, respectively. Obviously, $U_2 D_2 U_2^*$ is absent in (17) whenever K is nonnegative definite, and $U_1 D_1 U_1^*$ is absent whenever K is nonpositive definite. Since K of the form (17) satisfies $|K| = (K^*K)^{1/2} = U_1 D_1 U_1^* + U_2 D_2 U_2^*$, it is seen that

$$H - K = H - |K| + 2U_2 D_2 U_2^* \quad \text{and} \quad H + K = H - |K| + 2U_1 D_1 U_1^*. \tag{18}$$

In view of the nonnegative definiteness of $U_i D_i U_i^*$, $i = 1, 2$, an immediate consequence of (18) is that

$$H \geq_{\mathsf{L}} |K| \Rightarrow H \geq_{\mathsf{L}} \pm K. \tag{19}$$

On the other hand, it is clear that if K is a nonnegative or nonpositive definite matrix, then $|K| = K$ or $|K| = -K$, respectively, and thus $H \geq_{\mathsf{L}} \pm K$ trivially implies $H \geq_{\mathsf{L}} |K|$. In general, however, the converse of (19) is not true. A simple example is provided by the matrices

$$H = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy $H - K \geq_{\mathsf{L}} 0$ and $H + K \geq_{\mathsf{L}} 0$, but not $H - |K| \geq_{\mathsf{L}} 0$.

(b) Theorem 1 of Albert (1969) asserts that

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq_{\mathsf{L}} 0 \Leftrightarrow A \geq_{\mathsf{L}} 0,\ \mathcal{C}(B) \subseteq \mathcal{C}(A),\ C \geq_{\mathsf{L}} B^*A^+B, \tag{20}$$

where $\mathcal{C}(\cdot)$ stands for the column space of a matrix and A^+ denotes the Moore–Penrose inverse of A. Another result used in discussion concerning part (b) is the following.

LEMMA. *Let A and B be square complex matrices of the same order such that $A = A^*$ and $AB = BA$. Then $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow \mathcal{C}(B^*) \subseteq \mathcal{C}(A)$.*

PROOF. It is known that $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ if and only if $P_A B = B$, where P_A denotes the orthogonal projector onto $\mathcal{C}(A)$. Since one of possible representations of P_A is AA^+, it follows that $B^* = B^*(AA^+)^* = B^*AA^+ = (AB)^*A^+ = (BA)^*A^+ = AB^*A^+$, which shows that $\mathcal{C}(B^*) \subseteq \mathcal{C}(A)$. The proof of the converse implication is analogous. □

It can easily be verified that the commutativity of A and B entails the commutativity of A and B^*B, and therefore, since A and B^*B are nonnegative definite, also the commutativity of $A^{1/2}$ and B^*B. Consequently,

$$AB^*A^+BA = B^*AA^+AB = B^*AB = AB^*B = A^{1/2}B^*BA^{1/2},$$

and thus the last condition in (20) implies that

$$ACA \geq_{\mathsf{L}} A^{1/2}B^*BA^{1/2}. \tag{21}$$

In view of the equalities $(A^{1/2})^+A = A^{1/2} = A(A^{1/2})^+$ and $(A^{1/2})^+A^{1/2} = P_A = A^{1/2}(A^{1/2})^+$, premultiplying and postmultiplying in (21) by $(A^{1/2})^+$ leads to

$$A^{1/2}CA^{1/2} \geq_{\mathsf{L}} P_A B^* B P_A. \tag{22}$$

But, on account of the lemma above, the middle condition on the right-hand side of (20) ensures that $P_A B^* = B^*$ (and hence $BP_A = B$ as well). This observation transforms (22) to the inequality $A^{1/2}CA^{1/2} \geq_{\mathsf{L}} B^*B$, thus establishing the validity of (i).

On the other hand, the claim in (ii) appears incorrect. For example, if

$$A = C = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

then all the conditions on the right-hand side of (20) are fulfilled, thus ensuring that $M \geq_{\mathsf{L}} 0$. Moreover, $AC = CA$ and $B = B^*$. However,

$$A^{1/2}CA^{1/2} - B^*B = A^2 - B^2 = \begin{pmatrix} 9 & 12 \\ 12 & 9 \end{pmatrix}$$

is not a nonnegative definite matrix. In view of this observation, the additional question in (ii) concerning necessity of the condition $B = B^*$ is negligible.

Reference

A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics,* 17, 434–440.

IMAGE Problem Corner: New Problems

Problems 29-1 through 29-8 are on page 36.

Problem 29-9: Equality of Two Nonnegative Definite Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two nonnegative definite Hermitian matrices of the same order, and let $(\cdot)^\dagger$ denote the Moore–Penrose inverse of the matrix $(\cdot)$. Show that the following five statements are equivalent:

$$\text{(a) } A = B, \quad \text{(b) } A + AA^\dagger = B + BB^\dagger, \quad \text{(c) } AB^\dagger A = B,$$

$$\text{(d) } \operatorname{rank}(A) = \operatorname{rank}(B) \text{ and } 2A(A+B)^\dagger A = A, \quad \text{(e) range} \begin{pmatrix} A \\ B \end{pmatrix} = \text{range} \begin{pmatrix} B \\ A \end{pmatrix}.$$

Problem 29-10: Equivalence of Three Reverse-Order Laws

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Show that

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [\,(A^\dagger)^* B\,]^\dagger = B^\dagger A^* \Leftrightarrow [\,A(B^\dagger)^*\,]^\dagger = B^* A^\dagger,$$

where $(\cdot)^\dagger$ and $(\cdot)^*$ denote the Moore–Penrose inverse and the conjugate transpose, respectively.

Problem 29-11: The Minimal Rank of a Block Matrix with Generalized Inverses

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Show that

$$\min_{A^-, B^-, C^-} \operatorname{rank} \begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} = \max\{\operatorname{rank}(A), \quad \operatorname{rank}(B) + \operatorname{rank}(C)\},$$

where $(\cdot)^-$ denotes generalized inverse.

Problem 29-12: Matrices Commuting with the Vector Cross Product

Proposed by Dietrich TRENKLER, *Universität Osnabrück, Osnabrück, Germany:* dtrenkler@nts6.oec.uni.osnabrueck.de
and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let a nonzero vector $a \in \mathbb{R}^3$ be given. Find all square matrices A with real entries such that (a) for all $x \in \mathbb{R}^3$, it follows that $a \times Ax = A(a \times x)$ and (b) for all $x \in \mathbb{R}^3$, it follows that $a \times Ax = (Aa) \times x$. Here $\times$ denotes the vector cross product in $\mathbb{R}^3$.

Problem 29-13 : Normal Matrices with Prescribed Diagonal Elements and Their Differences Elsewhere

Proposed by Lajos LÁSZLÓ, *Eötvös Loránd University, Budapest, Hungary:* laszlo@numanal.inf.elte.hu

Show that there are normal matrices of any order with prescribed diagonal elements and their differences elsewhere. More precisely, show that for any n, there exist $n \times n$ "index" matrices P and Q such that the $n \times n$ matrix A defined according to

$$a_{i,j} = \begin{cases} z_i, & i = j \\ z_{p_{i,j}} - z_{q_{i,j}}, & i \neq j \end{cases}$$

is normal for any given complex sequence $(z_i)_{i=1}^n$. Let $p_{i,i} = i,\ q_{i,i} = 0,\ 1 \le i \le n,$ and assume that $p_{i,j} < q_{i,j},\ i \neq j$. Is there only a unique pair of such matrices P and Q? If so, characterize these matrices! For example, with $n = 3$ we find that

$$A = \begin{pmatrix} z_1 & z_1 - z_3 & z_2 - z_3 \\ z_1 - z_3 & z_2 & z_1 - z_2 \\ z_2 - z_3 & z_1 - z_2 & z_3 \end{pmatrix}, \text{ i.e., } P = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to werner@united.econ.uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 29-9 through 29-13 are on page 35.*

Problem 29-1: A Condition for an EP Matrix to be Hermitian

Proposed by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

Let A be an EP matrix, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$, where A^* and $\mathcal{R}(A)$ denote the conjugate transpose and range of A. Show that A is Hermitian if and only if there exists a matrix B having a generalized inverse B^- (i.e., a solution to $BB^-B = B$), for which both B^- and $(B^-)^*$ are also generalized inverses of A, i.e., $AB^-A = A$ and $A(B^-)^*A = A$. From this property it follows, in particular, that every EP-matrix which is a predecessor of a Hermitian matrix with respect to the minus partial ordering is necessarily Hermitian.

Problem 29-2: Triangle with Vertices Circumscribing an Ellipse

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Let A be a 2×2 complex matrix which is not normal. Then, it is well known that the numerical range $W(A)$ of A is a solid ellipse. Let $z_1, z_2, z_3 \in \mathbb{C}$. Show that a necessary and sufficient condition for A to possess a 3×3 normal dilation with eigenvalues z_1, z_2, z_3 is that the triangle with vertices z_1, z_2, z_3 circumscribe the ellipse.

Problem 29-3: Isometric Realization of a Finite Metric Space

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Show that every finite metric space can be realized isometrically as a subset of a normed vector space.

Problem 29-4: Normal Matrix and a Commutator

Proposed by S. W. DRURY and George P. H. STYAN, *McGill University, Montréal (Québec), Canada:*
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Show that every $n \times n$ complex matrix A can be written in the form $A = N + [H, N]$, where N is normal and H is Hermitian, and the commutator $[H, N] = HN - NH$.

Problem 29-5: Product of Two Hermitian Nonnegative Definite Matrices

Proposed by Jürgen GROß and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:*
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Let A and B be two Hermitian nonnegative definite matrices of the same order. Show that the column space $\mathcal{R}(AB)$ and the null space $\mathcal{N}(AB)$ of the product AB are complementary subspaces.

Problem 29-6: Product of Companion Matrices

Proposed by Eric S. KEY, *University of Wisconsin-Milwaukee, Wisconsin, USA:* ericskey@csd.uwm.edu

Suppose that $A_1, \ldots, A_k$ are $n \times n$ companion matrices with common eigenvalue a. Show that a^k is an eigenvalue for the product $A_1 A_2 \cdots A_k$.

Problem 29-7: Complementary Principal Submatrices and Their Eigenvalues

Proposed by Chi-Kwong LI, *The College of William and Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu

Let $n = 2k$ and let A be a real symmetric or complex Hermitian idempotent matrix (i.e., $A^2 = A$) of rank k. If the leading $k \times k$ principal submatrix has eigenvalues $a_1, \ldots, a_k$, show that the complementary principal submatrix has eigenvalues $1 - a_1, \ldots, 1 - a_k$.

Problem 29-8: A Range Equality for Idempotent Matrix

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Suppose that the matrix P of order m satisfies $P^2 = P$. Show that $\text{range}(\, I_m - PP^* \,) = \text{range}(\, 2I_m - P - P^* \,)$, where P^* is the conjugate transpose of P.

Problems 29-9 through 29-13 are on page 35.