The Bulletin of the International Linear Algebra Society

[IL AS] IMAGE [IL AS]

Serving the International Linear Algebra Community

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ILAS President's and Vice President's Annual Report: April 2003

1. The following have been elected to ILAS offices with terms that began on March 1, 2003:

Secretary/Treasurer: Jeff Stuart (2nd three-year term ending February 28, 2006)

Board of Directors: Rafael Bru and Hugo Woerdeman (three-year terms ending February 28, 2006).

The following continue in their offices to which they were previously elected:

President: Daniel Hershkowitz (term ends February 28, 2005).

Vice President: Roger A. Horn (term ends February 29, 2004).

Board of Directors: Ravi Bapat (term ends February 28, 2005), Tom Markham (term ends February 29, 2004), Michael Neumann (term ends February 28, 2005), and Daniel Szyld (term ends February 29, 2004).

2. This fall there will be elections for Vice President (Roger Horn's term as Vice President ends on February 29, 2004) and for two members of the Board of Directors (to replace retiring members Tom Markham and Daniel Szyld). The President has appointed Harm Bart to chair the Nominating Committee. Other members of the committee, as selected by the Board of Directors and by the ILAS Advisory Committee, are LeRoy Beasley, Raphael Loewy, Dale Olesky and Michael Overton.

3. Bryan L. Shader (University of Wyoming, USA) has been appointed to a three-year term (2003–2006) as an Editor-in-Chief of IMAGE: *The Bulletin of the International Linear Algebra Society* (ISSN 1533-8991) and joins Hans Joachim Werner in that position. Bryan replaces George P. H. Styan who concludes after almost 10 years of devoted service. We thank George for his magnificent work that has upgraded IMAGE and raised it to impressive heights. Under his leadership, IMAGE has become a model for many professional newsletters. George's ongoing initiatives have turned IMAGE into a lively and attractive journal.

4. The 10th ILAS Conference took place on June 10–13, 2002, at Auburn University, Alabama, USA. The chair of the organizing committee was Frank Uhlig. There were 152 registered participants. Tsuyoshi Ando (Sapporo, Japan) was awarded the Hans Schneider prize and delivered his Prize Lecture. Michele Benzi (Emory University) and Misha Kilmer (Tufts University) were the SIAM SIAG/LA Speakers. Hans Schneider was the After Dinner Speaker. The conference organizers offered an excursion consisting of a tour of Tuskegee University (Tuskegee, Alabama) and the Carver Museum there. The tour was followed by a trip to the Alabama Shakespeare Festival in Montgomery for a choice of two plays. The conference was preceded by The 6th Workshop on "Numerical Ranges and Numerical Radii".

5. The following ILAS conferences are planned:

(a) The 11th ILAS Conference, Coimbra, Portugal, summer 2004. At this conference Peter Lancaster (University of Calgary, Canada) will be awarded the ILAS Hans Schneider Prize in Linear Algebra and will deliver his Prize Lecture.

(b) The 12th ILAS Conference, Regina, Saskatchewan, Canada, June 26–29, 2005.

(c) The 13th ILAS Conference, Amsterdam, The Netherlands, July 19–22, 2006.

(d) The 14th ILAS Conference, Shanghai, China, July or August 2007.

(e) The 15th ILAS Conference, Cancún, Mexico, June 16–20, 2008.

6. ILAS has recently endorsed these conferences of particular interest to ILAS members:

(a) The 12th International Workshop on Matrices and Statistics (IWMS-2003), August 5–8, 2003, Dortmund, Germany.

(b) International Conference on Matrix Analysis and Applications, December 14–16, 2003, Fort Lauderdale, USA.

(c) The Two-Day Workshop on "Directions in Combinatorial Matrix Theory", Banff International Research Station (BIRS), May 6–8, 2004, Banff, Alberta, Canada.

(d) The 13th International Workshop on Matrices and Statistics (IWMS-2004), August 19–21, 2004, Będlewo, near Poznań, Poland.

(e) The Householder Meeting on Numerical Linear Algebra: Householder Symposium XVI, May 23–27, 2005, Seven Springs Mountain Resort, Campion, Pennsylvania, USA.

7. ILAS has selected Bryan L. Shader and Judi MacDonald as the ILAS Lecturers at the 2003 SIAM SIAG/LA Williamsburg meeting (College of William and Mary, July 15–19, 2003).

8. ILAS has continued to consider requests for the sponsorship of an ILAS Lecturer at a conference which is of substantial interest to ILAS members. ILAS is sponsoring three Lecturers in 2003:

(a) Hans Schneider at the one-day meeting on "Matrix Analysis and Applied Linear Algebra" in celebration of the 60th birthday of Carl Dean Meyer, Jr. The meeting was held in Raleigh, North Carolina, May 15, 2003.

(b) Jerzy K. Baksalary at The 12th International Workshop on Matrices and Statistics (IWMS-2003), Dortmund, Germany, August 5–8, 2003.

(c) Roger A. Horn at the Matrix Analysis and Applications Conference, Nova Southeastern University, Fort Lauderdale, Florida, USA, December 14–16, 2003.

9. *The Electronic Journal of Linear Algebra* (ELA), ISSN 1081-3810: Volume 1, published in 1996, contained 6 papers. Volume 2, published in 1997, contained 2 papers. Volume 3, the Hans Schneider issue, published in 1998, contained 13 papers. Volume 4, published in 1998 as well, contained 5 papers. Volume 5, published in 1999, contained 8 papers. Volume 6, Proceedings of the Eleventh Haifa Matrix Theory Conference, published in 1999 and 2000, contained 8 papers. Volume 7, published in 2000, contained 14 papers. Volume 8, published in 2001, contained 12 papers. Volume 9, published in 2002, contained 24 papers. Volume 10, is being published now; as of June 8, 2003, Volume 10 contains 12 papers.

ELA's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz, in the University of Lisbon, in The European Mathematical Information Service (EMIS) offered by the European Mathematical Society, and in the 36 EMIS Mirror Sites.

A complimentary copy of the CDROM for ELA (vol. 1–8, 1996–2001) was distributed to ILAS members at The 10th ILAS Conference (Auburn University, Alabama, USA, June 10–13, 2002); a complimentary copy of this CDROM is being sent to all other ILAS members with IMAGE 30 (April 2003).

Volumes 1–7 (1996–2000) of ELA are in print, bound as two separate books: vol. 1–4 and 5–7. Copies can be ordered from Jim Weaver: jweaver@uwf.edu

10. ILAS-NET: As of June 9, 2003, we have circulated 1282 ILAS-NET announcements. ILAS-NET currently has 496 subscribers.

11. The primary site of ILAS INFORMATION CENTER (IIC) is in Regina, Saskatchewan, Canada. Mirror sites are located in the Technion, in Temple University, in the University of Chemnitz, and in the University of Lisbon.

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Indexing IMAGE: 1–30 (1988–2003)

We have started to make an index to IMAGE: 1–30 (1988–2003) and we welcome any help readers may wish to offer.

The first issue of IMAGE (Vol. 1, No. 1, 8 pp., January 1988) was edited by Robert C. Thompson and carried the subtitle "The Bulletin of the International Matrix Group serving the International Linear Algebra Community" and announced that the International Matrix Group (IMG) was constituted in Victoria, British Columbia, Canada, May 1987. The second issue (No. 2, 14 pp., January 1989), edited jointly by Jane M. Day & Robert C. Thompson, carried the subtitle "The Bulletin of the International Linear Algebra Society (formerly the International Matrix Group) serving the International Linear Algebra Community" and announced the Inaugural Meeting of the International Linear Algebra Society (ILAS) at Brigham Young University (Provo, Utah, USA, August 12–15, 1989).

The subsequent 28 issues, no. 3–30 (July 1989–April 2003), carry the title "IMAGE: The Bulletin of the International Linear Algebra Society, serving the International Linear Algebra Community"; from no. 26 (April 2001) on, with ISSN 1533-8991.

Issues no. 3–10 (July 1989–January 1993) were edited jointly by Steven J. Leon and R. C. Thompson and no. 11–12 (July 1993–January 1994) by S. J. Leon. These 10 issues appeared twice a year, each with total pagination ranging between 14 and 24 pages. Issues no. 13–18 (July 1994–Winter/Spring 1997) were edited jointly by S. J. Leon & George P. H. Styan; no. 19–24 (Summer/Fall 1997–April 2000) by G. P. H. Styan, and no. 25–30 (October 2000–April 2003) jointly by G. P. H. Styan and Hans Joachim Werner. Issues no. 27–30 each ran 36 pp.

Our preliminary findings in making this index indicate:

- ILAS officers' reports and news items
  - (a) 14 President's/Vice President's & Treasurer's reports
  - (b) 18 ILAS news items
- Articles
  - (a) 23 feature articles
  - (b) 14 obituaries
  - (c) 67 short communications
- Problem Corner
  - (a) 88 problems with solutions
  - (b) 13 problems without solutions
- Conferences and workshops
  - (a) 139 announcements of individual forthcoming events
  - (b) 79 reports on individual events already held
  - (c) 18 lists of forthcoming events
- New and forthcoming books
  - (a) 32 signed book reviews
  - (b) 16 lists of new and forthcoming books
- Photographs and pictures
  - (a) Photographs or pictures of 27 individuals
  - (b) Photographs of 51 groups (at meetings)
  - (c) 5 miscellaneous other photographs
- 14 postage stamps depicting 5 mathematicians.

The obituaries are of Patricia James Eberlein, Dennis Ray Estes, Vlad Ionescu, John Maybee, Bill Larry Neal, Vlastimil Pták, Norman J. Pullman, Arthur Asquith Rayner, Sally Rear, Kermit Sigmon, Richard D. Sinkhorn, Robert Charles Thompson, Olga Taussky Todd, and Albert William Tucker.

The photographs or pictures of individuals are of Luís de Albuquerque, Tsuyoshi Ando, Jerzy K. Baksalary, Patricia James Eberlein, Dennis Ray Estes, Feliks Ruvimovich Gantmakher, Daniel Hershkowitz, Vlad Ionescu, John Stanley Maybee, Pedro Nunes, Graciano de Oliveira, Simo Puntanen, Sally Rear, Hans Schneider, Miriam Schneider, Peter Šemrl, Kermit Sigmon, Alexander Spotswood, William Spottiswoode, George P. H. Styan, Olga Taussky Todd, Hüsein Tevfik Paşa, Robert Charles Thompson, Yongge Tian, Olga Taussky Todd, James R. Weaver, and Hans Joachim Werner.

The images of postage stamps depict Tadeusz Banachiewicz, Charles Dodgson (Lewis Carroll), Sir William Rowan Hamilton, Gottfried Wilhelm von Leibniz, and Takakazu Seki Kôwa.

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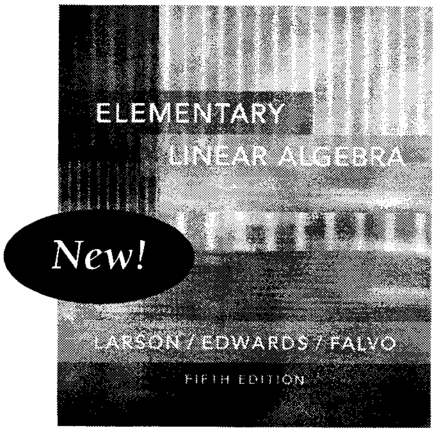
ILAS Treasurer's Annual Report: March 1, 2002–February 28, 2003

Net Account Balances on February 28, 2002			
Vanguard (ST Fed. Bond Fund 1111.153 Shares)			
(72% Schneider Fund and 28% Todd Fund)		11,578.21	
Checking account		69,281.21	**$80,859.42**
General Fund		34,402.18	
Conference Fund		10,038.94	
ILAS/LAA Fund		4,840.00	
Olga Taussky Todd/John Todd Fund		8,267.14	
Frank Uhlig Education Fund		3,475.98	
Hans Schneider Prize Fund		19,835.18	**$80,859.42**
March 1, 2002 through February 28, 2003			
Income: Interest	267.51		
Dues	6140.00		
Corporate Dues	1000.00		
Book Sales	222.00		
General Fund	1068.56		
Conference Fund	480.00		
ILAS\LAA Fund	3000.00		
Taussky-Todd Fund	530.25		
Uhlig Education Fund	210.00		
Schneider Prize Fund	1111.37	14,029.69	
Expenses: IMAGE (2 issue)	3263.49		
Speakers (3)	1800.00		
Schneider Prize	1200.00		
Elsevier UW Madison Refund	2000.00		
Executive Board Travel	1400.00		
ELA Copyedit & CD	1172.00		
Fees	70.00		
Labor - Mailing & Conference	292.00		
Postage	599.10		
Supplies and Copying	180.78		
Bad Checks	360.00	12,337.37	
Net Account Balances on February 28, 2003			
Vanguard (ST Fed. Bond Fund 1165.096 Shares)			
(72% Schneider Fund and 28% Todd Fund)		12,489.83	
Checking account		68,997.91	
Pending checks		940.00	
Pending VISA/Mastercard		2124.00	
Outstanding check to UW Madison		(2,000.00)	**$82,551.74**
General Fund		33,962.88	
Conference Fund		10,518.94	
ILAS/LAA Fund		5,840.00	
Olga Taussky Todd/John Todd Fund		8,797.39	
Frank Uhlig Education Fund		3,685.98	
Hans Schneider Prize Fund		19,746.55	**$82,551.74**

Jeffrey L. STUART, *ILAS Treasurer:* jeffrey.stuart@plu.edu
Pacific Lutheran University, Tacoma, Washington, USA

HOUGHTON MIFFLIN Mathematics 2003

HOUGHTON MIFFLIN MATHEMATICS



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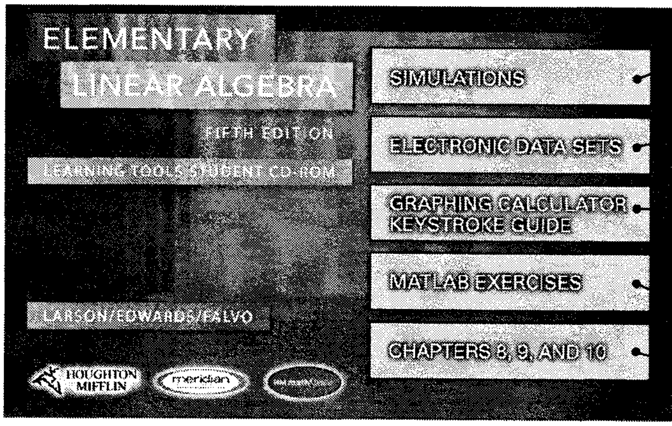

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Up With Determinants!

Garry J. Tee

Determinants were applied in 1683 by the Japanese mathematician Takakasu Seki Kôwa (1642–1708) in the construction of the resolvent of a system of polynomial equations, see Mikami (1913, pp. 191–199; 1977) and the IMAGE Philatelic Corner (IMAGE 23, October 1999, page 8). Determinants were independently invented in 1693 by Gottfried Wilhelm von Leibniz (1646–1716). Sir Thomas Muir (1844–1934) gave a magisterial survey of publications about determinants from 1693–1920, published in five volumes from 1890–1930; see also the recent article by Farebrother, Jensen & Styan (2002), which includes a list of 131 nineteenth-century books on determinants and an extensive biography of Muir[1].

Matrices were first formalized in 1858 by Arthur Cayley (1821–1895), but matrices remained little known until the theoretical physicist Werner Karl Heisenberg (1901–1976) reinvented matrices in 1925 for quantum physics. Most of the work on determinants which was surveyed by Muir makes more sense in terms of matrices than in terms of determinants. In fact Muir himself, at the age of 87 in 1931, wrote that he "welcomed the light matrix proofs in contrast to the heavy footed method of thirty–five years ago" [Turnbull (1934, page 79)]. Sheldon Axler urges, in his 1995 polemic article entitled "Down with determinants!", that linear algebra should be done without determinants. He asserts that "Determinants are needed in one place in the undergraduate mathematics curriculum: the change of variables formula for multi-variable integrals". Accordingly, he defines the determinant of a matrix to be the product of its eigenvalues (counting multiplicities) and then proceeds to "derive the change-of-variables formula for multi-variable integrals in a fashion that makes the appearance of the determinant there seem natural".

I agree with Axler that actual numerical evaluation of the determinant of a matrix is very rarely required. I have written many procedures based on the ALGOL 60 procedures [Wilkinson & Reinsch (1971)] which form the basis of the NAG Library of Mathematical Software. Several of those matrix procedures produce the value of the determinant as a by-product, but I have always deleted that feature from my own versions since I have never required it.

In 1958, when I was a consultant mathematician with the English Electric Company (at Whetstone in England), I found that one of the computing laboratory assistants was spending a great deal of time in punching data onto cards for an engineer. Those data consisted of many square matrices of order 6, each of the form $\mathbf{A} - \lambda\mathbf{I}$ for various values of λ. The engineer told me that he intended to use a subroutine in the DEUCE library (written mostly by James H. Wilkinson and his colleagues) to evaluate the determinant of each of those matrices and that he would then apply inverse interpolation to find those values of λ for which the determinant equals zero! I explained (tactfully) to the engineer that there are better ways of tackling that problem, and referred him to the subroutines in the DEUCE library for computing eigenvalues.

The Necessity of Determinants

In 1963 I attended a conference on Numerical Linear Algebra at the National Physical Laboratory in England. My colleague there, Charles G. Broyden, delivered an impassioned appeal for the elimination of determinants from linear algebra, and declared that he would write a text on matrix computations in which determinants would never be mentioned. But I responded that determinants need to be kept as a small but essential part of linear algebra; it seems to me that any text such as his would require at least half a page of fine print about the theory of determinants. And indeed, the text by Broyden (1977) does contain a 3-page Appendix on determinants.

Only elementary algebra is needed in developing the theory of determinants, and much of it can be understood and used by high-school students. The only part of the standard definition of a determinant [Aitken (1939, page 31)] which high-school students might find difficult is the classification of permutations as even or odd. Indeed, that was not actually proved until circa 1870, and in 1871 James Joseph Sylvester (1814–1897) apparently got highly excited about that advance in the theory.

Axler (1995) uses the language of linear operators T on an n-dimensional complex vector space V; but I prefer to use the alternative language of square ($n \times n$) matrices $\mathbf{A}$, since there are interesting relations between the elements of a matrix and its eigenvectors and eigenvalues. For example, every eigenvalue of $\mathbf{A}$ has modulus less than or equal to any norm of $\mathbf{A}$, and the row-sum and column-sum norms of matrices are easy to compute. And a matrix can be handled numerically directly on a computer, but a linear operator has first to be converted to a matrix representation in some chosen basis before it can be represented on a computer. To a numerical analyst, much of the material in Axler (1995) appears as unnecessarily abstract, since it cannot readily be programmed on a computer.

In the standard definition, det($\mathbf{A}$) is computed from the elements of $\mathbf{A}$ by means of a finite number of multiplications, additions and subtractions. Hence, if all elements of $\mathbf{A}$ are integer (or rational, algebraic, real, complex), then the value of det($\mathbf{A}$) is integer (or rational, algebraic, real, complex).

In a recent text, Hoppensteadt & Peakin (2002) remark (in Appendix A) that "The determinant is defined in a complicated way that we do not present here, but MATLAB can often com-

[1] A Special Issue of *Linear Algebra and its Applications* on "Determinants and the Legacy of Sir Thomas Muir" is in progress with Special Issue Editors: Wayne Barrett, Samad Hedayat, Christian Krattenthaler & Raphael Loewy. *–Ed.*

pute it quickly"! Readers of that text should realise that numerical computation of determinants is very rarely desirable.

Adjugate and Inverse Matrices

In 1750, Gabriel Cramer (1704–1752) of "Cramer's rule" used determinants to prove a major theorem [Muir (1906, pp. 11–14)]. In matrix notation:

$$\mathbf{A}\ \mathrm{adj}(\mathbf{A}) = \mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}, \tag{1}$$

where the elements of the adjugate matrix adj(**A**) are signed determinants of submatrices of **A** of order $n-1$. Therefore, **A** has the (left and right) inverse

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\ \mathrm{adj}(\mathbf{A}), \tag{2}$$

unless det(**A**) = 0 and then **A** is not invertible.

Hence, if all elements of **A** are rational (or algebraic, real, complex etc.) with $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}^{-1}$ has rational (or algebraic, real, complex etc.) elements. If **A** is unimodular, i.e., $\det(\mathbf{A}) = \pm 1$, then $\mathbf{A}^{-1} = \pm\,\mathrm{adj}(\mathbf{A})$; and hence if also all elements of **A** are integers then so are the elements of $\mathbf{A}^{-1}$ (Tee 1972, 1994).

In my opinion, the most important property of determinants is the theorem which follows from (1), that every square matrix **A** is invertible, unless its determinant equals 0 [Axler (1995, Th. 9.1)]. Every number is the determinant of some matrices — for the determinant to have the particular value 0 is a *singular* occurrence, and hence such a matrix is aptly called singular. That theorem was used by Seki in 1683 [Mikami (1913, pp. 191–199; 1977)], but the first published proof that if det(**A**) = 0 then the homogeneous equation $\mathbf{Av} = \mathbf{0}$ has a vector solution $\mathbf{v} \neq \mathbf{0}$ was given in 1851 by William Spottiswoode (1825–1883) [Muir (1911, pp. 54–58)]. Muir described Spottiswoode's Theorem 10, proving that result, as "new but unimportant"! For a genealogy of William Spottiswoode see Farebrother (1999) and for a genealogy of the Spottiswoode family see Farebrother & Styan (2000).

The explicit expression (2) for $\mathbf{A}^{-1}$ is useful for the theory of matrices, but it is not an efficient method for computing the inverse of **A** for large n. Moreover, matrix inverses should very rarely be actually computed. Matrix expressions involving inverses can be computed (in rounded arithmetic) more efficiently by various other algorithms; e.g., the Schur complement $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ can be evaluated efficiently by Aitken's algorithm [Fox (1964, pp. 75–78)].

An important class of matrices is that of alternant matrices [Aitken (1939, p. 42)], where

$$a_{i,j} = \mu_i^{j-1} \quad (1 \leq i, j \leq n). \tag{3}$$

An important property is that the alternant matrix **A** is singular if and only if two or more of the μ_i are equal. That can be proved using polynomial factorization, but the standard textbook proof using the formula

$$\det(\mathbf{A}) = \prod_{i>j}(\mu_i - \mu_j)$$

is simpler. We note that alternant matrices have been called Vandermonde matrices. There is nothing corresponding to alternant matrices in the important 1771 paper by Alexandre-Théophile Vandermonde (1735–1796) on determinants [Muir (1906, pp. 17–24 & 306)]. But Abraham de Moivre (1667–1754) published the inverse of a general alternant matrix in 1738, and hence alternant matrices could well be called de Moivre matrices [Tee (1993, pp. 89–90)].

From the definition of determinant, it is easy to prove that $\det(\mathbf{A}^T) = \det(\mathbf{A})$, from which it readily follows that the row rank of every (rectangular) matrix equals its column rank. Can that important theorem be proved as simply without determinants?

Eigenvectors and Eigenvalues

The problem of a nonzero vector **v** being an eigenvector of the $n \times n$ matrix **A** (with its associated eigenvalue λ)

$$\mathbf{Av} = \lambda\mathbf{v}, \tag{4}$$

reduces (by Spottiswoode's Theorem) to the characteristic polynomial equation for the eigenvalue λ

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0; \tag{5}$$

and thus the problem of the existence of an eigenvector is equivalent to the Fundamental Theorem of Algebra. That theorem is a rather deep theorem of analysis, and hence no simpler proof of the existence of eigenvectors and eigenvalues can be expected. Thus, if **A** has complex (or real) elements then it has exactly n complex eigenvalues, counting multiplicities.

The characteristic polynomial could actually be constructed in terms of matrix elements, from the definition of the determinant. For a matrix **A** of order n, denote the characteristic polynomial

$$\begin{aligned} P(\lambda) &\stackrel{\text{def}}{=} \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= (-1)^n(\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \cdots - c_n). \end{aligned} \tag{6}$$

From this definition, it is clear that the coefficients of the characteristic polynomial are composed from elements of the matrix by multiplications, additions and subtractions. Hence, if the elements of the matrix are integers, then so are the coefficients of the monic characteristic polynomial, scaled by $(-1)^n$; and similarly if the elements of the matrix are rational, algebraic, real or complex.

A standard theorem, based on the determinantal definition (6), gives the coefficients of the characteristic polynomial as the sums of determinants of principal minors of $\mathbf{A}$, with the simplest instances being that:

$$c_1 = -\text{trace}(\mathbf{A}), \quad c_n = (-1)^{n-1}\det(\mathbf{A}). \tag{7}$$

It follows immediately, from Vieta's Relations [François Vieta, Seigneur de La Bigottière (1540–1603)] for the characteristic polynomial, that the sum and the product of eigenvalues are trace(**A**) and det(**A**), respectively. In this manner, all symmetric functions of the eigenvalues can be expressed (through the coefficients of the characteristic polynomial) in terms of elements of **A** [Tee (1994)]. For example, the simplest algorithm for constructing the characteristic polynomial of **A** is Le Verrier's 1840 algorithm [Urbain Jean Joseph Le Verrier (1811–1877)], see, e.g., Faddeev & Faddeeva (1963), which is based on the well-known result that

$$\text{trace}(\mathbf{A}^k) = \sum_{i=1}^{n} \lambda_i^k, \tag{8}$$

which depends upon the determinantal relation (7).

Let λ be an eigenvalue of **A**, satisfying the determinantal equation (5). Define

$$\mathbf{B} = \text{adj}(\mathbf{A} - \lambda\mathbf{I}), \tag{9}$$

so that it follows from (4) and (5) that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{B} = \mathbf{0}. \tag{10}$$

Thus, every nonzero column $\mathbf{v}$ of $\mathbf{B}$ is an eigenvector of **A**, with eigenvalue λ. This method for constructing eigenvectors will fail only when $\mathbf{B} = \mathbf{0}$, i.e., when $\mathbf{A} - \lambda\mathbf{I}$ has rank less than $n - 1$ (i.e., nullity greater than 1); and that can happen only when λ is a multiple eigenvalue of **A** which occurs in more than one Jordan box in the Jordan canonical form of **A**.

This method is sometimes useful for giving an explicit expression for an eigenvector $\mathbf{v}$ with eigenvalue λ, although it is not efficient for large n.

Axler (1995) defines eigenvalues thus: "A complex number λ is called an eigenvalue of the linear operator T on V if $T - \lambda I$ is not injective". And in his Theorem 2.1, he purports to prove that: *Every linear operator on a finite-dimensional complex vector space has an eigenvalue.* Axler's proof is as follows: To show that T has an eigenvalue, fix any nonzero vector $v \in V$. The vectors $v,\ Tv,\ T^2v, \ldots,\ T^n v$ cannot be linearly independent, because V has dimension n and we have $n + 1$ vectors. Thus there exist complex numbers $a_0, \cdots, a_n$, not all 0, such that

$$a_0 v + a_1 Tv + \cdots + a_{n-1}T^{n-1}v + a_n T^n v = 0.$$

Make the a's the coefficients of a polynomial, which can be written in factored form as

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - r_1)\cdots(z - r_m),$$

where c is a nonzero complex number, each r_j is complex, and the equation holds for all complex z. We then have

$$\begin{aligned} 0 &= (a_0 I + a_1 T + \cdots + a_{n-1}T^{n-1} + a_n T^n)v \\ &= c(T - r_1 I)\cdots(T - r_m I)v, \end{aligned}$$

which means that $T - r_j I$ is not injective for at least one j. In other words, T has an eigenvalue.

Axler confidently declares (p. 154) that "The simple proof of the existence of eigenvalues given in [his] Theorem 2.1 [above] should be the one imprinted on our minds, written on our blackboards, and published in our textbooks". That proof may seem simple to him — but it uses mathematical concepts which are *much* more complicated than the high-school algebra used in the standard proof (5) with determinants. His definition of eigenvalue is far from simple, his proof does not give any construction of the characteristic polynomial, and it is more difficult to comprehend than the standard proof.

Moreover, it seems to be wrong!

The coefficient c equals a_n, which can be zero. For, let $\mathbf{v}$ be an eigenvector of **A** (which does exist, by the standard determinantal proof (5) above) with eigenvalue λ, as in (4). Then Axler's linear dependence holds with coefficients

$$a_0 = -\lambda,\ a_1 = 1,\ a_2 = \cdots = a_n = 0, \tag{11}$$

and in particular $c = a_n = 0$, unless $n = 1$. [We note that Axler (twice) writes r_m instead of r_n — but since he says nothing about m, it can only be regarded as a misprint.]

Axler defines a vector $v \in V$ as "an *eigenvector* of T if $Tv = \lambda v$ for some eigenvalue λ", and in Proposition 2.2 he speaks of "nonzero eigenvectors". But the standard definition of eigenvector for a square matrix **A** of order n is that it is a *nonzero* vector $\mathbf{v}$ such that (4) holds for some scalar λ. If $\mathbf{v} = \mathbf{0}$ were accepted as an eigenvector, then $\mathbf{v} = \mathbf{0}$ would be an eigenvector of every matrix of order n, and every scalar would be an eigenvalue for $\mathbf{v} = \mathbf{0}$.

I feel that Axler's definition of the multiplicity of an eigenvalue is more complicated than the standard definition in terms of the linear factorization of the characteristic polynomial, which is defined as $\det(\mathbf{A} - \lambda\mathbf{I})$. If R is any rational function, then $\mathbf{v}$ is an eigenvector of $R(\mathbf{A})$ with eigenvalue $R(\lambda)$, whose multiplicity is determined from the linear factorization of the characteristic polynomial of **A**. Is there any simple way of doing that without determinants?

One practical way of solving a polynomial equation $q(z) = 0$ is to construct the companion matrix $\mathbf{Q}$ of q and then compute its eigenvalues. But the proof that the characteristic polynomial of $\mathbf{Q}$ is q consists of expanding $\det(\mathbf{Q} - z\mathbf{I})$ by its last row and obtaining $q(z)$.

How would one show, in Axler's version, that every eigenvalue r has at least one eigenvector $\mathbf{v}$ (and hence the eigenspace has dimension at least 1)? How would one relate the coefficients of the characteristic polynomial to the elements of the matrix, as above, without determinants? How could one construct symmetric functions of the eigenvalues, in terms of elements of the matrix, without determinants? It might be possible to do these things without determinants, but I cannot see how that could be simpler than the standard approach using determinants.

The eigenvalues of $\mathbf{A}$ are continuous functions of the coefficients of the characteristic polynomial, which are continuous functions of the elements of $\mathbf{A}$; and hence the eigenvalues are continuous functions of the elements of $\mathbf{A}$. That continuity is important in perturbation analysis, including round-off analysis; and it is required for proving Gerschgorin's very important 1931 theorem [Semeon Aranovich Gerschgorin (1901–1933)] that the union of k Gerschgorin disks of $\mathbf{A}$ (disjoint from the other $n-k$ disks) contains exactly k eigenvalues of $\mathbf{A}$ (counting multiplicities). With the determinant definition of characteristic polynomial, the continuity of the eigenvalues (as $\mathbf{A}$ is perturbed) follows from first–year analysis. Can it be proved as simply without determinants?

Eigenvalues (and eigenvectors) of real symmetric $\mathbf{A}$ are best computed by first applying Householder's similarity transformation to convert $\mathbf{A}$ to symmetric tridiagonal form:

$$\mathbf{T} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & & \\ & \beta_2 & \alpha_3 & \beta_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \beta_{n-3} & \alpha_{n-2} & \beta_{n-2} & \\ & & & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

Without loss of generality we can take each $\beta_j \neq 0$; for if any $\beta_j = 0$ then $\mathbf{T}$ splits (after row and column j) into a direct sum of tridiagonal submatrices, and the eigenvalues of each of those submatrices can be found independently of the others. Expanding by row j the determinant of the submatrix of $\mathbf{T} - x\mathbf{I}$ consisting of rows and columns 1 to j, to obtain the characteristic polynomial

$$p_j(x) = (\alpha_j - x)p_{j-1}(x) - \beta_{j-1}^2 p_{j-2}(x); \quad j = 2, \ldots, n,$$

where $p_0(x) = 1$, $p_1(x) = \alpha_1 - x$, and $p_n(x)$ is the characteristic polynomial of $\mathbf{T}$. This 3-term recurrence relation shows that the sequence $p_0(x), p_1(x), \cdots, p_n(x)$ is a Sturm sequence, whose sign-changes give the number of eigenvalues less than x. Hence, the eigenvalues of $\mathbf{T}$ can be found by a bisection method, and this is a reliable and practically useful method for computing eigenvalues (and eigenvectors) of a real symmetric matrix $\mathbf{A}$. Could this be proved readily without using determinants?

Polynomial Resultants

The resultant of 2 or more polynomials (which equals 0 if and only if the polynomials have a common zero) is most simply represented as a determinant. For example, the two polynomials

$$\begin{aligned} p(x) &= ax^3 + bx^2 + cx + d, \\ q(x) &= ex^4 + fx^3 + gx^2 + hx + i, \end{aligned}$$

have a common zero if and only if the resultant $R(p, q) = 0$, where $R(p, q)$ is the determinant of the matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & a & b & c & d & 0 \\ 0 & a & b & c & d & 0 & 0 \\ a & b & c & d & 0 & 0 & 0 \\ 0 & 0 & e & f & g & h & i \\ 0 & e & f & g & h & i & 0 \\ e & f & g & h & i & 0 & 0 \end{pmatrix}.$$

The matrix $\mathbf{R}$ displays a clearly comprehensible pattern; whereas the expanded form of the determinant $R(p, q) = \det(\mathbf{R})$ has hundreds of terms, with no such clear pattern. The alternative definition of $R(p, q)$ as the product of the squares of the differences between zeros of p and of q gives no indication of the nature of the coefficients of the expanded form of $R(p, q)$, but the determinant definition shows immediately that if the coefficients of p and q are integers (or real, etc.), then so are the coefficients of $R(p, q)$.

The Cayley–Hamilton Theorem

Let P denote the characteristic polynomial of the square matrix $\mathbf{A}$. Then the well-known Cayley–Hamilton Theorem is that $P(\mathbf{A}) = \mathbf{0}$. Axler's proof uses a lengthy sequence of theorems on linear operators, which many undergraduates would find quite difficult. But the Cayley–Hamilton Theorem can be proved quite simply with determinants, see, e.g., Faddeeva (1959, pp. 154–155). We define

$$\mathbf{B} \stackrel{\text{def}}{=} \operatorname{adj}(\mathbf{A} - \lambda\mathbf{I}), \tag{12}$$

and so each element of $\mathbf{B}$ is the signed determinant of a submatrix (order $n-1$) of $\mathbf{A} - \lambda\mathbf{I}$, and hence is a polynomial in λ of degree $n-1$ or less. And so the matrix $\mathbf{B}$ can be written in the form

$$\mathbf{B} = \mathbf{B}_{n-1} + \mathbf{B}_{n-2}\lambda + \cdots + \mathbf{B}_0\lambda^{n-1}, \tag{13}$$

where the matrices $\mathbf{B}_{n-1}, \cdots, \mathbf{B}_0$ are independent of λ. Then, from (1),

$$
\begin{aligned}
&(\mathbf{B}_{n-1} + \mathbf{B}_{n-2}\lambda + \cdots + \mathbf{B}_0\lambda^{n-1})(\mathbf{A} - \lambda\mathbf{I}) \\
&= \mathbf{B}(\mathbf{A} - \lambda\mathbf{I})\det(\mathbf{A} - \lambda\mathbf{I})\mathbf{I} \\
&= (-1)^n(\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \cdots - c_n)\mathbf{I}.
\end{aligned}
$$

Equating the (matrix) coefficients of λ on left and right (which could be done, element by element), we get a system of $n+1$ matrix equations:

$$
\begin{aligned}
-\mathbf{B}_0 &= (-1)^n\mathbf{I}, \\
\mathbf{B}_0\mathbf{A} - \mathbf{B}_1 &= (-1)^{n+1}c_1\mathbf{I}, \\
\cdots \quad &\cdots \quad \cdots \\
\mathbf{B}_{n-2}\mathbf{A} - \mathbf{B}_{n-1} &= (-1)^{n+1}c_{n-1}\mathbf{I}, \\
\mathbf{B}_{n-1}\mathbf{A} &= (-1)^{n+1}c_n\mathbf{I}\,.
\end{aligned}
$$

Premultiplying these equations by $\mathbf{A}^{n-1}$, $\mathbf{A}^{n-2}, \cdots$, $\mathbf{A}$, $\mathbf{I}$ and adding, we get the matrix polynomial equation:

$$
\mathbf{0} = (-1)^n(\mathbf{A}^n - c_1\mathbf{A}^{n-1} - c_2\mathbf{A}^{n-2} - \ldots - c_n\mathbf{I}) = P(\mathbf{A}),
$$

where P is the characteristic polynomial of $\mathbf{A}$, as defined by (6).

This determinantal proof of the Cayley–Hamilton Theorem is much simpler than Axler's proof of Theorem 5.2 — it uses only elementary algebra and does not even require the Fundamental Theorem of Algebra. Axler develops the theory of generalized eigenvectors, minimal polynomial, Jordan canonical form and orthonormal bases, to show that linear algebra can be done without determinants. But various texts, including Faddeeva (1959) and Faddeev & Faddeeva (1963), use determinants for establishing basic results about inversion, singularity, characteristic polynomial, eigenvectors, eigenvalues and the Cayley–Hamilton Theorem as above and thereafter develop linear algebra with little or no subsequent explicit use of determinants.

Even though Axler acknowledges that determinants have their uses in mathematics at the research level, he concludes his paper with the slogan "Down with Determinants!". But we have shown here that several significant parts of undergraduate mathematics do indeed require the use of determinants. Hence we say: *Up with Determinants!*

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Gottfried Wilhelm von Leibniz: 1646–1716

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As Tee (2003) noted in his article "Up with determinants!" in this issue of IMAGE, "Determinants were applied in 1683 by the Japanese mathematician Takakasu Seki Kôwa (1642–1708) in the construction of the resolvent of a system of polynomial equations [but] were independently invented in 1693 by Gottfried Wilhelm von Leibniz (1646–1716)".

In Smith (1929, pp. 267–270) there appear English translations (from the French and Latin by Thomas Freeman Cope) of two extracts of writings by Leibniz on determinants. The first extract, which contains the system of equations below, is from a letter by Leibniz to Guillaume François Antoine Marquis de L'Hôpital (1661–1704), dated 28 April 1693, and published for the first time in Gerhardt (1850, pp. 238–240; see also pp. 229 & 245). The second extract is from a manuscript on eliminating unknowns, published for the first time in Gerhardt (1863, pp. 5–6) and which "bears no date, but it was probably written before 1693 and possibly goes back to 1678". See also Muir (1890, pp. 6–10).

In the 1693 letter by Leibniz to L'Hôpital, Leibniz explained that the equations

$$\begin{aligned} 10 + 11x + 12y &= 0 \\ 20 + 21x + 22y &= 0 \\ 30 + 31x + 32y &= 0 \end{aligned}$$

have a solution because

$$\begin{aligned} &10 \times 21 \times 32 + 11 \times 22 \times 30 + 12 \times 20 \times 31 \\ &\quad = 10 \times 22 \times 31 + 11 \times 20 \times 32 + 12 \times 21 \times 30, \end{aligned}$$

which is exactly the condition that the coefficient matrix has determinant 0.

Leibniz here denoted general numbers by double subscripts. In modern notation, his equations may be rewritten as the vector equation $\mathbf{Gz} = \mathbf{0}$ with a square matrix $\mathbf{G}$, where

$$\mathbf{G} = \begin{pmatrix} g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \\ g_{30} & g_{31} & g_{32} \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\mathbf{z} \neq \mathbf{0}$, it follows at once that the determinant $\det(\mathbf{G}) = 0$, which is equivalent to Leibniz's condition above.

Leibniz was born in Leipzig on 1 July 1646. His father, Friedrich Leibniz, was a professor of moral philosophy at the Universität Leipzig; his mother, Catharina Schmuck, was the daughter of a lawyer and Friedrich's third wife. However, Friedrich Leibniz died when Leibniz was only six years old and he was brought up by his mother, who died when Leibniz was 17.

At the age of 7, Leibniz entered the Nicolai School in Leipzig, and at the age of 14, he entered the Universität Leipzig. He studied philosophy, which was well taught there, and mathematics, which was very poorly taught. Among the other topics included in his two-year general degree course were rhetoric, Latin, Greek and Hebrew. He graduated with a Bachelor's degree in 1663 and then "a Master's degree in philosophy for a dissertation in which he combined aspects of philosophy and law; he studied relations in these subjects with mathematical ideas. A few days after Leibniz presented his dissertation, his mother died" (O'Connor & Robertson 1998).

Leibniz worked on his habilitation in philosophy to be published in 1666 as "Dissertatio de Arte Combinatoria" (Dissertation on the Combinatorial Art). According to O'Connor & Robertson (1998) "In this work Leibniz aimed to reduce all reasoning and discovery to a combination of basic elements such as numbers, letters, sounds and colours. Despite his growing reputation and acknowledged scholarship, Leibniz was refused the doctorate in law at Leipzig." And so Leibniz went to the University of Altdorf, receiving a doctorate in law in February 1667 for his dissertation "De Casibus Perplexis" (On Perplexing Cases).

One of Leibniz's lifelong aims was to collate all human knowledge. MacDonald Ross (1984) noted that "Although Leibniz's interests were clearly developing in a scientific direction, he still hankered after a literary career. All his life he prided himself on his poetry (mostly Latin), and boasted that he could recite the bulk of Virgil's *Aeneid* by heart." In *The Cambridge Biographical Encyclopedia*, Crystal (1994) observes that "Leibniz was a man of remarkable breadth of knowledge and made original contributions to optics, mechanics, statistics, logic, and probability theory. He conceived the idea of calculating machines and of a universal language. He wrote on history, law, and political theory."

Leibniz went to Paris on a diplomatic mission in the autumn of 1672 and studied mathematics and physics under Christiaan Huygens (1629–1695) there, see Hofmann (1978, page 12). In Paris, Leibniz developed the basic features of his version of the calculus. The Royal Society (of London) elected Leibniz a Fellow on 19 April 1673. In October 1675, in the middle of a paper about double integration, "Leibniz replaces the abbreviation omn. by the sign $\int$ (a 'long s', the initial letter of the word summa whose place it takes), at first writing $\int y$ where we would set $\int_0^x y.dx$: all integrals are understood to be definite, but no special notation for the limits is used. It is particularly noted by Leibniz that the operation $\int$ raises the dimension by one degree. Where $\int y = z$, he puts, conversely,

$$y = \frac{z}{d}$$

(Hofmann, page 192). The notation $f(x)$ only came into use around the end of the 18th century.

By autumn 1676 Leibniz discovered the familiar $d(x^n) = nx^{n-1}dx$ for both integral and fractional n. "In 1684 Leibniz published details of his differential calculus in "Nova Methodus pro Maximis et Minimis, itemque Tangentibus ..." in *Acta Eruditorum*, a journal established in Leipzig in 1682. The paper contained the familiar d notation, the rules for computing the derivatives of powers, products and quotients. In 1686 Leibniz published, also in *Acta Eruditorum*, a paper dealing with the integral calculus with the first appearance in print of the $\int$ notation" (O'Connor & Robertson 1998). For English translations (from the Latin by Evelyn Walker) of Leibniz's first publications on calculus, see Smith (1929, pp. 619–626); see also Struik (1969, ch. V, pp. 270–284).

According to Schaaf (1978, pp. 65–66), Sir Isaac Newton (1643–1727) and Leibniz developed the calculus independently but "Newton unhappily devised a rather clumsy notation". Both men "were seeking general methods of finding maximum and minimum values of a curve" and were "superb mathematicians".

The extensive survey by Sir Thomas Muir of publications about determinants starts with the item headed "Leibnitz (1693)". Muir uses the spelling "Leibnitz" but the spelling "Leibniz" appears on all eight postage stamps that we have found in his honour. For publications about him, we searched OCLC First Search (WorldCat) on 31 May 2003 to find 1781 entries with "Leibniz" in the title and 296 with "Leibnitz" in the title. We believe that "Leibnitz" is a British spelling, but according to Mates (1986, page 17), "For several generations before Leibniz's father, the family spelled its name Leubnitz. Leibniz's father Friedrich and Leibniz's half brother Johann Friedrich used Leibnütz, Leibnüz, and Leibnitz. Leibniz himself used Leibnütz until his mother died (in 1673), then for a time Leibnüz, and after 1671 Leibniz. Correspondingly he shifted the Latin form from Leibnuzius and Leibnuezius to Leibnitius. Etymologically the name probably derives from the Slavic 'Lipnice', which refers to a certain kind of grass that grows in river bottoms; variants on this appear as names of rivers and places all over Eastern Europe." Leibniz is sometimes called "Gottfried Wilhelm *Freiherr* von Leibniz" but although he occasionally employed this title himself, he was never officially raised to the peerage (Mates, 1986, page 17).

Illustrated here are eight postage stamps issued in honour of Leibniz. The stamps are arranged clockwise starting with the oldest top left. Technical details are given in the table below. Ten colour jpeg images of all eight stamps are available on Jeff Miller's "Images of Mathematicians on Postage Stamps" Web site http://jeff560.tripod.com/ Colour prints of the St. Vincent stamp [6] and the 1996 German stamp [8] are in the new book by Wilson (2001, page 59), and colour prints of [1, 3, 4] are in the book by Schaaf (1978, page 66).

	year	country	face value	series or anniversary	colour	Scott catalogue	Stanley Gibbons catalogue
[1]	1926	Germany [Deutsches Reich]	40 pfennig	Portraits of famous Germans	violet	360	410
[2]	1950	German Democratic Republic	24 pfennig	250th anniversary of the German Academy of Sciences in Berlin	red	66	E28
[3]	1966	Federal Republic of Germany	30 pfennig	250th death anniversary	black & mauve	962	1423
[4]	1966	Romania [Romana]	1.35 lei	Portraits: cultural anniversaries	olive, black & blue	1855	3387
[5]	1980	Federal Republic of Germany	60 pfennig	Europa	multicoloured	1329	1928
[6]	1991	St. Vincent	$2	Anniversaries and events	multicoloured	1557	1758
[7]	1996	Albania [Shqiperia]	10 leke	Famous philosophers & mathematicians	multicoloured	2515	2638
[8]	1996	Federal Republic of Germany	100 pfennig	350th birth anniversary	red & black	1933	2719

In Paris in 1672, Leibniz had examined specimens of Pascal's adding machine (of 1642), and he designed a much more powerful calculating machine to perform addition, subtraction, multiplication and division; see "Leibniz: on his calculating machine" translated from the Latin by Mark Kormes in Smith (1929, pp. 173–181: a picture of the machine is on page 173). The major feature was the moveable accumulator, such that, with a positive integer x in the setting register, one forward turn of the handle would add x into the accumulator and one backward turn of the handle would subtract x from the accumulator, when the accumulator was in its standard position. Thus, n turns of the handle would add nx into the accumulator, where n is any integer. But the accumulator could be shifted past the setting register, so that the unit digit of the setting register added into the tens digit of the accumulator, and then one turn of the handle added $10x$ into the accumulator. And with k shifts, each turn of the handle added $10^k x$ into the accumulator. Thus, to multiply x by a p-digit positive integer y, for each of the p digits of y the handle is turned at most 9 times, followed by a further shift of the accumulator. And similarly for division by y, with the handle being turned backward.

When Leibniz first visited London in 1673 he brought with him the first version of his calculating machine. That prototype did not operate reliably, but the Royal Society very promptly elected him as a Fellow. For the rest of his life, Leibniz employed the most skilled clockmakers in Europe to make successive versions of his calculating machines. But the engineering problems of "transmitting carry through successive digits"[2] could not then be overcome, and no reliably operating Leibniz calculator was made in his lifetime. By 1877, engineering technology had been advanced to the extent that the first reliable Leibniz calculator was made by Willgodt Theophil Odhner (1845–1905), a Swedish engineer working in Russia, see Maĭstrov & Sokolov (1981). Odhner's factory in Sankt-Peterburg (Petrograd, Leningrad) manufactured Odhner calculators from 1886 to about 1982, and in 1892 he licensed a German firm to manufacture them in Braunschweig, see the chapter entitled "Brunsviga Calculating Machine" apparently written by the manufacturer Grimme, Natalis & Co. Ltd. (at Braunschweig), in Horsburgh (1914, pp. 84–91). Those Odhner and Brunsviga calculators, based on Leibniz's design, were the machines most used for scientific computing until about 1957.

Leibniz might have liked to remain in Paris at the Academy of Sciences, but apparently further invitations to "foreigners" were not forthcoming. And so Leibniz accepted the position from the Duke of Hannover, Johann Friedrich, of Head Librarian and of Court Councillor in Hannover. Interestingly, the 1991 stamp [6] from St. Vincent has the inscription: "Gottfried Wilhelm Leibniz, Head librarian for the electors of Hannover (& co-inventor of the calculus): 750th Anniversary of Hannover".

From December 1676 until his death in 1716, Leibniz lived in Hannover although he travelled frequently. His duties at Hannover "... as librarian were onerous, but fairly mundane: general administration, purchase of new books and second-hand libraries, and conventional cataloguing" (MacDonald Ross 1984). In 1700 Leibniz founded the "Deutsche Akademie der Wissenschaften zu Berlin" and became its first President; to mark its 250th anniversary in 1950, the German Democratic Republic issued a postage stamp [2] depicting Leibniz.

According to Aiton (1985, page ix) "the townspeople of Hannover ... erected a circular temple with a bust in white marble and the simple inscription 'Genio Leibnitii'. The Leibniztempel was established by Hofrat Johann Daniel Ramberg in the Waterlooplatz in 1790 but was moved in 1934 to the Georgengarten in Hannover's Nordstadt; the Georgengarten is one of four gardens in the Herrenhäuser Gärten, which were laid out in the 17th century. A picture of the Leibniztempel appears in the background of the St. Vincent stamp [6] and on the Web site www.nordstadt-online.de/info/sights/leibnitz.htm where it is noted that the dome is supported by twelve Ionian columns. The bust of Leibniz was created by the Irish sculptor Christopher Hewetson (1739–1798) from Carrara marble and is now in the Leibnizhaus in the Holzmarkt; the Leibnizhaus is the Conference Centre and Guest Residence for Visiting Scientists of the Universities and Academies in Hannover.

In the *Encyclopaedia Britannica*, Leibniz is described as "A man of medium height with a stoop, broad-shouldered but bandy-legged, as capable of thinking for several days sitting in the same chair as of travelling the roads of Europe summer and winter. He was an indefatigable worker, a universal letter writer (he had more than 600 correspondents), a patriot and cosmopolitan, a great scientist, and one of the most powerful spirits of Western civilisation." Mates (1986, page 33) notes that "Leibniz remained a bachelor all his life. Once, in his 50th year, he was eager to get married. But the intended person asked for time to think it over, and meanwhile he lost the inclination. He sometimes said that he had always thought there was plenty of time, but one day he realized that now it was too late."

Acknowledgements. Much of the information about Leibniz is extracted from the excellent article on the Web by O'Connor & Robertson (1998), which includes 229 references and 8 jpeg images of Leibniz; see also Aiton (1985, page facing the title page), and Smith (1929, page facing page 619). We are also very grateful to Götz Trenkler for introducing us to, and for providing us with a copy of, the 1996 German stamp [8]. Many thanks to Monty Strauss for supplying us with a jpeg image of the stamps [5] and [6] and to Jeff Miller for allowing us to reproduce his jpeg image of the stamp [2].

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[2] "Transmitting carry through successive digits" is a standard phrase in accounts of calculating machinery. It is simple to design a machine which will add (say) 26+54 to give 80. But, engineering had to be developed over 2 centuries (from Leibniz's first calculator) before general integers (of 10 or more digits) could be mechanically added reliably, rapidly and repeatedly. Slowly turn the handle of an Odhner calculator as it adds 1 to 9999999999999999, and we see the carry being transmitted, converting each successive 9 to 0 with carry to the next 9.

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Defining the Determinant

What is your favorite definition of determinant? The customary one springs like Athena from the head of Zeus with no motivation visible till later. — Go the volume route? Define as a multiplicative homomorphism to the base field? Product of eigenvalues? Bi-linear map? ...? Ken Ireland, a colleague now deceased, told me Gauss said something like: First find a proof; then find the right proof. — I am still looking for the right definition of determinants.

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Another Matrix Representation of Quaternions

Let a, b, c, d be real numbers and let $i = \sqrt{-1}$. Then it is well known that the complex numbers $a + ib$ and $c + id$ may be represented by the 2×2 matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & d \\ -d & c \end{pmatrix},$$

respectively. Whence we may deduce that the second representation in Farebrother (2002) of a typical quaternion $q = a + bh + cj + dk$ by the 4×4 real matrix

$$P = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

may be replaced by the 2×2 complex matrix

$$Q = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}.$$

That is, as $P = aI + bL + cM + dN$ where I is the 2×2 identity matrix and

$$L = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Clearly these 2×2 matrices satisfy the Hamiltonian conditions

$$L^2 = m^2 = N^2 = -I,$$

$$LM = N = -ML, \; MN = L = -NM, \; NL = M = -LN.$$

A similar representation of the first expression in Farebrother (2002) may be achieved by interchanging the third and fourth rows and the third and fourth columns of the 4×4 matrix P.

Finally, as Hawkins (1972, page 245) has pointed out, this complex representation of Hamilton's quaternions was suggested by Cayley (1858, page 31), although he did not specify the matrices involved; for related work on the origins of group theory, see Hawkins (1971).

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Forthcoming Conferences and Workshops in Linear Algebra

Fields Institute Special Session on Matrices and Statistics

Halifax, Nova Scotia: 10 June 2003

The 2003 annual meeting of the Statistical Society of Canada was held at Dalhousie University in Halifax, Nova Scotia, Canada, June 8–11, 2003. A Special Session on Matrices and Statistics, sponsored by The Institute for Research in Mathematical Sciences (Toronto) and organized by George P. H. Styan, was held on June 10, 2003, and featured three invited speakers:

Jerzy K. Baksalary (Zielona Góra, Poland): A revisitation of formulae for the Moore–Penrose inverse of modified matrices (joint work with Oskar Maria Baksalary & Götz Trenkler),

Simo Puntanen (Tampere, Finland): Matrix tricks for teaching linear statistical models—Our personal Top Ten (joint work with George P. H. Styan), and

Hans Joachim Werner (Bonn, Germany): In the Year of the Matrix: Prediction techniques in the general Gauß–Markov model.

12th International Workshop on Matrices and Statistics

Dortmund, Germany: 5–8 August 2003

The 12th International Workshop on Matrices and Statistics (IWMS-2003) will be held at the Universität Dortmund (Dortmund, Germany), 5–8 August 2003, during the week immediately before the 54th Biennial Session of the International Statistical Institute (ISI) in Berlin. Dortmund is a city of over half a million inhabitants in the Ruhr Valley; it is an ancient walled city first mentioned *c.* 885 AD as "Throtmannia"; in the 12th century it became "Tremonia" as a member of the Hanseatic League, and later ""Trutmunia", "Trutmenni", and "Dorpmunde". The nearest major airport is Düsseldorf (DUS), with a direct train connection from DUS to Dortmund of about 45 minutes.

This Workshop, which will be hosted by the Department of Statistics at the Universität Dortmund, is cosponsored by the Bernoulli Society as an ISI satellite meeting, and is endorsed by the International Linear Algebra Society (ILAS).

Jerzy K. Baksalary (Zielona Góra, Poland) will be the ILAS Lecturer. Other invited speakers include Adi Ben-Israel, Narasanga Rao Chaganty, Ludwig Elsner, Bjarne Kjær Ersbøll, R. William Farebrother, Patrick Groenen, Stephen Pollock, Júlia Volaufová, and Roman Zmyślony.

A special series of invited lectures in celebration of Götz Trenkler's 60th birthday will be held in the afternoon of Monday, 4 August 2003; those invited include Jerzy K. Baksalary, Herbert Büning, Iris Pigeot, Bernhard Schipp, Peter Stahlecker, and George P. H. Styan.

The International Organizing Committee for this Workshop comprises R. William Farebrother, Simo Puntanen, George P. H. Styan (vice-chair), and Hans Joachim Werner (chair). The Local Organizing Committee at the University of Dortmund consists of Jürgen Groß, Götz Trenkler (chair), and Claus Weihs. The Workshop Secretary is Eva Brune: iwms2003@statistik.uni-dortmund.de For up-to-date information on this Workshop please visit the Web site www.statistik.uni-dortmund.de/IWMS/main.html

This Workshop will include the presentation of both invited and contributed papers on matrices and statistics. We also plan to have a special session for papers presented by graduate students as well as a session of lectures for students. It is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and Its Applications.*

On Wednesday, 6 August 2003, there will be an excursion to the Mining Museum Bochum, followed in the evening by the Workshop Dinner at Hövels Brauhaus in downtown Dortmund.

From left to right: Jochen Werner, Jerzy Baksalary, George Styan, Yongge Tian, and Simo Puntanen, putting the final touches on IMAGE 30 *in Halifax, Tuesday, 10 June 2003. Photograph by Oskar Maria Baksalary.*

Special Meeting on Linear Algebra and Applications

Caparica, near Lisbon, Portugal: 8–10 September 2003

A Special Meeting on Linear Algebra and Applications (Encontro de Álgebra Linear e Aplicações: EALA-2003) will be held from 8 to 10 September 2003 on the occasion of Graciano de Oliveira's 65th birthday. This Meeting will take place on the campus of the Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa, which is located at Caparica on the south side of the Tagus river near Lisbon and the Atlantic Ocean. It is being organized by the Centro de Estruturas Lineares e Combinatórias and follows the one held in Sevilla, Spain (10–12 September 1997), in a series of joint Portuguese–Spanish conferences.

Confirmed invited speakers include Itziar Baragaña (Universidad del País Vasco, Spain), Wayne Barrett (Brigham Young University, USA), Cristina Caldeira (Universidade de Coimbra, Portugal), Purificação Coelho (Universidade de Lisboa, Portugal), António Leal Duarte (Universidade de Coimbra, Portugal), Susana Furtado (Universidade do Porto, Portugal), Vakhtang Lomadze (Institute of Mathematics, Republic of Georgia), Alberto Márquez (Universidad de Sevilla, Spain), Juan Manuel Peña (Universidad de Zaragoza, Spain), and Xavier Puerta (Universidad Politécnica de Valencia, Spain).

The Scientific Committee comprises Isabel Cabral (Universidade Nova de Lisboa, Portugal), Juan Miguel Gracia (Universidad del País Vasco, Spain), Fernando Puerta (Universidad de Barcelona, Spain), João Filipe Queiró (Universidade de Coimbra, Portugal), Fernando C. Silva (Universidade de Lisboa, Portugal), J. A. Dias da Silva (Universidade de Lisboa, Portugal), Ana Urbano (Universidad de Valencia, Spain), and Ion Zaballa (Universidad del País Vasco, Spain). The Organizing Committee comprises Isabel Cabral (Universidade Nova de Lisboa, Portugal), Cecília Perdigão (Universidade Nova de Lisboa, Portugal), Carlos Saiago (Universidade Nova de Lisboa, Portugal), and Fernando C. Silva (Universidade de Lisboa, Portugal).

For further details contact Isabel Cabral by e-mail at ice@fct.unl.pt or please visit the Web site http://hermite.cii.fc.ul.pt/eala03/

International Conference on Matrix Analysis and Applications

Fort Lauderdale, Florida: 14–16 December 2003

An International Conference on Matrix Analysis and Applications will be held at Nova Southeastern University, Fort Lauderdale, Florida, USA, 14–16 December 2003. The aim of this mathematical meeting is to stimulate research and interaction of researchers interested in all aspects of linear and multilinear algebra, matrix analysis and applications and to provide an opportunity for researchers to exchange ideas and recent developments on these subjects. The conference is sponsored by Nova Southeastern University and endorsed by the International Linear Algebra Society (ILAS).

The organizing committee consists of Tsuyoshi Ando (Sapporo, Japan), Chi-Kwong Li (College of William and Mary, USA), George P. H. Styan (McGill University, Canada), Hugo Woerdeman (College of William and Mary, USA), and Fuzhen Zhang (Nova Southeastern University, USA).

Roger A. Horn (University of Utah, USA) will be the ILAS Lecturer. Confirmed speakers and participants include (in addition to the organizers): Rafig Agaev, Koenraad Audenaert, Jaspal Singh Aujla, Jerzy K. Baksalary, Oskar Maria Baksalary, Ham Bart, Natália Bebiano, Man-Duen Choi, Chandler Davis, Luz Maria DeAlba, Mingzhou Ding, Dragomir Ž. Đoković, Driss Drissi, Hossein Teimoori Faan, Shaun Fallat, Takayuki Furuta, Armenak Gasparyan, Frank Hall, Matthew He, Daniel Hershkowitz, Jinchuan Hou, Erxiong Jiang, Sang-Gu Lee, Wen Li, Zhongshan Li, Niloufer Mackey, Tom D. Morley, Hiroshi Nakazato, Peter Nylen, Vadim Olshevsky, Leiba Rodman, Mandeep Singh, Jai N. Singh, Mohammad Shakil, Ilya Spitkovsky, Tin-Yau Tam, Michael Tsatsomeros, William Watkins, Hans Joachim Werner, Changqing Xu, and Masahiro Yanagida.

We expect that many of the papers presented at this conference will be published, after refereeing, in a Special Issue of *Linear Algebra and Its Applications* associated with this conference. A reception and a pool party will take place in the evenings of Saturday 13 December and Monday 15 December 2003, respectively. There will be no registration fee. The conference hotel is Best Western Rolling Hills Resort: www.bestwestern.com/rollinghillsresort which is within walking distance to the conference site at Nova Southeastern University: www.nova.edu To register, contact Chi-Kwong Li: ckli@math.wm.edu For local information, contact Fuzhen Zhang: zhang@nova.edu The Web site is www.resnet.wm.edu/~cklixx/nova03.html

The Many Facets of Linear Algebra and Matrix Theory

Bangalore, India: 17–20 December 2003

The first joint meeting of the American Mathematical Society (AMS) and Indian mathematicians will take place in Bangalore, India, on December 17–20, 2003. Plenary lectures at this inaugural meeting will be given by Professors Balasubramanian, Papanicolaou, Raghunathan, Sarnak, Sinha, and Voevodsky.

At this meeting there will be a special 10-hour session on "The Many Facets of Linear Algebra and Matrix Theory" organized by Rajendra Bhatia and Richard Brualdi. We hope to showcase the broad and important contributions that have been made and are being made to linear algebra and matrix theory, and their key role in applications.

We have assembled a distinguished, eclectic, and international group to accomplish this. They are: Ravi Bapat, Adi Ben-Israel, Tirthankar Bhattacharyya, Francesco Brenti, Biswa Datta, Jose Dias da Silva, Anne Greenbaum, Ravi Kannan, Fuad Kittaneh, Tom Laffey, Raphi Loewy, Michael Overton, Dijen Ray-Chaudhuri, Peter Šemrl, Stefano Serra, and Pei Yuan Wu.

The 10-hour session will be divided into four sessions:

1. "Algebraic Linear Algebra": Bapat, Dias da Silva, Laffey, Šemrl
2. "Analytic Linear Algebra": Bhattacharyya, Kittaneh, Serra, Wu
3. "Applied & Computational Linear Algebra": Ben-Israel, Datta, Greenbaum, Overton.
4. "Combinatorial Linear Algebra": Brenti, Kannan, Loewy, Ray-Chaudhuri.

We hope that other ILAS members will consider attending this special meeting. The AMS Web site for the meeting is: http://www.ams.org/amsmtgs/internmtgs.html which will be updated over the next several months.

13th International Workshop on Matrices and Statistics

Będlewo, near Poznań, Poland: 19–21 August 2004

The 13th International Workshop on Matrices and Statistics (IWMS-2004), in Celebration of Ingram Olkin's 80th birthday, will be held in Będlewo, about 30 km. (18 miles) south of Poznań, Poland, from 19 to 21 August 2004. Będlewo is the Mathematical Research and Conference Center of the Polish Academy of Sciences; the setting is similar to Oberwolfach, with accommodation on site. For pictures of the Będlewo Center visit the Web site www.impan.gov.pl/Bedlewo/ Poznań is one of the oldest cities in Poland and has over half a million inhabitants; it is located about 300 km. (185 miles) west of Warsaw and about halfway between Warsaw and Berlin. It was here in Poznań that the first Polish state was created about a thousand years ago.

The Local Organizing Committee comprises Jan Hauke, Augustyn Markiewicz (chair): amark@owl.au.poznan.pl, Tomasz Szulc, and Waldemar Wołyński; the International Organizing Committee for this Workshop comprises R. William Farebrother, Simo Puntanen (chair), George P. H. Styan (vice-chair), and Hans Joachim Werner.

This Workshop will include both invited and contributed papers on matrices and statistics. Also a special session for graduate students will be arranged. It is expected that many of these papers will be published, after refereeing, in a Special Issue on Linear Algebra and Statistics of *Linear Algebra and its Applications*. Invited speakers include Rafael Bru, Carles M. Cuadras, Pierre Druilhet, Ludwig Elsner, Jürgen Groß, Joachim Kunert, Erkki P. Liski, Richard J. Martin, Volker Mehrmann, Joao Tiago Mexia, Herve Monod, PSSNVP Rao, Waldemar Ratajczak, Dietrich von Rosen, Bikas K. Sinha, and Haruo Yanai.

14th International Workshop on Matrices and Statistics

Auckland, New Zealand: 29 March–1 April 2005

The 14th International Workshop on Matrices and Statistics (IWMS-2005) will be held at Massey University (Albany Campus), Auckland, New Zealand, 29 March to 1 April 2005, just before the 55th Biennial Session of the International Statistical Institute (Sydney, Australia, 5–12 April 2005). This Workshop will be hosted by the Institute of Information and Mathematical Sciences at Massey University and will be cosponsored by the New Zealand Statistical Association.

The Local Organizing Committee will be chaired by Jeffrey J. Hunter (Massey University): J.Hunter@massey.ac.nz The International Organizing Committee comprises Simo Puntanen, George P. H. Styan (chair) and Hans Joachim Werner.

Eigenvalues and Latent Roots

Following the comments in IMAGE by Farebrother (1999), Schneider (2000) and Searle (2000), we note that although "eigenvalue" and "eigenvector" are to be reprobated as unfortunate hybrid words, the same criticism is valid for such familiar words as "automobile", "television" and "velodrome".

Schneider (2000) noted that "The obsolete German root *wurz* occurs in modern German as *Wurzel* (root) and *Würze* (spice)." Indeed, *Würze* (spice) occurs in the Alsatian grape variety "Gewürztraminer" and *Wurzel* (root) occurs in the English word "mangelwurzel": large white or yellow swollen roots[3] developed in the 1700s for feeding livestock; Wurzel is also used to indicate a rustic yokel as in "Wurzel Gummidge" (a BBC children's television character) and in "The Wurzels" (a popular singing group).

In view of the comments by Schneider (2000) and Searle (2000), it is clear that I should have continued my quotation from Grattan-Guinness & Ledermann (1994, page 785) in Farebrother (1999) to include two more sentences: "The properly English phrases 'latent root' and 'latent vector' have been employed in this article. The former was introduced in Sylvester (1883), a charming phrase: *Latent roots* of a matrix — latent in a somewhat similar sense as a vapour may be said to be latent in water or smoke in a tobacco leaf."

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[3]The English word "mangel-wurzel" is an alteration of the German word "Mangoldwurzel"; these roots are also known as "Dickwurz" (or Runkelrübe or Futterrübe); "Mangold" (from Middle High German "Mānegolt") is the German for "chard" (beta vulgaris) and "Rübe" is the German for "beet"; see also http://www.bartleby.com/61/75/M0077500.html as well as the "Mangelwurzel Mulled Wine Appreciation Page" www.luhc-alumni.org/features/mangelwurzel.html –*Ed.*

IMAGE Problem Corner: Old Problems, Many With Solutions

We present solutions to IMAGE Problems 29-1, 29-2, 29-5 through 29-10, 29-12 and 29-13 [IMAGE 29 (October 2002), pp. 36 & 35] and a corrected version of Solution 28-2.2; all references cited in these solutions are collected together on page 35. Problems 28-3, 29-3, 29-4 and 29-11 are repeated below without solutions; we are still hoping to receive further solutions to these problems (we do have solutions from the Proposers of Problems 29-3, 29-4 and 29-11). We introduce 7 new problems on pp. 36 & 35 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 28-2: Linear Combinations of Imaginary Units

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

Let $i,\ j,\ k$ denote the imaginary units of the algebra of quaternions. Then, it is well known that these units satisfy the conditions $i^2 = j^2 = k^2 = ijk = -1$. Let v denote the 3×1 matrix of imaginary units $v = [i\ j\ k]'$, and let $p,\ q,\ r$ be arbitrary 3×1 real matrices. Find conditions such that the linear combinations $i_o = p'v,\ j_o = q'v,\ k_o = r'v$ satisfy the conditions $i_o^2 = j_o^2 = k_o^2 = i_o j_o k_o = -1$.

Corrected Solution 28-2.2 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

Acknowledgement. I am very grateful to Richard William Farebrother, the Proposer of this problem, for correcting my original solution [IMAGE 29 (October 2002), page 26].

The solution is presented in the following form.

PROPOSITION. *Let i, j, k denote the imaginary units of the algebra of quaternions. Further, let $v = (i,j,k)'$, let $p = (p_1,p_2,p_3)'$, $q = (q_1,q_2,q_3)'$, and $r = (r_1,r_2,r_3)'$ be 3×1 real matrices, and let $i_0 = p'v$, $j_0 = q'v$, and $k_0 = r'v$. Then*

$$i_0^2 = -1,\ j_0^2 = -1,\ k_0^2 = -1,\ i_0 j_0 k_0 = -1 \tag{1}$$

if and only if the matrix

$$S = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}, \tag{2}$$

having p', q', and r' as its successive rows, is orthogonal, i.e., $SS' = I_3$, and the determinant $\det(S) = 1$.

PROOF. Since i,j,k satisfy $ij = k = -ji,\ ik = -j = -ki,\ jk = i = -kj$, it follows that the first three conditions in (1) are equivalent to

$$p'p = 1,\ q'q = 1,\ r'r = 1. \tag{3}$$

From (1) it also follows that $i_0 j_0 = k_0,\ i_0 k_0 = -j_0,\ j_0 k_0 = i_0$. It is straightforward to verify that

$$i_0 j_0 = -p'q + R_1 i + R_2 j + R_3 k, \tag{4}$$

where R_m denotes the cofactor of r_m in the matrix S of the form (2), $m = 1,2,3$. Hence the equality $i_0 j_0 = k_0$ implies $p'q = 0$, and similar arguments lead to $p'r = 0$ and $q'r = 0$. Combining these observations with (3) shows that S must be orthogonal. Moreover, in view of (4) and $p'q = 0$, we see that $i_0 j_0 k_0 = -(r_1R_1 + r_2R_2 + r_3R_3) - (r_2R_3 - r_3R_2)i + (r_1R_3 - r_3R_1)j - (r_1R_2 - r_2R_1)k$. It is clear that $r_1R_1 + r_2R_2 + r_3R_3 = \det(S)$ and hence the additional condition is $\det(S) = 1$.

Conversely, the orthogonality of S implies the conditions in (3), as well as $p'q = 0$, $p'r = 0$, and $q'r = 0$. Consequently, in view of $\det(S) = 1$ and the representation of $i_0 j_0 k_0$ given above, it remains to show that

$$r_2R_3 = r_3R_2,\ r_1R_3 = r_3R_1,\ r_1R_2 = r_2R_1. \tag{5}$$

The first equality in (5) follows by noting that

$$r_2R_3 - r_3R_2 = r_2(p_1q_2 - p_2q_1) + r_3(p_1q_3 - p_3q_1) = p_1(q_2r_2 + q_3r_3) - q_1(p_2r_2 + p_3r_3) = p_1(-q_1r_1) - q_1(-p_1r_1) = 0,$$

and the remaining two are obtained similarly. □

Problem 28-3: Ranks of Nonzero Linear Combinations of Certain Matrices.

Proposed by Shmuel FRIEDLAND, *University of Illinois at Chicago, Chicago, Illinois, USA:* friedlan@uic.edu
and Raphael LOEWY, *Technion–Israel Institute of Technology, Haifa, Israel:* loewy@technunix.technion.ac.il

Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Show that any nonzero real linear combination of these four matrices has rank at least 3.

The Proposers of Problem 28-3 and the Editors of IMAGE *are still looking forward to receiving a solution to this problem; the Proposers prefer a solution which does not depend on the use of a computer package such as* MAPLE.

Problem 29-1: A Condition for an EP Matrix to be Hermitian

Proposed by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

Let A be an EP matrix, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$, where A^* and $\mathcal{R}(A)$ denote the conjugate transpose and range of A. Show that A is Hermitian if and only if there exists a matrix B having a generalized inverse B^- (i.e., a solution to $BB^-B = B$), for which both B^- and $(B^-)^*$ are also generalized inverses of A, i.e., $AB^-A = A$ and $A(B^-)^*A = A$. From this property it follows, in particular, that every EP matrix which is a predecessor of a Hermitian matrix with respect to the minus partial ordering is necessarily Hermitian.

Solution 29-1.1 by Néstor THOME, *Universidad Politécnica de Valencia, Valencia, Spain:* njthome@mat.upv.es

Necessity follows by setting $B := A = A^*$. In fact, from the definition of a generalized inverse we have $AB^-A = AA^-A = A$ and $A(B^-)^*A = (A^*)^*((A^*)^-)^*(A^*)^* = (A^*(A^*)^-A^*)^* = (A^*)^* = A$. Sufficiency follows from a result by Katz (1965) stating that a square matrix A is an EP matrix if and only if there exists a matrix Y such that $A^* = YA$, see also Ben-Israel and Greville (1974, p. 166). Then $A^*B^-A = YAB^-A = YA = A^*$, and hence $A = (A^*B^-A)^* = A^*(B^-)^*A = YA(B^-)^*A = YA = A^*$.

Solution 29-1.2 by William F. TRENCH, *Woodland Park, Colorado, USA:* wtrench@trinity.edu
and the Proposers Jerzy K. BAKSALARY and Oskar Maria BAKSALARY.

If A is Hermitian, then the Moore-Penrose inverse $A^\dagger$ of A satisfies $(A^\dagger)^* = (A^*)^\dagger = A^\dagger$, and therefore $B = A$ with $B^- = A^\dagger$ has the desired properties. For the converse, it suffices to show that if $\mathcal{R}(A) = \mathcal{R}(A^*)$, i.e., $AA^\dagger = A^\dagger A$, and if there is a matrix G such that $AGA = A$ and $A^*GA^* = A^*$, then A is Hermitian. Since these equalities are clearly equivalent to $A^\dagger AGAA^\dagger = A^\dagger$ and $AA^\dagger GA^\dagger A = (A^\dagger)^*$, it follows that $A^\dagger = (A^*)^\dagger$, and hence, by uniqueness of the Moore-Penrose inverse, $A = A^*$. Since the minus order $A \preceq_- B$ may be characterized by $AB^\dagger B = A$, $BB^\dagger A = A$, and $AB^\dagger A = A$, it follows that if B is Hermitian, then $B^\dagger$ satisfies $A(B^\dagger)^*A = A$ in addition to $AB^\dagger A = A$, thus forcing the EP matrix A to be Hermitian.

Solution 29-1.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

We prove the following slightly more informative theorem.

THEOREM. *Let $A \in \mathbb{C}^{n\times n}$ be an EP matrix. Then:*

(i) *A is Hermitian if and only if there exists a Hermitian matrix B such that $ABA = A$.*

(ii) *A is nonnegative definite Hermitian if and only if there exists a nonnegative definite Hermitian B such that $ABA = A$.*

PROOF. If A is Hermitian, then $A^\dagger$, the Moore–Penrose inverse of A, is also Hermitian. Since $AA^\dagger A = A$, necessity follows. To prove sufficiency, let A be EP and let B be a Hermitian matrix with $ABA = A$. Let r denote the rank of A. Since A is EP, A can be written as $A = UCU^*$ for some nonsingular $r \times r$ matrix C and some column-unitary $n \times r$ matrix U, see (5.11.15) in Meyer (2000, p. 408). From $U^*U = I_r$, the equation $ABA = A$ reduces to $C = CU^*BUC$ or, equivalently, to U^*BU being nonsingular and $C = (U^*BU)^{-1}$. Since B is Hermitian, so is $(U^*BU)^{-1}$, which completes the proof of (i). To prove (ii), if A is Hermitian and nonnegative definite, so is $A^\dagger$. Since $AA^\dagger A = A$, necessity is shown. Sufficiency follows along similar lines as that of (i). □

The result in Problem 29-1 follows at once from (i) of our Theorem in that the matrix $H := [B^- + (B^-)^*]/2$ is Hermitian with $AHA = A$. From part (ii) of our Theorem it follows, in particular, that every EP matrix which is a predecessor of a nonnegative definite Hermitian matrix with respect to the minus partial ordering is necessarily nonnegative definite Hermitian.

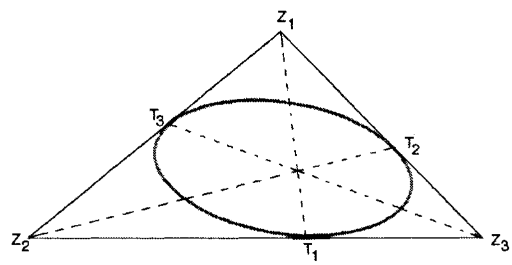
Problem 29-2: Triangle with Vertices Circumscribing an Ellipse

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Let A be a 2×2 complex matrix which is not normal. Then, it is well known that the numerical range $W(A)$ of A is a solid ellipse. Let $z_1, z_2, z_3 \in \mathbb{C}$. Show that a necessary and sufficient condition for A to possess a 3×3 normal dilation with eigenvalues z_1, z_2, z_3 is that the triangle with vertices z_1, z_2, z_3 circumscribe the ellipse.

Solution 29-2.1 by Bernardete RIBEIRO: bribeiro@dei.uc.pt and Alexander KOVAČEC: kovacec@mat.uc.pt
Universidade de Coimbra, Coimbra, Portugal.

PROOF OF NECESSITY. Let A possess a normal dilation N with eigenvalues z_1, z_2, z_3. We assume w.l.o.g. that (in MATLAB-notation) $A = N(1:2, 1:2)$. By well-known properties of the numerical range, see Horn & Johnson (1991, p. 9), we can assume, if necessary multiplying N with a suitable $e^{i\theta}$, that $z_1 = a + ib_1$, $z_2 = a + ib_2$ for some $a, b_1, b_2 \in \mathbb{R}$. There is a unitary U such that $N = U^*\text{diag}(a + ib_1, a + ib_2, z_3)U$, and consequently $A = U^*(1:2, :)\text{diag}(a + ib_1, a + ib_2, z_3)U(:, 1:2)$. It is easy to see that there is an $x \in \mathbb{C}^2$ with $x^*x = 1$ such that $w = U(:, 1:2)x$ has its third component $w_3 = 0$. Hence $\Re(x^*Ax) = \Re((a + ib_1)|w_1|^2 + (a + ib_2)|w_2|^2) = a$. Since $W(A) \subseteq W(N) = \text{co}\{z_1, z_2, z_3\} = \Delta z_1 z_2 z_3$, see Horn & Johnson (1991, p. 13), it follows that the ellipse is necessarily tangent to the line through z_1, z_2. Analogous reasoning for the other sides $z_1 z_3$ and $z_2 z_3$ of Δ proves necessity.



PROOF OF SUFFICIENCY. Let $\Delta = \Delta z_1 z_2 z_3$ be the triangle spanned by given three noncolinear complex numbers and let the ellipse $\mathcal{E} = W(A)$ be inscribed in Δ. Let T_i denote the point at which $\mathcal{E}$ touches the side opposed to z_i. There are reals α_i, α_i' so that $T_1 = \alpha_1 z_2 + \alpha_1' z_3$, $T_2 = \alpha_2 z_3 + \alpha_2' z_1$, $T_3 = \alpha_3 z_1 + \alpha_3' z_2$, $\alpha_i + \alpha_i' = 1$, $\alpha_i, \alpha_i' > 0$. Since Δ and $\mathcal{E}$ can be viewed as a projection of another triangle with an inscribed circle, the cevians $T_i z_i$ pass through the so-called Gergonne point (but aliases are also found); see, e. g., Berger (1987, p. 330). So by Ceva's theorem, see Berger (1987, p. 64), $\alpha_1'\alpha_2'\alpha_3'/(\alpha_1\alpha_2\alpha_3) = 1$. It follows that the points $p_1 = (0, \sqrt{\alpha_1}, \sqrt{\alpha_1'})$, $p_2 = (\sqrt{\alpha_2'}, 0, \sqrt{\alpha_2})$, $p_3 = (\sqrt{\alpha_3}, 0, -\sqrt{\alpha_3'})$ lie coplanar with the origin O of $\mathbb{R}^3$ on a unit circle. Let Ouv be an orthonormal frame in that plane and let U be an orthogonal matrix having u, v as the first two columns. There are reals c_i, s_i so that $c_i u + s_i v = p_i$, and $c_i^2 + s_i^2 = 1$, for $i = 1, 2, 3$. The normal matrix $N' = U^*\text{diag}(z_1, z_2, z_3)U$ satisfies $W(N') = \Delta$ and for $A' = N(1:2, 1:2)$, we find that $[c_i, s_i]A'[c_i, s_i]^T = [c_i, s_i]U^*(1:2, :)\text{diag}(z_1, z_2, z_3)U(:, 1:2)[c_i, s_i]^T = p_i\text{diag}(z_1, z_2, z_3)p_i^T = T_i$. This shows that the points T_i are in the numerical range of A' and $W(A') = \mathcal{E} = W(A)$. From the discussion leading to Theorem 1.3.6 in Horn & Johnson (1991, p. 23) it follows that there is a 2×2 unitary V such that $A = V^*A'V$. Thus $N = (V^* \oplus [1])N'(V \oplus [1])$ is a normal dilation for A as desired.

Solution 29-2.2 by the Proposer S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

If A possesses such a normal dilation N, then it is easy to prove that $W(A)$ lies in the convex hull of the points z_1, z_2, z_3. Now, the direct sum of the eigenspaces of N corresponding to z_2 and z_3 has dimension 2 and therefore meets the linear span of the first two coordinate vectors in a subspace of dimension at least 1. Thus, there is a vector v such that v^*Av is a convex combination of z_2 and z_3. So, a point of $W(A)$ meets the line segment joining z_2 to z_3. Cyclically permuting the eigenvalues shows that the triangle with vertices z_1, z_2, z_3 circumscribes $W(A)$.

In the other direction, we observe first that the problem is unaffected by translation and rescaling. Thus, we can assume without loss of generality that the points z_1, z_2, z_3 lie on the unit circle. Next, using barycentric coordinates, we may write $A = z_1 A_1 + z_2 A_2 + z_3 A_3$, where A_j are nonnegative definite 2×2 matrices with $I = A_1 + A_2 + A_3$, Bhatia (1996, p. 25). Now let w be the point of contact of the line segment joining z_2 to z_3 with the ellipse $W(A)$ and let v be a unit vector such $w = v^\star A v$ and therefore also $\overline{w} = v^\star A^\star v$. Substituting into Bhatia's definition of A_1, we find that $v^\star A_1 v = 0$, and it follows that A_1 (and similarly A_2 and A_3) have rank one. So, we write $A_k = w_k \otimes w_k^\star$ for $k = 1, 2, 3$. Next, we define a linear map K from $\mathbb{C}^2$ to $\mathbb{C}^3$ by $K^\star e_j = w_j$, where e_j denotes the jth coordinate vector in $\mathbb{C}^3$. We have $A_k = w_k \otimes w_k^\star = K^\star e_k \otimes e_k^\star K$ and $I = \sum_{k=1}^3 A_k = \sum_{k=1}^3 K^\star e_k \otimes e_k^\star K = K^\star K$ so that K is an isometry. We further obtain that $A = \sum_{k=1}^3 z_k A_k = K^\star \left(\sum_{k=1}^3 z_k e_k \otimes e_k^\star \right) K = K^\star N K$, where N is normal with eigenvalues z_1, z_2 and z_3, as required.

Problem 29-3: Isometric Realization of a Finite Metric Space

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Show that every finite metric space can be realized isometrically as a subset of a normed vector space.

While we have received a solution from its Proposer, we look forward to receiving further solutions to Problem 29-3.

Problem 29-4: Normal Matrix and a Commutator

Proposed by S. W. DRURY: drury@math.mcgill.ca and George P. H. STYAN: styan@math.mcgill.ca
McGill University, Montréal (Québec), Canada.

Show that every $n \times n$ complex matrix A can be written in the form $A = N + [H, N]$, where N is normal and H is Hermitian, and the commutator $[H, N] = HN - NH$.

While we have received a solution from its Proposers, we look forward to receiving further solutions to Problem 29-4.

Problem 29-5: Product of Two Hermitian Nonnegative Definite Matrices

Proposed by Jürgen GROß: gross@statistik.uni-dortmund.de and Götz TRENKLER: trenkler@statistik.uni-dortmund.de
Universität Dortmund, Dortmund, Germany.

Let A and B be two Hermitian nonnegative definite matrices of the same order. Show that the column space $\mathcal{R}(AB)$ and the null space $\mathcal{N}(AB)$ of the product AB are complementary subspaces.

Solution 29-5.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@main.amu.edu.pl

We will show that Problem 29-5 is a corollary to a more general result. In what follows, $\mathbb{C}_{m,n}$ denotes the set of $m \times n$ complex matrices, and K^*, $\mathcal{R}(K)$, $\mathcal{N}(K)$, and $\mathrm{rank}(K)$ denote the conjugate transpose, range (column space), null space, and rank of a given $K \in \mathbb{C}_{m,n}$, respectively. We have the following lemma.

LEMMA. *Let $K \in \mathbb{C}_{n,n}$. Then*

$$\mathbb{C}_{n,1} = \mathcal{R}(K) \oplus \mathcal{N}(K) \Leftrightarrow \mathcal{R}(K) \cap \mathcal{N}(K) = \{0\} \Leftrightarrow \dim[\mathcal{R}(K) + \mathcal{N}(K)] = n \Leftrightarrow \mathrm{index}(K) \leq 1, \tag{6}$$

where the last condition means that $\mathrm{rank}(K) = \mathrm{rank}(K^2)$.

PROOF. This lemma includes the part (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) of Exercise 5.10.12 in Meyer (2000). The first two equivalences in the Lemma are immediate consequences of the rank-plus-nullity theorem stating that $\dim \mathcal{R}(K) + \dim \mathcal{N}(K) = n$; see (4.4.15) in Meyer (2000). It follows that for any choice of generalized inverse K^- of K, i.e., for any K^- satisfying

$KK^-K = K$, see (2.30) and (2.35) in Marsaglia & Styan (1974), $\dim[\mathcal{R}(K) + \mathcal{N}(K)] = \dim[\mathcal{R}(K) + \mathcal{R}(I_n - K^-K)] = \operatorname{rank}((K : I_n - K^-K)) = \operatorname{rank}(I_n - K^-K) + \operatorname{rank}(K^-K^2) = n - \operatorname{rank}(K) + \operatorname{rank}(K^-K^2)$. Consequently, since $\operatorname{rank}(K^2) = \operatorname{rank}(KK^-K^2) \leq \operatorname{rank}(K^-K^2) \leq \operatorname{rank}(K^2)$, it follows that $\dim[\mathcal{R}(K) + \mathcal{N}(K)] = n$ if and only if $\operatorname{rank}(K) = \operatorname{rank}(K^2)$, which concludes the proof of the lemma. □

We now observe that if A and B are Hermitian nonnegative definite matrices, i.e., if $A = SS^*$ and $B = TT^*$ for some $S \in \mathbb{C}_{n,p}$ and $T \in \mathbb{C}_{n,q}$, then $\operatorname{rank}[(AB)^2] = \operatorname{rank}[SS^*T(S^*T)^*S^*TT^*] = \operatorname{rank}(S^*T) = \operatorname{rank}(SS^*TT^*) = \operatorname{rank}(AB)$, see (2.12) and (2.13) in Marsaglia & Styan (1974, Th. 2). This shows that the product of any two nonnegative definite Hermitian matrices is of index 0 or 1, and therefore $K = AB$ satisfies the first equality in (6), as claimed in Problem 29-5.

Solution 29-5.2 by Roger A. HORN, *University of Utah, Salt Lake City, Utah, USA:* rhorn@math.utah.edu

It is known that the product of two positive semidefinite matrices is diagonalizable and has nonnegative eigenvalues; see Hong & Horn (1991, Corollary 2.3). The range and nullspace of any diagonalizable matrix are complementary subspaces, so the assertion follows. The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ show that the assertion need not be correct for the product of a positive semidefinite matrix A and a Hermitian matrix B. However, the nilpotent Jordan blocks of such a product are at most 2×2, so $(AB)^2$ is diagonalizable and hence its range and null space are complementary subspaces; see Hong & Horn (1991, Prop. 3.3).

Solution 29-5.3 by Denis SERRE, *École Normale Supérieure de Lyon, Lyon, France:* serre@umpa.ens-lyon.fr

Let x belong to $R(AB) \cap \ker(AB)$. Let also S be a nonnegative square root of A. Then $SBx \in R(S) \cap \ker S$, hence $SBx = 0$. But $x \in R(A) = R(S)$, say $x = Sy$. Thus $SBSy = 0$, which implies $(Sy)^*B(Sy) = 0$. Since B is nonnegative, this means $BSy = 0$, that is $Bx = 0$. Actually, the assumption is $x = ABz$, and so $BABz = 0$, which implies $(Bz)^*A(Bz) = 0$. Again, this means $ABz = 0$ since A is nonnegative. Since the sum of dimensions is n, this yields $R(AB) \cap \ker(AB) = \{0\}$, or $\mathbb{C}^n = R(AB) \oplus \ker(AB)$, as desired.

Solution 29-5.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Our elementary proof is based on the following well-known result.

THEOREM. *Let $H \in \mathbb{C}^{m,m}$ be a Hermitian nonnegative definite matrix, and let $\mathcal{M}$ be a linear subspace of $\mathbb{C}^m$. Then $(H\mathcal{M}) \cap \mathcal{M}^\perp = \{0\}$, with $\mathcal{M}^\perp$ denoting the orthogonal complement of $\mathcal{M}$ with respect to the usual standard inner product in $\mathbb{C}^m$.*

PROOF. Let $H^{1/2}$ denote the square root of H. Then $(H^{1/2})^* = H^{1/2}$ and, for each $x \in \mathbb{C}^m$, $x^*Hx = x^*H^{1/2}H^{1/2}x = \|H^{1/2}x\|^2$. Therefore, $x^*Hx = 0 \Rightarrow H^{1/2}x = 0 \Rightarrow Hx = 0$. Since $\mathcal{M}^\perp = \{y \,|\, \forall\, x \in \mathcal{M} : y^*x = 0\}$, the theorem is proved. □

Let $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ denote the null space and the column space (range), respectively, of the matrix $(\cdot)$. Since $\mathcal{R}(B)^\perp = \mathcal{N}(B)$, according to our Theorem, $[A\mathcal{R}(B)] \cap \mathcal{N}(B) = \{0\}$. Likewise $\mathcal{R}(A)^\perp = \mathcal{N}(A)$ and $[B\mathcal{R}(A)] \cap \mathcal{N}(A) = \{0\}$, and so, in view of $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, in particular $[B\mathcal{R}(AB)] \cap \mathcal{N}(A) = \{0\}$. Combining all these observations results in $ABABx = 0 \Rightarrow BABx = 0 \Rightarrow ABx = 0$ or, equivalently, $\mathcal{R}(AB) \cap \mathcal{N}(AB) = \{0\}$. Since for an arbitrary matrix $C \in \mathbb{C}^{m,n}$ we also have $\dim \mathcal{N}(C) = n - \operatorname{rank}(C)$, it is now clear that $\mathcal{R}(AB)$ and $\mathcal{N}(AB)$ are complementary subspaces.

Solution 29-5.5 by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, USA:* zhang@nova.edu

It is sufficient to show that $\mathcal{R}(AB) \cap \mathcal{N}(AB) = \{0\}$. Let y be in the intersection and write $y = (AB)x$ for some x. Then $(AB)y = (AB)^2x = 0$. We claim $(AB)x = 0$ as follows:

$$(AB)^2x = 0 \;\Rightarrow\; (ABAB)x = 0 \;\Rightarrow\; (x^*B)(ABAB)x = 0 \;\Rightarrow\; (x^*BAB^{1/2})(B^{1/2}ABx) = 0 \;\Rightarrow\; (B^{1/2}AB)x = 0$$
$$\Rightarrow\; B^{1/2}(B^{1/2}AB)x = (BAB)x = 0 \;\Rightarrow\; (x^*BA^{1/2})(A^{1/2}Bx) = 0 \;\Rightarrow\; (A^{1/2}B)x = 0 \;\Rightarrow\; (AB)x = 0\,.$$

Solutions to Problem 29-5 were also received from William F. Trench *and from the Proposers* Jürgen Groß & Götz Trenkler.

Problem 29-6: Product of Companion Matrices

Proposed by Eric S. KEY, *University of Wisconsin–Milwaukee, Wisconsin, USA:* ericskey@csd.uwm.edu

Let $A_1, \ldots, A_k$ be $n \times n$ companion matrices with common eigenvalue a. Show that a^k is an eigenvalue of the product $A_1 A_2 \cdots A_k$.

Solution 29-6.1 by Bernardete RIBEIRO: bribeiro@dei.uc.pt and Alexander KOVAČEC: kovacec@mat.uc.pt
Universidade de Coimbra, Coimbra, Portugal.

LEMMA. *If A is an $n \times n$ companion matrix and $Ax = ax$, then $x = x_1(1, a, a^2, \ldots, a^{n-1})^T$.*
PROOF. From Hungerford (1974, p. 359), we see that $A = (a_{ij})$ satisfies $a_{ij} = \delta_{i+1,j}$, $i = 1, 2, \ldots, n-1$. Let $x = (x_1, \ldots, x_n)^T$. Then $(A - aI)x = 0$ yields $0 = \sum_j (\delta_{i+1,j} - a\delta_{i,j})x_j = x_{i+1} - ax_i$, for $i = 1, 2, \ldots, n-1$, and the lemma is proved. □

Applying the lemma inductively yields $A_1 A_2 \ldots A_k x = a^k x$.

Solution 29-6.2 by Iwona WRÓBEL, *Warsaw University of Technology, Warsawa, Poland:* wrubelki@wp.pl

The companion matrix of a monic polynomial $p(z) = z^n + c_n z^{n-1} + \ldots + c_2 z + c_1$ is defined by

$$C(p) = \begin{pmatrix} -c_n & -c_{n-1} & \ldots & -c_2 & -c_1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

It is known that if a is an eigenvalue of a $n \times n$ companion matrix, there exists a corresponding eigenvector of a form $x = (a^{n-1}, a^{n-2}, \ldots, a, 1)^T$. By assumption, a is a common eigenvalue of matrices $A_1, A_2, \ldots, A_k$. So x is the eigenvector of each matrix A_i, i.e., $A_i x = ax$ for $i = 1, \ldots, k$. Let A denote the product $A_1 \cdot A_2 \cdot \ldots \cdot A_k$. Then $Ax = A_1 \cdot \ldots \cdot A_k x = A_1 \cdot \ldots \cdot A_{k-1} ax = aA_1 \cdot \ldots \cdot A_{k-1} x = a^2 A_1 \cdot \ldots \cdot A_{k-2} x = a^k x$. Thus a^k is an eigenvalue of A and x is an eigenvector associated with a^k.

Solutions to Problem 29-6 were also received from Denis Serre *and from the Proposer* Eric S. Key; *see also* Key (1984).

Problem 29-7: Complementary Principal Submatrices and Their Eigenvalues

Proposed by Chi-Kwong LI, *The College of William and Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu

Let $n = 2k$ and let A be a real symmetric or complex Hermitian idempotent matrix (i.e., $A^2 = A$) of rank k. If the leading $k \times k$ principal submatrix has eigenvalues $a_1, \ldots, a_k$, show that the complementary principal submatrix has eigenvalues $1 - a_1, \ldots, 1 - a_k$.

Solution 29-7.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Roger A. HORN, *University of Utah, Salt Lake City, Utah, USA:* rhorn@math.utah.edu

Let A be an $n \times n$ complex idempotent matrix of rank r, so $A^2 = A$ and A is diagonalizable; r of the eigenvalues of A are one and $n - r$ are zero. Suppose $1 \le p \le q < n$ and $p + q = n$, and partition A as a 2×2 block matrix $A = [A_{ij}]_{i,j=1}^2$ in which A_{11} is $p \times p$ and A_{22} is $q \times q$. There are three cases:

(a) If $r \le p$, then for some $\{\nu_1, ..., \nu_r\} \subset \mathbb{C}$ the eigenvalues of A_{11} are $\nu_1, ..., \nu_r$, and 0 with multiplicity $p - r$; the eigenvalues of A_{22} are $1 - \nu_1, ..., 1 - \nu_r$, and 0 with multiplicity $q - r$.

(b) If $p \le r \le q$ and the eigenvalues of A_{11} are $\lambda_1, ..., \lambda_p$, then the eigenvalues of A_{22} are $1 - \lambda_1, ..., 1 - \lambda_p$, 1 with multiplicity $r - p$, and 0 with multiplicity $q - r$.

(c) If $q \le r$, then for some $\{\nu_1, ..., \nu_{n-r}\} \subset \mathbb{C}$ the eigenvalues of A_{11} are $\nu_1, ..., \nu_{n-r}$ and 1, with multiplicity $r - q$; the eigenvalues of A_{22} are $1 - \nu_1, ..., 1 - \nu_{n-r}$, and 1 with multiplicity $r - p$.

We write $A = S\Lambda S^{-1}$, in which $\Lambda = I_r \oplus 0_{n-r}$, $S = (S_1,\ S_2)$ is nonsingular, $S_1 = \begin{pmatrix} X \\ Y \end{pmatrix}$ is $n\times r$, $S^{-1} = \begin{pmatrix} (S^{-1})_1 \\ (S^{-1})_2 \end{pmatrix}$, X is $p\times r$, Y is $q\times r$, $(S^{-1})_1 = [\Gamma\ \Theta]$ is $r\times n$, Γ is $r\times p$, and Θ is $r\times q$. Then $A_{11} = X\Gamma$, $A_{22} = Y\Theta$, and $I_r = (S^{-1})_1 S_1 = \Gamma X + \Theta Y$. Denote the eigenvalues of A_{11} by $\{\lambda_1, ..., \lambda_p\}$, those of A_{22} by $\{\mu_1, ..., \mu_q\}$, those of ΓX by $\{\gamma_1, ..., \gamma_r\}$, and those of ΘY by $\{\eta_1, ..., \eta_r\}$. Since $\Theta Y = I_r - \Gamma X$, we know that $\{\eta_1, ..., \eta_r\} = \{1-\gamma_1, ..., 1-\gamma_r\}$. We also know that the eigenvalues of $X\Gamma$ and ΓX are essentially the same: the eigenvalues of the larger matrix are just the eigenvalues of the smaller one, together with 0 with multiplicity $|r-p|$; the eigenvalues of the larger of $Y\Theta$ and ΘY are just the eigenvalues of the smaller, together with 0 with multiplicity $|r-q|$. See Horn & Johnson (1985, Th. 1.3.20). In (a), the eigenvalues of $A_{11} = X\Gamma$ are $\{\gamma_1, \ldots, \gamma_r\} \cup \{p-r \text{ zeros}\}$. The eigenvalues of $A_{22} = Y\Theta$ are $\{\eta_1, \ldots, \eta_r\} \cup \{q-r \text{ zeros}\} = \{1-\gamma_1, \ldots, 1-\gamma_r\} \cup \{q-r \text{ zeros}\}$. In (b), $\{\gamma_1, \ldots, \gamma_r\} = \{\lambda_1, \ldots, \lambda_p\} \cup \{r-p \text{ zeros}\}$, and $\{\mu_1, \ldots, \mu_q\} = \{\eta_1, \ldots, \eta_r\} \cup \{q-r \text{ zeros}\}$. Thus, $\{\mu_1, \ldots, \mu_q\} = \{1-\gamma_1, \ldots, 1-\gamma_r\} \cup \{q-r \text{ zeros}\} = \{1-\lambda_1, \ldots, 1-\lambda_p\} \cup \{r-p \text{ ones}\} \cup \{q-r \text{ zeros}\}$. Case (c) follows by applying case (a) to the idempotent matrix $I-A$. The original problem is the special case $r = p = q$ with the additional assumption that A is Hermitian. In this case, Cauchy's Interlacing Theorem ensures that all of the parameters ν_i, λ_i, μ_i, γ_i, and η_i are real and in the interval $[0, 1]$.

Solution 29-7.2 by William F. TRENCH, *Woodland Park, Colorado, USA:* wtrench@trinity.edu

The assumptions imply that $A = \sum_{i=1}^{k} \phi_i \phi_i^*$, where $\{\phi_1, \ldots, \phi_k\}$ is an orthonormal basis for the range of A. Let $\phi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ where u_i and v_i are k-vectors, and denote $U = (u_1, \ldots, u_k)$, $V = (v_1, \ldots, v_k)$. Then $A = \begin{pmatrix} UU^* & UV^* \\ VU^* & VV^* \end{pmatrix}$. Since UU^* and VV^* have the same eigenvalues as U^*U and V^*V respectively and $U^*U + V^*V = (u_i^* u_j + v_i^* v_j)_{i,j=1}^n = (\phi_i^* \phi_j)_{i,j=1}^n = I$, the conclusion follows. Moreover, since UU^* and VV^* are both positive semidefinite, $0 \le a_i \le 1$, $1 \le i \le n$.

Solutions to Problem 29-7 were also received from
Bernardete Ribeiro & Alexander Kovačec, Denis Serre, Alicja Smoktunowicz, *and from* Fuzhen Zhang.

Problem 29-8: A Range Equality Involving an Idempotent Matrix
Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Suppose that the matrix P of order m satisfies $P^2 = P$. Show that range$(I_m - PP^*) =$ range$(2I_m - P - P^*)$, where P^* is the conjugate transpose of P.

Solution 29-8.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Xiaoji LIU, *Xidian University, Xian, China:* xiaojiliu72@yahoo.com.cn

If $P = 0$, then the solution is trivial. For $P \neq 0$, we offer an elementary proof based on Schur's unitary triangularization theorem; see, e.g., Horn & Johnson (1985, Theorem 2.3.1). Since the only nonsingular idempotent matrix of order m is I_m, in which case the equality in question is trivial, we assume that rank$(P) = r < m$. This means that $P = P^2$ has r eigenvalues equal to one and $m-r$ eigenvalues equal to zero, and thus may be represented in the form $P = U \begin{pmatrix} T & X \\ 0 & N \end{pmatrix} U^*$, where U is a unitary matrix of order m, and T and N are upper triangular matrices of order r and $m-r$ with the diagonal elements $t_{ii} = 1$, $i = 1, ..., r$, and $n_{jj} = 0$, $j = 1, ..., m-r$, respectively. Since the idempotency of P further implies $T^2 = T$ and $N^2 = N$, and hence $T = I_r$ and $N = 0$, it follows that

$$I_m - PP^* = U \begin{pmatrix} -XX^* & 0 \\ 0 & I_{m-r} \end{pmatrix} U^* \quad \text{and} \quad 2I_m - P - P^* = U \begin{pmatrix} 0 & -X \\ -X^* & 2I_{m-r} \end{pmatrix} U^*.$$

Then it can easily be verified that, with X^+ denoting the Moore–Penrose inverse of X,

$$(2I_m - P - P^*)U \begin{pmatrix} 2I_r & -(X^+)^* \\ X^* & \frac{1}{2}(I_{m-r} - X^+X) \end{pmatrix} U^* = I_m - PP^* \quad \text{and} \quad (I_m - PP^*)U \begin{pmatrix} 0 & (X^+)^* \\ -X^* & 2I_{m-r} \end{pmatrix} U^* = 2I_m - P - P^*.$$

This shows, with $\mathcal{R}(.)$ denoting range that $\mathcal{R}(I_m - PP^*) \subseteq \mathcal{R}(2I_m - P - P^*)$ and $\mathcal{R}(2I_m - P - P^*) \subseteq \mathcal{R}(I_m - PP^*)$, respectively, thus leading to the required equality $\mathcal{R}(I_m - PP^*) = \mathcal{R}(2I_m - P - P^*)$.

Solution 29-8.2 by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Since $P = P^2$, we may write $P = U\begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix}U^*$, where U is an $m \times m$ unitary matrix, I_r is the identity matrix of order $r = \operatorname{rank}(P)$ and K is $r\times(m-r)$; see Hartwig & Loewy (1992). Hence $I_m - PP^* = U\begin{pmatrix} -KK^* & 0 \\ 0 & I_{m-r} \end{pmatrix}U^*$ and $2I_m - P - P^* = U\begin{pmatrix} 0 & -K \\ -K^* & 2I_{m-r} \end{pmatrix}U^*$. It then follows that $2I_m - P - P^* = (I_m - PP^*)U\begin{pmatrix} 0 & K^{\dagger *} \\ -K^* & 2I_{m-r} \end{pmatrix}U^*$ and therefore $I_m - PP^* = (2I_m - P - P^*)U\begin{pmatrix} 2I_r & -K^{\dagger *} \\ K^* & \frac{1}{2}(I_{m-r} - K^\dagger K) \end{pmatrix}U^*$, where $K^\dagger$ is the Moore–Penrose inverse of K. The asserted identity is established.

Solution 29-8.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

The claim is an immediate consequence of the following more informative theorem.

THEOREM. *Let P be an idempotent matrix of order m, and let $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the range and null space, respectively. Then:*

(a) $\mathcal{N}(P+P^*) = \mathcal{N}(P)\cap\mathcal{N}(P^*)$, (b) $\mathcal{N}(I-P+I-P^*) = \mathcal{R}(P)\cap\mathcal{R}(P^*)$, (c) $\mathcal{N}(I-PP^*) = \mathcal{N}(I-P^*P) = \mathcal{R}(P)\cap\mathcal{R}(P^*)$.

PROOF: If $P = 0$, then the theorem is trivial. Therefore, let $P \neq 0$. According to the singular value decomposition, P can then be written as $P = S_0D_0T_0^* + S_1D_1T_1^* + S_2D_2T_2^*$, where $S = (S_0, S_1, S_2)$ and $T = (T_0, T_1, T_2)$ are unitary $m \times m$ matrices (i.e., $S^*S = I_m$ and $T^*T = I_m$) composed of left and right singular vectors of P, and where $D = \operatorname{diag}(D_0, D_1, D_2)$ is an accordingly partitioned nonnegative diagonal matrix consisting of the associated singular values of P. Without loss of generality, we assume that $D_0 = 0$, $D_1 = I$, and all the diagonal elements of D_2 are different from zero and unity. If some of the blocks in D do not exist, we interpret the corresponding summands and terms in the decomposition $P = S_0D_0T_0^* + S_1D_1T_1^* + S_2D_2T_2^*$, and in expressions which follow as absent. It is well known that the singular values of P are the nonnegative square roots of the eigenvalues of PP^* and that the nonnegative definite Hermitian matrices PP^* and P^*P have the same eigenvalues with the same (algebraic) multiplicities. Clearly, $S_1T_1^* + S_2D_2T_2^*$ is a full rank factorization of P, and so P is idempotent if and only if $\begin{pmatrix} I & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} T_1^* \\ T_2^* \end{pmatrix}(S_1 \;\; S_2) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ or, equivalently, if and only if $T_1^*S_1 = I$, $T_1^*S_2 = 0$, $T_2^*S_1 = 0$, $T_2^*S_2 = D_2^{-1}$. In view of $T_1^*T_1 = I$ and $S_1^*S_1 = I$, we therefore get $(T_1 - S_1)^*(T_1 - S_1) = T_1^*T_1 - T_1^*S_1 - S_1^*T_1 + S_1^*S_1 = 0$ or, equivalently, $S_1 = T_1$ provided S_1 and T_1 exist.

(a): We note that $(P + P^*)(P + P^*) = P + P^* + PP^* + P^*P$ is a nonnegative definite Hermitian matrix. Since PP^* and P^*P are also nonnegative definite Hermitian matrices, it follows that $\mathcal{N}(P + P^*) = \mathcal{N}((P + P^*)^2) \subseteq \mathcal{N}(PP^* + P^*P) = \mathcal{N}(PP^*) \cap \mathcal{N}(P^*P) = \mathcal{N}(P^*) \cap \mathcal{N}(P) \subseteq \mathcal{N}(P + P^*)$, and so (a) is established.

(b): Since a matrix M is idempotent if and only if $I - M$ is idempotent, (b) follows from (a) since $\mathcal{N}(I - M) = \mathcal{R}(M)$.

(c): Clearly, $0 \neq x \in \mathcal{N}(I - PP^*)$ if and only if x belongs to the eigenspace of the eigenvalue 1 of PP^*. This eigenspace is spanned by the columns of S_1 and is trivially contained in the range of P. Likewise, $0 \neq x \in \mathcal{N}(I - P^*P)$ if and only if $x \in \mathcal{R}(T_1)$, which is the eigenspace of the eigenvalue 1 of P^*P which is contained in $\mathcal{R}(P^*)$. In which case $S_1 = T_1$, and so it is clear that in any case $\mathcal{N}(I - PP^*) = \mathcal{N}(I - P^*P)$ and $\mathcal{N}(I - PP^*) \subseteq \mathcal{R}(P) \cap \mathcal{R}(P^*)$. The converse inclusion is trivial. □

Solutions to Problem 29-8 were also received from Johanns de Andrade Bezerra *and from the Proposer* Yongge Tian.

Problem 29-9: Equality of Two Nonnegative Definite Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two nonnegative definite Hermitian matrices of the same order, and let $(\cdot)^\dagger$ denote the Moore–Penrose inverse of the matrix $(\cdot)$. Show that the following five statements are equivalent:

$$\text{(a) } A = B, \quad \text{(b) } A + AA^\dagger = B + BB^\dagger, \quad \text{(c) } AB^\dagger A = B,$$
$$\text{(d) } \operatorname{rank}(A) = \operatorname{rank}(B) \text{ and } 2A(A + B)^\dagger A = A, \quad \text{(e) range}\begin{pmatrix} A \\ B \end{pmatrix} = \text{range}\begin{pmatrix} B \\ A \end{pmatrix}.$$

Solution 29-9.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

The first observation is that the condition $AB^\dagger A = B$ alone, as given in (c), is in general not suffficient to entail $A = B$. A simple example is provided by the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It becomes sufficient, however, when accompanied by its counterpart $BA^\dagger B = A$ or by the range equality $\mathcal{R}(A) = \mathcal{R}(B)$. The lemma below reveals relationships between these three conditions themselves and the condition (e). They are established under the assumption that the matrices involved are EP, which is obviously weaker than the requirement that they are Hermitian nonnegative definite.

LEMMA. *Let $A, B \in \mathbb{C}_{n,n}$ be EP matrices, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$ and $\mathcal{R}(B) = \mathcal{R}(B^*)$. Then any two of the following three conditions:* (i) $AB^\dagger A = B$, (ii) $BA^\dagger B = A$, (iii) $\mathcal{R}(A) = \mathcal{R}(B)$, *imply the third. Moreover, the three conditions together are equivalent to $\mathcal{R}((A^* : B^*)^*) = \mathcal{R}((B^* : A^*)^*)$.*

PROOF. The part "(i), (ii) $\Rightarrow$ (iii)" is obvious. Further, if $AB^\dagger A = B$ holds along with $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$, which follow from (iii) and the assumption that A and B are EP matrices, then $BA^\dagger B = AB^\dagger AA^\dagger B = AB^\dagger B = A$. Similarly, combining $BA^\dagger B = A$ with $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$ yields $AB^\dagger A = BA^\dagger BB^\dagger A = BA^\dagger A = B$. Moreover, the subspaces $\mathcal{R}((A^* : B^*)^*)$ and $\mathcal{R}((B^* : A^*)^*)$ are identical if and only if $A = BK$ and $B = AK$ for some $K \in \mathbb{C}_{n,n}$. We see that both $K = A^\dagger B$ and $K = B^\dagger A$ are suitable choices of K. On the other hand, if $A = BK$ and $B = AK$, then $\mathcal{R}(A) = \mathcal{R}(B)$ and hence the assumption that A and B are EP leads to $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$, which entails $AB^\dagger A = AB^\dagger BK = AK = B$. □

In Propositions 1 and 2 below it is shown that the equality $A = B$ is implied by conditions (d) and (b) proposed in Problem 29-9 within wider classes of matrices than Hermitian nonnegative definite ones.

PROPOSITION 1. *For any parallel summable matrices $A, B \in \mathbb{C}_{m,n}$, if $2A(A+B)^\dagger A = A$ and* $\operatorname{rank}(A) = \operatorname{rank}(B)$, *then $A = B$.*

PROOF. It is clear that if $2A(A+B)^\dagger A = A$, then $2(A+B)(A+B)^\dagger A - 2B(A+B)^\dagger A = A = 2A(A+B)^\dagger(A+B) - 2A(A+B)^\dagger B$. Immediate consequences of the inclusions $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*)$, which according to Rao & Mitra (1971, p. 189) constitute the definition of parallel summability, are the equalities $(A+B)(A+B)^\dagger A = A = A(A+B)^\dagger(A+B)$, and thus it follows that $2B(A+B)^\dagger A = A = 2A(A+B)^\dagger B$. This shows that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$, and combining these conditions with $\operatorname{rank}(A) = \operatorname{rank}(B)$ yields $\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{R}(A+B)$ and $\mathcal{R}(A^*) = \mathcal{R}(B^*) = \mathcal{R}(A^* + B^*)$. Hence $A^\dagger A(A+B)^\dagger = (A+B)^\dagger = (A+B)^\dagger AA^\dagger$, and thus premultiplying and postmultiplying $2A(A+B)^\dagger A = A$ by $A^\dagger$ leads to $2(A+B)^\dagger = A^\dagger$. Consequently, in view of the uniqueness of the Moore–Penrose inverse, $A + B = 2A$, i.e., $A = B$. □

We note that the assumption of parallel summability of A and B in Proposition 1 entails the possibility of replacing the Moore–Penrose inverse $(A+B)^\dagger$ in the expression $A(A+B)^\dagger A$ by any generalized inverse of $A + B$. We see immediately that condition (b) implies $A = B$ whenever A and B are nonsingular.

Another class of matrices having the desired property is revealed in the following proposition.

PROPOSITION 2. *For any normal matrices $A, B \in \mathbb{C}_{n,n}$ not containing -1 in their spectra, if $A + AA^\dagger = B + BB^\dagger$, then $A = B$.*

PROOF. According to Theorem 2.5.4 in Horn & Johnson (1985), a matrix is normal if and only if it is unitarily diagonalizable. Consequently, if A is of rank a, say, then $A = UDU^*$, where $U \in \mathbb{C}_{n,a}$ satisfies $U^*U = I_a$ and D is an $a \times a$ diagonal matrix, with diagonal elements $d_i \neq 0$ and $d_i \neq -1$, $i = 1, \ldots, a$. Then $D + I_a$ is nonsingular, and therefore $\mathcal{R}(A + AA^\dagger) = \mathcal{R}[U(D + I_a)U^*] = \mathcal{R}(U) = \mathcal{R}(A)$. Similarly, $\mathcal{R}(B + BB^\dagger) = \mathcal{R}(B)$ and, consequently, $A + AA^\dagger = B + BB^\dagger$ implies $\mathcal{R}(A) = \mathcal{R}(B)$. Hence $AA^\dagger = BB^\dagger$, thus leading to $A = B$. □

It is interesting to notice that the result of Proposition 2 is not valid within the set of all normal, or even within the set of all Hermitian, matrices. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $A + AA^\dagger = B + BB^\dagger$, but $A \neq B$. Within the same set, also the pair of equalities $AB^\dagger A = B$ and $BA^\dagger B = A$ (which constitute a corrected version of (c) equivalent to (e)) is not sufficient to imply $A = B$. A trivial example is provided by $A = (1)$ and $B = (-1)$. The assumption concerning matrices involved in this part of our solution to Problem 29-9 is as in its original formulation.

PROPOSITION 3. *For any Hermitian nonnegative definite matrices $A, B \in \mathbb{C}_{n,n}$, if $AB^\dagger A = B$ and $BA^\dagger B = A$, then $A = B$.*

PROOF. From our Lemma and with the use of the notation $C = (B^\dagger)^{1/2}A(B^\dagger)^{1/2}$, the equation $AB^\dagger A = B$ takes the form $C^2 = BB^\dagger$. Hence, if a spectral decomposition of C is $C = UDU^*$, then in view of $\mathcal{R}(C) = \mathcal{R}(B)$, which is equivalent to $BB^\dagger = CC^\dagger = UU^*$, it follows that $D^2 = I_a$, thus leading to $C = UU^*$. Consequently, $(B^\dagger)^{1/2}A(B^\dagger)^{1/2} = (B^\dagger)^{1/2}B(B^\dagger)^{1/2}$, and hence, in view of the equality $\mathcal{R}(A) = \mathcal{R}(B)$, by premultiplying and postmultiplying by $B^{1/2}$ we arrive at the conclusion that $A = B$. □

Solution 29-9.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Problem 29-9 was not correctly stated, in that condition (a) is not equivalent to condition (c). Of course, (c) is trivially necessary for (a) to hold. However, it is not sufficient, as the scalars (1×1 matrices) $A = 1$ and $B = 0$ illustrate. Clearly, $A = A^\dagger$ and $B = B^\dagger$. Moreover, $AB^\dagger A = B$, although $A \neq B$. Next, we prove the following corrected version of Problem 29-9.

THEOREM. *Let A, $B \in \mathbb{C}^{m,m}$ be nonnegative definite Hermitian matrices, and let $\mathcal{R}(\cdot)$ denote range (column space). Then the following five conditions are equivalent:*

$$\text{(a) } A = B; \quad \text{(b) } A + AA^\dagger = B + BB^\dagger; \quad \text{(c) } \operatorname{rank}(A) = \operatorname{rank}(B) \textit{ and } AB^\dagger A = B;$$

$$\text{(d) } \operatorname{rank}(A) = \operatorname{rank}(B) \textit{ and } 2A(A+B)^\dagger A = A; \quad \text{(e) } \mathcal{R}\begin{pmatrix} A \\ B \end{pmatrix} = \mathcal{R}\begin{pmatrix} B \\ A \end{pmatrix}.$$

PROOF. It suffices to show that each of the conditions (b)–(e) is sufficient for (a) to hold.

(b) $\Rightarrow$ (a): It is well known that for any given matrix M the matrix $MM^\dagger$ is the orthogonal projector onto $\mathcal{R}(M)$, denoted by $P_{\mathcal{R}(M)}$. Therefore, if N is another matrix with $\mathcal{R}(M) = \mathcal{R}(N)$, then clearly $MM^\dagger = NN^\dagger$. If M is a nonnegative definite Hermitian matrix, then the orthogonal projector onto $\mathcal{R}(M)$, i.e., $MM^\dagger$, is also nonnegative definite and Hermitian. Since the range of the sum of nonnegative definite Hermitian matrices is the sum of the ranges of the summands, clearly $\mathcal{R}(M + MM^\dagger) = \mathcal{R}(M)$. By means of these observations it is clear that condition (b) implies $\mathcal{R}(A) = \mathcal{R}(B)$ or, equivalently, $AA^\dagger = BB^\dagger$. Eq. $A + AA^\dagger = B + BB^\dagger$ therefore reduces, as claimed, to $A = B$.

(c) $\Rightarrow$ (a): Let $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $AB^\dagger A = B$. Then evidently $\mathcal{R}(B) = \mathcal{R}(A)$. Moreover, $(AB^\dagger)^2 = BB^\dagger$. Since $BB^\dagger$ is idempotent, its eigenvalues are all equal to one or zero; see Lancaster (1969, Ex. 4, p. 65). Following Lancaster (1969, Ex. 12, p. 104), $AB^\dagger$ and $A^{1/2}B^\dagger A^{1/2}$ have the same characteristic polynomials and hence the same eigenvalues. Clearly, since $A^{1/2}B^\dagger A^{1/2}$ is a nonnegative definite Hermitian matrix, these eigenvalues are all nonnegative. From Lancaster (1969, Th. 2.5.2, p. 64) we know that if $\mu_1, \cdots, \mu_n$ are the eigenvalues of any $n \times n$ matrix M and p is any scalar polynomial, then the eigenvalues of $p(M)$ are $p(\mu_1), \cdots, p(\mu_n)$. Hence by combining our observations it follows that the eigenvalues of $AB^\dagger$ as well as those of the matrix $A^{1/2}B^\dagger A^{1/2}$ are all equal to zero and one. The nonnegative definite Hermitian matrix $A^{1/2}B^\dagger A^{1/2}$ is then necessarily idempotent, i. e., $A^{1/2}B^\dagger A^{1/2}A^{1/2}B^\dagger A^{1/2} = A^{1/2}B^\dagger A^{1/2}$. Pre- and postmultiplying this eq. by $A^{1/2}$ and $A^{1/2}A^\dagger$, respectively, we obtain $AB^\dagger AB^\dagger AA^\dagger = AB^\dagger AA^\dagger$. But this equation readily reduces to $(AB^\dagger)^2 = AB^\dagger$, for in view of $\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{R}(B^\dagger)$ we have $B^\dagger AA^\dagger = B^\dagger P_{\mathcal{R}(A)} = B^\dagger$. Hence, since $AB^\dagger$ is idempotent, eq. $(AB^\dagger)^2 = BB^\dagger$ becomes $AB^\dagger = BB^\dagger$. Postmultiplying this equation by B results in $0 = (A - B)B^\dagger B = A - B$. As desired, we thus arrive at $A = B$.

(d) $\Rightarrow$ (a): Let $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $2A(A+B)^\dagger A = A$. Then $A\left(\frac{1}{2}(A+B)\right)^\dagger A = A$ or, equivalently, $A(A+B)^\dagger A = \frac{1}{2}A$. Since $\mathcal{R}(A+B) = \mathcal{R}(A)+\mathcal{R}(B)$ holds for nonnegative definite Hermitian matrices A and B, it follows that $(A+B)(A+B)^\dagger A = A$. Therefore $B(A+B)^\dagger A = \frac{1}{2}A = A(A+B)^\dagger B$, and hence, in view of $\operatorname{rank}(A) = \operatorname{rank}(B)$, clearly $\mathcal{R}(A) = \mathcal{R}(B)$ or, equivalently, $\mathcal{N}(A) = \mathcal{N}(B)$. Consequently, $\mathcal{R}(\frac{1}{2}(A+B)) = \mathcal{R}(A) = \mathcal{R}((\frac{1}{2}(A+B))^\dagger)$ and $\mathcal{N}(\frac{1}{2}(A+B)) = \mathcal{N}(A) = \mathcal{N}((\frac{1}{2}(A+B))^\dagger)$, and $\left(\frac{1}{2}(A+B)\right)^\dagger$ is thus a reflexive g-inverse of A with the same column space and the same null space as $A^\dagger$. But then necessarily $\left(\frac{1}{2}(A+B)\right)^\dagger = A^\dagger$ or, equivalently, $\frac{1}{2}(A+B) = A$. This in turn easily reduces to $A = B$.

(e) $\Rightarrow$ (a): When (e) holds, there obviously exists a matrix X with $B = AX$ and $A = BX$, which in turn implies $\mathcal{R}(A) = \mathcal{R}(B)$. If X is a solution to $A = BX$, then X can be written as $B^- A$ for some suitable g-inverse B^- of B. Hence $B = AX = AB^- A$. Since $\mathcal{R}(A) = \mathcal{R}(B)$, we have $AB^- A = B$ for any choice of B^-. Therefore, in particular, $AB^\dagger A = B$ and so the conditions (c) hold. Since we have already shown that (c) $\Rightarrow$ (a), our proof is complete. □

A solution to the corrected Problem 29-9 was also received from the Proposer Yongge Tian, *with apologies for the error in the original statement of Problem 29-9.*

Problem 29-10: Equivalence of Three Reverse-Order Laws

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Show that $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [(A^\dagger)^* B]^\dagger = B^\dagger A^* \Leftrightarrow [A(B^\dagger)^*]^\dagger = B^* A^\dagger$, where $(\cdot)^\dagger$ and $(\cdot)^*$ denote the Moore–Penrose inverse and the conjugate transpose, respectively.

Solution 29-10.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl

A solution will be obtained with the use of another general property of a pair of matrices satisfying the reverse order law.

LEMMA. *For any* $K \in \mathbb{C}_{m,n}$ *and* $L \in \mathbb{C}_{n,p}$, $(KL)^\dagger = L^\dagger K^\dagger \;\Rightarrow\; L^*L(KL)^\dagger = [K(L^\dagger)^*]^\dagger$ *and* $(KL)^\dagger KK^* = [(K^\dagger)^* L]^\dagger$.

PROOF. Let $F_1 = K(L^\dagger)^*$ and $G_1 = L^*L(KL)^\dagger$. Then $F_1G_1 = K(LL^\dagger)^*L(KL)^\dagger = KL(KL)^\dagger$, and hence it is clear that $G_1F_1G_1 = G_1$ and $F_1G_1 = (F_1G_1)^*$. Since $F_1 = KLL^\dagger(L^\dagger)^*$, another consequence is that $F_1G_1F_1 = F_1$. Similarly, let $F_2 = (K^\dagger)^*L$ and $G_2 = (KL)^\dagger KK^*$. Then $G_2F_2 = (KL)^\dagger KL$, and hence $G_2F_2G_2 = G_2$ and $G_2F_2 = (G_2F_2)^*$. Further, the representation $F_2 = (K^\dagger)^*K^\dagger KL$ shows that $F_2G_2F_2 = F_2$. Moreover, under the assumption $(KL)^\dagger = L^\dagger K^\dagger$, it follows that

$$G_1F_1 = L^*LL^\dagger K^\dagger K(L^\dagger)^* = L^*K^\dagger K(L^\dagger)^* = (L^\dagger K^\dagger KL)^* = (KL)^\dagger KL \tag{7}$$

and

$$F_2G_2 = (K^\dagger)^*LL^\dagger K^\dagger KK^* = (K^\dagger)^*LL^\dagger K^* = (KLL^\dagger K^\dagger)^* = KL(KL)^\dagger. \tag{8}$$

From (7) and (8) it is seen that $G_1F_1 = (G_1F_1)^*$ and $F_2G_2 = (F_2G_2)^*$, which completes the sets of conditions defining G_i to be the Moore–Penrose inverse of F_i, $i = 1, 2$. □

We note that the implication in this lemma cannot be reversed. A counterexample is provided by the orthogonal projectors $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $L = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, in which case $KK^* = K = (K^\dagger)^*$ and $L^*L = L = (L^\dagger)^*$, thus transforming the conditions on the right-hand side to $L(KL)^\dagger = (KL)^\dagger = (KL)^\dagger K$. Both these conditions are fulfilled, but $(KL)^\dagger \neq L^\dagger K^\dagger$.

On the other hand, the proof above shows that the assumption $(KL)^\dagger = L^\dagger K^\dagger$ was actually used only for establishing (7) and (8). This observation leads to the remark below, in which generalized inverses of matrices are denoted according to Definition 1 in Ben-Israel & Greville (1974, p. 8).

REMARK. Let $K \in \mathbb{C}_{m,n}$ and $L \in \mathbb{C}_{n,p}$. Then, for any $(KL)^{(1,2,3)}$ and $L^{(1,2,3)}$, the matrix $L^*L(KL)^{(1,2,3)}$ is a $\{1,2,3\}$-generalized-inverse of $K(L^{(1,2,3)})^*$ and, for any $(KL)^{(1,2,4)}$ and $K^{(1,2,4)}$, the matrix $(KL)^{(1,2,4)}KK^*$ is a $\{1,2,4\}$-generalized-inverse of $(K^{(1,2,4)})^*L$.

The implication in the lemma easily leads to a solution of Problem 29-10. Postmultiplying $(AB)^\dagger = B^\dagger A^\dagger$ by AA^* yields $(AB)^\dagger AA^* = B^\dagger A^*$, while from the lemma it follows that $(AB)^\dagger AA^* = [(A^\dagger)^*B]^\dagger$, thus showing that $(AB)^\dagger = B^\dagger A^\dagger \Rightarrow [(A^\dagger)^*B]^\dagger = B^\dagger A^*$. Conversely, postmultiplying the last equality by $(A^\dagger)^*A^\dagger$ yields $[(A^\dagger)^*B]^\dagger(A^\dagger)^*A^\dagger = B^\dagger A^\dagger$, and from the lemma it follows that $[(A^\dagger)^*B]^\dagger(A^\dagger)^*A^\dagger = (AB)^\dagger$, thus completing the proof that $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [(A^\dagger)^*B]^\dagger = B^\dagger A^*$. Similarly, premultiplying $(AB)^\dagger = B^\dagger A^\dagger$ by B^*B yields $B^*B(AB)^\dagger = B^*A^\dagger$, while from the lemma it follows that $B^*B(AB)^\dagger = [A(B^\dagger)^*]^\dagger$. On the other hand, premultiplying $[A(B^\dagger)^*]^\dagger = B^*A^\dagger$ by $B^\dagger(B^\dagger)^*$ yields $B^\dagger(B^\dagger)^*[A(B^\dagger)^*]^\dagger = B^\dagger A^\dagger$, while from the lemma it follows that $B^\dagger(B^\dagger)^*[A(B^\dagger)^*]^\dagger = (AB)^\dagger$, thus completing the proof that $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [A(B^\dagger)^*]^\dagger = B^*A^\dagger$.

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We present an elementary solution based directly on the definition of the Moore–Penrose inverse of a given complex matrix $K \in \mathbb{C}_{m,n}$ as the unique matrix $K^\dagger \in \mathbb{C}_{n,m}$ satisfying the conditions $KK^\dagger K = K$, $K^\dagger KK^\dagger = K^\dagger$, $KK^\dagger \in \mathcal{H}$, and $K^\dagger K \in \mathcal{H}$, where $\mathcal{H}$ stands for the set of Hermitian matrices of an appropriate order. It can easily be verified that

$$AB(B^\dagger A^\dagger)AB = AB \;\Leftrightarrow\; (A^\dagger)^*A^\dagger ABB^\dagger(A^\dagger A)^*B = (A^\dagger)^*A^\dagger AB \;\Leftrightarrow\; (A^\dagger)^*B(B^\dagger A^*)(A^\dagger)^*B = (A^\dagger)^*B,$$

$$(B^\dagger A^\dagger)AB(B^\dagger A^\dagger) = B^\dagger A^\dagger \;\Leftrightarrow\; B^\dagger(A^\dagger A)^*BB^\dagger A^\dagger AA^* = B^\dagger A^\dagger AA^* \;\Leftrightarrow\; (B^\dagger A^*)(A^\dagger)^*B(B^\dagger A^*) = B^\dagger A^*,$$

$$AB(B^\dagger A^\dagger) \in \mathcal{H} \;\Leftrightarrow\; [A(BB^\dagger)A^\dagger]^* \in \mathcal{H} \;\Leftrightarrow\; (A^\dagger)^*B(B^\dagger A^*) \in \mathcal{H},$$

$$(B^\dagger A^\dagger)AB \in \mathcal{H} \;\Leftrightarrow\; B^\dagger(A^\dagger A)^*B \in \mathcal{H} \;\Leftrightarrow\; (B^\dagger A^*)(A^\dagger)^*B \in \mathcal{H},$$

thus establishing that

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [(A^\dagger)^* B]^\dagger = B^\dagger A^*. \tag{9}$$

Similar arguments show that $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow [A(B^\dagger)^*]^\dagger = B^* A^\dagger$. However, a solution to the problem can also be completed by combining (9) with the observation that, in view of $(K^\dagger)^* = (K^*)^\dagger$ and $(K^\dagger)^\dagger = K$, we see that $[(A^\dagger)^* B]^\dagger = B^\dagger A^* \Leftrightarrow (B^* A^\dagger)^\dagger = A(B^\dagger)^* \Leftrightarrow [A(B^\dagger)^*]^\dagger = B^* A^\dagger$.

Solution 29-10.3 by Shizhen CHENG, *Tianjin Polytechnical University, Tianjin, China:* csz@mail.tjpu.edu.cn

We first show that

$$(AB)^\dagger = [(A^\dagger)^* B]^\dagger (A^\dagger)^* (B^\dagger)^* [A(B^\dagger)^*]^\dagger. \tag{10}$$

Let $X = (A^\dagger)^* B$ and $Y = A(B^\dagger)^*$, and let $\mathcal{R}$ denote range. Then $\mathcal{R}(X^*) = \mathcal{R}(B^* A^\dagger) = \mathcal{R}(B^* A^*) = \mathcal{R}[(AB)^*]$ and $\mathcal{R}(Y) = \mathcal{R}[A(B^\dagger)^*] = \mathcal{R}(AB)$. Hence $X^\dagger X = (AB)^\dagger (AB)$ and $YY^\dagger = (AB)(AB)^\dagger$. Thus $X^\dagger (A^\dagger)^* (B^\dagger)^* Y^\dagger = X^\dagger (A^\dagger)^* B B^\dagger (B^\dagger)^* Y^\dagger = X^\dagger X B^\dagger (B^\dagger)^* Y^\dagger = (AB)^\dagger AB B^\dagger (B^\dagger)^* Y^\dagger = (AB)^\dagger A (B^\dagger)^* Y^\dagger = (AB)^\dagger Y Y^\dagger = (AB)^\dagger (AB)(AB)^\dagger = (AB)^\dagger$, and so (10) holds. If $(AB)^\dagger = B^\dagger A^\dagger$, then $(B^\dagger A^\dagger)^\dagger = AB$ and $[(A^\dagger)^*(B^\dagger)^*]^\dagger = B^* A^*$. Hence $[(A^\dagger)^* B]^\dagger = (AB)^\dagger A (B^\dagger)^* [(A^\dagger)^* (B^\dagger)^*]^\dagger = B^\dagger A^\dagger A (B^\dagger)^* B^* A^* = B^\dagger A^\dagger AB B^\dagger A^* = B^\dagger A^\dagger AB B^\dagger A^\dagger A A^* = B^\dagger A^\dagger A A^* = B^\dagger A^*$. This shows that $(AB)^\dagger = B^\dagger A^\dagger$ implies $[(A^\dagger)^* B]^\dagger = B^\dagger A^*$. By symmetry, $[(A^\dagger)^* B]^\dagger = B^\dagger A^*$ also implies $(AB)^\dagger = B^\dagger A^\dagger$. Thus $(AB)^\dagger = B^\dagger A^\dagger$ and $[(A^\dagger)^* B]^\dagger = B^\dagger A^*$ are equivalent. The equivalence of $(AB)^\dagger = B^\dagger A^\dagger$ and $[A(B^\dagger)^*]^\dagger = B^* A^\dagger$ can be shown similarly.

Solution 29-10.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Let A and B be complex matrices such that AB exists. One of the well-known shortcomings of the Moore–Penrose inverse is that *the reverse order law* does not always hold. That is, for some pairs of matrices A, B the relation $(AB)^\dagger = B^\dagger A^\dagger$ holds, and for others it does not. This observation suggests the question, when does $(AB)^\dagger = B^\dagger A^\dagger$? Arghiriade (1963) and Greville (1966) have already given two criteria for distinguishing the cases for which $(AB)^\dagger = B^\dagger A^\dagger$ holds. Several other authors have also contributed to this question and they have investigated a similar question for other (more general) special classes of generalized inverses of A and B; see Werner (1992) for more details and a list of references. According to Greville (1966), $(AB)^\dagger = B^\dagger A^\dagger$ holds if and only if the conditions $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ are simultaneously satisfied; here and below we denote by $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ the range (column space) and the null space of the matrix $(\cdot)$, respectively. A solution to the present problem is found immediately just by combining Greville's conditions for the different *reverse order equations* with the equivalences of the corresponding conditions (a) and (g) in the following lemma.

LEMMA. *Let M and N be any complex matrices such that the product MN exists. Then the following seven conditions are equivalent:*

(a) $\mathcal{R}(M^*MN) \subseteq \mathcal{R}(N)$; (b) $\mathcal{R}(M^*MN) \subseteq \mathcal{R}(N) \cap \mathcal{R}(M^*)$; (c) $(M^*M)\mathcal{R}(N) = \mathcal{R}(N) \cap \mathcal{R}(M^*)$;

(d) $(M^*M)[\mathcal{R}(N) \cap \mathcal{R}(M^*)] = \mathcal{R}(N) \cap \mathcal{R}(M^*)$ *and* $\mathcal{R}(N) = [\mathcal{R}(N) \cap \mathcal{R}(M^*)] \oplus [\mathcal{R}(N) \cap \mathcal{N}(M)]$;

(e) $(M^*M)^\dagger [\mathcal{R}(N) \cap \mathcal{R}(M^*)] = \mathcal{R}(N) \cap \mathcal{R}(M^*)$ *and* $\mathcal{R}(N) = [\mathcal{R}(N) \cap \mathcal{R}(M^*)] \oplus [\mathcal{R}(N) \cap \mathcal{N}(M)]$;

(f) $(M^*M)^\dagger \mathcal{R}(N) = \mathcal{R}(N) \cap \mathcal{R}(M^*)$; (g) $\mathcal{R}(M^\dagger M^{\dagger *} N) \subseteq \mathcal{R}(N)$.

PROOF. Since M^*M is a nonnegative definite Hermitian matrix with $\mathcal{R}(M^*M) = \mathcal{R}(M^*)$ and $\mathcal{N}(M^*M) = \mathcal{N}(M)$, it is not difficult to see that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). That (d) is equivalent to (e) is a direct consequence of the fact that $(M^*M)^\dagger(M^*M) = (M^*M)(M^*M)^\dagger$ is the orthogonal projector onto $\mathcal{R}(M^*)$ along $\mathcal{N}(M)$. Since $(M^*M)^\dagger = M^\dagger (M^\dagger)^*$, $\mathcal{R}(M^\dagger) = \mathcal{R}(M^*)$ and $\mathcal{N}(M^{\dagger *}) = \mathcal{N}(M)$, it is also clear that (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g), and so our proof is complete. □

A solution to Problem 29-10 was also received from the Proposer Yongge Tian.

Problem 29-11: The Minimal Rank of a Block Matrix with Generalized Inverses

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Let $(\cdot)^-$ denote generalized inverse. Show that

$$\min_{A^-,\,B^-,\,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} = \max\{\operatorname{rank}(A),\quad \operatorname{rank}(B)+\operatorname{rank}(C)\}.$$

While we have received a solution from its Proposer Yongge Tian, *we look forward to receiving further solutions to Problem 29-11.*

Problem 29-12: Matrices Commuting with the Vector Cross Product

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Let a nonzero vector $a \in \mathbb{R}^3$ be given. Find all square matrices A with real entries such that (a) for all $x \in \mathbb{R}^3$, it follows that $a \times Ax = A(a \times x)$ and (b) for all $x \in \mathbb{R}^3$, it follows that $a \times Ax = (Aa) \times x$. Here $\times$ denotes the vector cross product in $\mathbb{R}^3$.

Solution 29-12.1 by William F. TRENCH, *Woodland Park, Colorado, USA:* wtrench@trinity.edu

Without essential loss of generality, let $a = (\,a_1 \quad a_2 \quad a_3\,)^T$ be a unit vector. Let $b = (\,b_1 \quad b_2 \quad b_3\,)^T$ be an arbitrary unit vector perpendicular to a, $c = a \times b = (\,c_1 \quad c_2 \quad c_3\,)^T$, and $Q = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Let L be the linear transformation with matrix A relative to the natural basis for $\mathbb{R}^3$.

(a) From the assumptions, we have (i) $a \times La = L(a \times a) = 0$, (ii) $a \times Lb = L(a \times b) = Lc$, and (iii) $a \times Lc = L(a \times c) = -Lb$. From (i), $La = \lambda a$ for some real λ. From (ii) and (iii), Lc and Lb are in the plane of b and c; i.e., $Lb = p_1 b + q_1 c$ and $Lc = p_2 b + q_2 c$. From (ii), $p_2 b + q_2 c = p_1(a \times b) + q_1(a \times c) = -q_1 b + p_1 c$, so $q_2 = p_1$ and $p_2 = -q_1$. (These conclusions also follow from (iii).) We drop the subscripts and write $Lb = pb + qc$ and $Lc = -qb + pc$. Therefore the matrix of L with respect to $\{a, b, c\}$ is $B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & p & -q \\ 0 & q & p \end{pmatrix}$ and $A = QBQ^T$. Conversely, it is straightforward to verify that any matrix A of this form has the desired property.

(b) Since $a \times La = (La) \times a = -(a \times La)$, it follows that $a \times La = 0$, so $La = \lambda a$ for some real λ. Hence $a \times Lb = (La) \times b = \lambda(a \times b) = \lambda c$, and $a \times Lc = (La) \times c = \lambda(a \times c) = -\lambda b$ so Lb is in the plane of a and b and Lc is in the plane of a and c; thus, $Lb = pa + rb$ and $Lc = qa + sc$. Therefore $\lambda c = a \times Lb = p(a \times a) + r(a \times b) = rc$ and $-\lambda b = a \times Lc = q(a \times a) + s(a \times c) = -sb$, so $Lb = pa + \lambda b$ and $Lc = qa + \lambda c$. Therefore the matrix of L with respect to $\{a, b, c\}$ is $C = \begin{pmatrix} \lambda & p & q \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, and $A = QCQ^T$. Conversely, it is straightforward to verify that any matrix A of this form has the desired property.

Solution 29-12.2 by the Proposers Dietrich TRENKLER, *Universität Osnabrück, Germany:* dtrenkler@nts6.oec.uni.osnabrueck.de and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

PROOF of (a). Since the cross product $a \times x$ is linear in the second component, there exists a unique matrix T_a such that $T_a x = a \times x$ for all $x \in \mathbb{R}^3$. When $a = (\,a_1 \quad a_2 \quad a_3\,)'$, the matrix T_a is readily seen to be of the form $T_a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$, see Noble (1969). Hence our problem is equivalent to the search for all matrices A commuting with T_a. By some straightforward calculations, it follows that the characteristic polynomial $\mathsf{P}(\lambda)$ of T_a is $\mathsf{P}(\lambda) = \det(T_a - \lambda I) = -\lambda^3 - \lambda a'a$. Hence T_a is nonderogatory, i.e., every eigenvalue of A has geometric multiplicity 1. By Theorem 3.2.4.2 in Horn & Johnson (1985), A is a polynomial in T_a of degree at most 2. Thus we conclude $A = \alpha_1 I_3 + \alpha_2 T_a + \alpha_3 T_a^2$, or equivalently, by noting that $T_a^2 = aa' - a'aI_3$, $A = \beta_1 I_3 + \beta_2 T_a + \beta_3 aa'$, where β_1, β_2, β_3 are real constants. Observe that this class of matrices comprises the rotations by a certain angle about an axis in $\mathbb{R}^3$; see Noble (1969, p. 421) or Room (1952). The above problem can be modified by setting brackets differently. Then we might wish to identify all real square matrices A satisfying $a \times Ax = (Aa) \times x$ for all $x \in \mathbb{R}^3$.

To prove (b) we find all matrices A such that $T_a A = T_{Aa}$. Clearly, this equation has at least one solution, namely $A = I_3$. To determine all solution matrices A we compare the left-hand and right-hand sides of the above equation. Then we arrive at a homogenous linear equation system of 9 equations with 9 unknowns. Using the Gaussian elimination method and *Mathematica* we see that the solution subspace is three dimensional with a possible basis set consisting of the matrices A_1, A_2 and A_3, where $A_1 = I_3$. The other two basis matrices depend on the coefficients of a, as follows, and this completes our proof of (b).

$$\text{(i) } a_3 \neq 0: \; A_2 = \begin{pmatrix} a_2a_3 & a_1a_3 & -a_1a_2 \\ 0 & 2a_2a_3 & -a_2^2 \\ 0 & a_3^2 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 2a_1a_3 & 0 & -a_1^2 \\ a_2a_3 & a_1a_3 & -a_1a_2 \\ a_3^2 & 0 & 0 \end{pmatrix};$$

$$\text{(ii) } a_3 = 0, a_1 \neq 0, a_2 \neq 0: \; A_2 = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} -a_1a_2 & a_1^2 & 0 \\ -a_2^2 & a_1a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\text{(iii) } a_3 = a_1 = 0: \; A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \text{(iv) } a_3 = a_2 = 0: \; A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A solution to Problem 29-12 was also received from Jerzy K. Baksalary & Oskar Maria Baksalary. *See also* Trenkler (2001).

Problem 29-13 : Normal Matrices with Prescribed Diagonal Elements and Their Differences Elsewhere

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Show that there are normal matrices of any order with prescribed diagonal elements and their differences elsewhere. More precisely, show that for any n, there exist $n \times n$ "index" matrices P and Q such that the $n \times n$ matrix $A = \{a_{ij}\}$, defined according to $a_{ii} = z_i$ and $a_{ij} = z_{p_{ij}} - z_{q_{ij}}$ when $i \neq j$.

Solution 29-13.1 by the Proposer Lajos LÁSZLÓ, *Eötvös Loránd University, Budapest, Hungary:* laszlo@numanal.inf.elte.hu

We only sketch the proof, by giving the most important observations. The key is that given a commuting family $(H_i)_{i=1}^n$ of Hermitian matrices, the matrix $A = \sum_{i=1}^n z_i H_i$ is normal for arbitrary complex numbers $(z_i)_{i=1}^n$. Let us define first H_1, a real symmetric tridiagonal (0,1) matrix having ones in the sub- and superdiagonal and in the (1,1) position ($2n-1$ in all). We then compute $(H_i)_{i=2}^n$, with 1 in the (i,i) position and 0 in all other diagonal positions, to commute with H_1. For $n = 3$, e.g., we have

$$H_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

It turns out that all the H_i are $(0,1,-1)$ matrices. Moreover, $\sum_{i=1}^n z_i H_i$ has the desired difference form, and $p_{i,j} < q_{i,j}$ also holds for the off-diagonal elements. As for the characterisation of P and Q, a detailed examination shows that they are sums of suitable Hankel and Toeplitz matrices, e.g., for $n = 5$ we have the decompositions

$$P = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 1 & 4 \\ 2 & 3 & 1 & 4 & 1 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 4 \\ 0 & 3 & 0 & 4 & 0 \\ 3 & 0 & 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 5 & 5 & 4 & 4 \\ 5 & 0 & 4 & 5 & 3 \\ 5 & 4 & 0 & 3 & 5 \\ 4 & 5 & 3 & 0 & 2 \\ 4 & 3 & 5 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 4 & 0 \\ 5 & 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 3 & 0 \\ 4 & 0 & 3 & 0 & 2 \\ 0 & 3 & 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 5 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 5 & 0 & 0 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

Also, in both P and Q, the diagonals parallel with the main diagonal contain arithmetic sequences with difference $0, 1$ or -1. Note finally, that $q_{i,j} = p_{i,j} + n + 1 - \max(i,j)$ for all $i \neq j$.

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IMAGE Problem Corner: More New Problems

Problem 30-6: A Matrix Related to an Idempotent Matrix

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix from $\mathbb{C}^{n\times n}$. What can be said about the matrix $R = P(P+P^*)^-P^*$, where $(P+P^*)^-$ is a generalized inverse of $P+P^*$ and P^* denotes the conjugate transpose of P?

Problem 30-7: A Condition for an Idempotent Matrix to be Hermitian

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix from $\mathbb{C}^{n\times n}$. Show that P is Hermitian if and only if the Moore–Penrose inverse of $P(I-P^*)$ is idempotent, where P^* denotes the conjugate transpose of P.

Problems 30-1 through 30-5 are on page 36.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to werner@united.econ.uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 30-6 and 30-7 are on page 35.*

Problem 30-1: Star Partial Ordering, Left-star Partial Ordering, and Commutativity

Proposed by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
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For any $A, B \in \mathbb{C}_{m,n}$, the star partial ordering $A \overset{*}{\leq} B$, defined by $A^*A = A^*B$ and $AA^* = BA^*$, clearly implies the left-star partial ordering $A \ {*}{\leq} \ B$, defined by $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, where $\mathcal{R}(.)$ denotes the range of a given matrix. Show that if $m = n$ and A or B is an EP matrix, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$ or $\mathcal{R}(B) = \mathcal{R}(B^*)$, then the implication $A \ {*}{\leq} \ B \Rightarrow AB = BA$ cannot hold unless $A \ {*}{\leq} \ B$ is strengthened to $A \overset{*}{\leq} B$.

Problem 30-2: Class of (0, 1)-Matrices Containing Constant Column-Sum Submatrices

Proposed by Bernardete RIBEIRO: bribeiro@dei.uc.pt and Alexander KOVAČEC: kovacec@mat.uc.pt
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For given $k_1, \ldots, k_n \in [n] = \{1, 2, \ldots, n\}$ define the $\{0,1\}$-matrix $A = A(k_1, \ldots, k_n) = (a_{ij})$ by putting $a_{ij} = 1$ iff j is one of the first k_i entries of the n-tuple $(i, i+1, \ldots, n, 1, 2, \ldots, i-1)$. Show that there exists a $\{0,1\}$-row x and a $k \in [n-1]$ such that $xA = k1_n$, where $1_n = (1, \ldots, 1)$.

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

Problem 30-4: The Similarity of Two Block Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two idempotent matrices of the same size and let $M := A+B$. Show that $\begin{pmatrix} M & A \\ 0 & -M \end{pmatrix}$ is similar to $\begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}$.

Problem 30-5: A Range Equality for the Difference of Orthogonal Projectors

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two orthogonal projectors of the same size. Show that range $[\,(A-B)^\dagger - (A-B)\,] =$ range $(AB - BA)$, where $(A-B)^\dagger$ is the Moore–Penrose inverse of $A - B$. Hence show that $(A-B)^\dagger = A - B$ if and only if $AB = BA$.

Problems 30-6 and 30-7 are on page 35.