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SIAG LA Prize Winners Speed up the QR Algorithm

by Nicholas J. Higham*

Karen Braman (University of Kansas), Ralph Byers (University of Kansas), and Roy Mathias (College of William and Mary) received the 2003 SIAM Activity Group on Linear Algebra Prize for their paper "The multishift QR algorithm. Part II: Aggressive early deflation" [1] at the SIAM Conference on Applied Linear Algebra, held at the The College of William and Mary, Williamsburg, July 15-19, 2003. The citation of the Prize Committee, comprising Ludwig Elsner (University of Bielefeld), Anne Greenbaum (University of Washington, Seattle), Bo Kagstrom (Umeå University), Nick Trefethen (University of Oxford), and chair Steve Vavasis (Cornell University), reads "This elegant paper on solution of large dense eigenvalue problems blends theory and computational experiments to significantly improve one of the best established numerical algorithms."

The QR algorithm for solving the nonsymmetric eigenvalue problem is one of the jewels in the crown of matrix computations. Nominated by Jack Dongarra and Francis Sullivan [2] as one of the "10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century," the QR algorithm has allowed routine solution of the eigenvalue problem since its invention in the early 1960s. As Beresford Parlett [3] points out, the QR algorithm's eminence stems from the fact that it is a "genuinely new contribution to the field of numerical analysis and not just a refinement of ideas given by Newton, Gauss, Hadamard, or Schur."

Anyone who computes eigenvalues by typing "eig(A)" in MATLAB is invoking the QR algorithm, or more precisely the LAPACK implementation, and for matrices up to 300-by-300 they will obtain the result within less than a second on a fast modern workstation. Dense eigenvalue problems of much larger sizes arise in various applications, and for dimensions up to 10,000 or so the QR algorithm is still the method of choice for computing all the eigenvalues. Unfortunately, since the number of floating point operations is proportional to the cube of the dimension, execution times for matrices at the upper end of this range are measured in hours. But thanks to recent work by Braman, Byers, and Mathias, execution times of the QR algorithm for matrices of dimension a few hundred upwards are set to decrease substantially.

Since the QR algorithm was first developed it has been understood that deflation is essential to its success. Deflation is the process of splitting the problem into smaller pieces during QR iterations on the upper Hessenberg matrix. (For efficiency, a full matrix is reduced to Hessenberg form before carrying out the QR iteration.) Previously, deflation was accomplished by zeroing tiny elements on the subdiagonal. The key idea in this new work is to introduce carefully chosen perturbations to reveal deflations that are not yet evident on the subdiagonal. Braman, Byers and Mathias have developed clever analysis and algorithmics to understand and make practical this idea. Important to the success is strategically expending some computational effort to look for early deflations and carefully exploiting modern computer architectures in the implementation. Their well-designed numerical experiments present convincing evidence of the improvements that aggressive early deflation can bring. In extreme cases, the cost of the QR algorithm on a matrix of size 10,000 already in Hessenberg form is reduced by two orders of magnitude.

The three prizewinners gave a joint presentation on their work at the conference. Organized by a committee co-chaired by Roy Mathias and Hugo Woerdeman (College of William and Mary), and in cooperation with the International Linear Algebra Society, the conference was the eighth in a successful series of meetings that began in Raleigh, N.C. in 1982.

The next SIAM Conference on Applied Linear Algebra will take place in 2006 in Germany in collaboration with Gesellschaft für Angewandte Mathematik und Mechanik (GAMM).

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Research Experiences with Undergraduates

by
Chi-Kwong Li
Department of Mathematics
College of William & Mary

Prelude

When first approached to write an article for IMAGE about the REU program at William and Mary, I wasn't sure there was anything new for me to say, as the paper [JL] already clearly described the program. But then, Hurricane Isabel hit Virginia. The College was closed and computer systems were down. This gave me some more time to think about the project. I came up with the idea to focus the article on some of my personal experience in doing research with undergraduates. In the last 14 years, I have worked with 24 undergraduate students on a number of research projects; see the reference list. For whatever it is worth, here is my story.

What types of undergraduate research programs have I participated in?

I have participated in several different types of undergraduate research programs including: National Science Foundation (NSF) Research Experiences for Undergraduate (REU) programs conducted in the summer, NSF supplementary REU programs conducted during the academic year, Honors projects for mathematics majors and the Wilson Interdisciplinary research program at William and Mary. Accordingly, I selected students or was selected by students in a variety of ways.

For each of the summer NSF REU programs, eight to nine students were recruited from different institutions. In the first two days of the eight-week program, several potential advisors would present their research projects. Students would then have a meeting among themselves to determine a matching between advisors and advisees. It is amazing that it always worked well with students spread rather evenly among the advisors.

The NSF supplementary REU opportunities were limited to William and Mary students. Sometimes I invited outstanding students who were taking my courses to participate, and other times I offered the vacancies to good students who inquired about possible research opportunities. The latter approach is the standard way to get students for Honors projects and other research programs at our College. Knowing that I am interested in advising Honors projects and other undergraduate research projects, students would talk to me about such possibilities. Usually, they were encouraged to talk to other potential advisors as well. In any event, I did get a number of good students working with me in this way.

What kind of research have I done with students?

It is not hard for readers, especially for those who know me, to guess the answer: matrix analysis! Instead of boring the readers with the technical details of various students' projects, I will only touch upon some of them later when I discuss why I think that matrix analysis is a good theme for undergraduate research. Here let me mention the few exceptional cases, that is, those research projects with undergraduates with topics other than matrix analysis.

In [LN], a student and I studied coding theory related to the familiar Tower of Hanoi puzzle. This was actually an extension of the student's summer REU project at another university. When I filled out the recommendation form of the other university's REU program for the student, one of the questions was whether a faculty member at the student's home university would continue to work with the student after the summer if the student would be interested in doing so. I said yes to the question and the student was admitted to the REU program. After she came back to William and Mary, she expressed interest in continuing the research. So, I kept my promise, and worked with her in the following academic year. The research led to [LN], which contains a short proof of the result and the answer to an open problem posed in a paper of the student and her REU advisor.

In the spring of 1997, I taught a course in applied abstract algebra covering topics including some coding theory and cryptology. A student in my class was a double mathematics and computer science major. The student was concurrently enrolled in a computer science class concerning the implementation of crypto systems. He was very interested in both the theoretical and practical aspects of cryptology, and ended up doing an Honors project on cryptology under the joint supervision of a colleague in the computer science department and me. When he graduated, he was hired by a software security company—of course, with a salary much higher than mine. He later learned that he was selected over many applicants with Masters degrees because of his course work and research in cryptology. Two years later, he and his colleagues made CNN news for cracking an online casino by showing that the pseudo-random number generator used to deal the poker game was very insecure. They illustrated how one could predict the poker hands after observing the game for an hour or so. This remains one of my favorite stories for those abstract algebra students who do not find abstract algebra interesting and useful!

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Linear Algebra Titles *from SIAM*

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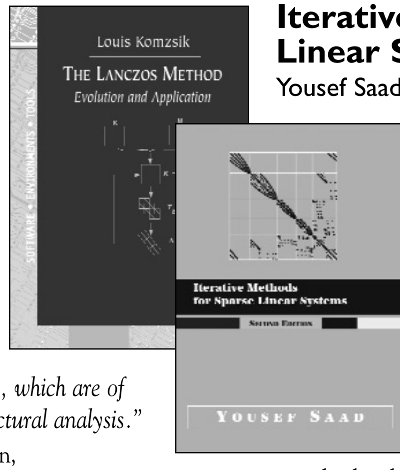
The Lanczos Method: *Evolution and Application*

Louis Komzsik



"…I recommend this book to anyone who wants to appreciate the often subtle interactions between algorithm research and engineering applications. For the engineer, it comprehensively summarizes 25 years of intellectual development in the understanding of the basic Lanczos algorithm and its many variants. For the numerical analysts, it describes the variety of practical considerations, which are of critical importance in such applications as structural analysis."

—Horst Simon, Director, NERSC Division, Berkeley National Laboratory.

"Detailed development of the method and its algorithmic structure, illustrations by industrial grade examples make this book a welcome addition to the literature on this important subject. The book is a valuable reference…"

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The Lanczos Method: Evolution and Application is divided into two distinct parts. The first part reviews the evolution of one of the most widely used numerical techniques in the industry. The development of the method, as it became more robust, is demonstrated through easy-to-understand algorithms. The second part contains industrial applications drawn from the author's experience. These chapters provide a unique interaction between the numerical algorithms and their engineering applications.

This is a valuable reference for numerical analysts and engineers and can be used as a supplementary or reference text at the graduate level. Readers should be familiar with basic linear algebra and numerical analysis.

Contents: Preface; **Part I: Evolution**. Chapter 1: *The classical Lanczos method*; Chapter 2: *The Lanczos method in exact arithmetic*; Chapter 3: *The Lanczos method in finite precision*; Chapter 4: *Block real symmetric Lanczos method*; Chapter 5: *Block unsymmetric Lanczos method*; **Part II: Applications**. Chapter 6: *Industrial implementation of the Lanczos method*; Chapter 7: *Free undamped vibrations*; Chapter 8: *Free damped vibrations*; Chapter 9: *Forced vibration analysis*; Chapter 10: *Linear systems and the Lanczos method*; Closing Remarks; A Brief Biography of Cornelius Lanczos; Bibliography; Index.

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Yousef Saad

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Research Experiences, cont'd from page 3

The next case is just half exceptional because the story started with chaos and ended in matrices. In the spring of 1996, an economics student approached me about the possibility of doing a Wilson interdisciplinary research project in the following summer. Based on a Time magazine article that had piqued her interest, she wanted to work on chaos theory and economics. I frankly told her that I knew nothing about chaos, but I was willing to learn some chaos theory with her in addition to learning some economics theory from her! She obviously realized that it would be too heavy a burden for both of us. So, she looked into other possibilities, and found another article about game theory and auctions in Forbes magazine. When she asked me about that, I told her that at least I knew matrix games. So, we ended up doing a project in game theory and economics. In fact, I was so fond of the subject that I taught a topics course in game theory in the following semester in which I discussed some applications of game theory in biology. Some results obtained in that summer and the following semester led to [LNa]. It was quite an educational experience for the student as well as myself.

Why is matrix analysis a good theme for undergraduate research?

In my opinion, matrix analysis is an excellent topic for undergraduate research. It does not require a lot of background to understand some research questions, yet it is linked to different topics such as group theory, operator theory, operator algebras, and numerical analysis, and it offers endless opportunities for further research. In fact, the many different aspects of matrix analysis can attract students with different backgrounds. In my work, for students with strong abstract algebra background, we studied homomorphisms or linear/additive maps that leave invariant symmetric groups, alternating groups, semi-groups of stochastic matrices, and other related nonnegative matrix sets [AM,ChL1,ChL2]; for students interested in complex analysis and functional analysis, we studied numerical range [LMR2,LSS] or isometry problems [CL2,Cet,KL,LM]; for students interested in combinatorics, we studied topics in combinatorial matrix theory [CP,LLR,SS]; for students interested in convex analysis, we studied geometrical structure of matrix sets [HL]; and for students with a computer science background, we used scientific computation to study matrix problems [CL1,CP,He]. In fact, advising undergraduate research projects in matrix analysis well manifests the theory of Confucius that "students should be educated and trained according to their strength". (This is truly from Confucius and not from a fortune cookie!)

According to the nature of the research problems, students may need to use or develop techniques in group theory, combinatorial theory, functional analysis or scientific computation, in the matrix analysis research projects. This exposed students to different research areas in addition to matrix analysis, and might influence their future choices of research topics in graduate studies. Moreover, the techniques acquired in the projects might be useful in their future research in mathematics or other subjects. For example, the matrix techniques developed in [LNa] were later used in the graduate study in economics by the student (see [Na]).

What have students and I gained by doing undergraduate research projects?

Students received stipends for their summer research, and Honors project students graduated with honors. Students acquired some experience in mathematical research and got a glimpse of how professional mathematicians work. In some cases, the research led to the excitement of their first publication. In any event, students at least learned some mathematics that might be useful for their future study. On the one hand, I am glad to see that most of my undergraduate research students have gone on to graduate school to study mathematics and related subjects. On the other hand, as long as the students have seen a real picture of what mathematical research is about, I do not have any problem of seeing them pursue directions other than mathematics.

I received a stipend for doing the summer REU projects. Other projects had no financial compensation. Nevertheless, successful research projects led to research papers, a better CV for tenure, promotion, and even for faculty award nominations. Similar to my other research projects, it was most enjoyable to develop with collaborators new ideas to solve problems. Moreover, I have acquired a lot of knowledge through studying new topics with students or through consultation with colleagues on problems arising in the research. All of these are good. But there is a more primitive motivation for me to do research with undergraduates. Researchers, educators, and grant agencies may emphasize that undergraduate experiences can help train young scientists. In comparison, I have a more elementary goal: to let more young people know what mathematics research is about.

I like mathematics, I like mathematics research, I like to share my research experiences with others, and I feel that appreciating mathematics should not be restricted to a small group of people. Not everyone has to be a musician, but many people can appreciate good music. Similarly, I would like to see that more people can appreciate mathematics and mathematical research work—though not every one has to be a research mathematician!

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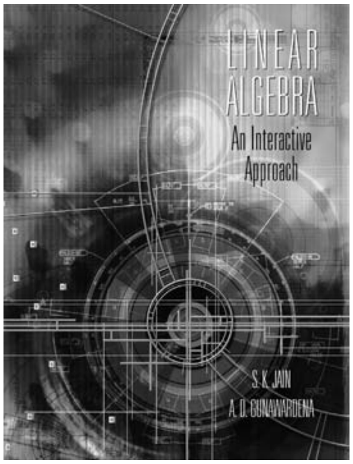

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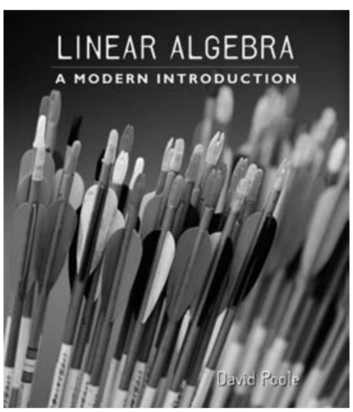

Linear Algebra: A Modern Introduction

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The Eighth SIAM Conference on Applied Linear Algebra Williamsburg, VA: 15-19 July 2003

The 8th SIAM Conference on Applied Linear Algebra was held July 15-19, 2003, at the College of William and Mary, Williamsburg, VA, USA. There were 246 registered participants from academia, government laboratories and industry. Geographically, the participants were from North America, Europe and Asia. The co-chairs of the meeting were Roy Mathias and Hugo Woerdeman.

The keynote speakers (including two ILAS speakers) were George Cybenko (Dartmouth College), Heike Fassbender (TU Braunschweig), Andreas Frommer (Bergische Universität-Gesamthochschule Wuppertal), Rich Lehoucq (Sandia National Laboratories), Judith McDonald (Washington State University; ILAS Speaker), James G. Nagy (Emory University), Michael Overton (New York University), Bryan Shader (University of Wyoming; ILAS Speaker), G. W. (Pete) Stewart (University of Maryland), and Gilles Villard (CNRS/EcoleNormale Superieure de Lyon).

In addition there were 26 minisymposia with a wide range of topics, including Combinatorics in Linear Algebra, Linear Algebra in Computational Biomedicine, Matrix Inequalities and Applications, Recent Developments in Sparse Matrix Algorithms, Indefinite Inner Products and Applications, Linear Algebra in Data Mining and Information Retrieval. The wide variety of topics and the wide variety of backgrounds of the participants resulted in a scientifically exciting atmosphere.

The SIAM Activity Group on Linear Algebra (SIAG LA) Prize was awarded to the paper by K. Braman, R. Byers, and R. Mathias, "The multishift QR algorithm. II. Aggressive early deflation." SIAM J. Matrix Anal. Appl. 23 (2002), 948--973. The three authors gave an excellent joint presentation on their awarded work.

The SIAG LA business meeting was held over lunch and was attended by approximately 40% of the participants, which led to some lively discussions. The business meeting featured also Junping Wang, NSF, Computational Mathematics and Applied Mathematics.

The social events included a welcome reception and a banquet featuring Roger Horn. Roger did an excellent job of entertaining the crowd and made thankful use of some of the snafus in the organization, which included having two keynote presentations with the same title.

For the first time in this series of conferences, the proceedings were published online http://www.siam.org/meetings/la03/proceedings/.

The next SIAM Conference on Applied Linear Algebra will be held in Düsseldorf, Germany, in 2006. It will be the first time the meeting will be held outside of the US.

Twelfth International Workshop on Matrices and Statistics IWMS-2003

Dortmund, Germany: 5–8 August 2003

Report by Hans Joachim Werner

The Twelfth International Workshop on Matrices and Statistics (IWMS-2003) was held at the University of Dortmund (Dortmund, Germany), 5–8 August 2003, during the week immediately before the 54th Biennial Session of the International Statistical Institute (ISI) in Berlin. This Workshop, which was hosted by the Department of Statistics at the University of Dortmund, had been cosponsored by the Bernoulli Society as an ISI satellite meeting, and had been endorsed by the International Linear Algebra Society (ILAS).

The International Organizing Committee for this workshop consisted of R. William Farebrother (Shrewsbury, England), Simo Puntanen (University of Tampere, Finland), George P. H. Styan (McGill University, Montréal, Québec, Canada; vice-chair), and Hans Joachim Werner (University of Bonn, Germany; chair). The Local Organizing Committee (LOC) at the University of Dortmund comprised Jürgen Groß, Götz Trenkler (chair) and Claus Weihs. The Workshop Secretary was Eva Brune.

The purpose of the workshop was to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. More than 45 participants from 15 different countries joined this workshop. The Workshop was opened by Professor Dr. Eberhard Becker, Rector of the University of Dortmund. This was followed by plenary sessions of invited, short course and contributed papers. The invited and short course speakers were Jerzy K. Baksalary, Adi Ben-Israel, N. Rao Chaganty, Ludwig Elsner, Bjarne Kjær Ersbøll, Richard William Farebrother, Patrick J. F. Groenen, Alexander Guterman, Stephen Pollock, Simo Puntanen, George P. H. Styan, Júlia Volaufová and Roman Zmyślony. The ILAS-Lecturer was Jerzy K. Baksalary. Another 25 papers were presented in several contributed paper sessions, and 3 further papers were presented just by title. It is expected that many of these papers will be published, after refereeing, in *Linear Algebra and Its Applications*. The Workshop Programme can still be downloaded from the Workshop website: www.statistik.uni-dortmund.de/IWMS/main.html.

On Wednesday, August 6, there was an Afternoon Outing to Bochum which is a neighboring city of Dortmund. There,

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visiting the famous Mining Museum Bochum, we had a few hours to imagine the hard work of coal miners. We started our visit with an excellent guided tour down in a mine, followed by some free time to walk on our own through the many exhibition halls of this museum and to climb up a winding tower. Up to the mid 1970's the Ruhr region, Dortmund lies on the north-west edge of the Ruhr, was one-sidedly characterized by mining, steel and iron-making industries. In the evening of the same day there was a Beer Taste and Test at Hövels Brewery in downtown Dortmund. Afterwards a delicious Workshop Dinner was served at the same place. Like our Workshop in Lyngby (Denmark) last year, this Workshop in Dortmund again provided an extremely good atmosphere to stimulate contacts and exchange ideas.

The 13th International Workshop on Matrices and Statistics (IWMS-2004), in Celebration of Ingram Olkin's 80th Birthday, will be held at Będlewo, about 30 km (18 miles) south of Poznán, Poland, from 18 to 21 August 2004. For further details visit http://matrix04.amu.edu.pl.

The 14th International Workshop on Matrices and Statistics (IWMS-2005) will be held at Massey University (Albany Campus), Auckland, New Zealand, 29 March to 1 April 2005, just before the 55th Biennial Session of the International Statistical Institute (Sydney, Australia, 5–12 April 2005).

Photo of Participants in the 2003 Workshop on Matrices and Statistics

Photo by N. Rao Chaganty

John William Strutt and the Rayleigh Quotient

by Richard William Farebrother

The name of Lord Rayleigh, although frequently misspelled, is well-known to computational linear algebraists for the Rayleigh quotient. In his contribution to a panel discussion at the 1995 AMS-SIAM Summer Seminar in Applied Mathematics, Beresford Parlett (1995) noted that Lord Rayleigh made this discovery when working on the first draft of his *Theory of Sound* (1877-78) during a six-month trip to Egypt in 1872-73. However, Parlett also noted that:

> "There is an interation called the Raleigh quotient interation but I don't think he [Lord Rayleigh] ever used it. He did use a Raleigh quotient and he did do inverse interation with a Raleigh quotient shift for the first time."

[I have retained the AMS-SIAM secretary's misspellings of `interaction' and `Rayleigh'].

John William Strutt was born in Langford Grove, Malden, Essex, England on 12 November 1842, he succeeded his father as the third Baron Rayleigh in June 1873, and died at his home Terling Place, Witham, Essex, England on 30 June 1919. Because of his social background, he was not able to follow a conventional academic career, but undertook numerous scientific experiments in a private laboratory at Terling Place. He accepted the posts of Professor of Experimental Physics at the University of Cambridge from 1879 to 1884 and that of Professor of Natural Philosophy at the Royal Institution in London in 1887. He was elected President of the Royal Society in 1905 and Chancellor of the University of Cambridge in 1908. Rayleigh was a member of all the leading scientific societies and received many awards. In particular, he was a founding member of the (British) Order of Merit in 1902 and he and Sir William Ramsay were awarded a Nobel Prize in 1904 for their discovery of the inert gas argon.

As noted above, Lord Rayleigh's title is sometimes misspelled. For myself, I cannot recollect having had any trouble with the spelling of his name—all of the milk delivered to my parents' home during the first 25 years of my life was supplied by the local branch of *Lord Rayleigh's Dairies*.

In 1885 Lord Rayleigh's younger brother Edward Strutt went into partnership with his friend Charles Parker to found a farm management and land agent company that still continues today. In 1886 the Strutt brothers set up *Lord Rayleigh's Farms*. In 1887, they bought a London retail outlet that formed the nucleus of *Lord Rayleigh's Dairies*. *Lord Rayleigh's Farms* continued as an independent concern until 1996 when it became part of Mejeriselsk and Danmark Foods, and in June 2000 they, in turn, merged with the Swedish-based Arla Group.

For those interested in such matters, the glass milk bottles of my childhood were marked with the words "Lord Rayleigh's Dairies" set one above another and enclosed in a truncated rhombus. By contrast, the logo on the waxed cardboard milk carton illustrated in anonymous (1986) consists of the words "Lord Rayleigh's Farms" set one above the other but without a surround.

For further details of Lord Rayleigh's life, see the (British) *Dictionary of National Biography*, the *Dictionary of Scientific Biography*, or "The MacTutor: History of Mathematics Archive" website: wwwhistory.mcs.standrews.ac.uk/History/Mathematicians/Rayleigh.html. For further details of the history of Lord Rayleigh's Farms and Lord Rayleigh's Dairies, see Anonymous (1986) and Wormall (1999, pp. 111-119). See Wilkinson (1965) for a discussion of the Rayleigh quotient.

Acknowledgement: I am indebted to Margaret Irvine for downloading some of the material cited above, and to Richard Shackle, the Local Studies Librarian at Colchester Library, for identifying and supplying copies of Anonymous (1986) and Chapter 19 of Wormall (1999).

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Beresford Parlett (1995), Contribution to "Does Numerical Analysis Need a Model of Computation?" AMS-SIAM Summer Seminar in Applied Mathematics, Park City, July 17-August 11, 1995.

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James H. Wilkinson (1965), *The algebraic eigenvalue problem*, Oxford University Press, Oxford.

Peter Wormall (1999). *Essex Farming 1900-2000*, Abbercon Books.

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Book Review

Developments and Applications of Block Toeplitz Iterative Solvers by Xiao-Qing Jin, Science Press (Beijing-New York) & Kluwer Academic Publishers (Dordrecht-Boston-London), 2002, Series on Combinatorics and Computer Science, Vol. 2, ISBN 7-03-010719-5 (Science Press, Beijing) ISBN 1-4020-0830-9 (Kluwer Academic Publishers), US $103 or EUR $105, xiii+218 pp., hard cover.

Reviewed by Yimin Wei

This book includes the latest developments on iterative methods for solving block Toeplitz systems. Such systems have been widely used in the field of image processing, numerical differential equations and integral equations, time-series analysis and control theory. Iterative methods make it possible to solve a large class of *mn*-by-*mn* block Toeplitz systems in $O(mn \log(mn))$ operations.

The book is divided into twelve chapters. Chapter 1 introduces some basics about matrix computations and some good circulant preconditioners for solving Toeplitz systems. Chapter 2 studies block circulant preconditioners and their use in solving systems block $T_{mn}u=b$, where T_{mn} is an *m*-by-*m* block Toeplitz matrix with *n*-by-*n* blocks, via the preconditioned conjugate gradient method. Chapter 3 discusses obtaining block circulant preconditioners for block Toeplitz systems from the viewpoint of kernels. Chapter 4 proposes a fast algorithm with two preconditioners to solve block Toeplitz systems with tensor structure and gives an application to the inverse heat problem. Chapters 5 and 6 discuss the constrained and weighted Toeplitz least squares problem, and ill-conditioned block Toeplitz systems, respectively. Non-circulant preconditioners are studied in Chapter 7 and multigrid methods are used for solving block Toeplitz systems in Chapter 8. Chapters 9, 10 and 11 propose some block preconditioners for partial differential equations and ordinary differential equations with Krylov subspace methods. Both theoretical analysis and numerical results are given. Chapter 12 applies the preconditioning technique to image restoration problems. Finally, the Bibliography of the book contains many recent papers in the related area.

This book is the first on Toeplitz iterative solvers. Since the book contains current developments and applications, it should be of benefit to anybody with research interests in block Toeplitz systems. Overall, I really enjoy this book and I am sure that it will be useful to students and researchers alike for many years to come.

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Math Books Published in 2003

As a service to IMAGE readers, below is a listing of mathematics books published in 2003. The list was complied from the Mathematics Online Bookshelf™. Additional information about these books is available online at

http://www.mathbookshelf.com.

The titles are sorted by subject.

Applied Math

*Theory of Scheduling***.** Conway,. R.; Maxwell, W.; Miller, L., Dover 2003, 0-486-42817-6.

Finite Element Methods for Structures with Large Stochastic Variations. Elishakoff, I., Oxford University Press 2003, 0-19-852631-8.

The Universality of the Radon Transform, Ehrenpreis, Leon, Oxford University Press 2003, 0-19-850978-2, 860 pp.

The Lanczos Method: Evolution and Application. Komzsik, L., SIAM 2003, 0-89871-537-7.

Chaos

Chaos: A Mathematical Introduction. Banks, J., Cambridge University Press, 2003, 0-521-53104-7.

Collected Works

The Collected Papers of William Burnside. Neumann, Peter, Oxford University Press, 2003, 0-19-850585-X.

Combinatorics

Surveys in Combinatorics 2003, Wensley, C., Cambridge University Press 2003, 0-521-54012-7.

Discrete Convex Analysis. Murota, Kazuo. SIAM 2003, 0-89871-540-7, 389 pp.

Discrete Mathematics: Elementary and Beyond. Lovasz, Laszlo; Pelikan, Jozsef; Vesztergombi, Katalin L., Springer 2003, 0-387-95585-2, 296 pp.

Automatic Sequences. Haeseler, Friedrich von, Walter de Gruyter 2003, 3-11-015629-6, 191 pp.

Combinatorics. Merris, R., John Wiley. 2 ed., 2003, 0-471-26296-X.

Complex Analysis

Complex Variables: Introduction and Applications. Ablowitz, M.; Fokas, A., Cambridge University Press.

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Books, cont'd from page 12

Complex Analysis. Kodaira, Kunihiko, Cambridge University Press 2003, 0-521-80937-1.

Complex Analysis. Stein, E., Princeton University Press 2003, 0-691-11385-8, 392 pp.

Control Theory
Adaptive Control Design and Analysis. Tao, G., John Wiley 2003, 0-471-27452-6.

Real-Time Optimization by Extreme-Seeking Control. Ariyur, K.; Krstic, M., John Wiley 2003, 0-471-46859-2.

Differential Equations
Differential Equations. King, A. C., Cambridge University Press 2003, 0-521-81658-0, 500 pp.

Numerical Methods for Delay Differential Equations. Bellen, A.; Zennaro, M., Oxford University Press 2003, 0-19-850654-6.

Differential Geometry
Introduction to Mobius Differential Geometry. Hertrich-Jeromin, U., Cambridge University Press 2003, 0-521-53569-7.

Abstract Algebra
An Introduction to Abstract Algebra. Robinson, D., Walter de Gruyter, 2003, 3-11-017544-4, 282 pp.

Galois Groups and Fundamental Groups. Schneps, L., Cambridge University Press 2003, 0-521-80831-6, 470 pp.

Finance
Financial Markets in Continuous Time. Dana, Rose-Anne; Jeanblanc-Picque, Monique, Springer 2003, 3-540-43403-8, 330 pp.

Weak Convergence of Financial Markets. Prigent, J.-L., Springer 2003, 3-540-42333-8.

The Statistical Mechanics of Financial Markets. Voit, J., Springer 2 ed., 2003, 3-540-00978-7.

Fluid Dynamics
Generalized Riemann Problems in Computational Fluid Dynamics. Ben-Artzi, M., Cambridge University Press 2003, 0-521-77296-6, 392 pp.

Fourier Analysis
Fourier Analysis: An Introduction. Stein, E., Princeton University Press 2003, 0-691-11384-X, 320 pp.

Functional Analysis
An Introduction to the Theory of Operator Spaces. Pisier, Gilles, Cambridge University Press 2003, 0-521-81165-1, 300 pp.

Geometry
The Changing Shape of Geometry. Pritchard, C., Cambridge University Press 2003, 0-521-53162-4, 550 pp.

Dissections: Plane & Fancy. Frederickson, G., Cambridge University Press 2003, 0-521-52582-9.

Graph Theory
Random Geometric Graphs. Penrose, M., Oxford University Press 2003, 0-19-850626-0.

Four Colors Suffice: How the Map Problem Was Solved. Wilson, Robin, Princeton University Press 2003, 0-691-11533-8, 280 pp.

Group theory
Elementary Number Theory, Group Theory, and Ramanujan Graphs. Davidoff, G., Cambridge University Press 2003, 0-521-53143-8.

Finite Structures with Few Types. Cherlin, Gregory; Hrushovski, Ehud, Princeton University Press 2003, 0-691-11332-7, 192 pp.

Information theory
Information Theory, Inference and Learning Algorithms. MacKay, David, Cambridge University Press, 2003, 0-521-64444-5, 550 pp.

Linear Algebra/Matrix Theory
*Iterative Krylov Methods for Large Linear Systems.*Vorst, H., Cambridge University Press 2003, 0-521-81828-1.

Linear Algebra and Geometry: A Second Course. Kaplansky, I.. Dover 2003, 0-486-43233-5, 146 pp.

*Iterative Solution of Large Linear Systems.*Young, D., Dover 2003, 0-486-42548-7, 570 pp.

Fast Algorithms for Structured Matrices: Theory and Applications. Olshevsky, Vadim, SIAM 2003, 0-89871-543-1, 433 pp.

Iterative Methods for Sparse Linear Systems. Saad, Yousef, SIAM 2 ed., 2003, 0-89871-534-2, 528 pp.

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Books, cont'd from page 13

Mathematical Physics
Perturbation Techniques in Mathematics, Engineering and Physics. Bellman, R., Dover 2003, 0-486-43258-0, 118 pp.

The Theory of Relativity. Pathria, R., Dover 2003, 0-486-42819-2.

Topics in Quantum Mechanics. Williams, Floyd, Springer 2003, 0-8176-4311-7, 416 pp.

Miscellaneous
Mathematical Constants, Finch, S., Cambridge University Press 2003, 0-521-81805-2.

Number Theory
The Riemann Zeta-Function: Theory and Applications. Ivic, A., Dover 2003, 0-486-42813-3.

The Art of the Infinite. Kaplan, Robert, Oxford University Press.

Numerical Analysis
Practical Extrapolation Methods. Sidi, Avram, Cambridge University Press 2003, 0-521-66159-5.

Optimization
Real-Time Optimization by Extreme-Seeking Control. Ariyur, K.; Krstic, M., John Wiley 2003, 0-471-46859-2.

PDE's
Soliton Equations and their Alegbro-Geometric Solutions. Gesztesy, F.; Holden, H., Cambridge University Press 2003, 0-521-75307-4.

A Tutorial on Elliptic PDE Solvers and Their Parallelization. Douglas, Craig, SIAM 2003, 0-89871-541-5, 135 pp.

Probability
Probability Theory. Jaynes, E. T., Cambridge University Press, 2003, 0-521-59271-2, 650 pp.

Real Analysis
A Course in Modern Analysis and its Applications. Cohen, G. Cambridge University Press 2003, 0-521-52627-2.

Counterexamples in Analysis. Gelbaum, B.; Olmsted, H. Dover 2003, 0-486-42875-3.

A Concise Approach to Mathematical Analysis. Robdera, M.A., Springer 2003, 1-85233-552-1, 374 pp.

Statistics
Data Analysis and Graphics Using R: An Example-Based Approach. Maindonald, J.; Braun, J., Cambridge University Press 2003, 0-521-81336-0.

Statistical Models. Davison, A. C., Cambridge University Press 2003, 0-521-77339-3, 680 pp.

Radial Basis Functions. Buhmann, M., Cambridge University Press 2003, 0-521-63338-9.

Statistical Inference. Rohatgi, V., Dover 2003, 0-486-42812-5, 948 pp.

Bayesian Statistics 7. Bernardo, J., Oxford University Press 2003, 0-19-852615-6, 768 pp.

Statistical Thought: A Perspective and History. Chatterjee, S., Oxford University Press 2003, 0-19-852531-1.

Proceedings of the Third SIAM International Conference on Data Mining. Barbará, Daniel, SIAM 2003, 0-89871-545-8, 347 pp.

Mathematical Statistics. Shao, Jun, Springer 2 ed., 2003, 0-387-95382-5.

Statistical Methods for Rates and Proportions. Fleiss, J. John Wiley 2003, 0-471-52629-0.

Quantitative Methods in Population Health: Extensions of Ordinary Regression. Palta, M., John Wiley 2003, 0-471-45505-9.

A Primer on Statistical Distributions. Balakrishnan, N; Nevzorov, V., John Wiley 2003, 0-471-42798-5.

Order Statistics. David, H.; Nagaraja, H., John Wiley 3 ed., 2003, 0-471-38926-9.

An Introduction to Multivariate Statistical Analysis. Anderson, T., John Wiley, 3 ed., 2003, 0-471-36091-0.

Probability and Statistics for Computer Science, Johnson, J., John Wiley, 2003, 0-471-32672-0.

Statistical Size Distributions in Economics and Actuarial Sciences., Kleiber, C.; Kotz, S., John Wiley 2003, 0-471-15064-9.

Forthcoming Conferences and Workshops in Linear Algebra

The SVG Meeting: A Celebration

Stanford University
9-10 January, 2004

Alan George, Michael Saunders, and Jim Varah are turning 60 between July 2003 and January 2004. During the mid- and late 1960s, these three young men decided to pursue their doctoral studies in scientific computing at Stanford University. In the approximately π decades that followed (up to a modest rounding error), they have become close colleagues and are well-established individuals in the field of numerical computing. Alan is well known for his work on sparse matrix computation, much of which is fundamental in the area. Mike is one of the leading experts in large-scale numerical optimization. Jim's work has covered a wide spectrum of numerical analysis and scientific computing.

A two-day workshop will take place at Stanford University on January 9-10, 2004 to celebrate their birthdays and accomplishments. Students, friends, and colleagues are welcome to attend. Additional information about the workshop is available at http://sccm.stanford.edu/svg/. If you have any questions, comments, or suggestions about the workshop, please feel free to contact one of the organizers. In particular, if you would like to speak about our three prospective senior citizens, feel free to volunteer; talks as well as personal observations are equally welcome!

Also, we would like to fill a photo page with remembrances of the honorees; we welcome the entire gamut, from toddler to graduation days to the present. Please send hard copies (we'll be sure to return them!) or JPEGs, etc, to Michael Friedlander.

The organizing committee is Gene Golub, Stanford University (golub@sccm.stanford.edu), Michael Friedlander, Argonne National Laboratory (michael@mcs.anl.gov), Chen Greif, University of British Columbia (greif@cs.ubc.ca), and Esmond G. Ng, Lawrence Berkeley National Laboratory (egng@lbl.gov).

SIAM Workshop on Combinatorial Scientific Computing (CSC04)

San Francisco, CA
27-28 February, 2004

Combinatorial algorithms play a key, supporting role in many aspects of scientific computing. Examples include orderings for sparse direct methods, graph coloring and partitioning for parallel computing, geometric algorithms in mesh generation and string algorithms in computational biology. The enabling importance of combinatorial algorithms in scientific computing is often overlooked, and sub-communities of researchers with overlapping interests are often unaware of each other. To address this fragmentation and to strengthen the ties between the scientific computing and discrete algorithms communities, SIAM is sponsoring a workshop on Combinatorial Scientific Computing (CSC04).

CSC04 will be organized following the 11th SIAM Conference on Parallel Processing for Scientific Computing (PP04) on February 27 and 28, 2004. The workshop aims to bring together researchers interested in applications of combinatorial mathematics and algorithms to scientific computing.

Plenary speakers include Richard Brualdi (University of Wisconsin, Madison), Shang-hua Teng (University of Illinois, Champaign-Urbana), and Dan Gusfield (University of California, Davis).

Funds have been requested to provide partial travel support for graduate students, post-doctoral fellows, and faculty in the early stages of their careers. Further details are available at www.siam.org/meetings/pp04/cscworkshop.htm

The organizing committee is comprised of John Gilbert (University of California, Santa Barbara), Bruce Hendrickson (Sandia National Laboratories), Alex Pothen (Old Dominion University), Horst Simon (Lawrence Berkeley National Laboratory), and Sivan Toledo (Tel-Aviv University).

Directions in Combinatorial Matrix Theory

Banff International Research Station
Banff, Alberta, Canada
6-8 May, 2004

A two-day workshop *Directions in Combinatorial Matrix Theory* will be held May 6-8, 2004 at the recently opened Banff International Research Station (BIRS). This Oberwolfach-style workshop, participation in which is by invitation only, will include up to 40 researchers whose interests lie at the interface of combinatorics and matrix theory.

The workshop will provide researchers working in combinatorial matrix theory an opportunity to present accounts of their current research, to identify challenges for the discipline to undertake, and to suggest new approaches to explore. A refereed proceedings of the workshop will appear in the *Electronic Journal of Linear Algebra.* The organizers of the workshop hope that *Directions in Combinatorial Matrix Theory* will serve to establish connections between both individual researchers and between research areas, and so will also promote collaboration and new research in this exciting discipline.

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Directions in Combinatorial Matrix Theory, cont'd from page 15

The organizing committee for this workshop comprises Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Regina), Bryan Shader (University of Wyoming), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria).

International Algebraic Conference

Moscow, Russia
26 May-2 June, 2004

Moscow State University was founded by M.V. Lomonosov on January 25, 1755. The Department of Algebra in the Moscow State University was founded in 1929 by Professor Otto Yu. Schmidt. In connection with these events the Department of Algebra of Moscow State University is organizing an International Algebraic Conference. The conference will be held in the Main Building of Moscow State University on Vorobievy Hills, Moscow, Russia from May 26 till June 2, 2004. The campus of Moscow State University is located on the southwest of Moscow in one of the best regions of the city. It can be reached from the international airport Sheremetyevo-2 by taxi in less than an hour.

The topics of the conference are:

- rings and modules, homological algebra, K-theory;
- quantum groups and Hopf algebras;
- group theory;
- computer algebra;
- invariants and algebraic transformation groups;
- algebraic geometry;
- commutative algebra and algebraic number theory;
- linear algebra;
- general algebraic systems.

If you plan to attend the conference, please send an e-mail with the following information

1. Full name
2. Title
3. Affiliation
4. Mailing address
5. e-mail address
6. Title of your talk
7. Necessity of Russian visa

to artamon@mech.math.msu.su.

If you plan to give a talk, please also send by e-mail the LATEX2e file of your abstract (up to 1 page).

Deadline for submission of an abstract is January 15, 2004.
Deadline for registration is February 15, 2004.

All information is also available at :
http://mech.math.msu.su/department/algebra/IAC04

The participants of the conference can stay at the Hotel of Moscow University. The price at the moment is 10–25 USD per night. There are also some hotels close to the university campus. We regret that travel and daily expenses cannot be paid by the organizing committee. The registration fee is 100 USD.

The organizing committee consists of: co-chairs V. N. Latyshev, A. V. Mikhalev, E. B. Vinber, and V. A. Iskovskih; vice-chairs M. V. Zaicev and A. A. Mikhalev; members Yu. A. Bahturin, K. Brown, A. Facchini, E. S. Golod, A. Giambruno, V. A. Iskovskih, V. V. Kirichenko, S. Liu, R. McKenzie, A. Yu. Olshansky, F. V. Oystaeyen, B. Plotkin, C. Ringel, A. V. Yakovlev, V. I. Yanchevski, and R. Wisbauer.

The program committee consists of: co-chairs V. A. Artamonov and A. L. Shmelkin; members J. Alev, L. Avramov, A. Bak, L. B. Beasley, L. A. Bokut, R. A. Brualdi, A. Conte, C. DeConcini, V. Dlab, K. Denecke, K. Goodearl, J. Kollar, O. Kraft, Yu. I. Manin, V. T. Makrov, A. A. Nechaev, C. Procesi, Yu. P. Rasmyslov, A. Roiter, V. N. Remeslennikov, P. Šemrl, G. B. Shabat, and I. P. Shestakov.

11th ILAS Conference

Coimbra, Portugal
19-22 July, 2004

The 11th Conference of the International Linear Algebra Society will be held at the University of Coimbra, Portugal, July 19–22, 2004. The conference is dedicated to Richard Brualdi in honor of his 65th birthday and his numerous contributions to Linear Algebra, ILAS, and Mathematics.

The members of the organizing committee are: Danny Hershkowitz (ILAS President), Hans Schneider, Thomas Laffey, Raphael Loewy, Ion Zaballa, Bryan Shader, Graciano de Oliveira, José Dias da Silva, Eduardo Marques de Sá and João Filipe Queiró (Chair).

The members of the local organizing committee are A. P. Santana, A. L. Duarte, C. Caldeira, J. C. Gallardo, O. Azenhas and J. F. Queiró.

The plenary speakers are: Rajendra Bhatia (Indian Statistical Institute New Delhi), Hal Caswell (Woods Hole Oceanographic Institution), George Cybenko (Dartmouth College), Erik Elmroth (Umeå University), Shmuel Friedland (University of Illinois, Chicago), Peter Gritzmann (Technical University Munich), Robert Guralnick (University of Southern

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11th ILAS Conference, cont'd from page 16

California), Uwe Helmke (Würzburg University), William Helton (University of California-San Diego), Christian Krattenthaler (Université Claude Bernard Lyon), Matjaz Omladic (University of Ljubljana), Xavier Puerta (Polytechnic University Catalonia), Arun Ram (University of Wisconsin-Madison), Joachim Rosenthal (Notre Dame University), Siegfried Rump (Technical University Hamburg-Harburg), Fernando Silva (Lisbon University).

In addition there will be special lectures by the Hans Schneider Prize winner Peter Lancaster (University of Calgary), the SIAG LA speakers Beatrice Meini (University of Pisa) and Julio Moro (Carlos III University Madrid), and the Taussky-Todd speaker Peter Šemrl (University of Ljubljana).

The following mini-symposia will take place: *Group representations*, organized by Ana Paula Santana and Carlos André; *Combinatorial Matrix Theory,* organized by Bryan Shader; *Markov methods for search engines*, organized by Ilse Ipsen and Steve Kirkland; and *Non-negative matrices*, organized by Thomas J. Laffey.

The organizing committee will consider additional suggestions for mini-symposia, as the scheduling constraints allow.

The deadline for submission of contributed papers is April 30, 2004. The pre-registration deadline is May 31, 2004. Information concerning accommodation, abstract submission and registration will be posted at a later stage at the site http://www.mat.uc.pt/ilas2004 and

Clock Tower at University of Coimbra

The Thirteenth International Workshop on Matrices and Statistics

Będlewo, Poland

18-21 August, 2004

The 13th International Workshop on Matrices and Statistics will be held at the Mathematical Research and Conference Center of the Polish Academy of Sciences in Będlewo (near Poznán) Poland, 18-21 August 2004. The workshop is in celebration of Ingram Olkin's 80th Birthday.

The purpose of this workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. This workshop will include the presentation of both invited and contributed papers on matrices and statistics. Also a special session for graduate students will be arranged. It is expected that many of these papers will be published, after refereeing, in a special issue of *Linear Algebra and its Applications* associated with this workshop.

For further information contact Augustyn Markiewicz by e-mail amark@owl.au.poznan.pl or please visit the web site http://matrix04.amu.edu.pl.

COMPSTAT 2004
16th Symp. of IASC

Prague, Czech Republic
23-27 August, 2004

Statistical computing provides the link between statistical theory and applied statistics. As at previous COMPSTATs, the scientific program will cover all aspects of this link, from the development and implementation of new statistical ideas to user experiences and software evaluation. The program should appeal to anyone working in statistics and using computers, whether in universities, industrial companies, government agencies, research institutes or as software developers. A brief synopsis of the scientific program for COMPSTAT 2004 is as follows.

The Keynote Lectures are: S. Van Huffel (Katholieke Universiteit Leuven) *Bridging the gap between statistics, computational mathematics and engineering*; A. Barron (Yale University) *Function fitting with many variables: Neural networks and beyond*; Chun-houh Chen (Academia Sinica Taipei) *Dimension free data visualization and information mining*; W. Grossmann (Universitat Wien), M. Schimek (Universitat Graz) and P. Sint (Austrian Academy of Sciences) *Thirty years of COMPSTAT and key steps of statistical computing*.

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COMPSTAT 2004, cont'd from page 17

Invited minisymposia topics are: Advances in multiple time series modeling: present impact and future potential; Applications of computational statistics methods; Computational aspects in risk calculation and risk assessment; Computational aspects of optimum model based design of experiments; Computational aspects of robust statistical methods; Computational search in classification and clustering; Data visualization; E-statistics; Functional data: modeling and applications; High-dimensional data analysis; Machine learning and neural networks; Modern trends of teaching statistics for the information society; New approaches to model based cluster methods; PLS tools for regressions and structural modeling; and Statistical biocomputing.

There will be two tutorials: G. Golub (Stanford University) *Numerical methods for statisticians*; and K. Hornik (Vienna University of Technology) (R): *The next generation*.

Participants are encouraged to present contributed talks, or to submit posters on following topics: Algorithms, Graphics, Partial least squares, Applications image analysis, Resampling methods, Bayesian methods, Internet-based methods, Robustness, Biostatistics, Machine learning, Simulations, Classification, Metadata smoothing, Clustering, MCMC, Spatial statistics, Data imputation, Model selection, Statistical data mining, Data mining, Multivariate analysis, Statistical software, Data visualization, Neural networks, Teaching statistics, Design of experiments, Nonparametrical statistics, Time series analysis, Dimensional reduction, Numerical methods for statistics, Tree-based methods, E-statistics, Official statistics, Web mining, Functional data analysis, Optimization.

February 2, 2004 is the deadline for electronic submission of contributed and invited papers. For more information see http://compstat2004.cuni.cz or write to compstat2004@cuni.cz.

Gini–Lorenz Conference

Sienna, Italy
23-26 May, 2005

The University of Siena, Italy, will host the International C. Gini and M. O. Lorenz Centenary Scientific Research Conference from May 23 to May 26, 2005. The Organizing Committee invites specialists to present papers in the fields of Income and Wealth Distributions, Lorenz Curve, Human Capital, Inequality and Poverty. A proposal should include: title of the paper, abstract, names of the participants, institutional affiliation, address, e-mail, phone and fax number, and should be submitted to:

C.R.I.D.I.R.E.
Department of Quantitative Methods
Piazza San Francesco 8 - 53100
SIENA, ITALY

or electronically to: ginilorenz05@unisi.it.

The language of the Meeting will be English, and the abstract should also be submitted in English. It is planned to publish a book with the papers selected after refereeing.

The scientific committee is: S. Kotz (Chairman), B. Arnold, L. Biggeri, F. Cowell, C. Dagum, G. M. Giorgi, C. Kleiber, A. Lemmi, E. Maasoumi, P. Moyes, J. Silber, D. J. Slottje.

The organizing committee is comprised of A. Lemmi (Chairman), G. Betti, L. D'Alessandro, F. Farina, L. Fattorini, L. Greco, M. Marcheselli, S. Naddeo, L. Neri, C. Pisani, S. Vannucci, A. Vercelli.

The scientific secretariat is: C. Carmignani, A. Giannini, V. Mazza.

Call for Submissions to IMAGE

As always, IMAGE welcomes announcements of upcoming meetings, reports on past conferences, historical essays on linear algebra, book reviews, essays on the development of Linear Algebra in a certain country or region, and letters to the editor or signed columns of opinion. IMAGE would like to slightly expand its scope by including general audience articles that highlight emerging applications and topics in Linear Algebra. Contributions for IMAGE should be sent to Bryan Shader (bshader@uwyo.edu) or Hans Joachim Werner (werner@united.econ.uni-bonn.de). The deadlines are October 1 for the fall issue, and April 1 for the spring issue.

Electronic Journal of Linear Algebra

The Electronic Journal of Linear Algebra (ELA), a publication of the International Linear Algebra Society (ILAS), is a refereed all-electronic journal that welcomes mathematical articles of high standards that contribute new information and new insights to matrix analysis and the various aspects of linear algebra and its applications. Refereeing of articles is conventional and of high standards, and is being carried out electronically. The Editors-in-Chief are Ludwig Elsner and Daniel Hershkowitz. The web page is http://www.math.technion.ac.il/iic/ela.

Recent Papers Published in ELA
Volume 10 (2003)

1. Stephen J. Kirkland, *Conditioning properties of the stationary distribution for a Markov chain,* pp. 1-15.

2. Ravindra B. Bapat and Bing Zheng, *Generalized inverses of bordered matrices,* pp. 16-30.

3. Tedja Santanoe Oepomo, *A contribution to Collatz's eigenvalue inclusion theorem for nonnegative irreducible matrices,* pp. 31-45.

4. Ji Young Choi, Luz Maria DeAlba, Leslie Hogben, Benard M. Kivunge, Sandra K. Nordstrom and Mike Shedenhelm, *The nonnegative P_0-matrix completion problem,* pp. 46-59.

5. Hari Bercovici, *Spectral versus classical Nevanlinna-Pick interpolation in dimension two,* pp. 60-64.

6. Zewen Zhu, Daniel C. Coster and Leroy B. Beasley, *Properties of a covariance matrix with an application to D-optimal design,* pp. 65-76.

7. Geir Dahl, *A note on linear discrepancy,* pp. 77-80.

8. Daniel Hershkowitz and Hans Schneider, *One-sided simultaneous inequalities and sandwich theorems for diagonal similarity and diagonal equivalence of nonnegative matrices,* pp. 81-101.

9. Walter D. Morris, *Recognition of hidden positive row diagonally dominant matrices,* pp. 102-105.

10. D. Steven Mackey, Niloufer Mackey and Francoise Tisseur, *Structured tools for structured matrices,* pp. 106-145.

11. Masaya Matsuura, *A generalization of Moore-Penrose biorthogonal systems*, pp. 146-154.

12. C.M. da Fonseca, *The path polynomial of a complete graph,* pp. 155-162.

13. Michael Neumann and Nic Ormes, *Bounds for graph expansions via elasticity,* pp. 163-178.

14. Jan Snellman, *The maximal spectral radius of a digraph with $(m+1)^{2-s}$ edges,* pp. 179-189.

15. Charles R. Johnson, Yonatan Harel, Christopher J. Hillar, Jonathan M. Groves and Patrick X. Rault, *Absolutely flat idempotents,* pp. 190-200.

16. Jean-Daniel Rolle, *Optimal subspaces and constrained principal component analysis,* pp. 201-211.

17. Felix Goldberg and Gregory Shapiro, *The Merris index of a graph,* pp. 212-222.

18. Michael Marks, Rick Norwood and George Poole, *The maximum number of 2 by 2 odd submatrices in (0,1)-matrices,* pp. 223-231.

19. Randall J. Elzinga, *Strongly regular graphs: Values of λ and μ for which there are only finitely many feasible (v, k, λ, μ),* pp. 232-239.

20. Gilbert J. Groenewald and Mark A. Petersen, *J-spectral factorization for rational matrix functions with alternative realization,* pp. 240-256.

21. Luz Maria DeAlba, Timothy L. Hardy, Leslie Hogben and Amy Wangsness, *The (weakly) sign symmetric P-matrix completion problems,* pp. 257-271.

22. K.A.M. Sayyed, M.S. Metwally and Raed S. Batahan, *On generalized Hermite matrix polynomials,* pp. 272-279.

23. S.W. Drury, J.K. Merikoski, V. Laakso and T. Tossavainen, *On nonnegative matrices with given row and column sums,* pp. 280-290.

24. Robert M. Guralnick, Chi-Kwong Li and Leiba X. Rodman, *Multiplicative maps on invertible matrices that preserve matricial properties,* pp. 291-319.

IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 29-3, 29-4, 29-11 [IMAGE 29 (October 2002), pp. 36 & 35], 30-1, 30-2, and 30-4 through 30-7 [IMAGE 30 (April 2003), pp. 36 & 35]. Problems 28-3 and 30-3 are repeated below without solutions; we are still hoping to receive solutions to these problems. We introduce 8 new problems on pp. 44 & 43 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to werner@united.econ.uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 28-3: Ranks of Nonzero Linear Combinations of Certain Matrices.

Proposed by Shmuel FRIEDLAND, *University of Illinois at Chicago, Chicago, Illinois, USA:* friedlan@uic.edu
and Raphael LOEWY, *Technion–Israel Institute of Technology, Haifa, Israel:* loewy@technunix.technion.ac.il

Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Show that any nonzero real linear combination of these four matrices has rank at least 3.

The Proposers of Problem 28-3 and the Editors of IMAGE *are still looking forward to receiving a solution to this problem; the Proposers prefer a solution which does not depend on the use of a computer package such as* MAPLE.

Problem 29-3: Isometric Realization of a Finite Metric Space

Proposed by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Show that every finite metric space can be realized isometrically as a subset of a normed vector space.

Solution 29-3.1 by Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Blumenthal & Menger (1970, p. 240, Exercise 6) claim that an n-point metric space $M = (\{1, 2, \ldots, n\}, d)$ can be isometrically embedded into the normed space $(\mathbb{R}^{n-1}, |\cdot|_\infty)$. Indeed, let $d_{ij} = d(i, j)$, for $i, j = 1, \ldots, n$. Define points $p^k = (d_{k2}, d_{k3}, \ldots, d_{kn}) \in \mathbb{R}^{n-1}$, for $k = 1, \ldots, n$. By definition of $|\cdot|_\infty$, we have $|p^k - p^j|_\infty = \max\{|d_{k2} - d_{j2}|, \ldots, |d_{kn} - d_{jn}|\}$. Now, the traditional triangle inequalities in the metric space M are actually equivalent to $|d_{ki} - d_{ji}| \le d_{jk}$ for every set $\{i, j, k\}$ of not necessarily distinct points of M. If $k \ne j$, then k or j is in $M \setminus \{1\} = \{2, \ldots, n\}$. Hence $|p^k - p^j|_\infty = d_{kj}$. Clearly this is also true if $k = j$. With this exercise solved, so is Problem 29-3.

NOTES. It would be interesting to know from colleagues having Blumenthal (1953) available whether this exercise is there - even solved? Kelly (1975) is a more modern source having material and many references on isometric embeddability of metric spaces.

References

L. M. Blumenthal & K. Menger (1970). *Studies in Geometry.* W. H. Freeman, San Francisco.
L. M. Blumenthal (1953). *Theory and Applications of Distance Geometry.* Clarendon Press, Oxford.
L. M. Kelly, (Ed.) (1975). *The Geometry of Metric and Linear Spaces.* Lecture Notes in Mathematics 490, Springer-Verlag, Berlin.

A Solution to Problem 29-3 was also received from the Proposer S. W. Drury.

Problem 29-4: Normal Matrix and a Commutator

Proposed by S. W. DRURY and George P. H. STYAN, *McGill University, Montréal (Québec), Canada:*
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Show that every $n \times n$ complex matrix A can be written in the form $A = N + [H, N]$, where N is normal and H is Hermitian, and $[H, N]$ denotes the commutator $HN - NH$.

Solution 29-4.1 by the Proposers S. W. DRURY and George P. H. STYAN, *McGill University, Montréal (Québec), Canada:*

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We consider the supremum of the function $B \mapsto \beta_n(B) = \sum_{j=1}^n |b_{jj}|^2$ on the (compact) orbit of A under unitary similarity. Let us suppose that this continuous function attains its maximum value at B. We argue by making variations of B of the form $U^\star BU$, where $U = \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}$ and V is a 2×2 unitary block. It is easy to see that B_{11} must be a maximum point on the unitary similarity orbit $\{V^\star B_{11}V; V \in U(2)\}$ for the function β_2.

We start by considering the case where V is a variation of the 2×2 identity matrix

$$V = \begin{pmatrix} 1 & p \\ -\overline{p} & 1 \end{pmatrix} + O(|p|^2),$$

where p is a small complex number. This leads to

$$\begin{aligned} V^\star B_{11} V &= \begin{pmatrix} 1 & -p \\ \overline{p} & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & p \\ -\overline{p} & 1 \end{pmatrix} + O(|p|^2) \\ &= \begin{pmatrix} b_{11} - \overline{p}b_{12} - pb_{21} & b_{12} + p(b_{11} - b_{22}) \\ b_{21} + \overline{p}(b_{11} - b_{22}) & b_{22} + \overline{p}b_{12} + pb_{21} \end{pmatrix} + O(|p|^2). \end{aligned}$$

In turn, this gives

$$\beta_2(V^\star B_{11}V) = |b_{11}|^2 + |b_{22}|^2 + 2\Re p((\overline{b_{22}} - \overline{b_{11}})b_{21} + (b_{22} - b_{11})\overline{b_{12}}) + O(|p|^2).$$

Since $\beta_2(V^\star B_{11}V)$ takes its maximum value when $p = 0$, it follows that

$$(\overline{b_{22}} - \overline{b_{11}})b_{21} + (b_{22} - b_{11})\overline{b_{12}} = 0. \tag{1}$$

In the case $b_{11} = b_{22}$, a more detailed analysis (which we omit) using $V = \begin{pmatrix} \cos\theta & \omega\sin\theta \\ -\overline{\omega}\sin\theta & \cos\theta \end{pmatrix}$, where θ is real and ω is a complex number of absolute value 1, reveals that B_{11} cannot be local maximum point unless the off-diagonal elements b_{12} and b_{21} both vanish.

The pair $\{1, 2\}$ can be replaced by an arbitrary pair $\{j, k\}$. We therefore define

$$h_{jk} = \begin{cases} 0 & \text{if } b_{jj} = b_{kk}, \\ \dfrac{b_{jk}}{b_{kk} - b_{jj}} & \text{otherwise.} \end{cases} \tag{2}$$

It follows from the generalized form of (1) that $h_{kj} = \overline{h_{jk}}$ and from (2) that $h_{jj} = 0$. We can therefore write $b_{j,k} = b_{jj}\delta_{jk} + h_{jk}(b_{kk} - b_{jj})$, effectively $B = D + [H, D]$, where D is diagonal and H is Hermitian. Applying a unitary similarity now allows us to write A in the desired form.

Solution 29-4.2 by Lajos LÁSZLÓ, *Eötvös Loránd University, Budapest, Hungary:* laszlo@numanal.inf.elte.hu

The statement is nothing else than the first order necessary condition for N to be the best normal approximation to A in the Frobenius norm, as can e. g. be found in Ruhe (1987).

Reference

A. Ruhe (1987). Closest normal matrix finally found! *BIT*, **27**, 585–598.

Problem 29-11: The Minimal Rank of a Block Matrix with Generalized Inverses

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let $(\cdot)^-$ denote generalized inverse. Show that

$$\min_{A^-,\,B^-,\,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} = \max\{\operatorname{rank}(A),\ \operatorname{rank}(B)+\operatorname{rank}(C)\}.$$

Solution 29-11.1 by the Proposer Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

A matrix X is a generalized inverse of M if $MXM = M$, and is henceforth denoted by M^-. The general expression for M^- can be written as $M^- = M^\dagger + F_M U + V E_M$, where $M^\dagger$ is the Moore-Penrose inverse of M, $F_M = I - M^\dagger M$, $E_M = I - MM^\dagger$, and U and V are two arbitrary matrices of appropriate size. Because $MM^-M = M$, it follows that $\operatorname{rank}(M^-) \geqslant \operatorname{rank}(M)$. Note that for any bordered matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where A, B and C are $m \times n$, $m \times k$ and $l \times n$ matrices, respectively, $\operatorname{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \geqslant \max\{\operatorname{rank}(A),\ \operatorname{rank}(B)+\operatorname{rank}(C)\}$. Hence

$$\min_{A^-,B^-,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} \geqslant \max\{\operatorname{rank}(A^-),\ \operatorname{rank}(B^-)+\operatorname{rank}(C^-)\} \geqslant \max\{\operatorname{rank}(A),\ \operatorname{rank}(B)+\operatorname{rank}(C)\}. \tag{3}$$

We next show that the lower bound on the right-hand side of (3) is attainable. Let $B^- = B^\dagger + VE_B$ and $C^- = C^\dagger + F_C U$, where U and V are arbitrary, and substitute them into $\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix}$ to get

$$\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} = \begin{pmatrix} A^- & C^\dagger + F_C U \\ B^\dagger + VE_B & 0 \end{pmatrix} = \begin{pmatrix} A^- & C^\dagger \\ B^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_k \end{pmatrix} V \begin{pmatrix} E_B & 0 \end{pmatrix} + \begin{pmatrix} F_C \\ 0 \end{pmatrix} U \begin{pmatrix} 0 & I_l \end{pmatrix}. \tag{4}$$

It is shown in Tian (2002b) that

$$\begin{aligned} \min_{X_1,\,X_2} \operatorname{rank}(\,A - B_1X_1C_1 - B_2X_2C_2\,) &= \operatorname{rank}\begin{pmatrix} A \\ C_1 \\ C_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A & B_1 & B_2 \end{pmatrix} \\ &+\ \max\left\{ \operatorname{rank}\begin{pmatrix} A & B_1 \\ C_2 & 0 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix}, \right. \\ &\left. \operatorname{rank}\begin{pmatrix} A & B_2 \\ C_1 & 0 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Applying this result to (4) with any fixed A^- and simplifying gives

$$\begin{aligned} \min_{B^-,\,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} &\leqslant \min_{U,V} \operatorname{rank}\left(\begin{pmatrix} A^- & C^\dagger \\ B^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_k \end{pmatrix} V \begin{pmatrix} E_B & 0 \end{pmatrix} + \begin{pmatrix} F_C \\ 0 \end{pmatrix} U \begin{pmatrix} 0 & I_l \end{pmatrix} \right) \\ &= \operatorname{rank}(B) + \operatorname{rank}(C) + \max\{0,\ \operatorname{rank}(A^-) - \operatorname{rank}(A^-B) - \operatorname{rank}(CA^-)\}. \end{aligned}$$

Therefore

$$\min_{A^-,\,B^-,\,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} \leqslant \operatorname{rank}(B) + \operatorname{rank}(C) + \max\{0,\ \min_{A^-}[\operatorname{rank}(A^-) - \operatorname{rank}(A^-B) - \operatorname{rank}(CA^-)]\}. \tag{5}$$

Notice that A^-AA^- is also a generalized inverse of A. Replace A^- in $\operatorname{rank}(A^-) - \operatorname{rank}(A^-B) - \operatorname{rank}(CA^-)$ in (5) by A^-AA^-. Since $\operatorname{rank}(A^-AA^-) = \operatorname{rank}(A)$, $\operatorname{rank}(A^-AA^-B) = \operatorname{rank}(AA^-B)$, and $\operatorname{rank}(CA^-AA^-) = \operatorname{rank}(CA^-A)$, it follows that

$$\min_{A^-}[\operatorname{rank}(A^-) - \operatorname{rank}(A^-B) - \operatorname{rank}(CA^-)] \leqslant \operatorname{rank}(A) - \max_{A^-}[\operatorname{rank}(AA^-B) + \operatorname{rank}(CA^-A)]. \tag{6}$$

Note that

$$AA^-B = AA^\dagger B + AVE_AB, \quad CA^-A = CA^\dagger A + CF_AUA,$$

where U and V are arbitrary. Applying

$$\max_X \operatorname{rank}(A + BXC) = \min\left\{\operatorname{rank}(A \quad B),\ \operatorname{rank}\begin{pmatrix} A \\ C \end{pmatrix}\right\}$$

[see Tian (2002a)] to AA^-B and CA^-A and simplifying gives

$$\max_{A^-} \operatorname{rank}(AA^-B) = \max_V \operatorname{rank}(AA^\dagger B + AVE_AB) = \min\{\operatorname{rank}(A),\ \operatorname{rank}(B)\}, \tag{7}$$

$$\max_{A^-} \operatorname{rank}(CA^-A) = \max_U \operatorname{rank}(CA^\dagger A + CF_AUA) = \min\{\operatorname{rank}(A),\ \operatorname{rank}(C)\}. \tag{8}$$

Combining (6) with (7) and (8) gives

$$\min_{A^-}[\operatorname{rank}(A^-) - \operatorname{rank}(A^-B) - \operatorname{rank}(CA^-)] \leqslant \max\{-\operatorname{rank}(A),\ -\operatorname{rank}(B),\ -\operatorname{rank}(C),\ \operatorname{rank}(A) - \operatorname{rank}(B) - \operatorname{rank}(C)\}. \tag{9}$$

Substituting (9) into (5) and simplifying gives

$$\min_{A^-,\,B^-,\,C^-} \operatorname{rank}\begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} \leqslant \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\},$$

and so, in view of (3), the claimed result.

References

Y. Tian (2002a). The maximal and minimal ranks of some expressions of generalized inverses of matrices. *Southeast Asian Bulletin of Mathematics*, **25**, 745–755.

Y. Tian (2002b). The minimal rank completion of a 3×3 partial block matrix. *Linear and Multilinear Algebra*, **50**, 125–131.

Solution 29-11.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

THEOREM 1. *Let*

$$H := \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

be a given block partitioned real matrix. Define

$$G(A^-, B^-, C^-) := \begin{pmatrix} A^- & C^- \\ B^- & 0 \end{pmatrix} \qquad \textit{and} \qquad g(A^-, B^-, C^-) := \operatorname{rank}(G(A^-, B^-, C^-)).$$

Then

$$\min_{A^-, B^-, C^-} g(A^-, B^-, C^-) = \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\}.$$

Our proof of this result will be based on the geometry of generalized inversion. For the sake of clarity as well as for easier reference, we therefore begin with introducing some notation and stating some auxiliary results.

Let $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of n-dimensional real column vectors and the set of $m \times n$ real matrices, respectively. Given $A \in \mathbb{R}^{m\times n}$, let A', $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\operatorname{rank}(A)$ denote the transpose, the range (column space), the null space, and the rank, respectively, of A.

Let $\mathcal{M}$ and $\mathcal{N}$ be linear subspaces in $\mathbb{R}^n$. Then $\mathcal{M}^\perp$ will stand for the orthogonal complement of $\mathcal{M}$ in $\mathbb{R}^n$ (with respect to the usual inner product), and if $\mathcal{M} \cap \mathcal{N} = \{0\}$, then $\mathcal{M} \oplus \mathcal{N}$ will denote the direct sum of $\mathcal{M}$ and $\mathcal{N}$. Next, if $\mathcal{N}$ is a direct complement of $\mathcal{M}$ (i.e., $\mathbb{R}^n = \mathcal{M} \oplus \mathcal{N}$), then $P_{\mathcal{M},\mathcal{N}}$ will denote the well-defined (generally oblique) projector onto $\mathcal{M}$ along $\mathcal{N}$. We note that $P_{\mathcal{M},\mathcal{N}}$ may be defined by $P_{\mathcal{M},\mathcal{N}}u = u$ if $u \in \mathcal{M}$ and $P_{\mathcal{M},\mathcal{N}}u = 0$ if $u \in \mathcal{N}$; see, e.g., Rao and Mitra (1971, pp. 106–113). We recall that any projector $P_{\mathcal{M},\mathcal{N}}$ is an idempotent matrix, i.e. $(P_{\mathcal{M},\mathcal{N}})^2 = P_{\mathcal{M},\mathcal{N}}$, and that conversely every idempotent matrix P is a projector, namely $P = P_{\mathcal{R}(P),\mathcal{N}(P)}$. It is also pertinent to mention that $(P_{\mathcal{M},\mathcal{N}})' = P_{\mathcal{N}^\perp,\mathcal{M}^\perp}$. If $\mathcal{N} = \mathcal{M}^\perp$, then we briefly write $P_\mathcal{M}$ for the orthogonal projector onto $\mathcal{M}$, i.e. $P_\mathcal{M} := P_{\mathcal{M},\mathcal{M}^\perp}$. The dimension of $\mathcal{M}$ will be denoted by $\dim(\mathcal{M})$.

For a given $A \in \mathbb{R}^{n\times m}$ and for given linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{R}^m$, it is convenient to denote by $\mathcal{M}+\mathcal{N}$, $A\mathcal{M}$, $\mathcal{N}_c(A)$, and $\mathcal{R}_c(A)$, respectively, the Minkowski sum of $\mathcal{M}$ and $\mathcal{N}$, the image of $\mathcal{M}$ under A, the set of all direct complements of $\mathcal{N}(A)$, and the set of all direct complements of $\mathcal{R}(A)$. We note that $(\mathcal{M}+\mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp$, $(\mathcal{M}\cap\mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$, and that $\mathcal{N}^\perp \subseteq \mathcal{M}^\perp$ whenever $\mathcal{M} \subseteq \mathcal{N}$. We further recall that $\mathcal{R}(A)^\perp = \mathcal{N}(A')$ and $\mathcal{N}(A)^\perp = \mathcal{R}(A')$.

For given $A \in \mathbb{R}^{n\times m}$, $\mathcal{M} \in \mathcal{N}_c(A)$ and $\mathcal{S} \in \mathcal{R}_c(A)$, consider the matrix equations

$$(G1) \quad AXA = A, \qquad (G2) \quad XAX = X, \qquad (G\mathcal{M}) \quad XA = P_{\mathcal{M},\mathcal{N}(A)}, \qquad (G\mathcal{S}) \quad AX = P_{\mathcal{R}(A),\mathcal{S}}.$$

Suppose that $\emptyset \neq \eta \subseteq \{1, 2, \mathcal{M}, \mathcal{S}\}$. Then $A\eta$ will denote the set of all those matrices X which satisfy equations (Gi) for all $i \in \eta$. A matrix $X \in A\eta$ is called an *η-inverse* of A and is denoted by A^η. $\{1\}$-inverses are usually called *generalized inverses* or *g-inverses* and are also denoted by A^-. For an extensive discussion of the theory of g-inverses, we refer, e.g., to the books by Ben-Israel and Greville (1974, 1980, 2003), Hartung and Werner (1984), Pringle and Rayner (1971), Rao and Mitra (1971); for a geometric approach, to Werner (1977, Chapter 1) and Rao and Yanai (1985); and for a projector theoretical one e.g. to the paper by Langenhop (1967). Only for the sake of clarity and for easier reference, a few basic results are summarized in Theorem 2 (cf. Werner (1986), see also Werner and Yapar (1996)).

THEOREM 2. *For $A \in \mathbb{R}^{n\times m}$, $\mathcal{M} \in \mathcal{N}_c(A)$ and $\mathcal{S} \in \mathcal{N}_c(A)$ we have the following results.*

(a) *The $\{2, \mathcal{M}, \mathcal{S}\}$-inverse of A exists uniquely. The $\{2, \mathcal{R}(A'), \mathcal{N}(A')\}$-inverse of A coincides with the Moore-Penrose inverse of A and is usually denoted by $A^\dagger$.*

(b) *Any $\{\mathcal{M}\}$-inverse of A and likewise any $\{\mathcal{S}\}$-inverse of A is always a $\{1\}$-inverse of A. Conversely, for each $\{1\}$-inverse X of A there uniquely exist an $\mathcal{M} \in \mathcal{N}_c(A)$ and an $\mathcal{S} \in \mathcal{R}_c(A)$ such that $X \in A\{\mathcal{M}, \mathcal{S}\}$. Moreover, if $X \in A\{\mathcal{M}, \mathcal{S}\}$, then $XAX = A^{\{2,\mathcal{M},\mathcal{S}\}}$.*

(c) *If $X \in A\{\mathcal{M}, \mathcal{S}\}$, then $\mathcal{M} = \mathcal{R}(XA) \subseteq \mathcal{R}(X)$, and $\mathcal{N}(X) \subseteq \mathcal{S} = \mathcal{N}(AX)$. In particular, $X\mathcal{S} \subseteq \mathcal{N}(A)$. Moreover, $X = A^{\{2,\mathcal{M},\mathcal{S}\}}$ if and only if $\mathcal{R}(X) = \mathcal{M}$ and $\mathcal{N}(X) = \mathcal{S}$. Hence $\operatorname{rank}(A^-) \geq \operatorname{rank}(A)$, and $X \in A\{1, 2\}$ if and only if $X \in A\{1\}$ and $\operatorname{rank}(X) = \operatorname{rank}(A)$.*

(d) *If $X \in A\{\mathcal{M}, \mathcal{S}\}$, then $X' \in A'\{\mathcal{S}^\perp, \mathcal{M}^\perp\}$, where $\mathcal{S}^\perp \in \mathcal{N}_c(A')$ and $\mathcal{M}^\perp \in \mathcal{R}_c(A')$.*

(e) *If A is nonsingular, then the only $\{1\}$-inverse of A is its regular inverse, i.e., $A\{1\} = \{A^{-1}\}$.*

For given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times k}$, it is well-known that $\operatorname{rank}(A,\ B) = \operatorname{rank}(A) + \operatorname{rank}(B) - \dim[\mathcal{R}(A) \cap \mathcal{R}(B)]$. Applying this result twice and observing that the rank of a matrix coincides with the rank of its transpose, we readily obtain the following result.

LEMMA 3. *For the rank of the partitioned matrix H of Theorem 1 we have*

$$\operatorname{rank}(H) = \operatorname{rank}(A) + \operatorname{rank}(B) + \operatorname{rank}(C) - \dim[\mathcal{R}(C') \cap \mathcal{R}(A')] - \dim[(A\mathcal{N}(C)) \cap \mathcal{R}(B)].$$

COROLLARY 4. *For the partitioned matrix H of Theorem 1 we have the following results:*

(a) $\operatorname{rank}(H) \geq \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\}$,

(b) $\operatorname{rank}(H) = \operatorname{rank}(A)$ *if and only if* $\mathcal{R}(B) \subseteq A\mathcal{N}(C)$ *and* $\mathcal{R}(C') \subseteq \mathcal{R}(A')$,

(c) $\operatorname{rank}(H) = \operatorname{rank}(B) + \operatorname{rank}(C)$ *if and only if* $\operatorname{rank}(A) = \dim[\mathcal{R}(C') \cap \mathcal{R}(A')] + \dim[\mathcal{R}(B) \cap (A\mathcal{N}(C))]$ *or, equivalently, if and only if* $A\mathcal{N}(C) \subseteq \mathcal{R}(B)$,

(d) $\operatorname{rank}(H) = \operatorname{rank}(B)$ *if and only if* $C = 0$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$,

(e) $\operatorname{rank}(H) = \operatorname{rank}(C)$ *if and only if* $B = 0$ *and* $\mathcal{R}(A') \subseteq \mathcal{R}(C')$,

(f) $\operatorname{rank}(H) = \operatorname{rank}(A) + \operatorname{rank}(B) + \operatorname{rank}(C)$ *if and only if* $\mathcal{R}(A') \cap \mathcal{R}(C') = \{0\}$ *and* $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.

PROOF. The claimed results follow easily from Lemma 3. We only prove (c). The rest is left to the reader. According to Lemma 3, $\operatorname{rank}(H) = \operatorname{rank}(B) + \operatorname{rank}(C)$ if and only if $\operatorname{rank}(A) = \dim[\mathcal{R}(C') \cap \mathcal{R}(A')] + \dim[(A\mathcal{N}(C)) \cap \mathcal{R}(B)]$. So it suffices to show that this is equivalent to $A\mathcal{N}(C) \subseteq \mathcal{R}(B)$. Clearly, $\dim[A\mathcal{N}(C)] = \dim\left[\mathcal{R}(AP_{\mathcal{N}(C)})\right] = \operatorname{rank}(AP_{\mathcal{N}(C)}) = \operatorname{rank}\left((AP_{\mathcal{N}(C)})'\right) = \operatorname{rank}(P_{\mathcal{N}(C)}A') = \operatorname{rank}(A') - \dim[\mathcal{R}(A') \cap \mathcal{R}(C')] = \operatorname{rank}(A) - \dim[\mathcal{R}(A') \cap \mathcal{R}(C')]$ or, equivalently, $\operatorname{rank}(A) = \dim[A\mathcal{N}(C)] + \dim[\mathcal{R}(A') \cap \mathcal{R}(C')]$. Therefore, $\operatorname{rank}(A) = \dim[\mathcal{R}(C') \cap \mathcal{R}(A')] + \dim[(A\mathcal{N}(C)) \cap \mathcal{R}(B)]$ if and only if $A\mathcal{N}(C) \subseteq \mathcal{R}(B)$. □

Applying Corollary 4(a) to the block partitioned matrix $G(A^-, B^-, C^-)$ of Theorem 1 and recalling Theorem 2(c) yields

$$g(A^-, B^-, C^-) \geq \max\{\operatorname{rank}(A^-),\ \operatorname{rank}(B^-) + \operatorname{rank}(C^-)\} \geq \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\}. \qquad (10)$$

In order to verify Theorem 1, we only have to show that the lower bound on the right-hand side of (10) is actually attainable for some suitably chosen generalized inverses. Below we therefore wish to construct such a triplet of g-inverses that minimizes our objective function $g(A^-, B^-, C^-)$. For that purpose, the following three results prove useful.

LEMMA 5. *Let A and B be matrices with the same number of rows, say m. Then $\operatorname{rank}(A) \leq \operatorname{rank}(B)$ if and only if there exist $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$ such that $\mathcal{S}_B \subseteq \mathcal{S}_A$.*

PROOF. Let the block partitioned matrix $[R_{A,B}, R_{rA}, R_{rB}, S_{A,B}]$, whose set of columns constitute a basis for $\mathbb{R}^m$, be such that $\mathcal{R}(R_{A,B}) = \mathcal{R}(A) \cap \mathcal{R}(B)$, $\mathcal{R}(R_{A,B}, R_{rA}) = \mathcal{R}(A)$ and $\mathcal{R}(R_{A,B}, R_{rB}) = \mathcal{R}(B)$. Then $\mathcal{S}_{A,B} := \mathcal{R}(S_{A,B})$ is a direct complement of $\mathcal{R}(A) + \mathcal{R}(B)$. Since $\operatorname{rank}(A) \leq \operatorname{rank}(B)$, it is possible to partition $R_{rB} = (R_{1rB}, R_{2rB})$ such that R_{1rB} and R_{rA} have the same number of column vectors. Define $\mathcal{S}_B := \mathcal{S}_{A,B} \oplus \mathcal{R}(R_{rA} + R_{1rB})$ and $\mathcal{S}_A := \mathcal{S}_{A,B} \oplus [\mathcal{R}(R_{rA} + R_{1rB}) \oplus \mathcal{R}(R_{2rB})]$. Since $\mathcal{S}(B) \in \mathcal{R}_c(B)$, $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \subseteq \mathcal{S}_A$, the proof of necessity is complete. The converse implication is trivial. □

On similar lines we obtain the following.

LEMMA 6. *Let A and C be matrices with the same number of columns. Then $\operatorname{rank}(A) \leq \operatorname{rank}(C)$ if and only if there exist $\mathcal{M}_A \in \mathcal{N}_c(A)$ and $\mathcal{M}_C \in \mathcal{N}_c(C)$ such that $\mathcal{M}_A \subseteq \mathcal{M}_C$.*

LEMMA 7. *Let A and B be matrices with the same number of rows and let $\operatorname{rank}(A) > \operatorname{rank}(B)$. Then there exist $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$ such that $\mathcal{S}_A \subseteq \mathcal{S}_B$, in which case*

$$P := P_{\mathcal{S}_B \cap \mathcal{R}(A), \mathcal{S}_A \oplus \mathcal{R}(B)} \tag{11}$$

is a well-defined (generally oblique) projector for which we have

$$A^{\{\mathcal{S}_A\}} P \in (PA)\{1,2\}$$

as well as $\operatorname{rank}(A^{\{\mathcal{S}_A\}} P) = \operatorname{rank}(PA) = \operatorname{rank}(A) - \operatorname{rank}(B)$, irrespective of the choice of $A^{\{\mathcal{S}_A\}} \in A\{\mathcal{S}_A\}$.

PROOF. Since $\operatorname{rank}(A) > \operatorname{rank}(B)$, the existence of $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$ with $\mathcal{S}_A \subseteq \mathcal{S}_B$ is guaranteed by Lemma 5. Then, in view of $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$, clearly $\mathcal{S}_B = \mathcal{S}_A \oplus [\mathcal{R}(A) \cap \mathcal{S}_B]$, so that P is indeed a well-defined projector. Since $\mathcal{S}_A \subseteq \mathcal{N}(P)$, we get $PAA^{\{\mathcal{S}_A\}} = P$, so $PAA^{\{\mathcal{S}_A\}}PPA = P^3A = PA$ and $A^{\{\mathcal{S}_A\}}PPAA^{\{\mathcal{S}_A\}}P = A^{\{\mathcal{S}_A\}}P^3 = A^{\{\mathcal{S}_A\}}P$, thus showing that $A^{\{\mathcal{S}_A\}}P$ is as claimed a $\{1,2\}$-inverse of PA. Therefore, in view of Theorem 2(c), $\operatorname{rank}(A^{\{\mathcal{S}_A\}}P) = \operatorname{rank}(PA)$. Since by construction $\operatorname{rank}(PA) = \operatorname{rank}(P) = \operatorname{rank}(A) - \operatorname{rank}(B)$, our proof is complete. □

We are now in the position to prove Theorem 1 just by making use of all our auxiliary observations.

PROOF OF THEOREM 1. We consider the following three exhaustive cases: (i) $\operatorname{rank}(B) \geq \operatorname{rank}(A)$, (ii) $\operatorname{rank}(C) \geq \operatorname{rank}(A)$, and (iii) $\operatorname{rank}(A) > \max\{\operatorname{rank}(B), \operatorname{rank}(C)\}$.

Case (i): Let $\operatorname{rank}(B) \geq \operatorname{rank}(A)$, in which case $\operatorname{rank}(B) + \operatorname{rank}(C) = \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\}$. Then, according to Theorem 2(c), clearly $\operatorname{rank}(B) = \operatorname{rank}(B^{\{1,2\}}) \geq \operatorname{rank}(A^{\{1,2\}}) = \operatorname{rank}(A)$. Lemma 5 allows us to choose $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$ such that $\mathcal{S}_B \subseteq \mathcal{S}_A$, in which case, in the light of Theorem 2,

$$A^{\{2,\mathcal{S}_A\}} \mathcal{N}(B^{\{2,\mathcal{S}_B\}}) = A^{\{2,\mathcal{S}_A\}} \mathcal{S}_B = \{0\} \subseteq \mathcal{R}(C^{\{1,2\}}),$$

whence, by means of Corollary 4(c) and Theorem 2(c), we get

$$g(A^{\{2,\mathcal{S}_A\}},\ B^{\{2,\mathcal{S}_B\}},\ C^{\{1,2\}}) = \operatorname{rank}(B^{\{2,\mathcal{S}_B\}}) + \operatorname{rank}(C^{\{1,2\}}) = \operatorname{rank}(B) + \operatorname{rank}(C)$$

irrespective of the choices of $A^{\{2,\mathcal{S}_A\}} \in A\{2, \mathcal{S}_A\}$, $B^{\{2,\mathcal{S}_B\}} \in B\{2, \mathcal{S}_B\}$ and $C^{\{1,2\}} \in C\{1,2\}$.

Case (ii): Let $\operatorname{rank}(C) \geq \operatorname{rank}(A)$, in which case $\operatorname{rank}(C) + \operatorname{rank}(B) \geq \max\{\operatorname{rank}(A),\ \operatorname{rank}(B) + \operatorname{rank}(C)\}$. Since $g(A^{\{1,2\}},\ B^{\{1,2\}},\ C^{\{1,2\}}) = g((A^{\{1,2\}})',\ (C^{\{1,2\}})',\ (B^{\{1,2\}})')$, it follows from Case (i) that there exist some $\{1,2\}$-inverses $A^{\{1,2\}}$, $B^{\{1,2\}}$ and $C^{\{1,2\}}$ such that

$$g(A^{\{1,2\}},\ B^{\{1,2\}},\ C^{\{1,2\}}) = \operatorname{rank}(C) + \operatorname{rank}(B).$$

Case (iii): Let $\operatorname{rank}(A) > \max\ \{\operatorname{rank}(B),\ \operatorname{rank}(C)\}$. According to Lemma 5, choose $\mathcal{S}_A \in \mathcal{R}_c(A)$ and $\mathcal{S}_B \in \mathcal{R}_c(B)$ such that $\mathcal{S}_A \subseteq \mathcal{S}_B$ or, equivalently, $\mathcal{S}_B^{\perp} \subseteq \mathcal{S}_A^{\perp}$. Then $\mathcal{S}_B = \mathcal{S}_A \oplus [\mathcal{S}_B \cap \mathcal{R}(A)]$ and $\mathcal{S}_A^{\perp} \cap \mathcal{S}_B^{\perp} = \mathcal{S}_B^{\perp}$. Applying Lemma 3 to the block partitioned matrix $G(A^{\{2,\mathcal{S}_A\}},\,B^{\{2,\mathcal{S}_B\}},\,C^{\{1,2\}})$ and making repeatedly use of Theorem 2 results in

$$\begin{aligned} g(A^{\{2,\mathcal{S}_A\}},\,B^{\{2,\mathcal{S}_B\}},\,C^{\{1,2\}}) &= \operatorname{rank}(A^{\{2,\mathcal{S}_A\}})+\operatorname{rank}(B^{\{2,\mathcal{S}_B\}})+\operatorname{rank}(C^{\{1,2\}})-\dim[\mathcal{R}((B^{\{2,\mathcal{S}_B\}})')\cap\mathcal{R}((A^{\{2,\mathcal{S}_A\}})')] \\ &\quad -\dim[\mathcal{R}(C^{\{1,2\}})\cap(A^{\{2,\mathcal{S}_A\}}\mathcal{N}(B^{\{2,\mathcal{S}_B\}}))] \\ &= \operatorname{rank}(A)+\operatorname{rank}(B)+\operatorname{rank}(C)-\dim[\mathcal{S}_B^{\perp}\cap\mathcal{S}_A^{\perp}]-\dim[\mathcal{R}(C^{\{1,2\}})\cap(A^{\{2,\mathcal{S}_A\}}\mathcal{S}_B)] \\ &= \operatorname{rank}(A)+\operatorname{rank}(B)+\operatorname{rank}(C)-\dim\,\mathcal{S}_B^{\perp}-\dim[\mathcal{R}(C^{\{1,2\}})\cap(A^{\{2,\mathcal{S}_A\}}\mathcal{S}_B)] \\ &= \operatorname{rank}(A)+\operatorname{rank}(C)-\dim[\mathcal{M}_C\cap(A^{\{2,\mathcal{S}_A\}}\mathcal{S}_B)], \end{aligned} \tag{12}$$

where $\mathcal{M}_C := \mathcal{R}(C^{\{1,2\}})$. Since $A^{\{2,\mathcal{S}_A\}}\mathcal{S}_B = A^{\{2,\mathcal{S}_A\}}[\mathcal{S}_B\cap\mathcal{R}(A)] = \mathcal{R}(A^{\{2,\mathcal{S}_A\}}P)$, where P is defined as in (11), we know from Lemma 7 that $\dim(A^{\{2,\mathcal{S}_A\}}\mathcal{S}_B) = \operatorname{rank}(PA) = \operatorname{rank}(A)-\operatorname{rank}(B)$. For convenience, put $\tilde{A} := PA$. Since $\mathcal{N}(A)\subseteq\mathcal{N}(PA)=\mathcal{N}(\tilde{A})$, it is possible to choose for any given $\mathcal{M}_{\tilde{A}}\in\mathcal{N}_c(\tilde{A})$ an $\mathcal{M}_A\in\mathcal{N}_c(A)$ with $\mathcal{M}_{\tilde{A}}\subseteq\mathcal{M}_A$, in which case, as a consequence of Lemma 7 and Theorem 2, $\mathcal{R}(A^{\{2,\mathcal{M}_A,\mathcal{S}_A\}}P)=\mathcal{M}_{\tilde{A}}\subseteq\mathcal{M}_A$. Then

$$A^{\{2,\mathcal{M}_A,\mathcal{S}_A\}}\mathcal{S}_B=\mathcal{M}_{\tilde{A}} \quad\text{and}\quad \dim(\mathcal{M}_{\tilde{A}})=\operatorname{rank}(A)-\operatorname{rank}(B). \tag{13}$$

We now proceed with considering two complementary subcases, namely (iii$_1$) $\operatorname{rank}(\tilde{A})\le\operatorname{rank}(C)$ and (iii$_2$) $\operatorname{rank}(C)<\operatorname{rank}(\tilde{A})$.

(iii$_1$): Let $\operatorname{rank}(\tilde{A})\le\operatorname{rank}(C)$. Then $\operatorname{rank}(A)\le\operatorname{rank}(B)+\operatorname{rank}(C)$ or, equivalently, $\max\{\operatorname{rank}(A),\,\operatorname{rank}(B)+\operatorname{rank}(C)\}=\operatorname{rank}(B)+\operatorname{rank}(C)$. According to Lemma 6, choose $\mathcal{M}_{\tilde{A}}\in\mathcal{N}_c(\tilde{A})$ and $\mathcal{M}_C\in\mathcal{N}_c(C)$ such that $\mathcal{M}_{\tilde{A}}\subseteq\mathcal{M}_C$. Then $\mathcal{M}_C\cap\mathcal{M}_{\tilde{A}}=\mathcal{M}_{\tilde{A}}$, and so it follows from (12) and (13) that

$$g(A^{\{2,\mathcal{M}_A,\mathcal{S}_A\}},\,B^{\{2,\mathcal{S}_B\}},\,C^{\{2,\mathcal{M}_C\}})=\operatorname{rank}(B)+\operatorname{rank}(C)$$

holds for each $\mathcal{M}_A\in\mathcal{N}_c(A)$ for which $\mathcal{M}_{\tilde{A}}\subseteq\mathcal{M}_A$.

(iii$_2$): Let $\operatorname{rank}(C)<\operatorname{rank}(\tilde{A})$. Then $\operatorname{rank}(B)+\operatorname{rank}(C)<\operatorname{rank}(A)$, and so $\max\{\operatorname{rank}(A),\,\operatorname{rank}(B)+\operatorname{rank}(C)\}=\operatorname{rank}(A)$. According to Lemma 6, choose $\mathcal{M}_C\in\mathcal{N}_c(C)$ and $\mathcal{M}_{\tilde{A}}\in\mathcal{N}_c(\tilde{A})$ such that $\mathcal{M}_C\subseteq\mathcal{M}_{\tilde{A}}$. Then $\mathcal{M}_C\cap\mathcal{M}_{\tilde{A}}=\mathcal{M}_C$. Consequently, $\dim(\mathcal{M}_C\cap\mathcal{M}_{\tilde{A}})=\operatorname{rank}(C)$, and so it follows from (12) and (13) that

$$g(A^{\{2,\mathcal{M}_A,\mathcal{S}_A\}},\,B^{\{2,\mathcal{S}_B\}},\,C^{\{2,\mathcal{M}_C\}})=\operatorname{rank}(A)+\operatorname{rank}(C)-\operatorname{rank}(C)=\operatorname{rank}(A)$$

holds for each $\mathcal{M}_A\in\mathcal{N}_c(A)$ for which $\mathcal{M}_{\tilde{A}}\subseteq\mathcal{M}_A$. This completes the proof of Theorem 1. □

We conclude with mentioning that our solution can obviously be extended to the case of complex matrices just by replacing transposition by conjugate transposition.

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Problem 30-1: Star Partial Ordering, Left-star Partial Ordering, and Commutativity

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For any $A, B\in\mathbb{C}_{m,n}$, the star partial ordering $A\overset{*}{\le}B$, defined by $A^*A=A^*B$ and $AA^*=BA^*$, clearly implies the left-star partial ordering $A\ *\!\le B$, defined by $A^*A=A^*B$ and $\mathcal{R}(A)\subseteq\mathcal{R}(B)$, where $\mathcal{R}(\cdot)$ denotes the range of a given matrix. Show that if $m = n$ and A or B is an EP matrix, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$ or $\mathcal{R}(B) = \mathcal{R}(B^*)$, then the implication $A \ {*\!\leq}\ B \Rightarrow AB = BA$ cannot hold unless $A \ {*\!\leq}\ B$ is strengthened to $A \overset{*}{\leq} B$.

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In fact we will establish a somewhat more general result, whose one part refers to the notion of the minus partial ordering instead of the left-star partial ordering, the former admitting a characterization through the rank subtractivity property

$$A \overset{-}{\leq} B \ \Leftrightarrow\ \mathrm{r}(B - A) = \mathrm{r}(B) - \mathrm{r}(A). \tag{14}$$

The generalization mentioned above is a consequence of the relationships

$$A \overset{*}{\leq} B \ \Rightarrow\ A \ {*\!\leq}\ B \ \Rightarrow\ A \overset{-}{\leq} B; \tag{15}$$

cf. Theorem 2.1 of Baksalary and Mitra (1991).

THEOREM. *Under the assumption that $A, B \in \mathbb{C}_{n,n}$ satisfy the commutativity condition $AB = BA$, the following statements hold:*

(a) *when A is an EP matrix, then $A \overset{-}{\leq} B \Leftrightarrow A \overset{*}{\leq} B$,*

(b) *when B is an EP matrix, then $A \ {*\!\leq}\ B \Leftrightarrow A \overset{*}{\leq} B$.*

PROOF. In the case where the ranks $\mathrm{r}(A) = a$ and $\mathrm{r}(B) = b$ are equal, it follows from (14) that each of the orders in (15) holds merely when $A = B$. In nontrivial situations, where $a < b$, Theorems 1 and 2 of Hartwig and Styan (1986) and Theorem 2.1 of Baksalary, Baksalary, and Liu (2003a) assert that A and B are ordered as in the succsessive parts of (15) if and only if

$$A = U \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} V^* \tag{16}$$

and, correspondingly to the cases $A \overset{*}{\leq} B$, $A \ {*\!\leq}\ B$, $A \overset{-}{\leq} B$,

$$B = U \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} V^*, \quad B = U \begin{pmatrix} D_1 & 0 \\ D_2 S & D_2 \end{pmatrix} V^*, \quad B = U \begin{pmatrix} D_1 + RD_2S & RD_2 \\ D_2 S & D_2 \end{pmatrix} V^* \tag{17}$$

for some $U \in \mathbb{C}_{m,b}$ and $V \in \mathbb{C}_{n,b}$ such that $U^*U = I_b = V^*V$, some positive definite diagonal matrices D_1 and D_2 of degree a and $b - a$, respectively, and some $R \in \mathbb{C}_{a,b-a}$, $S \in \mathbb{C}_{b-a,a}$.

Now, assuming that $m = n$ and U and V are partitioned as $U = (U_1 : U_2)$ and $V = (V_1 : V_2)$, where $U_1, V_1 \in \mathbb{C}_{n,a}$, $U_2, V_2 \in \mathbb{C}_{n,b-a}$, let the product V^*U be partitioned accordingly as

$$V^*U = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{18}$$

with $W_{ij} = V_i^* U_j$, $i, j = 1, 2$. Referring to notation (18), A of the form (16) commutes with B of the form given in the third part of (17) if and only if

$$\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} D_1 + RD_2S & RD_2 \\ D_2 S & D_2 \end{pmatrix} = \begin{pmatrix} D_1 + RD_2S & RD_2 \\ D_2 S & D_2 \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

A straightforward analysis of this equality shows that when $A \overset{-}{\leq} B$, then

$$AB = BA \ \Leftrightarrow\ W_{11}R + W_{12} = 0 \ \text{ and } \ SW_{11} + W_{21} = 0. \tag{19}$$

Noting that the matrix B in the second part of (17) is obtainable from that in the third part by substituting $R = 0$ leads to the corollary that when $A \ {*\!\leq}\ B$, then

$$AB = BA \ \Leftrightarrow\ W_{12} = 0 \ \text{ and } \ SW_{11} + W_{21} = 0; \tag{20}$$

cf. Theorem 2.1 of Baksalary, Baksalary, and Liu (2003b).

Let $\mathbb{C}_n^{\mathsf{EP}}$ denote the subset of $\mathbb{C}_{n,n}$ consisting of EP matrices. It is known that $K \in \mathbb{C}_n^{\mathsf{EP}}$ if and only if $KK^\dagger = K^\dagger K$, where $K^\dagger$ denotes the Moore-Penrose inverse of K, i.e., the unique solution to the equations

$$KK^\dagger K = K,\ K^\dagger KK^\dagger = K^\dagger,\ KK^\dagger = (KK^\dagger)^*,\ K^\dagger K = (K^\dagger K)^*. \tag{21}$$

Referring to (21), it can easily be verified that if A is of the form (16) and B of the form given in the middle part of (17), then

$$A^\dagger = V\begin{pmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* \quad \text{and} \quad B^\dagger = V\begin{pmatrix} D_1^{-1} & 0 \\ -SD_1^{-1} & D_2^{-1} \end{pmatrix}U^*. \tag{22}$$

Premultiplying and postmultiplying the equality $AA^\dagger = A^\dagger A$, which on account of (16) and the first part of (22) is expressible as

$$U\begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix}U^* = V\begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix}V^*,$$

firstly by U^* and U and then by V^* and V, respectively, shows that

$$A \in \mathbb{C}_n^{\mathsf{EP}} \ \Rightarrow\ W_{11}^*W_{11} = I_a = W_{11}W_{11}^*,\ \ W_{12}^*W_{12} = 0,\ \text{ and }\ W_{21}W_{21}^* = 0. \tag{23}$$

Since the conditions on the right-hand side of (23) obviously imply the nonsingularity of $W_{11} \in \mathbb{C}_{a,a}$ and $W_{12} = 0$, $W_{21} = 0$, it is seen that combining (19) with (23) leads to $R = 0$ and $S = 0$. Then B takes the form as in the first part of (17), which means that $A \overset{*}{\leq} B$, thus concluding the proof of part (a) of the theorem. Further, premultiplying and postmultiplying the equality $BB^\dagger = B^\dagger B$, which on account of the second parts of (17) and (22) is expressible as $UU^* = VV^*$, by V^* and V, respectively, shows that

$$B \in \mathbb{C}_n^{\mathsf{EP}} \ \Rightarrow\ W_{11}W_{11}^* + W_{12}W_{12}^* = I_a \ \text{ and }\ W_{11}W_{21}^* + W_{12}W_{22}^* = 0. \tag{24}$$

Consequently, combining (20) with (24) yields $W_{11}W_{11}^* = I_a$ and $W_{11}W_{21}^* = 0$. On account of the nonsingularity of W_{11}, the latter of these equalities entails $W_{21} = 0$, and then from (20) it follows that $S = 0$. It is seen, therefore, that B takes again the form as in the first part of (17), which means that $A \overset{*}{\leq} B$. □

We conclude our solution by pointing out that the assumption of the left-star order $A\ {*\!\leq}\ B$ in part (b) of the theorem cannot in general be weakened to the minus order $A \overset{-}{\leq} B$ as in part (a). A counterexample is provided by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

obviously satisfying the conditions $AB = BA$ and $B \in \mathbb{C}_2^{\mathsf{EP}}$ along with $A \overset{-}{\leq} B$, but not satisfying the equality $A^*A = A^*B$, and therefore not being even left-star ordered, which according to (15) is necessary for $A \overset{*}{\leq} B$.

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Solution 30-1.2 by Nir COHEN, *Campinas State University, Campinas, Brazil:* nir@ime.unicamp.br

A part of the assertion in Problem 30-1 can be seen by checking that both

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are EP matrices which satisfy $A* \leq B$, but not the commutativity property $[A,B]=0$. The second example, in which

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

shows that if B is EP but A is not, then $A \overset{*}{\leq} B$ does not necessarily imply $[A,B]=0$. It is correct, however, that if A is EP and $A* \leq B$, then $[A,B]=0$. We shall show a stronger result.

PROPOSITION. *If A is EP and $A \overset{*}{\leq} B$, then $\Re_A$ is a reducing subspace for B and $B|\Re_A = A|\Re_A$.* (This implies that $AB = BA = A^2$, hence in particular A and B commute.)

PROOF. Indeed, since A is EP, there exist a unitary $n \times n$ matrix U and an invertible $r \times r$ matrix A_{11} (with $r = \operatorname{rank} A$) such that $A = U^*(A_{11} \oplus 0)U$ (easy). Writing $B = U^* \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} U$, we get

$$A^*(B-A) = 0 \quad \Rightarrow \quad B_{11} = A_{11}, \quad B_{12} = 0,$$

$$(B-A)A^* = 0 \quad \Rightarrow \quad B_{11} = A_{11}, \quad B_{21} = 0.$$

Hence $B = U^*(A_{11} \oplus B_{22})U$, proving the assertion. □

Solution 30-1.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

For a complex matrix C, let $\operatorname{rank}(C)$, C^*, $\mathcal{R}(C)$, $\mathcal{N}(C)$, and $P_{\mathcal{R}(C)}$ denote the rank, the conjugate transpose, the range (column space), the null space, and the orthogonal projector onto $\mathcal{R}(C)$ [along its usual orthogonal complement $\mathcal{N}(C^*)$], respectively, of C. Recall that any orthogonal projector is Hermitian and that $P_{\mathcal{R}(C)}$ may be defined by $P_{\mathcal{R}(C)}x = x$ if $x \in \mathcal{R}(C)$ and $P_{\mathcal{R}(C)}x = 0$ if $x \in \mathcal{N}(C^*)$.

We offer the following slightly more informative solution to the problem under study.

THEOREM. *For square complex matrices A, A_1 and A_2 with $A = A_1 + A_2$ we have the following results:*

(a) *$A_1 * \leq A$ if and only if $P_{\mathcal{R}(A)} = P_{\mathcal{R}(A_1)} + P_{\mathcal{R}(A_2)}$, in which case, in particular, $A_1 \overset{-}{\leq} A$, i.e., $\mathcal{R}(A) = \mathcal{R}(A_1) \oplus \mathcal{R}(A_2)$ or, equivalently, $\mathcal{R}(A^*) = \mathcal{R}(A_1^*) \oplus \mathcal{R}(A_2^*)$, where $\oplus$ indicates a direct sum.*

(b) *When $A_1 * \leq A$, then $A_1A = AA_1$ if and only if $A_1A_2 = 0 = A_2A_1$.*

(c) *When $A_1 * \leq A$ and $A_1A = AA_1$, then A is EP if and only if A_1 and A_2 are both EP.*

(d) *When A_1 is EP and $A_1A = AA_1$, then $A_1 * \leq A$ if and only if $A_1 \overset{*}{\leq} A$.*

(e) *When A is EP and $A_1A = AA_1$, then $A_1 * \leq A$ if and only if $A_1 \overset{*}{\leq} A$.*

PROOF. (a): By definition, $A_1 * \leq A$ if and only if $A_1^*A_2 = 0$ and $\mathcal{R}(A_1) \subseteq \mathcal{R}(A)$. Clearly, $A_1^*A_2 = 0 \Leftrightarrow \mathcal{R}(A_2) \subseteq \mathcal{N}(A_1^*)$, in which case $\mathcal{R}(A_2) \cap \mathcal{R}(A_1) = \{0\}$. Hence, whenever $A_1 * \leq A$, then necessarily $\mathcal{R}(A) = \mathcal{R}(A_1) \oplus \mathcal{R}(A_2)$. According to, e.g., Theorem 2.3 in Jain, Mitra & Werner (1996),

$$\mathcal{R}(A) = \mathcal{R}(A_1) \oplus \mathcal{R}(A_2) \Leftrightarrow A_1 \overset{-}{\leq} A \Leftrightarrow \operatorname{rank}(A) = \operatorname{rank}(A_1) + \operatorname{rank}(A_2) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A_1^*) \oplus \mathcal{R}(A_2^*).$$

For completing the proof of (a), observe first that $A_1^*A_2 = 0 \Leftrightarrow P_{\mathcal{R}(A_1)}A_2 = 0 \Leftrightarrow P_{\mathcal{R}(A_1)}P_{\mathcal{R}(A_2)} = 0 \Leftrightarrow P_{\mathcal{R}(A_2)}P_{\mathcal{R}(A_1)} = 0 \Leftrightarrow A_2^*A_1 = 0$. Recall next that $P_{\mathcal{R}(A_1)} + P_{\mathcal{R}(A_2)}$ is an orthogonal projector if and only if $P_{\mathcal{R}(A_1)}P_{\mathcal{R}(A_2)} = 0$, in which case the sum of these orthogonal projectors is the orthogonal projector onto $\mathcal{R}(A_1) \oplus \mathcal{R}(A_2)$; see, e.g., Theorem 5.12 in Rao & Mitra (1971). Consequently, $P_{\mathcal{R}(A)} = P_{\mathcal{R}(A_1)} + P_{\mathcal{R}(A_2)}$ if and only if $\mathcal{R}(A_1) \subseteq \mathcal{R}(A)$ and $A_1^*A_2 = 0$, and so the proof of part (a) is complete.

(b): According to (a), whenever $A_1 * \leq A$, then $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$. Therefore, in view of $A_1A = AA_1 \Leftrightarrow A_1A_2 = A_2A_1$, clearly $A_1A = A_1A$ if and only if $A_1A_2 = 0 = A_2A_1$.

(c): Let $A_1 * \leq A$ and $A_1A = AA_1$. Then $A_1^*A_2 = 0$ or, equivalently, $A_2^*A_1 = 0$. Furthermore, in view of (a), $P_{\mathcal{R}(A)} = P_{\mathcal{R}(A_1)} + P_{\mathcal{R}(A_2)}$ and $\mathcal{R}(A^*) = \mathcal{R}(A_1^*) \oplus \mathcal{R}(A_2^*)$. Finally, according to (b), $A_2A_1 = 0 = A_1A_2$ or, equivalently, $A_1^*A_2^* = 0 = A_2^*A_1^*$. Consequently, A is EP $\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^*) \Leftrightarrow P_{\mathcal{R}(A)}A^* = A^* \Leftrightarrow P_{\mathcal{R}(A)}A_i^* = A_i^* \ (i = 1,2) \Leftrightarrow P_{\mathcal{R}(A_i)}A_i^* = A_i^* \ (i = 1,2) \Leftrightarrow \mathcal{R}(A_i^*) = \mathcal{R}(A_i) \ (i = 1,2) \Leftrightarrow A_1$ and A_2 are both EP.

(d): Let $A_1 * \leq A$ and $A_1A = AA_1$. Then $A_1^*A_2 = 0$ and, in view of (b), $A_2A_1 = 0$. Needless to say, if A_1 is EP, i.e., if $\mathcal{R}(A_1) = \mathcal{R}(A_1^*)$, then $A_2A_1 = 0 \Leftrightarrow A_2A_1^* = 0$. In such a case we therefore have $A_1^*A_2 = 0$ and $A_2A_1^* = 0$ or, equivalently, $A_1 \overset{*}{\leq} A$. The converse implication is trivial.

(e): This result follows directly from (c) and (d). □

We conclude with mentioning the following Corollary which is easy to prove by means of our Theorem.

COROLLARY. *Let $A := A_1 + A_2$ be such that $A_1 * \leq A$. Then any two of the following three conditions imply the remaining one:*

(i) A *is* EP, (ii) A_1 *and* A_2 *are* EP, (iii) $A_1A = AA_1$.

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Problem 30-2: Class of (0, 1)-Matrices Containing Constant Column-Sum Submatrices

Proposed by Bernardete RIBEIRO and Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:*
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For given $k_1, \ldots, k_n \in [n] = \{1, 2, \ldots, n\}$ define the $\{0, 1\}$-matrix $A = A(k_1, \ldots, k_n) = (a_{ij})$ by putting $a_{ij} = 1$ iff j is one of the first k_i entries of the $n-$tuple $(i, i+1, \ldots, n, 1, 2, \ldots, i-1)$. Show that there exists a $\{0, 1\}-$row x and a $k \in [n-1]$ such that $xA = k1_n$, where $1_n = (1, \ldots, 1)$.

Solution 30-2.1 by Nir COHEN, *Campinas State University, Campinas, Brazil:* nir@ime.unicamp.br

Define a function $f : [n] \to [n]$ by $f(i) \equiv i + k_i \; mod(n)$. We shall call a set $C \subset [n]$ "stable" if $f(C) = C$. Obviously, the minimal stable sets are closed chains of the form $C = \{i_1, \ldots, i_m\}$ with $f(i_j) = i_{j+1}$ $(j = 1, \cdots, m-1)$ and $f(i_m) = i_1$, with $|C| = m$. A singleton may be a stable set. The existence of minimal stable sets is easily established by following a chain $i_{k+1} = f(i_k)$ until it repeats itself. The minimal stable sets are pairwise disjoint.

With any minimal stable set C define the (0,1)-vector $x_C = \sum_{i \in C} e_i$, with the usual canonical basis $\{e_i\}$ in $\mathbf{R}^n$. Cyclicity of the chain implies that $\sum_{i \in C} f(i) = k_C n$ for some positive integer k_C, implying that $x_C A = k_C 1_n$.

This settles affirmatively the question raised, but more can be said: Every (0,1)-vector y with $yA = t1_n$ is supported on a disjoint union of minimal stable subsets.

Indeed, let S be the support of y. We claim that $f(S) \subset S$, hence S contains a minimal stable subset, unless $S = \emptyset$.

Indeed, if i but not $f(i)$ were in S then $(y_1A)_{i+k_i}$ would be smaller than $(y_1A)_{i+k_i-1}$, since the sequence of 1's in row i ends in column $f(i)$, while no new sequence of ones would start there. But this would contradict the identity $y_1A = t_1 1_n$.

Let now C be the (non-trivial, disjoint) union of minimal stable subsets in S, and call x the vector supported on it. We have $xA = k1_n$. The vector $z := y - x$ is a (0,1) vector with support $S \setminus C$ and satisfies $zA = (t-k)1_n$. However, the support of z contains no minimal stable subsets, hence by the previous claim, it is empty, implying that $S = C$. □

Solution 30-2.2 by the Proposers Bernardete RIBEIRO and Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:*
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We define the involutive map op : $\mathbb{R}^n \to \mathbb{R}^n$ by op$(a) = 1_n - a$ and first prove the following combinatorial lemma of interest in its own right.

LEMMA. *Let $n \in \mathbb{Z}_{\geq 1}$, $\emptyset \neq X \subset \mathbb{R}^n$ be finite, and $Y =$ op(X). Let $f : X \to Y$ be any map. Then there is a nonempty subset X' of X on which $f|X'$ is injective and such that*

$$\sum_{x \in X'} (x + f(x)) = |X'| 1_n.$$

PROOF. Inductively define $X_0 = X$, $Y_0 = f(X_0)$, and $X_{k+1} = \text{op}(Y_k)$, $Y_{k+1} = f(X_{k+1})$, for $k = 0, 1, 2, \ldots$. These sets are nonempty. Note that $X_1 \subseteq X_0$, hence $Y_1 \subseteq Y_0$, hence $X_2 \subseteq X_1$, hence $Y_2 \subseteq Y_1$, etc. . By the finiteness of X, Y and the injectivity of op, there is a k such that $|X_k| = |X_{k+1}|$. Thus for $X' = X_k$ and $Y' = Y_k$ we have $X' = X_{k+1}$, $f|X' : X' \to Y'$ is a bijection, and $X' = \text{op}(Y')$. Consequently,

$$\sum_{x \in X'} (x + f(x)) = \sum_{x \in X'} x + \sum_{y \in Y'} y = \sum_{y \in Y'} (y + \text{op}(y)) = \sum_{y \in Y'} 1_n = |X'| 1_n,$$

as claimed. □

We borrow from Matlab the notational devices to write A(I,J) for the submatrix obtained by restricting the set of row and column indices to the sets I, J, assumed in their natural order, $A(i,:)$ for the $i-$th row of a matrix A, $s(j : j')$ for the $(j' - j + 1)-$tuple made from entries in positions $j, j+1, \ldots, j'$ of an $n-$tuple s, etc.

The reader hopefully will get a rough idea of what is actually going on in the proof below, by following it with the example given in Figure 1, where all blanks are zeroes. There $n = 7$, $(k_1, \ldots, k_7) = (4, 5, 6, 4, 5, 3, 4)$ respectively. With the definitions

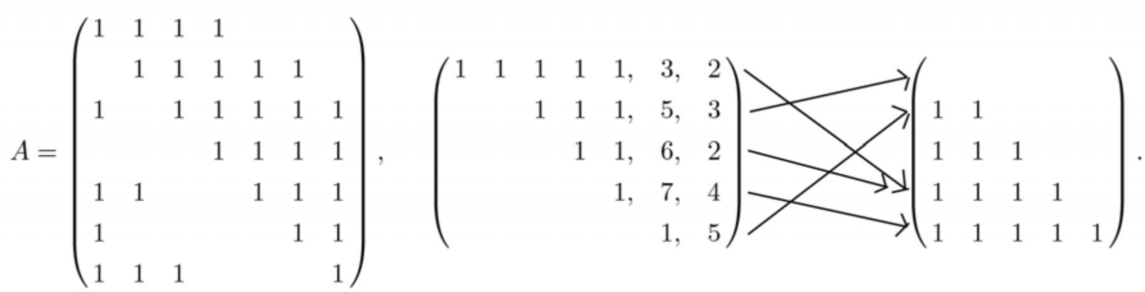

Figure 1: Example

given in the proof below, we have $R = \{1, 3, 5, 6, 7\}$, $R^c = \{2, 4\}$, $r_0 = 3$. The set X should be thought of essentially as being the rows $A(R, r_0 : 7)$ that arise by replacing ones that come from left blocks of ones in A by zeroes, *except* the first row of that matrix, and instead the row 0_{7-r_0+1} added. The rows of the middle matrix give the set $\{(x, r(x), j(r(x)) : x \in X\}$, and the rows of the right matrix the set Y of the present case. An idea is to add rows from $A(R^c,:)$ to the rows of $A(R,:)$ in such a manner that the positions in $R \times [r_0 - 1]$ become filled with ones; the overflow of ones that can arise (or degenerate to be 'empty') to the positions in $R \times (r_0 : n)$ define a map $f : X \to Y$ detailed in the proof. The arrows are intended to indicate that map.

PROOF OF CLAIMED PROPERTY OF MATRIX $A(k_1, \ldots, k_n)$. Let $R = \{r : a_{r1} = 1\}$. Clearly $1 \in R$. If A has some row equal to 1_n, then we are done. So we assume from now on that each row of A has a 0 and a 1. Then $1 \leq |R| < n \geq 2$.

For each $r \in R$ we have a $j(r) \geq 2$ so that

$$A(1,:) = (\underbrace{1, 1, \ldots 1}_{j(1)-1}, 0, 0, \ldots, 0), \text{ and } A(r,:) = (\underbrace{1, 1, \ldots 1}_{j(r)-1}, 0, 0, \ldots, 0, \underset{r}{1}, 1, \ldots, 1), \text{ if } r \neq 1. \tag{25}$$

Given any $j \in \{1, 2, \ldots, n\}$ as an input consider the following algorithm:

$s = (1_{j-1}, 0_{n-j+1})$; $I := R^c$; $R' = \emptyset$;

while there is an $r' \in I$ so that the leftmost 1 of $A(r',:)$ is at the position of the leftmost 0 of s **do**

$s = s + A(r',:)$; $I = I \setminus \{r'\}$; $R' = R' \cup \{r'\}$;

end

The algorithm returns a certain n-tuple $s = s^j$ and set $R' = R'^j$.

Claim: Let $j \geq 2$. Then:

(i). $R'^j \subseteq R^c$ and $s^j = (1_{j-1}, 0_{n-j+1}) + \sum\{A(r',:) : r' \in R'^j\}$.

(ii). Either $s^j = 1_n$ or there exists an $r \in R \setminus \{1\}$ so that $s^j = (1_{r-1}, 0_{n-r+1}) \in \{0,1\}^n$.

(iii). If $s^j \neq s^{j'}$ then $R'^j \cap R'^{j'} = \emptyset$.

(i). is an immediate consequence of the algorithm's code. (ii). The leftmost 0 of an s^j exists iff $s^j \neq 1_n$. In this case the definition of the algorithm prohibits for its position an integer in R^c. Clearly $r \geq j \geq 2$. So (ii) follows. (iii). If R'^j and $R'^{j'}$ would have an element, say r', in common, then $A(r', :)$ was added to some intermediate s in the production of s^j as well as to some such s in the production of $s^{j'}$. But this is seen to imply that the two referred s are equal, and from there on all the following corresponding s will be equal; in particular $s^j = s^{j'}$, contradicting the hypothesis of (iii).

In the case $R = \{1\}$, claim (ii) implies $s^j = 1_n$ for every $j \geq 2$. Hence $s^{j(1)} = 1_n$ and from (i) and (25) we get that 1_n is $A(1, :)+$a sum of rows of $A(R^c, :)$ and are done. So we assume from now on $|R| \geq 2$, put $r_0 := \min(R \setminus \{1\})$, and define

$$X = \{(0_{r-r_0}, 1_{n-r+1}) : r \in R \setminus \{1\} \cup \{n+1\}\},\ Y = \{(1_{r-r_0}, 0_{n-r+1}) : r \in R \setminus \{1\} \cup \{n+1\}\}.$$

Clearly $X, Y \subseteq \{0,1\}^{n-r_0+1}$ and $Y = \mathrm{op}(X)$. For an $x \in X \setminus \{0_{n-r_0+1}\}$ consider the $r = r(x) \in R \setminus \{1\}$ that defines it, and put $r(0_{n-r_0+1}) := 1$. By claim (ii), $x \mapsto f(x) := s^{j(r(x))}(r_0 : n)$ yields a map $f : X \to Y$. Therefore, by the lemma, there exist $X' \subseteq X,\ Y' \subseteq Y$ so that $f|X' : X' \to Y'$ is a bijection and $\sum_{x \in X'}(x + f(x)) = |X'|1_{n-r_0+1}$. Then for every $x \in X$ we have

$$\begin{aligned} A(r(x), :) + \sum\{A(\nu, :) : \nu \in R'^{j(r(x))}\} &= (0_{r_0-1}, x) + (1_{j(r(x))-1}, 0_{n-j(r(x))+1}) + \sum\{A(\nu, :) : \nu \in R'^{j(r(x))}\} \\ &= (0_{r_0-1}, x) + s^{j(r(x))} \\ &= (s^{j(r(x))}(1 : r_0 - 1), x + s^{j(r(x))}(r_0 : n)) \\ &= (1_{r_0-1}, x + f(x)). \end{aligned}$$

Since the $f(x),\ x \in X'$, are all distinct, so are the $n-$tuples $s^{j(r(x))}$, so that by claim (iii), the $R'^{j(r(x))}$ are all disjoint. Thus summing above expressions over all $x \in X'$, the left hand side yields a sum of rows of A, while the right hand side yields $|X'|1_n$. Since $|X'| \leq |X| = |R| < n$, the claim concerning $A(k_1, \ldots, k_n)$ is proved. □

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

We look forward to receiving solutions to Problem 30-3!

Problem 30-4: The Similarity of Two Block Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two idempotent matrices of the same size m and let $M := A + B$. Show that

$$\begin{pmatrix} M & A \\ 0 & -M \end{pmatrix} \quad \text{is similar to} \quad \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}.$$

Solution 30-4.1 by Robert REAMS, *College of William and Mary, Williamsburg, Virginia, USA:* reams@math.wm.edu

The given block matrices are similar, since

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} M & A \\ 0 & -M \end{pmatrix},$$

where $X = \frac{1}{4}(I + A - B)$.

Solution 30-4.2 by the Proposer Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

It is easy to verify that

$$(A+B)(A-B) + (A-B)(A+B) = 2A - 2B = 4A - 2(A+B).$$

Hence $MX + XM = A$, where $X = \frac{1}{4}(I_m + A - B)$. Thus

$$\begin{aligned} \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \begin{pmatrix} M & A \\ 0 & -M \end{pmatrix} \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix}^{-1} &= \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \begin{pmatrix} M & A \\ 0 & -M \end{pmatrix} \begin{pmatrix} I_m & -X \\ 0 & I_m \end{pmatrix} \\ &= \begin{pmatrix} M & A - MX - XM \\ 0 & -M \end{pmatrix} \\ &= \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}. \end{aligned}$$

The proof is complete.

Solution 30-4.3 by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Define $Z := \begin{pmatrix} I & \frac{1}{4}(A - B + I) \\ 0 & -I \end{pmatrix}$. Then Z is nonsingular with the inverse $Z^{-1} = Z$. Some straightforward calculations show that

$$Z^{-1} \begin{pmatrix} M & A \\ 0 & -M \end{pmatrix} Z = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix},$$

which proves the asserted similarity.

Solutions to Problem 30-4 were also received from Nir Cohen and from Alicja Smoktunowicz.

Problem 30-5: A Range Equality for the Difference of Orthogonal Projectors
Proposed by Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

Let A and B be two orthogonal projectors of the same size. Show that $\mathrm{range}\,[\,(\,A - B\,)^\dagger - (\,A - B\,)\,] = \mathrm{range}\,(\,AB - BA\,)$, where $(\,A - B\,)^\dagger$ is the Moore–Penrose inverse of $A - B$. Hence show that $(\,A - B\,)^\dagger = A - B$ if and only if $AB = BA$.

Solution 30-5.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
and Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl

We establish a more general result, in which the assumption that A and B are orthogonal projectors is relaxed by referring to the concept of an EP matrix. Let us recall that the set of EP matrices (range-Hermitian matrices) of order n is specified as

$$\mathbb{C}_n^{\mathsf{EP}} = \{K \in \mathbb{C}_{n,n}\colon\, \mathcal{R}(K) = \mathcal{R}(K^*)\} = \{K \in \mathbb{C}_{n,n}\colon\, KK^\dagger = K^\dagger K\}, \tag{26}$$

where K^*, $K^\dagger$, and $\mathcal{R}(K)$ denote the conjugate transpose, Moore-Penrose inverse, and range of K, respectively.

THEOREM. *Any idempotent matrices $A, B \in \mathbb{C}_{n,n}$ such that $A - B \in \mathbb{C}_n^{\mathsf{EP}}$ and $AB - BA \in \mathbb{C}_n^{\mathsf{EP}}$ satisfy*

$$\mathcal{R}[(A-B)^\dagger - (A-B)] = \mathcal{R}(AB - BA). \tag{27}$$

PROOF. It can easily be verified that if $A-B \in \mathbb{C}_n^{\mathsf{EP}}$, then

$$\mathcal{R}[(A-B)^\dagger - (A-B)] = \mathcal{R}[(A-B)^\dagger (A-B)^2 - (A-B)^3] = \mathcal{R}[(A-B)-(A-B)^3].$$

Consequently, since the assumption of the idempotency of A and B entails

$$(A-B)^3 = (A-B) - (ABA - BAB), \tag{28}$$

it follows that (27) is equivalent to

$$\mathcal{R}(ABA - BAB) = \mathcal{R}(AB - BA). \tag{29}$$

Hence, by referring to the orthogonal complements of the subspaces involved in (29), the proof reduces to showing that, for any $x \in \mathbb{C}_{n,1}$,

$$x^*AB = x^*BA \;\Leftrightarrow\; x^*ABA = x^*BAB.$$

But this is indeed the case. If $x^*AB = x^*BA$, then

$$x^*ABA = x^*BA^2 = x^*BA = x^*AB = x^*AB^2 = x^*BAB.$$

Conversely, since in view of (26) the assumption $AB - BA \in \mathbb{C}_n^{\mathsf{EP}}$ can be expressed in the form $(AB-BA)^* = (AB-BA)L$ for some $L \in \mathbb{C}_{n,n}$, it follows that

$$ABA = BAB \;\Rightarrow\; (AB-BA)(AB-BA)^* = (AB-BA)^2 L = (BAB^2 - ABA - BAB + ABA^2)L = 0. \tag{30}$$

Hence it is seen that if $x^*ABA = x^*BAB$, then $x^*(AB-BA)(AB-BA)^* = 0$, which is obviously equivalent to $x^*AB = x^*BA$. □

It is clear that if A and B are orthogonal projectors, then $A-B = (A-B)^*$ and $AB-BA = -(AB-BA)^*$, and thus the conditions $A-B \in \mathbb{C}_n^{\mathsf{EP}}$ and $AB-BA \in \mathbb{C}_n^{\mathsf{EP}}$ are fulfilled trivially. In addition to this observation it should be pointed out that a generalization of the claim in Problem 30-5 given in the theorem above is substantial. For example, if A and B are projectors of the form

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

then the matrices

$$A - B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad AB - BA = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

are both EP, and therefore A and B satisfy equality (27) although neither of them is an orthogonal projector.

From Thorem it immediately follows that, under the assumptions involved therein, the equality $(A-B)^\dagger = A-B$ holds if and only if the projectors A and B commute. We extend this statement by referring to the set $(A-B)\{1\}$ of all generalized inverses of $A-B$, i.e., matrices $G \in \mathbb{C}_{n,n}$ satisfying $(A-B)G(A-B) = A-B$.

COROLLARY. *For any idempotent matrices $A, B \in \mathbb{C}_{n,n}$ such that $A-B \in \mathbb{C}_n^{\mathsf{EP}}$ and $AB-BA \in \mathbb{C}_n^{\mathsf{EP}}$, the following statements are equivalent:*

(a) $A-B = (A-B)^\dagger$, (b) $A-B \in (A-B)\{1\}$, (c) $AB = BA$.

PROOF. The part (a) ⇒ (b) is an obvious consequence of the definitions of $(A-B)^\dagger$ and $(A-B)\{1\}$. Further, on account of specification of $(A-B)\{1\}$ and (28), condition (b) is equivalent to $ABA = BAB$, and then from (30) it is seen that $AB - BA = 0$, which is (c). Finally, as already mentioned, the part (c) ⇒ (a) follows straightforwardly from Theorem. □

We conclude by pointing out that the result on commutativity of projectors given in Corollary is an interesting supplement to several other characteristics of such a type derived by Baksalary and Baksalary (2002, Section 4).

Reference

J. K. Baksalary & O. M. Baksalary (2002). Commutativity of projectors. *Linear Algebra and Its Applications*, **341**, 129–142.

Solution 30-5.2 by the Proposer Yongge TIAN, *Queen's University, Kingston, Ontario, Canada:* ytian@mast.queensu.ca

We first show that if M is Hermitian, then

$$\operatorname{range}(M - M^\dagger) = \operatorname{range}(M - M^3). \tag{31}$$

Recall that if M is Hermitian, then $MM^\dagger = M^\dagger M$. Hence

$$(M - M^\dagger)M^2 = M^3 - M^\dagger M^2 = M^3 - M \ \text{ and } \ (M^3 - M)M^\dagger M^\dagger = M - M^\dagger.$$

These two equalities imply (31). If A and B are two orthogonal projectors of the same size, then the matrix $A - B$ is Hermitian and $A - B - (A - B)^3 = ABA - BAB$. Thus by (31)

$$\text{range}[(A - B)^\dagger - (A - B)] = \text{range}(ABA - BAB). \tag{32}$$

For any two idempotent matrices A and B of order m, it is easy to verify the following two identities:

$$AB - BA = (A - B)(A + B - I_m),$$

$$ABA - BAB = (AB - BA)(A + B - I_m) = (A - B)(A + B - I_m)^2.$$

If A and B are two orthogonal projectors of the same size, then the matrix $A + B - I_m$ is Hermitian. Thus

$$\text{range}[(A - B)(A + B - I_m)^2] = \text{range}[(A - B)(A + B - I_m)] = \text{range}(AB - BA).$$

Hence

$$\text{range}(ABA - BAB) = \text{range}(AB - BA). \tag{33}$$

Combining (32) and (33) yields the desired result.

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We note that a matrix $A \in \mathbb{C}^{n\times n}$ is an orthogonal projector if and only if $A^2 = A = A^*$, where A^* denotes the conjugate transpose of A. We further recall that for any $B \in \mathbb{C}^{m\times n}$ we have $\mathcal{R}(B^\dagger) = \mathcal{R}(B^*)$ and $\mathcal{N}(B^\dagger) = \mathcal{N}(B^*)$, where $B^\dagger$ indicates the Moore-Penrose inverse of B and $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ stand for the range (column space) and the null space, respectively, of $(\cdot)$. If B is Hermitian, i.e., if $B = B^*$, then trivially $\mathcal{R}(B) = \mathcal{R}(B^*)$, i.e., B is an EP-matrix. Finally, we mention that for an EP-matrix C we have $C^\dagger C = CC^\dagger$, whence we get $C^2C^\dagger = C = C^\dagger C^2$ and $\left(C^\dagger\right)^2 C = C^\dagger$. With these observations in mind, it is not difficult to prove the following theorem.

THEOREM. *Let A and B be orthogonal projectors such that AB is defined. Then the following conditions are equivalent:*

(a) $x \in \mathcal{N}((A - B)^\dagger - (A - B))$,

(b) $x \in \mathcal{N}((A - B)^3 - (A - B))$,

(c) $x \in \mathcal{N}(BAB - ABA)$,

(d) $x \in \mathcal{N}(AB - BA)$.

PROOF. (a) ⇔ (b): First, let $x \in \mathcal{N}((A - B)^\dagger - (A - B))$, i.e., let $(A - B)x = (A - B)^\dagger x$. Premultiplying by $(A - B)^2$ yields $(A-B)^3x = (A-B)^2(A-B)^\dagger x = (A-B)x$ or, equivalently, $x \in \mathcal{N}((A-B)^3-(A-B))$. Conversely, let $(A-B)x = (A-B)^3x$. Premultiplying by $((A - B)^\dagger)^2$ results in $(A - B)^\dagger x = (A - B)x$.

(b) ⇔ (c): This is a direct consequence of $(A - B)^3 - (A - B) = BAB - ABA$.

(c) ⇔ (d): First, let $x \in \mathcal{N}(BAB - ABA)$, i.e., let $BABx = ABAx$. Premultiplying by B yields $BABx = BABAx$ or, equivalently, $BAB(I-A)x = 0$. Since BAB is nonnegative definite and Hermitian, $BAB(I-A)x = 0$ implies that $AB(I-A)x = 0$ or, equivalently, $ABx = ABAx$. Analogously, premultiplying $BABx = ABAx$ by A, we obtain $BAx = BABx$. Since $ABAx = BABx$, we now obtain $ABx = BAx$. To prove the converse, let $ABx = BAx$. Premultiplying this equality by A and B, respectively, results in $ABx = ABAx$ and $BABx = BAx$. Hence $ABAx = BABx$, and so the proof is complete. □

Since the range of a matrix coincides with the orthogonal complement of the null space of its conjugate transpose (where the orthogonal complement is with respect to the usual standard inner product), the claim in Problem 30-5 follows directly from our theorem above. We conclude with emphasizing that (a) ⇔ (b) holds for EP-matrices, while (b) ⇔ (c) holds for idempotent matrices. Only our proof of (c) ⇔ (d) is based on the assumption that A and B are orthogonal projectors.

Problem 30-6: A Matrix Related to an Idempotent Matrix

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Let P be an idempotent matrix from $\mathbb{C}^{n\times n}$. What can be said about the matrix $R = P(P+P^*)^-P^*$, where $(P+P^*)^-$ is a generalized inverse of $P+P^*$ and P^* denotes the conjugate transpose of P?

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Our contribution to answering the question posed in Problem 30-6 is concerned with the concept of parallel summability. Let us recall, following Rao and Mitra (1971, p. 188), that for matrices $A, B \in \mathbb{C}^{m\times n}$ the term "parallel sum" is attributed to the expression $A(A+B)^-B$ whenever it is independent of the choice of a generalized inverse $(A+B)^-$, i.e., any matrix satisfying $(A+B)(A+B)^-(A+B) = A+B$.

PROPOSITION. *For any idempotent $P \in \mathbb{C}^{n\times n}$, the matrix $P(P+P^*)^-P^*$ is the parallel sum of P and P^*.*

PROOF. It is known that the product $A(A+B)^-B$ is invariant with respect to the choice of $(A+B)^-$ if and only if $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^*+B^*)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$, where $\mathcal{R}(.)$ denotes the range of a given matrix; cf., e.g., Rao and Mitra (1971, pp. 21 and 43). Consequently, a necessary and sufficient condition for parallel summability of P and P^* is

$$\mathcal{R}(P^*) \subseteq \mathcal{R}(P+P^*). \qquad (34)$$

By referring to the orthogonal complements of the subspaces involved in (34), the proof consists in showing that, for any $x \in \mathbb{C}^{n\times 1}$,

$$x^*(P+P^*) = 0 \;\Rightarrow\; x^*P^* = 0.$$

But this follows by noting that

$$x^*(P+P^*) = 0 \;\Leftrightarrow\; x^*(P+P^*)^2 = 0 \;\Leftrightarrow\; x^*(P+PP^*+P^*P+P^*) = 0,$$

and hence

$$x^*(P+P^*) = 0 \;\Rightarrow\; x^*(PP^*+P^*P) = 0 \;\Leftrightarrow\; x^*(P:P^*)(P:P^*)^* = 0 \;\Leftrightarrow\; x^*(P:P^*) = 0 \;\Rightarrow\; x^*P^* = 0,$$

as desired. □

Reference

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications*. Wiley, New York.

Solution 30-6.2 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

We show that $2R$ is the orthogonal projector on $\mathcal{R}(P) \cap \mathcal{R}(P^*)$, where $\mathcal{R}(\cdot)$ denotes the column space of a matrix. This follows trivially if P is nonsingular, since then it must be the identity matrix of order n. To see this in general, we write P in the form

$$P = U \begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix} U^*,$$

where U is an $n\times n$ unitary matrix, I_r is the identity matrix of order $r = \operatorname{rank}(P)$, and K is an $r\times(n-r)$ matrix (see Hartwig and Loewy, 1992). This implies

$$P+P^* = U \begin{pmatrix} 2I_r & K \\ K^* & 0 \end{pmatrix} U^*.$$

Using Theorem 3.5.2 from Campbell and Meyer (1979), we get the Moore-Penrose inverse of $P+P^*$ as

$$(P+P^*)^+ = U \begin{pmatrix} \frac{1}{2}(I_r - KK^+) & K^{+*} \\ K^+ & -2(K^*K)^+ \end{pmatrix} U^*$$

and consequently

$$(P+P^*)^+(P+P^*) = (P+P^*)(P+P^*)^+ = U \begin{pmatrix} I_r & 0 \\ 0 & K^+K \end{pmatrix} U^*.$$

It is readily established that $(P+P^*)(P+P^*)^+P = P$ and $(P+P^*)(P+P^*)^+P^* = P^*$, which implies the somewhat surprising result $\mathcal{R}(P) \subset \mathcal{R}(P+P^*)$ and $\mathcal{R}(P^*) \subset \mathcal{R}(P+P^*)$. Hence $P(P+P^*)^-P^*$ is invariant under the choice of the g-inverse $(P+P^*)^-$, that is, we get the parallel sum of P and P^* as $P \mp P^* = P(P+P^*)^-P^* = P(P+P^*)^+P^*$. According to Rao and Mitra (1971, Theorem 10.1.8e), we have $\mathcal{R}(P \mp P^*) = \mathcal{R}(\mathcal{P}) \cap \mathcal{R}(P^*)$. However, further calculations give

$$P(P+P^*)^+P^* = U \begin{pmatrix} \frac{1}{2}(I_r - KK^+) & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

such that $2P \mp P^*$ is the orthogonal projector on $\mathcal{R}(P) \cap \mathcal{R}(P^*)$.

References

S. L. Campbell & C. D. Meyer (1979). *Generalized Inverses of Linear Transformations.* Pitman, London.

R. E. Hartwig & R. Loewy (1992). Maximal elements under the three partial orders. *Linear Algebra and Its Applications*, **175**, 39–61.

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

Solution 30-6.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

For a complex $m \times n$ matrix A, let A^*, A^-, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, a generalized inverse, the range (column space), and the null space, respectively, of A. If $\mathcal{M}$ is a linear subspace of $\mathbb{C}^n$, then we denote by $P_{\mathcal{M}}$ the orthogonal projector onto $\mathcal{M}$ along $\mathcal{M}^\perp$, with the orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ being defined with respect to the usual standard inner product in $\mathbb{C}^n$. We note that the projector $P_{\mathcal{M}}$ may be defined by $P_{\mathcal{M}}x = x$ if $x \in \mathcal{M}$ and $P_{\mathcal{M}}x = 0$ if $x \in \mathcal{M}^\perp$. If $\mathcal{M}$ and $\mathcal{N}$ are two linear subspaces in $\mathbb{C}^n$, then we recall that $[\mathcal{M} \cap \mathcal{N}]^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$. We also mention that $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$.

THEOREM. *Let P be a (complex) idempotent matrix, i.e., let $P^2 = P$. Then:*

(a) $\mathcal{N}(P+P^*) = \mathcal{N}(P) \cap \mathcal{N}(P^*)$ *and* $\mathcal{R}(P+P)^* = \mathcal{R}(P) + \mathcal{R}(P^*)$.

(b) $P(P+P^*)^-P^* = P(P+P^*)^\dagger P^*$, *irrespective of the choice of* $(\cdot)^-$, *with* $(\cdot)^\dagger$ *indicating the Moore-Penrose inverse of* $(\cdot)$. *Moreover,* $P(P+P^*)^-P^*$ *is Hermitian and* $P(P+P^*)^-P^* = P^*(P+P^*)^-P$.

(c) $2P(P+P^*)^-P^* = P_{\mathcal{R}(P)\cap\mathcal{R}(P^*)}$.

PROOF. (a): From Theorem in Werner (2003) we know that $\mathcal{N}(P+P^*) = \mathcal{N}(P) \cap \mathcal{N}(P^*)$. Taking orthogonal complements $(\cdot)^\perp$ in this set equation results in $\mathcal{R}(P+P^*) = \mathcal{R}(P) + \mathcal{R}(P^*)$.

(b): In view of (a), clearly $\mathcal{R}(P) \subseteq \mathcal{R}(P+P^*)$ and $\mathcal{R}(P^*) \subseteq \mathcal{R}(P+P^*)$. From the theory of generalized inversion, see, e. g., Rao & Mitra (1971), we then know that $P(P+P^*)^-P^*$ is independent of the choice of the g-inverse $(P+P^*)^-$. Therefore, $P(P+P^*)^-P^* = P(P+P^*)^\dagger P^*$. Since $(P+P^*)(P+P^*)^-$ is a projector onto $\mathcal{R}(P) + \mathcal{R}(P^*)$ whereas $(P+P^*)^-(P+P^*)$ is a projector along $\mathcal{N}(P) \cap \mathcal{N}(P^*)$, it is now not difficult to see that

$$P(P+P^*)^-P^* = P^* - P^*(P+P^*)^-P^* = P - P(P+P^*)^-P = P^*(P+P^*)^-P, \tag{35}$$

where all expressions are again independent of the choice of the g-inverse $(P+P^*)^-$. Hence, in particular, $P(P+P^*)^-P^* = P^*(P+P^*)^-P$. Since $(P+P^*)^\dagger$ is Hermitian, it is further clear that $P(P+P^*)^-P^*$ is also Hermitian.

(c): For convenience, put $A := P(P+P^*)^-P^*$. By means of (b) and (35) check that $A^2 = A - A^2$. Hence $A^2 = \frac{1}{2}A$ or, equivalently, $(2A)^2 = 2A$. Since $2A$ is also Hermitian, $2A$ is an orthogonal projector, and so it remains to show that $\mathcal{R}(2A) = \mathcal{R}(P) \cap \mathcal{R}(P^*)$. In view of $P(P+P^*)^-P^* = P^*(P+P^*)^-P$, clearly $\mathcal{R}(2A) \subseteq \mathcal{R}(P) \cap \mathcal{R}(P^*)$. That the converse inclusion, namely $\mathcal{R}(P) \cap \mathcal{R}(P^*) \subseteq \mathcal{R}(2A)$ is also true is seen as follows. Let $x \in \mathcal{R}(P) \cap \mathcal{R}(P^*)$. Then $x = Px = P^*x$ and so $2Ax = A(P+P^*)x = P(P+P^*)^-(P+P^*)x = Px = x$. This completes our proof. □

We conclude with mentioning that in the literature the matrix $P(P+P^*)^-P^*$ is called the parallel sum of P and P^* and is often denoted by $P \mp P^*$; cf. Rao & Mitra (1971, pp. 188-192). That $\mathcal{R}(P \mp P^*) = \mathcal{R}(P) \cap \mathcal{R}(P^*)$ is shown in Theorem 10.1.8(e) in Rao & Mitra for more general classes of matrices.

References

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

H. J. Werner (2003). A range equality involving an idempotent matrix. Solution 29-8.3 *IMAGE: The Bulletin of the International Linear Algebra Society*, no. **30** (April 2003), 28.

Problem 30-7: A Condition for an Idempotent Matrix to be Hermitian

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Let P be an idempotent matrix from $\mathbb{C}^{n\times n}$. Show that P is Hermitian if and only if the Moore–Penrose inverse of $P(I-P^*)$ is idempotent, where P^* denotes the conjugate transpose of P.

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Let $A = P - PP^*$. Since $P = P^2$, it follows that $P = P^* \Leftrightarrow A = 0$. Moreover, for $A^\dagger$ denoting the Moore-Penrose inverse of A,

$$A^\dagger = (A^\dagger)^2 \;\Leftrightarrow\; A(A^\dagger AA^*) = AA^\dagger(A^\dagger AA^*) \;\Leftrightarrow\; AA^* = AA^\dagger A^*.$$

Consequently, the statement in Problem 30-7 may be reformulated as

$$AA^* = AA^\dagger A^* \;\Leftrightarrow\; A = 0. \tag{36}$$

The ”$\Leftarrow$ part” is trivial, as well as the converse implication when P is nonsingular (in which case P must be equal to the identity matrix I_n). For establishing the ”$\Rightarrow$ part” in general, it is therefore assumed that P is a singular idempotent matrix (of rank p, say, $p < n$) having a representation of the form

$$P = U \begin{pmatrix} I_p & K \\ 0 & 0 \end{pmatrix} U^*, \tag{37}$$

where $U^*U = I_n$ and K is any matrix of order $p \times (n-p)$; see Hartwig and Loewy (1992) and comments in Trenkler (1994, p. 260). From (37) it follows that

$$A = U \begin{pmatrix} I_p & K \\ 0 & 0 \end{pmatrix} U^* - U \begin{pmatrix} I_p + KK^* & 0 \\ 0 & 0 \end{pmatrix} U^* = U \begin{pmatrix} -KK^* & K \\ 0 & 0 \end{pmatrix} U^*. \tag{38}$$

Hence it is seen that the range of A satisfies

$$\mathcal{R}(A) = \mathcal{R}(U \begin{pmatrix} K \\ 0 \end{pmatrix} (-K^* : I_{n-p})U^*) = \mathcal{R}(U \begin{pmatrix} K \\ 0 \end{pmatrix}),$$

and therefore the orthogonal projector $AA^\dagger$ onto $\mathcal{R}(A)$ is of the form

$$AA^\dagger = U \begin{pmatrix} KK^\dagger & 0 \\ 0 & 0 \end{pmatrix} U^*. \tag{39}$$

On account of (38) and (39),

$$AA^* = AA^\dagger A^* \;\Leftrightarrow\; U \begin{pmatrix} (KK^*)^2 + KK^* & 0 \\ 0 & 0 \end{pmatrix} U^* = U \begin{pmatrix} -KK^* & 0 \\ 0 & 0 \end{pmatrix} U^* \;\Leftrightarrow\; (KK^*)^2 + 2KK^* = 0. \tag{40}$$

Since KK^* is obviously a nonnegative definite matrix, it is clear from (40) that $AA^* = AA^\dagger A^* \Leftrightarrow KK^* = 0 \Leftrightarrow K = 0$, which in view of (38) means that $A = 0$, as required in (36).

References

R. E. Hartwig & R. Loewy (1992). Maximal elements under the three partial orders. *Linear Algebra and Its Applications*, **175**, 39–61.

G. Trenkler (1994). Characterizations of oblique and orthogonal projectors. In *Proceedings of the International Conference on Linear Statistical Inference LINSTAT'93* (T. Caliński & R. Kala, eds.), Kluwer, Dordrecht, pp. 255–270.

Solution 30-7.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Let $Q = [P(I-P^*)]^\dagger$. If $P = P^*$, then $Q = 0 = Q^\dagger$, so $Q^\dagger$ is idempotent. For the converse note that the conditions defining the Moore-Penrose inverse imply

$$P(I-P^*)Q = Q^*(I-P)P^*, \tag{41}$$

$$QP(I-P^*) = (I-P)P^*Q^*, \tag{42}$$

$$QP(I - P^*)Q = Q, \tag{43}$$

$$P(I - P^*)QP(I - P^*) = P(I - P^*). \tag{44}$$

From (42), $PQP(I - P^*) = 0$, so (43) implies that $PQ = 0 = Q^*P^*$. Hence, (44) simplifies to

$$-PP^*QP(I - P^*) = P(I - P^*) \tag{45}$$

and (41) simplifies to $PP^*Q = (PP^*Q)^*$. Therefore, if $Q^2 = Q$, then $PP^*Q = [(PP^*Q)Q]^* = Q^*PP^*Q$ is positive semidefinite, so (45) implies that $P(I - P^*) = 0$. Hence $P = PP^*$ is Hermitian.

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Necessity is trivial. Conversely, let the Moore-Penrose inverse of $P(I - P^*)$ be idempotent. Since P is idempotent, it can be written as

$$P = U \begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix} U^*,$$

where I_r is the identity matrix of order $r =$ rank P, K is an $r \times (n - r)$ matrix and U is a unitary matrix (see Hartwig and Loewy, 1992). By some straightforward computations one finds that

$$[P(I - P^*)]^+ \ = \ U \begin{pmatrix} -G(KK^*)^+ & 0 \\ K^+G(KK^*)^+ & 0 \end{pmatrix} U^*,$$

where $G = [I_r + (KK^*)^+]^{-1}$. Since G and $(KK^*)^+$ commute, the idempotency of $[P(I - P^*)]^+$ entails $K = 0$, so that P is Hermitian.

Reference

R.E. Hartwig & R. Loewy (1992). Maximal elements under the three partial orders. *Linear Algebra and its Applications*, **175**, 39–61.

Solution 30-7.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* werner@united.econ.uni-bonn.de

Our solution offers additional insights into the theory of projectors. We begin with characterizing $(A^\dagger)^2 = A^\dagger$ in terms of the matrix A and its conjugate transpose.

THEOREM 1. *Let A be a square complex matrix. Then the Moore-Penrose inverse $A^\dagger$ of A is idempotent, i.e., $(A^\dagger)^2 = A^\dagger$, if and only if $A^2 = AA^*A$.*

PROOF. We note that the Moore-Penrose inverse of A satisfies the following well-known properties: (a) $A^\dagger AA^\dagger = A^\dagger$, (b) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, and (c) $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, with $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denoting the range (column space) and the null space, respectively, of $(\cdot)$; cf. Theorem 2 in Werner (2003b). By means of (a), (b), and (c) it is now easy to see that $(A^\dagger)^2 = A^\dagger \Leftrightarrow A^\dagger(I - A)A^\dagger = 0 \Leftrightarrow A^*(I - A)A^* = 0 \Leftrightarrow A(I - A^*)A = 0 \Leftrightarrow A^2 = AA^*A$. □

This powerful characterization has a series of direct implications. Here we only mention the following.

COROLLARY 2. *Let A be a square complex matrix. Then we have:*

(a) *If A is an EP-matrix, i.e., if $\mathcal{R}(A) = \mathcal{R}(A^*)$, then $A^\dagger$ is idempotent if and only if A is idempotent and Hermitian, in which case $A^2 = A = A^* = A^\dagger$.*

(b) *If A is idempotent, then $A^\dagger$ is idempotent if and only if A is a partial isometry, i.e., if and only if $A = AA^*A$, in which case $A^2 = A = A^* = A^\dagger$.*

(c) *$A^\dagger$ is idempotent only if $\operatorname{index}(A) \leq 1$. Moreover, if $A^\dagger$ is idempotent and $A^2 = 0$, then necessarily $A = 0$.*

PROOF. (a): Let A be an EP-matrix. Since $A^\dagger A$ is the orthogonal projector onto $\mathcal{R}(A^*)$, clearly $A^\dagger AA^*A = A^*A$. Since $A^\dagger A = AA^\dagger$ is equivalent to A being an EP-matrix, also $A^\dagger A^2 = AA^\dagger A = A$. According to Theorem 1, therefore, $(A^\dagger)^2 = A^\dagger \Leftrightarrow A^2 = AA^*A \Leftrightarrow A^\dagger A^2 = A^\dagger AA^*A \Leftrightarrow A = A^*A \Leftrightarrow A = A^* = A^2$.

(b): Let A be idempotent, i. e., $A^2 = A$. Then, in view of Theorem 1, clearly $(A^\dagger)^2 = A^\dagger \Leftrightarrow A = AA^*A$. From Theorem in Werner (2003a) it follows that this is equivalent to $A = A^*$.

(c): According to Theorem 1, $(A^\dagger)^2 = A^\dagger$ if and only if $A^2 = AA^*A$. Since $\mathcal{R}(AA^*A) = \mathcal{R}(A)$, necessarily $\mathcal{R}(A^2) = \mathcal{R}(A)$ or, equivalently, $\mathrm{index}(A) \leq 1$. Hence, whenever $(A^\dagger)^2 = A^\dagger$ and $A^2 = 0$, then necessarily $A = 0$. □

Part (c) of Corollary 2 enables us now to give a very brief solution to the stated problem.

THEOREM 3. *Let P be an idempotent matrix, and let $Q := P(I - P^*)$. Then $(Q^\dagger)^2 = Q^\dagger$ if and only if $Q^2 = 0$ or, equivalently, if and only if $P = P^*$.*

PROOF. According to Theorem 1, $(Q^\dagger)^2 = Q^\dagger \Leftrightarrow Q^2 = QQ^*Q \Leftrightarrow Q(I - Q^*)Q = 0$. Check that $Q(I - Q^*)Q = (I + PP^*)Q^2$. Since $I + PP^*$ is a positive definite Hermitian matrix, it is nonsingular, and so we get $Q(I - Q^*)Q = 0 \Leftrightarrow Q^2 = 0$, which, in virtue of Corollary 2(c), can happen only if $Q = 0$. But then $P = PP^*$ or, equivalently, $P = P^*$. □

We conclude with mentioning that a completely different proof for the characterization given in part (b) of Corollary 2 can be found in Werner (2003c).

References

H. J. Werner (2003a). Partial isometry and idempotent matrices. Solution 28-7.5. *IMAGE: The Bulletin of the International Linear Algebra Society*, no. **30** (April 2003), 31–32.

H. J. Werner (2003b). The minimal rank of a block partitioned matrix with generalized inverses. Solution 29-11.2. *IMAGE: The Bulletin of the International Linear Algebra Society*, no. **31** (October 2003), 26–29.

H. J. Werner (2003c). 02.6.1 Oblique Projectors – Solution. *Econometric Theory*, **19**, 1196–1197.

IMAGE Problem Corner: More New Problems

Problem 31-6: A Full Rank Factorization of a Skew-Symmetric Matrix

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Determine a full rank factorization of the matrix

$$C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

with real entries $c_i, i = 1, 2, 3$. (Observe that for $x = (x_1, x_2, x_3)' \in \mathbb{R}^3$ the identity $Cx = c \times x$, where $c = (c_1, c_2, c_3)'$, defines the vector cross product in $\mathbb{R}^3$.)

Problem 31-7: On the Product of Orthogonal Projectors

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be orthogonal projectors of the same order with complex entries and let A denote their product. Show that the following conditions are equivalent:

(i) A is an orthogonal projector, i.e. $A = AA^*$,
(ii) A is Hermitian, i.e. $A = A^*$,
(iii) A is normal, i.e. $AA^* = A^*A$,
(iv) A is EP, i.e. $AA^+ = A^+A$,
(v) A is bi-EP, i.e. $AA^+A^+A = A^+AAA^+$,
(vi) A is bi-normal, i.e. $AA^*A^*A = A^*AAA^*$,
(vii) A is bi-dagger, i.e. $(A^+)^2 = (A^2)^+$.

Problem 31-8: Eigenvalues and Eigenvectors of a Particular Tridiagonal Matrix

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA:* zhang@nova.edu

Let A be the n-by-n tridiagonal matrix with 2 on diagonal and 1 on super- and sub-diagonals. That is, $a_{ii} = 2$, $a_{ij} = 1$ if $j = i + 1$ or $j = i - 1$, and $a_{ij} = 0$ otherwise, $i, j = 1, 2, \cdots, n$. Find all eigenvalues and corresponding eigenvectors of A.

Problems 31-1 through 31-5 are on page 44.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to werner@united.econ.uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 31-6 through 31-8 are on page 43.*

Problem 31-1: A Property of Linear Subspaces

Proposed by Jürgen GROß and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:*
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In Groß (1999, Corollary 2) the following is stated: If U and V are linear subspaces of $\mathbb{C}^m$, then

$$\mathbb{C}^m = [U \cap (U^\perp + V^\perp)] \oplus [V \oplus (U^\perp \cap V^\perp)],$$

where "$\oplus$" indicates the direct sum of two subspaces and "$\perp$" denotes the orthogonal complement. Is this decomposition also valid in a Hilbert space? The Proposers of the problem have no answer to this question.

Reference

J. Groß (1999). On oblique projection, rank additivity and the Moore-Penrose inverse of the sum of two matrices. *Linear and Multilinear Algebra*, **46**, 265–275.

Problem 31-2: Matrices Commuting with All Nilpotent Matrices

Proposed by Henry RICARDO, *Medgar Evers College (CUNY) Brooklyn, New York, New York, USA:* odedude@yahoo.com

If an $n \times n$ matrix A commutes with all $n \times n$ nilpotent matrices, must A be nilpotent? Determine the whole class of these matrices. (We recall that a square matrix N is said to be nilpotent whenever $N^k = 0$ for some positive integer k.)

Problem 31-3: A Range Equality for Block Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca

Let A and B be two nonnegative definite complex matrices of the same size. Show that

$$\text{range}\begin{pmatrix} A & B & & \\ & \ddots & \ddots & \\ & & A & B \end{pmatrix}_{n\times(n+1)} = \text{range}\begin{pmatrix} A+B & & \\ & \ddots & \\ & & A+B \end{pmatrix}_{n\times n},$$

where all blanks are zero matrices.

Problem 31-4: Two Equalities for Ideals Generated by Idempotents

Proposed by Yongge TIAN, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca

Let R be a ring with unity 1 and let $a,\ b \in R$ be two idempotents, i.e., $a^2 = a$ and $b^2 = b$. Show that

$$(\,ab - ba\,)R = (\,a - b\,)R \cap (\,a + b - 1\,)R \ \text{ and } \ R(\,ab - ba\,) = R(\,a - b\,) \cap R(\,a + b - 1\,).$$

Problem 31-5: A Norm Inequality for the Commutator $AA^* - A^*A$

Proposed by Yongge TIAN, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca
and Xiaoji LIU, *University of Science and Technology of Suzhou, Suzhou, China:* xiaojiliu72@yahoo.com.cn

Let A be a square matrix and let A^* and $A^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse of A, respectively. A well-known result asserts that $AA^* = A^*A$ if and only if $AA^\dagger = A^\dagger A$ and $A^*A^\dagger = A^\dagger A^*$, that is, A is normal if and only if A is both EP and star-dagger. Show that in general

$$||\,AA^* - A^*A\,|| \leqslant ||A||^2(\,2||\,AA^\dagger - A^\dagger A\,|| + ||\,A^*A^\dagger - A^\dagger A^*\,||\,),$$

where $||\cdot||$ denotes the spectral norm of a matrix. This inequality shows that if $A^*A^\dagger - A^\dagger A^* \to 0$, $AA^\dagger - A^\dagger A \to 0$, and A is bounded, then $AA^* - A^*A \to 0$.