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IMAGE

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SEMYON ARONOVICH GERSHGORIN

by
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Introduction

Several people have asked me for information about Gershgorin–nothing about him seems to have been published in English. The standard reference work [1] for mathematics in the USSR is *History of our Nation's Mathematics* (in Russian) produced by the Academy of Sciences of the USSR and the Academy of Sciences of the Ukrainian SSR, published in 4 volumes by Naukova Dumka, Kiev, 1966-1970.

The biographical article on Semyon Aronovich Gershgorin [1, Volume 4, part 2, p.568] tells that he was born on 1901–8–24 at Pruzhany (in the Brest district), and that he died on 1933–5–30. He studied at Petrograd Technological Institute starting in 1923, became Professor in 1930, and from 1930 he worked in the Leningrad Mechanical Engineering Institute on algebra, theory of functions of complex variable, approximate and numerical methods, and differential equations.

Three papers by Gershgorin [7, 11, 15] are discussed in [1], and his 1931 paper [13] on eigenvalues was cited by Olga Taussky [17, p.296] and by D. K. Faddeyev and V. N. Faddeyeva [2, p.679]. Nine other papers are listed here, from the bibliography in Richard S. Varga's forthcoming treatise [19].

In 1925, Gershgorin proposed [1, Volume 4, part 2, p.378] an original and intricate mechanism for solving the Laplace equation, and he described such a device in detail [3]. J. J. Sylvester had proved that any algebraic relation between real variables could be modelled by linkage mechanisms, but he had not mentioned the possibility of actually constructing such mechanisms. In Gershgorin's 1926 paper [6], he described linkage mechanisms implementing the complex arithmetic operations of addition, subtraction, multiplication and division. He described mechanisms for constructing the complex relations $w = z^2$ and $w = z^3$, which could also be applied for extracting square roots and cube roots. Gershgorin proposed that linkage mechanisms be constructed for various standard functions, which could then be assembled into larger mechanisms for more complicated functions. Later he became the first person to construct analogue devices applying complex variables to the theory of mechanisms [1, Volume 4, part 2, p.326]. In 1928 he described devices modelling the aerofoil profiles of Zhukovskii and von Mises [10], and those analogue devices had practical value.

In 1910, Lewis Fry Richardson founded the finite-difference method for numerical approximation to the solution of partial differential equations [16]. In 1927, Gershgorin greatly advanced finite-difference methods [7]. For the 2-dimensional Poisson equation in u over a plane region, with the solution specified on the boundary Γ as a function of position x,

$$\Delta u = -f, \quad u|_\Gamma = \mu(x),$$

he used finite-difference approximations Λ to the Laplace operator Δ on regular nets ω_h with mesh-size h over the region:

$$\Lambda y = -\varphi, \quad y|_{\Gamma_h} = \mu(x),$$

where $\varphi = f$ at internal mesh-points, and the mesh-points on the boundary are denoted by Γ_h. He used a method of majorants to prove that the truncation error between the analytical solution u and the finite-difference solution y is $O(h)$ for regular hexagonal nets with 4-point finite-difference arrays, is $O(h^2)$ for regular square nets with 5-point finite-difference arrays, and is $O(h^4)$ for regular triangular nets with 7-point finite-difference arrays [1, Volume 4, part 2, pages 85-86]. Gershgorin's method of majorants was later generalized to dimensions greater than 2, and to other types of boundary condition [1, Volume 4, part 2, p.88].

In Gershgorin's 1929 paper [11], he first proposed solving finite-difference approximations to partial differential equations, by modelling them with networks of electrical components. [1, Volume 4, part 2, p.378.]

Above all, in Gershgorin's 1931 paper 'Über die Abgrenzung der Eigenwerte einer Matrix' [13, in German], he gave very powerful estimates for eigenvalues of matrices:

THEOREM 1. *For every square matrix A of order n, every eigenvalue lies in at least one of the n circular disks with centres a_{ii} and radii $\sum_{i \neq j} |a_{ij}|$.*

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THEOREM 2. *If s of the Gershgorin disks in Theorem 1 form a connected domain which is isolated from the other $n-s$ disks, then there are exactly s eigenvalues of A within that connected domain.*

A significant refinement was made by Olga Taussky[1] [17, p.286], which can sometimes be used to prove that a matrix is nonsingular:

THEOREM 3. *If A is irreducible then all eigenvalues lie inside the union of the Gershgorin disks, except that any eigenvalue on the boundary of any Gershgorin disk is on the boundary of all n disks.*

Hence, for irreducible A, if any Gershgorin disk has 2 distinct eigenvalues on its boundary, then the boundaries of all n disks pass through those 2 eigenvalues; and if any Gershgorin disk has 3 distinct eigenvalues on its boundary, then all n disks coincide.

James H. Wilkinson made very effective use of Gershgorin's Theorem 2 for refined estimation of eigenvalues, by applying similarity transforms to A (as Gershgorin had suggested) to isolate a single disk from the others, so that exactly one eigenvalue is contained in that isolated disk [20, pp. 71-81 & 638-646]. Gershgorin's seminal work on eigenvalues is cited in my recent paper [18, p.10].

Gershgorin's final paper [15] 'On conformal transformation of a simply-connected region onto a circle' (in Russian) was published in 1933. L. Lichtenstein had reduced that important problem to the solution of a Fredholm integral equation. Independently of Lichtenstein, Gershgorin utilised Nyström's method and reduced that conformal transformation problem to the same Fredholm integral equation. Later, A. M. Banin solved the Lichtenstein-Gershgorin integral equation approximately, by reducing it to a finite system of linear differential equations. [1, Volume 4 Part 1, p.365, & Volume 4, Part 2, p.146].

Semyon Aronovich Gershgorin died on 1933-5-30, at the age of 31.

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Cont'd on page 5

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ILAS Information Center

The electronic ILAS INFORMATION CENTER (IIC) provides current information on international conferences in linear algebra, other linear algebra activities, linear algebra journals, and ILAS-NET notices. The primary website can be found at:

http://www.ilasic.math.uregina.ca/iic/index1.html

and mirror sites are located at:

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htpp://wftp.tu-chemnitz.de/pub/iic/index1.html
htpp://hermite.cii.fc.ul.pt/iic/index1.html
htpp://www.math.temple.edu/iic/index1.html

Call for Submissions to IMAGE

As always, IMAGE welcomes announcements of upcoming meetings, reports on past conferences, historical essays on linear algebra, book reviews, essays on the development of Linear Algebra in a certain country or region, and letters to the editor or signed columns of opinion. IMAGE would like to slightly expand its scope by including general audience articles that highlight emerging applications and topics in Linear Algebra. Contributions for IMAGE should be sent to Bryan Shader (bshader@uwyo.edu) or Hans Joachim Werner (werner@united.econ.uni-bonn.de). The deadlines are October 1 for the fall issue, and April 1 for the spring issue.

ILAS 2003 - 2004 Treasurer's Report

Net Account Balances on February 28, 2003

Vanguard (ST Fed. Bond Fund 1165.096 Shares)		
(72% Schneider Fund and 28% Todd Fund)	12,489.83	
Checking account	68,997.91	
Pending checks	940.00	
Pending VISA/Mastercard	2124.00	
Outstanding check to UW Madison	(2,000.00)	**$82,551.74**

General Fund	33,962.88	
Conference Fund	10,518.94	
ILAS/LAA Fund	5,840.00	
Olga Taussky Todd/John Todd Fund	8,797.39	
Frank Uhlig Education Fund	3,685.98	
Hans Schneider Prize Fund	19,746.55	$**82,551.74**

March 1, 2003 through February 29, 2004

Income:

Dues	2750.00	
Corporate Dues	1000.00	
Book Sales	31.00	
General Fund	388.10	
Conference Fund	80.71	
ILAS\LAA Fund	1037.10	
Taussky-Todd Fund	947.24	
Uhlig Education Fund	33.35	
Schneider Prize Fund	490.70	6,758.20

Expenses:

IMAGE (2 issues)	3442.51	
Speakers (2)	800.00	
Credit Card Fees	251.60	
License Fees	61.25	
Labor - Mailing & Conference	257.00	
Postage	518.06	
Supplies and Copying	565.32	5,895.74

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February 29, 2004 Checking Account Balance

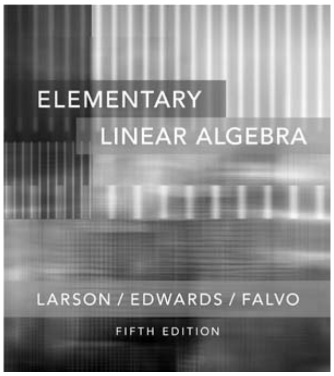
Net Account Balances on February 29, 2004

Vanguard (ST Fed. Bond Fund 3554.076 Shares)		
(10.60% Each: General Fund, Conference Fund and ILAS/LAA Fund, 17.40% Taussky Todd Fund, 7.95% Uhlig Fund, 42.85% Schneider Fund)		
	$37,815.37	
Checking account	$43,756.83	
Pending checks	$ 1,600.00	
Pending VISA/Mastercard/AMEX	$ 200.00	
Cash	$ 42.00	**$83,414.20**

General Fund	$32,236.24	
Conference Fund	$10,599.65	
ILAS/LAA Fund	$ 6,877.10	
Olga Taussky Todd/John Todd Fund	$ 9,744.63	
Frank Uhlig Education Fund	$ 3,719.33	
Hans Schneider Prize Fund	$20,237.25	**$83,414.20**

HOUGHTON MIFFLIN 2004
MATHEMATICS

NEW!



LARSON/EDWARDS/FALVO
Elementary Linear Algebra, 5/e
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This text offers a clear and concise presentation of linear algebra, balancing theory with examples, applications, and geometric intuition.

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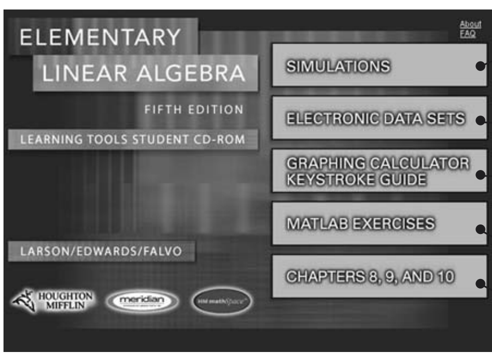

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ILAS President/Vice President Annual Report: April 2004

1) The following were elected in the ILAS fall, 2003 elections to offices with terms that began on March 1, 2004 and end on February 28, 2007:

Vice President: Roger Horn (second term)
Board of Directors: Roy Mathias and Joao Filipe Queiro

2) The following continue in ILAS offices to which they were previously elected:

President: Daniel Hershkowitz (term ends February 28, 2005)

Secretary/Treasurer: Jeff Stuart (term ends February 28, 2006)

Board of Directors:

Ravindra Bapat (term ends February 28, 2005)
Rafael Bru (term ends February 28, 2006)
Michael Neumann (term ends February 28, 2005)
Hugo Woerdeman (term ends February 28, 2006)

Tom Markham and Daniel Szyld completed their three-year terms on the ILAS Board of Directors on February 29, 2004.

President Hershkowitz appointed Jane Day as Chair of the Education Committee, replacing Guershon Harel, who resigned for personal reasons.

3) With the advice of the ILAS Executive Board, President Hershkowitz appointed a committee to select a recipient of the Hans Schneider Prize in Linear Algebra to be awarded at the 12th ILAS Conference, Regina, Canada, June 26-29, 2005. Chaired by Michael Neumann, the committee consists of Heike Fassbender, Miroslav Fiedler, Robert Guralnick, Danny Hershkowitz (ex-officio), and Eduardo Marques de Sá. Nominations may be made by any ILAS member and should be sent to the Chair (neumann@math.uconn.edu) before November 15, 2004.

4) Three ILAS-endorsed meetings took place during the last year:

The SIAM SIAG\LA Conference on Applied Linear Algebra, July 15-19, 2003, Williamsburg, Virginia, USA. Judi MacDonald and Bryan Shader were the ILAS Lecturers.

The 12th International Workshop on Matrices and Statistics (IWMS-2003), August 5-8, 2003 Dortmund, Germany. Jerszy Baksalary was the ILAS Lecturer. The International Conference on Matrix Analysis and Applications, December 14-16, 2003 Fort Lauderdale, USA. Roger Horn was the ILAS Lecturer.

5) The 11th ILAS Conference will take place in Coimbra, Portugal, July 19-22, 2004. Professor Peter Lancaster will be presented with the 2002 ILAS Hans Schneider Prize in Linear Algebra, and he will deliver the Prize Lecture. T. Ando was also a recepient of the 2002 H.S. prize, and gave his lecture at the Atlanta 2002 meeting. Professor Peter Šemrl will present the Olga Taussky-John Todd Lecture. The two SIAM SIAM\LA speakers will be Beatrice Meini and Julio Moro. Sixteen additional plenary speakers and several mini-symposia are scheduled. The chairman of the organizing committee is Joao Filipe Queiró. For more information visit http://www.mat.uc.pt/ilas2004/Body.html.

6) ILAS has endorsed the following conferences of interest to ILAS members:

The Directions in Combinatorial Matrix Theory, a two-day workshop at the Banff International Research Station (BIRS), Banff, Canada, May 6-8, 2004.

The 13th International Workshop on Matrices and Statistics, Poznan, Poland, August 18-21, 2004.

The 2005 Haifa Matrix Theory Conference to be held at The Technion during January 3-7, 2005. The ILAS Lecturer will be Michael Neumann.

The Householder Meeting on Numerical Linear Algebra, Campion, USA, May 23-27, 2005

7) The following ILAS conferences are scheduled:

The 12th ILAS Conference, Regina, Saskatchewan, Canada, June 26-29, 2005 (for details see http://www.math.uregina.ca/~ilas2005/).

The 13th ILAS Conference, Amsterdam, The Netherlands, July 19-22, 2006 (Chairman of the organizing committee is Andre Ran. Local organizers: Andre Ran, Andre Klein, Peter Spreij and Jan Brandts).

The 14th ILAS Conference, Shanghai, China, summer, 2007 (Organizing Committee: Richard Brualdi - co-chair, Erxiong Jiang - co-chair, Raymond Chan, Chuanqing Gu, Danny Hershkowitz - ILAS President, Roger Horn, Ilse Ipsen, Julio Moro, Peter Šemrl, Jia-yu Shao and Pei Yuan Wu).

The 15th ILAS Conference, Cancun, Mexico, June 16-20, 2008 (Chairman of the organizing committee is Luis Verde).

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ILAS Report, cont'd from page 9

8) ELA : The Electronic Journal of Linear Algebra is now in its eleventh volume. Its editors-in-chief are Ludwig Elsner and Danny Hershkowitz.

Volume 1, published in 1996, contained 6 papers.

Volume 2, published in 1997, contained 2 papers.

Volume 3, the Hans Schneider issue, published in 1998, contained 13 papers.

Volume 4, published in 1998 as well, contained 5 papers.

Volume 5, published in 1999, contained 8 papers.

Volume 6, Proceedings of the Eleventh Haifa Matrix Theory Conference, published in 1999 and 2000, contained 8 papers.

Volume 7, published in 2000, contained 14 papers.

Volume 8, published in 2001, contained 12 papers.

Volume 9, published in 2002, contained 24 papers.

Volume 10, published in 2003, contained 25 papers.

Volume 11, is being published now. As of April 13, 2004, it contains 7 papers.

The rejection rate in ELA is currently 39%. ELA's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz, in the University of Lisbon, in EMIS - The European Mathematical Information Service offered by the European Mathematical Society, and in EMIS' 36 mirror sites.

Volumes 1-7 (1996-2000) of ELA are in print, bound as two separate books: vol. 1-4, and 5-7. Copies can be ordered from Jim Weaver.

9) ILAS-NET is managed by Shaun Fallat, and now has 485 subscribers. As of April 12, 2004, we have circulated 1342 ILAS-NET announcements.

10) The primary site of ILAS INFORMATION CENTER (IIC) is at Regina. Mirror sites are locate in the Technion, in Temple University, in the University of Chemnitz and in the University of Lisbon.

Respectfully submitted,

Daniel Hershkowitz, ILAS President, hershkow @tx.technion.ac.il; Roger Horn, ILAS Vice-President, rhorn@math.utah.edu.

Call for Papers
Special Issue of LAA
11th ILAS Conference

Linear Algebra and its Applications will publish a special issue devoted to papers presented at the 11th ILAS Conference, Coimbra, 19–22 July 2004. Papers should be submitted by 31 October 2004 to one of the special editors whose names and addresses are listed below. The usual standards of LAA will apply.

Graciano de Oliveira
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For details of the conference see http://www.mat.uc.pt/ilas2004 .

Book Report

Introduction to Linear Algebra (3rd edition), by Gilbert Strang, Wellesley-Cambridge Press, 2003.
ISBN 0961408898

This book is the text for an introductory course in Linear Algebra at MIT. The course is offered primarily to students in disciplines other than mathematics. For this purpose it is admirably suited. It is clear and interesting to read. It has excellent treatments of things that are difficult to explain. I found every section contains a charming example or a fresh way of looking at something. Unlike many math books, the author does not strive to remove all evidence of the book being written by a human being. However, not everyone will enjoy this book. For example, a proponent of the Theorem-Proof-QED style of writing for introductory texts will be disappointed by its informality. The price of the text's chattiness is a lack on concision. This would be a poor reference book. Those who find "cute" comments annoying will be annoyed. However, in my experience, students tend to enjoy books that are written more informally. I think that particularly for a lower level course, taught to students in many disciplines, this is quite an appropriate text.

Each section of each chapter obeys the following structure: informal and often interesting comments; the body of the section; a concise summary of key ideas worked example; and problems, some of which have solutions in the back of the text.

Chapter 1 is a review of the basic properties of vectors: addition, scalar multiplication, dot products, the Schwarz inequality. However, much of it is only done in two or three dimensions. Chapter 2 covers basic matrix properties, operations for square matrices, along with the solution of square linear systems. This contains Strang's exemplary exposition of the connection between Gaussian Elimination and LU Factorization: my favorite bit in the book.

Chapter 3 introduces vector spaces in R^n for arbitrary n. He discusses the vector spaces associated with a rectangular matrix. Along with this he tackles the solution of consistent rectangular linear systems. Chapter 4 discusses orthogonality of vectors and subspaces of vectors, segueing into orthogonal projections, least-squares problems and the QR decomposition. Chapter 5 is devoted to determinants. I do not know if I agree with the opening comment: "The determinant contains an amazing amount of information about the matrix." I suppose if you were forced to summarize a matrix with a single scalar you could do worse. Granted, determinants must be discussed somewhere in such a course, but perhaps they could be postponed till after eigenvalues, as in Axler's text on Linear Algebra. Chapter 6 covers eigenvalues, diagonalization, and linear differential equations. One aspect of the treatment is a discussion of the matrix exponential, something that I appreciate greatly and is missing from other introductory texts. He goes on to symmetric and positive definite matrices, similarity and the SVD. An excellent example of Strang's relaxed style is given by his treatment of the spectral theorem. He states the theorem, gives intuitive proofs for special cases, and a compelling argument for why you can extend it to the general case. Chapter 7 is devoted to the concept of Linear transformations. Here the fiddly topic of change of basis matrices is covered. (An eminent group theorist once told me he found this subject more difficult to get straight than the most difficult issues in his research.) He uses the Haar wavelets as a motivating example. I am not sure if this is a stroke of genius—in that it is an important and interesting application—or rather a confusing digression in an already confusing topic. Other interesting items in this chapter are the polar decomposition and the pseudoinverse.

Chapter 8 has six sections each of which covers an application of linear algebra. The selection is good, covering both the usual topics (Markov matrices and computer graphics) but also some less common ones such as linear programming. In case you were wondering what "the most fundamental law of applied mathematics" is, according to Strang it is the "balance equations" (total force on a static object is zero). Even if you do not agree with this, you may still enjoy the section in which he discusses this as part of an interesting introduction to structural mechanics. In Chapter 9 Strang delves into numerical linear algebra in more detail than he does elsewhere in the book. Though I personally like this subject, I found this short chapter to be not very interesting. Most of the topics usually placed under the rubric of numerical linear algebra are covered elsewhere and most readers could skip this chapter.

The final chapter considers issues related to complex numbers, that is, both real matrices with complex eigenvalues and matrices that are complex to begin with. There is a section on the properties of complex numbers that a lecturer may want to refer to earlier in the course if need be. Of particular interest to some users of the book is a section on the Fast Fourier Transform.

Strang ends the book by thanking the reader for studying linear algebra. My impression is that this is a great text for teaching scientists and engineers. I have some misgivings about this being used as a text for mathematicians, applied or otherwise. It is important that mathematics students are exposed to an axiomatic treatment of linear algebra at some point, and this text does not do a thorough job of that–nor is it intended to. On the other hand, an introductory course based on this book would be far more interesting than a more rigorously oriented course, and would give mathematics students a much-needed introduction to applied mathematics early on.

Reviewed by Paul Tupper
Department of Mathematics and Statistics
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Montreal, QC H3A 2K6 CANADA

The Hans Schneider Prize in Linear Algebra

Call for Nominations

The Hans Schneider Prize in Linear Algebra is awarded by The International Linear Algebra Society for research contributions, and achievements at the highest level of Linear Algebra. The Prize may be awarded for either an outstanding scientific achievement or for a lifetime contribution.

According to its specifications, the Prize is awarded every three years at an appropriate ILAS conference. The last prize was awarded in June 2002 at the ILAS Meeting in Auburn jointly to Tsuyoshi Ando and Peter Lancaster and thus it is appropriate to award the prize again at the ILAS Regina, Canada meeting, June 26-29, 2005. The prize guidelines can be found at

http://www.ilasic.math.uregina.ca/iic/ILASPRIZE.html
or
http://www.math.technion.ac.il/iic/ILASPRIZE.html

The committee appointed by the ILAS president upon the advice of the ILAS Executive Board consists of Heike Fassbender, Mirek Fiedler, Bob Guralnick, Danny Hershkowitz (ILAS president - ex-officio member), Miki Neumann (chair), and Eduardo Marques de Sá.

Nominations, of distinguished individuals judged worthy of consideration for the Prize, are now being invited from members of ILAS and the linear algebra community in general. In nominating an individual, the nominator should include:

(1) a brief biographical sketch of the nominee, and

(2) a statement explaining why the nominee is considered worthy of the prize, including references to publications or other contributions of the nominee which are considered most significant in making this assessment.

Nominations are open until November 15, 2004 and should be sent to the Chair, Michael Neumann, of the committee at the address below. The committee may ask the nominator to supply additional information.

Professor Michael Neumann
Department of Mathematics
University of Connecticut
Storrs, Connecticut 06269-3009 USA
email: neumann@math.uconn.edu

Recent Releases of Interest

Dover Publications have recently published a new edition of the classic *Lambda-Matrices and Vibrating Systems* by Peter Lancaster. It was first published by Pergamon press in 1966 and has been out of print for many years.

Jonathan Golan has recently written a book entitled *The Linear Algebra a Beginning Graduate Student Ought to Know (*Kluwer Academic Publishes, 2004, ISBN: 1-4020-1824-X). The book is intended either as a textbook for an advanced-undergraduate or first-year graduate course in linear algebra, or as a reference and self-study guide for preliminary exams in linear algebra, and contains both theoretical material and material on computational matrix theory. A review of this book will appear in the next issue of *IMAGE.*

Morris Newman Conference

Report by **Fuzhen Zhang**

A math conference in honor of Dr. Morris Newman's 80th birthday was held on April 17th and 18th, 2004, at the University of California-Santa Barbara. Dr. Newman well known for his research in number theory, linear algebra, scientific computation, and group theory.

The following people attended the conference: Doug Moore, Ben Fine, Fuzhen Zhang, Ion Zaballa, Karl Rubin, Montserrat Alsina, Charles Johnson, Edward Ordman, Wasin So, Matt Boylan, Charles Ryavec Larry Gerstein, Marvin Knopp, Ahmad El-Guindy, Adrian Stanger, Russell Merris,Timothy Redmond, Cindy Wyels, Steve Pierce, Chris Agh, Markus Sandy, Jeffrey Stopple, Basil Gordon, and Bob Guralnick.

Doug Moore, Morris Newman and Charlie Johnson

International Conference on Matrix Analysis and Applications

Report by **Fuzhen Zhang**

The International Conference on Matrix Analysis and Applications was held on the main campus of Nova Southeastern University (NSU), Fort Lauderdale, Florida, December 14-16, 2003. Eighty-five mathematicians participated in the three-day event and sixty-eight contributed talks were presented.

The conference was co-sponsored by NSU's Farquhar College of Arts and Sciences and the International Linear Algebra Society (ILAS). The featured guest lecturer Roger Horn, Research Professor of Mathematics at the University of Utah, and one of the most respected and renowned mathematicians in the field of matrix analysis.

The organizing committee for the conference consisted of Tsuyoshi Ando (Hokkaido University), Chi-Kwong Li (College of William and Mary), George P.H. Styan (McGill University), Hugo Woerdeman (College of William and Mary, and Catholic University) and Fuzhen Zhang (Nova Southeastern University).

The goals of the conference were to stimulate research and interaction of researchers interested in all aspects of linear and multilinear algebra, matrix analysis and applications, as well as to provide an opportunity to exchange ideas, recent results and developments on the subjects. The pool party in the evening of the 15th was a great joy.

For more information and conference photos, please visit the website at www.resnet.wm.edu/~cklixx/nova03.html.

First Workshop on Matrix Analysis

Report by **Mohammad Sal Moslehian**

The First Workshop on Matrix Analysis sponsored by the Ferdowsi University was held on March 11-12, 2004. This workshop took place in the Mathematics Department of Ferdowsi University of Mashhad in Iran, and was organized to benefit graduate students. The following eight talks on matrix norms and related topics were presented:

Dr. Madjid Mirzavaziri: "Vector Norms, Matrix Norms and Induced Norm Problem," and "Absolute Norms, Monotone Norms and Symmetric Norms."

Dr. Shirin Hejazian: "Spectral Radius, Numerical Radius and Matrix Norm," and "Dual Norms and Self-adjoint Norms."

Dr. Mohammad Sal Moslehian (Organizer): "Minimal Matrix Norms," and "Unitarily invariant Norms."

Dr. Assad Niknam: "Contraction Matrix Norms," and "Matrix Norms and Graph Theory. "

There were 32 participants, some of whom were supported. Participants actively exchanged many ideas on the subject in a good atmosphere, and all look forward to future workshops.

Nova Matrix Conference, Dec. 14-16, 2003, Ft. Lauderdale

Forthcoming Conferences and Workshops in Linear Algebra

4th GAMM Workshop on Applied and Numerical Linear Algebra

Hagen, Germany: 2-3 July, 2004

The special emphasis of this workshop is on "Linear Algebra in Systems and Control Theory", but all other aspects of applied and numerical linear algebra are most welcome. The workshop follows up the closely related 5th International Workshop on Accurate Solution of Eigenvalue Problems (IWASEP 5, June 28-July 1) at the same location.

Confirmed invited speakers are: Chris Beattie (Virginia Tech, USA), Ralph Byers (University of Kansas, USA), Diederich Hinrichsen (Universitat Bremen, Germany).

The workshop will consist of three invited talks and contributed talks of 25 minutes. Abstracts can be submitted via the conference webpage:

http://www.math.tu-berlin.de/~kressner/GAMM04

More information on the conference location and registration can be found on this web page.

Important dates:

Submission of abstracts:	15.05.04
Notification of acceptance:	01.06.04
Registration:	15.05.04

The organizers are: Volker Mehrmann (TU Berlin, mehrmann@math.tu.-berlin.de) and Heike Fassbender (TU Braunschweig, Germany, h.fassbender@tu-bs.de).

6th International Conference on Matrix Theory and Its Applications in China

Harbin, China: 17–22 July 2004

The 6th International Conference on Matrix Theory and Its Applications in China will be held July 17–22, 2004, at Helongjiang University in Harbin, Helongjiang Province, China.

The meeting is an international conference on Matrix Theory and its Applications held in China every even year. The conference provides a forum for researchers from various countries to exchange new ideas, recent developments and results on Matrix Theory and its Applications, including traditional linear algebra, combinational linear algebra, numerical linear algebra and related areas.

The Honorary Conference Chairs are Professor Erxiong Jiang (Shanghai University, China), and Professors Chongguang Cao, and Prof. Shaowu Liu (Heilongjiang University, China). The Program Chairs are Professors Chongguang Cao, Prof. Shaowu Liu, Dayuan Zheng (Heilongjiang University, China). The Conference Secretary-generals are Dr. Kun Jiang, Dr. Xiaomin Tang, and Yahong Guo (Heilongjiang University, China)

Registration

The registration fee is US$100 per person for faculty, and US$80 per for students. The registration fee includes 5 breakfasts, 5 lunches and 5 dinners (at the University restaurants), as well as a local tour and conference materials.

Call for papers

Papers of outstanding quality that are presented at the conference will be selected for publication in Journal of Natural Science of Heilongjiang University.

Full papers in English containing original and unpublished results are solicited. The maximum length of each paper is limited to 6 double spaced pages. Electronic submission is required. Acceptable formats for submission are Word, PDF, and Postcript.

The cover page must include the name, address, telephone number, and e-mail address of the corresponding author, and the affiliation of all authors. The information on the cover page must also be submitted by e-mail in a plain text file.

To submit a paper, send the paper by e-mail to matrix2004@hlju.edu.cn by May 15, 2004. Submission will be acknowledged within seven days.

Deadlines

Full paper submission: May 15, 2004
Notification of Acceptance: May 31, 2004
Camera-ready copy due: June 20, 2004

California Matrix Meeting

San Jose, CA: 13 November 2004

A California Matrix Meeting will be held at San Jose State University, San Jose, CA on Saturday, Nov. 13, 2004. There is no registration fee, and contributed papers are welcome. More details will be provided later on the ILAS-net, and other venues. The organizers are Wasin So (so@math.sjsu.edu) and Jane Day (day@math.sjsu.edu).

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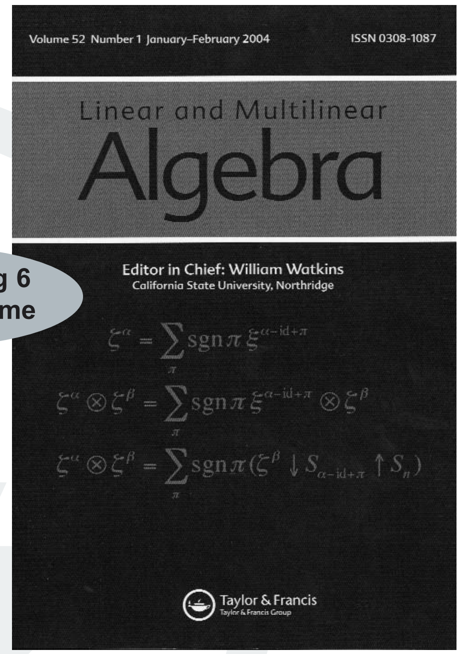

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The 2004 NZIMA Conference in Combinatorics and its Applications and The 29th Australasian Conference in Combinatorial Mathematics and Combinatorial Computing

Lake Taupo, New Zealand: 13-18 December 2004

The 2004 New Zealand Institute of Mathematics and its Applications (NZIMA) Conference in Combinatorics and its Applications, and the 20th Australasian Conference in Combinatorial Mathematics and Combinatorial Computing will be jointly held 13–18 December 2004 in the Lake Taupo district of New Zealand.

Conference topics include: Graph Theory, Matroid Theory, Design Theory, Coding Theory, Enumerative Combinatorics, Combinatorial Optimization, Combinatorial Computing and Theoretical Computer Science, and Combinatorial Matrix Theory.

A tentative list of invited speakers includes: Dan Archdeacon (University of Vermont), Richard Brualdi (University of Wisconsin), Darryn Bryant (University of Queensland), Peter Cameron (Queen Mary, University of London), Bruno Courcelle (Bordeaux University), Catherine Greenhill (University of New South Wales), Bojan Mohar (University of Ljubljana), Bruce Richter (University of Waterloo), Neil Robertson (Ohio State University), Robin Thomas (Georgia Institute of Technology), Carsten Thomassen (Technical University of Denmark), Mark Watkins (University of Syracuse) and Dominic Welsh (Oxford University).

There will be slots in the program for contributed talks by participants. It is expected that this slots will be 20 minutes in length with a limited number of 30-minute slots available on request. Deadlines for registration, titles and abstracts of contributed talks will be announced shortly.

Additional information about the conference can be found on the conference web page: http://www.nzima.auckland.ac.nz/combinatorics/conference.html

The 2005 Haifa Matrix Theory Conference

Haifa, Israel: 3-7 January, 2005

The conference plans to cover all aspects of matrix theory, linear algebra, and their applications.

The following have confirmed speaking at the conference: Ron Adin, Daniel Alpay, Jonathan Arazy, Ravindra Bapat, Harm Bart, Genrich Belitsky, Adi Ben-Israel, Alfred Bruckstein, Yair Censor, David Chillag, Harry Dym, Ludwig Elsner, Yuly Eidelman, Karl-Heinz Foerster, Shmuel Friedland, Paul Fuhrman, Israel Gohberg, Roger Horn, Tomas Kosir, Thomas Laffey, Yuri Lyubich, Alexander Markus, Volker Mehrmann, Roy Meshulam, Michael Neumann (ILAS speaker), Vadim Olshevsky, Allan Pinkus, Robert Plemmons, Leiba Rodman, Uriel Rothblum, Hans Schneider, Bryan Shader, Naomi Shaked-Monderer, Robert Shorten, Avram Sidi, Bit-Shun Tam, Michael Tsatsomeros, Eugene Tyrtyshnikov, Victor Vinnikov, William Watkins, Hans Joachim Werner, and Hugo Woerdeman.

The organizing committee consists of Abraham Berman (Chair), Moshe Goldberg, Daniel Hershkowitz, Leonid Lerer, and Raphael Loewy.

Call for papers

Titles and abstracts should be submitted to Ms. Sylvia Schur, conference secretary, at the address below, no later than October 1, 2004. Abstracts should be up to one page in length, and can be sent either by e-mail in Tex/Latex, or by mail.

Proceedings

The journal *Linear Algebra and Its Applications* will publish a special issue devoted to papers presented at the conference. The special editors are Abraham Berman, Leonid Lerer and Raphael Loewy. The usual standards of LAA will apply. The submission deadline is April 30, 2005. Further details will appear in due course at http://www.math.wisc.edu/~hans/speciss.html

For further information, please contact:
Ms. Sylvia Schur (Secretary)
Department of Mathematics
Technion-Israel Institute of Technology
Haifa 32000, Israel
email: cms@math.technion.ac.il
Phone: 972 4 829 4278
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Selected Papers

Estimating the Approximation Error in Learning Theory
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Selected Papers

Infinite Cogalois Theory, Clifford Extensions, and Hopf Algebras
T Albu

Profinite Identities for Finite Semigroups Whose Subgroups Belong to a Given Pseudovariety
J Almeida & *M V Volkov*

Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore – Vol. 2

REPRESENTATIONS OF REAL AND *p*-ADIC GROUPS

by Eng-Chye Tan & Chen-Bo Zhu *(National University of Singapore, Singapore)*

The Institute for Mathematical Sciences at the National University of Singapore hosted a research program on "Representation Theory of Lie Groups" from July 2002 to January 2003. As part of the program, tutorials for graduate students and junior researchers were given by leading experts in the field.

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981-238-779-X	US$72	£44

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LECTURES ON FINITE FIELDS AND GALOIS RINGS

by Zhe-Xian Wan *(Chinese Academy of Sciences, China)*

This is a textbook for graduate and upper level undergraduate students in mathematics, computer science, communication engineering and other fields. The explicit construction of finite fields and the computation in finite fields are emphasised. In particular, the construction of irreducible polynomials and the normal basis of finite fields are included. The essentials of Galois rings are also presented. This invaluable book has been written in a friendly style, so that lecturers can easily use it as a text and students can use it for self-study. A great number of exercises have been incorporated.

352pp	Aug 2003	
981-238-504-5	US$68	£50
981-238-570-3(pbk)	US$38	£28

COMPLETELY POSITIVE MATRICES

by Abraham Berman (*Technion – Israel Institute of Technology*) & Naomi Shaked-Monderer (*Emek Yezteel College, Israel*)

A real matrix is positive semidefinite if it can be decomposed as $A=BB'$. In some applications the matrix B has to be elementwise nonnegative. If such a matrix exists, A is called completely positive. The smallest number of columns of a nonnegative matrix B such that $A=BB'$ is known as the c*p*-rank of A.

This invaluable book focuses on necessary conditions and sufficient conditions for complete positivity, as well as bounds for the c*p*-rank. The methods are combinatorial, geometric and algebraic. The required background on nonnegative matrices, cones, graphs and Schur complements is outlined.

216pp	Apr 2003	
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IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 28-3 [IMAGE 28 (April 2002), p. 36], and 31-1 through 31-8 [IMAGE 31 (October 2003), pp. 44 & 43]. Problem 30-3 is repeated below without solution; we are still hoping to receive a solution to this problem. We introduce 7 new problems on pp. 40 & 39 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to ujw902@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 28-3: Ranks of Nonzero Linear Combinations of Certain Matrices.

Proposed by Shmuel FRIEDLAND, *University of Illinois at Chicago, Chicago, Illinois, USA:* friedlan@uic.edu
and Raphael LOEWY, *Technion–Israel Institute of Technology, Haifa, Israel:* loewy@technunix.technion.ac.il

Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Show that any nonzero real linear combination of these four matrices has rank at least 3.

Solution 28-3.1 by S. W. DRURY, *McGill University, Montréal (Québec), Canada:* drury@math.mcgill.ca

Let $B = t_1B_1+t_2B_2+t_3B_3+t_4B_4$ and let C be the classical adjoint of B. The entries of C are cubic polynomials in (t_1, t_2, t_3, t_4). Now, consider

$$Q = \begin{pmatrix} 998 & 401 & 213 & 560 \\ 401 & 600 & 459 & 296 \\ 213 & 459 & 484 & 303 \\ 560 & 296 & 303 & 614 \end{pmatrix}$$

which is easily checked to be a positive definite matrix and let

$$q = 998{t_1}^2 + 802t_1t_2 + 426t_3t_1 + 1120t_4t_1 + 600{t_2}^2 + 918t_3t_2 + 592t_2t_4 + 484{t_3}^2 + 606t_3t_4 + 614{t_4}^2$$

be the quadratic form that it defines. Then, calculations show that

$$t_2\, q = 36C_{1,1} + 94C_{1,2} - 58C_{1,3} + 58C_{2,2} + 130C_{2,3} - 94C_{2,4} + 246C_{3,3} - 108C_{3,4} + 36C_{4,4}$$

and

$$t_3\, q = -94C_{1,1} + 94C_{1,2} + 58C_{1,3} - 36C_{1,4} - 130C_{2,2} - 188C_{2,3} + 72C_{2,4} - 94C_{3,3} - 94C_{4,4}.$$

Now assume that B has rank strictly less than 3 and that not all the t_j are zero. Then C is identically zero and q is strictly positive. We conclude that $t_2 = t_3 = 0$. But now we have

$$C_{1,1} = -t_4({t_4}^2 + {t_1}^2 + t_4t_1) \text{ and } C_{4,4} = -t_1({t_4}^2 + {t_1}^2 + t_4t_1).$$

Repeating the above idea on a smaller scale, we see that ${t_4}^2 + {t_1}^2 + t_4t_1 > 0$ unless $t_1 = t_4 = 0$. But again since C is identically zero, we are forced to conclude that $t_1 = t_4 = 0$ anyway.

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

We look forward to receiving solutions to Problem 30-3!

Problem 31-1: A Property of Linear Subspaces

Proposed by Jürgen GROß and Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:*
gross@statistik.uni-dortmund.de trenkler@statistik.uni-dortmund.de

In Groß (1999, Corollary 2) the following is stated: If U and V are linear subspaces of $\mathbb{C}^m$, then

$$\mathbb{C}^m = [U \cap (U^\perp + V^\perp)] \oplus [V \oplus (U^\perp \cap V^\perp)],$$

where "$\oplus$" indicates the direct sum of two subspaces and "$\perp$" denotes the orthogonal complement. Is this decomposition also valid in a Hilbert space? The Proposers of the problem have no answer to this question.

Reference

J. Groß (1999). On oblique projection, rank additivity and the Moore-Penrose inverse of the sum of two matrices. *Linear and Multilinear Algebra*, **46**, 265–275.

Solution 31-1.1 by Leo LIVSHITS, *Colby College, Waterville, Maine, USA:* llivshi@colby.edu

The theorem is not true as stated in the infinite-dimensional Hilbert space setting. The obstruction is due to the fact that the sum of two closed subspaces in an infinite-dimensional Hilbert space is a subspace that may not be closed. For a concise discussion of this phenomena see Problem 52 in "A Hilbert Space Problem Book" by P. R. Halmos. We shall base our counterexample on it.

The strategy for constructing a counterexample becomes apparent when one notes that

$$U + V = \left[[U \cap (U \cap V)^\perp] \oplus (U \cap V)\right] + \left[[V \cap (U \cap V)^\perp] \oplus (U \cap V)\right],$$

so that

$$U + V = [U \cap (U \cap V)^\perp] \dotplus V,$$

and consequently

$$H = \left[[U \cap (U \cap V)^\perp] \dotplus V\right] \oplus (U + V)^\perp,$$

where U, V are closed subspaces of the Hilbert space H, $\oplus$ stands for orthogonal direct sum, and $\dotplus$ stands for linear direct sum.

Furthermore, $(U + V)^\perp = U^\perp \cap V^\perp$, and $U^\perp + V^\perp \subset (U \cap V)^\perp$, and the last inclusion may be strict since $U^\perp + V^\perp$ may not be closed.

Let $A : \ell^2 \to \ell^2$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).$$

Then A is a continuous linear function whose range contains all finitely non-zero sequences, but not the sequence $h = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$. Therefore $\mathrm{range}(A)$ is a proper dense subspace of ℓ^2. Let $U = \{(0, y) | y \in \ell^2\}$ and $V = \{(x, Ax) | x \in \ell^2\}^\perp$. Clearly U and V are closed subspaces of the Hilbert space $\ell^2 \oplus \ell^2$. Consequently (making use of the Closed Graph Theorem) one concludes that

$$U^\perp + V^\perp = \{(x, y) | x \in \ell^2, y \in \mathrm{range}(A)\}$$

$$U \cap (U^\perp + V^\perp) = \{(0, y) | y \in \text{range}(A)\}$$
$$U^\perp \cap V^\perp = \{(0, 0)\},$$

so that

$$[U \cap (U^\perp + V^\perp)] + V + (U^\perp \cap V^\perp) = [U \cap (U^\perp + V^\perp)] + V$$

In particular, $[U \cap (U^\perp + V^\perp)] + V + (U^\perp \cap V^\perp)$ does not contain $(0, h)$ (and hence is a proper subspace of $\ell^2 \oplus \ell^2$). Indeed, if $(0, h) - (0, y) \in \{(x, Ax) | x \in \ell^2\}^\perp$ for some $y \in \text{range}(A)$, then $h - y \in (\text{range}(A))^\perp = \{0\}$, so that $h = y \in \text{range}(A)$, which is a contradiction.

Reference

P. R. Halmos (1967). *A Hilbert Space Problem Book.* Van Nostrand Comp., Princeton, N. J.

Problem 31-2: Matrices Commuting with All Nilpotent Matrices

Proposed by Henry RICARDO, *Medgar Evers College (CUNY) Brooklyn, New York, New York, USA:* odedude@yahoo.com

If an $n \times n$ matrix A commutes with all $n \times n$ nilpotent matrices, must A be nilpotent? Determine the whole class of these matrices. (We recall that a square matrix N is said to be nilpotent whenever $N^k = 0$ for some positive integer k.)

Solution 31-2.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and Xiaoji LIU, *University of Science and Technology of Suzhou, Suzhou, People's Republic of China:* xiaojiliu72@yahoo.com.cn

Let $\mathbb{C}_{n,n}$ be the set of $n \times n$ complex matrices and let a_{ij}, $i, j = 1, ..., n$, denote the successive entries of $A \in \mathbb{C}_{n,n}$. The answer to the first question is obviously negative. The identity matrix $A = I_n$ constitutes a trivial counterexample. The answer to the second question will be obtained as a simple corollary to the theorem below, which characterizes $A \in \mathbb{C}_{n,n}$ satisfying $AN = NA$ for all nilpotent $N \in \mathbb{C}_{n,n}$ and in addition shows that this property is actually equivalent to the commutativity of A with suitably selected n nilpotent matrices $N_{ij} \in \mathbb{C}_{n,n}$ only. Here N_{ij} (with $i \in \{1, ..., n\}$, $j \in \{1, ..., n\}$, and $i \neq j$) stands for the matrix whose the (i, j)th entry is equal to one and all the remaining entries are zeros, so that it is nilpotent of index 2.

THEOREM. *For any $A \in \mathbb{C}_{n,n}$, the following statements are equivalent:*

(a) $AN = NA$ *for every nilpotent* $N \in \mathbb{C}_{n,n}$,

(b) $AN_{ij} = N_{ij}A$ *for every* N_{ij} *from a given set of n nilpotent matrices* $\{N_{i_1 j_1}, ..., N_{i_n j_n}\}$ *indexed by the pairs* (i_m, j_m), *which are selected so that* $\{i_1, ..., i_n\} = \{1, ..., n\}$ *or* $\{j_1, ..., j_n\} = \{1, ..., n\}$ *and that at least* $n - 1$ *of them satisfy* $(i_m, j_m) \neq (j_m, i_m)$, $m \in \{1, ..., n\}$,

(c) $A = \alpha I_n$ *for some* $\alpha \in \mathbb{C}$.

PROOF. It is obvious that (a) ⇒ (b) and (c) ⇒ (a), and thus the proof reduces to establishing the part (b) ⇒ (c). It can easily be observed that the jth column of the matrix AN_{ij} coincides with the ith column of A and the ith row of the matrix $N_{ij}A$ coincides with the jth row of A, with all the remaining entries of AN_{ij} and $N_{ij}A$ being equal to zero. This means that, for any given i and j, the (k, l)th entry of AN_{ij} is a_{ki} when $l = j$ and zero otherwise, while the corresponding entry of $N_{ij}A$ is a_{jl} when $k = i$ and zero otherwise, $k, l = 1, ..., n$. Hence if follows that the equality $AN_{ij} = N_{ij}A$ holds if and only if

$$a_{ii} = a_{jj}, \tag{1}$$

$$a_{ki} = 0 \;\text{ for every } k = 1, ..., n; k \neq i, \tag{2}$$

$$a_{jl} = 0 \;\text{ for every } l = 1, ..., n; l \neq j. \tag{3}$$

From any set of $n(n-1)$ conditions obtained by replacing i in (2) by $i_1, ..., i_n$ such that $\{i_1, ..., i_n\} = \{1, ..., n\}$ or by replacing j in (3) by $j_1, ..., j_n$ such that $\{j_1, ..., j_n\} = \{1, ..., n\}$ it follows that all the off-diagonal entries of A are equal to zero. Consequently, to complete the proof it remains to notice that if the equations

$$a_{i_m i_m} = a_{j_m j_m} \text{ (with } i_m \neq j_m), \; m = 1, ..., n, \tag{4}$$

implied by (1) contain no more than one reduplication, then they are fulfilled simultaneously for $\{i_1, ..., i_n\} = \{1, ..., n\}$ or $\{j_1, ..., j_n\} = \{1, ..., n\}$ if and only if $a_{11} = ... = a_{nn}$ ($= \alpha$, say). The assumption restricting the number of reduplications corresponds to the latter part of the description of a set of indices involved in (b) and shows, for instance, that for $n = 4$ the choice of $\{N_{12}, N_{21}, N_{34}, N_{43}\}$ is not a proper one, for then it only follows that $a_{11} = a_{22}$ and $a_{33} = a_{44}$, which in general is insufficient for $a_{11} = a_{22} = a_{33} = a_{44}$. On the other hand, it seems noteworthy to point out that a simple example of the choice of matrices N_{ij} in (b) which imply (c) is $\{N_{12}, N_{21}, ..., N_{n1}\}$.

In general, under the condition $\{i_1, ..., i_n\} = \{1, ..., n\}$ the set (4) can clearly be reexpressed as

$$a_{ii} = a_{j_i j_i} \text{ (with } i \neq j_i),\ i = 1, ..., n;\ j_i \in \{1, ..., n\}. \tag{5}$$

It is obvious that for $n = 2$ the two equations in (5) become reduplications one of the other, and lead to $a_{11} = a_{22}$, as desired. Now assume that if the equations in (5) hold for $i = 1, ..., n-1$ and $j_i \in \{1, ..., n-1\}$, then

$$a_{11} = ... = a_{n-1,n-1}, \tag{6}$$

and consider the full set of equations given therein. If $j_1, ..., j_{n-1} \in \{1, ..., n-1\}$, then the assumption above entails (6), and since the nth equation must be of the form $a_{nn} = a_{j_n j_n}$ with $j_n \in \{1, ..., n-1\}$, it follows that $a_{11} = ... = a_{nn}$. Otherwise, if the set

$$a_{11} = a_{j_1 j_1},\ ...,\ a_{n-1,n-1} = a_{j_{n-1} j_{n-1}} \tag{7}$$

contains an equation (or equations) of the form $a_{ii} = a_{nn}$ for some $i \in \{1, ..., n-1\}$, then replacing a_{nn} in (7) by $a_{j_n j_n}$ where j_n must belong to $\{1, ..., n-1\}$, leads to the situation considered above, and hence to (6). Combining (6) and $a_{nn} = a_{j_n j_n}$ with $j_n \in \{1, ..., n-1\}$ yields $a_{11} = ... = a_{nn}$. Clearly, analogous arguments lead to the same conclusion when the condition $\{i_1, ..., i_n\} = \{1, ..., n\}$ is replaced by $\{j_1, ..., j_n\} = \{1, ..., n\}$. □

COROLLARY. *When $A \in \mathbb{C}_{n,n}$ commutes with all nilpotent matrices, then it is nilpotent itself if and only if $A = 0$.*

PROOF. The result follows straightforwardly by noting that A of the form $A = \alpha I_n$ cannot be nilpotent unless $\alpha = 0$. □

Solution 31-2.2 by Leo LIVSHITS, *Colby College, Waterville, Maine, USA:* llivshi@colby.edu

Since any scalar multiple of the $n \times n$ identity I_n commutes with every $n \times n$ matrix, the commutant of the set $\mathcal{N}_n$ of nilpotent $n \times n$ matrices contains every scalar multiple of I_n (and hence non-nilpotent members). In fact these are the only elements of the commutant of $\mathcal{N}_n$. Indeed, assuming that $n \geq 2$ for non-triviality, each element A of the commutant commutes with every matrix of the form xy^T, where $x, y \in \mathbb{C}^n$ are column vectors and $y^T x = 0$. In particular, $(Ax)y^T = x(A^T y)^T$ for any such pair x, y. It follows that each non-zero $x \in \mathbb{C}^n$ is an eigenvector of A. Hence A is a scalar multiple of I_n.

Solution 31-2.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* ujw902@uni-bonn.de

Our offered solution to this problem is based on the following two interesting observations. Their elementary proofs are left to the reader.

THEOREM 1. *Let N be the $n \times n$ matrix with one on the super-diagonal and zeros everywhere else, that is, $n_{ij} = 1$ if $j = i + 1$ and $i = 1, 2, \cdots, n-1$ and $n_{ij} = 0$ in all the remaining cases. Then $TN = NT$ if and only if T is an upper-diagonal Toeplitz matrix, i.e., if and only if*

$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ 0 & t_0 & t_1 & \cdots & t_{n-2} \\ 0 & 0 & t_0 & \cdots & t_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_0 \end{pmatrix}.$$

THEOREM 2. *Let T be an upper-diagonal $n \times n$ Toeplitz matrix and, for $j = 1, 2$, let e_j denote the jth unit column vector with a one in the jth position and zeros everywhere else. Consider the matrix $M = e_2 e_1'$. Then $TM = MT$ if and only if T is a scalar diagonal matrix, i.e., if and only if $T = t_0 I_n$ for some scalar t_0, with I_n denoting as usual the identity matrix of order n.*

The matrices N and M defined in Theorems 1 and 2, respectively, are both nilpotent. Whereas N is nilpotent of index n, the matrix

M is nilpotent of index 2. With the above two theorems in mind, it is therefore clear that the set of $n \times n$ matrices commuting with all $n \times n$ nilpotent matrices consists of all $n \times n$ scalar diagonal matrices.

A solution to Problem 31-2 was also received from Julio Benítez *and* Néstor Thome.

Problem 31-3: A Range Equality for Block Matrices

Proposed by Yongge TIAN, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca

Let A and B be two nonnegative definite complex matrices of the same size. Show that

$$\operatorname{range}\begin{pmatrix} A & B & & \\ & \ddots & \ddots & \\ & & A & B \end{pmatrix}_{n\times(n+1)} = \operatorname{range}\begin{pmatrix} A+B & & \\ & \ddots & \\ & & A+B \end{pmatrix}_{n\times n},$$

where all blanks are zero matrices.

Solution 31-3.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl

Let $\mathbb{C}_{p,q}$ be the set of $p \times q$ complex matrices. The symbols K^*, $K^\dagger$, $\mathcal{R}(K)$, and $\mathcal{N}(K)$ will stand throughout for the conjugate transpose, Moore-Penrose inverse, range (column space), and null space, respectively, of $K \in \mathbb{C}_{p,q}$. Moreover, let $\mathbb{C}_m^{\mathsf{H}}$ and $\mathbb{C}_m^{\geq}$ denote the subsets of $\mathbb{C}_{m,m}$ consisting of Hermitian and Hermitian nonnegative definite matrices, and let $\mathcal{S}$ be the set of pairs of Hermitian matrices defined by

$$\mathcal{S} = \{(A,B)\colon A,B \in \mathbb{C}_m^{\mathsf{H}},\ A+B \in \mathbb{C}_m^{\geq},\ \mathcal{R}(A) \subseteq \mathcal{R}(A+B),\ \mathcal{R}(B) \subseteq \mathcal{R}(A+B)\}. \tag{8}$$

If $A, B \in \mathbb{C}_m^{\geq}$, then clearly $(A,B) \in \mathcal{S}$, but not the other way around. A simple counterexample is provided by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix},$$

which form a pair contained in $\mathcal{S}$ although neither of them is nonnegative definite. This shows that establishing the result under the assumption $(A,B) \in \mathcal{S}$ instead of $A, B \in \mathbb{C}_m^{\geq}$ strengthens the statement in Problem 31-3 essentially.

Given $(A,B) \in \mathcal{S}$, the matrices $M \in \mathbb{C}_{nm,(n+1)m}$ and $M_0 \in \mathbb{C}_{nm}^{\geq}$ are specified as

$$M = \begin{pmatrix} A & B & & \\ & \ddots & \ddots & \\ & & A & B \end{pmatrix} \quad \text{and} \quad M_0 = \begin{pmatrix} A+B & & \\ & \ddots & \\ & & A+B \end{pmatrix},$$

where all blanks are null matrices. It is clear that if $n = 1$, then $\mathcal{R}(M) = \mathcal{R}(M_0)$ reduces to the equality $\mathcal{R}((A \quad B)) = \mathcal{R}(A+B)$, whose validity is a simple consequence of the assumption $(A,B) \in \mathcal{S}$. In the proof for $n \geq 2$ we will refer to the following auxiliary result, which seems to be also of independent interest.

LEMMA. *Let $(A,B) \in \mathcal{S}$ and let $x_i \in \mathbb{C}_{m,1}$. Then the set of equations*

$$Ax_1 = 0,\ Ax_i + Bx_{i-1} = 0 \text{ for } i = 2, \ldots, n \tag{9}$$

is satisfied if and only if

$$Ax_i = 0 \text{ for } i = 1, \ldots, n \quad \text{and} \quad Bx_i = 0 \text{ for } i = 1, \ldots, n-1. \tag{10}$$

PROOF. The sufficiency is obvious and the necessity is proved by the principle of mathematical induction.

Notice that the two inclusions in the definition of $\mathcal{S}$ in (8) are equivalent to

$$A(A+B)^\dagger(A+B) = A \quad \text{and} \quad B(A+B)^\dagger(A+B) = B. \tag{11}$$

Actually, these conditions are necessary and sufficient for the parallel summability of Hermitian A and B; cf. Rao and Mitra (1971, p. 189). If $n = 2$, then the set (9) reduces to

$$Ax_1 = 0,\ Ax_2 + Bx_1 = 0. \tag{12}$$

Hence

$$(A + B)x_1 = -Ax_2 \tag{13}$$

and, on account of the first parts of (11) and (12), premultiplying (13) by $A(A + B)^\dagger$ leads to $A(A + B)^\dagger Ax_2 = 0$. Under the assumption $(A, B) \in \mathcal{S}$, which in particular implies that $A \in \mathbb{C}^{\mathsf{H}}_m$ and $A + B \in \mathbb{C}^{\geq}_m$, this equation simplifies to $Ax_2 = 0$, and then the second equation in (12) entails $Bx_1 = 0$, thus completing (10).

Now assume that $n \geq 3$ and that the statement in the lemma is valid for $i = 1, ..., n - 1$. Then it follows that $Ax_{n-1} = 0$, and combining this equation with $Ax_n + Bx_{n-1} = 0$ leads to the analogue of (12) with the subscripts "1" and "2" replaced by "$n - 1$" and "n", respectively. Consequently, the same arguments as above show that $Ax_n = 0$ and $Bx_{n-1} = 0$, which concludes the proof. □

THEOREM. *Let $(A, B) \in \mathcal{S}$ and let M and M_0 be the matrices specified in* (2)*. Then $\mathcal{R}(M) = \mathcal{R}(M_0)$.*

PROOF. The equality $\mathcal{R}(M) = \mathcal{R}(M_0)$ can be established quite simply by transforming it into the form $\mathcal{N}(M^*) = \mathcal{N}(M_0^*)$. Let $x \in \mathbb{C}_{nm,1}$ and let $x_i \in \mathbb{C}_{m,1}$, $i = 1, ..., n$, be the successive subvectors of x. Then

$$x \in \mathcal{N}(M^*) \ \Leftrightarrow\ Ax_1 = 0,\ Ax_i + Bx_{i-1} = 0 \text{ for } i = 2, ..., n,\ Bx_n = 0,$$

and hence, on account of the lemma above, $x \in \mathcal{N}(M^*)$ if and only if

$$Ax_i = 0,\ Bx_i = 0 \text{ for } i = 1, ..., n. \tag{14}$$

On the other hand,

$$x \in \mathcal{N}(M_0^*) \ \Leftrightarrow\ (A + B)x_i = 0 \text{ for } i = 1, ..., n. \tag{15}$$

In view of (11), premultiplying the conditions on the right-hand side of (15) first by $A(A + B)^\dagger$ and then by $B(A + B)^\dagger$ shows that the equalities in (14) are necessary and sufficient also for $x \in \mathcal{N}(M_0^*)$, thus completing the proof. □

Reference

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and Its Applications.* Wiley, New York.

Solution 31-3.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Let $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote range and nullspace respectively. We assume only that

$$\mathcal{N}(A^* + B^*) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*), \tag{16}$$

which holds if A and B are nonnegative definite. Let $\mathcal{N}_0 = \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$ and $\ell = \dim(\mathcal{N}_0)$.

Suppose $A, B \in \mathbb{C}^{m\times m}$. Let

$$U_n = \begin{pmatrix} A & B & & \\ & \ddots & \ddots & \\ & & A & B \end{pmatrix}_{n\times(n+1)} \quad \text{and} \quad V_n = \begin{pmatrix} A+B & & \\ & \ddots & \\ & & A+B \end{pmatrix}_{n\times n}.$$

If $x, y \in \mathbb{C}^m$ then $Ax \perp \mathcal{N}(A^*)$ and $By \perp \mathcal{N}(B^*)$, so $(Ax + By) \perp (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))$. This and (16) imply that $(Ax + By) \perp \mathcal{N}(A^* + B^*)$, so $Ax + By \in \mathcal{R}(A + B)$. Therefore $\mathcal{R}(U_n) \subset \mathcal{R}(V_n)$. To complete the proof we will show by induction that $\text{nullity}(U_n^*) = n\ell$, which implies that $\text{rank}(U_n) = \text{rank}(U_n^*) = (m - \ell)n = \text{rank}(V_n)$.

Note that $\text{nullity}(U_n^*) = n\ell$ if and only if $\mathcal{N}(U_n^*)$ is the set of vectors $(\, z_1^* \ \cdots \ z_n^* \,)^*$ such that $z_i \in \mathcal{N}_0$, $1 \leq i \leq n$.

Clearly $U_1^* z = 0$ if and only if $z \in \mathcal{N}_0$; hence $\text{nullity}(U_1^*) = \ell$. Now suppose $n > 1$ and $\text{nullity}(U_{n-1}^*) = (n - 1)\ell$. We note that $U_n^* (\, z_1^* \ \cdots \ z_n^* \,)^* = 0$ if and only if

$$A^* z_1 = 0, \quad B^* z_{i-1} + A^* z_i = 0, \quad 2 \leq i \leq n, \quad \text{and} \quad B^* z_n = 0. \tag{17}$$

Let $\zeta_i = z_{i+1} + \cdots + z_n$, $1 \le i \le n-1$. Summing the equalities in (17) shows that $(A^* + B^*)(z_1 + \zeta_1) = 0$, so (16) implies that $z_1 + \zeta_1 \in \mathcal{N}_0$. Since $A^* z_1 = 0$, it follows that $A^* \zeta_1 = 0$. If $2 \le i \le n-1$, then summing the last $n-i+1$ equalities in (17) yields $B^*\zeta_i + A^*\zeta_{i+1} = 0$. Since $\zeta_{n-1} = z_n$, the last equality in (17) is equivalent to $B^*\zeta_{n-1} = 0$. Thus, $U_{n-1}^* \begin{pmatrix} \zeta_1^* & \cdots & \zeta_{n-1}^* \end{pmatrix}^* = 0$, so the induction assumption implies that $\zeta_i \in \mathcal{N}_0$, $1 \le i \le n-1$. Since $\zeta_{n-1} = z_n$, a simple repetitive argument shows that z_n, $z_{n-1}, \ldots, z_2 \in \mathcal{N}_0$. Then the first two equalities in (17) imply that $z_1 \in \mathcal{N}_0$, so nullity$(U_n^*) = n\ell$, which completes the induction.

Solutions to Problem 31-3 were also received from Leo Livshits *and from the Proposer* Yongge Tian.

Problem 31-4: Two Equalities for Ideals Generated by Idempotents

Proposed by Yongge Tian, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca

Let R be a ring with unity 1 and let $a,\ b \in R$ be two idempotents, i.e., $a^2 = a$ and $b^2 = b$. Show that

$$(\,ab - ba\,)R = (\,a - b\,)R \cap (\,a + b - 1\,)R \;\text{ and }\; R(\,ab - ba\,) = R(\,a - b\,) \cap R(\,a + b - 1\,).$$

Solution 31-4.1 by the Proposer Yongge Tian, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca

Let $S = (\,ab - ba\,)R$, $S_1 = (a - b)R$, and $S_2 = (a + b - 1)R$. It is easy to verify that

$$ab - ba = (\,a - b\,)(\,a + b - 1\,) = -(\,a + b - 1\,)(\,a - b\,).$$

Hence

$$(\,ab - ba\,)x = (\,a - b\,)(\,a + b - 1\,)x = (\,a + b - 1\,)(\,b - a\,)x \;\text{ for all }\; x \in R.$$

This equality implies that $S \subseteq S_1$ and $S \subseteq S_2$. Hence $S \subseteq S_1 \cap S_2$. This set inclusion also implies that $S_1 \cap S_2$ is a nonempty set. Suppose $x \in S_1 \cap S_2$. Then x can be represented as

$$x = (\,a - b\,)p = (\,a + b - 1\,)q, \;\text{ where }\; p,\ q \in R. \tag{18}$$

The equality $(\,a - b\,)p = (\,a + b - 1\,)q$ can be written as

$$(\,a - b\,)(\,p + q\,) = (\,2a - 1\,)q. \tag{19}$$

Since $a^2 = a$, it follows that $(\,2a - 1\,)^2 = 1$. This implies that $2a - 1$ is invertible and $(\,2a - 1\,)^{-1} = 2a - 1$. In this case, q in (19) can be expressed as

$$q = (\,2a - 1\,)^{-1}(\,a - b\,)(\,p + q\,) = (\,2a - 1\,)(\,a - b\,)(\,p + q\,). \tag{20}$$

Also note that $(\,2a - 1\,)(\,a - b\,) = (\,a - b\,)(\,1 - 2b\,) = a - 2ab + b$. Hence q in (20) takes the form

$$q = (\,2a - 1\,)(\,a - b\,)(\,p + q\,) = (\,a - b\,)(\,1 - 2b\,)(\,p + q\,).$$

Substituting this q into (18) gives

$$x = (\,a + b - 1\,)q = (\,a + b - 1\,)(\,a - b\,)(\,1 - 2b\,)(\,p + q\,) = (\,ab - ba\,)(\,2b - 1\,)(\,p + q\,) \in S.$$

This implies that $S_1 \cap S_2 \subseteq S$. Thus $S_1 \cap S_2 = S$. The equality $R(\,ab - ba\,) = R(\,a - b\,) \cap R(\,a + b - 1\,)$ can be shown similarly.

Solution 31-4.2 by William F. Trench, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

It is straightforward to verify that

$$(a + b - 1)(a - b) = ba - ab, \quad (a - b)(a + b - 1) = ab - ba \tag{21}$$

and

$$(a + b - 1)^2 + (a - b)^2 = 1. \tag{22}$$

From (21),

$$(ab-ba)R \subset (a-b)R \cap (a+b-1)R \quad \text{and} \quad R(ab-ba) \subset R(a-b) \cap R(a+b-1). \tag{23}$$

Now suppose that $x \in (a-b)R \cap (a+b-1)R$, i.e., $x = (a-b)r_1 = (a+b-1)r_2$. Then (22) implies that

$$x = (a+b-1)^2x + (a-b)^2x = (a+b-1)^2(a-b)r_1 + (a-b)^2(a+b-1)r_2. \tag{24}$$

However, from (21),

$$\begin{aligned}(a+b-1)^2(a-b) &= (a+b-1)(a+b-1)(a-b) = -(a+b-1)(a-b)(a+b-1)\\ &= (ab-ba)(a+b-1)\end{aligned}$$

and

$$\begin{aligned}(a-b)^2(a+b-1) &= (a-b)(a-b)(a+b-1) = -(a-b)(a+b-1)(a-b)\\ &= -(ab-ba)(a-b).\end{aligned}$$

From (24) it follows that

$$x = (ab-ba)((a+b-1)r_1 - (a-b)r_2) \in (ab-ba)R,$$

which implies the first of the inclusions

$$(a-b)R \cap (a+b-1)R \subset (ab-ba)R \quad \text{and} \quad R(a-b) \cap R(a+b-1)R \subset R(ab-ba). \tag{25}$$

Similar arguments yield the second inclusion. Now (23) and (25) imply the conclusion.

Problem 31-5: A Norm Inequality for the Commutator $AA^* - A^*A$

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Let A be a square matrix and let A^* and $A^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse of A, respectively. A well-known result asserts that $AA^* = A^*A$ if and only if $AA^\dagger = A^\dagger A$ and $A^*A^\dagger = A^\dagger A^*$, that is, A is normal if and only if A is both EP and star-dagger. Show that in general

$$||\, AA^* - A^*A \,|| \leqslant ||A||^2 (\, 2||\, AA^\dagger - A^\dagger A \,|| + ||\, A^*A^\dagger - A^\dagger A^* \,||\,),$$

where $||\cdot||$ denotes the spectral norm of a matrix. This inequality shows that if $A^*A^\dagger - A^\dagger A^* \to 0$, $AA^\dagger - A^\dagger A \to 0$, and A is bounded, then $AA^* - A^*A \to 0$.

Solution 31-5.1 by the Proposers Yongge TIAN, *Queen's University, Kingston, Canada:* ytian@mast.queensu.ca
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It is easy to verify that

$$AA^*(\, AA^\dagger - A^\dagger A\,) = AA^* - AA^*A^\dagger A,$$

$$(\, AA^\dagger - A^\dagger A\,)A^*A = AA^\dagger A^*A - A^*A,$$

$$A(A^*A^\dagger - A^\dagger A^*)A = AA^*A^\dagger A - AA^\dagger A^*A.$$

Hence

$$AA^* - A^*A = AA^*(\, AA^\dagger - A^\dagger A\,) + (\, AA^\dagger - A^\dagger A\,)A^*A + A(A^*A^\dagger - A^\dagger A^*)A.$$

Taking the spectral norm on both sides of the above equality and noting that $||AA^*|| = ||A||^2$ gives

$$||AA^* - A^*A|| \leqslant 2||A||^2\, ||\, AA^\dagger - A^\dagger A\,|| + ||A||^2\, ||\, A^*A^\dagger - A^\dagger A^*\,||,$$

as required.

Solution 31-5.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If $A = 0$, the assertion is trivial, so we assume that $A \neq 0$. Let $A = PSQ^*$ be a singular value decomposition of A and define $\Omega = P^*Q$. Then $A^* = QSP^*$ and $A^\dagger = QS^\dagger P^*$, so

$$AA^* - A^*A = PUQ^* \quad \text{with} \quad U = S^2\Omega - \Omega S^2,$$

$$AA^\dagger - A^\dagger A = PVQ^* \quad \text{with} \quad V = SS^\dagger\Omega - \Omega S^\dagger S,$$

and

$$A^*A^\dagger - A^\dagger A^* = QWP^* \quad \text{with} \quad W = S\Omega S^\dagger - S^\dagger\Omega S.$$

Hence,

$$\|A\| = \|S\|, \|AA^* - A^*A\| = \|U\|, \|AA^\dagger - A^\dagger A\| = \|V\|, \text{ and } \|A^*A^\dagger - A^\dagger A^*\| = \|W\|. \tag{26}$$

If $\operatorname{rank}(A) = n$, then $S^\dagger = S^{-1}$, $V = 0$, and $U = SWS$, so $\|U\| \leq \|S\|^2\|W\|$ and (26) implies the assertion. If $\operatorname{rank}(A) = k < n$, let $\Sigma = \operatorname{diag}(\sigma_1(A), \ldots, \sigma_k(A))$ with $\sigma_1(A) \geq \cdots \geq \sigma_k(A) > 0$. Then we may assume that

$$S = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad S^\dagger = \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \Omega = \begin{pmatrix} \Phi & X \\ Y & \Psi \end{pmatrix}$$

with $\Phi \in \mathbb{C}^{k\times k}$. Routine computations yield

$$U = \begin{pmatrix} \Sigma^2\Phi - \Phi\Sigma^2 & \Sigma^2 X \\ -Y\Sigma^2 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & X \\ -Y & 0 \end{pmatrix}, \quad W = \begin{pmatrix} \Sigma\Phi\Sigma^{-1} - \Sigma^{-1}\Phi\Sigma & 0 \\ 0 & 0 \end{pmatrix},$$

and $U = SWS + S^2V + VS^2$. Hence $\|U\| \leq \|S\|^2(2\|V\| + \|W\|)$, and (26) implies the assertion.

Problem 31-6: A Full Rank Factorization of a Skew-Symmetric Matrix
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Determine a full rank factorization of the matrix

$$C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

with real entries $c_i, i = 1, 2, 3$. (Observe that for $x = (x_1, x_2, x_3)' \in \mathbb{R}^3$ the identity $Cx = c \times x$, where $c = (c_1, c_2, c_3)'$, defines the vector cross product in $\mathbb{R}^3$.)

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Let $\mathcal{R}(\cdot)$ and $\mathrm{r}(\cdot)$ denote the range and rank of a given matrix, respectively. It is easily seen that if C is of the form given in Problem 31-6, then $\mathrm{r}(C) = 2$ except only for the trivial case where $c_1 = c_2 = c_3 = 0$, which is excluded from further considerations. The problem consists, therefore, in specifying 3×2 real matrices A and 2×3 real matrices B such that $\mathrm{r}(A) = \mathrm{r}(B) = 2$ and $C = AB$, in which case $\mathcal{R}(C) = \mathcal{R}(A)$ and $\mathcal{R}(C') = \mathcal{R}(B')$. There are infinitely many choices of such matrices. In our solution we provide representations of complete sets of them referring to the property that if some A_0 and B_0 satisfy the conditions above, then all desired pairs (A, B) can be expressed as (A_0M^{-1}, MB_0) with M varying freely over the set of all nonsingular matrices of order 2. Indeed, it is trivially seen that if $A = A_0M^{-1}$ and $B = MB_0$, then $AB = A_0M^{-1}MB_0 = A_0B_0 = C$. Conversely, from $\mathcal{R}(B') = \mathcal{R}(B_0')$ it follows that $B = MB_0$ for some 2×2 matrix M, which on account of $\mathrm{r}(B) = 2$ must be nonsingular. Then the equality $AB = A_0B_0$ takes the form $AMB_0 = A_0B_0$, and hence, in view of the fact that B_0 is of full row rank, $AM = A_0$ or, equivalently, $A = A_0M^{-1}$, thus concluding a proof of the property formulated above.

The procedure of constructing A_0 and B_0 proposed by us, which seems to be among the simplest possible, can be described as follows: under the assumption that $c_i \neq 0$ for a fixed $i \in \{1, 2, 3\}$ choose A_0 as the submatrix of C consisting of these two columns (jth and kth, say, where $j < k$) which contain entries c_i or $-c_i$ and then take B_0 as the matrix having the transpose of the kth column of A_0 multiplied by $(-1)^i c_i^{-1}$ and the transpose of the jth column of A_0 multiplied by $(-1)^{i+1}c_i^{-1}$ as its first and second rows. This procedure leads to the factorizations

$$C = \begin{pmatrix} -c_3 & c_2 \\ 0 & -c_1 \\ c_1 & 0 \end{pmatrix} \begin{pmatrix} -c_2/c_1 & 1 & 0 \\ -c_3/c_1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c_2 \\ c_3 & -c_1 \\ -c_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -c_1/c_2 & 0 \\ 0 & -c_3/c_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -c_3 \\ c_3 & 0 \\ -c_2 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -c_1/c_3 \\ 0 & 1 & -c_2/c_3 \end{pmatrix},$$

which are valid when $c_1 \neq 0$, $c_2 \neq 0$, and $c_3 \neq 0$, respectively.

Representing the set of 2×2 real nonsingular matrices as

$$\{M = \begin{pmatrix} s & t \\ u & v \end{pmatrix} : w = sv - tu \neq 0\}$$

and noting that any such M has the inverse expressible as

$$M^{-1} = (1/w) \begin{pmatrix} v & -t \\ -u & s \end{pmatrix},$$

we can summarize our considerations in the following form.

THEOREM. *Matrices A and B provide a full rank factorization $C = AB$ of*

$$C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

where c_1, c_2, c_3 are any real numbers with at least one of them being nonzero, if and only if

$$A = \begin{pmatrix} -(uc_2 + vc_3)/w & (sc_2 + tc_3)/w \\ uc_1/w & -sc_1/w \\ vc_1/w & -tc_1/w \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -(sc_2 + tc_3)/c_1 & s & t \\ -(uc_2 + vc_3)/c_1 & u & v \end{pmatrix} \quad \text{whenever } c_1 \neq 0,$$

$$A = \begin{pmatrix} -uc_2/w & sc_2/w \\ (uc_1 + vc_3)/w & -(sc_1 + tc_3)/w \\ -vc_2/w & tc_2/w \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} s & -(sc_1 + tc_3)/c_2 & t \\ u & -(uc_1 + vc_3)/c_2 & v \end{pmatrix} \quad \text{whenever } c_2 \neq 0,$$

$$A = \begin{pmatrix} uc_3/w & -sc_3/w \\ vc_3/w & -tc_3/w \\ -(uc_1 + vc_2)/w & (sc_1 + tc_2)/w \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} s & t & -(sc_1 + tc_2)/c_3 \\ u & v & -(uc_1 + vc_2)/c_3 \end{pmatrix} \quad \text{whenever } c_3 \neq 0,$$

where the choice of real numbers s, t, u, v is restricted merely by the condition that the difference $w = sv - tu$ is nonzero.

Solution 31-6.2 by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England:* R.W.Farebrother@man.ac.uk

If c_1, c_2, or c_3 are nonzero then C has a nontrivial full rank factorization. in particular, if $c_3 \neq 0$ then C may be written as

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c_3 \\ c_3 & 0 \\ -c_2 & c_1 \end{pmatrix} \begin{pmatrix} 0 & -c_3 \\ c_3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \end{pmatrix}$$

where all three matrices on the right have rank 2 as they each contain the same 2×2 nonsingular matrix.

Similar expressions are available for $c_1 \neq 0$ and for $c_2 \neq 0$, but if $c_1 = c_2 = c_3 = 0$ then $C = 0$ is null and has a trivial full rank factorization.

Solution 31-6.3 by Lajos LÁSLÓ, *Eötvös Loránd University, Budapest, Hungary:* laszlo@numanal.inf.elte.hu

The rank of C is 2, except when all three c_i's vanish. So $C = ab' - ba'$ for some vectors a and b, with $(\cdot)'$ indicating the transpose of $(\cdot)$. If $c_1 = 0$, then $a' = (1 \quad 0 \quad 0)$, $b' = (0 \quad -c_3 \quad c_2)$, else $a' = (-\frac{c_2}{c_1} \quad 1 \quad 0)$, $b' = (c_3 \quad 0 \quad -c_1)$.

Solution 31-6.4 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

We assume that $C \neq 0$. It is straightforward to verify that if $a, b \in \mathbb{R}^3$, then

$$C = (a \quad b) \begin{pmatrix} -b^T \\ a^T \end{pmatrix}. \tag{27}$$

if and only if $a \times b = c$. Moreover, we will show that any full rank factorization

$$C = (x \quad y) \begin{pmatrix} u^T \\ v^T \end{pmatrix} = xu^T + yv^T \tag{28}$$

can be rewritten as in (27). Since $C^T = -C$ and $Cc = 0$, (28) implies that $(u^T c)x + (v^T c)y = -(x^T c)u - (y^T c)v = 0$. Since $\{x, y\}$ and $\{u, v\}$ are both linearly independent sets, it follows that x, y, u, and v are all perpendicular to c. Hence, (28) can be rewritten as

$$C = (x \quad y) \begin{pmatrix} c_1 x^T + c_2 y^T \\ k_1 x^T + k_2 y^T \end{pmatrix} = c_1 xx^T + c_2 xy^T + k_1 yx^T + k_2 yy^T.$$

Therefore,

$$C = -C^T = -c_1 xx^T - c_2 yx^T - k_1 xy^T - k_2 yy^T,$$

so $C = (c_2 - k_1)(xy^T - yx^T)/2$, which implies (27) with $a = (c_2 - k_1)x/2$ and $b = y$.

In particular, if a is a unit vector perpendicular to c, then $a \times (c \times a) = c$, so

$$C = (a \quad c \times a) \begin{pmatrix} -(c \times a)^T \\ a^T \end{pmatrix}.$$

Solution 31-6.5 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

If all c_i are zero, such a decomposition is trivial. Let now $c = (c_1, c_2, c_3)' \neq 0$. Since $\dim \mathcal{N}(c') = 2$, where $\mathcal{N}(\cdot)$ denotes the null space, it is possible to choose two nonzero vectors a and b from $\mathbb{R}^3$ such that $a'b = 0$, $a'c = 0$ and $b'c = 0$. The 3×2 matrix $A = (a : b)$ is of full column rank with Moore-Penrose inverse

$$A^+ = \begin{pmatrix} a^+ \\ b^+ \end{pmatrix}.$$

Let now $B = A^+ C$, i.e.

$$B = \begin{pmatrix} a^+ C \\ b^+ C \end{pmatrix}.$$

It is easy to verify that the rows of the 2×3 matrix B are linearly dependent and $AB = C$. For the latter identity note that $\mathcal{R}(C) = \mathcal{N}(c') = \mathcal{R}(a) \oplus \mathcal{R}(b)$, and $aa^+ + bb^+$ is the orthogonal projector on $\mathcal{R}(a) \oplus \mathcal{R}(b)$ with $\mathcal{R}(\cdot)$ being the column space of a matrix. Hence $C = AB$ is the desired full rank decomposition.

A solution to Problem 31-6 was also received from Julio Benítez *and* Néstor Thome.

Problem 31-7: On the Product of Orthogonal Projectors

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be orthogonal projectors of the same order with complex entries and let A denote their product. Show that the following conditions are equivalent:

(i) A is an orthogonal projector, i.e. $A = AA^*$,

(ii) A is Hermitian, i.e. $A = A^*$,

(iii) A is normal, i.e. $AA^* = A^*A$,

(iv) A is EP, i.e. $AA^+ = A^+A$,

(v) A is bi-EP, i.e. $AA^+A^+A = A^+AAA^+$,

(vi) A is bi-normal, i.e. $AA^*A^*A = A^*AAA^*$,

(vii) A is bi-dagger, i.e. $(A^+)^2 = (A^2)^+$.

Solution 31-7.1 by Jerzy K. BAKSALARY, *Zielona Góra University, Zielona Góra, Poland:* J.Baksalary@im.uz.zgora.pl
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Let $\mathbb{C}_{n,n}$ be the set of all $n \times n$ complex matrices and let $\mathbb{C}_n^{\mathsf{OP}}$ denote the subset of $\mathbb{C}_{n,n}$ consisting of orthogonal projectors, i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{A \in \mathbb{C}_{n,n}: A = A^2 = A^*\} = \{A \in \mathbb{C}_{n,n}: A = AA^*\} = \{A \in \mathbb{C}_{n,n}: A = A^*A\}$$
$$= \{A \in \mathbb{C}_{n,n}: A = A^\dagger A\} = \{A \in \mathbb{C}_{n,n}: A = AA^\dagger\},$$

where A^* and $A^\dagger$ stand for the conjugate transpose and the Moore-Penrose inverse of A, respectively. It is easily seen that if both $P \in \mathbb{C}_{n,n}$ and $Q \in \mathbb{C}_{n,n}$ are projectors (i.e., idempotent matrices), then the equality $PQ = QP$ is sufficient for the products PQ and QP to be projectors as well. This commutativity condition becomes also necessary when $P, Q \in \mathbb{C}_n^{\mathsf{OP}}$; cf., e.g., Baksalary (1987, Theorem 1) and Ben-Israel and Greville (2003, p. 80). Hence it follows that the statements (i) and (ii) in Problem 31-7 are equivalent, and thus the proof can be reduced to establishing the mutual equivalence between the conditions (ii)–(vii). In the solution proposed below, this list of six conditions is extended by five additional ones. A motivation for introducing them is provided in the last part of our considerations.

THEOREM. *Let $P, Q \in \mathbb{C}_n^{\mathsf{OP}}$ and let $A = PQ$. Then $A \in \mathbb{C}_n^{\mathsf{OP}}$ if and only if any of the following equivalent conditions is fulfilled:*

$$\text{(a) } A = A^*, \quad \text{(b) } AA^* = A^*A, \quad \text{(c) } (AA^*)(A^*A) = (A^*A)(AA^*),$$
$$\text{(d) } A = A^\dagger, \quad \text{(e) } AA^\dagger = A^\dagger A, \quad \text{(f) } (AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger),$$
$$\text{(g) } (AA^*)(A^\dagger A) = (A^\dagger A)(AA^*), \quad \text{(h) } (AA^\dagger)(A^*A) = (A^*A)(AA^\dagger),$$
$$\text{(i) } (A^2)^\dagger = (A^\dagger)^2, \quad \text{(j) } \mathcal{R}[A(A^*)^2] \subseteq \mathcal{R}(A^*), \quad \text{(k) } \mathcal{R}(A^*A^2) \subseteq \mathcal{R}(A),$$

where $\mathcal{R}(.)$ in (j) *and* (k) *denotes the range (column space) of a given matrix.*

PROOF. It is trivially seen that (a) $\Rightarrow$ (b), (c), (j), (k), that (d) $\Rightarrow$ (e), (f), (i), and that (a), (d) $\Rightarrow$ (g), (h). Moreover, it is known that

$$(PQ)^\dagger P = (PQ)^\dagger = Q(PQ)^\dagger \quad \text{and} \quad P(QP)^\dagger = (QP)^\dagger = (QP)^\dagger Q. \tag{29}$$

The first of these equalities further leads to

$$PA^\dagger A = PQ(PQ)^\dagger PQ = PQ = A, \quad AA^\dagger Q = PQ(PQ)^\dagger PQ = PQ = A, \tag{30}$$

and

$$(A^\dagger)^2 = (PQ)^\dagger PQ(PQ)^\dagger = (PQ)^\dagger = A^\dagger. \tag{31}$$

From (29) it follows that if (a) holds, i.e., if $PQ = QP$, then

$$(PQ)^\dagger = Q(PQ)^\dagger P = Q(QP)^\dagger P = QP(QP)^\dagger QP = QP = PQ,$$

which is (d). Conversely, if (d) holds, i.e., if $PQ = (PQ)^\dagger$, then

$$PQ = Q(PQ)^\dagger = QPQ,$$

and hence $PQ = QP$. Consequently, (d) $\Leftrightarrow$ (a) and therefore the proof reduces to establishing that each of the conditions (b), (c), and (e)–(k) implies the commutativity of P and Q.

It is clear that

$$\text{(b)} \;\Rightarrow\; \text{(c)} \;\Rightarrow\; PQPQPQ = QPQPQP. \tag{32}$$

Further, on account of (31), the matrix product on the left-hand side of (f) is actually equal to $AA^\dagger A$, i.e., to A. Consequently, premultiplying and postmultiplying (f) by A leads to $A^3 = A^2$, thus showing that

$$\text{(e)} \;\Rightarrow\; \text{(f)} \;\Rightarrow\; PQPQPQ = PQPQ. \tag{33}$$

In view of (29) and (30),

$$A^*A^\dagger = QPQ(PQ)^\dagger = (AA^\dagger Q)^* = A^* \quad \text{and} \quad A^\dagger A^* = (PQ)^\dagger PQP = (PA^\dagger A)^* = A^*.$$

These relationships enable to reexpress the conditions (g) and (h) in the forms $AA^*A = A^\dagger A^2 A^*$ and $AA^*A = A^*A^2A^\dagger$, respectively. Premultiplying by A in the first case and postmultiplying by A in the second leads to

$$\text{(g)} \;\Rightarrow\; PQPQPQ = PQPQP \quad \text{and} \quad \text{(h)} \;\Rightarrow\; PQPQPQ = QPQPQ. \tag{34}$$

By referring again to (31) it is seen that (i) is equivalent to $(A^2)^\dagger = A^\dagger$, and hence, due to the uniqueness of the Moore-Penrose inverse,

$$\text{(i)} \;\Rightarrow\; PQPQ = PQ. \tag{35}$$

Finally, since the orthogonal projectors onto $\mathcal{R}(A^*)$ and $\mathcal{R}(A)$ admit the representations $A^\dagger A$ and $AA^\dagger$, it follows that the inclusions (j) and (k) can be replaced by the equalities $A^\dagger A^2(A^*)^2 = A(A^*)^2$ and $AA^\dagger A^*A^2 = A^*A^2$, respectively. Premultiplying the first of them by P, postmultiplying the conjugate transpose of the second by Q, and applying (30) yields $A^2(A^*)^2 = A(A^*)^2$ and $(A^*)^2A^2 = (A^*)^2A$, which means that

$$\text{(j)} \;\Rightarrow\; PQPQPQP = PQPQP \quad \text{and} \quad \text{(k)} \;\Rightarrow\; QPQPQPQ = QPQPQ. \tag{36}$$

Part (a) $\Leftrightarrow$ (b) of Theorem in Baksalary, Baksalary, and Szulc (2002), which generalizes Lemma in Baksalary and Baksalary (2002), asserts that any product composed with orthogonal projectors P and Q is equal to another such product if and only if P and Q commute. Consequently, from the equalities in (32)–(36) it is immediately seen that every condition involved therein implies (a), as desired. □

We supplement our solution by pointing out that matrices $A \in \mathbb{C}_{n,n}$ satisfying $(AA^*)(A^*A) = (A^*A)(AA^*)$ and $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$, called in the statement of Problem 31-7 bi-normal and bi-EP, are in Baksalary, Baksalary, and Liu (2004) referred to as weakly normal and weakly EP. Further, it is seen that the equalities (g) and (h), which have been added to the original list, can be viewed as specific modifications of the conditions in (c) and (f) defining $A \in \mathbb{C}_{n,n}$ to be bi-normal (weakly normal) and bi-EP (weakly EP), respectively. Finally, it is clear that the condition (i) actually expresses the reverse order law for the Moore-Penrose inverse of the product AA. According to Greville (1966) [see also Ben-Israel and Greville (2003, p. 160)], this law is equivalent to the conjunction of the inclusions (j) and (k), while our theorem shows that in the particular case, where A is a product of two orthogonal projectors, these inclusions are mutually equivalent and thus each of them is necessary and sufficient for $(AA)^\dagger = A^\dagger A^\dagger$.

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Solution 31-7.2 by Jerzy K. BAKSALARY and Anna KUBA, *Zielona Góra University, Zielona Góra, Poland:*

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Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. For a given $K \in \mathbb{C}_{m,n}$, the symbols K^* and $K^\dagger$ denote the conjugate transpose and Moore-Penrose inverse of K, respectively. Moreover, $K \in \mathbb{C}_{n,n}$ is called orthogonal projector whenever $K = K^2 = K^*$ or, equivalently, $K = KK^*$ or, in still another version, $K = KK^\dagger$. The main tool used in our solution is the following compilation of Theorem 1 and Lemma 1 given by Groß (1999).

LEMMA. *Let $A = PQ$, where $P \in \mathbb{C}_{n,n}$ and $Q \in \mathbb{C}_{n,n}$ are orthogonal projectors. Then there exists a unitary $U \in \mathbb{C}_{n,n}$ such that*

$$A = U \begin{pmatrix} D & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix} U^*, \tag{37}$$

where $D \in \mathbb{C}_{n_1,n_1}$ is a diagonal matrix with the diagonal entries d_{jj} $(j = 1, ..., n_1)$ in the open interval $(0, 1)$ which satisfies the equation $D - D^2 = XX^*$, while the subscripted I denotes the identity matrix of the indicated order. Furthermore, if A is of the form (37)*, then its Moore-Penrose inverse has the representation*

$$A^\dagger = U \begin{pmatrix} D^{-1}Y & 0 & 0 \\ S & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix} U^*, \tag{38}$$

where $S = [I_{n_2} + (D^{-1}X)^* D^{-1}X]^{-1}(D^{-1}X)^* D^{-1} = (I_{n_2} + X^* D^{-2} X)^{-1} X^* D^{-2}$ and $Y = I_{n_1} - XS$. In (37) *and* (38)*, $n = n_1 + n_2 + n_3$ with $0 \leq n_i \leq n$ $(i = 1, 2, 3)$ and the submatrices in the ith row and column being absent when $n_i = 0$.*

We will employ this lemma for solving a generalized version of Problem 31-7, with generalizations which consist in replacing the concepts of bi-EP, bi-normal, and bi-dagger matrices by m-EP, m-normal, and m-dagger matrices, respectively, and in referring additionally to the concepts of idempotent and m-potent matrices.

THEOREM. *Let $A = PQ$, where $P \in \mathbb{C}_{n,n}$ and $Q \in \mathbb{C}_{n,n}$ are orthogonal projectors, and let m be an integer not less than 2. Then the following statements are equivalent:*

(a) *A is an orthogonal projector,*
(b) *A is idempotent, i.e., $A = A^2$,*
(c) *A is Hermitian, i.e., $A = A^*$,*
(d) *A is normal, i.e., $AA^* = A^*A$,*
(e) *A is EP, i.e.,* $\text{range}(A) = \text{range}(A^*)$ *or, equivalently, $AA^\dagger = A^\dagger A$,*
(f) *A is m-potent, i.e., $A = A^m$,*
(g) *A is m-normal, i.e., $[(AA^*)(A^*A)]^k = [(A^*A)(AA^*)]^k$ when $m = 2k$ and $[(AA^*)(A^*A)]^k(AA^*) = [(A^*A)(AA^*)]^k(A^*A)$ when $m = 2k + 1$, where k is a positive integer,*
(h) *A is m-EP, i.e., $[(AA^\dagger)(A^\dagger A)]^k = [(A^\dagger A)(AA^\dagger)]^k$ when $m = 2k$ and $[(AA^\dagger)(A^\dagger A)]^k(AA^\dagger) = [(A^\dagger A)(AA^\dagger)]^k(A^\dagger A)$ when $m = 2k + 1$, where k is a positive integer,*
(i) *A is m-dagger, i.e., $(A^m)^\dagger = (A^\dagger)^m$.*

PROOF. If $n_1 = 0$, then (37) simplifies to

$$A = U \begin{pmatrix} 0 & 0 \\ 0 & I_{n_3} \end{pmatrix} U^*, \tag{39}$$

in which case A trivially satisfies all the conditions (a)–(i). Further, consider the case where $n_1 > 0$, but $n_2 = 0$. Then (37) reduces to

$$A = U \begin{pmatrix} D & 0 \\ 0 & I_{n_3} \end{pmatrix} U^*,$$

which is nonsingular, thus implying that both P and Q must also be nonsingular. Since the only nonsingular idempotent matrix of order n is I_n, it follows that $P = I_n$, $Q = I_n$, and hence $A = I_n$, which is in a contradiction with the specification of D. Consequently, it is henceforth assumed that if $n_1 > 0$, then neccessarily $n_2 > 0$, the presence or absence of I_{n_3} in (37) and (38) having no influence on further considerations.

It is clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (f). Suppose that A is of the form (37) with $n_1 > 0$. Then the north-west $n_1 \times n_1$ submatrix of U^*A^mU is D^m, and therefore $A = A^m$ entails $D = D^m$. However, since D is required to be diagonal with the diagonal entries $d_{jj} \in (0,1)$, the equality $D = D^m$ cannot be achieved. This observation leads to the conclusion that a representation of A in (37) must be reduced to (39), thus strengthening the chain of implications above to (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (f).

Further, from the condition $D - D^2 = XX^*$ it is seen that if $X = 0$, then $D = D^2$, which is irreconcilable with the assertion that all diagonal entries of D are in $(0,1)$. Hence it is clear that in each case where $X = 0$, the first n_1 rows and columns in the partitioned matrix occurring in (37) must vanish, thus reducing A to the form (39). Since clearly (a) $\Rightarrow$ (d) $\Rightarrow$ (g), proving that (g) $\Rightarrow$ (a) will close this chain. If D and X were present in the representation (37), then it can quite straightforwardly be verified that A would be m-normal if and only if

$$D^{3k-1}X = 0 \tag{40}$$

when $m = 2k$, and if and only if

$$D^{3k+1} = D^{3k+2}, \quad D^{3k+1}X = 0, \quad \text{and} \quad X^*D^{3k}X = 0 \tag{41}$$

when $m = 2k+1$, with k being in both cases a positive integer. According to the lemma above, D is a nonsingular matrix, and thus from the condition (40) as well as from the second condition in (41) it follows immediately that $X = 0$, which forces A to take the desired reduced form (39).

Similarly, since (a) $\Rightarrow$ (c) $\Rightarrow$ (e) $\Rightarrow$ (h), establishing that (h) $\Rightarrow$ (a) will ensure the equivalence of these four conditions. Again, if D and X were present in (37) and, consequently, in (38), then with the notation $W = D^{-1}YD$ and the rule $W^0 = I_{n_1}$ the matrix A would be m-EP if and only if

$$W^{k-1}D^{-1}YX = 0 \quad \text{and} \quad SDW^{k-1} = 0 \tag{42}$$

when $m = 2k$, and if and only if

$$W^k = W^{k+1}, \quad W^kD^{-1}YX = 0, \quad SDW^k = 0, \quad \text{and} \quad SDW^{k-1}D^{-1}YX = 0 \tag{43}$$

when $m = 2k+1$, with k being in both cases a positive integer.

From the specification of S in the lemma it is quite easily seen that

$$I_{n_2} - SX = (I_{n_2} + X^*D^{-2}X)^{-1} \tag{44}$$

and an immediate consequence of (44) is

$$SY = 0 \;\Leftrightarrow\; S = SXS \;\Leftrightarrow\; (I_{n_2} + X^*D^{-2}X)^{-1}S = 0 \;\Leftrightarrow\; S = 0.$$

But $S = 0$ is further equivalent to $X = 0$, which entails the desired reduction of (37) to (39). Consequently, it follows that this part of the proof reduces to showing that $SY = 0$ holds in both cases (42) and (43). If k in (42) is equal to 1, then the second condition therein leads immediately to $S = 0$ (and hence, obviously, to $SY = 0$). In the remaining cases, we utilize the formula

$$SDW^k = SY^kD = (I_{n_2} - SX)^{k-1}SYD, \tag{45}$$

whose validity for any integer $k \geq 1$ can easily be established by the principle of mathematical induction adopting the rule $(I_{n_2} - SX)^0 = I_{n_2}$. Since according to (44) the matrix $(I_{n_2} - SX)^{k-1}$ is nonsingular also for any $k \geq 2$, it follows from (45) that $SDW^k = 0 \Leftrightarrow SY = 0$. In view of the second condition in (42) and the third condition in (43), this observation leads to (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e) $\Leftrightarrow$ (h).

Finally, if A is an orthogonal projector, then $(A^m)^\dagger = [(AA^\dagger)^m]^\dagger = (AA^\dagger)^\dagger = AA^\dagger = (AA^\dagger)^m = (A^\dagger)^m$, and thus the last lacking point in the proof is the implication (i) $\Rightarrow$ (a). It can quite straightforwardly be verified that if A and $A^\dagger$ are of the forms (37) and (38) with $n_1 > 0$ (and thus $n_2 > 0$), then

$$(A^m)^\dagger = U \begin{pmatrix} D^{-1}YD^{-m+1} & 0 & 0 \\ SD^{-m+1} & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix} U^* \quad \text{and} \quad (A^\dagger)^m = U \begin{pmatrix} (D^{-1}Y)^m & 0 & 0 \\ S(D^{-1}Y)^{m-1} & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix} U^*,$$

where S and Y are as specified in the lemma. In such a case, A is m-dagger if and only if

$$D^{-1}YD^{-m+1} = (D^{-1}Y)^m \quad \text{and} \quad SD^{-m+1} = S(D^{-1}Y)^{m-1}. \tag{46}$$

With the use of notation

$$Z = I_{n_1} - D^{-1}XSD = I_{n_1} - D^{-1}X(I_{n_2} + X^*D^{-2}X)^{-1}X^*D^{-1} \tag{47}$$

the matrix $D^{-1}Y$ can be reexpressed as ZD^{-1}, and hence the former condition in (46) takes the form $ZD^{-m} = (ZD^{-1})^m$. On account of the nonsingularity of Z, it can further be transformed to $D^{-m} = D^{-1}(ZD^{-1})^{m-1}$, and hence, by premultiplying and postmultiplying by $D^{1/2}$, to

$$(D^{-1})^{m-1} = (D^{-1/2}ZD^{-1/2})^{m-1}. \tag{48}$$

Since both D^{-1} and $D^{-1/2}ZD^{-1/2}$ are positive definite matrices, it follows from (48) that $D^{-1} = D^{-1/2}ZD^{-1/2}$ or, equivalently, $Z = I_{n_1}$. In view of (47), this is possible if and only if $X = 0$, which concludes the proof. □

Reference

J. Groß (1999). On the product of orthogonal projectors. *Linear Algebra and Its Applications*, **289**, 141–150.

Solution 31-7.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Obviously, (i) implies (ii) and (ii) implies (iii) in general; i.e., without the stated assumption on A. If (iii) holds then $A = \Omega D \Omega^*$ with D diagonal and Ω unitary. Then $A^\dagger = \Omega D^\dagger \Omega^*$, so $AA^\dagger = \Omega D D^\dagger \Omega^*$ and $A^\dagger A = \Omega D^\dagger D \Omega^*$. Therefore, since $D^\dagger D = DD^\dagger$, (iii) implies (iv) in general. Obviously, (iv) implies (v) in general.

Now let $p = \operatorname{rank}(P)$ and $q = \operatorname{rank}(Q)$. Then

$$P = \Omega_P \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \Omega_P^* \quad \text{and} \quad Q = \Omega_Q \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} \Omega_Q^*$$

with Ω_P and Ω_Q unitary. If we write

$$\Omega_P^* \Omega_Q = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

with $X \in \mathbb{C}^{p\times q}$, then

$$XX^* + YY^* = I_p, \tag{49}$$

$$A = PQ = \Omega_P \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \Omega_Q^*, \tag{50}$$

$$A^\dagger = \Omega_Q \begin{pmatrix} X^\dagger & 0 \\ 0 & 0 \end{pmatrix} \Omega_P^*,$$

$$(A^\dagger)^2 = \Omega_Q \begin{pmatrix} X^\dagger X X^\dagger & 0 \\ 0 & 0 \end{pmatrix} \Omega_P^* = \Omega_Q \begin{pmatrix} X^\dagger & 0 \\ 0 & 0 \end{pmatrix} \Omega_P^* = A^\dagger, \tag{51}$$

$$A^2 = \Omega_P \begin{pmatrix} XX^*X & 0 \\ 0 & 0 \end{pmatrix} \Omega_Q^*, \tag{52}$$

and

$$AA^* = \Omega_P \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix} \Omega_P^* \Omega_Q \Omega_Q^* = \Omega_P \begin{pmatrix} XX^*X & XX^*Y \\ 0 & 0 \end{pmatrix} \Omega_Q^*. \tag{53}$$

From (51),

$$AA^\dagger A^\dagger A = AA^\dagger A = A \quad \text{and} \quad A^\dagger AAA^\dagger = (AA^\dagger A^\dagger A)^* = A^*,$$

so (v) implies that $A = A^*$, which implies (vi).

Since $A = PQ$, $P^2 = P$, and $Q^2 = Q$, (vi) implies that $(PQ)^3 = (QP)^3$. Multiplying on the left by P and the right by Q shows that $A^3 = A^4$ or, equivalently, $A^2(A^2 - A) = 0$. Therefore, (50) and (52) imply that if $U = XX^*X - X$, then

$$\Omega_P \begin{pmatrix} XX^*XX^*U & 0 \\ 0 & 0 \end{pmatrix} \Omega_Q^* = 0,$$

so $XX^*XX^*U = 0$. Since, in general, $F^*FG = 0$ implies that $FG = 0$, it follows that $XX^*U = 0$, and therefore that $X^*U = 0$, or, equivalently, $X^*X(X^*X - I_p) = 0$. Therefore $XX^*X = X$, so (50) and (52) imply that $A = A^2$. Hence $A^\dagger = (A^2)^\dagger$, so (51) implies (vii). Thus, (vi) implies (vii).

If (vii) holds then (51) implies that $A = A^2$, so (50) and (52) imply that $XX^*X = X$ and therefore $YY^*X = 0$, from (49). Hence $Y^*X = 0$ and (50) and (53) imply that $A = AA^*$. Thus, (vii) implies (i).

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We show (vi) ⇒ (vii) ⇒ (v) ⇒ (iv) ⇒ (i) ⇒ (ii) ⇒ (iii) ⇒ (vi).

The chain of implications (vi) ⇒ (vii) ⇒ (v) is well-known and does not require the assumption that P and Q are orthogonal projectors [see Hartwig and Spindelböck (1984, p. 246)].

(v) ⇒ (iv): According to Gross (1999, Corollary 1), as a product of orthogonal projectors, the matrix A is similar to a diagonal matrix. Hence we get $\operatorname{rank}(A) = \operatorname{rank}(A^2)$, or equivalently, $\mathcal{R}(A) = \mathcal{R}(A^2)$, where $\mathcal{R}(\cdot)$ denotes the column space of a matrix. Using Theorem 7 from Campbell and Meyer (1975) we find that A is EP.

(iv) ⇒ (i): Consulting again Corollary 1 from Gross (1999) we conclude that A is an orthogonal projector.

The chain of assertions (i) ⇒ (ii) ⇒ (iii) ⇒ (vi) is trivial.

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Solution 31-7.5 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* ujw902@uni-bonn.de

For a complex $m \times n$ matrix C, let C^*, C^+, C^-, $\mathcal{R}(C)$, $\mathcal{N}(C)$, and $P_{\mathcal{R}(C)}$ denote the conjugate transpose, the Moore-Penrose inverse, a g-inverse, the range (column space), the null space, and the orthogonal projector onto $\mathcal{R}(C)$ [along its usual orthogonal complement $\mathcal{R}(C)^\perp = \mathcal{N}(C^*)$], respectively, of C. By $\{C^-\}$ we denote the set of all g-inverses of C. Recall that the orthogonal projector $P_{\mathcal{R}(C)}$ may be defined by $P_{\mathcal{R}(C)}x = x$ if $x \in \mathcal{R}(C)$ and $P_{\mathcal{R}(C)}x = 0$ if $x \in \mathcal{N}(C^*)$. Clearly, $\mathbb{C}^m = \mathcal{R}(C) \oplus \mathcal{N}(C^*)$, with $\oplus$ indicating a direct sum. It is pertinent to mention that any orthogonal projector $P_{\mathcal{R}(C)}$ is Hermitian [i.e., $(P_{\mathcal{R}(C)})^* = P_{\mathcal{R}(C)}$] and idempotent [i.e., $(P_{\mathcal{R}(C)})^2 = P_{\mathcal{R}(C)}$], and that conversely, every idempotent Hermitian matrix P is an orthogonal projector, namely $P = P_{\mathcal{R}(P)}$, i.e., P projects onto $\mathcal{R}(P)$ along its orthogonal complement $\mathcal{R}(P)^\perp = \mathcal{N}(P^*) = \mathcal{N}(P)$. We further recall that $(C^+)^+ = C$, $\mathcal{R}(C^+) = \mathcal{R}(C^*)$ and $\mathcal{N}(C^+) = \mathcal{N}(C^*)$. Since $P_{\mathcal{R}(C)} = CC^+$ and $P_{\mathcal{R}(C^*)} = C^+C$, we also have $\mathcal{R}(CC^+) = \mathcal{R}(C)$, $\mathcal{N}(CC^+) = \mathcal{N}(C^*)$, $\mathcal{R}(C^+C) = \mathcal{R}(C^*)$ and $\mathcal{N}(C^+C) = \mathcal{N}(C)$. We finally recall that $P^+ = P$ holds for any orthogonal projector P.

The following two auxiliary results are useful in establishing a more informative solution to Problem 31-7. Although the result of Lemma 1 is well known [cf. Werner (2003a)], we present an alternative proof.

LEMMA 1. *For any matrix $B \in \mathbb{C}^{m\times n}$ we have $\mathcal{R}(BB^*) = \mathcal{R}(B)$ and $\mathcal{N}(BB^*) = \mathcal{N}(B^*)$.*

PROOF. Since $\mathbb{C}^n = \mathcal{R}(B^*) \oplus \mathcal{N}(B)$, $\mathcal{R}(BB^*) = B\mathcal{R}(B^*) = B[\mathcal{R}(B^*) \oplus \mathcal{N}(B)] = B\mathbb{C}^n = \mathcal{R}(B)$. By taking orthogonal complements on both sides of $\mathcal{R}(BB^*) = \mathcal{R}(B)$ we obtain $\mathcal{N}(BB^*) = \mathcal{N}(B^*)$. □

LEMMA 2. *Let P and Q be two complex $n \times n$ matrices and let $A := PQ$. If P and Q are two orthogonal projectors, then $\operatorname{index}(A) \le 1$ and $\operatorname{index}(A^*) \le 1$, in which case $\mathcal{R}(A^2) = \mathcal{R}(A)$ and $\mathcal{R}((A^*)^2) = \mathcal{R}(A^*)$ or, equivalently, $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathbb{C}^n$ and $\mathcal{R}(A^*) \oplus \mathcal{N}(A^*) = \mathbb{C}^n$.*

PROOF. Trivially, $\mathcal{N}(A) \subseteq \mathcal{N}(A^2)$. Conversely, by means of Lemma 1, $\mathcal{N}(A^2) = \mathcal{N}(PQPQ) \subseteq \mathcal{N}(QPQPQ) = \mathcal{N}(QPQQPQ) = \mathcal{N}(QPQ) = \mathcal{N}(QPPQ) = \mathcal{N}(PQ) = \mathcal{N}(A)$. Therefore, $\mathcal{N}(A) = \mathcal{N}(A^2)$ or, equivalently, $\mathcal{R}(A^*) = \mathcal{R}((A^*)^2)$. Hence $\operatorname{index}(A^*) \le 1$ or, equivalently, $\mathcal{R}(A^*) \oplus \mathcal{N}(A^*) = \mathbb{C}^n$. The remaining results are obtained now by replacing A by $A^* = QP$. □

We continue with citing with Theorem 1 from Werner (2003b) an extremely powerful result characterizing $(A^+)^2 = A^+$ in terms of A and its conjugate transpose.

THEOREM 3. *Let A be a square complex matrix. Then the Moore-Penrose inverse A^+ of A is idempotent, i.e., $(A^+)^2 = A^+$, if and* *only if* $A^2 = AA^*A$.

This characterization has a series of direct implications. From Werner (2003b, Corollary 2) we already know the following.

COROLLARY 4. *Let* A *be a square complex matrix. Then we have:*

(i) *If* A *is an EP-matrix, i.e., if* $\mathcal{R}(A) = \mathcal{R}(A^*)$*, then* A^+ *is idempotent if and only if* A *is idempotent and Hermitian, in which case* $A^2 = A = A^* = A^+$.

(ii) *If* A *is idempotent, then* A^+ *is idempotent if and only if* A *is a partial isometry, i.e., if and only if* $A = AA^*A$*, in which case* $A^2 = A = A^* = A^+$.

(iii) A^+ *is idempotent only if* $\mathrm{index}(A) \leq 1$. *Moreover, if* A^+ *is idempotent and* $A^2 = 0$*, then necessarily* $A = 0$.

In this paper we further add the following two corollaries which also illuminate the beauty of Theorem 3.

COROLLARY 5. *Let* A *be a square complex matrix. Then we have:*

(i) *If* A^+ *is idempotent, then* A *is idempotent if and only if* A *is a partial isometry, i.e., if and only if* $A = AA^*A$.

(ii) *If* A^+ *is idempotent, then* A *is EP, i.e.,* $\mathcal{R}(A) = \mathcal{R}(A^*)$ *or, equivalently,* $AA^+ = A^+A$*, if and only if* A *is Hermitian, in which case* $A^2 = A = A^* = A^+$.

(iii) *If* A *is an EP-matrix with* A^+ *being idempotent, then* A *is necessarily a partial isometry.*

PROOF. (i): This is an immediate consequence of the characterization in Theorem 3.
(ii): First, let $(A^+)^2 = A^+$ and $AA^+ = AA^+$. Then, according to Theorem 3, $A^2 = AA^*A$. Consequently, $A = AA^+A = A^+A^2 = A^+AA^*A = A^*A$ or, equivalently, $A = A^*A = A^2$. So, in particular, as claimed $A = A^*$. Conversely, if $A = A^*$, then A is trivially EP, and so the proof of (ii) is complete.
(iii): Combining (i) and (ii) directly results in (iii). □

COROLLARY 6. *If* $A = PQ$*, where* P *and* Q *are two orthogonal projectors of the same order, then* A^+ *is idempotent.*

PROOF. Since $A^2 = PQPQ = PQQPPQ = AA^*A$, the claim is again a straightforward consequence of Theorem 3. □

The preceding observations enable us now to give a succinct proof to the following more informative solution to Problem 31-7.

THEOREM 7. *Let* $A = PQ$*, where* P *and* Q *are orthogonal projectors of the same order. Then* A^+ *is idempotent and the following conditions are all equivalent to each other:*

(i) A *is an orthogonal projector, i.e.,* $A = A^* = A^2$ *or, equivalently,* $A = AA^*$,

(ii) A *is Hermitian, i.e.,* $A = A^*$,

(iii) A *is normal, i.e.,* $AA^* = A^*A$,

(iv) A *is EP, i.e.,* $\mathcal{R}(A) = \mathcal{R}(A^*)$ *or, equivalently,* $AA^+ = A^+A$,

(v) $\mathcal{R}(A) = \mathcal{R}(P) \cap \mathcal{R}(Q)$,

(vi) $\mathcal{R}(Q) = [\mathcal{R}(Q) \cap \mathcal{R}(P)] \oplus [\mathcal{R}(Q) \cap \mathcal{N}(P)]$,

(vii) $\mathcal{R}(PQ) \subseteq \mathcal{R}(Q)$,

(viii) $A^+ = A^*$,

(ix) A *is a partial isometry, i.e.,* $AA^*A = A$ *or, equivalently,* $A^* \in \{A^-\}$,

(x) A *is idempotent, i.e.,* $A^2 = A$,

(xi) $A^+ = A$,

(xii) A *is bi-EP, i.e.,* $AA^+A^+A = A^+AAA^+$,

(xiii) $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$ *is EP, i.e.,* $\mathcal{R}(P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}) = \mathcal{R}(P_{\mathcal{R}(A^*)}P_{\mathcal{R}(A)})$,

(xiv) A *is bi-normal, i.e.,* $AA^*A^*A = A^*AAA^*$,

(xv) AA^*A^*A *is EP, i.e.,* $\mathcal{R}(AA^*A^*A) = \mathcal{R}(A^*AAA^*)$,

(xvi) A *is bi-dagger, i.e.,* $(A^+)^2 = (A^2)^+$,

(xvii) $A^+ = (A^2)^+$.

PROOF. Corollary 6 tells us that A^+ is idempotent. Trivially, (i) ⇒ (ii) ⇒ (iii), and, in view of Lemma 1, (iii) ⇒ (iv). Since $A = PQ$, $A^* = QP$, and Q and P are orthogonal projectors, it is easy to see that (iv) ⇒ (v). In view of $\mathcal{R}(P) \oplus \mathcal{N}(P) = \mathbb{C}^n$, clearly (v) ⇔ (vi) ⇔ (vii). Theorem 5.4 in Werner (1992) tells us that (vi) ⇔ (viii). Evidently, (viii) ⇔ (ix). From Corollary 5(i) we know that (ix) ⇔ (x). Since $Q = Q^2 = Q^* = Q^+$ and $P = P^2 = P^* = P^+$, it follows from Corollary 5.8 in Werner (1992) that (viii) ⇔ (iv). For proving (x) ⇒ (i), let $A = PQ$ be idempotent, i.e., let $PQPQ = PQ$. As seen before, (x) ⇒ (viii) ⇒ (iv). Since A is therefore also an EP-matrix, it follows from Corollary 5(ii) that A is indeed an orthogonal projector. It is now clear that the conditions (i) through (x) are all equivalent to each other. Trivially, (i) ⇒ (xi) ⇒ (xii). Furthermore, since $AA^+ = P_{\mathcal{R}(A)}$ and $A^+A = P_{\mathcal{R}(A^*)}$, also (xii) ⇒ (xiii). By means of Lemma 2, $\mathcal{R}(P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}) = P_{\mathcal{R}(A)}\mathcal{R}(A^*) = P_{\mathcal{R}(A)}[\mathcal{R}(A^*) \oplus \mathcal{N}(A^*)] = P_{\mathcal{R}(A)}\mathbb{C}^n = \mathcal{R}(A)$. Since on similar lines we get $\mathcal{R}(P_{\mathcal{R}(A^*)}P_{\mathcal{R}(A)}) = \mathcal{R}(A^*)$, it is clear that (xiii) ⇒ (iv). That (ii) ⇒ (xiv) ⇒ (xv) is again straightforward. Next, let condition (xv) hold, i.e., let $\mathcal{R}(AA^*A^*A) = \mathcal{R}(A^*AAA^*)$. By applying Lemma 1 and Lemma 2 repeatedly we obtain $\mathcal{R}(AA^*A^*A) = AA^*\mathcal{R}(A^*A) = AA^*\mathcal{R}(A^*) = \mathcal{R}(AA^*A^*) = A\mathcal{R}((A^*)^2) = A\mathcal{R}(A^*) = \mathcal{R}(AA^*) = \mathcal{R}(A)$, and likewise $\mathcal{R}(A^*AAA^*) = \mathcal{R}(A^*)$. Consequently, $\mathcal{R}(A) = \mathcal{R}(A^*)$, and the proof of (xv) ⇒ (iv) is complete. If A is an orthogonal projector, then $A = A^2 = A^+$ and so (i) ⇒ (xvi) should be clear. Since A^+ is idempotent, (xvi) reduces to (xvii). Taking the Moore-Penrose inverse of both sides in condition (xvii) gives (x), and so our proof is complete. □

We conclude with mentioning that by making use of the results in Werner (1992) it would be easy to add a myriad of further (equivalent) conditions to those in the Theorem 7.

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Problem 31-8: Eigenvalues and Eigenvectors of a Particular Tridiagonal Matrix

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Let A be the n-by-n tridiagonal matrix with 2 on diagonal and 1 on super- and sub-diagonals. That is, $a_{ii} = 2$, $a_{ij} = 1$ if $j = i + 1$ or $j = i - 1$, and $a_{ij} = 0$ otherwise, $i, j = 1, 2, \cdots, n$. Find all eigenvalues and corresponding eigenvectors of A.

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A solution to the problem is actually known in the literature for a general real $n \times n$ tridiagonal Toeplitz matrix A, having b (say) as its diagonal entries and nonzero a and c (say) of the same sign as superdiagonal and subdiagonal entries, respectively, i.e., $a_{ii} = b$, $a_{ij} = a$ whenever $j = i + 1$, $a_{ij} = c$ whenever $j = i - 1$, and $a_{ij} = 0$ otherwise, $i, j = 1, ..., n$. If (λ_j, x_j) denote the jth eigenpair of A, then, according to Meyer (2000, pp. 514-516),

$$\lambda_j = b + 2a\sqrt{c/a}\cos(j\pi/(n + 1)) \tag{54}$$

and the components of x_{kj} of the eigenvector x_j are expressible as

$$x_{kj} = (c/a)^{k/2}\sin(kj\pi/(n + 1)), \quad k = 1, ..., n. \tag{55}$$

Clearly, in the case where $b = 2$ and $a = c = 1$, which corresponds to the original version of Problem 31-8, the formulae (54) and (55) simplify to $\lambda_j = 2 + 2\cos(j\pi/(n + 1))$ and $x_{kj} = \sin(kj\pi/(n + 1))$. An additional remark is the quotation of Meyer's (2000, p. 516) observation that since λ_j's are all different, A is diagonalizable, with the diagonalization being achieved with the use of the matrix having $x_1, ..., x_n$ as its succesive columns.

Reference

C. D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia, PA.

Solution 31-8.2 by C. M. da Jerzy K. FONSECA, *Universidade de Coimbra, Portugal:* cmf@mat.uc.pt

Consider a set of polynomials $\{P_k\}_{k\geq 0}$, such that each P_k is of degree exactly k, satisfying the recurrence relations

$$P_{k+1}(x) = (x - a)P_k(x) - bP_{k-1}(x)\ , \quad k \geq 0\ , \tag{56}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, with $b > 0$. Consider also the set of polynomials $\{U_k\}_{k\geq 0}$, which satisfy the three-term recurrence relations

$$2xU_k(x) = U_{k+1}(x) + U_{k-1}(x) \ , \quad k \geq 1 \ ,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Each U_k is called the Chebyshev polynomial of second kind of degree k and have the explicit form:

$$U_k(x) = \frac{\sin(k+1)\theta}{\sin\theta} \ , \quad \text{where} \quad \cos\theta = x \ ,$$

when $|x| < 1$.

There is a natural relation between the polynomials defined above:

$$P_k(x) = (\sqrt{b})^k U_k\left(\frac{x-a}{2\sqrt{b}}\right) \ .$$

On the other hand, the recurrence relation (56) is equivalent to

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} = \begin{pmatrix} a & 1 & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & b & a \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} + P_n(x)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \ .$$

Therefore the zeros of $P_n(x)$

$$\lambda_\ell = a + 2\sqrt{b}\cos\frac{\ell\pi}{n+1} \ , \quad \ell = 1, \ldots, n \ ,$$

are the eigenvalues of the tridiagonal matrix of order n

$$A = \begin{pmatrix} a & 1 & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & b & a \end{pmatrix} \ ,$$

and the vector column

$$\begin{pmatrix} P_0(\lambda_\ell) \\ P_1(\lambda_\ell) \\ \vdots \\ P_{n-1}(\lambda_\ell) \end{pmatrix} = \left(\sin\frac{\ell\pi}{n+1}\right)^{-1} \begin{pmatrix} \sin\frac{\ell\pi}{n+1} \\ \sqrt{b}\,\sin\frac{2\ell\pi}{n+1} \\ \vdots \\ (\sqrt{b})^{n-1}\sin\frac{n\ell\pi}{n+1} \end{pmatrix}$$

is an eigenvector associated to the eigenvalue λ_ℓ.

If $a = 2$ and $b = 1$, then we get the solution to Problem 31-8.

Solution 31-8.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Apply the following known result [see, e.g., Grenander & Szegö (1958), Haley (1980), or Trench (1985)]: Let c_{-1}, c_0, c_1 be complex numbers with $c_1c_{-1} \neq 0$, and let A be the $n \times n$ tridiagonal matrix such that $a_{ii} = c_0$, $1 \leq i \leq n$, $a_{i,i-1} = c_{-1}$, $2 \leq i \leq n$, and $a_{i,i+1} = c_1$, $1 \leq i \leq n-1$. Then the eigenvalues of A are

$$\lambda_q = c_0 + 2\sqrt{c_1c_{-1}}\cos\left(\frac{q\pi}{n+1}\right), \quad 1 \leq q \leq n,$$

with associated eigenvectors $X_q = \begin{pmatrix} x_{1q} & x_{2q} & \cdots & x_{nq} \end{pmatrix}^T$, where

$$x_{mq} = \left(\frac{c_{-1}}{c_1}\right)^{m/2} \sin\left(\frac{qm\pi}{n+1}\right), \quad 1 \leq m \leq n.$$

References

U. Grenander & G. Szegö (1958). *Toeplitz Forms and Their Applications.* University of California Press, Berkeley.

S. B. Haley (1980). Solution of band matrix equations by projection-recurrence. *Linear Algebra and Its Applications*, **32**, 33–48.

W. F. Trench (1985). On the eigenvalue problem for Toeplitz band matrices. *Linear Algebra and Its Applications*, **64**, 199–214.

Solution 31-8.4 by Iwona WRÓBEL, *Warsaw University of Technology, Warsaw, Poland:* wrubelki@wp.pl
and Marcin MAŹDZIARZ, *Polish Academy of Sciences, Warsaw, Poland:* mmazdz@ippt.gov.pl

There exist explicit formulae for eigenvalues and corresponding eigenvectors of the $n \times n$ tridiagonal matrix B, with 2 on diagonal and -1 on sub- and super-diagonals, see for example Golub and Ortega (1992, pp. 130, 132). The eigenvalues of B are given by $\lambda_k^B = 2 - 2\cos kh$, where $h = \frac{\pi}{n+1}$, with corresponding eigenvectors $x_k = [\sin kh, \sin 2kh, \ldots, \sin nkh]^T$. The matrix A that appears in Problem 31-8 can be expressed in terms of B in the following way: $A = 4I - B$, where I denotes the $n \times n$ identity matrix. Now using the spectral mapping theorem we obtain the formulae for the eigenvalues of A, namely $\lambda_k^A = 4 - \lambda_k^B = 2 + 2\cos kh$, with h defined as before. Moreover, the equality $A = 4I - B$ implies that A and B have the same eigenvectors.

Reference

G. H. Golub & J. M. Ortega (1992). *Scientific Computing and Differential Equations. An Introduction to Numerical Methods.* Academic Press, New York.

Solutions to Problem 31-8 were also received from Robert B. Reams *and from* Lajos Laśzló.

IMAGE Problem Corner: More New Problems

Problem 32-6: A Vector Cross Product Property in $\mathbb{R}^3$
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

In Milne (1965, Ex. 22, p. 26) the following problem is posed: *"If a, b are given non-parallel vectors, and x and y vectors satisfying $x \times a = y \times b$, show that x and y are linear functions of a and b, and obtain their most general forms."* Generalize this problem as follows: For given vectors a, b, and c from $\mathbb{R}^3$, where a and b are linearly independent, show that there always exist vectors x, $y \in \mathbb{R}^3$ such that

$$x \times a + y \times b + c = 0.$$

Determine the general solution (x, y) to this equation. Note that "$\times$" denotes the vector cross product in $\mathbb{R}^3$.

Reference

E. A. Milne (1965). *Vectorial Mechanics.* Methuen, London.

Problem 32-7: Invariance of the Vector Cross Product
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de
and Dietrich TRENKLER, *University of Osnabrück, Osnabrück, Germany:* dtrenkler@nts6.oec.uni-osnabrueck.de

For a given nonzero vector $a \in \mathbb{R}^3$ determine a wide class of matrices A of order 3×3 such that

$$A(a \times b) = (Aa) \times (Ab)$$

for all $b \in \mathbb{R}^3$. Here "$\times$" denotes the common vector cross product in $\mathbb{R}^3$. Such equations play a role in robotics, see Murray, Lee, and Sastry (1994).

Reference

R. M. Murray, Z. Lee & S. S. Sastry (1994). *A Mathematical Introduction to Robotic Manipulation.* CRC Press, Boca Raton, FL.

Problems 32-1 through 32-5 are on page 40.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to ujw902@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 32-6 and 32-7 are on page 39.*

Problem 32-1: Factorizations of Nonsingular Matrices by Means of Corner Matrices

Proposed by Richard W. FAREBROTHER, *Bayston Hill, Shrewsbury, England:* R.W.Farebrother@man.ac.uk

Show that any nonsingular $n \times n$ matrix A may be expressed as the product of

(a) two southwest and one northeast corner matrices,
(b) two northeast and one southwest corner matrices,
(c) three northwest corner matrices, and
(d) three southeast corner matrices,

where an $n \times n$ matrix A is called a southwest corner (or lower triangular) matrix if it satisfies $a_{ij} = 0$ for $i < j$, a northeast corner (or upper triangular) matrix if it satisfies $a_{ij} = 0$ for $i > j$, a northwest corner matrix if it satisfies $a_{ij} = 0$ for all i, j satisfying $i + j > n + 1$, and a southeast corner matrix if it satisfies $a_{ij} = 0$ for all i, j satisfying $i + j < n - 1$.

Problem 32-2: A Property of Plane Triangles – Eadem Resurgo

Proposed by Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Let $\lambda \in \mathbb{R}_{>0}$. Apply to a plane triangle Δ the following process: go clockwise around Δ and divide its sides in the ratio $\lambda : 1$. Use the distances from the division points to the opposite vertices as side-lengths for a new triangle Δ' (cyclically again). Repeat the process with Δ' but divide with the ratio $1 : \lambda$ to obtain a triangle Δ''. Show that Δ'' is similar to Δ with the ratio $\rho = \sqrt{1 + 2\lambda + 3\lambda^2 + 2\lambda^3 + \lambda^4}/(1 + \lambda)^2$.

Problem 32-3: Jacobians for the Square-Root of a Positive Definite Matrix

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au
and Heinz NEUDECKER, *University of Amsterdam, Amsterdam, The Netherlands:* H.Neudecker@uva.nl

Establish the following Jacobian matrices:

$$\frac{\partial \mathrm{v}(\mathrm{X}^{1/2})}{\partial \mathrm{v}'(\mathrm{X})} = D^+(X^{1/2} \otimes I + I \otimes X^{1/2})^{-1} D, \quad \frac{\partial \mathrm{v}(\mathrm{X}^{-1/2})}{\partial \mathrm{v}'(\mathrm{X})} = -D^+(X^{1/2} \otimes X + X \otimes X^{1/2})^{-1} D,$$

where X is an $n \times n$ positive definite matrix, $X^{1/2}$ is its positive definite square root, D is the $n^2 \times n(n+1)/2$ duplication matrix, D^+ is its Moore-Penrose inverse, I is the $n \times n$ identity matrix, $\mathrm{v}'(\cdot)$ denotes the transpose of $\mathrm{v}(\cdot)$, $\mathrm{v}(\cdot)$ denotes the $n(n+1)/2 \times 1$ vector that is obtained from $\mathrm{vec}(\cdot)$ by eliminating all supradiagonal elements of the matrix and $\mathrm{vec}(\cdot)$ transforms the matrix into a vector by stacking the columns of the matrix one underneath the other.

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j}X_1 + u_{2j}X_2 + u_{3j}X_3$, $j = 1, 2, 3$, such that the six matrices $W_j^{-1}W_k$, $j \neq k$, have zero traces. Equivalently, we want $(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for real scalars a_{jk}. These scalars also define the eigenvalues of $W_j^{-1}W_k$ as a_{jk}, $-a_{jk}(1+i\sqrt{3})/2$, and $-a_{jk}(1-i\sqrt{3})/2$, respectively. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

Problem 32-5: Diagonal Matrices Solving a Matrix Equation

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $A \in \mathbb{R}^{l\times m}$, $B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{l\times n}$ be given matrices. Find all vectors $x = (x_1, \ldots, x_m)' \in \mathbb{R}^m$ such that $A\,\mathrm{diag}(x_1, \ldots x_m)B = C$.