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Forthcoming Conferences and Workshops in Linear Algebra

IMAGE Problem Corner: Old Problems, Many Solutions

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THE USE OF LINEAR ALGEBRA BY WEB SEARCH ENGINES

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1. Introduction. Nearly all the major Web search engines today use link analysis to improve their search results. That's exciting news for linear algebraists because link analysis, the use of the Web's hyperlink structure, is built from fundamentals of matrix theory. Link analysis and its underlying linear algebra have helped revolutionize Web search, so much so that the pre-link analysis search (before 1998) pales in comparison to today's remarkably accurate search.

HITS [13] and PageRank [2, 3] are two of the most popular link analysis algorithms. Both were developed around 1998 and both have dramatically improved the search business. In order to appreciate the impact of link analysis, recall for a minute the state of search prior to 1998. Because of the immense number of pages on the Web, a query to an engine often produced a very long list of relevant pages, sometimes thousands of pages long. A user had to sort carefully through the list to find the most relevant pages. The order of presentation of the pages was little help because spamming was so easy then. In order to trick a search engine into producing rankings higher than normal, spammers used meta-tags liberally, claiming their page used popular search terms that never appeared in the page. Meta-tags became useless for search engines. Spammers also repeated popular search terms in invisible text (white text on a white background) to fool engines.

2. The HITS Algorithm. HITS [13], a link analysis algorithm developed by Jon Kleinberg from Cornell University during his postdoctoral studies at IBM Almaden, aimed to focus this long, unruly query list. The HITS algorithm is based on a pattern Kleinberg noticed among Web pages. Some pages serve as hubs or portal pages, i.e., pages with many outlinks. Other pages are authorities on topics because they have many inlinks. Kleinberg noticed that *good hubs seemed to point to good authorities and good authorities were pointed to by good hubs.* So he decided to give each page i both an hub score h_i and an authority score a_i. In fact, for every page i he defined the hub score at iteration k, $h_i^{(k)}$, and the authority score, $a_i^{(k)}$, as

$$a_i^{(k)} = \sum_{j:e_{ji}\in E} h_j^{(k-1)} \quad \text{and} \quad h_i^{(k)} = \sum_{j:e_{ij}\in E} a_j^{(k)} \quad \text{for } k = 1, 2, 3, \ldots,$$

where e_{ij} represents a hyperlink from page i to page j and E is the set of hyperlinks. To compute the scores for a page, he started with uniform scores for all pages, i.e., $h_i^{(1)} = 1/n$ and $a_i^{(1)} = 1/n$ where n is the number of pages in a so-called neighborhood set for the query list. The neighborhood set consists of all pages in the query list plus all pages pointing to or from the query pages. Depending on the query, the neighborhood set could contain just a hundred pages or a hundred thousand pages. (The neighborhood set allows latent semantic associations to be made.) The hub and authority scores are iteratively refined until convergence to stationary values.

Using linear algebra we can replace the summation equations with matrix equations. Let $\mathbf{h}$ and $\mathbf{a}$ be column vectors holding the hub and authority scores. Let $\mathbf{L}$ be the adjacency matrix for the neighborhood set. That is, $\mathbf{L}_{ij} = 1$ if page i links to page j, and 0, otherwise. These definitions show that

$$\mathbf{a}^{(k)} = \mathbf{L}^T\mathbf{h}^{(k-1)} \quad \text{and} \quad \mathbf{h}^{(k)} = \mathbf{L}\mathbf{a}^{(k)}.$$

Using some algebra, we have

$$\mathbf{a}^{(k)} = \mathbf{L}^T\mathbf{L}\mathbf{a}^{(k-1)}$$
$$\mathbf{h}^{(k)} = \mathbf{L}\mathbf{L}^T\mathbf{h}^{(k-1)}.$$

These equations make it clear that Kleinberg's algorithm is really the power method applied to the positive semi-definite matrices $\mathbf{L}^T\mathbf{L}$ and $\mathbf{L}\mathbf{L}^T$. $\mathbf{L}^T\mathbf{L}$ is called the hub matrix and $\mathbf{L}\mathbf{L}^T$ is the authority matrix. Thus, HITS amounts to solving the eigenvector problems $\mathbf{L}^T\mathbf{L}\mathbf{a} = \lambda_1\mathbf{a}$ and $\mathbf{L}\mathbf{L}^T\mathbf{h} = \lambda_1\mathbf{h}$, where λ_1 is the largest eigenvalue of $\mathbf{L}^T\mathbf{L}$ (and $\mathbf{L}\mathbf{L}^T$), and $\mathbf{a}$ and $\mathbf{h}$ are corresponding eigenvectors.

While this is the basic linear algebra required by the HITS method, there are many more issues to be considered. For example, important issues include convergence, existence, uniqueness, and numerical computation of these scores

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[5, 7, 14]. Several modifications to HITS have been suggested, each bringing various advantages and disadvantages [4, 6, 8]. A variation of Kleinberg's HITS concept is at the base of the search engine TEOMA (`http://www.teoma.com`), which is owned by Ask Jeeves, Inc.

3. The PageRank Algorithm. PageRank, the second link analysis algorithm from 1998, is the heart of Google. Both PageRank and Google were conceived by Sergey Brin and Larry Page while they were computer science graduate students at Stanford University. Brin and Page use a recursive scheme similar to Kleinberg's. Their original idea was that a *page is important if it is pointed to by other important pages.* That is, they decided that the importance of your page (its PageRank score) is determined by summing the PageRanks of all pages that point to yours. In building a mathematical definition of PageRank, Brin and Page also reasoned that when an important page points to several places, its weight (PageRank) should be distributed proportionately. In other words, if YAHOO! points to your Web page, that's good, but you shouldn't receive the full weight of YAHOO! because they point to many other places. If YAHOO! points to 999 pages in addition to yours, then you should only get credit for 1/1000 of YAHOO!'s PageRank.

This reasoning led Brin and Page to formulate a recursive definition PageRank. They defined

$$r_i^{(k+1)} = \sum_{j \in I_i} \frac{r_j^{(k)}}{|O_j|},$$

where $r_i^{(k)}$ is the PageRank of page i at iteration k, I_i is the set of pages pointing into page i and $|O_j|$ is the number of outlinks from page j. Like HITS, PageRank starts with a uniform rank for all pages, i.e., $r_i^{(0)} = 1/n$ and successively refines these scores, where n is the total number of Web pages.

Like HITS, we can write this process using matrix notation. Let the row vector $\boldsymbol{\pi}^{(k)T}$ be the PageRank vector at the k^{th} iteration. As a result, the summation equation for PageRank can be written compactly as

$$\boldsymbol{\pi}^{(k+1)T} = \boldsymbol{\pi}^{(k)T}\mathbf{H},$$

where $\mathbf{H}$ is a row normalized hyperlink matrix, i.e., $h_{ij} = 1/|O_i|$, if there is a link from page i to page j, and 0, otherwise. Unfortunately, this iterative procedure has convergence problems—it can cycle or the limit may be dependent on the starting vector.

To fix these problems, Brin and Page revised their basic PageRank concept. Still using the hyperlink structure of the Web, they build an irreducible aperiodic Markov chain characterized by a primitive (irreducible with only one eigenvalue on the spectral circle) transition probability matrix. The irreducibility guarantees the existence of a unique stationary distribution vector $\boldsymbol{\pi}^T$, which becomes the PageRank vector. The power method with a primitive stochastic iteration matrix will always converge to $\boldsymbol{\pi}^T$ independent of the starting vector, and the asymptotic rate of convergence is governed by the magnitude of the subdominant eigenvalue λ_2 of the transition matrix [19].

Here's how Google turns the hyperlink structure of the Web into a primitive stochastic matrix. If there are n pages in the Web, let $\mathbf{H}$ be the $n \times n$ matrix whose element h_{ij} is the probability of moving from page i to page j in one click of the mouse. The simplest model is to take $h_{ij} = 1/|O_i|$, which means that starting from any Web page we assume that it is equally likely to follow any of the outgoing links to arrive at another page.

However, some rows of $\mathbf{H}$ may contain all zeros, so $\mathbf{H}$ is not necessarily stochastic. This occurs whenever a page contains no outlinks; many such pages exist on the Web and are called *dangling nodes.* An easy fix is to replace all zero rows with $\mathbf{e}^T/n$, where $\mathbf{e}^T$ is the row vector of all ones. The revised (now stochastic) matrix $\mathbf{S}$ can be written as a rank-one update to the sparse $\mathbf{H}$. Let $\mathbf{a}$ be the dangling node vector in which

$$a_i = \begin{cases} 1 & \text{if page } i \text{ is a dangling node,} \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\mathbf{S} = \mathbf{H} + \mathbf{a}\mathbf{e}^T/n.$$

Actually, any probability vector $\mathbf{p}^T > 0$ with $\mathbf{p}^T\mathbf{e} = 1$ can be used in place of the uniform vector $\mathbf{e}^T/n$.

We're not home yet because the adjustment that produces the stochastic matrix $\mathbf{S}$ isn't enough to insure the existence of a *unique* stationary distribution vector (needed to make PageRank well defined). Irreducibility on top of

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stochasticity is required. But the link structure of the Web is reducible—the Web graph is not strongly connected. Consequently, an adjustment to make $\mathbf{S}$ irreducible is needed. This last adjustment brings us to the *Google matrix*, which is defined to be

$$\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\mathbf{E},$$

where $0 \le \alpha \le 1$ and $\mathbf{E} = \mathbf{e}\mathbf{e}^T/n$. Google eventually replaced the uniform vector $\mathbf{e}^T/n$ with a more general probability vector $\mathbf{v}^T$ (so that $\mathbf{E} = \mathbf{e}\mathbf{v}^T$) to allow them the flexibility to make adjustments to PageRanks as well as to personalize them. See [10, 15] for more about the personalization vector $\mathbf{v}^T$.

Because $\mathbf{G}$ is a convex combination of the two stochastic matrices $\mathbf{S}$ and $\mathbf{E}$, it follows that $\mathbf{G}$ is both stochastic and irreducible. Furthermore, every node is now directly connected to every other node (although the probability of transition may be very small in some cases), so $\mathbf{G} > 0$. Consequently, $\mathbf{G}$ is a primitive matrix, and this insures that the power method $\boldsymbol{\pi}^{(k+1)T} = \boldsymbol{\pi}^{(k)T}\mathbf{G}$ will converge, independent of the starting vector, to a unique stationary distribution $\boldsymbol{\pi}^T$ [19]. This is the mathematical part of Google's PageRank vector.

3.1. The Power Method. While it doesn't always excite numerical analysts, the power method has been Google's computational method of choice, and there are some good reasons for this. First, consider iterates of the power method applied to $\mathbf{G}$ (a completely dense matrix, were it to be formed explicitly). If we take $\mathbf{E} = \mathbf{e}\mathbf{v}^T$, then

$$\boldsymbol{\pi}^{(k)T} = \boldsymbol{\pi}^{(k-1)T}\mathbf{G} = \alpha\boldsymbol{\pi}^{(k-1)T}\mathbf{S} + (1-\alpha)\mathbf{v}^T = \alpha\boldsymbol{\pi}^{(k-1)T}\mathbf{H} + (\alpha\boldsymbol{\pi}^{(k-1)T}\mathbf{a} + (1-\alpha))\mathbf{v}^T,$$

$\boldsymbol{\pi}^{(k-1)T}\mathbf{e} = 1$. Written in this way, it becomes clear that the power method applied to $\mathbf{G}$ can be implemented with vector-matrix multiplications on the extremely sparse $\mathbf{H}$, and $\mathbf{G}$ and $\mathbf{S}$ are never formed or stored. A matrix-free method such as the power method is required due to the size of the matrices and vectors involved (Google's index is currently 4.3 billion pages). Fortunately, since $\mathbf{H}$ is sparse, each vector-matrix multiplication required by the power method can be computed in $nnz(\mathbf{H})$ flops, where $nnz(\mathbf{H})$ is the number of nonzeros in $\mathbf{H}$. And since the average number of nonzeros per row in $\mathbf{H}$ is significantly less than 10, we have $O(nnz(\mathbf{H})) \approx O(n)$. Furthermore, at each iteration, the power method only requires the storage of one vector, the current iterate, whereas other accelerated matrix-free methods, such as restarted GMRES or BiCGStab, require storage of at least several vectors, depending on the size of the subspace chosen. Finally, the power method applied in this way converges quickly. Brin and Page report success using only 50 to 100 power iterations [3]. This is due in large part to the fact that it can be proven [15] that the subdominant eigenvalue of $\mathbf{G}$ satisfies $|\lambda_2| \le \alpha$, and Google originally set $\alpha = .85$.

Like HITS, the basic concepts of PageRank are simple, but there are many subtle issues that lurk just below the surface. For example, there are complicated and unresolved issues concerning personalization, computation, accelerated computation, sensitivity, and updating—more information is available in [7, 11, 12, 21, 15, 17, 18, 20].

This brief introduction describes only the mathematical component of Google's ranking system. However, it's known that there are non-mathematical "metrics" that are also considered when Google responds to a query, so the results seen by a user are in fact PageRank tempered by other metrics. While Google is secretive about these other metrics, they state on their Web site (`http://www.google.com/technology`) that "The heart of our software is PageRank... ."

4. Books. SIAM is publishing a second edition of the popular *Understanding Search Engines: Mathematical Modeling and Text Retrieval* [1] by Michael W. Berry and Murray Browne in 2005. The new edition contains a chapter devoted to link analysis. As a result, readers can see how link analysis and ranking algorithms fit into the overall search process.

Also due out in 2005 is our book, *Understanding Web Search Engine Rankings: Google's PageRank, Teoma's HITS, and other ranking algorithms* [16]. This book from Princeton University Press will contain over 250 pages devoted to link analysis algorithms with several introductory chapters, examples, and code, as well as chapters dealing with more advanced issues in Web search ranking.

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Tidbits From Our Members

In his article "Research Experiences with Undergraduates," Chi-Kwong Li could have mentioned that research into matrix methods is particularly appropriate for students of the College of William and Mary as it was rebuilt by Lieutenant Governor Alexander Spotswood, the first cousin of the great, great, great, great grandfather of William Spottiswoode (pronounced Spotswood) the author of one of the earliest books on the theory of determinants. For details, see:

R.W. Farebrother, A genealogy of William Spottiswoode (1825-1883), *IMAGE* 23, 1999, pp. 3-4.

R.W. Farebrother and G.P.H. Styan, A Genealogy of the Spottiswoode Family (1510-1900), *IMAGE* 25, 2000, pp. 19-21 and *IMAGE* 27, 2001, p. 2.

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Further Early Statistical Applications of the Theory of Determinants

by **Richard William Farebrother**

Readers of Farebrother (2002a,b) and Searle (2000) may be interested to learn from Aldrich (1998, p. 74) that

> determinants were used in other important work on least squares and correlation—including Pearson (1896), Fisher (1922), Frisch and Waugh (1933) and David and Neyman (1938). Yule [1907] did not use them but Pearson (1916) went on to obtain Yule's [correlation] results by direct determinantal analysis, a task that required new results on determinants.

And, again, Aldrich (1998, p. 76),

> David and Neyman (1938, p. 105) judged Aitkens [1935] paper remarkably clear and elegant but found it

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Farebrother article cont'd

worthwhile to present a result using reasoning of a more elementary character—reasoning based on determinants. They saw their work as an extension of a neglected Markov theorem on least squares. Plackett (1949) pointed out that all the results were variations on a neglected result of Gauss (1821).

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Two ILAS Members Recently Honored

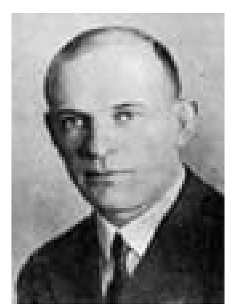
Miki Neumann is one of four recipients of the University of Connecticut 2004 Provost's Research Excellence Award. The award recognizes excellence in research at Storrs and its regional campuses. Factors considered by the review committee when evaluating the nominations include: Is the research seminal? What is the significance of the research in a global context? How effective has the nominee been in promoting research at the University of Connecticut, such as mentoring students and colleagues?

Richard Brualdi is one of four professors awarded a University of Wisconsin Bascom Professorship. These professorships recognize contributions to the university's teaching, research and service. Richard's appointment was based on his research in combinatorics, graph theory and linear algebra/matrix theory, his leadership as chair of the University of Wisconsin's Department of Mathematics from 1993-1996, his excellence in teaching as evidenced by a Chancellor's Award for Excellence in Teaching in 1986, and his service as a member of the College of Letters and Science Academic Planning Council and as a chair of the Letters and Science Curriculum Committee.

Electronic Journal of Linear Algebra

The Electronic Journal of Linear Algebra (ELA), a publication of the International Linear Algebra Society (ILAS), is a refereed all-electronic journal that welcomes mathematical articles of high standards that contribute new information and new insights to matrix analysis and the various aspects of linear algebra and its applications. Refereeing of articles is conventional and of high standards, and is being carried out electronically. The Editors-in-Chief are Ludwig Elsner and Daniel Hershkowitz. The web page is http://www.math.technion.ac.il/iic/ela/

After reading the article on Gershgorin in the last issue, several readers requested that *IMAGE* include a photograph. Here it is. A book review of Richard Varga's recently published book "*Gershgorin and His Circles*" will appear in a forthcoming issue of *IMAGE*.



11th ILAS CONFERENCE
Coimbra, Portugal: 19-22 July 2004

Report by **João Queiró**

The 11th Conference of the International Linear Algebra Society was held at the University of Coimbra, Portugal, July 19-22, 2004. The conference was dedicated to Richard Brualdi in honor of his 65th birthday. There were 204 participants, from 30 countries (in all five continents)

The scientific program included plenary talks by Rajendra Bhatia, Hal Caswell, George Cybenko, Erik Elmroth, Shmuel Friedland, Peter Gritzmann, Robert Guralnick, Uwe Helmke, William Helton, Christian Krattenthaler, Matjaz Omladic, Xavier Puerta, Arun Ram, Joachim Rosenthal, Fernando Silva, Siegfried Rump, Peter Lancaster (Hans Schneider Prize speaker), Beatrice Meini (SIAG/LA speaker), Julio Moro (SIAG/LA speaker) and Peter Šemrl (Taussky-Todd speaker).

There were five mini-symposia: Combinatorial Matrix Theory (organized by Bryan Shader), Group Representations (organized by Ana Paula Santana and Carlos André); Markov Methods for Search Engines (organized by Ilse Ipsen and Steve Kirkland); Nonnegative Matrices (organized by Thomas Laffey); and Matrix Inequalities (organized by Chi-Kwong Li). Apart from this, there were around 115 contributed talks.

The ILAS Business Meeting was held on Monday, July 19. On Tuesday, there was a visit to the old buildings of the University (which was founded in 1290), followed by a reception at the 12th century Old Church cloisters, hosted by the Coimbra Mayor. On Wednesday, the Conference banquet took place at the S. Marcos Palace, outside Coimbra. Before the dinner, the Hans Schneider Prize was presented to Peter Lancaster, with the laudatio given by prize committee chairman Harm Bart. After the dinner, there were speeches honoring Richard Brualdi, by Bryan Shader and Alan Hoffman. The evening ended with several songs performed by a local fado music group.

In the days before and after the Conference, other scientific events took place in the Coimbra Mathematics Department: the 7th Workshop on Numerical Ranges and Numerical Radii, and short courses by Persi Diaconis, Arun Ram, Erik Elmroth and Beatrice Meini.

Overall, the meeting went well, with excellent plenary talks, informative mini-symposia and varied contributed talks. The main objectives of the meeting, to display the depth and breadth of Linear Algebra and its applications, and to allow personal contacts and interactions between people with diversified interests, seem to have been attained.

The Organizing Committee members were Danny Hershkowitz (ILAS President), Hans Schneider, Thomas Laffey, Raphael Loewy, Ion Zaballa, Bryan Shader, Graciano de Oliveira. José Dias da Silva, Eduardo Marques de Sá and João Filipe Queiró (chair). The Local Organizing Committee was constituted by Olga Azenhas, Cristina Caldeira, Jesus Clemente Gallardo, António Leal Duarte, João Filipe Queiró and Ana Paula Santana.

The journal LAA will publish a special issue with papers presented at the Conference. The editors of this issue are Graciano de Oliveira, João Filipe Queiró, Bryan Shader, and Ion Zaballa.

Numerical Ranges and Radii—WONRA04
Coimbra, Portugal: 16-17 July 2004

Report by **Natália Bebiano**

The 7th Workshop on Numerical Ranges and Radii—WONRA04 was held at the University of Coimbra (Coimbra, Portugal), July 16-17, 2004. This Workshop was sponsored by the Centre of Mathematics of the University of Coimbra (CMUC), as a satellite meeting of the 11th Conference of the International Linear Algebra Society (ILAS), which was held in Coimbra during the week immediately after WONRA04. The Organizing Committee for this workshop consisted of N. Bebiano (University of Coimbra), R. Lemos (University of Aveiro) and G. Soares (University of Trás-os-Montes e Alto Douro).

The purpose of the workshop was to stimulate research on this topic, with many applications in different fields of mathematics, physics, architecture etc, and, in an informal setting, to foster the interaction of researchers from different areas of research. There were 50 registered participants, geographically, from North America, Europe, Asia and Africa. The wide variety of topics and backgrounds of the participants resulted in a extremely pleasant atmosphere to foster contacts, exchange ideas, and in a scientifically exciting meeting.

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[20% discount valid till 15 Jan 2005]

Highlights in Linear Algebra

from World Scientific and Imperial College Press

REPRESENTATIONS OF FINITE AND LIE GROUPS

by **Charles B Thomas** *(University of Cambridge, UK)*

This book provides an introduction to representations of both finite and compact groups. The proofs of the basic results are given for the finite case, but are so phrased as to hold without change for compact topological groups with an invariant integral replacing the sum over the group elements as an averaging tool. Among the topics covered are the relation between representations and characters, the construction of irreducible representations, induced representations and Frobenius reciprocity. Special emphasis is given to exterior powers, with the symmetric group S_n as an illustrative example. The book concludes with a chapter comparing the representations of the finite group $SL_2(\Phi_p)$ and the non-compact Lie group SL_2(p).

Readership: Advanced undergraduates and graduate students in algebra.

156pp Oct 2004 1-86094-482-5 US$48
1-86094-484-1(pbk) US$28

Published by Imperial College Press and distributed by World Scientific Publishing Co.

AN INTRODUCTION TO COMMUTATIVE ALGEBRA

From the Viewpoint of Normalization
by **Huishi Li** *(Jiaying University, China)*

Designed for a one-semester course in mathematics, this textbook presents a concise and practical introduction to commutative algebra in terms of normal (normalized) structure. It shows how the nature of commutative algebra has been used by both number theory and algebraic geometry. Many worked examples and a number of problems (with hints) can be found in the volume.

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188pp Aug 2004 981-238-951-2 US$46

LINEAR ALGEBRA

by **C Y Hsiung** *(Wuhan University)* & **G Y Mao** *(Wuhan University of Technology)*

Linear Algebra constitutes a foundation course for those specializing in the fields of mathematics, engineering and science. The course normally takes one semester, but for those needing a more rigorous study of the subject, it involves up to two semesters.

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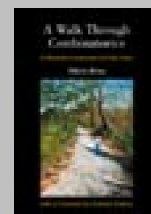
A WALK THROUGH COMBINATORICS

An Introduction to Enumeration and Graph Theory
by **Miklós Bóna** *(University of Florida, USA)*



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Readership: Upper level undergraduates and graduate students in the field of combinatorics and graph theory.

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With Applications to Business and Economics
by **Ferenc Szidarovszky** *(University of Arizona, USA)* &
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512pp Jun 2002 981-02-4504-1 US$94
981-02-4513-0(pbk) US$58

MATRIX ALGEBRA AND ITS APPLICATIONS TO STATISTICS AND ECONOMETRICS

by **C Radhakrishna Rao** *(Pennsylvania State University, USA)* &
M Bhaskara Rao *(North Dakota State University, USA)*

"… there is a category of readers that would certainly benefit from this book: statisticians who have obtained their knowledge of matrix algebra in a piecemeal fashion and now want to systematically improve their knowledge of the theory and their skills in manipulating matrices."
Journal of the American Statistical Association

Readership: Graduate students, researchers and scientists in economics, biology, engineering and physics.

556pp Jul 1998 981-02-3268-3 US$80

Numerical Ranges and Radii cont'd

The social events included a banquet and a performance of the student fado group, Lacrima, from Coimbra. The lunches and coffee breaks led to some lively discussions.

The Workshop Programme can still be downloaded from the Workshop website: http:// www.mat.uc.pt/ilas2004/ Body.html (click "associated events").

The next meeting on Numerical Ranges and Radii will be held in 2006, probably in Canada. The previous workshops of this series were held at the United States, Portugal, Japan and Greece.

Directions in Combinatorial Matrix Theory Workshop

Report by **Bryan Shader**

The Directions in Combinatorial Matrix Theory workshop was held at the Banff International Research Station May 7-8, 2004, and attracted 29 researchers (10 from Canada, 15 from the U.S. and 4 from abroad) and 7 post-doctoral or graduate students. Talks discussed current developments and open problems in the following emerging themes in Combinatorial Matrix Theory: Spectral properties of families of matrices associated with graphs; Matrix theory and graph theory in the service of Euclidean geometry; Algebraic tools for combinatorial problems; and Spectral properties of classes of matrices. Titles and abstracts of the talks presented can be found at http://www.pims.math.ca birs/workshops/2004/04w2525/abstracts.pdf.

Participants were: Mahmoud Akelbek, Richard Anstee, Francesco Barioli, Wayne Barrett, Majid Behbahani, Avi Berman, Thomas Britz, Richard Brualdi, Robert Craigen, Michael Doob, Shaun Fallat, Miroslav Fiedler, Peter Gibson, Bob Grone, Leslie Hogben, Yury Ionin, Charlie Johnson, Hadi Kharaghani, In-Jae Kim, Steve Kirkland, Peter Lancaster, Hien Le, Chi-Kwong Li, Zhongsan Li, XiaoPing Liu, Raphi Loewy, Judi McDonald, Dale Olesky, Alex Pothen, Hans Schneider, Bryan Shader, Wasin So, Jeff Stuart, Michael Tsatsomeros, Kevin Vander Meulen, and Rokas Varaneckas.

The organizers were: Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Regina), Bryan Shader (University of Wyoming), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria).

Miroslav Fiedler was the ILAS speaker, and he spoke on "*Matrices and Graphs in Euclidean Geometry*." The Workshop's Open Problem sessions were highly successful, and a list of problems presented are posted at http://www.pims.math.ca/birs/workshops/2004/04w2525/ openprobs.pdf. New collaborative efforts resulting from the workshop are already noticeable. Results presented at the conference will be disseminated through a special 2005 issue of the Electronic Journal of Linear Algebra.

The workshop strengthened the participants' beliefs that the directions for research in Combinatorial Matrix Theory are abundant, promising, and central to mathematics.

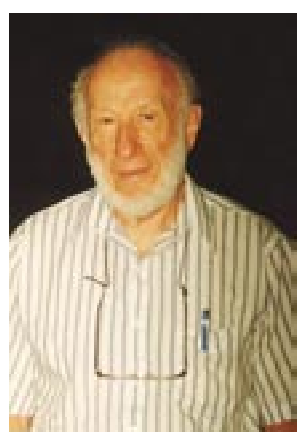
13th International Workshop on Matrices and Statistics, in Celebration of Ingram Olkin's 80th Birthday

http://matrix04.amu.edu.pl

Report by **Simo Puntanen**

The 13th International Workshop on Matrices and Statistics (IWMS-2004) was be held in Będlewo, about 30 km south of Poznań, Poland, from 18-21 August 2004. Będlewo is the Mathematical Research and Conference Center of the Polish Academy of Sciences; the setting is similar to Oberwolfach with accommodations on-site. The participants had a nice opportunity to enjoy the interesting meeting environment and facilities. The workshop was endorsed by the International Linear Algebra Society.

The center of attention was naturally Ingram Olkin,



Cont'd on page 12

Ingram Olkin's 80th Birthday cont'd

who gave the Nokia Lecture entitled "*Inequalities: some probabilistic, some matric, and some both*." Gene H. Golub gave the Opening Address on "*Numerical methods for solving least squares problems with constraints*." Other invited speakers were Theodore W. Anderson, Jerzy K. Baksalary, Rafael Bru, Carles M. Cuadras, Pierre Druilhet, Ludwig Elsner, Jürgen Gross, Jeffrey J. Hunter, Charles R. Johnson, Joachim Kunert, Teresa Ledwina, Erkki P. Liski, Peter Major, Nripesh K. Mandal, Richard J. Martin, Volker Mehrmann, Joao Tiago Mexia, Waldemar Ratajczak, Dietrich von Rosen, Bikas K. Sinha, George P.H. Styan, Gabor Tusnady, Béla Uhrin, Hans Joachim Werner, and Haruo Yanai. The total number of participants was 82, comprising participants from 18 countries.

The Local Organizing Committee was Jan Hauke, Augustyn Markiewicz (chair), Tomasz Szulc, and Waldemar Wołyński. The International Organizing Committee for this Workshop was R. William Farebrother (Shrewsbury, England), Simo Puntanen (Tampere, Finland; chair), George P. H. Styan (Montréal, Canada; vice-chair), and Hans Joachim Werner (Bonn, Germany).

Organizing institutes were Stefan Banach International Mathematical Center (Warsaw); Committee of Mathematics of the Polish Academy of Sciences (Warsaw); Faculty of Mathematics and Computer Science, Adam Mickiewicz University (Poznań); Institute of Socio-Economic Geography and Spatial Management, Faculty of Geography and Geology, Adam Mickiewicz University (Poznań); and Department of Mathematical and Statistical Methods, Agricultural University (Poznań). The workshop was sponsored by GlaxoSmithKline, Nokia, and SAS.

Augustyn Markiewicz and Waldemar Wołyński did an excellent job of editing the booklet "Program and Abstracts": the result is an unusually nice collector's item. It includes, in addition to the program and abstracts, biographies, an article and an interview of Ingram Olkin reprinted (from *Student* and from *Festschrift*) as well as his updated bibliography, and an article by R. W. Farebrother.

The organizers are pleased to announce a special issue of *Linear Algebra and Its Applications* devoted to this workshop. It will include selected papers strongly correlated to the talks of the conference. Submissions on the theory of matrices and methods of linear algebra with statistical origin or possible applications in statistics are encouraged.

All papers submitted must meet the publication standards of *Linear Algebra and Its Applications* and will be subject to normal refereeing procedure. The deadline for submission of papers is February 28, 2005, and the special issue should be published in 2006. Papers should be sent to any of the special editors, preferably by email in a PDF or PostScript format:

Ludwig Elsner, University of Bielefeld, Faculty of Mathematics, Postfach 100131, 33501 Bielefeld, Germany,
e-mail: elsner@Mathematik.Uni-Bielefeld.DE

Augustyn Markiewicz, Agricultural University of Poznań, Department of Mathematical and Statistical Methods, ul. Wojska Polskiego 28, 60-637 Poznań, Poland,
e-mail: amark@owl.au.poznan.pl

Tomasz Szulc, Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Umultowska 87, 61-614 Poznań, Poland, email: tszulc@amu.edu.pl.de

Volker Mehrmann (Editor-in-Chief for this special issue), Institut für Mathematik, MA4-5 Strasse des 17. Juni 136, D10623 Berlin, Germany, email: mehrmann@tu-berlin.de

IWMS-2004 group photo taken by Hazel Hunter in Będwelo

Hamilton Workshop on Nonnegative Matrix Theory
Maynooth Ireland

Report by **Avi Berman**

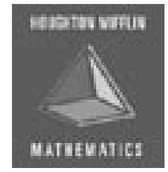
The Hamilton Institute at the National University of Ireland, Maynooth is an Applied Mathematics research center whose activities span System Theory, Communication Networks, Machine Learning and Cognitive Neuroscience, Mathematical Biology and Human Computer Interaction. These areas of research require significant interdisciplinary interaction between engineers, computer scientists, physical scientists and mathematicians, and part of the remit of the Institute is to promote such cooperation.

One of the mathematical disciplines that plays an important role in the research interests of the Hamilton Institute is Matrix Theory; in particular, the theory of Nonnegative Matrices. As part of this activity the Institute organized, with the support of the Science Foundation Ireland, a workshop on Nonnegative Matrices and their Applications that brought together matrix theorists and users of nonnegative matrices. The conference was held between July 11 and July 14, 2004 in Barberstown Castle and at the National University of Ireland, Maynooth. The organising committee consisted of A. Berman, T. Laffey (chair), O. Mason, R. Shorten (co-chair) and F. Wirth.

The program consisted of 29 talks given by speakers from Belgium, England, Germany, Ireland, Israel, Serbia, Taiwan and the United States. The past and present ILAS presidents, Hans Schneider, Richard Brualdi and Daniel Hershkowitz were the banquet speakers. The proceedings of the workshop will be published in a special volume of the *Electronic Journal of Linear Algebra*.

The workshop has already resulted in a number of collaborative projects between practitioners and mathematicians and the second Hamilton workshop on Nonnegative Matrices and their Applications is now currently being planned to further support and promote these activities.

HOUGHTON MIFFLIN 2004 MATHEMATICS



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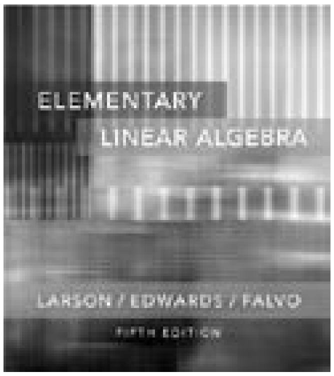

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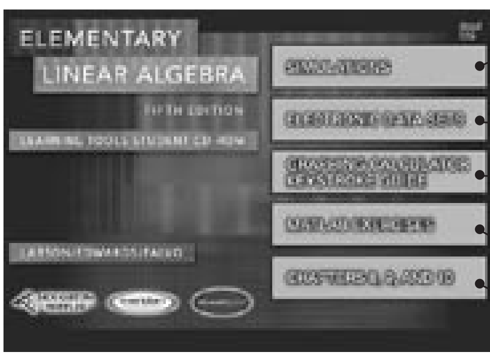

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Southern California Matrix Meeting

Report by **Jane Day and Wasin So**

The last "Southern California (SoCal) Matrix Meeting" was held on November 13, 2004, at San Jose State University, in honor of R.C. Thompson. Those present agreed to rename this event as the "R.C. Thompson Matrix Meeting," in memory of Bob's excellent research and love for matrix theory, as well as his generous mentoring of students interested in matrix theory.

The SoCal meetings have always been very informal one-day conferences. They were inspired by a mini conference on matrix theory organized by Steve Pierce in the early 1980's, parallel to an AMS meeting in Toronto. The mini was very well attended. After Pierce relocated to San Diego State University, they continued to hold an informal one-day matrix theory meeting each year. This happening soon was dubbed the "Southern California Matrix Meeting," or SoCal for short. Over the years, San Jose State University, Cal State Northridge, and even the University of Utah have hosted the meeting. The special charms found in each of these meetings have been good talks, plenty of time to discuss mathematics, no registration fee, and no support for speakers, but a free dinner afterwards!

The meeting on Nov. 13 at San Jose State University had about 40 participants from at least 4 countries and 6 states. The speakers were Charlie Johnson, Fuzhen Zhang, Fernando Szechtman, Leslie Foster, Hongbin Guo, Aaron Melman, Ilya Spitkovsky, C.K. Li, Roger Alperin, Maribel Bueno, Tin Yau Tam, Huajun Huang, and Lidia Elena Kozma. The next R.C. Thompson Matrix Meeting will be held in November 2005 at San Francisco State University.

Forthcoming Conferences and Workshops in Linear Algebra

14th International Workshop on Matrices and Statistics IWMS-2005

Auckland, New Zealand:
March 29-April 1, 2005
http://iwms2005.massey.ac.nz

The Local Organizing Committee has finalized details of IWMS-2005 to be held at Massey University, Albany Campus, Auckland, New Zealand over the period March 29-April 1, 2005. All intending participants are advised to consult the website http://iwms2005.massey.ac.nz and pre-register online.

The conference will have a wide variety of invited and contributed papers involving matrices in statistics, including applied probability. The following have accepted invitations to present Keynote Lectures: C.R. Rao (USA), Shayle Searle (USA), Eugene Seneta (Australia) and George Seber (New Zealand).

Invited Speakers include Gene Golub (USA), Stephen Haslett (New Zealand), Moshe Haviv (Israel), Nye John (New Zealand), Charles Johnson, (USA), A. Krishnamoorthy (India), Tõnu Kollo (Estonia), Alan Lee (New Zealand), Shuangzhe Liu (Australia), Bryan Manly (USA), Augustyn Markiewicz (Poland), Carl D. Meyer (USA), Robin Milne (Australia), Alastair Scott (New Zealand), Jennifer Seberry (Australia), Garry Tee (New Zealand), Götz Trenkler, (Germany), Hans Joachim Werner (Germany), and Keith Worsley (Canada).

The deadline for early registration and the submission of titles and abstracts of invited and contributed papers is February 11, 2005. A LATEX template is available on the workshop website.

The early registration fee is $NZ395 (includes participation in all sessions, workshop materials including book of abstracts, refreshments during the session breaks, lunches, opening reception, Workshop dinner). There is a reduced fee for students of $NZ145. Payment by Visa or Mastercard credit card is available with details being accepted by secure fax.

An excursion will be held on the afternoon of Wednesday, March 30th. This will involve a ferry trip to Waiheke with participants choosing either a visit to a Vineyard for wine tasting or a visit to a local Museum. The fee for the excursion (which includes transport from the Workshop venue to the ferry, the ferry fare, afternoon refreshments and return to the conference hotels) is $NZ85.00 An accompanying persons program has also been arranged involving an outing on the first day of the workshop with a guided tour of the Auckland Museum, a Maori cultural concert, a lunch, and returning to the conference venue for the opening reception. The program also includes the Workshop dinner.

IWMS-2005 is a Satellite meeting for 55th Biennial Session of the ISI, Sydney, April 5–12, 2005. It is being supported by Massey University, the New Zealand Institute of Mathematics and its Applications, the Royal Society of

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Book Review: *The Linear Algebra a Beginning Graduate Student Ought to Know* by Jonathan S. Golan, Kluwer, 2004. ISBN 140201824X

The author has undertaken an ambitious project: to write a book which assumes no previous knowledge of linear algebra, and includes all the standard material taught to beginning graduate students. According to the introduction, the book is intended to be used both as a textbook and for independent study.

The book begins with necessary preliminaries such as set theory and fields, then proceeds to vector spaces. The fourth chapter is a surprising addition: an introduction to algebras over a field, presented from an axiomatic point of view at a reasonably sophisticated level. Algebras form an integral part of the rest of the book.

The remaining chapters cover most of the usual topics: bases and dimension, linear transformations and matrices, systems of equations and Gaussian elimination, determinants, eigenvalues and eigenvectors, diagonalizing, Jordan form, the dual space, inner products, unitary similarity, and self-adjoint, unitary, and normal transformations. The more elementary topics are covered thoroughly but rapidly. I would hate to try to learn Gaussian elimination from this book, but it is suitable for reviewing the material.

Some less usual topics are also included. The last three chapters introduce the Singular Value Decomposition, pseudoinverses and their relation to least-squares, and bilinear forms. Quotient spaces are not covered, or even mentioned.

The book is an odd combination of the elementary and the sophisticated. Chapter 4 introduces algebras, including Lie and Jordan algebras. This is a pretty high level of abstraction to someone who has just learned about vector spaces. There is a point to this: algebras are used later to give a rigorous development of (for example) matrices of polynomials, something which usually treated more informally. Many of the proofs also require a high level of sophistication and experience to realize that there are steps which are left out and to be able to fill in the details. At other points, very little is assumed of the reader: some proofs show all steps of calculations, and some of the exercises seem intended for the clueless. I can't imagine a student who would need the elementary material and be able to understand the advanced ideas which precede it.

There are a number of interesting and unusual features of the book. Vector spaces are not assumed (in general) to be finite dimensional, and the author points out situations in which the theory differs for finite and infinite dimensions. Throughout most of the book, the theory is developed for general fields and examples using finite fields are included. The examples are one of the strongest points of the book: both in the text and in the exercises, they cover a broader spectrum of ideas than any other linear algebra texts that I know.

There are an impressive number of exercises—almost a thousand—which emphasize examples and which are quite different from those found in more traditional textbooks. The author has made an effort to integrate ideas from other parts of mathematics, particularly analysis. However, there are relatively few problems requiring proofs of general results.

There are many historical footnotes, emphasizing mathematicians who contributed to the subject and including photographs. These help give the subject a human face, but at times the descriptions seem superficial and the connections forced.

The book includes material which gives linear algebra a larger context. For various topics, the author discusses computational issues, algorithms, related advanced topics, and applications. It's unfortunate that there are no suggestions for further reading, and no references.

There are a number of problems with the book. The most troubling of these is the number of errors both in the text and in the exercises. Some of these appear to be a matter of poor proofreading, such as the statement on page 50 where "independent'' should be replaced by "dependent." However others are simply incorrect. For example, on page 127 it is stated that the product of upper triangular matrices is not necessarily upper triangular; and on page 343, exercise 857 states that $\|\alpha(v)\| \leq \|v\|$ for a self-adjoint linear transformation α. The book would have benefited from a careful mathematical proofreading, checking of theorem numbers, and being copy-edited for correct English.

The introduction says that the book assumes a "modicum of mathematical sophistication on the part of the reader." It should have gone on to warn that many statements will require the reader to stop and supply the proof. Since the book is intended for beginning graduate students (and encourages self-directed study), it should give some guidance at least at the beginning as to what reading the book entails. The exposition is often terse and at times hard to follow. There are a number of places where proofs have something missing, and it is not clear if the author didn't notice or is merely expecting a lot (at times, I would say too much) of the reader. The supplementary paragraphs on applications and so on include statements which the reader cannot be expected to prove as they are beyond the scope of the book, and so it can be hard to tell if a statement should be proved, or just accepted.

I think that this is a book which any teacher of advanced linear algebra will be glad to own, particularly as a source of unusual problems and examples. I would not recommend it to a student as a way to learn the subject on their own, although I might suggest it to a good student as a way to review the material (with suitable warnings about reading with a pencil and paper, and not believing everything you read). There are both pros and cons for using this book as a text for a course, as this review has tried to demonstrate.

Reviewed by **Sylvia Hobart**
Department of Mathematics
University of Wyoming - Laramie, Wyoming

Workshop on Matrices and Statistics cont'd

New Zealand, Statistics New Zealand, and by SAS. IWMS-2005 is also being supported by the New Zealand Statistical Association through the participation of the New Zealand Statistical Association Visiting Lecturer for 2005, C.R. Rao. The workshop has been endorsed by ILAS, the International Linear Algebra Society.

The International Organising Committee (IOC) consists of George P.H. Styan (Canada; Chair), styan@math.mcgill.ca, Hans Joachim Werner (Germany; Vice Chair), and Simo Puntanen (Finland). The Local Organising Committee (LOC) is chaired by Jeffrey Hunter (New Zealand), j.hunter@massey.ac.nz. The Workshop secretary is Freda Anderson, F.Anderson@massey.ac.nz.

Brualdi-fest: Linear Algebra, Graph Theory and Combinatorics Conference

University of Wisconsin-Madison:
30 April-1 May, 2005

The University of Wisconsin-Madison will host an informal, two-day conference, Brualdi-fest: Linear Algebra, Graph Theory and Combinatorics, from April 30 to May 1, 2005. The conference will be in honor of Richard Brualdi and his numerous contributions to mathematics. The keynote speaker will be Richard Wilson (CalTech).

The goals of the conference are to disseminate some of the recent successes of linear algebra, graph theory, and combinatorics to the mathematical community; discuss various emerging mathematical topics, applications and technologies for which linear algebraic and combinatorial techniques are needed; and bring together Brualdi enthusiasts to celebrate Richard's career.

Several sessions of contributed talks are planned. If you would like to present a talk, please send a title and abstract to Bryan Shader (bshader@uwyo.edu) by March 15, 2005.

The conference will be held at the Pyle Conference Center, http://conferencing.uwex.edu/pyle.cfm, on the University of Wisconsin-Madison campus. A block of rooms has been reserved at the nearby Lowell Inn, http://conferencing.uwex.edu/lowell.cfm.

Reservations for these rooms must be made by March 29, 2005. Additional lodging possibilities can be found at http://www.math.wisc.edu/~lemmpp/info/hotels.html.

Further details about the conference will be posted on the conference web page, http://math.uwyo.edu/~bshader/rabconf.html.

The organizing committee consists of John Goldwasser (University of West Virginia), Hans Schneider (University of Wisconsin-Madison), Bryan Shader (University of Wyoming), and Bob Wilson (University of Wisconsin-Madison).

Conference in Honor of Heydar Radjavi's 70th Birthday

Bled, Slovenia:
14-15 May 2005

The special emphasis of this conference is on topics in linear algebra and operator theory related to H. Radjavi's work. It will consist of invited and contributed talks.

The main invited speaker is P. Rosenthal. The following are invited participants: E.A. Azoff, R. Bhatia, P. Binding, L. Grunenfelder, R. Guralnick, D. Hadwin, J. Holbrook, T. J. Laffey, J. Leech, C. K. Li, L. Livshits, R. Loewy, V. Lomonosov, G. MacDonald, B. Mathes, M. Mathieu, R. Meshulam, V. Müller, J. Okniński, M. Radjabalipour, H. Radjavi, L. Rodman, B. A. Sethuraman, V. Shulman, A. Sourour, Y. Turovskii, A. Villena, and J. Zemanek.

The organizing committee consists of M. Brešar, L. Grunenfelder, T. Košir, M. Omladič (Chair), P. Rosenthal, and P. Šemrl.

For further information please contact the Secretary of the conference:

Damjana Kokol Bukovšek
Institute of Mathematics, Physics, and Mechanics
Jadranska 19, SI-1000 Ljubljana, Slovenia
Phone: +386 1 476 65 50
Fax: +386 1 251 72 81
E-mail: Damjana.Kokol@fmf.uni-lj.si

Additional information about the conference can be found on the conference web page: http://www.law05.si/hrc.

Householder Symposium XVI

Seven Springs Mountain Resort,
Champion Pennsylvania:
23-27 May, 2005

The Householder Symposium on Numerical Linear Algebra will be held May 23-27, 2005 at the Seven Springs Mountain Resort in Champion, Pennsylvania, http://www.7springs.com/. The resort is located about one hour (by car) southeast of Pittsburgh. This meeting is the sixteenth in a series, previously called the Gatlinburg Symposia. The name honors Alston S. Householder, one of the pioneers in numerical linear algebra and organizer of the first four meetings. The meeting has traditionally been held in an isolated location and is very informal in style. Each attendee is given the opportunity to present a talk, but a talk is not mandatory. The format of the meeting includes scheduled presentations during the day and more informal evening sessions that are organized electronically shortly before the meeting. Spirited discussion is encouraged.

Cont'd on page 18

Householder Symposium cont'd

At the meeting, the twelfth Householder prize will be awarded for the best thesis in numerical algebra written since January 1, 2002. We hope that the meeting will be attended by recent entrants into numerical linear algebra as well as more experienced researchers. We encourage attendance by core numerical linear algebra researchers, matrix theoreticians, and researchers in applications such as optimization, signal processing, control, etc.

The Program Committee welcomes your contribution. The meeting facility holds only 125 people, however, so attendance may need to be limited. The committee is seeking funding to provide financial assistance to recent PhDs and others who might need it. For full consideration, the committee must receive your abstract by 1 December 2004. Information concerning the application process may be found at the URL listed above. Please use the format provided at the conference website: http://www.cse.psu.edu/~zha/householder. The committee expects to complete the list of attendees and scheduled presentations by 7 January 2005.

If you have any questions about local arrangements, please contact the local arrangements committee at: householder2005@cse.psu.edu.

The local arrangements committee is Jesse Barlow (Penn State University), Hongyuan Zha (Penn State University), Daniel Szyld (Temple University). Other questions can be directed to: house-request@cs.cornell.edu

The program committee consists of Angelika Bunse-Gerstner (Bremen), Tony Chan (UCLA), Alan Edelman (MIT), Nick Higham (University of Manchester), Roy Mathias (College of William and Mary), Dianne O'Leary (University of Maryland), Michael Overton (New York University), Henk van der Vorst (Utrecht), Paul Van Dooren (Louvain-la-Neuve), and Charles Van Loan (Chair, Cornell University).

Second International Workshop on Combinatorial Scientific Computing (CSC05)

Toulouse, France:
June 21-23, 2005
http://www.cerfacs.fr/algor/CSC05

The Second International Workshop on Combinatorial Scientific Computing (CSC05) will be organized at CERFACS, Toulouse, June 21-23, 2005. The workshop, organized in cooperation with SIAM, CERFACS, ENSEEIHT-IRIT, and INRIA, will provide a forum for researchers interested in the interaction of combinatorial mathematics and algorithms with scientific computing to discuss current developments in research.

CSC05 follows the pioneering SIAM Workshop on Combinatorial Scientific Computing (CSC04) held at San Francisco in February 2004. The CSC04 Workshop, attended by close to a hundred participants, featured three plenary talks and 21 selected talks on the themes of sparse matrix computations, high-performance algorithms, combinatorial problems in optimization and automatic differentiation, mesh generation, computational biology, and combinatorial matrix theory. The CSC05 Workshop aims to bring together researchers interested in these themes as well as other aspects of combinatorial mathematics and algorithms in scientific computing, broadly interpreted. Researchers in emerging application areas as well as theoretical areas that intersect with combinatorial scientific computing, e.g., information science, networks, bioinformatics, and combinatorial optimization, are invited to participate.

Contributed presentations in lecture format are invited in all areas consistent with the workshop themes. A 2-page extended abstract, in PDF format, of a proposed talk should be submitted by February 21, 2005. The submission procedure is described on the web page for the conference (listed above). Authors will be notified of acceptance of their talks by mid March.

Toulouse, the capital of the Midi-Pyrenees region of France, is a European center for science and technology, and is close to the Pyrenees mountains, the Mediterranean Sea, Carcassone, and many tourist attractions. The Workshop will be held at the Conference Center of the Meteo France campus, where CERFACS is located. A limited number of rooms (single and double) will be available at the Meteo.

Further details on workshop registration, hotels and housing options, etc., will be posted as they become available at the conference web page.

The local committee is: Patrick R. Amestoy (IRIT, Toulouse and ScAlApplix, INRIA), Iain Duff (CERFACS, France and Rutherford Appleton Lab, UK), Luc Giraud (CERFACS, France), Serge Gratton (CERFACS, France) and Brigitte Yzel (Secretary).

The organizing committee is: Iain Duff, co-chair (CERFACS and Rutherford Laboratory), John Gilbert, co-chair, (University of California, Santa Barbara), Alex Pothen, co-chair (Old Dominion University), Patrick R. Amestoy (IRIT, Toulouse and ScAlApplix, INRIA), Rob Bisseling (University of Utrecht), Andreas Griewank (Humboldt University, Berlin), Jean-Yves L'Excellent (INRIA, Lyon), Cynthia A. Phillips (Sandia National Labs), and Bryan Shader, (University of Wyoming).

12th ILAS Conference
Regina, Saskatchewan, Canada:
26-29 June, 2005

The 12th meeting of the International Linear Algebra Society will be held at the Regina Inn and Conference Centre, June 26-29, 2005, in Regina, Canada. The Scientific Organizers for the meeting are Rajendra Bhatia, R. Guralnick, Danny Hershkowitz (ILAS President), Steve Kirkland (Chair), Volker Mehrmann, Bit-Shun Tam, Pauline van den Driessche and Henry Wolkowicz. The local organizing committee consists of Shaun Fallat, Doug Farenick, Chun-Hua Guo and Steve Kirkland.

The invited speakers are:

Avi Berman (Technion, Israel)
Jim Demmel (University of California Berkeley, USA)
Dragomir Djokovic (University of Waterloo, Canada)
Anne Greenbaum (University of Washington, USA)
Olga Holtz (University of California Berkeley, USA)
Ilse Ipsen (North Carolina State University, USA)
Charles Johnson (College of William and Mary, USA)
Peter Lax (Courant Institute, USA)
Adrian Lewis (Cornell University, USA)
Ren-Cang Li (University of Kentucky, USA)
Raphael Loewy (Technion, Israel)
João Queiró (University of Coimbra, Portugal)
Peter Rowlinson (University of Stirling, Scotland)
Bryan Shader (University of Wyoming, USA)
Christiane Tretter (Universität Bremen, Germany)
Pei Yuan Wu (National Chiao Tung University, Taiwan)
Xingzhi Zhan (East China Normal University, China)

In addition, the meeting will feature the following special lectures:

The LAMA Lecture:
Chi Kwong Li (College of William and Mary, USA)
The ILAS Education Lecture:
Anna Sierpinska (Concordia University, Canada)
The LAA Lecture: T.B.A.,
The Hans Schneider Prize Lecture: T.B.A.

Several mini-symposia will also take place, including Preserver Problems, organized by Chi-Kwong Li and Peter Šemrl, and Spectral Properties of Families of Matrices Described by Patterns or Graphs, organized by Leslie Hogben.

Linear Algebra and its Applications will publish a special issue devoted to papers presented at the conference; the editors for the special issue are Rajendra Bhatia, Robert Guralnick, Steve Kirkland and Henry Wolkowicz.

Social activities include a box lunch, courtesy of ILAS, on June 26, and a banquet to be held on June 28, featuring Chandler Davis as the after-dinner speaker.

The deadline for submission of abstracts for contributed papers is May 1, 2005, and the deadline for on-line conference registration is May 25, 2005. For more information regarding registration, accommodation, travel and support for students and post-docs, please visit the conference webpage at http://www.math.uregina.ca/~ilas2005

International Workshop on Operator Theory and Applications
University of Connecticut, Storrs, USA
24-27 July, 2005

The purpose of IWOTA 2005 is to bring together mathematicians and engineers interested in operator theory and its applications. Adhering to a tradition started at the previous IWOTA meetings, the meeting will be focused on a few special themes, without losing sight of the general IWOTA mission. Our special interest areas are:

* operator theory and function theory.
* system theory and control theory,
* structured matrices and efficient computations

Apart from these, we welcome proposals on special sessions, especially in traditional IWOTA areas.

This IWOTA meeting will be the sixteenth in a series of highly successful IWOTA meetings. The previous IWOTA meetings were held in Santa Monica (1981), Rehovot (1983), Amsterdam (1985), Mesa (1987), Rotterdam (1989), Sapporo (1991), Vienna (1993), Regensburg (1995), Bloomington (1995), Groningen (1998), Bordeaux (2000), Faro (2000), Blacksburg (2002), Cagliari (2003), and Newcastle (2004). The organizers of the present meeting intend to adhere to the high standards set by these previous meetings.

The IWOTA 2005 steering committee includes I. Gohberg-President (Tel Aviv), T. Ando (Sapporo), J.A. Ball (Blacksburg), H. Bart (Rotterdam), H. Bercovici (Bloomington), A.F. dos Santos (Lisbon), A. Dijksma (Groningen), M. Dritschel (Newcastle), H. Dym (Rehovot), C. Foias (Bloomington), J.W. Helton-Vice President (La Jolla), M.A. Kaashoek-Vice President (Amsterdam), M. Klaus (Blacksburg), H. Langer (Vienna), C.V.M. van der Mee (Cagliari), R. Mennicken (Regensburg), N.K. Nikolskii (Bordeaux), L. Rodman (Williamsburg), S. Seatzu (Cagliari), G. Stampfli (Bloomington), and N. Young (Newcastle).

It is a great honor for us, the local organizers, Vadim Olshevsky, Israel Koltracht, Michael Neumann, William Abikoff and Ron Blei to invite you to come to IWOTA 2005. We also invite you to make suggestions as to the program and the topics to be discussed.

We look forward to seeing you next summer in Storrs!

IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 23-1 [IMAGE 23 (October 1999), p. 28], 32-1 through 32-3, and 32-5 through 32-7 [IMAGE 32 (April 2004), pp. 40 & 39]. Problems 30-3 [IMAGE 30 (April 2003), p. 36] and 32-4 [IMAGE 32 (April 2004), p. 40] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 8 new problems on pp. 36 & 35 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to ujw902@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 23-1: The Expectation of the Determinant of a Random Matrix

Proposed by Moo Kyung CHUNG, *University of Wisconsin–Madison, Madison, Wisconsin, USA:* mchung@stat.wisc.edu

Let the $m \times n$ random matrix X be such that $\mathrm{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where vec indicates the vectorization operator for a matrix, the $m \times m$ matrix A is symmetric non-negative definite, $\otimes$ stands for the Kronecker product, $m > n$, and I_n is the $n \times n$ identity matrix. For a given $m \times m$ symmetric matrix C, find $\mathsf{E}\{\det(X'CX)\}$ in a closed form involving only C and A. Is this possible? (Finite summation would also be fine.)

Solution 23-1.1 by William KNIGHT, *University of New Brunswick, Fredericton, New Brunswick, Canada:* knight@unb.ca

THEOREM. *Let the $m \times n$ random matrix X be such that $\mathrm{vec}(X)$ is distributed as multivariate normal $\mathrm{N}(0, A \otimes I_n)$, where A is a given $m \times m$ nonnegative definite and symmetric matrix. For a given $m \times m$ symmetric matrix C, let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$ be the eigenvalues of $A^{1/2}CA^{1/2}$. Then*

$$\mathsf{E}\{|X'CX|\} = (n!)\psi_n(\lambda_1, \lambda_2, \ldots, \lambda_m),$$

where $\psi_n(\cdot)$ is nth elementary symmetric function of $\lambda_1, \cdots, \lambda_m$, i.e., the sum of all products of n distinct λ's, and $|\cdot|$ denotes the determinant. For example,

$$\begin{aligned}
\psi_1(\lambda_1, \lambda_2, \ldots, \lambda_m) &= \mathsf{trace}(A^{1/2}CA^{1/2}), \\
\psi_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \\
\psi_m(\lambda_1, \lambda_2, \ldots, \lambda_m) &= |A^{1/2}CA^{1/2}|.
\end{aligned}$$

PROOF. We proceed in three steps:
1. Reduction to the special case where $A = I_m$ and C is diagonal.
2. Solving the special case for $m = n$.
3. Induction on m for $m > n$.

1. Reduction: Let Y be an $m \times n$ random matrix whose elements are independent N(0,1). $A^{1/2}Y$ has the same multivariate normal distribution as X and so

$$\mathsf{E}\{|Y'A^{1/2}CA^{1/2}Y|\} = \mathsf{E}\{|(X'CX|\}\ .$$

Some orthogonal matrix, Q say, diagonalizes $A^{1/2}CA^{1/2}$, i.e., $A^{1/2}CA^{1/2} = Q'\Lambda\, Q$. After orthogonal transformation, a spherical normal distribution remains spherical normal, so $Z = QY$ still has independent N(0,1) elements, and

$$\mathsf{E}\{|X'CX|\} = \mathsf{E}\{|Y'A^{1/2}CA^{1/2}Y|\} = \mathsf{E}\{|Y'Q'\Lambda QY|\} = \mathsf{E}\{|Z'\Lambda Z|\}.$$

2. The special case $m = n$: ZZ' is an $n \times n$ Wishart matrix, and as such, its expected determinant is $n!$ [see Anderson (1960, Chapter 7.5.2, formula (19))], and so

$$\mathsf{E}\{|Z'\Lambda Z|\} = \mathsf{E}\{|ZZ'\Lambda|\} = \mathsf{E}\{|ZZ'|\}|\Lambda| = n!\,(\prod_{i=1}^{n}\lambda_i) = n!\,\psi_n(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\ .$$

3. Induction on m over $m > n$: There is an $n \times n$ Householder matrix, H, which rotates the last row of Z into its last coordinate:

$$ZH = W = W_{(m+1,n)} = \begin{pmatrix} W_{(m,n-1)} & w \\ 0 & \omega \end{pmatrix}.$$

Here, subscripts of the matrices W denote their sizes, w is a column vector, and the scalar ω, the norm of the last row of Z, is the square root of a χ^2 random variable with n degrees of freedom. Define $S_{(\mu,\nu)}$ as $[W_{(\mu,\nu)}]'\Lambda_{(\mu)}W_{(\mu,\nu)}$ (subscripts of S match those of the associated W and $\Lambda_{(\mu)}$ is the leading $\mu\times\mu$ submatrix of Λ). For induction, partition S as follows (these equations also define vector s and scalar σ):

$$S_{(m,n-1)} = [W_{(m,n-1)}]'\Lambda_{(m)}W_{(m,n-1)},$$

$$S_{(m,n)} = [W_{(m,n)}]'\Lambda_{(m)}W_{(m,n)} = \begin{pmatrix} [W_{(m,n-1)}]' \\ w' \end{pmatrix} \Lambda_{(m)} \left(W_{(m,n-1)}, w\right) = \begin{pmatrix} S_{(m,n-1)} & s \\ s' & \sigma \end{pmatrix},$$

$$S_{(m+1,n)} = [W_{(m+1,n)}]'\Lambda_{(m+1)}W_{(m+1,n)} = \begin{pmatrix} [W_{(m,n-1)}]' & 0 \\ w' & \omega \end{pmatrix}' \Lambda_{(m+1)} \begin{pmatrix} W_{(m,n-1)} & w \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} S_{(m,n-1)} & s \\ s' & \sigma+\lambda_{m+1}\omega^2 \end{pmatrix}.$$

Since H depends only on the last row of Z, and $W_{(m,n)}$ depends only on the first m rows of Z, the conditional distribution of $W_{(m,n)}$, given the last row of Z, is spherically normal. It follows that the conditional distributions of $W_{(m,n)}$ and of $S_{(m,n)}$, given ω, are spherically normal. Expand the determinant of $S_{(m+1,n)}$ on its last row,

$$|S_{(m+1,n)}| = \begin{vmatrix} S_{(m,n-1)} & s \\ s' & \sigma+\lambda_{m+1}\omega^2 \end{vmatrix} = \begin{vmatrix} S_{(m,n-1)} & s \\ s' & \sigma \end{vmatrix} + \begin{vmatrix} S_{(m,n-1)} & s \\ 0 & \lambda_{m+1}\omega^2 \end{vmatrix} = |S_{(m,n)}| + \lambda_{m+1}\,\omega^2\,|S_{(m,n-1)}|\,,$$

and take its expectation: $\mathsf{E}\{|S_{(m+1,n)}|\} = \mathsf{E}\{|S_{(m,n)}|\} + \lambda_{m+1}n\mathsf{E}\{|S_{(m,n-1)}|\} = n!\psi_n(\lambda_1,\lambda_2,\ldots,\lambda_m) + \lambda_{m+1}n(n-1)!\psi_{n-1}(\lambda_1,\lambda_2,\ldots,\lambda_m) = n!\psi_n(\lambda_1,\lambda_2,\ldots,\lambda_{m+1})$. □

Reference

T. W. Anderson (1960). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1,\ldots,c_{n-1} \in \mathbb{C}\setminus\{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

We look forward to receiving solutions to Problem 30-3!

Problem 32-1: Factorizations of Nonsingular Matrices by Means of Corner Matrices

Proposed by Richard W. FAREBROTHER, *Bayston Hill, Shrewsbury, England:* R.W.Farebrother@manchester.ac.uk

Show that any nonsingular $n \times n$ matrix A may be expressed as the product of

(a) two southwest and one northeast corner matrices,

(b) two northeast and one southwest corner matrices,

(c) three northwest corner matrices, and

(d) three southeast corner matrices,

where an $n \times n$ matrix A is called a southwest corner (or lower triangular) matrix if it satisfies $a_{ij} = 0$ for $i < j$, a northeast corner (or upper triangular) matrix if it satisfies $a_{ij} = 0$ for $i > j$, a northwest corner matrix if it satisfies $a_{ij} = 0$ for all i, j satisfying $i + j > n + 1$, and a southeast corner matrix if it satisfies $a_{ij} = 0$ for all i, j satisfying $i + j < n - 1$.

Solution 32-1.1 by the Proposer Richard W. FAREBROTHER, *Bayston Hill, Shrewsbury, England:* R.W.Farebrother@manchester.ac.uk

(a): It is well known that the Gaussian elimination procedure may be employed in an attempt to reduce any given square matrix A to upper triangular form by premultiplying it by a sequence of elementary lower triangular matrices. If, at any stage, we encounter a zero element on the diagonal of the partially reduced matrix, then we may often eliminate it by postmultiplying the matrix by a suitable elementary lower triangular matrix (where elementary triangular matrices are identity matrices with a single nonzero off-diagonal element).

In the exceptional case when all the elements in a particular row of the reduced matrix take zero values, then the matrix is singular and has a determinant of zero.

In the case in which no such row of zeroes occurs, the matrix is nonsingular and has a nonzero determinant. Further, since each of the elementary lower triangular matrices is invertible, we deduce that any nonsingular matrix may be expressed as the product of a unit lower triangular matrix, an upper triangular matrix, and a unit lower triangular matrix $A = L_1 U_2 L_3$, where the unit lower triangular matrices L_1, L_3 have unit elements on their diagonals and the upper triangular matrix U_2 is not restricted in this way.

(b): Now, applying the result of part (a) to the transpose $B = A'$ of A we have $A' = L_1 U_2 L_3$, whence we deduce that A may be expressed as the product of a unit upper triangular matrix, a lower triangular matrix, and a unit upper triangular matrix, $A = L'_3 U'_2 L'_1$.

(c): Let J represent the $n \times n$ matrix with unit elements on its secondary diagonal $i + j = n + 1$ and zeroes elsewhere. Then $JJ = I$ is the identity matrix, and $L_2 = JU_2J$ is a $n \times n$ lower triangular matrix if U_2 is an upper triangular matrix.

In part (a) we established that any nonsingular matrix $C = JA$ may be expressed as the product of a lower triangular matrix, an upper triangular matrix, and a lower triangular matrix $JA = L_1 U_2 L_3$. Thus $A = JL_1 U_2 L_3 = JL_1 JL_2 JL_3$ may be expressed as the product of three northwest corner matrices JL_1, JL_2, JL_3.

(d): Similarly, any nonsingular matrix $D = AJ$ may be expressed in the form $AJ = L_1 U_2 L_3$. Thus, $A = L_1 U_2 L_3 J = L_1 J L_2 J L_3 J$, where $L_1 J, L_2 J, L_3 J$ are now southeast corner matrices.

Supplementary Example. For example, the 2×2 order-reversing matrix may be expressed as such a product:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Problem 32-2: A Property of Plane Triangles – Eadem Mutata Resurgo

Proposed by Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Let $\lambda \in \mathbb{R}_{>0}$. Apply to a plane triangle Δ the following process: go clockwise around Δ and divide its sides in the ratio $\lambda : 1$. Use the distances from the division points to the opposite vertices as side-lengths for a new triangle Δ' (cyclically again). Repeat the process with Δ' but divide with the ratio $1 : \lambda$ to obtain a triangle Δ''. Show that Δ'' is similar to Δ with the ratio $\rho = \sqrt{1 + 2\lambda + 3\lambda^2 + 2\lambda^3 + \lambda^4}/(1 + \lambda)^2$.

Solution 32-2.1 by the Proposer Alexander KOVAČEC, *Universidade de Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Let a, b, c be the sides of an original triangle $\Delta = \Delta ABC$ (in traditional notation) and let a', b', c' be the sides of Δ' obtained as distances from A to the division point on side a, etc. Then by Stewart's theorem, see e.g., Berger (1987, Fact 9.14.35, p. 277), we can set up the following matrix relation:

$$\begin{pmatrix} a'^2 \\ b'^2 \\ c'^2 \end{pmatrix} = \frac{1}{1+\lambda} \begin{pmatrix} -\frac{\lambda}{1+\lambda} & \lambda & 1 \\ 1 & -\frac{\lambda}{1+\lambda} & \lambda \\ \lambda & 1 & -\frac{\lambda}{1+\lambda} \end{pmatrix} \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}.$$

So we can identify the process of passing from Δ to Δ' as multiplying with the matrix P_λ shown here (including the factor). Similarly, passing from Δ' to Δ'' is multiplying with $P_{\frac{1}{\lambda}}$. Now, noting that $P_{\frac{1}{\lambda}} = P_\lambda^T$ and verifying $P_\lambda^T P_\lambda = \rho^2 I_3$ concludes the proof.

Reference

M. Berger (1987). *Geometrie I.* Springer-Verlag, Berlin.

Solution 32-2.2 by Leo LIVSHITS, *Colby College, Waterville, Maine, USA:* llivshi@colby.edu

Our notation is simplified if we say that the division of sides in the process described is done in the ratio $\alpha : \beta$ with $\alpha + \beta = 1$ and $\lambda = \alpha/\beta$. In the second process we divide in the ratio $\beta : \alpha$.

Given a plane triangle $\triangle OPQ$ with alphabetic order followed counterclockwise, let us treat O as the origin and write $U = \vec{OP}$ and $W = \vec{OQ}$. After the first iteration of divisions is performed on $\triangle OPQ$, the lengths of the sides of the triangle $\triangle'$ are $||U + \beta(W - U)||$, $||\alpha W - U||$ and $||W - \beta U||$, and the order matters. Since

$$(U + \beta(W - U)) + (\alpha W - U) = W - \beta U,$$

we can assume without loss of generality that $\triangle' = \triangle OP'Q'$ with alphabetic order followed counterclockwise, where $U + \beta(W - U) = \vec{OP'}$ and $W - \beta U = \vec{OQ'}$. Let us express this as

$$\begin{pmatrix} \vec{OP'} \\ \vec{OQ'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \vec{OP} \\ \vec{OQ} \end{pmatrix}.$$

Performing the second process on $\triangle OP'Q'$ results in a triangle $\triangle OP''Q''$, where

$$\begin{pmatrix} \vec{OP''} \\ \vec{OQ''} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} \vec{OP'} \\ \vec{OQ'} \end{pmatrix};$$

simply interchange α and β. Hence

$$\begin{pmatrix} \vec{OP''} \\ \vec{OQ''} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \vec{OP} \\ \vec{OQ} \end{pmatrix} = \begin{pmatrix} 0 & \beta^2 + \alpha \\ -\beta - \alpha^2 & 1 - \alpha\beta \end{pmatrix} \begin{pmatrix} \vec{OP} \\ \vec{OQ} \end{pmatrix}.$$

Since $\beta^2 + \alpha = 1 - \alpha\beta = \beta + \alpha^2$ whenever $\alpha + \beta = 1$, we get

$$\begin{pmatrix} \vec{OP''} \\ \vec{OQ''} \end{pmatrix} = \begin{pmatrix} 0 & 1 - \alpha\beta \\ -(1 - \alpha\beta) & 1 - \alpha\beta \end{pmatrix} \begin{pmatrix} \vec{OP} \\ \vec{OQ} \end{pmatrix}$$

so that $\vec{OP''} = (1 - \alpha\beta)\vec{OQ}$, $\vec{OQ''} = (1 - \alpha\beta)(\vec{OQ} - \vec{OP}) = (1 - \alpha\beta)\vec{PQ}$, $\vec{P''Q''} = \vec{P''O} + \vec{OQ''} = (1 - \alpha\beta)(\vec{PQ} + \vec{QO}) = (1 - \alpha\beta)\vec{PO}$, which shows that $\triangle'' = \triangle OP''Q''$ is similar to triangle $\triangle = \triangle OPQ$ with the ratio $1 - \alpha\beta$. A quick symbolic calculation (I recommend use of a computer algebra package here!) shows that this equals the expression for ρ given in the problem, which incidentally simplifies to

$$\rho = \frac{1 + \lambda + \lambda^2}{(1 + \lambda)^2},$$

after the substitutions $\lambda = \alpha/\beta$ and $\alpha + \beta = 1$ are made.

Problem 32-3: Jacobians for the Square-Root of a Positive Definite Matrix

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au
and Heinz NEUDECKER, *University of Amsterdam, Amsterdam, The Netherlands:* H.Neudecker@uva.nl

Establish the following Jacobian matrices:

$$\frac{\partial \mathrm{v}(\mathrm{X}^{1/2})}{\partial \mathrm{v}'(\mathrm{X})} = D^+(X^{1/2} \otimes I + I \otimes X^{1/2})^{-1} D, \quad \frac{\partial \mathrm{v}(\mathrm{X}^{-1/2})}{\partial \mathrm{v}'(\mathrm{X})} = -D^+(X^{1/2} \otimes X + X \otimes X^{1/2})^{-1} D,$$

where X is an $n \times n$ positive definite matrix, $X^{1/2}$ is its positive definite square root, D is the $n^2 \times n(n+1)/2$ duplication matrix, D^+ is its Moore-Penrose inverse, I is the $n \times n$ identity matrix, $\mathrm{v}'(\cdot)$ denotes the transpose of $\mathrm{v}(\cdot)$, $\mathrm{v}(\cdot)$ denotes the $n(n+1)/2 \times 1$ vector that is obtained from $\mathrm{vec}(\cdot)$ by eliminating all supradiagonal elements of the matrix and $\mathrm{vec}(\cdot)$ transforms the matrix into a vector by stacking the columns of the matrix one underneath the other.

Solution 32-3.1 by the Proposers Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au and Heinz NEUDECKER, *University of Amsterdam, Amsterdam, The Netherlands:* H.Neudecker@uva.nl

(1) Taking the differential of $X^{1/2}X^{1/2} = X$, we get $(dX^{1/2})X^{1/2} + X^{1/2}(dX^{1/2}) = dX$. Using the vec operator and D so that $\mathrm{vec}(X^{1/2}) = D\mathrm{v}(X^{1/2})$, we get

$$(X^{1/2} \otimes I + I \otimes X^{1/2})d\mathrm{vec}(X^{1/2}) = d\mathrm{vec}(X), \quad (X^{1/2} \otimes I + I \otimes X^{1/2})Dd\mathrm{v}(X^{1/2}) = Dd\mathrm{v}(X).$$

Rearranging the terms and using $D^+D = I_{n(n+1)/2}$, we get

$$\frac{\partial \mathrm{v}(\mathrm{X}^{1/2})}{\partial \mathrm{v}'(\mathrm{X})} = [D^+(X^{1/2} \otimes I + I \otimes X^{1/2})D]^{-1} = D^+(X^{1/2} \otimes I + I \otimes X^{1/2})^{-1}D.$$

(2) Taking the differential of $X^{-1/2}X^{-1/2} = X^{-1}$, we get $(dX^{-1/2})X^{-1/2} + X^{-1/2}(dX^{-1/2}) = -X^{-1}(dX)X^{-1}$. Using the vec operator we get

$$(X^{-1/2} \otimes I + I \otimes X^{-1/2})Dd\mathrm{v}(X^{-1/2}) = -(X^{-1} \otimes X^{-1})Dd\mathrm{v}(X).$$

Rearranging the terms we get

$$\begin{aligned} \frac{\partial \mathrm{v}(\mathrm{X}^{-1/2})}{\partial \mathrm{v}'(\mathrm{X})} &= -[D^+(X^{-1/2} \otimes I + I \otimes X^{-1/2})D]^{-1}[D^+(X^{-1} \otimes X^{-1})D] \\ &= -D^+(X^{-1/2} \otimes I + I \otimes X^{-1/2})^{-1}DD^+(X^{-1} \otimes X^{-1})D \\ &= -D^+(X^{-1/2} \otimes I + I \otimes X^{-1/2})^{-1}(X^{-1} \otimes X^{-1})D \\ &= -D^+(X^{1/2} \otimes X + X \otimes X^{1/2})^{-1}D. \end{aligned}$$

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j}X_1 + u_{2j}X_2 + u_{3j}X_3$, $j = 1, 2, 3$, such that each of the six matrices $W_j^{-1}W_k$, $j \neq k$, has zero trace. Equivalently, we want $(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We look forward to receiving solutions to Problem 32-4!

Problem 32-5: Diagonal Matrices Solving a Matrix Equation

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $A \in \mathbb{R}^{l\times m}$, $B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{l\times n}$ be given matrices. Find all vectors $x = (x_1, \ldots, x_m)' \in \mathbb{R}^m$ such that $A\,\mathrm{diag}(x_1, \ldots, x_m)B = C$.

Solution 32-5.1 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let e_i denote the i-th column of the identity matrix I_m. Then $\mathrm{diag}(x_1, \ldots, x_m) = \sum_{i=1}^m (e_i'x)(e_ie_i') = \sum_{i=1}^m (e_ie_i')(xe_i')$. It follows that $A\mathrm{diag}(x_1, \ldots, x_m)B = \sum_{i=1}^m (Ae_ie_i')x(e_i'B)$, and the equation $A\mathrm{diag}(x_1, \ldots, x_m)B = C$ is equivalent to

$\sum_{i=1}^{m} \mathrm{vec}\left((Ae_ie_i')x(e_i'B)\right) = \mathrm{vec}(C)$ or $\left(\sum_{i=1}^{m} B'e_i \otimes Ae_ie_i'\right)x = \mathrm{vec}(C)$. Note that $(B'e_i)'$ is the i-th row of B and Ae_i is the i-th column of A. The latter equation is consistent if and only if $DD^+\mathrm{vec}(C) = \mathrm{vec}(C)$, where $D = \sum_{i=1}^{m} B'e_i \otimes Ae_ie_i'$ and D^+ denotes the Moore-Penrose inverse of D. In that case the general solution x is given by $x = D^+\mathrm{vec}(C) + (I_m - D^+D)z$, where $z \in \mathbb{R}^m$ is arbitrary.

Problem 32-6: A Vector Cross Product Property in $\mathbb{R}^3$

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

In Milne (1965, Ex. 22, p. 26) the following problem is posed: *"If a, b are given non-parallel vectors, and x and y vectors satisfying $x \times a = y \times b$, show that x and y are linear functions of a and b, and obtain their most general forms."* Generalize this problem as follows: For given vectors a, b, and c from $\mathbb{R}^3$, where a and b are linearly independent, show that there always exist vectors x, $y \in \mathbb{R}^3$ such that

$$x \times a + y \times b + c = 0.$$

Determine the general solution (x, y) to this equation. Note that "$\times$" denotes the vector cross product in $\mathbb{R}^3$.

Reference

E. A. Milne (1965). *Vectorial Mechanics.* Methuen, London.

Solution 32-6.1 by Leo LIVSHITS, *Colby College, Waterville, Maine, USA:* llivshi@colby.edu

Since under the given hypothesis $\{a, b, a \times b\}$ is a basis of $\mathbb{R}^3$, it is necessary and sufficient to find all $x, y \in \mathbb{R}^3$ such that the dot product of $x \times a + y \times b + c$ with each of $a, b, a \times b$ is zero. In other words, the task is to solve the system

$$y \times b \bullet a + c \bullet a = 0, \quad x \times a \bullet b + c \bullet b = 0, \quad x \times a \bullet a \times b + y \times b \bullet a \times b + c \bullet a \times b = 0. \tag{1}$$

We can write $x = \alpha a + \beta b + \gamma a \times b$ and $y = \delta a + \epsilon b + \lambda a \times b$ with $\alpha, \beta, \gamma, \delta, \epsilon, \lambda \in \mathbb{R}$, so that the system (1) becomes (after simplification) the system

$$\lambda(a \times b) \times a \bullet b = c \bullet a, \quad \gamma(a \times b) \times a \bullet b = -c \bullet b, \quad (\delta - \beta)||a \times b||^2 = -c \bullet a \times b,$$

in six real variables $\alpha, \beta, \gamma, \delta, \epsilon, \lambda$.

Recall that $v_1 \bullet v_2 \times v_3$ is the determinant of the matrix whose i-th column is v_i. In particular, $(a \times b) \times a \bullet b$ is the determinant of the invertible matrix with columns $b, a \times b, a$, and so is not zero. Hence the general solution to system (1) is:

$$(x, y) = \left(\alpha a + \beta b - \frac{c \bullet b}{(a \times b) \times a \bullet b}\, a \times b, \quad \left(\beta - \frac{c \bullet a \times b}{||a \times b||^2}\right)a + \epsilon b + \frac{c \bullet a}{(a \times b) \times a \bullet b}\, a \times b\right),$$

where α, β, ϵ are free real variables.

Solution 32-6.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Since a and b are linearly independent, $|a|^2|b|^2 - (a \cdot b)^2 > 0$ and $\{a, b, a \times b\}$ is a basis for $\mathbb{R}^3$. Let

$$x = \alpha_1 a + \alpha_2 b + \alpha_3(a \times b), \quad y = \beta_1 a + \beta_2 b + \beta_3(a \times b), \quad c = \gamma_1 a + \gamma_2 b + \gamma_3(a \times b).$$

Then

$$x \times a = -\alpha_2(a \times b) + \alpha_3[|a|^2 b - (a \cdot b)a] \quad \text{and} \quad y \times b = \beta_1(a \times b) + \beta_3[(a \cdot b)b - |b|^2 a].$$

So $x \times a + y \times b + c = 0$ if and only if

$$\alpha_2 - \beta_1 = \gamma_3 \quad \text{and} \quad \begin{pmatrix} -(a \cdot b) & -|b|^2 \\ |a|^2 & (a \cdot b) \end{pmatrix} \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} = -\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Solving for α_3 and β_3 yields

$$x = \alpha_1 a + \alpha_2 b - \frac{\gamma_1(a \cdot b) + \gamma_2|b|^2}{|a|^2|b|^2 - (a \cdot b)^2}(a \times b), \quad y = \beta_1 a + \beta_2 b + \frac{\gamma_1|a|^2 + \gamma_2(a \cdot b)}{|a|^2|b|^2 - (a \cdot b)^2}(a \times b),$$

where $\alpha_2 - \beta_3 = \gamma_3$ and α_1 and β_2 are arbitrary.

Solution 32-6.3 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

For a vector $u = (u_1, u_2, u_3)' \in \mathbb{R}^3$ consider the skew-symmetric matrix

$$T_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

It is well-known that for any vector $v \in \mathbb{R}^3$, the vector cross product may be expressed as $u \times v = T_u v$; see Rao and Mitra (1971) or Room (1952). Hence the equation $x \times a + y \times b + c = 0$ may be written as

$$T_x a + T_y b + c = 0. \tag{2}$$

However, since $T_x a = -T_a x$ and $T_y b = -T_b y$, (2) can be reexpressed as

$$(T_a, T_b) \begin{pmatrix} x \\ y \end{pmatrix} = c. \tag{3}$$

Thus, our problem reduces to the question whether equation (3) is consistent, and further to the identification of the general solution set belonging to (3). For this purpose write $T = (T_a, T_b)$ and $z = (x', y')'$. Then (3) takes the form

$$Tz = c, \tag{4}$$

and (4) is consistent if and only if $TT^- c = c$, where T^- is a generalized inverse of T. Following Campbell and Meyer (1979, p. 98), we find

$$T^- = \begin{pmatrix} T_a^+[I - T_b(aa^+T_b)^+aa^+] \\ (aa^+T_b)^+aa^+ \end{pmatrix}$$

as a generalized inverse of T. Here use is made of the identity $I - T_a T_a^+ = aa^+$ [see Trenkler (2001)]. Now $(aa^+T_b)^+ = (a^+T_b)^+a^+$, so that T^- becomes

$$T^- = \begin{pmatrix} T_a^+[I - T_b(aa^+T_b)^+] \\ (aa^+T_b)^+ \end{pmatrix}.$$

Furthermore, we have $(aa^+T_b)^+ = \gamma T_b aa'$, where $\gamma = 1/a'T_b^2 a$. Note that $a'T_b^2 a \neq 0$, since a and b are linearly independent. Hence we arrive at

$$T^- = \begin{pmatrix} T_a^+[I - \gamma T_b^2 aa'] \\ \gamma T_b aa' \end{pmatrix}.$$

It follows that

$$TT^- = T_a T_a^+(I - \gamma T_b^2 aa') + \gamma T_b^2 aa' = (I - aa^+)(I - \gamma T_b^2 aa') + \gamma T_b^2 aa' = I - aa^+ + \gamma aa^+ T_b^2 aa' = I.$$

Hence $TT^- = I$, and (3) is solvable for every c. This is not surprising since by the linear independence of a and b, the null space of $T = (T_a, T_b)$ must be zero.

The general solution of (3) is given by

$$z = T^- c + (I - T^- T)w,$$

where w is an arbitrary vector.

Now we have

$$T^- T = \begin{pmatrix} T_a^+ T_a & T_a^+ T_b(I - \gamma T_b aa' T_b) \\ 0 & \gamma T_b aa' T_b \end{pmatrix}.$$

For a vector u introduce the notation $P(u) = uu^+$ and $Q(u) = I - uu^+$. Thus $P(u)$ is the orthogonal projector on the column space $\mathcal{L}(u)$ of u, and $Q(u)$ is the orthogonal projector on the orthogonal complement of $\mathcal{L}(u)$, respectively.

It follows that $I - \gamma T_b aa' T_b = I - (T_b a)(T_b a)^+ = Q(T_b a)$ and $\gamma T_b aa' T_b = P(T_b a)$, and hence

$$T^- T = \begin{pmatrix} T_a^+ T_a & T_a^+ T_b Q(T_b a) \\ 0 & P(T_b a) \end{pmatrix}$$

so that

$$I - T^- T = \begin{pmatrix} P(a) & -T_a^+ T_b Q(T_b a) \\ 0 & Q(T_b a) \end{pmatrix}.$$

Consequently, the general solution of (3) is

$$z = \begin{pmatrix} T_a^+ c - \gamma a' c T_a^+ T_b^2 a \\ \gamma(a'c) T_b a \end{pmatrix} + \begin{pmatrix} P(a) & -T_a^+ T_b Q(T_b a) \\ 0 & Q(T_b a) \end{pmatrix} w.$$

We finally note that the general solution of equation (3) can also be found by using the Moore-Penrose inverse of $T = (T_a, T_b)$. Since T is of full row rank we have

$$T^+ = T'(TT')^{-1}$$

or, equivalently,

$$T^+ = -\begin{pmatrix} T_a \\ T_b \end{pmatrix} Z, \tag{5}$$

where $Z = (T_a T_a' + T_b T_b')^{-1}$. Some straightforward calculations show that Z turns out to be $Z = (\alpha I - aa' - bb')^{-1}$, where $\alpha = a'a + b'b$.

In addition we get

$$T^+ T = -\begin{pmatrix} T_a Z T_a & T_a Z T_b \\ T_b Z T_a & T_b Z T_b \end{pmatrix}. \tag{6}$$

Then identities (5) and (6) can be used to write down the general solution of (3) in a different way.

References

S. L. Campbell & C. D. Meyer (1979). *Generalized Inverses of Linear Transformations.* Dover, New York.

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

T. G. Room (1952). The composition of rotations in Euclidian three-space. *American Mathematical Monthly* **59**, 688–692.

G. Trenkler (2001). The vector cross product from an algebraic point of view. *Discussiones Mathematicae, General Algebra and Applications* **21**, 67–82.

Solution 32-6.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* ujw902@uni-bonn.de

For a real matrix A, let A', $\mathrm{rank}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the transpose, the rank, the range (column space), and the null space, respectively, of A. For $a = (a_1, a_2, a_3)' \in \mathbb{R}^3$, let

$$T_a := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Observe that for $x = (\, x_1 \quad x_2 \quad x_3 \,)'$, the identity $T_a x = a \times x$ defines the vector cross product in $\mathbb{R}^3$. We note that $0 \neq a \in \mathbb{R}^3$ $\Leftrightarrow \mathrm{rank}(T_a) = 2 \Leftrightarrow \mathcal{N}(T_a) = \mathcal{R}(a)$. Moreover, $\mathcal{R}(T_a) = \mathcal{R}(T_b) \Leftrightarrow \mathcal{R}(a) = \mathcal{R}(b)$. If $a, b \in \mathbb{R}^3$ are linearly independent, then it is easy to see that $\mathrm{rank}((\, T_a \quad T_b \,)) = 3$ and

$$\mathcal{N}((\, T_a \quad T_b \,)) = \mathcal{R}(\begin{pmatrix} a & 0 & b \\ 0 & b & a \end{pmatrix}) \tag{7}$$

is of dimension 3; observe that $T_a a = 0 = T_b b$ and $T_a b = -T_b a$. For given vectors $a, b, c \in \mathbb{R}^3$, where a and b are linearly independent, consider

$$(x \times a) + (y \times b) + c = 0$$

or, equivalently,

$$(\, T_a \quad T_b \,) \begin{pmatrix} x \\ y \end{pmatrix} = c. \tag{8}$$

Then, by virtue of $\operatorname{rank}((\, T_a \quad T_b \,)) = 3$, $\mathcal{R}((\, T_a \quad T_b \,)) = \mathbb{R}^3$ and it is clear that (8) is solvable in x and y for each $c \in \mathbb{R}^3$. The general solution to (8) is explicitly given in the following theorem in terms of the vectors a, b, c.

THEOREM. *For given vectors $a, b, c \in \mathbb{R}^3$, where a and b are linearly independent, the general solution to* (8) *is given by*

$$\begin{aligned} x &= -\frac{1}{a'a + b'b}\left[(a \times c) + \frac{b'c \cdot b'b + a'b \cdot a'c}{a'a \cdot b'b - (a'b)^2} \cdot (a \times b)\right] + \alpha a + \gamma b, \\ y &= -\frac{1}{a'a + b'b}\left[(a \times c) - \frac{a'a \cdot a'c + b'a \cdot b'c}{a'a \cdot b'b - (a'b)^2} \cdot (a \times b)\right] + \beta b + \gamma a, \end{aligned}$$

where α, β, and γ are free to vary in $\mathbb{R}$.

PROOF. Evidently, $\operatorname{rank}((\, T_a \quad T_b \,)) = 3 \Rightarrow H := (\, T_a \quad T_b \,)(\, T_a \quad T_b \,)'$ is nonsingular. It is hence clear that

$$x_0 := T_a' H^{-1} c, \quad y_0 := T_b' H^{-1} c \tag{9}$$

provide us with a particular solution to (8). We note that $T_a' = -T_a$, $T_b' = -T_b$, $T_a' T_a = (a'a)I - aa'$, and $T_b' T_b = (b'b)I - bb'$, where I stands for the identity matrix of order 3. Therefore $H = (a'a + b'b)I - aa' - bb'$. By applying the famous Sherman-Morrison Formula [cf. Meyer (2000, p. 124)] twice, first to $W := (a'a + b'b)I - aa'$ and afterwards to $H = W - bb'$, we obtain

$$W^{-1} = \frac{1}{a'a + b'b}\left(I + \frac{1}{b'b} \cdot aa'\right) \tag{10}$$

and

$$H^{-1} = W^{-1} + \frac{1}{1 - b'W^{-1}b} W^{-1} bb' W^{-1}. \tag{11}$$

Inserting (10) into (11) readily yields

$$H^{-1} = \frac{1}{a'a + b'b}\left[I + \frac{1}{b'b} \cdot aa' + \frac{b'b}{a'a \cdot b'b - (a'b)^2} \cdot \left(b + \frac{a'b}{b'b} \cdot a\right)\left(b + \frac{a'b}{b'b} \cdot a\right)'\right]. \tag{12}$$

By means of (12) it is now possible to rewrite the particular solution given in (9) as follows

$$x_0 := T_a' H^{-1} c = -T_a H^{-1} c = -\frac{1}{a'a + b'b}\left[(a \times c) + \frac{b'c \cdot b'b + a'b \cdot a'c}{a'a \cdot b'b - (a'b)^2} \cdot (a \times b)\right], \tag{13}$$

$$y_0 := T_b' H^{-1} c = -T_b H^{-1} c = -\frac{1}{a'a + b'b}\left[(b \times c) - \frac{a'a \cdot a'c + b'a \cdot b'c}{a'a \cdot b'b - (a'b)^2} \cdot (a \times b)\right]. \tag{14}$$

Since the set of all solutions to equation (8) consists of all sums of one particular solution to (8) plus any vector in $\mathcal{N}((\, T_a \quad T_b \,))$, our claim follows from (13) and (14) by virtue of (7). □

EXAMPLE. Let $a = (1, -1, 1)'$, $b = (-1, 0, -1)'$, and $c = (1, 0, 0)'$. Then

$$(x \times a) + (y \times b) + c = 0$$

if and only if

$$x = \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad y = \frac{1}{10}\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Reference

C. D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia.

Problem 32-7: Invariance of the Vector Cross Product

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For a given nonzero vector $a \in \mathbb{R}^3$ determine a wide class of matrices A of order 3×3 such that

$$A(a \times b) = (Aa) \times (Ab)$$

for all $b \in \mathbb{R}^3$. Here "$\times$" denotes the common vector cross product in $\mathbb{R}^3$. Such equations play a role in robotics, see Murray, Lee, and Sastry (1994).

Reference

R. M. Murray, Z. Lee & S. S. Sastry (1994). *A Mathematical Introduction to Robotic Manipulation.* CRC Press, Boca Raton.

Solution 32-7.1 by Leo LIVSHITS, *Colby College, Waterville, Maine, USA:* llivshi@colby.edu

It is possible to determine *all* real matrices satisfying the condition. It is clear that one can assume that $||a||$, the Euclidean length of a, is 1. Furthermore,

$$A(a \times b) = A(a) \times A(b) \tag{15}$$

for all $b \in \mathbb{R}^3$, exactly when A satisfies this equation for all b in some basis of $\mathbb{R}^3$. Of course equation (15) holds trivially for any A when $b = a$.

Since every matrix A in $\mathcal{SO}_3$ satisfies $A(x \times y) = A(x) \times A(y)$ whenever x and y are orthonormal vectors, we see that every such A satisfies equation system (15) for any unit vector a, since a is an element of some orthonormal basis.

The primer is a standard fact, but here is a proof for completeness' sake. Since orthogonal matrices preserve lengths and angles, it must be that $A(x \times y) = \pm A(x) \times A(y)$. Yet if $(\, u \quad v \quad w\,)$ is the matrix of A with respect to the ordered basis $\mathcal{B} = \{x, y, x \times y\}$, where u, v, w are column vectors, then $[Ax]_{\mathcal{B}} = u,\ [Ay]_{\mathcal{B}} = v,\ [A(x \times y)]_{\mathcal{B}} = w$, and

$$1 = \det(A) = \det((\, u \quad v \quad w\,)) = u \bullet (v \times w),$$

so that

$$A(x \times y) \bullet (A(x) \times A(y)) = w \bullet (u \times v) = \det((\, w \quad u \quad v\,)) = \det((\, u \quad v \quad w\,)) = 1.$$

This forces the required equality. It follows that

$$A(e_1) \times A(e_2) = A(e_3),\ \ A(e_1) \times A(e_3) = -A(e_2),\ \ A(e_2) \times A(e_3) = A(e_1)$$

whenever $A \in \mathcal{SO}_3$. For each *unit* vector $a \in \mathbb{R}^3$, let us define

$$\mathcal{F}_a := \{A \in \mathbb{R}^{3\times 3} \mid A(a \times b) = A(a) \times A(b) \text{ for all } b \in \mathbb{R}^3\},$$

$$\mathcal{G}_a := \{S \operatorname{diag}(1, \lambda, \lambda)\, T \mid T, S \in \mathcal{SO}_3,\ T^T(e_1) = a,\ 0 \neq \lambda \in \mathbb{R}\} \cup \{xa^T \mid x \in \mathbb{R}^3\}.$$

THEOREM. $\mathcal{F}_a = \mathcal{G}_a$.

PROOF OF $\mathcal{G}_a \subset \mathcal{F}_a$: If $A \in \{xa^T \mid x \in \mathbb{R}^3\}$, then $A(z) = 0$ for every z orthogonal to a. Pick any orthogonal basis $\mathcal{B}$ of $\mathbb{R}^3$ containing a, and observe that A satisfies equation system (15) trivially for each $b \in \mathcal{B}$ and hence for all b. Let $A = S \operatorname{diag}(1, \lambda, \lambda)\, T$, where $T, S \in \mathcal{SO}_3,\ T^T(e_1) = a$ and $\lambda \neq 0$. We can write $T^T = (\, a \quad v \quad a \times v\,)$ and it is sufficient to verify that

$$A(a \times v) = A(a) \times A(v), \tag{16}$$

$$A(a \times (a \times v)) = A(a) \times A(a \times v). \tag{17}$$

Since $a \times (a \times v) = -v$, equation (17) becomes

$$A(v) = A(a \times v) \times A(a). \tag{18}$$

Proof of equality (16): $A(a \times v) = AT^T(e_3) = \lambda S(e_3)$, $A(a) \times A(v) = (AT^T(e_1)) \times (AT^T(e_2)) = S(e_1) \times \lambda S(e_2) = \lambda S(e_1) \times S(e_2) = \lambda S(e_3)$. Proof of equality (18): $A(v) = \lambda S(e_2)$, $A(a \times v) \times A(a) = \lambda S(e_3) \times S(e_1) = \lambda S(e_2)$.

PROOF OF $\mathcal{F}_a \subset \mathcal{G}_a$: Let $\mathcal{B} = \{a, c, a \times c\}$ be an orthonormal basis of $\mathbb{R}^3$. If $0 \neq A \in \mathcal{F}_a$, then

$$\begin{aligned} A(a) \times A(c) &= A(a \times c), \\ A(a) \times A(a \times c) &= A(a \times (a \times c)) = -A(c). \end{aligned}$$

In particular $A(a), A(c)$, and $A(a \times c)$ are mutually orthogonal. Hence

$$||A(a \times c)|| = ||A(a)|| \cdot ||A(c)|| \quad \text{and} \quad ||A(c)|| = ||A(a)|| \cdot ||A(a \times c)||.$$

It follows that $||A(a)|| = 1$ and $||A(c)|| = ||A(a \times c)|| \neq 0$, or $A(c) = A(a \times c) = 0 \neq A(a)$. A is invertible in the first case and singular in the second.

Case 1: A is invertible. In this case $||A(a)|| = 1$ and $||A(c)|| = ||A(a \times c)|| \neq 0$, and consequently $[A]_{\mathcal{E}\leftarrow\mathcal{B}} = U\,\mathrm{diag}(1, \lambda, \lambda)$ for some $\lambda \neq 0$ and $U \in \mathcal{SO}_3$; ($\mathcal{E}$ stands for the standard basis e_1, e_2, e_3). Since $A[I]_{\mathcal{E}\leftarrow\mathcal{B}} = [A]_{\mathcal{E}\leftarrow\mathcal{B}}$ and $[I]_{\mathcal{E}\leftarrow\mathcal{B}}(e_1) = a$, $[I]_{\mathcal{E}\leftarrow\mathcal{B}}(e_2) = c$, $[I]_{\mathcal{E}\leftarrow\mathcal{B}}(e_3) = a \times c$, so that $[I]_{\mathcal{E}\leftarrow\mathcal{B}} \in \mathcal{SO}_3$, we obtain

$$A = U\,\mathrm{diag}(1, \lambda, \lambda)\,([I]_{\mathcal{E}\leftarrow\mathcal{B}})^T \in \mathcal{G}_a$$

as needed.

Case 2: A is a non-zero singular matrix. In this case $A(c) = A(a \times c) = 0 \neq A(a)$, whenever $\{a, c, a \times c\}$ is an orthonormal basis of $\mathbb{R}^3$; in other words whenever c is a unit vector perpendicular to a. Since $\mathrm{Ker}(A) = a^\perp$, A is a rank one matrix which can be written as xa^T for some non-zero $x \in \mathbb{R}^3$. □

Solution 32-7.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Let

$$A = (\,\phi_1 \quad \phi_2 \quad \phi_3\,) \quad \text{with} \quad \{\phi_1, \phi_2, \phi_3\} \subset \mathbb{R}^3$$

and $a = (\,a_1 \quad a_2 \quad a_3\,)^T$. Let $\{e_1, e_2, e_3\}$ be the natural basis for $\mathbb{R}^3$. Then A satisfies the stated condition if and only if

$$A(a \times e_i) = Aa \times Ae_i, \quad i = 1, 2, 3. \tag{19}$$

Since $a \times e_1 = (\,0 \quad a_3 \quad -a_2\,)^T$, $a \times e_2 = (\,-a_3 \quad 0 \quad a_1\,)^T$, and $a \times e_3 = (\,a_2 \quad -a_1 \quad 0\,)^T$,

$$A(a \times e_1) = a_3\phi_2 - a_2\phi_3, \quad A(a \times e_2) = a_1\phi_3 - a_3\phi_1, \quad A(a \times e_3) = a_2\phi_1 - a_1\phi_2. \tag{20}$$

Since $Aa = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$ and $Ae_i = \phi_i$, $i = 1, 2, 3$,

$$Aa \times Ae_1 = a_2(\phi_2 \times \phi_1) + a_3(\phi_3 \times \phi_1), Aa \times Ae_2 = a_3(\phi_3 \times \phi_2) + a_1(\phi_1 \times \phi_2), Aa \times Ae_3 = a_1(\phi_1 \times \phi_3) + a_2(\phi_2 \times \phi_3). \tag{21}$$

Now define

$$B = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi = (\,\phi_1 - \phi_2 \times \phi_3 \quad \phi_2 - \phi_3 \times \phi_1 \quad \phi_3 - \phi_1 \times \phi_2\,).$$

From (20) and (21), it is straightforward to verify that (19) holds if and only if $B\Phi^T = 0$. Since $Bx = a \times x$ for any vector x, (19) holds if and only if $\Phi = ca^T$ for some $c \in \mathbb{R}^3$. It seems difficult to characterize $\{\phi_1, \phi_2, \phi_3\}$ such that this holds; however, it holds if $\phi_i = a_i c$, $i = 1, 2, 3$, for some $c \neq 0 \in \mathbb{R}^3$, in which case $A(x \times y) = Ax \times Ay$ for all $x, y \in \mathbb{R}^3$.

Solution 32-7.3 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

For a vector $c = (c_1, c_2, c_3)' \in \mathbb{R}^3$, let

$$T_c = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

It is well-known that T_c can be used to express the vector cross product in $\mathbb{R}^3$. In fact $c \times x = T_c x$ for all $x \in \mathbb{R}^3$. Our problem then reduces to finding matrices A such that $AT_a = T_{Aa}A$.

We first try the class of matrices given by $A = \alpha T_a + \beta I + \gamma aa'$, where α, β, γ are real scalars and I is the identity matrix. Since $T_a a = 0$ and $aa'T_a = 0$, one finds that $Aa = (\beta + \gamma a'a)a$, $T_{Aa} = (\beta + \gamma a'a)T_a$, and $AT_a = \beta T_a - \alpha(a'a)I + \alpha aa'$, the latter equation by $T_a^2 = aa' - a'aI$. Furthermore, we obtain $T_{Aa}A = (\beta + \gamma a'a)T_a A = (\beta + \gamma a'a)[\beta T_a - \alpha(a'a)I + \alpha aa']$. Then, equation $AT_a = T_{Aa}A$ to be valid is equivalent to the condition $\beta + \gamma a'a = 1$. Thus, a class of matrices A satisfying $A(a \times b) = (Aa) \times (Ab)$ for all $b \in \mathbb{R}^3$ is

$$\mathfrak{A} = \{\alpha T_a + \beta I + \gamma aa' \mid \beta + \gamma a'a = 1\}.$$

Note that a matrix of the form $\alpha T_a + \beta I + \gamma aa'$ is orthogonal if and only if (i) $\alpha^2 a'a + \beta^2 = 1$ and (ii) $(\beta + \gamma a'a)^2 = 1$.

Let us now consider the problem of finding the general solution to the equation $AT_a = T_{Aa}A$. For this purpose, let

$$A = (\, A_1 \quad A_2 \quad A_3 \,)'$$

be the row representation of A. It follows that

$$AT_a = \begin{pmatrix} A_1'T_a \\ A_2'T_a \\ A_3'T_a \end{pmatrix} \quad \text{and} \quad T_{Aa} = \begin{pmatrix} 0 & -A_3'a & A_2'a \\ A_3'a & 0 & -A_1'a \\ -A_2'a & A_1'a & 0 \end{pmatrix} \quad \text{so that} \quad T_{Aa}A = \begin{pmatrix} (A_2'a)A_3' - (A_3'a)A_2' \\ (A_3'a)A_1' - (A_1'a)A_3' \\ (A_1'a)A_2' - (A_2'a)A_1' \end{pmatrix}.$$

Thus, the equation $AT_a = T_{Aa}A$ is equivalent to

$$T_a A_1 + (A_3 A_2' - A_2 A_3')a = 0, \;\; T_a A_2 + (A_1 A_3' - A_3 A_1')a = 0, \;\; T_a A_3 + (A_2 A_1' - A_1 A_2')a = 0.$$

Taking into account that $T_c d = -T_d c$ we obtain the equivalent conditions

$$(-T_{A_1} + A_3 A_2' - A_2 A_3')a = 0, \;\; (-T_{A_2} + A_1 A_3' - A_3 A_1')a = 0, \;\; (-T_{A_3} + A_2 A_1' - A_1 A_2')a = 0.$$

However, it was not possible to obtain the general solution $A = (A_1, A_2, A_3)'$ for this set of equations.

Solution 32-7.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* ujw902@uni-bonn.de

Our offered solution to this problem is based on the famous singular value decomposition (SVD): we recall that for each matrix $A \in \mathbb{R}^{m \times n}$ of rank r, there are orthogonal matrices $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, and a diagonal matrix $D = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ such that

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V' \quad \text{with} \quad \sigma_i > 0 \quad \text{for} \quad i = 1, 2, \cdots, r. \tag{22}$$

The factorization in (22) is called a singular value decomposition of A, and the columns of U and V are called left-hand and right-hand singular vectors of A, respectively. The σ_i's are called the nonzero singular values of A. When $r < p := \min\{m, n\}$, A has $p - r$ additional singular values. The nonzero singular values of A are precisely the positive square roots of the nonzero (positive) eigenvalues of $A'A$, and the right-hand singular vectors are *particular* eigenvectors of $A'A$. This establishes the uniqueness of the nonzero singular values of A but not of the matrix V of right-hand singular vectors of A. The nonzero singular values of A are also the positive square roots of the nonzero eigenvalues of AA', and the columns of the matrix U are *particular* eigenvectors of AA'. For more details on SVD, see, e.g., Lancaster (1969) or Meyer (2000). We further note that a square matrix, say $B \in \mathbb{R}^{m \times m}$, is called simple if for each distinct eigenvalue of B the algebraic multiplicity is equal to the geometric multiplicity or, equivalently, if B is similar to a diagonal matrix, i.e., if there exists a nonsingular matrix X such that

$$X^{-1}BX = \mathrm{diag}(\lambda_1, \cdots, \lambda_m),$$

in which case the columns of X can be interpreted as m linearly independent right eigenvectors of B with the λ's as corresponding eigenvalues. Square matrices of order m with m distinct eigenvalues are automatically simple; cf. Lancaster (1969, p. 61). Hermitian matrices and so all real symmetric nonnegative definite matrices are simple; cf. Lancaster (1969, p. 76 or p. 78). The following result on commuting matrices opens the door to solve the problem posed by Trenkler.

LEMMA 1. [See, e.g., Lancaster (1969, p. 266).] *Let $B, C \in \mathbb{R}^{m \times m}$ be matrices which commute, i.e., let $BC = CB$. If B and C are simple, then there is a polynomial p of degree not exceeding $m - 1$ for which $C = p(B)$.*

Throughout, for a real matrix A, let A', $\mathrm{rank}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the transpose, the rank, the range (column space), and the null space, respectively, of A. If A is square, then we denote by $\mathrm{spectrum}(A)$ the set of eigenvalues of A. The (algebraic) multiplicity of $\lambda \in \mathrm{spectrum}(A)$ is denoted by $\mathrm{mult}(\lambda)$. For $c = (c_1, c_2, c_3)' \in \mathbb{R}^3$, let

$$T_c := \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

The matrix T_c is skew-symmetric, i.e., $T_c' = -T_c$. Moreover, $T_c^2 = (c'c)I - cc'$. We note that for $d = (d_1, d_2, d_3)'$ the identity $T_c d = c \times d$ defines the vector cross product in $\mathbb{R}^3$. Consequently, $A(a \times b) = (Aa) \times (Ab) \Leftrightarrow AT_a b = T_{Aa}Ab$. Therefore, for a given $a \in \mathbb{R}^3$, $AT_a b = T_{Aa}Ab$ for all $b \in \mathbb{R}^3$ if and only if $AT_a = A_{Ta}A$. If $c \neq 0$, then $\mathrm{spectrum}(T_c) = \{0, (c'c)^{1/2}i, -(c'c)^{1/2}i\}$, with i indicating the imaginary unit, and so T_c is simple. We further note that $\mathcal{N}(T_c) = \mathcal{R}(c)$ if and only if $c \neq 0$.

THEOREM 2. *For a given nonzero vector $a \in \mathbb{R}^3$, a nonzero matrix $A \in \mathbb{R}^{3\times 3}$ satisfies*

$$A(a \times b) = (Aa) \times (Ab) \quad \textit{for all} \quad b \in \mathbb{R}^3 \tag{23}$$

or, equivalently,

$$AT_a = T_{Aa}A \tag{24}$$

if and only if A is either a nonsingular matrix with a SVD of the form

$$A = U\mathrm{diag}(1, \sigma, \sigma)V', \tag{25}$$

where $\sigma > 0$ and the orthogonal (column partitioned) matrices $U = (\,U_1 \quad U_2 \quad U_3\,)$ and $V = (\,V_1 \quad V_2 \quad V_3\,)$ are such that $V_1 \in \mathcal{R}(a)$ and $\{U_1, U_2, U_3\}$ and $\{V_1, V_2, V_3\}$ define both right-handed coordinate systems [i.e., $V_1 \times V_2 = V_3$ and $U_1 \times U_2 = U_3$] or both left-handed coordinate systems [i.e., $V_1 \times V_2 = -V_3$ and $U_1 \times U_2 = -U_3$], or a rank 1 matrix with a SVD of the form

$$A = U\mathrm{diag}(\sigma, 0, 0)V', \tag{26}$$

where $\sigma > 0$ and V_1 is such that $V_1 \in \mathcal{R}(a)$. Needless to say, if $A = 0$, then trivially $AT_a = T_{Aa}A$.

PROOF. Clearly, $AT_a = T_{Aa}A \Rightarrow A'AT_a = A'T_{Aa}A \Rightarrow A'AT_a = T_aA'A$, i.e., $A'A$ and T_a are commuting matrices. Since $A'A$ and T_a are simple matrices, it follows from Lemma 1 that

$$A'A = \alpha T_a^2 + \beta T_a + \gamma I \quad \text{for some} \quad \alpha, \beta, \gamma \in \mathbb{R}, \tag{27}$$

where I is the identity matrix of order 3. Because $A'A$ is symmetric, necessarily $\beta = 0$, and so (27) reduces to

$$A'A = \alpha T_a^2 + \gamma I \quad \text{for some} \quad \alpha, \gamma \in \mathbb{R}. \tag{28}$$

Clearly, since $\mathcal{N}(T_a) = \mathcal{R}(a)$, $A'Aa = \gamma a$, i.e., (γ, a) is an eigenpair for $A'A$. It is well-known that if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the $n \times n$ matrix B and p is a scalar polynomial, then the eigenvalues of $p(B)$ are $p(\lambda_1), \cdots, p(\lambda_n)$; cf. Lancaster (1969, Theorem 2.5.2). Therefore, $\mathrm{spectrum}(\alpha T_a^2 + \gamma I) = \{\gamma, \gamma - \alpha(a'a)\}$ with

$$\mathrm{mult}(\gamma - \alpha(a'a)) = \begin{cases} 2 & \text{if } \alpha \neq 0, \\ 3 & \text{if } \alpha = 0. \end{cases}$$

Since $A'A$ is nonnegative definite and symmetric, necessarily $\gamma \geq 0$ and $\gamma - \alpha(a'a) \geq 0$. Therefore, (28) becomes

$$A'A = \alpha T_a^2 + \gamma I \quad \text{for some } \gamma \geq 0 \text{ and some } \alpha \text{ such that } \gamma - \alpha(a'a) \geq 0. \tag{29}$$

Next, we consider the following three disjoint and exhaustive cases: (i) $\gamma = 0$, (ii) $\gamma > 0$, $\gamma - \alpha(a'a) > 0$, and (iii) $\gamma > 0$, $\gamma - \alpha(a'a) = 0$.

Case (i): Let $\gamma = 0$, in which case $a \in \mathcal{N}(A) = \mathcal{N}(A'A)$ and (29) becomes $A'A = \alpha T_a^2$. Then clearly $A'Aa = \alpha T_a^2 a = 0 \Rightarrow Aa = 0 \Rightarrow T_{Aa} = 0 \Rightarrow T_{Aa}A = 0$. Hence, in case of $AT_a = T_{Aa}A$, $AT_a = 0$ or, equivalently, $\mathcal{R}(T_a) \subseteq \mathcal{N}(A)$. Since $\mathcal{R}(T_a) + \mathcal{R}(a) = \mathbb{R}^3$, it follows that $A = 0$ or, equivalently, $\alpha = 0$.

Case (ii): Let $\gamma > 0$ and $\gamma - \alpha(a'a) > 0$, in which case $A'A = \alpha T_a^2 + \gamma I$ and so A are nonsingular. Then $AT_a = T_{Aa}A \Leftrightarrow T_{Aa} = AT_aA^{-1}$, i.e., T_{Aa} and T_a are similar matrices. Since similar matrices have the same eigenvalues with identical (algebraic) multiplicities [cf. Lancaster (1969, Theorem 2.4.1)], it follows that $a'A'Aa = a'a$. We have already seen that (γ, a) is an eigenpair for $A'A$. Therefore, necessarily $\gamma = 1$. Then $\mathrm{spectrum}(A'A) = \{1, 1-\alpha(a'a)\}$ and $\mathrm{mult}(1-\alpha(a'a)) = 2$. It is now clear that the matrix A has the singular value 1 with multiplicity 1 and the singular value $\sigma := {}^{+}\sqrt{1-\alpha(a'a)}$ with multiplicity 2. A SVD of the matrix A is hence necessarily of the form

$$A = U\mathrm{diag}(1,\sigma,\sigma)V', \text{ where } V_1 \text{ in } V = (\, V_1 \quad V_2 \quad V_3 \,) \text{ is such that } V_1 \in \mathcal{R}(a). \tag{30}$$

So it remains to show that any matrix A with a SVD of the form (30) satisfies $AT_a = T_{Aa}A$ if and only if $\{U_1, U_2, U_3\}$ and $\{V_1, V_2, V_3\}$ define either both right-handed coordinate systems or both left-handed coordinate systems in $\mathbb{R}^3$. So, let A be a matrix with a SVD as in (30), where $U = (\, U_1 \quad U_2 \quad U_3 \,)$ is the column partitioning of U. Without loss of generality, we can assume that V_1 points in the same direction as a. Then $Aa = \tau U_1$ with τ denoting the positive square root of $a'a$. Therefore $T_{Aa} = \tau T_{U_1}$ and $T_{Aa}A = \tau T_{U_1}A = \tau T_{U_1}(U_1V_1' + \sigma[U_2V_2' + U_3V_3']) = \sigma\tau[T_{U_1}U_2V_2' + T_{U_1}U_3V_3'] = \sigma\tau[(U_1 \times U_2)V_2' + (U_1 \times U_3)V_3']$ and $AT_a = \sigma(U_2V_2' + U_3V_3')T_a = -\sigma U_2(a \times V_2)' - \sigma U_3(a \times V_3)' = -\sigma\tau[U_2(V_1 \times V_2)' + U_3(V_1 \times V_3)']$. Consequently, $AT_a = T_{Aa}A$ if and only if

$$-U_2(V_1 \times V_2)' - U_3(V_1 \times V_3)' = (U_1 \times U_2)V_2' + (U_1 \times U_3)V_3'. \tag{31}$$

If $\{V_1, V_2, V_3\}$ defines a right-handed coordinate system, i.e., if $V_1 \times V_2 = V_3$, then equation (31) holds if and only if $U_1 \times U_3 = -U_2$ and $U_1 \times U_2 = U_3$ or, equivalently, if and only if the coordinate system defined by $\{U_1, U_2, U_3\}$ is also right-handed. If $\{V_1, V_2, V_3\}$ is left-handed, i.e., if $V_1 \times V_2 = -V_3$, then on similar lines it is seen that equation (31) holds if and only if $\{U_1, U_2, U_3\}$ is left-handed.

Case (iii): Let $\gamma > 0$ and let $\gamma - \alpha(a'a) = 0$. Then $\mathrm{spectrum}(A'A) = \{\gamma, 0\}$ with $\mathrm{mult}(0) = 2$. Consequently, $\mathrm{rank}(A) = \mathrm{rank}(A'A) = 1$. Consider a SVD of A. Clearly, $A = \sigma U_1V_1'$, where $\sigma = {}^{+}\sqrt{\gamma}$ is the only positive singular value of A with U_1 and V_1' as associated left-hand and right-hand singular vectors, respectively. For such a matrix A, we get $Aa = \sigma(V_1'a)U_1$. Hence $T_{Aa}A = \gamma(V_1'a)T_{U_1}U_1V_1' = 0$ and $AT_a = \sigma U_1V_1'T_a = -\sigma U_1(a \times V_1)'$. Therefore, $AT_a = T_{Aa}A \Leftrightarrow a \times V_1 = 0 \Leftrightarrow V_1 \in \mathcal{R}(a)$. □

We conclude with emphasizing the interesting and somehow surprising observation that if $a \neq 0$, then there does not exist a matrix A of rank 2 satisfying $AT_a = T_{Aa}A$. Of course, if $a = 0$, then trivially $AT_a = 0 = T_{Aa}A$ for each matrix $A \in \mathbb{R}^{3\times 3}$.

References

P. Lancaster (1969). *Theory of Matrices.* Academic Press, New York.

C. D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia.

IMAGE Problem Corner: More New Problems

Problem 33-7: Property of the Cross Product

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let a, b, c, and d be vectors from $\mathbb{R}^3$. For the matrix

$$A = (c \times d)(b \times a)' + (a \times b)(c \times d)',$$

find the unique vector e such that $Ax = e \times x$ for all $x \in \mathbb{R}^3$. Here, "$\times$" denotes the usual cross product in $\mathbb{R}^3$.

Problem 33-8: Singular Value Decomposition of a Skew-Symmetric Real Matrix

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a 3×3 skew-symmetric matrix with real entries. Find a singular value decomposition of A, i.e., provide a representation $A = UDV'$, where U and V are orthogonal, and D is a diagonal matrix of singular values of A.

Problems 33-1 through 33-6 are on page 36.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to ujw902@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 33-7 and 33-8 are on page 35.*

Problem 33-1: A Range Equality for the Kronecker Product of Matrices

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Let A and B be $m_1 \times n_1$ and $m_2 \times n_2$ matrices, respectively. Show that

$$\operatorname{range}(A \otimes I_{m_2}) \cap \operatorname{range}(I_{m_1} \otimes B) = \operatorname{range}(A \otimes B),$$

where $\otimes$ denotes the Kronecker product of matrices and I is the identity matrix of the indicated order.

Problem 33-2: Similarity of Two Block Matrices

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Suppose that the two square matrices A and B satisfy $A^2 = 0$ and $B^2 = 0$. Show that the two block matrices

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

are similar if and only if

$$\operatorname{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B) \quad \text{and} \quad AC + CB = 0.$$

Problem 33-3: Two Characterizations of an EP Matrix

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Show that the following statements are equivalent:

(a) a complex square matrix A is EP, i.e., $\operatorname{range}(A) = \operatorname{range}(A^*)$,

(b) $\operatorname{range}(\, A - A^\dagger\,) = \operatorname{range}(\, A - A^3\,)$,

(c) $\operatorname{range}(\, A + A^\dagger\,) = \operatorname{range}(\, A + A^3\,)$,

where A^* and $A^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse of A, respectively.

Problem 33-4: An Euclidean Norm Property in $\mathbb{R}^3$

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $a = (a_1, a_2, a_3)'$ and $b = (b_1, b_2, b_3)'$ be vectors from $\mathbb{R}^3$ such that $a'b = 0$. Show that

$$\|a + b\| \leq \max\{|a_3| + |a_2| + |b_1|, |a_3| + |a_1| + |b_2|, |a_2| + |a_1| + |b_3|, |b_1| + |b_2| + |b_3|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

Problem 33-5: Factorization of a Projector

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix with possibly complex entries. Show that P can be written as $P = RS$, where R is positive definite and S is nonnegative definite.

Problem 33-6: Projectors and Similarity

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix with possibly complex entries. Show that P is Hermitian if and only if P and P^+ are similar, where P^+ denotes the Moore-Penrose inverse of P.