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NONNEGATIVE MATRIX FACTORIZATION AND APPLICATIONS

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1. Introduction. Data analysis is pervasive throughout science, engineering and business applications. Very often the data to be analyzed is nonnegative, and it is very often preferable to take this constraint into account in the analysis process. In this paper we provide a survey of some aspects of nonnegative matrix factorization and its applications to nonnegative matrix data analysis. In general the problem is the following: given a nonnegative data matrix Y find reduced rank nonnegative matrices U and V so that

$$Y \approx UV.$$

Here, U is often thought of as the source matrix and V as the mixing matrix associated with the data in Y. A more formal definition of the problem is given below. This approximate factorization process is an active area of research in several disciplines (a Google search on this topic recently provided over 250 references to papers involving nonnegative matrix factorization and applications written in the past ten years), and the subject is certainly a fertile area of research for linear algebraists.

An indispensable task in almost every discipline is to analyze a certain data to search for relationships between a set of exogenous and endogenous variables. There are two special concerns in data analysis. First, most of the information gathering devices or methods at present have only finite bandwidth. One thus cannot avoid the fact that the data collected often are not exact. For example, signals received by antenna arrays often are contaminated by instrumental noises; astronomical images acquired by telescopes often are blurred by atmospheric turbulence; database prepared by document indexing often are biased by subjective judgment; and even empirical data obtained in laboratories often do not satisfy intrinsic physical constraints. Before any deductive sciences can further be applied, it is important to first reconstruct or represent the data so that the inexactness is reduced while certain feasibility conditions are satisfied. Secondly, in many situations the data observed from complex phenomena represent the integrated result of several interrelated variables acting together. When these variables are less precisely defined, the actual information contained in the original data might be overlapping and ambiguous. A reduced system model could provide a fidelity near the level of the original system. One common ground in the various approaches for noise removal, model reduction, feasibility reconstruction, and so on, is to replace the original data by a lower dimensional representation obtained via subspace approximation. The notion of low rank approximations therefore arises in a wide range of important applications. Factor analysis and principal component analysis are two of the many classical methods used to accomplish the goal of reducing the number of variables and detecting structures among the variables.

However, as indicated above, often the data to be analyzed is nonnegative, and the low rank data are further required to be comprised of nonnegative values only in order to avoid contradicting physical realities. Classical tools cannot guarantee to maintain the nonnegativity. The approach of low-rank **nonnegative matrix factorization** (NNMF) thus becomes particularly appealing. The NNMF problem, probably due originally to Paatero and Tapper [21], can be stated in generic form as follows:

(NNMF) *Given a nonnegative matrix* $Y \in \mathbb{R}^{m\times n}$ *and a positive integer* $p < \min\{m,n\}$, *find nonnegative matrices* $U \in \mathbb{R}^{m\times p}$ *and* $V \in \mathbb{R}^{p\times n}$ *so as to minimize the functional*

$$f(U,V) := \frac{1}{2}\|Y - UV\|_F^2. \tag{1.1}$$

The product UV of the least squares solution is called a nonnegative matrix factorization of Y, although Y is not necessarily equal to the product UV. Clearly the product UV is of rank at most p. An appropriate decision on the value of p is critical in practice, but the choice of p is very often problem dependent. The objective function (1.1) can be modified in several ways to reflect the application need. For example, penalty terms can be added to $f(U,V)$ in order to enforce sparsity or to enhance smoothness in the solution U and V [13, 24]. Also, because $UV = (UD)(D^{-1}V)$ for any invertible matrix $D \in \mathbb{R}^{p\times p}$, sometimes it is desirable to "normalize" columns of U. The question of uniqueness of the nonnegative factors U and V also arises, which is easily seen by considering case where the matrices D and D^{-1} are nonnegative. For simplicity, we shall concentrate on (1.1) only in this essay, but the metric to be minimized in the NNMF problem can certainly be generalized and constraints beyond nonnegativity are sometimes imposed for specific situations, e.g., [5, 13, 14, 15, 18, 19, 24, 25, 26, 27]. In many applications, we will see that the p factors, interpreted as either sources, basis elements, or concepts, play a vital role in data analysis. In practice, there is a need to determine as few factors as possible and, hence the need for a low rank NNMF of the data matrix Y arises.

2. Some Applications. The basic idea behind the NNMF is the linear model. The matrix $Y = [y_{ij}] \in \mathbb{R}^{m\times n}$ in the NNMF formulation denotes the "observed" data whereas each entry y_{ij} represents, in a broad sense, the *score* obtained by entity j on variable i. One way to characterize the interrelationships among multiple variables that contribute to the observed data Y is to assume that y_{ij} is a linearly weighted score by entity j based on several "factors". We shall temporarily assume that there are p factors, but often it is precisely the point that the factors are to be retrieved in the mining process. A linear model, therefore, assumes the relationship

$$Y = AF, \tag{2.1}$$

where $A = [a_{ik}] \in \mathbb{R}^{m\times p}$ is a matrix with a_{ik} denoting the *loading* of variable i to factor k or, equivalently, the *influence* of factor k on variable i, and $F = [f_{kj}] \in \mathbb{R}^{p\times n}$ with f_{kj} denoting the *score* on factor k by entity j or the *response* of entity j to factor k. Depending on the applications, there are many ways to interpret the meaning of the linear model. We briefly describe a few applications below.

2.1. Air Emission Quality. In the air pollution research community, one observational technique makes use of the ambient data and source profile data to apportion sources or source categories [12, 15]. The fundamental principle in this model is that mass conservation can be assumed and a mass balance analysis can be used to identify and apportion sources of airborne particulate matter in the atmosphere. For example, it might be desirable to determine a large number of chemical constituents such as elemental concentrations in a number of samples. The relationships between p sources which contribute m chemical species to n samples, therefore, lead to a *mass balance equation*,

$$y_{ij} = \sum_{k=1}^{p} a_{ik} f_{kj}, \tag{2.2}$$

where y_{ij} is the elemental concentration of the ith chemical measured in the jth sample, a_{ik} is the gravimetric concentration of the ith chemical in the kth source, and f_{kj} is the airborne mass concentration that the kth source has contributed to the jth sample. In a typical scenario, only values of y_{ij} are observable whereas neither the sources are known nor the compositions of the local particulate emissions are measured. Thus, a critical question is to estimate the number p, the compositions a_{ik}, and the contributions f_{kj} of the sources.

Tools that have been employed to analyze the linear model include principal component analysis, factor analysis, cluster analysis, and other multivariate statistical techniques. In this receptor model, however, there is a physical constraint imposed upon the data. That is, the source compositions a_{ik} and the source contributions f_{kj} must all be nonnegative. The identification and apportionment, therefore, becomes a nonnegative matrix factorization problem of Y.

2.2. Image and Spectral Data Processing. Digital images are represented as nonnegative matrix arrays, since pixel intensity values are nonnegative. It is sometimes desirable to process data sets of images represented by column vectors as composite objects in many articulations and poses, and sometimes as separated parts in, for example, biometric identification applications such as face or iris recognition. It is suggested that the factorization in the linear model would enable the identification and classification of intrinsic "parts" that make up the object being imaged by multiple observations [7, 16, 26]. More specifically, each column $\mathbf{y}_j$ of a nonnegative matrix Y now represents m pixel values of one image. The columns $\mathbf{a}_k$ of A are basis elements in $\mathbb{R}^m$. The columns of F, belonging to $\mathbb{R}^p$, can be thought of as coefficient sequences representing the n images in the basis elements. In other words, the relationship,

$$\mathbf{y}_j = \sum_{k=1}^{p} \mathbf{a}_k f_{kj}, \tag{2.3}$$

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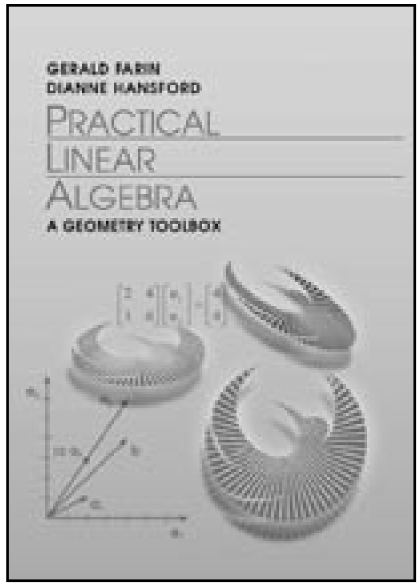
Practical Linear Algebra: A Geometry Toolbox

by **Gerald Farin**
and **Dianne Hansford**

$67.00; 1-56881-234-5
2005; Hardcover; 394 pp.



Practical Linear Algebra introduces students in math, science, engineering, and computer science to Linear Algebra from an intuitive and geometric viewpoint, creating a level of understanding that goes far beyond mere matrix manipulations.

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NNMF article cont'd from page 3

can be thought of as that there are *standard parts* $\mathbf{a}_k$ in a variety of positions and that each image represented as a vector $\mathbf{y}_j$, making up the factor U of basis elements is made by superposing these parts together in specific ways by a mixing matrix represented by V in (1.1). Those parts, being images themselves, are necessarily nonnegative. The superposition coefficients, each part being present or absent, are also necessarily nonnegative. A related application to the identification of object materials from spectral reflectance data at different optical wavelengths has been investigated in [25].

2.3. Text Mining. Assume that the textual documents are collected in an *indexing matrix* $Y = [y_{ij}] \in \mathbb{R}^{m\times n}$. Each document is represented by one column in Y. The entry y_{ij} represents the *weight* of one particular *term* i in document j whereas each term could be defined by just one single word or a string of phrases. To enhance discrimination between various documents and to improve retrieval effectiveness, a term-weighting scheme of the form,

$$y_{ij} = t_{ij} g_i d_j, \tag{2.4}$$

is usually used to define Y [2], where t_{ij} captures the relative importance of term i in document j, g_i weights the overall importance of term i in the entire set of documents, and $d_j = \left(\sum_{i=1}^m t_{ij} g_i\right)^{-1/2}$ is the scaling factor for normalization. The normalization by d_j per document is necessary because, otherwise, one could artificially inflate the prominence of document j by padding it with repeated pages or volumes. After the normalization, the columns of Y are of unit length and usually nonnegative.

The indexing matrix contains lot of information for retrieval. In the context of latent semantic indexing (LSI) application [2, 10], for example, suppose a query represented by a row vector $\mathbf{q}^\top = [q_1, \ldots, q_m] \in \mathbb{R}^m$, where q_i denotes the weight of term i in the query $\mathbf{q}$, is submitted. One way to measure how the query $\mathbf{q}$ matches the documents is to calculate the row vector $\mathbf{s}^\top = \mathbf{q}^\top Y$ and rank the relevance of documents to $\mathbf{q}$ according to the *scores* in $\mathbf{s}$.

The computation in the LSI application seems to be merely the vector-matrix multiplication. This is so only if Y is a "reasonable" representation of the relationship between documents and terms. In practice, however, the matrix Y is never exact. A major challenge in the field has been to represent the indexing matrix and the queries in a more compact form so as to facilitate the computation of the scores [6, 23]. The idea of representing Y by its NNMF approximation seems plausible. In this context, the standard parts $\mathbf{a}_k$ indicated in (2.3) may be interpreted as subcollections of some "general concepts" contained in these documents. Like images, each document can be thought of as a linear composition of these general concepts. The column-normalized matrix A itself is a term-concept indexing matrix.

Nonnegative matrix factorization has many other applications, including linear sparse coding [13, 29], chemometric analysis [11, 21], image classification [9], neural learning process [20], sound recognition [14], remote sensing and object characterization [25, 30]. We stress that, in addition to low-rank and nonnegativity, there are applications where other conditions need to be imposed on U and V. Some of these constraints include sparsity, smoothness, specific structures, and so on. The NNMF formulation and resulting computational methods need to be modified accordingly, but it will be too involved to include that discussion in this brief survey.

3. Optimality. Quite a few numerical algorithms have been developed for solving the NNMF. The methodologies adapted are following more or less the principles of alternating direction iterations, the projected Newton, the reduced quadratic approximation, and the descent search. Specific implementations generally can be categorized into alternating least squares algorithms [21], multiplicative update algorithms [16, 17, 13], gradient descent algorithm, and hybrid algorithm [24, 25]. Some general assessments of these methods can be found in [5, 18, 28]. It appears that there is much room for improvement of numerical methods. Although schemes and approaches are different, any numerical method is essentially centered around satisfying the first order optimality conditions derived from the Kuhn-Tucker theory. Recall that the computed factors U and V may only be local minimizers of (1.1).

THEOREM 3.1. *Necessary conditions for $(U, V) \in \mathbb{R}_+^{m\times p} \times \mathbb{R}_+^{p\times n}$ to solve the nonnegative matrix factorization problem (1.1) are*

$$U .* \left((Y - UV)V^\top\right) = 0 \in \mathbb{R}^{m\times p}, \tag{3.1}$$

$$V .* \left(U^\top (Y - UV)\right) = 0 \in \mathbb{R}^{p\times n}, \tag{3.2}$$

$$(Y - UV)V^\top \leq 0, \tag{3.3}$$

$$U^\top (Y - UV) \leq 0, \tag{3.4}$$

where $.*$ denotes the Hadamard product.

4. Conclusions and Some Open Problems. We have attempted to outline some of the major concepts related to nonnegative matrix factorization and to briefly discuss a few of the many practical applications. Several open problems remain, and we list just a few of them.

- Preprocessing the data matrix Y. It has been observed, e.g. [25, 27], that noise removal or a particular basis representation for Y can improve the effectiveness of algorithms for solving (1.1). This is an active area of research and is unexplored for many applications.
- Initializing the factors. Methods for choosing, or seeding, the initial matrices U and V for various algorithms (see, e.g., [30]) is a topic in need of further research.
- Uniqueness. Sufficient conditions for uniqueness of solutions to the NNMF problem can be considered in terms of simplicial cones [1], and have been studied in [7]. Algorithms for computing the factors U and V generally produce local minimizers of $f(U, V)$, even when constraints are imposed. It would thus be interesting to apply global optimization algorithms to the NNMF problem.
- Updating the factors. Devising efficient and effective updating methods when columns are added to the data matrix Y in (1.1) appears to be a difficult problem and one in need of further research.

Our survey in this short essay is of necessity incomplete, and we apologize for resulting omission of other material or references. Comments by readers to the authors on the material are welcome.

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ILAS President/Vice President Annual Report: April 2004

1) The following were elected in the ILAS 2004 elections to offices with terms that began on March 1, 2005 and end on February 29, 2008:

President: Daniel Hershkowitz (second term)
Board of Directors: Ilse Ipsen and Reinhard Nabben

The following continue in ILAS offices to which they were previously elected:

Vice President: Roger Horn (term ends February 28, 2007)
Secretary/Treasurer Jeff Stuart (term ends February 28, 2006)

Board of Directors:

Rafael Bru (term ends February 28, 2006)
Roy Mathias (term ends February 28, 2007)
Joao Filipe Queiro (term ends February 28, 2007)
Hugo Woerdeman (term ends February 28, 2006)

Ravindra Bapat and Michael Neumann completed their three-year terms on the ILAS Board of Directors on February 28, 2005.

We thank the members of the nomination committee (Hans Schneider–chair, Ludwig Elsner, Shmuel Friedland, Steve Kirkland and Andre Ran) for their efforts on behalf of ILAS, and to Richard A. Brualdi for his help in counting the ballots.

2) With the advice of the ILAS Committee for the Hans Schneider Prize in Linear Algebra, the ILAS Executive decided that the 2005 Prize will be given to Richard A. Brualdi and Richard S. Varga, each on the basis of an outstanding lifetime contribution to the field of Linear Algebra and related fields. Professor Varga will be awarded the prize at the 12th ILAS Conference in Regina, June 26-29, 2005, and will give his Hans Schneider Prize Lecture there. Professor Brualdi will be awarded the prize at the 13th ILAS Conference in Amsterdam, July 18-21, 2006, and will give his Hans Schneider Prize Lecture there.

We thank the members of the Prize Committee (Eduardo de Sa, Heike Fassbender, Miroslav Fiedler, Robert Guralnick, Michael Neumann - Chair, Danny Hershkowitz - ex officio) for their efforts on behalf of ILAS.

3) Four ILAS-endorsed meetings took place since our last report:

2-day Workshop “Directions in Combinatorial Matrix Theory”, Banff International Research Station (BIRS), May 6-8, 2004, Banff, Canada (Miroslav Fiedler was an ILAS Lecturer in that conference)

13th International Workshop on Matrices and Statistics (IWMS-2004), August 18-21, 2004, Poznań, Poland.

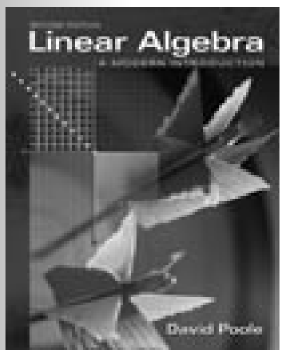
Haifa 2005 Matrix Meeting, Haifa, Israel, January 3-7, 2005 (Michael Neumann was an ILAS Lecturer in that conference)

14th International Workshop on Matrices and Statistics (IWMS-2005), March 29-April 1, 2005, Auckland, New Zealand

4) ILAS has endorsed the following conference of interest to ILAS members:

Cont’d on page 9

New Linear Algebra Texts from Thomson Brooks/Cole!



Linear Algebra: A Modern Introduction, Second Edition

David Poole, Trent University

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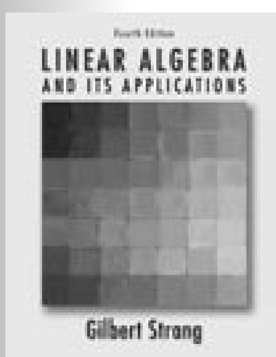

Linear Algebra and Its Applications, Fourth Edition

Gilbert Strang, Massachusetts Institute of Technology

544 pages. 7 3/8 x 9 1/4. 2-color. Casebound. © 2006. ISBN: 0-03-010567-6.

Available July 2005!

Gilbert Strang demonstrates the beauty of linear algebra and its crucial importance. Strang's emphasis is always on understanding. He explains concepts, rather than concentrating entirely on proofs. The informal and personal style of the text teaches students real mathematics. Throughout the book, the theory is motivated and reinforced by genuine applications, allowing every mathematician to teach both pure and applied mathematics. Applications to physics, engineering, probability and statistics, economics, and biology are thoroughly integrated as part of the mathematics in the text.

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Also Available

Linear Algebra and Matrix Theory, Second Edition

Jimmie Gilbert and **Linda Gilbert**, both of University of South Carolina, Spartanburg

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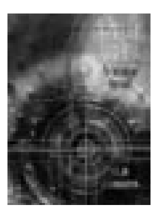

Linear Algebra: An Interactive Approach

S. K. Jain, Ohio University

A. D. Gunawardena, Carnegie Mellon University

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ILAS Report cont'd from page 7

The Householder Meeting on Numerical Linear Algebra, May 23-27, 2005, Campion, USA

5) The following ILAS conferences are scheduled:

12th ILAS Conference, Regina, Saskatchewan, Canada, June 26-29, 2005 (Organizing committee: R. Bhatia, R. Guralnick, D. Hershkowitz, S. Kirkland, V. Mehrmann, B-S. Tam, P. van den Driessche, and H. Wolkowicz. Local organizing committee: Shaun Fallat, Doug Farenick, Chun-Hua Guo, Steve Kirkland (University of Regina)). See http://www.math.uregina.ca/~ilas2005/.

13th ILAS Conference, Amsterdam, The Netherlands, July 18-21, 2006 (Chairman of the organizing committee is Andre C.M. Ran. Local organizers: Andre Ran, Andre Klein, Peter Spreij and Jan Brandts).

14th ILAS Conference, Shanghai, China, July 16-20, 2007 (Organizing Committee: Richard Brualdi - co-chair, Erxiong Jiang - co-chair, Raymond Chan, Chuanqing Gu, Danny Hershkowitz - ILAS President, Roger Horn, Ilse Ipsen, Julio Moro, Peter Šemrl, Jia-yu Shao and Pei Yuan Wu).

15th ILAS Conference, Cancun, Mexico, June 16-20, 2008 (Chairman of the organizing committee is Luis Verde).

6) ILAS is also a partner in organizing the joint GAMM-SIAM conference on Applied Linear Algebra, Dusseldorf, July 24-27, 2006 (which continues the SIAM series of conferences on Applied Linear Algebra). Ludwig Elsner and Olga Holtz will be ILAS Lecturers at that conference.

7) The Electronic Journal of Linear Algebra (ELA) is now in its 12th, 13th and 14th volumes. Its editors-in-chief are Ludwig Elsner and Danny Hershkowitz.

Volume 1, published in 1996, contained 6 papers.
Volume 2, published in 1997, contained 2 papers.
Volume 3, the Hans Schneider issue, published in 1998, contained 13 papers.
Volume 4, published in 1998 as well, contained 5 papers.
Volume 5, published in 1999, contained 8 papers.
Volume 6, Proceedings of the Eleventh Haifa Matrix Theory Conference, published in 1999 and 2000, contained 8 papers.
Volume 7, published in 2000, contained 14 papers.
Volume 8, published in 2001, contained 12 papers.
Volume 9, published in 2002, contained 24 papers.
Volume 10, published in 2003, contained 25 papers.
Volume 11, published in 2004, contained 22 papers.
Volume 12, Proceedings of the 2004 Workshop on Nonnegative Matrices, Maynooth, is being published now. As of March 31, 2004, it contains 5 papers.
Volume 13, is being published now. As of March 31, 2004, it contains 8 papers.
Volume 14, Proceedings of the 2004 workshop "Directions in Combinatorial Matrix Theory", Banff, is being published now. As of March 31, 2004, it contains 3 papers.

Acceptance percentage in ELA is currently 59%. The Science Citation Index now includes ELA among the more than 3700 scholarly science and technical journals in their citation database.

ELA's primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz, in the University of Lisbon, in EMIS - The European Mathematical Information Service offered by the European Mathematical Society, and in EMIS's more than 40 Mirror Sites.

8) IMAGE - The Bulletin of ILAS is edited by Bryan Shader and Hans Joachim Werner. ILAS members are now given the option to receive either a print version of IMAGE or an electronic version.

9) ILAS-NET is managed by Shaun Fallat. As of March 18, 2005, we have circulated 1424 ILAS-NET announcements.

10) The primary site of ILAS INFORMATION CENTER (IIC) is at Regina. Mirror sites are located in the Technion, in Temple University, in the University of Chemnitz and in the University of Lisbon

Respectfully submitted,

Daniel Hershkowitz, ILAS President, hershkow @tx.technion.ac.il; Roger Horn, ILAS Vice-President, rhorn@math.utah.edu

Call for Submissions to IMAGE

IMAGE welcomes expository articles on emerging applications and topics in Linear Algebra, announcements of upcoming meetings, reports on past conferences, historical essays on linear algebra, book reviews, essays on the development of Linear Algebra in a certain country or region, and letters to the editor or signed columns of opinion. Contributions for IMAGE should be sent to either Bryan Shader (bshader@uwyo.edu) or Hans Joachim Werner (hjw.de@uni-bonn.de). The deadlines are October 1 for the fall issue, and April 1 for the spring issue.

ILAS 2004 - 2005 Treasurer's Report
March 1, 2004 through February 28, 2005

Net Account Balances on February 29, 2004

Vanguard (ST Fed. Bond Fund 3554.076 Shares) (10.60% Each: General Fund, Conference Fund and ILAS/LAA Fund, 17.40% Taussky Todd Fund, 7.95% Uhlig Fund, 42.85% Schneider Fund)		$37, 815.37		
Checking account		$43,756.83		
Pending checks		$ 1,600.00		
Pending VISA/Mastercard/AMEX		$ 200.00		
Cash		$ 42.00		**$83,414.20**
General Fund		$32,236.24		
Conference Fund		$10,599.65		
ILAS/LAA Fund		$ 6,877.10		
Olga Taussky Todd/John Todd Fund		$ 9,744.63		
Frank Uhlig Education Fund		$ 3,719.33		
Hans Schneider Prize Fund		$20,237.25		**$83,414.20**
Income:				
Dues	6640.00			
Corporate Dues	400.00			
Book Sales	31.00			
General Fund	525.33			
Conference Fund	26.17			
ILAS\LAA Fund	1005.17			
Taussky-Todd Fund	259.49			
Uhlig Education Fund	182.38			
Schneider Prize Fund	376.90		9446.44	
Expenses:				
IMAGE (2 issues)	3018.52			
Speakers (3)	3015.00			
ILAS Board Travel	250.00			
Credit Card and Bank Fees	300.83			
License Fees	61.25			
Labor - Mailing & Conference	280.00			
Postage	489.45			
Supplies and Copying	393.92		7808.97	

Net Account Balances on February 28, 2005

Vanguard (ST Fed. Bond Fund 3554.076 Shares) (10.60% Each: General Fund, Conference Fund and ILAS/LAA Fund, 17.40% Taussky Todd Fund, 7.95% Uhlig Fund, 42.85% Schneider Fund)		$37,864.14	
Checking account		$44,323.09	
Pending checks (payable)		($ 685.56)	
Pending checks (receivable)		$ 2,200.00	
Pending VISA/Mastercard/AMEX		$ 1,350.00	**$85,051.67**
General Fund		$34,288.60	
Conference Fund		$10,625.82	
ILAS/LAA Fund		$ 7,882.27	
Olga Taussky Todd/John Todd Fund		$ 9,354.12	
Frank Uhlig Education Fund		$ 3,901.71	
Hans Schneider Prize Fund		$18,999.15	**$85,051.67**

Jeffrey L. Stuart, Secretary-Treasurer
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Book Review: *Geršgorin and his Circles* by Richard S. Varga, Springer, x+226pp, 2004. ISBN 3-540-21100-4.

Geršgorin and his circles by R. S. Varga is volume 36 in the Springer Series in Computational Mathematics, and is intended for upper level and graduate students and also researchers in Linear Algebra (both theoretical and applied). This book is very well-written and is a comprehensive treatment of eigenvalue location results, eigenvalue estimation problems, and conditions for invertibility. Varga's treatment of this topic is both elegant and extensive, and he clearly demonstrates a mastery of this subject.

This book begins with Geršgorin's original contribution, referred to as Geršgorin's circle theorem, concerning the location of the eigenvalues of matrix: If $A = [a_{ij}]$ is an $n \times n$ matrix and for each $i = 1, 2, \ldots, n$ let $r_i(A) = \sum_{i \neq j} |a_{ij}|$, then the set of eigenvalues of A is contained in the union of the n discs $\{z : |z - a_{ii}| \leq r_i(A)\}$. From this classical and pivotal observation Varga builds and extends it in a variety of ways (e.g., by including ovals and consideration of the graph of a matrix). Further along these lines, he considers eigenvalues regions given by radii based on certain partitions of the rows of a matrix. He then explores the issue of minimal Geršgorin sets in a effort to obtain more precise estimates of the eigenvalues of a matrix, and to address the issue of how sharp these various eigenvalue regions really are for certain classes of matrices related to a given matrix (via diagonal similarity, comparison matrices, and permutations). For some of the analysis in this chapter he makes use of a link to the theory of essentially nonnegative matrices, and he ends with a new look at comparing minimal Geršgorin sets and so-called Brualdi sets. The final part of this book uncovers the many extensions and generalizations of these inclusion results, including a widening of the definition of the radii (F-functions, G-functions), and the impact felt by considering partitioned matrices.

Varga spends considerable time comparing and contrasting the various types of inclusion results, often by example, but also by more definitive methods. There are numerous accompanying examples illustrating relevant concepts and engaging the reader about the many subtleties and issues surrounding this vast topic.

One aspect of this book, which was particularly attractive to me, was Appendix A. In this appendix, there is vital information about Geršgorin, a list of his significant papers, a translated obituary, and a copy of his original 1931 paper containing his circle theorem.

Some further of items of note are: (i) Each chapter is concluded with a discussion (by section number) highlighting the main points, indicating other useful connections, and sometimes including a tantalizing anecdote to accompany the topic addressed. I personally found these discussions interesting and useful; (ii) Throughout the book there is a constant theme pointing out the equivalence between "eigenvalue inclusion" type results and a corresponding "dominance condition" for invertibility. There are numerous places where this important theme impacts the particular issue at hand.

This book consists of six in depth chapters. The first chapter (entitled "Basic Theory") contains Geršgorin's original result, and extensions thereof (including Olga Tauskky-Todd's contributions incorporating irreducibility). Chapter 2 ("Geršgorin-Type Eigenvalue Inclusion Theorems") includes Brauer's ovals of Cassini result, and Brualdi's contributions taking into account the graph of a matrix (along with comparisons). Chapter 3 ("More Eigenvalue Inclusion Results") discusses other results about eigenvalue regions (Parodi-Schneider, and Pupkov-Solovev) with an emphasis on more recent advances. Chapter 4 ("Minimal Geršgorin Sets and Their Sharpness") explores how exact these various Geršgorin-type inclusion results are with respect to diagonally similarity and permutations. Chapter 5 ("G-functions") brings to light significant generalizations of the inclusion sets by extending the notion of radii. A nice connection to the theory of M-matrices is observed. Chapter 6 ("Geršgorin-Type Theorems for Partitioned Matrices") contains yet another extension via partitioning the rows and columns of a matrix, and also ties this together with the issue of allowing other norms.

In closing, this text represents a significant treatment of eigenvalue inclusion regions including a well deserved tribute to Geršgorin and his original contribution to this area. In my opinion, this book will end up on many bookshelves both for reference use and for general interest.

Reviewed by **S.M. Fallat**
University of Regina

Electronic Journal of Linear Algebra

The Electronic Journal of Linear Algebra (ELA), a publication of the International Linear Algebra Society (ILAS), is a refereed all-electronic journal that welcomes mathematical articles of high standards that contribute new information and new insights to matrix analysis and the various aspects of linear algebra and its applications. Refereeing of articles is conventional and of high standards, and is being carried out electronically. The Editors-in-Chief are Ludwig Elsner and Daniel Hershkowitz. The website is
http://www.math.technion.ac.il/iic/ela/

Book Review: *Generalized Inverses: Theory and Computations* by Wang Guorong, Wei Yimin, and Qiao Sanzheng, Science Press (Beijing/New York), 2004.
ISBN 7-03-012437-5. xi+294 pp.

This book is a graduate text in the *Graduate Series in Mathematics* published by the prestigious Chinese publisher *Science Press.* The book, written in English, contains eleven chapters covering topics ranging from basic theory on generalized inverses of matrices to the Drazin inverse of operators. Unlike other existing books on generalized inverses emphasizing application to statistics, this one stands out focusing on many aspects of computation.

Chapter 1 on Equation Solving Generalized Inverses shows basic properties of the Moore-Penrose and $\{i, j, k\}$ inverses and discusses the generalized inverses with prescribed range and null space. Much ink is also spent on the (generalized) Bott-Duffin inverses.

Chapter 2 studies the Drazin and the group inverses. Using generalized inverses, Chapter 3 investigates solutions to linear equations and Cramer's rule. Chapter 4 goes further on the properties of generalized inverses such as the reversed order law ($(AB)^{(r)} = B^{(r)}A^{(r)}$) and the forward order law ($(AB)^{(f)} = A^{(f)}B^{(f)}$).

The first four chapters may be considered as the basics on generalized inverses. The following chapters from 5 to 8 are on computation. Chapter 5 discusses the direct methods (i.e., computation of accurate solutions in finite steps) for computing a special type of generalized inverse. It also pays much attention to the generalized inverses of sums and partitioned matrices. Methods covered here include Greville's, Cline's, as well as Nobel's. Chapter 6 deals with parallel algorithms for computing generalized inverses, whereas Chapters 7 and 8 are devoted to the perturbation analysis of the (weighted) Moore-Penrose, Drazin and group inverses.

If the previous chapters are viewed as the generalized inverse theory for finite matrices, the rest of the book, Chapter 9, 10, and 11, are the extensions of the theory in Hilbert space. In the infinite dimensional setting, for instance, if T is a bounded linear operator from a Hilbert space to another Hilbert space with closed range $R(T)$, then $R(T^\dagger) = R(T^*) = R(T^\dagger T)$. Regarding computation of a generalized inverse of a linear operator, the authors present a few methods including the Euler-Knopp, Newton, and hyperpower methods. For the Drazin inverse, the existence and uniqueness theorems are shown, and perturbation bounds are discussed.

The book contains a great deal of research results of the authors as well as other mathematicians in the areas of perturbation theory, condition numbers, recursive algorithms, finite algorithms, imbedding algorithms, parallel algorithms, generalized inverses of rank-r modified matrices and Hessenberg matrices, extensions of the Cramer rules and the representation and approximation of generalized inverses of linear operators.

Although the book is written primarily as a text for a graduate course, there is no doubt it will serve as a valuable reference for the researchers in the fields of matrix theory, numerical linear algebra, parallel computations and particularly generalized inverses with applications. Prerequisites are basic linear algebra, matrix theory and functional analysis.

Reviewed by **Fuzhen Zhang**
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From the Editors

Recently Image lost one of our strongest supporters, contributors, and friends. Jerzy Baksalary read and revised nearly every problem submitted to the *Image* Problem Corner. In addition, he often provided his own (always elegant, and illuminating) solutions. Jerzy actively solicited lead articles, book reviews, and reports. Additional comments about Jerzy's mathematical contributions can be found in the tribute on the following pages. Jerzy, we (and the readers of Image) will miss you,

Bryan Shader & Hans Joachim Werner
(Editors-in-Chief)

ILAS-NET

ILAS operates ILAS-NET, an electronic news service that transmits announcements of ILAS activities and circulates other notices of interest to linear algebraists. Announcements for ILAS-NET or requests to be on the mailing list for ILAS-NET, should be sent to Shaun Fallat (sfallat@math.uregina.ca). Subscription to ILAS-NET is independent of membership in ILAS and is free.

ILAS INFORMATION CENTER

The electronic ILAS INFORMATION CENTER (IIC) provides current information on international conferences in linear algebra, other linear algebra activities, linear algebra journals, and ILAS-NET notices. The primary website can be found at:

http://www.ilasic.math.uregina.ca/iic/index1.html
http://www.math.technion.ac.il/iic/index1.html
http://wftp.tu-chemnitz.de/pub/iic/index1.html
http://hermite.cii.fc.ul.pt/iic/index1.html
http://www.math.temple.edu/iic/index1.html

Jerzy K. Baksalary (1944–2005) and his contributions to *Image*

Oskar Maria Baksalary & George P. H. Styan

Professor Jerzy K. Baksalary passed away in Poznań, Poland, on 8 March 2005. He was 60 years old. Although suffering, he remained active in his research work to the very end.


Photo by Simo Puntanen

Jerzy Baksalary was born in Poznań on 25 June 1944. He was awarded a Ph.D. degree in 1975 by the Adam Mickiewicz University, Poznań, for his dissertation written under the supervision of Tadeusz Caliński.

At the funeral service for Jerzy Baksalary held in Poznań on 15 March 2005, Tadeusz Caliński eulogized him (in Polish):

> "Let me express our feelings particularly on behalf of those who were close to you in the early years of your academic career, in the seventies and eighties of the past century, at the Agricultural University of Poznań. At that time you were for us an encouraging example of a person full of scientific ideas and willing to work hard. Your works in the theory and application of mathematical statistics and linear algebra drew us into the streams of worldwide scientific literature.
>
> Your personality stimulated younger colleagues and students, for whom you soon became a master and promoter of their careers. Among our joint scientific results of those years, your achievements shine with a particular brilliance. Your contributions to the Poznań school of mathematical statistics and biometry are highly esteemed at present and will be acknowledged by future generations."

For the years 1969–1988, Jerzy Baksalary was associated with the Department of Mathematical and Statistical Methods in the Agricultural University of Poznań, and he joined the academic community in Zielona Góra in 1988, first working in the Department of Mathematics of the Tadeusz Kotarbiński Pedagogical University and then in the Institute of Mathematics of the University of Zielona Góra, after it was founded in 2001. He was Rector of the Tadeusz Kotarbiński Pedagogical University from 1990 to 1996 and then Dean of its Faculty of Mathematics, Physics, and Technology from 1996 to 1999.

Jerzy Baksalary published extensively on matrix methods for statistics. He is the author or coauthor of 170 published research publications in linear algebra and statistics, including 45 papers published in *Linear Algebra and its Applications* (LAA). The Third Special Issue on Linear Algebra and Statistics of LAA (vol. 176, November 1992) was edited by Jerzy K. Baksalary and George P. H. Styan.

A Special Memorial Session for Jerzy Baksalary was organized by Oskar Maria Baksalary, Simo Puntanen, George P. H. Styan, and Götz Trenkler at the 14th International Workshop on Matrices and Statistics (Auckland, New Zealand, 29 March–1 April 2005). For this Memorial Session, Oskar Baksalary wrote about his father:

> "Although from the formal point of view I am a physicist and not a mathematician or statistician, with the death of JKB I have lost not only my father, but also my scientific master. On the one hand, this makes his passing away twice as hard for me to bear, but on the other hand I am very happy that for about the last four years I have been sharing with my father his great passion – mathematics. During this period we have been spending lots of time together, for instance travelling, visiting jazz clubs and art galleries, attending Thursday seminars on linear algebra organized at the Agricultural University of Poznań, chatting, and first of all . . . doing mathematics.
>
> JKB really loved his subject and especially he was in love with everything having to do with matrices. This means he also loved the International Workshops on Matrices and Statistics. My father and I have been participating in these Workshops since 2000, when the Workshop was held in Hyderabad, India, and thus the one organized this year in Auckland was to be the sixth which we would jointly attend."

The First Southern Ontario Matrices and Statistics Days, to be held in Windsor, Ontario, Canada, 9–10 June 2005, will be dedicated to Jerzy K. Baksalary. In preparation for distribution there is a handout, which will include a complete publication list[1] plus comments on the life and work of Jerzy K. Baksalary by Oskar Maria Baksalary, Tadeusz Caliński, R. William Farebrother, Jürgen Groß, Jan Hauke, Erkki Liski, Augustyn Markiewicz, Friedrich Pukelsheim, Tarmo Pukkila, Tomasz Szulc, Yongge Tian, Júlia Volaufová, Haruo Yanai, and Fuzhen Zhang. As Thomas Szulc points out there, Jerzy Baksalary was an extremely active contributor to the *Image* Problem Corner:

[1] Building on the list prepared for the "Session on the occasion of the 60th birthday of Jerzy K. Baksalary" held at the Mathematical Research & Conference Center, Polish Academy of Sciences, Będlewo, Poland, on 17 August 2004, just before the 13th International Workshop on Matrices and Statistics.

"I was deeply sadded to learn that Jerzy Baksalary has passed away. For the very last time we saw each other in mid-February at the seminar held at the Agricultural University of Poznań. This was a meeting in a series of seminars organized since 1999 every second Thursday. Participants of these seminars were: Jerzy K. Baksalary, Oskar Maria Baksalary, Jan Hauke, Augustyn Markiewicz, Tomasz Szulc and a group of Ph. D. students – our group was called by Jerzy: PLAG, an acronym for the Poznań Linear Algebra Group. Our meetings were instructive and fruitful and without any doubt, this was mainly due to Jerzy.

The activities of PLAG are well reflected in the problems and solutions in subsequent *Image* Problem Corners, for the problems posed therein were extensively discussed and analyzed during our seminars. The fruit of the cooperation between Jerzy and myself within PLAG meetings resulted in two joint papers published in *Linear Algebra and its Applications*. With the death of Professor Jerzy K. Baksalary, the linear algebra community has lost a truly great specialist in matrix analysis and PLAG has lost its leader."

Jerzy Baksalary's contributions to the *Image* Problem Corner include the following, listed here in order of publication:

1. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 25-1.1 (to Problem 25-1: "Moore–Penrose inverse of a skew-symmetric matrix" proposed by Jürgen Groß, Sven-Oliver Troschke, Götz Trenkler). *Image* 26, 2.
2. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 25-4.1 (to Problem 25-4: "Two rank equalities associated with blocks of an orthogonal projector" proposed by Yongge Tian). *Image* 26, 6–7.
3. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 25-5.1 (to Problem 25-5: "Three inequalities involving Moore–Penrose inverses" proposed by Yongge Tian). *Image* 26, 9–10.
4. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 25-6.1 (to Problem 25-6: "Generalized inverse of a matrix product" proposed by Yongge Tian). *Image* 26, 10–11.
5. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 26-4.1 (to Problem 26-4: "Commutativity of EP matrices" proposed by Yongge Tian). *Image* 27, 30.
6. Jerzy K. Baksalary, Oskar Maria Baksalary (2001). Solution 26-5.1 (to Problem 26-5: "Convex matrix inequalities" proposed by Bao-Xue Zhang). *Image* 27, 33–34.
7. Jerzy K. Baksalary, Richard William Farebrother (2002). Solution 27-1.1 (to Problem 27-1: "A class of square roots of involutory matrices" proposed by Richard William Farebrother). *Image* 28, 26–28.
8. Jerzy K. Baksalary, Oskar Maria Baksalary (2002). Solution 27-2.1 (to Problem 27-2: "Specific generalized inverses" proposed by Jürgen Groß, Götz Trenkler). *Image* 28, 29.
9. Jerzy K. Baksalary, Jan Hauke (2002). Solution 27-6.1 (to Problem 27-6: "Inequalities of Hadamard products of nonnegative definite matrices" proposed by Xingzhi Zhan). *Image* 28, 33.
10. Jerzy K. Baksalary, Oskar Maria Baksalary (2002). Solution 28-5.1 (to Problem 28-5: "A range equality for Moore–Penrose inverses" proposed by Yongge Tian). *Image* 29, 28–29.
11. Jerzy K. Baksalary (2002). Solution 28-6.1 (to Problem 28-6: "Square roots and additivity" proposed by Dietrich Trenkler, Götz Trenkler). *Image* 29, 30.
12. Jerzy K. Baksalary, Oskar Maria Baksalary (2002). Solution 28-7.2 (to Problem 28-7: "Partial isometry and idempotent matrices" proposed by Götz Trenkler). *Image* 29, 31.
13. Jerzy K. Baksalary, Jan Hauke (2002). Solution 28-10.1 (to Problem 28-10: "Inequalities involving square roots" proposed by Fuzhen Zhang). *Image* 29, 33–34.
14. Jerzy K. Baksalary, Oskar Maria Baksalary (2002). Problem 29.1: "A condition for an EP matrix to be Hermitian". *Image* 29, 36.
15. William F. Trench, Jerzy K. Baksalary, Oskar Maria Baksalary (2003). Solution 29-1.2 (to Problem 29-1: "A condition for an EP matrix to be Hermitian" proposed by Jerzy K. Baksalary, Oskar Maria Baksalary [14]). *Image* 30, 22.
16. Jerzy K. Baksalary, Oskar Maria Baksalary (2003). Solution 29-5.1 (to Problem 29-5: "Product of two Hermitian nonnegative definite matrices" proposed by Jürgen Groß, Götz Trenkler). *Image* 30, 24–25.
17. Jerzy K. Baksalary, Roger A. Horn (2003). Solution 29-7.1 (to Problem 29-7: "Complementary principal submatrices and their eigenvalues" proposed by Chi-Kwong Li). *Image* 30, 26–27.
18. Jerzy K. Baksalary, Xiaoji Liu (2003). Solution 29-8.1 (to Problem 29-8: "A range equality involving an idempotent matrix" proposed by Yongge Tian). *Image* 30, 27.
19. Jerzy K. Baksalary, Jan Hauke (2003). Solution 29-9.1 (to Problem 29-9: "Equality of two nonnegative definite matrices" proposed by Yongge Tian). *Image* 30, 29–30.
20. Jerzy K. Baksalary (2003). Solution 29-10.1 (to Problem 29-10: "Equivalence of three reverse-order laws" proposed by Yongge Tian). *Image* 30, 31.
21. Jerzy K. Baksalary, Oskar Maria Baksalary, Xiaoji Liu (2003). Problem 30-1: "Star partial ordering, left-star partial ordering, and commutativity". *Image* 30, 36.
22. Jerzy K. Baksalary, Oskar Maria Baksalary, Xiaoji Liu (2003). Solution 30-1.1 (to Problem 30-1: "Star partial ordering, left-star partial ordering, and commutativity" proposed by Jerzy K. Baksalary, Oskar Maria Baksalary, Xiaoji Liu [21]. *Image* 31, 30–31.
23. Jerzy K. Baksalary, Oskar Maria Baksalary (2003). Solution 30-5.1 (to Problem 30-5: "A range equality for the difference or orthogonal projectors" proposed by Yongge Tian). *Image* 31, 36–37.
24. Jerzy K. Baksalary, Oskar Maria Baksalary (2003). Solution 30-6.1 (to Problem 30-6: "A matrix related to an idempotent matrix" proposed by Götz Trenkler). *Image* 31, 39.
25. Jerzy K. Baksalary, Oskar Maria Baksalary (2003). Solution 30-7.1 (to Problem 30-7: "A condition for an idempotent matrix to be Hermitian" proposed by Götz Trenkler). *Image* 31, 41.
26. Jerzy K. Baksalary, Oskar Maria Baksalary, Xiaoji Liu (2004). Solution 31-2.1 (to Problem 31-2: "Matrices commuting with all nilpotent matrices" proposed by Henry Ricardo). *Image* 32, 21–22.
27. Jerzy K. Baksalary (2004). Solution 31-3.1 (to Problem 31-3: "A range equality for block matrices" proposed by Yongge Tian). *Image* 32, 23–24.
28. Jerzy K. Baksalary, Paulina Kik, Augustyn Markiewicz (2004). Solution 31-6.1 (to Problem 31-6: "A full rank factorization of a skew-symmetric matrix" proposed by Götz Trenkler). *Image* 32, 27–28.
29. Jerzy K. Baksalary, Oskar Maria Baksalary (2004). Solution 31-7.1 (to Problem 31-7: "On the product of orthogonal projectors" proposed by Götz Trenkler). *Image* 32, 30–31.
30. Jerzy K. Baksalary, Anna Kuba (2004). Solution 31-7.2 (to Problem 31-7: "On the product of orthogonal projectors" proposed by Götz Trenkler). *Image* 32, 31–34.

Photo by Magdala Werner

Participants of the 14th International Workshop on Matrices and Statistics: Auckland, New Zealand, 29 March- 1 April 2005

14th International Workshop on Matrices and Statistics Auckland, New Zealand: 29 March–1 April 2005

Jeffrey J. Hunter & George P. H. Styan

The 14th International Workshop on Matrices and Statistics (IWMS-2005) was held on the Albany campus of Massey University in Auckland, New Zealand, 29 March–1 April 2005. This Workshop was a Satellite meeting to the 55th Biennial Session of the International Statistical Institute held in Sydney, Australia, 5–12 April 2005 and was endorsed by the International Linear Algebra Society. The Workshop was supported by the New Zealand Statistical Association, Massey University, New Zealand Institute of Mathematics & its Applications, Royal Society of New Zealand, and Statistics New Zealand.

The International Organising Committee comprised Simo Puntanen (University of Tampere, Finland), George P. H. Styan (chair; McGill University, Canada) & Hans Joachim Werner (vice-chair; Universität Bonn, Germany). The Local Organising Committee was chaired by Jeffrey J. Hunter, and included Freda Anderson (Workshop Secretary), Merrill Bowers, Paul Cowpertwait, Marie Fitch, Stephen Ford, Beatrix Jones, Claire Jordan, Nikki Luke, Barry McDonald, Dennis Viehland, and Danny Walsh, all of Massey University, Auckland.

A Special Memorial Session in honour of Jerzy K. Baksalary (1944–2005) was organized by Oskar Maria Baksalary (Adam Mickiewicz University, Poland), Simo Puntanen, George P. H. Styan, and Götz Trenkler (Universität Dortmund, Germany).

The keynote speakers were:

C. Radhakrishna Rao, Pennsylvania State University, USA: "Statistical proofs of matrix theorems",

Shayle R. Searle, Cornell University, USA: "Reflections on a fifty year random walk midst matrices and statistics",

George A. F. Seber, University of Auckland, New Zealand: "Things my mother never told me about matrices", and

Eugene Seneta, University of Sydney, Australia: "Coefficients of ergodicity in a matrix setting".

C. Radhakrishna Rao was sponsored by Nokia as the IWMS-2005 Nokia Lecturer and was the New Zealand Statistical Association Visiting Lecturer for 2005 and a Massey University Distinguished Visitor. Shayle R. Searle was sponsored by SAS as the IWMS-2005 SAS Lecturer. Eugene Seneta was sponsored as the New Zealand Mathematical Society Lecturer for 2005.

The Workshop opened on 29 March 2005 with a powhiri, the Maori welcome ceremony. It removes the tapu of the Manuhiri (visitors) to make them one with the Tangata Whenua (home people) and is a gradual process of the Manuhiri and the Tangata Whenua coming together. The excursion on 30 March explored Waiheke Island and its beaches, wineries and a boutique brewery.

The invited lectures were:

S. Ejaz Ahmed: "Approximation-assisted estimation of eigenvectors under quadratic loss",

Anyue Chen: "Asymptotic birth-death processes: a matrix analysis approach",

Karl E. Gustafson: "The geometry of statistical efficiency",

Stephen Haslett & John Haslett: "What are the residuals for the linear model?",

Moshe Haviv: "On singularly perturbed Markov chains",

Jarkko Isotalo & Simo Puntanen: "Comparison of the ordinary least squares predictor and the best linear unbiased predictor in the general Gauss–Markov model",

J. A. 'Nye' John: "Inverse of the information matrix",

Estate Khmaladze: "Inverse matrices, Volterra operators and innovation processes: application to statistics",

Tõnu Kollo & Dietrich von Rosen: "Approximation of the parameter distributions of growth curve model",

Alexander Kukush: "Invariant estimator in a quadratic measurement error model",

Alan J. Lee & Alastair J. Scott: "Semi-parametric efficiency, projection and the Scott–Wild estimator",

Simo Puntanen, Ka Lok Chu, Jarkko Isotalo & George P. H. Styan: "Decomposing the Watson efficiency in partitioned linear models",

C. Radhakrishna Rao: "Anti-eigen and anti-singular values of a matrix and applications to problems in statistics",

George P. H. Styan, Ka Lok Chu, Jarkko Isotalo & Simo Puntanen: "Inequalities and equalities for the Watson efficiency in orthogonally partitioned full rank linear models",

Garry J. Tee: "Eigenvectors of block circulant matrices",

Götz Trenkler: "On the commutativity of orthogonal projectors",

Hans Joachim Werner & Ingram Olkin: "On permutations of matrix products".

The Workshop group photo is on the facing page; for other photos please visit http://iwms2005.massey.ac.nz/photos.html

The 15th International Workshop on Matrices and Statistics (IWMS-2006) will be held in Uppsala, Sweden, 13–17 June 2006: http://www.bt.slu.se/iwms2006/iwms06.html The International Organizing Committee is chaired by Hans Joachim Werner: ujw902@uni-bonn.de and the Local Organizing Committee is chaired by Dietrich von Rosen (Swedish University of Agricultural Sciences): iwms06@bt.slu.se

2004 NZIMA and 29th Australian CMCC Conference

Report by **Ian Wanless**

One of the most significant events in the combinatorics calendar for 2004 was held on the shores of beautiful Lake Taupo, New Zealand in 13th-18th December. It was a joint conference with the un-snappy title of "The 2004 NZIMA Conference in Combinatorics and its Applications and The 29th Australasian Conference in Combinatorial Mathematics and Combinatorial Computing". Details including a full programme and photos can be found at the conference webpage: http://www.nzima.auckland.ac.nz/combinatorics/conference.html.

The conference had an unusually large number of plenary speakers and all were of the highest quality. They were Dan Archdeacon (U. Vermont), Rosemary Bailey (Queen Mary, U. London), Richard Brualdi (U. Wisconsin), Darryn Bryant (U. Queensland), Peter Cameron (Queen Mary, U. London), Maria Chudnovsky (Princeton U. and CMI), Bruno Courcelle (Bordeaux U.), Jim Geelen (U. Waterloo), Bert Gerards (CWI and Eindhoven U. Technology), Catherine Greenhill (U. New South Wales), Bojan Mohar (U. Ljubljana), Bruce Richter (U. Waterloo), Neil Robertson (Ohio State U.), Paul Seymour (Princeton U.), Alan Sokal (New York U.), Robin Thomas (Georgia Institute of Technology), Carsten Thomassen (Technical U. Denmark), Tom Tucker (Colgate U.), Mark Watkins (U. Syracuse) and Dominic Welsh (Oxford U.).

This galaxy of stars attracted significant global interest; so much so, that the organisers were faced with the unusual problem of having to turn a few people away for fear of exceeding the venue's capacity. In the end over 150 participants from 19 countries took part, making it the largest pure mathematics conference to be held in New Zealand since 1978.

Over six days the attendees were treated to very interesting talks on a wide range of combinatorial themes including graph theory, matroids, designs, coding theory, enumeration, optimization, theoretical computer science and combinatorial matrix theory, to name just a few. The winner of the prize for best student talk was Shuji Kijima from the University of Tokyo. His talk was titled

cont'd on page 19

NZIMA and CMCC Conference Photo

CMCC Conference cont'd

"Perfect Sampler for Closed Jackson Networks".

Lake Taupo is the caldera of a truly enormous (but luckily dormant!) volcano, but there are still active volcanoes nearby. These volcanoes (which feature in the Lord of the Rings movies) got a new coat of snow during the week (a reminder to Northern hemispherians that December is, at least nominally, the height of summer in New Zealand). Together with the lake they formed a very impressive backdrop. Carsten Thomassen, who has given a few opening lectures in his time, was moved to observe that he has never had a more beautiful view for one of his plenary lectures.

The excursion on the Wednesday afternoon sampled just some of the local tourist highlights including the torrential Huka falls, lunch at a local prawn farm, a scenic geothermal hotspot called Orakei Korako and finally a geothermal power station near Taupo.

The conference organisers (Paul Bonnington and Geoff Whittle from NZ and Brendan Mckay and Ian Wanless from Australia) would like to acknowledge the sponsorship of the New Zealand Institute of Mathematics and its Applications (NZIMA), Australian Mathematical Society (AustMS) and the Centre of Discrete Mathematics and Theoretical Computer Science (CDMTCS).

MAT-TRIAD 2005 - Three Days Full of Matrices

Report by **Oskar Maria Baksalary**

MAT-TRIAD 2005 - three days full of matrices was held at the Mathematical Research and Conference Center in Będlewo, Poland, on March 3-5, 2005. This international workshop attracted 40 participants; 38 representing seven European countries, one from Canada, and one from the USA. The organizing committee, consisting of Jan Hauke, Augustyn Markiewicz (chair), Tomasz Szulc, and Waldemar Wołyński, did an excellent job in acquiring generous sponsors and thus there was no registration fee and local

cont'd on page 21

MAT-TRIAD 2005 Conference Photo

Linear Algebra

from World Scientific and Imperial College Press

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Textbook

(Volume One)
by **I-Hsiung Lin** *(National Taiwan Normal University, China)*

This accessible book for beginners uses intuitive geometric concepts to create abstract algebraic theory with a special emphasis on geometric characterizations. The book applies known results to describe various geometries and their invariants, and presents problems concerned with linear algebra, such as in real and complex analysis, differential equations, differentiable manifolds, differential geometry, Markov chains and transformation groups. The clear and inductive approach makes this book unique among existing books on linear algebra both in presentation and in content.

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376pp **Sept 2004**
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981-256-080-7(pbk) **US$42**

NONCOMMUTATIVE CHARACTER THEORY OF THE SYMMETRIC GROUP

by **Dieter Blessenohl** *(Christian-Albrechts-Universität Kiel, Germany)* &
Manfred Schocker *(University of Wales Swansea, UK)*

This is the first account in book form entirely devoted to the new "noncommutative method". As a modern and comprehensive survey of the classical theory the book contains such fundamental results as the Murnaghan–Nakayama and Littlewood–Richardson rules as well as more recent applications in enumerative combinatorics and in the theory of the free Lie algebra. But it is also an introduction to the vibrant theory of certain combinatorial Hopf algebras such as the Malvenuto–Reutenauer algebra of permutations.

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MAT-TRIAD cont'd

expenses, such as food and lodging, were covered. Unfortunately, the number of participants was limited to just 40 and this is why the workshop was not widely advertised beforehand.

The meeting brought together researchers interested in various aspects of matrix analysis and its applications. During the three days, 27 talks were given and later extensively discussed in a friendly and stimulating atmosphere. Since the attendants of MAT-TRIAD 2005 left Będlewo sharing the common opinion that the workshop was extremely fruitful, the participants along with the organizers are looking forward to the next meeting in the MAT-TRIAD series, which is initially planned for 2007. The list of attendants of MAT-TRIAD 2005, abstracts of talks given, and the photo gallery are available at http://mtriad05.amu.edu.pl.

Brualdi-Fest: Linear Algebra, Graph Theory and Combinatorics

Report by **Bryan L. Shader**

An informal two-day conference, Brualdi-fest: Linear Algebra, Graph Theory and Combinatorics, was held April 30-May 1, 2005 at the University of Wisconsin-Madison. The conference was in honor of Richard Brualdi and his numerous contributions to mathematics. The keynote speaker was Richard Wilson (CalTech).

Over 50 Brualdi-enthusiasts participated in the conference, and the 25 contributed talks illustrated the breadth, depth and impact of Richard's mathematical contributions. Participants included: Jackie Anderson, Dwight Bean, Rikki Bostelman, Keith Chavey, Han Hyuk Cho, Jason Darby, Luz DeAlba, George Dinolt, Cliff Ealy, Mike Engel, Thomas Foregger, Shmuel Friedland, John Goldwasser, Frank Hall, Sylvia Hobart, Leslie Hogben, Susan Hollingsworth, Suk-geun Hwang, Rebecca Kohler, Elliott Krop, Mark Lawrence, Doug Lepro, Zhongshan Li, Chi-Kwong Li, Ryan Martin, T. S. Michael, Nancy Neudauer, Arlene Pascasio, Yiu Poon, Dan Pritikin, Jim Propp, Jennifer Quinn, Hans Schneider, Bryan Shader, Jia-yu Shao, Jian Shen, Jonathan Smith, Wasin So, Paul Terwilliger, Kevin vander Meulen, Amy Wangsness, Rick Wilson, and Rober Wilson.

The conference was organized by John Goldwasser (U. West Virginia), Hans Schneider (U. Wisconsin-Madison), Bryan Shader (U. Wyoming), and Robert Wilson (U. Wisconsin-Madison). A photo-journal of the conference can be found at http://www.oldflutes.com/Kathy&Rick/brualdi.htm.

Richard with some of his past and present Ph.D. students
George Dinolt, Thomas Foregger, John Goldwasser, Dan Pritikin, Suk-geun Hwang, Han Cho, T.S. Michael
Bryan Shader, Keith Chavey, Jennifer Quinn, Mark Lawrence, Nancy Neudauer, Susan Hollingsworth
Adam Berliner and Louis Deatt

THE 2005 Haifa Matrix Theory Conference

Report by **Bryan L. Shader**

The Thirteenth Haifa Matrix Theory Conference was held January 3-7, 2005 at the Technion–Israel Institute of Technology. The Scientific Committee consisted of Abraham Berman (Chair), Moshe Goldberg, Daniel Hershkowitz, Leonid Lerer and Raphael Loewy. This conference is the thirteenth in a sequence, dating back to 1984, devoted to matrix analysis and the various aspects of linear algebra and its applications.

The program consisted of over 60 talks, given by speakers from China, the Czech Republic, Germany, India, Ireland, Israel, The Netherlands, Russia, Slovenia, Taiwan and the United States. The ILAS Lecture,*"Soules Matrices and the nonnegative matrix factorization"* was given by Miki Neumann (U. Connecticut).

Linear Algebra and its Applications will publish a special issue devoted to papers presented at the conference. The special editors are Abraham Berman, Leonid Lerer and Raphael Loewy. The submission deadline was April 30, 2005. The usual standards of LAA will apply.

In addition to the incredibly rich and broad slate of talks, participants were treated to numerous social events, and excursions. These included: a bus tour of the lovely city of Haifa, a reception hosted by the Mayor of Haifa, a reception hosted by the Center for Mathematical Sciences, a tour of the historical site Bet Shearim (During the 3rd and 4th Centuries CE, Bet Shearim was the seat of the Sanhedrin and the home to the famous Rabbi Yehuda Hanassi. This great scholar was buried among the many catacombs carved inside this limestone necropolis, an immense labyrinth of vaulted chambers and stone sarcophagi, most of which are still intact), a reception at the home of Izchak Lewkowicz, and a banquet feast in a hotel that overlooks Haifa.

Haifa Matrix Theory Conference Photo

Carl de Boor honored

ILAS member Carl de Boor was one of 14 scientists (the only one in mathematics) awarded the prestigious U.S. National Medal of Science during a recent White House Ceremony.

The award honors individuals for pioneering research that has led to a better understanding of the world, and to the development of significant innovations and technologies. Carl's work in numerical mathematics, and in particular in the spline theory, is truly outstanding, and continues to be the mathematical heart of computer aided design, computer graphics, and signal and image processing.

Congratulations Carl!

Forthcoming Conferences and Workshops in Linear Algebra

1st International Workshop on Matrix Analysis

Beijing Normal University, Beijing
9–10, June 2005

This workshop aims to stimulate research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis and their applications and to provide an opportunity for Chinese as well as for international researchers to exchange ideas and recent developments on these subjects. This will be a two-day meeting without parallel sessions.

Registration will be accepted from through June 6, 2005. There is no registration fee. Titles and abstracts should be submitted to Xiuping Zhang no later than June 6, 2005 by e-mail in LaTeX or TeX.

Talks can be presented in either English or Chinese, and slides must be in English.

Beijing Normal University (BNU) is located in Haidian District of Beijing. Transportation to and from BNU is available and very convenient. The University's Lanhui Hotel hosts guests and foreigners at decent rates. A Workshop Dinner may be scheduled per request by majority of the participants. No financial aid will be provided to general participants.

No excursion is planned for this Workshop. However, we will be happy to provide information for visiting the Great Wall, the Forbidden City, the Summer Palace, and other interesting spots.

The organizers are: Dr. Tiangang Lei (National Natural Science Foundation of China, Chinaleitg@mail.nsfc.gov.cn), Dr. Xiuping Zhang (Beijing Normal University, China, zhxp@bnu.edu.cn) and Dr. Fuzhen Zhang (Nova Southeastern University, USA, zhang@nova.edu).

The workshop website is: www.nova.edu/~zhang/1Matrix Workshop.html.

Combinatorial Matrix Theory

Lincoln, NE
21–23, October 2005

A special session on Combinatorial Matrix Theory, will be held in conjunction with the 2005 Fall AMS Sectional meeting (October 21-23) in Lincoln, Nebraska. The deadline for Contributed Papers for consideration in AMS Special Sessions is July 5, 2005. Submissions must be done electronically at the conference website.

There will also be a special session devoted to graph theory at this meeting, making it particularly attractive to those with interests in both linear algebra and graph theory. This meeting was chosen to host the major AMS public lecture that is held annually at a sectional meeting, and Sir Michael Atiyah will deliver this lecture on Friday, October 21, 2005 (most probably from 4:00-5:00 pm). More information about the meeting can be found on the AMS website: http://www.ams.org/amsmtgs/2117_program.html.

Confirmed participants include:

Francesco Barioli (Carelton U.),
Luz DeAlba (Drake U.),
Shaun Fallat (U. Regina),
Frank Hall (Georgia State U.),
Elliot Krop (U. Illinois-Chicago),
Chi-Kwong Li (College of William & Mary),
Zhongshan Li (Georgia State U.),
Yiu Poon (Iowa State U.),
Judi MacDonald (Washington State U.),
Wasin So (San Jose State U.),
Kevin Vander Meulen (Redeemer College), and
Amy Wangsness (Iowa State U.)

The special session is being organized by Leslie Hogben (lhogben@iastate.edu) and Bryan Shader (bshader@uwyo.edu).

Workshop on Graph Spectra

Aveiro, Portugal
10-12 April 2006

The recognition of the strong developments on spectral graph theory encouraged the organization of this workshop which will be a meeting point for many researchers around the world.

The main goal is to bring together the leading researchers on graph spectra and related topics, to establish the state of the art and to discuss the main current achievements and challenges in this topic.

We are proud with the participation of renown specialists in spectral graph theory as members of the Scientific Committee and invited speakers, delivering 10 plenary presentations. Additionally, we have planned a few parallel contributions and a problem session in which we expect the participation of international experts with their most recent results.

The contributors should submit a pdf file of the abstract (no

cont'd on page 23

Aveiro conference cont'd from page 23

more than two A4 size pages), attached to the registration form or to a message sent to awgs@mat.ua.pt, by January 15, 2006.

Furthermore, after notification of acceptance (by February 15, 2006), the contributors should also send the corresponding LaTex file.

A collection of selected papers on spectral graph theory and related topics will be published in a special issue of *Linear Algebra and Its Applications*, to be edited by: Dragos Cvetkovic, Willem Haemers and Peter Rowlinson.

Participants will be invited to submit their papers during the workshop or afterwards. The deadline for submission is July 15, 2006.

13th ILAS Conference
Amsterdam, The Netherlands
18–21, July 2006

The 13th ILAS conference will be held in Amsterdam during the week preceding the SIAM conference on Linear Algebra in Dusseldorf, Germany. The dates will be July 18-21 for the ILAS conference. The conference will be organized at the Vrije Universiteit, located in the southern part of the city of Amsterdam.

The organizing committee consists of: Harm Bart, Jan Brandts, Daniel Hershkowitz, Steve Kirkland, Andre Klein, Andre Ran, Peter Spreij, Henk van der Vorst and Paul VanDooren. The local organization will be done by Jan Brandts, Andre Klein, Andre Ran and Peter Sprey.

The conference will be mainly structured around a number of themes. For each of those themes an invited lecture will be combined with a mini-symposium. Themes selected so far include:

- Linear Algebra in Statistics
- Numerical Linear Algebra
- Matrices in Indefinite Scalar Product Spaces
- Structured Matrices
- Positive Linear Algebra

In addition, there will be the possibility for participants to present their work at the conference, even if it does not fall under one of the special themes. The website of the conference is

http://staff.science.uva.nl/~brandts/ILAS06.

The city of Amsterdam can be reached easily from around the globe. The main airport of the Netherlands, Schiphol airport, is only a short train ride away from both the city center and the Vrije Universiteit. In addition, Amsterdam can be reached from the major cities in Europe easily by train. Once inside the city, the excellent network of public transportation will guarantee you an easy trip to your hotel and the conference location.

Participants are expected to make their own arrangements for accommodation. This can be done easily online, see for instance the website www.travel-holland.com/amsterdam.

Although at this time the exact amount for the registration fee is still uncertain, our aim is to keep the registration cost as low as possible.

As usual for ILAS meetings, the proceedings will appear as a volume of *Linear Algebra and its Applications*. Editors for the volume will be Harm Bart, Jan Brandts, Andre Ran and Paul VanDooren.

Applied Linear Algebra
Düsseldorf, Germany
24–27, July 2006

A joint GAMM-SIAM Conference organized in cooperation with ILAS will be held at the Heinrich-Heine Universität in Düsseldorf, Germany from July 24–27, 2006.

Linear algebra problems and linear algebra algorithms for their solution are at the very heart of almost all numerical computations and play a prominent role in modern simulation methods in science and engineering. This conference, which belongs to a series of tri-annual meetings organized by SIAM in the US is the premier international conference on applied linear algebra.

Participants will present and discuss their latest results in the area of applied linear algebra, ranging from advances in the theory over the development and analysis of new precise and efficient algorithms to large scale supercomputer applications. Ludwig Elsner and Olga Holtz will be ILAS Lecturers at the conference.

The conference is organized jointly by Heinrich-Heine Universität Düsseldorf and Bergische Universität Wuppertal, and the co-organizers are Andreas Frommer, Marlis Hochbruck and Bruno Lang.

The Program Committee consists of: Michele Benzi, Zlatko Drmac, Heike Fassbender, Sven Hammarling, Daniel Hershkowitz, Ilse Ipsen, Bo Kagstrom, Steve Kirkland, Rich Lehoucq, Volker Mehrmann, Julio Moro and Jim Nagy.

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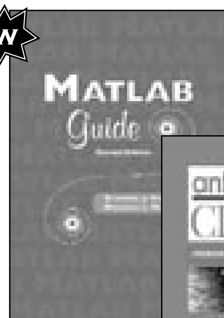

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— Steven H. Frankel, Purdue University.

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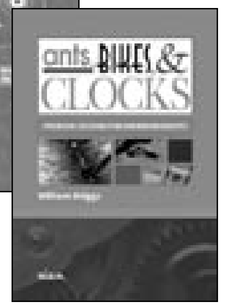

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IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 33-1 through 33-8 [IMAGE 33 (October 2004), pp. 36 & 35]. Problems 30-3 [IMAGE 30 (April 2003), p. 36] and 32-4 [IMAGE 32 (April 2004), p. 40] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 10 new problems on pp. 40 & 39 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to hjw.de@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

We look forward to receiving solutions to Problem 30-3!

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j}X_1 + u_{2j}X_2 + u_{3j}X_3$, $j = 1, 2, 3$, such that each of the six matrices $W_j^{-1}W_k$, $j \neq k$, has zero trace. Equivalently, we want $(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We look forward to receiving solutions to Problem 32-4!

Problem 33-1: A Range Equality for the Kronecker Product of Matrices

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Let A and B be $m_1 \times n_1$ and $m_2 \times n_2$ matrices, respectively. Show that

$$\text{range}(A \otimes I_{m_2}) \cap \text{range}(I_{m_1} \otimes B) = \text{range}(A \otimes B),$$

where $\otimes$ denotes the Kronecker product of matrices and I is the identity matrix of the indicated order.

Solution 33-1.1 by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Consider the orthogonal projectors $P_1 = P_{\text{range}(I_{m_1} \otimes B)}$ and $P_2 = P_{\text{range}(A \otimes I_{m_2})}$. Then $P_1 = (I_{m_1} \otimes B)(I_{m_1} \otimes B)^+ = (I_{m_1} \otimes B)(I_{m_1} \otimes B^+) = I_{m_1} \otimes BB^+$. Analogously, $P_2 = (AA^+ \otimes I_{m_2})$. It follows that $P_1P_2 = P_2P_1 = AA^+ \otimes BB^+ =$

$(A \otimes B)(A^+ \otimes B^+) = (A \otimes B)(A \otimes B)^+ = P_{\text{range}(A\otimes B)}$. Since P_1 and P_2 commute, $P_1P_2 = P_{\text{range}(A\otimes B)}$ is the orthogonal projector on range$(A \otimes B)$ = range(P_1P_2) = range$(P_1) \cap$ range(P_2) [cf. Rao & Mitra (1971, Theorem 5.1.4)]. This implies range$(A \otimes I_{m_2}) \cap$ range$(I_{m_1} \otimes B)$ = range$(A \otimes B)$.

Reference

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

Solution 33-1.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We offer an extremely elementary proof. First, we recall that $A \otimes B := (a_{ij}B)_{i,j}$, where $A = (a_{ij})_{i,j}$. As usual, let $\mathcal{R}(\cdot)$ denote the range (column space) of the matrix $(\cdot)$. From the definition of the Kronecker product we directly obtain

$$\mathcal{R}(A \otimes B) = \{(\sum_{j=1}^{n_1} a_{ij}b_j)_{i=1,\cdots,m_1} \mid \forall j : \, b_j \in \mathcal{R}(B)\} \tag{1}$$

and

$$\mathcal{R}(I_{m_1} \otimes B) \cap \mathcal{R}(A \otimes I_{m_2}) = \{(\sum_{j=1}^{n_1} a_{ij}c_j)_{i=1,\cdots,m_1} \mid \forall j : \, c_j \in \mathbb{C}^{m_2}; \, \forall i : \, \sum_{j=1}^{n_1} a_{ij}c_j \in \mathcal{R}(B)\} \tag{2}$$

So, trivially $\mathcal{R}(A \otimes B) \subseteq \mathcal{R}(I_{m_1} \otimes B) \cap \mathcal{R}(A \otimes I_{m_2})$. Conversely, let $x \in \mathcal{R}(I_{m_1} \otimes B) \cap \mathcal{R}(A \otimes I_{m_2})$ be represented as in (2). Clearly, $\sum_j a_{ij}c_j \in \mathcal{R}(B) \Leftrightarrow \sum_j a_{ij}c_j = BB^-(\sum_j a_{ij}c_j)$, where B^- indicates a generalized inverse of B. Needless to mention, $BB^-(\sum_j a_{ij}c_j) = \sum_j a_{ij}BB^-c_j$ and $BB^-c_j \in \mathcal{R}(B)$. Hence, by letting $b_j := BB^-c_j$, $x = \left(\sum_j a_{ij}b_j\right)_i$, and so, according to (1), $x \in \mathcal{R}(A \otimes B)$. Therefore, $\mathcal{R}(I_{m_1} \otimes B) \cap \mathcal{R}(A \otimes I_{m_2}) \subseteq \mathcal{R}(A \otimes B)$. So, Problem 33-1 is solved.

Solutions to Problem 33-1 were also received from Yongge Tian and from William F. Trench.

Problem 33-2: Similarity of Two Block Matrices
Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Suppose that the two square matrices A and B satisfy $A^2 = 0$ and $B^2 = 0$. Show that the two block matrices

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

are similar if and only if

$$\text{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \text{rank}\,(A) + \text{rank}\,(B) \quad \text{and} \quad AC + CB = 0.$$

Solution 33-2.1 by the Proposer Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Suppose the two block matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are similar, i.e., there is a nonsingular matrix P such that $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = P\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}P^{-1}$, then it turns out that $\text{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \text{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \text{rank}\,(A) + \text{rank}\,(B)$, and

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^2 = P\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^2 P^{-1} = P\begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix}P^{-1} = 0.$$

Expanding the left-hand side of this equality yields $AC + CB = 0$. According to the well-known Roth theorem [see Roth (1952)]: $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are similar if and only if there is an X such that $AX - XB = C$. Since both $A^2 = 0$ and $B^2 = 0$, they can be decomposed, from the similarity theory of matrices, as

$$A = P^{-1}\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}P, \quad B = Q\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}Q^{-1}, \tag{3}$$

where A_1 and B_1 are square matrices of appropriate sizes. In this case, $AX - XB = C$ becomes

$$\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix} Y - Y \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} = F. \tag{4}$$

Partition

$$Y = PXQ = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \text{ and } F = PCQ = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}. \tag{5}$$

Then (4) can be expressed as

$$\begin{pmatrix} A_1Y_3 & A_1Y_4 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & Y_1B_1 \\ 0 & Y_3B_1 \end{pmatrix} = \begin{pmatrix} A_1Y_3 & A_1Y_4 - Y_1B_1 \\ 0 & -Y_3B_1 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \tag{6}$$

that is, $A_1Y_3 = F_1$, $Y_3B_1 = -F_4$, $A_1Y_4 - Y_1B_1 = F_2$, $F_3 = 0$. According to Mitra (1984), the pair of matrix equations $A_1Y_3 = F_1$ and $Y_3B_1 = -F_4$ have a common solution if and only if

$$\text{range}\,(F_1) \subseteq \text{range}\,(A_1), \quad \text{range}\,(F_4^T) \subseteq \text{range}\,(B_1^T), \text{ and } A_1F_4 + F_1B_1 = 0. \tag{7}$$

The matrix equation $A_1Y_4 - Y_1B_1 = F_2$ is consistent if and only if [see Roth (1952)]

$$\text{rank}\begin{pmatrix} A_1 & F_2 \\ 0 & B_1 \end{pmatrix} = \text{rank}\,(A_1) + \text{rank}\,(B_1). \tag{8}$$

The condition $F_3 = 0$, the two range inclusions in (7) and the rank equality (8) are equivalent to the rank equality

$$\text{rank}\begin{pmatrix} 0 & A_1 & F_1 & F_2 \\ 0 & 0 & F_3 & F_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix} + \text{rank}\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix},$$

which, by (3) and (5), is further equivalent to

$$\text{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \text{rank}\,(A) + \text{rank}\,(B). \tag{9}$$

On the other hand, $A_1F_4 + F_1B_1 = 0$ is equivalent to

$$\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} + \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} = 0,$$

which, by (3) and (5), is also equivalent to

$$P^{-1}\begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix} PC + CQ\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} Q^{-1} = 0,$$

that is,

$$AC + CB = 0. \tag{10}$$

These results show that if (9) and (10) hold, then the matrix equation (6) is consistent. Correspondingly, the matrix equation (4) is consistent.

References

W. E. Roth (1952). The equations $AX - YB = C$ and $AX - XB = C$ in matrices. *Proceedings of the American Mathematical Society* **3**, 392–396.

S. K. Mitra (1984). The matrix equations $AX = C$, $XB = D$. *Linear Algebra and Its Applications* **59**, 171–181.

Solution 33-2.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We offer the following very informative solution to this nice and interesting problem.

THEOREM 1. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be both nilpotent of index ≤ 2, i.e., let $A^2 = 0$ and $B^2 = 0$. Moreover, let $C \in \mathbb{C}^{m\times n}$. For the matrices*

$$W_1 := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{and} \quad W_2 := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

the following conditions are then equivalent:

(a) $W_1 \sim W_2$, *i.e., there exists a nonsingular matrix R such that $RW_1R^{-1} = W_2$.*
(b) $\operatorname{rank}(W_1) = \operatorname{rank}(W_2)$ *and* $AC + CB = 0$.
(c) $C = AA^-C + CB^-B - AA^-CB^-B$ *and* $AC + CB = 0$, *with $(\cdot)^-$ denoting any generalized inverse of $(\cdot)$.*
(d) $C = AA^-M + MB^-B - AA^-MB^-B$ *for some matrix M and* $AC + CB = 0$.
(e) $AX - XB = C$ *for some matrix X.*

Our proof of this theorem will be based on the following powerful auxiliary results.

LEMMA 2. *Let W_1 and W_2 be block partitioned matrices as in Theorem 1, where, however, the matrices A and B need not necessarily be nilpotent. Then the following conditions are equivalent:*

(a) $\operatorname{rank}(W_1) = \operatorname{rank}(W_2)$.
(b) $C\mathcal{N}(B) \subseteq \mathcal{R}(A)$, *with $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ denoting the nullspace and the range (column space), respectively, of the matrix $(\cdot)$.*
(c) $C = AA^-C + CB^-B - AA^-CB^-B$.
(d) $C = AA^-M + MB^-B - AA^-MB^-B$ *for some matrix M.*

PROOF. Clearly, $\operatorname{rank}(W_2) = \operatorname{rank}(A) + \operatorname{rank}(B)$. Therefore, according to Corollary 4(c) in Werner (2003), (a) $\Leftrightarrow$ (b). We next note that

$$C\mathcal{N}(B) \subseteq \mathcal{R}(A) \;\Leftrightarrow\; (I - AA^-)C(I - B^-B) = 0 \;\Leftrightarrow\; C = AA^-C + CB^-B - AA^-CB^-B, \tag{11}$$

where $(\cdot)^-$ indicates a generalized inverse of $(\cdot)$ and where I stands for an identity matrix of appropriate order. In this context we note that $\mathcal{R}(I - B^-B) = \mathcal{N}(B)$ and $\mathcal{N}(I - AA^-) = \mathcal{R}(A)$. Observing that the general solution to $(I - AA^-)Z(I - B^-B) = 0$ is of the form $Z = M - (I - AA^-)M(I - B^-B) = AA^-M + (I - AA^-)MB^-B$, where M of suitable order can be arbitrarily chosen, completes the proof of this lemma. □

LEMMA 3. *Let $N \in \mathbb{C}^{n\times n}$ be nilpotent of index ≤ 2, i.e., let $N^2 = 0$. Moreover, let $\operatorname{rank}(N) = r$. Then*

$$N \;\sim\; \tilde{N} := \begin{pmatrix} 0_{r\times r} & 0_{r\times(n-2r)} & I_r \\ 0_{(n-2r)\times r} & 0_{(n-2r)\times(n-2r)} & 0_{(n-2r)\times r} \\ 0_{r\times r} & 0_{r\times(n-2r)} & 0_{r\times r} \end{pmatrix}, \tag{12}$$

where I_r stands for the identity matrix of order $r\times r$, and where $0_{s\times t}$ denotes the zero matrix of order $s \times t$. If $r = 0$ or, equivalently, if $N = 0$, then all blocks except $0_{(n-2r)\times(n-2r)} = 0_{n\times n}$ are interpreted as absent. If $r = n/2$, then all blocks except the four corner blocks are interpreted as absent.

PROOF. Let $N = TU^*$ be a full-rank factorization of N. The matrices T and U are then both of order $n\times r$ and of rank r. Moreover, since $N^2 = 0$, clearly $U^*T = 0_{r\times r}$. Consequently, $(T,\ U)$ is of order $n \times 2r$ and has rank $2r$. Needless to say, necessarily $r \leq n/2$. Moreover, if $n - 2r \geq 1$, then there exists a $n \times (n - 2r)$ matrix V of rank $n - 2r$ such that $U^*V = 0$ and $T^*V = 0$. Otherwise, i.e. when $n = 2r$, we interpret V as absent. Consequently, the nonsingular inverse S^{-1} of $S := (T(T^*T)^{-1},\ V,\ U)^*$ does exist and is given by $S^{-1} := (T,\ V(V^*V)^{-1},\ U(U^*U)^{-1})$. Since $SNS^{-1} = \tilde{N}$, $N \sim \tilde{N}$ and so our proof is complete. □

LEMMA 4. *For $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$, let W_1 and W_2 be defined as in Theorem 1. Let $\operatorname{rank}(A) = r_1$ and let $\operatorname{rank}(B) = r_2$. If A and B are nilpotent of index ≤ 2, i.e., if $A^2 = 0$ and $B^2 = 0$, then there exist nonsingular matrices S_1 and S_2 such that*

$$S_1AS_1^{-1} = \tilde{A} := \begin{pmatrix} 0 & 0 & I_{r_1} \\ 0 & 0 & 0 \\ 0_{r_1\times r_1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_2BS_2^{-1} = \tilde{B} := \begin{pmatrix} 0 & 0 & I_{r_2} \\ 0 & 0 & 0 \\ 0_{r_2\times r_2} & 0 & 0 \end{pmatrix} \tag{13}$$

i.e., $A \sim \tilde{A}$ and $B \sim \tilde{B}$. Moreover, $W_1 \sim W_2$ if and only if

$$\tilde{W}_1 := \begin{pmatrix} \tilde{A} & \tilde{C} \\ 0 & \tilde{B} \end{pmatrix} \sim \tilde{W}_2 := \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix}, \tag{14}$$

where $\tilde{C} := S_1 C S_2^{-1}$ and where $\tilde{A}$ and $\tilde{B}$ are as in (13). *In* (13) *the orders of all those zero matrices, which are denoted by* 0, *are clear from the context and are therefore not explicitly given. Needless to mention, as in Lemma 3, there are situations in which we interpret some of the blocks in* (13) *as absent.*

PROOF. Lemma 3 tells us that there exist nonsingular matrices S_1 and S_2 such that $S_1 A S_1^{-1} = \tilde{A}$ and $S_2 B S_2^{-1} = \tilde{B}$, i.e., we have (13). For the nonsingular block-diagonal matrix $S := \mathrm{diag}(S_1, S_2)$ it is easy to check that $SW_1S^{-1} = \tilde{W}_1$ and $SW_2S^{-1} = \tilde{W}_2$. Hence, as claimed, $W_1 \sim W_2 \Leftrightarrow \tilde{W}_1 \sim \tilde{W}_2$. □

LEMMA 5. [Cf. Theorem 4.4.22 in Horn & Johnson (1991, p. 279).] *Let W_1 and W_2 be block partitioned matrices as in Theorem 1, where, however, the matrices A and B need not necessarily be nilpotent. Then,*

$$W_1 \sim W_2 \ \ \Leftrightarrow \ \exists\, X : \ AX - XB = C.$$

In which case, $RW_1R^{-1} = W_2$ holds true for

$$R := \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

We are now able to give a first proof of Theorem 1 that allows some deep insights into Problem 33-2 and the results claimed in Theorem 1. An alternative extremely succinct proof of just the equivalence (a) ⇔ (b) will be given later.

PROOF OF THEOREM 1. Let A and B be such that $A^2 = 0$ and $B^2 = 0$. Clearly, if $W_1 \sim W_2$, then $\mathrm{rank}(W_1) = \mathrm{rank}(W_2)$. Moreover, since $W_2^2 = 0$, necessarily $W_1^2 = 0$ or, equivalently, $AC + CB = 0$. Hence, (a) ⇒ (b). In view of Lemma 1, (b) ⇔ (c) ⇔ (d). Moreover, according to Lemma 3, (e) ⇔ (a). So, (d) ⇒ (e) remains to be shown. For that purpose, we first note that, according to Lemma 4, $W_1 \sim W_2 \Leftrightarrow \tilde{W}_1 \sim \tilde{W}_2$, where $\tilde{W}_1$ and $\tilde{W}_2$ are as in (14). As in Lemma 4, let S_1 and S_2 be such that $S_1 A S_1^{-1} = \tilde{A}$ and $S_2 B S_2^{-1} = \tilde{B}$. Since $\tilde{A}\tilde{C} + \tilde{C}\tilde{B} = S_1AS_1^{-1}S_1CS_2^{-1} + S_1CS_2^{-1}S_2BS_2^{-1} = S_1(AC + CB)S_2^{-1}$, trivially $AC + CB = 0 \Leftrightarrow \tilde{A}\tilde{C} + \tilde{C}\tilde{B} = 0$. Needless to mention, $\mathrm{rank}(A) = \mathrm{rank}(\tilde{A})$, $\mathrm{rank}(B) = \mathrm{rank}(\tilde{B})$ and, for $i = 1, 2$, $\mathrm{rank}(W_i) = \mathrm{rank}(\tilde{W}_i)$. In view of these observations, it is now clear that it suffices to show that the conditions $\tilde{A}\tilde{C} + \tilde{C}\tilde{B} = 0$ and $\mathrm{rank}(\tilde{W}_1) = \mathrm{rank}(\tilde{W}_2)$ imply $\tilde{W}_1 \sim \tilde{W}_2$ or, by virtue of Lemma 5, equivalently, the existence of a matrix $\tilde{X}$ satisfying $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$. In what follows, let

$$\tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{pmatrix} \text{ and } \tilde{X} = \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} \\ \tilde{X}_{31} & \tilde{X}_{32} & \tilde{X}_{33} \end{pmatrix} \tag{15}$$

be partitioned into blocks, where the rows of $\tilde{C}$ and $\tilde{X}$ are partitioned in the same way as the columns of $\tilde{A}$ in (13) and where the columns of $\tilde{C}$ and $\tilde{X}$ are partitioned in the same way as the rows of $\tilde{B}$ in (13). Since

$$\tilde{A}\tilde{C} + \tilde{C}\tilde{B} = \begin{pmatrix} \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} + \tilde{C}_{11} \\ 0 & 0 & \tilde{C}_{21} \\ 0 & 0 & \tilde{C}_{31} \end{pmatrix}, \tag{16}$$

clearly

$$\tilde{A}\tilde{C} + \tilde{C}\tilde{B} = 0 \ \ \Leftrightarrow \ \ \tilde{C}_{21} = 0, \tilde{C}_{31} = 0, \tilde{C}_{32} = 0, \tilde{C}_{33} = -\tilde{C}_{11} \ \ \Leftrightarrow \ \ \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & \tilde{C}_{22} & \tilde{C}_{23} \\ 0 & 0 & -\tilde{C}_{11} \end{pmatrix}. \tag{17}$$

According to Lemma 2, $\mathrm{rank}(\tilde{W}_1) = \mathrm{rank}(\tilde{W}_2)$ if and only if

$$\tilde{C} = \tilde{A}\tilde{A}^-\tilde{C} + \tilde{C}\tilde{B}^-\tilde{B} - \tilde{A}\tilde{A}^-\tilde{C}\tilde{B}^-\tilde{B}, \tag{18}$$

where M^- denotes any generalized inverse of M, i.e. any matrix M^- satisfying $MM^-M = M$. It is easy to check that

$$\begin{pmatrix} 0 & 0 & 0_{r_1 \times r_1} \\ 0 & 0 & 0 \\ I_{r_1} & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0_{r_2 \times r_2} \\ 0 & 0 & 0 \\ I_{r_2} & 0 & 0 \end{pmatrix}$$

are generalized inverses of $\tilde{A}$ and $\tilde{B}$, respectively. By means of these generalized inverses, equation (18) can be rewritten in block-partitioned from as follows

$$\begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{pmatrix} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & \tilde{C}_{33} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \tilde{C}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\operatorname{rank}(\tilde{W}_1) = \operatorname{rank}(\tilde{W}_2) \;\Leftrightarrow\; \tilde{C}_{21} = 0,\ \tilde{C}_{22} = 0,\ \tilde{C}_{31} = 0,\ \tilde{C}_{32} = 0 \;\Leftrightarrow\; \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & \tilde{C}_{33} \end{pmatrix}. \tag{19}$$

Combining (17) with (19) results in

$$\tilde{A}\tilde{C} + \tilde{C}\tilde{B} = 0,\ \tilde{C} = \tilde{A}\tilde{A}^-\tilde{C} + \tilde{C}\tilde{B}^-\tilde{B} - \tilde{A}\tilde{A}^-\tilde{C}\tilde{B}^-\tilde{B} \;\Leftrightarrow\; \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & -\tilde{C}_{11} \end{pmatrix}. \tag{20}$$

We complete our proof by showing that if $\tilde{C}$ is of the form (20), then there exists a solution $\tilde{X}$ to the matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$. Actually, we even show that the matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ is consistent if and only if $\tilde{C}$ is as in (20). For doing so, let $\tilde{X}$ and $\tilde{C}$ be partitioned as in (15). Then,

$$\exists\, \tilde{X} :\ \tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C} \;\Leftrightarrow\; \begin{pmatrix} \tilde{X}_{31} & \tilde{X}_{32} & \tilde{X}_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \tilde{X}_{11} \\ 0 & 0 & \tilde{X}_{21} \\ 0 & 0 & \tilde{X}_{31} \end{pmatrix} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{pmatrix}. \tag{21}$$

Consequently, as just claimed,

$$\exists\, \tilde{X} :\ \tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C} \;\Leftrightarrow\; \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & -\tilde{C}_{11} \end{pmatrix}.$$

From (21) it is further clear that if $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ is consistent, then its general solution is given by

$$\tilde{X} = \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} \\ -\tilde{C}_{23} & \tilde{X}_{22} & \tilde{X}_{23} \\ \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} + \tilde{X}_{11} \end{pmatrix}, \tag{22}$$

where $\tilde{X}_{11}$, $\tilde{X}_{12}$, $\tilde{X}_{13}$, $\tilde{X}_{22}$ and $\tilde{X}_{14}$, all of suitable orders, can be chosen arbitrarily. In view of $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C} \Leftrightarrow A(S_1^{-1}\tilde{X}S_2) - (S_1^{-1}\tilde{X}S_2)B = C$, it is clear that the general solution to $AX - XB = C$ is then given by $X = S_1^{-1}\tilde{X}S_2$, with $\tilde{X}$ as in (22). □

Our solution is concluded with four remarks. First of all, we emphasize that, as our preceding proof has shown, neither condition $AC + CB = 0$ nor condition $\operatorname{rank}(W_1) = \operatorname{rank}(W_2)$ is redundant in characterizing $W_1 \sim W_2$ under the matrix set-up of Theorem 1. Secondly, we mention that Lemma 3 even enables us to present an alternative extremely succinct and more direct proof of (b) ⇒ (a) in Theorem 1. For observe that the conditions under (b) imply that $W_1^2 = 0$, $W_2^2 = 0$ and $\operatorname{rank}(W_1) = \operatorname{rank}(W_2)$. Lemma 3 thus tells us that W_1 and W_2 are both similar to the same matrix $\tilde{W}$, a matrix structured as $\tilde{N}$ in (12), but with $r := \operatorname{rank}(W_1)$ and with $m + n$ rows and columns. Therefore, because the similarity relation $\sim$ is transitive, also $W_1 \sim W_2$. Thirdly, by a similar reasoning as before it is seen that any given $m \times m$ matrices, all supposed to be nilpotent of index ≤ 2, are similar to each other if and only if their ranks coincide. Finally, we show how the following interesting result can be singled out from Theorem 1.

COROLLARY 6. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be such that $A^2 = 0$ and $B^2 = 0$. Consider the Kronecker sum $M_+ := (I_n \otimes A) + (B \otimes I_m)$ and the Kronecker difference $M_- := (I_n \otimes A) - (B \otimes I_m)$ of the nilpotent matrices A and B, with '$\otimes$' indicating the Kronecker product of matrices. Moreover, let $P := (I - BB^-) \otimes (I - AA^-)$. Then*

$$\mathcal{R}(M_-) = \mathcal{N}(M_+) \cap \mathcal{N}(P),$$

irrespective of the choices of A^- and B^-.

PROOF. We recall that if D, E and F are matrices, for which DEF is defined, then $\mathrm{vec}(DEF) = (F' \otimes D)\mathrm{vec}(E)$, with $(\cdot)'$ and $\mathrm{vec}(\cdot)$ indicating the transpose and the vectorization, respectively, of $(\cdot)$. We further recall that B^- is a generalized inverse of B if and only if $(B^-)'$ is a generalized inverse of B'. Hence, (i) $\exists X : C = AX - XB' \Leftrightarrow \mathrm{vec}(C) \in \mathcal{R}(M_-)$, (ii) $AC + CB' = 0 \Leftrightarrow \mathrm{vec}(C) \in \mathcal{N}(M_+)$, and (iii) $C = AA^-C + C(B')^-B' - AA^-C(B')^-B' \Leftrightarrow \mathrm{vec}(C) \in \mathcal{N}(P)$. Theorem 1, with the matrix B being replaced by B', now immediately tells us that $\mathcal{R}(M_-) = \mathcal{N}(M_+) \cap \mathcal{N}(P)$. □

References

R. A. Horn & Ch. R. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press, Cambridge.

H. J. Werner (2003). The minimal rank of a block matrix with generalized inverses. Solution 29-11.2 *IMAGE: The Bulletin of the International Linear Algebra Society* **31** (October 2003), 26–29.

A Solution to Problem 33-2 was also received from Johanns de Andrade Bezerra who mentions that under the 'if conditions' of the problem both block-partitioned matrices are similar to the same Jordan form and therefore also similar to each other.

Problem 33-3: Two Characterizations of an EP Matrix

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Show that the following statements are equivalent:

(a) a complex square matrix A is EP, i.e., $\mathrm{range}(A) = \mathrm{range}(A^*)$,

(b) $\mathrm{range}(\, A - A^\dagger \,) = \mathrm{range}(\, A - A^3 \,)$,

(c) $\mathrm{range}(\, A + A^\dagger \,) = \mathrm{range}(\, A + A^3 \,)$,

where A^* and $A^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse of A, respectively.

Solution 33-3.1 by the Proposer Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

It is well-known that A is EP if and only if $AA^\dagger = A^\dagger A$. In such case, $(\, A \pm A^\dagger \,)A^2 = A^3 \pm A$ and $(\, A^3 \pm A \,)A^\dagger A^\dagger = A \pm A^\dagger$. These two equalities imply that $\mathrm{range}(\, A \pm A^\dagger \,) = \mathrm{range}(\, A \pm A^3 \,)$. It is easy to find by the rank formula

$$\mathrm{rank}(D - CA^\dagger B) = \mathrm{rank}\begin{pmatrix} A^*AA^* & A^*B \\ CA^* & D \end{pmatrix} - \mathrm{rank}(A)$$

that

$$\mathrm{rank}(A \pm A^\dagger \mid A \pm A^3) = \mathrm{rank}(A \mid A^*) + \mathrm{rank}(A^* \pm A^*A^2) - \mathrm{rank}(A). \tag{23}$$

Hence, if $\mathrm{range}(A \pm A^\dagger) = \mathrm{range}(A \pm A^3)$, then

$$\mathrm{rank}(A \pm A^\dagger \mid A \pm A^3) = \mathrm{rank}(A \pm A^3). \tag{24}$$

The combination of (23) and (24) gives

$$\mathrm{rank}(A \mid A^*) - \mathrm{rank}(A) + \mathrm{rank}(A^* \pm A^*A^2) - \mathrm{rank}(A \pm A^3) = 0. \tag{25}$$

From $AA^\dagger(A^\dagger)^*(A^* \pm A^*A^2)A = A \pm A^3$, one can get $\mathrm{rank}(A^* \pm A^*A^2) \geqslant \mathrm{rank}(A \pm A^3)$. Also note that $\mathrm{rank}(A \mid A^*) \geqslant \mathrm{rank}(A)$. Thus (25) implies that $\mathrm{rank}(A \mid A^*) = \mathrm{rank}(A)$, which is equivalent to $\mathrm{range}(A) = \mathrm{range}(A^*)$.

Solution 33-3.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We recall that $A \in \mathbb{C}^{n\times n}$ is EP if and only if $AA^\dagger = A^\dagger A$. Now, let A be EP. Then $A^\dagger A^2 = A$, $A^3(A^\dagger)^2 = A$ and $A(A^\dagger)^2 = A^\dagger$. Therefore, $\mathrm{range}(A^3 - A) = \mathrm{range}((A - A^\dagger)A^2) \subseteq \mathrm{range}(A - A^\dagger) = \mathrm{range}((A^3 - A)(A^\dagger)^2) \subseteq \mathrm{range}(A^3 - A)$, so $\mathrm{range}(A^3 - A) = \mathrm{range}(A - A^\dagger)$. So, we have (a) $\Rightarrow$ (b). To prove the converse, let $\mathrm{range}(A - A^\dagger) = \mathrm{range}(A^3 - A)$. Then for each $x \in \mathbb{C}^n$ there exists a $y \in \mathbb{C}^n$ such that $(A - A^\dagger)x = (A^3 - A)y$ or, equivalently, $A^\dagger x = A[x - (A^2 - I)y]$. Consequently, $\mathrm{range}(A^\dagger) \subseteq \mathrm{range}(A)$. We recall that $\mathrm{range}(A^\dagger) = \mathrm{range}(A^*)$ and that the rank of a matrix always coincides with the rank of its conjugate transpose. Combining these observations now results in $\mathrm{range}(A) = \mathrm{range}(A^*)$. This completes the proof of (a) $\Leftrightarrow$ (b). The proof of (a) $\Leftrightarrow$ (c) follows on similar lines.

A Solution to Problem 33-3 was also received from Götz Trenkler.

Problem 33-4: An Euclidean Norm Property in $\mathbb{R}^3$

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $a = (a_1, a_2, a_3)'$ and $b = (b_1, b_2, b_3)'$ be vectors from $\mathbb{R}^3$ such that $a'b = 0$. Show that

$$\|a + b\| \leq \max\{|a_3| + |a_2| + |b_1|, |a_3| + |a_1| + |b_2|, |a_2| + |a_1| + |b_3|, |b_1| + |b_2| + |b_3|\},$$

where $\| \cdot \|$ denotes the usual Euclidean norm.

Solution 33-4.1 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Consider the matrix

$$A = \begin{pmatrix} 0 & -a_3 & a_2 & b_1 \\ a_3 & 0 & -a_1 & b_2 \\ -a_2 & a_1 & 0 & b_3 \\ -b_1 & -b_2 & -b_3 & 0 \end{pmatrix}.$$

Its is easily seen that the eigenvalues of A are 0, 0 and $\pm\sqrt{-a'a - b'b} = \pm i\sqrt{a'a + b'b}$. By Gerschgorin's Theorem [see Meyer (2000, p. 498)], it follows that all eigenvalues of A are contained in the union of the four circles of the complex plane with common midpoint 0 and radii r_i, $i = 1, 2, 3, 4$, where $r_1 = |a_3| + |a_2| + |b_1|$, $r_2 = |a_3| + |a_1| + |b_2|$, $r_3 = |a_2| + |a_1| + |b_3|$, and $r_4 = |b_1| + |b_2| + |b_3|$. Hence $| \pm i\sqrt{a'a + b'b}| \leq \max\{r_i :\ i = 1, 2, 3, 4\}$, and by $(a + b)'(a + b) = a'a + b'b$ the assertion follows.

REMARK: Observe that our upper bound can be sharper than $\|a\| + \|b\|$. To illustrate this, consider $a_1 = (1, -1, 0)'$ and $b_1 = (1, 1, 1)'$. Then $a_1'a_1 = 2$, $b_1'b_1 = 3$, $a_1'b_1 = 0$, $\sqrt{a_1'a_1 + b_1'b_1} = \sqrt{5} \approx 2.2361$, $\|a_1\| + \|b_1\| = \sqrt{2} + \sqrt{3} \approx 3.1463$ which is larger than our upper bound 3. The upper bound is attained in the following case: $a_2 = (1, 0, -1), b_2 = (1, 0, 1)$. Then $\|a_2 + b_2\| = 2$, which is also our upper bound. Note that in general the alternative upper bound $\|a\| + \|b\|$ is attained if and only if a and b are linearly dependent. This follows from the Cauchy-Schwarz inequality.

Reference

C. D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia.

Problem 33-5: Factorization of a Projector

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix with possibly complex entries. Show that P can be written as $P = RS$, where R is positive definite and S is nonnegative definite.

Solution 33-5.1 by Dario FASINO, *University of Udine, Udine, Italy:* fasino@dimi.uniud.it

Any idempotent matrix P is diagonalizable, i.e., $P = MDM^{-1}$ with D a $(0, 1)$-diagonal matrix. Let $R = MM^*$ and $S = (M^{-1})^*DM^{-1}$. Then $P = RS$, R is positive definite, and S is nonnegative definite, as requested.

Solution 33-5.2 by Chi-Kwong LI, *The College of William & Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu

Problem 33-5 can be solved by a canonical form of idempotent $n \times n$ matrices under unitary similarity [see Djoković (1991)], namely, if $P^2 = P$ then there is a unitary matrix U such that

$$U^*PU = I_p \oplus 0_q \oplus P_1 \oplus \cdots \oplus P_r,$$

where $p + q + 2r = n$, and

$$P_j = \begin{pmatrix} 1 & 0 \\ q_j & 0 \end{pmatrix} \quad \text{with } q_j > 0, \quad j = 1, \ldots, r.$$

Now, $P_j = R_jS_j$ with

$$R_j = \begin{pmatrix} 1 & q_j \\ q_j & 2q_j^2 \end{pmatrix} \quad \text{and} \quad S_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad j = 1, \ldots, r.$$

Let $R = U(I_{p+q} \oplus R_1 \oplus \cdots \oplus R_r)U^*$ and $S = U(I_p \oplus 0_q \oplus S_1 \oplus \cdots \oplus S_r)U^*$. We get the required factorization of P.

Reference
D. Ž. Djoković (1991). Unitary similarity of projectors. *Aequationes Mathematicae* **42**, no. 2-3, 220–224.

Solution 33-5.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If $P \in \mathbb{C}^{n\times n}$ and $P \neq 0,\, I$, then there are positive integers $r,\, s \in \{1, \ldots, n\}$ and matrices $U \in \mathbb{C}^{n\times r}$ and $V \in \mathbb{C}^{n\times s}$ such that $r + s = n$, $PU = U$, $PV = 0$, $U^*U = I_r$, $V^*V = I_s$, and

$$P = (\,U \quad V\,) \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (\,U \quad V\,)^{-1}.$$

In fact, it is straightforward to verify that

$$(\,U \quad V\,)^{-1} = \begin{pmatrix} \widehat{U} \\ \widehat{V} \end{pmatrix}, \quad \text{with} \quad \widehat{U} = U^*P \quad \text{and} \quad \widehat{V} = V^*(I - P).$$

Hence,

$$P = (\,U \quad V\,) \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{V} \end{pmatrix} = (\,U \quad V\,) \begin{pmatrix} U^* \\ V^* \end{pmatrix} (\,\widehat{U}^* \quad \widehat{V}^*\,) \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{V} \end{pmatrix} = RS, \tag{26}$$

with $R = UU^* + VV^*$ and $S = \widehat{U}^*\widehat{U} = P^*UU^*P$. If $z \in \mathbb{C}^n$, then $z^*Rz = |U^*z|^2 + |V^*z|^2$ and $z^*Sz = |U^*Pz|^2$, so R is positive definite and S is nonnegative definite. From (26), $P = U\widehat{U} = U(U^*P)$ is a full rank factorization of P; hence, $P^\dagger = P^*U(U^*PP^*U)^{-1}U^*$.

Solution 33-5.4 by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de
and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Our solution offers additional insights into the theory of complex idempotent matrices. For a complex matrix $A \in \mathbb{C}^{m\times n}$, let A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the range (column space), and the null space, respectively, of A. In what follows, we consider some subsets of $\mathbb{C}^{n\times n}$. For convenience, let $\mathcal{N}^n$, $\mathcal{H}^n$, $\mathcal{P}^n_>$, and $\mathcal{P}^n_\geq$ be the set of all square, the set of all nonsingular, the set of all Hermitian, the set of all positive definite and Hermitian, and the set of all nonnegative definite and Hermitian $n \times n$ matrices, respectively.

Since the results in our first theorem are straightforward, their easy proofs are left to the interesting reader.

THEOREM 1. *Let $(R, S) \in \mathcal{N}^n \times \mathbb{C}^{n\times n}$. The following conditions are then equivalent:*

(a) *$R^{-1}S$ is idempotent, i.e., $R^{-1}SR^{-1}S = R^{-1}S$.*
(b) *R^{-1} is a generalized inverse of S, i.e., $SR^{-1}S = S$.*
(c) *$(R - S)R^{-1}S = 0$ or, equivalently, $SR^{-1}(R - S) = 0$.*
(d) *$R^{-1}(R - S)$ is idempotent, i.e., $R^{-1}(R - S)R^{-1}(R - S) = R^{-1}(R - S)$.*
(e) *S and $R - S$ is a pair of bi-complementary matrices, i.e., $\mathcal{R}(S) \oplus \mathcal{R}(R - S) = \mathbb{C}^n$, where $\oplus$ indicates a direct sum. For more details concerning bi-complementarity, see Werner (1986).*

THEOREM 2. *Let $P \in \mathbb{C}^{n\times n}$ be idempotent, and let $R \in \mathcal{P}^n_>$. The following conditions are then equivalent:*

(a) *$RP \in \mathcal{P}^n_\geq$.*
(b) *$RP \in \mathcal{H}^n$.*
(c) *$RP = P^*RP$.*
(d) *$R(I_n - P) \in \mathcal{P}^n_\geq$.*
(e) *$R(I_n - P) \in \mathcal{H}^n$.*
(f) *$R(I_n - P) = (I_n - P)^*R(I_n - P)$.*

PROOF. <u>(a) ⇔ (b) ⇔ (c)</u>: Trivially, (a) ⇒ (b), and (c) ⇒ (a). Now, let RP be Hermitian, i.e, let $RP = P^*R$. Then, in view of $P^2 = P$, $RP = RPP = P^*RP \in \mathcal{P}^n_\geq$. So, we also have (b) ⇒ (c).
<u>(b) ⇔ (d) ⇔ (e) ⇔ (f)</u>: Trivially, (b) ⇔ (e), (d) ⇒ (e), and (f) ⇒ (d). The proof of (e) ⇒ (f) follows on similar lines as (b) ⇒ (c) and is therefore left to the reader. □

THEOREM 3. *Let $P \in \mathbb{C}^{n\times n}$ be idempotent, and let $(R,S) \in \mathcal{P}^n_> \times \mathcal{P}^n_\geq$ be such that $P = R^{-1}S$ or, equivalently, $RP = S$. Then*

$$R = P^*RP + (I-P)^*R(I-P) \quad \text{and} \quad S = P^*RP.$$

PROOF. Because $S = RP \in \mathcal{P}^n_\geq$, according to Theorem 2(c)&(f), clearly $S = P^*RP$ and $R = RP + R(I-P) = P^*RP + (I-P)^*R(I-P)$. □

THEOREM 4. *Let $P \in \mathbb{C}^{n\times n}$ be idempotent, and let $W \in \mathcal{P}^n_>$. For $R := P^*WP + (I-P)^*W(I-P)$ and $S := P^*WP$ we then have the following:*

(a) $R \in \mathcal{P}^n_>$, $S = P^*RP \in \mathcal{P}^n_\geq$ *and* $R - S = (I-P)^*R(I-P) \in \mathcal{P}^n_\geq$.

(b) $RP = S$ *or, equivalently,* $P = R^{-1}S$.

PROOF. (a): In view of $P^2 = P$, clearly $P^*RP = P^*SP = S$ and $(I-P)^*R(I-P) = (I-P)^*(R-S)(I-P) = R-S$. Since $P^*WP \in \mathcal{P}^n_\geq$ and $(I-P)^*W(I-P) \in \mathcal{P}^n_\geq$, trivially $R \in \mathcal{P}^n_\geq$. Evidently, $\mathcal{R}(P^*WP) = \mathcal{R}(P^*)$, $\mathcal{R}((I-P)^*W(I-P)) = \mathcal{R}((I-P)^*)$ and $\mathbb{C}^n = \mathcal{R}(P^*) \oplus \mathcal{R}(I-P^*)$. Therefore, needless to say, $R \in \mathcal{P}^n_>$.

(b): From the definitions of R and S we directly get $RP = S$ or, equivalently, $P = R^{-1}S$. □

For a given idempotent matrix $P \in \mathbb{C}^{n\times n}$, we now immediately obtain, just by combining our previous theorems, the following complete representation of the set of all those pairs $(R,S) \in \mathcal{P}^n_> \times \mathcal{P}^n_\geq$ for which $P = R^{-1}S$.

COROLLARY 5. *Let P be an idempotent complex $n\times n$ matrix. For $(R,S) \in \mathcal{P}^n_> \times \mathcal{P}^n_\geq$ we then have $R^{-1}S = P$ if and only if*

$$R = P^*WP + (I-P)^*W(I-P) \quad \text{and} \quad S = P^*WP$$

for some matrix $W \in \mathcal{P}^n_>$. In which case,

$$S = P^*RP \in \mathcal{P}^n_\geq,\ R - S = (I-P)^*R(I-P) \in \mathcal{P}^n_\geq,\ \text{and } S \text{ is bi-complementary to } R-S.$$

Reference

H. J. Werner (1986). Generalized inversion and weak bi-complementarity. *Linear and Multilinear Algebra* **19**, 357–372.

Solutions to Problem 33-5 were also received from Johanns de Andrade Bezarra, from Alicja Smoktunowicz and Iwona Wróbel, and from Pei Yuan Wu. Smoktunowicz & Wróbel used the same canonical form as Chi-Kwong Li in his Solution 33-5.2, but with a different reference.

Problem 33-6: Projectors and Similarity

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P be an idempotent matrix with possibly complex entries. Show that P is Hermitian if and only if P and P^+ are similar, where P^+ denotes the Moore-Penrose inverse of P.

Solution 33-6.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl

Let P be an idempotent matrix and let P^+ denote its Moore-Penrose inverse. It is well known that if P is Hermitian, then $P = P^+$ [see e.g., Trenkler (1994, Theorem 12, part (i))], and thus P and P^+ are trivially similar. Conversely, if there exists a nonsingular matrix Q such that $P^+ = Q^{-1}PQ$, then P^+ is idempotent. Hence, in view of part (v) of Theorem 12 in Trenkler (1994), it follows that P is Hermitian.

Reference

G. Trenkler (1994). Characterizations of oblique and orthogonal projectors. In: *Proceedings of the International Conference on Linear Statistical Inference LINSTAT'93* (T. Caliński & R. Kala, eds.), Kluwer, Dordrecht, pp. 255-270.

Solution 33-6.2 by Alicja SMOKTUNOWICZ and Iwona WRÓBEL, *Warsaw University of Technology, Warsaw, Poland:* smok@mini.pw.edu.pl; wrubelki@wp.pl

It is known [see Fuji & Furuta (1980)] that if $P \in \mathbb{C}^{n\times n}$ is idempotent, then there exists a unitary matrix U such that $P = UTU^*$, where

$$T = \begin{pmatrix} I_r & B \\ 0 & 0 \end{pmatrix}, \tag{27}$$

with $r = \operatorname{rank}(P)$. Here I_r denotes an $r \times r$ identity matrix. There is no loss of generality in assuming that $1 \leq r < n$. It is sufficient to prove the result only for T. We show that T is Hermitian if and only if T and T^+ are similar.

First let us recall two useful facts. The Moore-Penrose inverse of T is given by the formula [cf. Aleksiejczyk & Smoktunowicz (2000)]

$$T^+ = \begin{pmatrix} (I_r + BB^*)^{-1} & 0 \\ B^*(I_r + BB^*)^{-1} & 0 \end{pmatrix}.$$

To check it, one can easily verify the Moore-Penrose axioms. Notice also that as T is an idempotent and $1 \leq r = \operatorname{rank}(T) < n$, the spectra of T and T^+ are

$$\sigma(T) = \{0, 1\} \quad \text{and} \quad \sigma(T^+) = \{0, \frac{1}{1 + \sigma_i^2(B)}\}, \tag{28}$$

where $\sigma_i^2(B)$ are the singular values of B. The last equality can be easily verified with help of the characteristic polynomial of T^+ and the spectral mapping theorem.

To establish the desired equivalence, suppose first that T is Hermitian. Then $B = 0$ in (27) and

$$T^+ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = T.$$

Now let T be similar to T^+. Then T and T^+ have the same spectrum, i.e. $\sigma(T) = \sigma(T^+)$. This and (28) imply that $B = 0$. Then

$$T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = T^*.$$

References

J. I. Fuji & T. Furuta (1980). Holub's factorization and normal approximants of idempotent operators. *Mathematica Japonica* **25/1**, 143–145.

M. Aleksiejczyk & A. Smoktunowicz (2000). On properties of quadratic matrices. *Mathematica Pannonica* **11/2**, 239–248.

Solution 33-6.3 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Necessity is trivial, since $P = P^*$ implies $P = P^+$. Assume now $P = RP^+R^{-1}$ for some nonsingular R. Then $P^2 = RP^+R^{-1}RP^+R^{-1}$ and thus $P = RP^+P^+R^{-1}$ which implies $P^+ = P^+P^+$. Consulting complement 7(ii), Rao and Mitra (1971, p. 68), it follows that $\mathcal{R}(PP^*P^*) \subset \mathcal{R}(P^*)$ and $\mathcal{R}(P^*PP) \subset \mathcal{R}(P)$, where $\mathcal{R}(\cdot)$ denotes the column space of a matrix. Since both P and P^* are idempotents, we obtain $P^*PP^* = PP^*$ and $PP^*P = P^*P$. Then by the well-known cancellation rule we get $P = P^*P$, i.e. P is Hermitian.

Reference

C. R. Rao & S. K. Mitra (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

Solution 33-6.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

In what follows, let P be idempotent. If $P = P^*$, then $P^+ = P$, so that necessity is trivial. Conversely, let $P^+ = NPN^{-1}$ for some nonsingular matrix N. Then $(P^+)^2 = NP^2N^{-1} = NPN^{-1} = P^+$, and so sufficiency follows just by citing Corollary 2(b) in Werner (2003): *If P is idempotent, then P^+ is idempotent if and only if P is a partial isometry, i.e., if and only if $P = PP^*P$, in which case $P^2 = P = P^* = P^+$.*

Reference

H. J. Werner (2003). A condition for an idempotent matrix to be Hermitian. Solution 30.7-4. *IMAGE: The Bulletin of the International Linear Algebra Society* **31** (October 2003), 42–43.

A Solution to Problem 33-6 was also received from Chi-Kwong Li. He used the same canonical form as Smoktunowicz & Wróbel, but with a different reference.

Problem 33-7: Property of the Cross Product

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let a, b, c, and d be vectors from $\mathbb{R}^3$. For the matrix

$$A = (c \times d)(b \times a)' + (a \times b)(c \times d)',$$

find the unique vector e such that $Ax = e \times x$ for all $x \in \mathbb{R}^3$. Here, "$\times$" denotes the usual cross product in $\mathbb{R}^3$.

Solution 33-7.1 by Andres SÁEZ-SCHWEDT, *Universidad de León, León, Spain:* demass@unileon.es

Set $M = (c \times d)(b \times a)'$, so that $A = M - M'$ is skew-symmetric and there exists certainly a unique e such that $Ax = e \times x$ for all x in $\mathbb{R}^3$: if

$$A = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix},$$

then the associated vector is $e = (e_1, e_2, e_3)'$.

Suppose $a \times b = -b \times a = (a_1, a_2, a_3)', c \times d = (b_1, b_2, b_3)'$ and let $\{v_1, v_2, v_3\}$ be the standard basis of $\mathbb{R}^3$. Now e_1 is the element $(3, 2)$ of A, hence $e_1 = v_3' A v_2 = v_3'(c \times d)(b \times a)' v_2 + v_3'(a \times b)(c \times d)' v_2 = b_3(-a_2) + a_3 b_2$, and similarly $e_2 = v_1' A v_3 = -b_1 a_3 + b_3 a_1$ and $e_3 = v_2' A v_1 = b_1 a_2 - b_2 a_1$, which means that $e = (c \times d) \times (a \times b)$.

Solution 33-7.2 by Diego SÁEZ-SCHWEDT, *Universidad ORT, Montevideo, Uruguay:* saez@ort.edu.uy

First of all, as $b \times a = -a \times b$ we have $A = (a \times b)(c \times d)' - (c \times d)(a \times b)'$, hence for all $x \in \mathbb{R}^3$ one has

$$Ax = (a \times b)(c \times d)' x - (c \times d)(a \times b)' x.$$

If $<,>$ denotes the scalar product and we define $u = c \times d, v = a \times b$, then $Ax = v < u, x > -u < v, x >$. Now, by the triple cross product identity $(u \times v) \times x =< u, x > v- < v, x > u$ one has that the equality $Ax = e \times x$ holds for $e = u \times v = (c \times d) \times (a \times b)$. The vector e is unique because A is skew-symmetric (it is equal to a matrix minus its transpose).

Solution 33-7.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

We can write $A = \beta\alpha' - \alpha\beta'$, where $\alpha = b \times a = (\, \alpha_1 \quad \alpha_2 \quad \alpha_3 \,)'$ and $\beta = c \times d = (\, \beta_1 \quad \beta_2 \quad \beta_3 \,)'$. Therefore,

$$A = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix},$$

with $e_1 = \alpha_2\beta_3 - \alpha_3\beta_2$, $e_2 = \alpha_3\beta_1 - \alpha_1\beta_3$ and $e_3 = \alpha_1\beta_2 - \alpha_2\beta_1$. Let $e = (\, e_1 \quad e_2 \quad e_3 \,)' = \alpha \times \beta = (b \times a) \times (c \times d)$. Then $Ax = e \times x$ for all $x \in \mathbb{R}^3$.

A solution to Problem 33-7 was also received from Götz Trenkler, the proposer of this problem.

Problem 33-8: Singular Value Decomposition of a Skew-Symmetric Real Matrix
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a 3×3 skew-symmetric matrix with real entries. Find a singular value decomposition of A, i.e., provide a representation $A = UDV'$, where U and V are orthogonal, and D is a diagonal matrix of singular values of A.

Solution 33-8.1 by João R. CARDOSO, *Instituto Superior de Engenharia de Coimbra, Coimbra, Portugal:* jocar@isec.pt

Let

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

be a 3×3 skew-symmetric real matrix and $\alpha := \sqrt{a^2 + b^2 + c^2}$. The spectrum of A is $\{0, \pm\alpha i\}$. We will assume throughout that $\alpha \neq 0$ (i.e. $A \neq 0$) and that $\beta := \sqrt{a^2 + b^2} \neq 0$. We note that if $\beta = 0$, the problem is trivial.

First, we find an orthogonal diagonalization of the real symmetric matrix $A'A = -A^2$. Using the eigenvectors of A and the Gram-Schmidt orthogonalization process, we have $A'A = V\,\tilde{D}\,V'$, where

$$V := \begin{pmatrix} a/\alpha & -b/\beta & -ac/(\alpha\beta) \\ b/\alpha & a/\beta & -bc/(\alpha\beta) \\ c/\alpha & 0 & \beta/\alpha \end{pmatrix}$$

is orthogonal and $\tilde{D} = \operatorname{diag}(0, \alpha^2, \alpha^2)$. From the theory of singular values, we know that the matrix A allows the following singular value decomposition: $A = UDV'$, where V is the matrix above, D is a diagonal matrix with the singular values of A and $U = [u_1\ u_2\ u_3]$ is an 3×3 orthogonal matrix whose columns satisfy

$$u_1 \in N(A'),\ u_2 = \frac{1}{\alpha}Av_2,\ u_3 = \frac{1}{\alpha}Av_3. \tag{29}$$

Here $N(A')$ denotes the null space of A' and v_i $(i = 1, 2, 3)$ is the i-th column of V. Using (29) we can easily find the orthogonal matrix U and write

$$A = UDV' = \begin{pmatrix} a/\alpha & -ac/(\alpha\beta) & b/\beta \\ b/\alpha & -bc/(\alpha\beta) & -a/\beta \\ c/\alpha & \beta/\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a/\alpha & b/\alpha & c/\alpha \\ -b/\beta & a/\beta & 0 \\ -ac/(\alpha\beta) & -bc/(\alpha\beta) & \beta/\alpha \end{pmatrix}.$$

Solution 33-8.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If $A \neq 0$ we can write $A = \sigma A_0$ with $\sigma > 0$ and

$$A_0 = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \text{ where } a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$$

is a unit vector. Let $b \in \mathbb{R}^3$ be a unit vector perpendicular to a, and let $c = a \times b$. Since $A_0 y = a \times y$ for all $y \in \mathbb{R}^3$, $A_0 a = 0$, $A_0 b = c$, and $A_0 c = -b$. Since $\{a, b, c\}$ is an orthonormal set, it is straightforward to verify that if

$$A_1 = \begin{pmatrix} b & c & a \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -c' \\ b' \\ a' \end{pmatrix},$$

then $A_1 a = 0 = A_0 a$, $A_1 b = c = A_0 b$, and $A_1 c = -b = A_0 c$. Since $\{a, b, c\}$ is a basis for $\mathbb{R}^3$, it follows that $A_0 = A_1$. Hence,

$$A = \sigma A_1 = \begin{pmatrix} b & c & a \end{pmatrix} \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -c' \\ b' \\ a' \end{pmatrix}.$$

Moreover, A has the spectral representation

$$A = \begin{pmatrix} u & v & a \end{pmatrix} \begin{pmatrix} \sigma i & 0 & 0 \\ 0 & -\sigma i & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u^* \\ v^* \\ a' \end{pmatrix},$$

with $u = (ib + c)/\sqrt{2}$ and $v = (b + ic)/\sqrt{2}$.

Solution 33-8.3 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $A \neq 0$ (the case $A = 0$ is trivial). Write

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},$$

so that A is uniquely determined by the vector $a = (\,a_1 \quad a_2 \quad a_3\,)' \in \mathbb{R}^3$. Note that $a \times x = Ax$ for all $x \in \mathbb{R}^3$, where "$\times$" denotes the vector cross product in $\mathbb{R}^3$.

Assume first $\|a\| = 1$. Then $A'A = AA' = I - aa' = Q_a$, which is the orthogonal projector on the two-dimensional subspace of all vectors orthogonal to a. The eigenvalues of Q_a are 1, 1, and 0, coinciding with the singular values of A. Choose vectors b and c of norm 1 satisfying $Ab = a \times b = c$ and $Ac = a \times c = -b$. Define the matrices V and U by their column representations: $V = (\,b \quad c \quad a\,)$ and $U = (\,c \quad -b \quad a\,)$. Then V and U are orthogonal matrices, i.e. $U'U = I = V'V$. Let $D = \mathrm{diag}(1,1,0)$ be the diagonal matrix of the singular values of A. Then $UD = (\,c \quad -b \quad 0\,) = AV$, and consequently $A = UDV' = cb' - bc'$, which is the desired singular value decomposition of A.

For the general case of arbitrary norm of a write $A = \|a\|B$, where $B = \frac{1}{\|a\|}A$. Then the skew-symmetric matrix B is characterized by the vector $b = a/\|a\|$. Thus our preceding derivations apply, and consequently A may be written as $A = UD_aV'$ with $D_a = \mathrm{diag}(\|a\|, \|a\|, 0)$, where U and V are taken from the representation of B.

A Solution to Problem 33-8 was also received from Dario Fasino.

IMAGE Problem Corner: More New Problems

Problem 34-6: The Schur Complement in an Orthogonal Projector

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Suppose that $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ is an orthogonal projector, that is, $M^2 = M = M^*$. Show that

(a) The Schur complement $D - B^*A^\dagger B$ of A in M satisfies the rank subtractivity condition

$$\mathrm{rank}(D - B^*A^\dagger B) = \mathrm{rank}(D) - \mathrm{rank}(B^*A^\dagger B),$$

where $A^\dagger$ denotes the Moore-Penrose inverse of A.

(b) $\{D^-\} \subseteq \{(B^*A^\dagger B)^-\}$, where $(\cdot)^-$ denotes a g-inverse of a matrix.

(c) $D = B^*A^\dagger B \Leftrightarrow \mathrm{rank}(M) = \mathrm{rank}(A) \Leftrightarrow \mathrm{rank}(D) = \mathrm{rank}(B)$.

Problem 34-7: A Sufficient Condition for a Matrix to be Normal

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Suppose that A, $R \in \mathbb{R}^{n\times n}$, $R = R^{-1} \neq \pm I_n$, $RAR = A^T$, and $x^T(A^TA - AA^T)y = 0$ whenever $Rx = x$ and $Ry = -y$. Show that A is normal.

Problem 34-8: A Property for the Sum of a Matrix A and its Moore-Penrose Inverse A^+

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a square complex matrix. Show that the following two statements are equivalent:

(i) $A + A^+ = 2AA^+$.

(ii) $A + A^+ = AA^+ + A^+A$.

Verify that under (i) or (ii), A must be EP, i.e. the column spaces of A and A^* coincide.

Problem 34-9: A Sum Property for the Moore-Penrose Inverse of EP Matrices

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a $n \times n$ EP matrix with complex entries whose rows add up to the same sum s. Show that the Moore-Penrose inverse A^+ of A has the property that its rows add up to $1/s$ if $s \neq 0$ and 0 if $s = 0$.

Problem 34-10: On the Product of Orthogonal Projectors

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be orthogonal projectors with complex entries. Show that PQ is an orthogonal projector if and only if $PQP \leq_L Q$, with $\leq_L$ indicating the Löwner ordering.

Problems 34-1 through 34-5 are on page 40.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to hjw.de@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. *Problems 34-6 through 34-10 are on page 39.*

Problem 34-1: A Well-Known Matrix Equation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@man.ac.uk

Let V and W be given $n \times n$ and $m \times m$ positive semidefinite matrices and let $X,\ y$, and z be given $n \times m,\ n \times 1$ and $m \times 1$ matrices. Then the system of $n + m$ equations in $n + m$ unknowns

$$\begin{pmatrix} V & X \\ X' & W \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}$$

is well-known in Mathematical Statistics when $W = 0$ and $z = 0$ are both null. Readers are invited to identify at least one other well-known instance of this matrix equation.

Problem 34-2: Eigenvalues of a Class of Tridiagonal Matrices

Proposed by Steven J. LEON, *University of Massachusetts Dartmouth, Dartmouth, Massachusetts, USA:* sleon@umassd.edu

Let α and β be real scalars and let $A_n(\alpha, \beta)$ denote the $n \times n$ tridiagonal matrix whose entries on the main diagonal are all equal to α and whose entries on the other two diagonals are all equal to β except that the $(1, 2)$ entry is 2β. For example,

$$A_4(\alpha, \beta) = \begin{pmatrix} \alpha & 2\beta & 0 & 0 \\ \beta & \alpha & \beta & 0 \\ 0 & \beta & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}.$$

Show that the eigenvalues of $A_n(\alpha, \beta)$ are

$$\lambda_j = \alpha + 2\beta \cos \frac{(2j-1)\pi}{2n}, \quad j = 1, \dots n.$$

Problem 34-3: On the Spectral Radius

Proposed by Chi-Kwong LI and Sebastian J. SCHREIBER, *The College of William & Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu; sjschr@wm.edu

Suppose $A \in M_m$, $B \in M_n$ and R is an $m \times n$ matrix such that $AR = RB$.

(a) Suppose R has full column rank. Show that the spectrum of B is a subset of that of A, and hence the spectral radius of B is not larger than that of A.

(b) If A and B are nonnegative, and if R has no zero rows or zero columns, show that A and B have the same spectral radius.

Problem 34-4: A Range Equality for the Commutator with Two Involutory Matrices

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

Suppose that A and B are both involutory matrices of the same order, that is, $A^2 = B^2 = I$, where I is the identity matrix. Show that

$$\text{range}(AB - BA) = \text{range}(A - B) \cap \text{range}(A + B).$$

Problem 34-5: A Rank Equality for Sums of Two Outer Inverses of a Matrix

Proposed by Yongge TIAN, *University of Alberta, Edmonton, Canada:* ytian@stat.ualberta.ca

An $m \times n$ matrix X is called an outer inverse of an $n \times m$ matrix A if $XAX = X$. Show that if $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 + a_2 \neq 0$, then $\text{rank}(a_1X_1 + a_2X_2) = \text{rank}(X_1 + X_2)$ for any two outer inverses X_1 and X_2 of A.