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GENOMIC SIGNAL PROCESSING: FROM MATRIX ALGEBRA TO GENETIC NETWORKS*

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1. Introduction.

1.1. DNA Microarray Technology and Genome-Scale Molecular Biological Data. The Human Genome Project, and the resulting sequencing of complete genomes, fueled the emergence of the DNA microarray hybridization technology in the past decade. This novel experimental high-throughput technology makes it possible to assay the hybridization of fluorescently tagged DNA or RNA molecules, which were extracted from a single sample, with several thousand synthetic oligonucleotides [1] or DNA targets simultaneously. Different types of molecular biological signals, such as DNA copy number, RNA expression levels and DNA-bound proteins' occupancy levels, that correspond to activities of cellular systems, such as DNA replication, RNA transcription and binding of transcription factors to DNA, can now be measured on genomic scales [2, 3]. For the first time in human history it is possible to monitor the flow of molecular biological information, as DNA is transcribed to RNA, RNA is translated to proteins, and proteins bind to DNA, and thus to observe experimentally the global signals that are generated and sensed by cellular systems. Already laboratories all over the world are producing vast quantities of genome-scale data in studies of cellular processes and tissue samples [5–7].

Analysis of these new data promises to enhance fundamental understanding of life on the molecular level and might prove useful in medical diagnosis, treatment and drug design. Comparative analysis of these data among two or more organisms promises to give new insights into the universality as well as the specialization of evolutionary, biochemical and genetic pathways. Integrative analysis of different types of these global signals from the same organism promises to reveal cellular mechanisms of regulation, i.e., global causal coordination of cellular activities.

1.2. From Technology and Large-Scale Data to Discovery and Control of Basic Phenomena Using Mathematical Models: Analogy from Astronomy. Biology and medicine today, with these recent advances in DNA microarray technology, may very well be at a point similar to where physics was after the advent of the telescope in the 17th century. In those days, astronomers were compiling tables detailing observed positions of planets at different times, for navigation. Popularized by Galileo Galilei, telescopes were being used in these sky surveys, enabling more accurate and more frequent observations of a growing number of celestial bodies. One astronomer, Tycho Brahe, compiled some of the more extensive and accurate tables of such astronomical observations. Another astronomer, Johannes Kepler, used mathematical equations from analytical geometry to describe trends in Brahe's data, and to determine three laws of planetary motion, all relating observed time intervals with observed distances. These laws enabled the most accurate predictions of future positions of planets to date. Kepler's achievement posed the question: Why are the planetary motions such that they follow these laws? A few decades later, Isaac Newton considered this question in light of the experiments of Galileo, the data of Brahe and the models of Kepler. Using mathematical equations from calculus, he introduced the physical observables mass, momentum and force, and defined them in terms of the observables time and distance. With these postulates, the three laws of Kepler could be derived within a single mathematical framework, known as the universal law of gravitation, and Newton concluded that the physical phenomenon of gravitation is the reason for the trends observed in the motion of the planets [8]. Today, Newton's discovery and mathematical formulation of the basic phenomenon that is gravitation enables control of the dynamics of moving bodies, e.g., in exploration of outer space.

The rapidly growing number of genome-scale molecular biological datasets hold the key to the discovery of previously unknown molecular biological principles, just as the vast number of astronomical tables compiled by Galileo and Brahe enabled accurate prediction of planetary motions and later also the discovery of universal gravitation. Just as Kepler and Newton made their discoveries by using mathematical frameworks to describe trends in these large-scale astronomical data, also future predictive power, discovery and control in biology and medicine will come from the mathematical modeling of genome-scale molecular biological data.

1.3. From Complex Signals to Simple Principles Using Mathematical Models: Analogy from Neuroscience. Genome-scale molecular biological signals appear to be complex, yet they are readily generated and sensed by the cellular systems. For example, the division cycle of human cells spans an order of one day only of cellular activity.

*Excerpted from Alter, O., "Genomic Signal Processing: From Matrix Algebra to Genetic Networks, to be published in *Microarray Data Analysis: Methods and Applications*, Korenberg, M. J., ed. (Humana Press).

The period of the cell division cycle in yeast is of the order of an hour.

DNA microarray data or genomic scale molecular biological signals in general, may very well be similar to the input and output signals of the central nervous system, such as images of the natural world that are viewed by the retina and the electric spike trains that are produced by the neurons in the visual cortex. In a series of classic experiments, the neuroscientists Hubel and Wiesel [9] recorded the activities of individual neurons in the visual cortex in response to different patterns of light falling on the retina. They showed that the visual cortex represents a spatial map of the visual field. They also discovered that there exists a class of neurons, which they called "simple cells," each of which responds selectively to a stimulus of an edge of a given scale at a given orientation in the neuron's region of the visual field. These discoveries posed the question: What might be the brain's advantage in processing natural images with a series of spatially localized scale-selective edge detectors? Barlow [10] suggested that the underlying principle of such image processing is that of sparse coding, which allows only a few neurons out of a large population to be simultaneously active when representing any image from the natural world. Naturally, such images are made out of objects and surfaces, i.e., edges. Two decades later, Olshausen and Field [11] developed a novel algorithm, which separates or decomposes natural images into their optimal components, where they defined optimality mathematically as the preservation of a characteristic ensemble of images as well as the sparse representation of this ensemble. They showed that the optimal sparse linear components of a natural image are spatially localized and scaled edges, thus validating Barlow's postulate.

The sensing of the complex genomic scale molecular biological signals by the cellular systems might be governed by simple principles, just as the processing of the complex natural images by the visual cortex appear to be governed by the simple principle of sparse coding. Just as the natural images could be represented mathematically as superpositions, i.e., weighted sums, of images, which correlate with the measured sensory activities of neurons, also the complex genomic scale molecular biological signals might be represented mathematically as superpositions of signals, which might correspond to the measured activities of cellular elements.

1.4. Matrix Algebra Models for DNA Microarray Data. The first data-driven predictive models for DNA microarray data or genomic scale molecular biological signals in general use adaptations and generalizations of matrix algebra frameworks [12] in order to provide mathematical descriptions of the genetic networks that generate and sense the measured data [Fig. 1]. The singular value decomposition (SVD) model formulates a dataset as the result of a simple linear network: The measured gene patterns are expressed mathematically as superpositions of the effects of a few independent sources, biological or experimental, and the measured sample patterns – as superpositions of the corresponding cellular states [13–15]. The comparative generalized SVD (GSVD) model formulates two datasets, e.g., from two different organisms such as yeast and human, as the result of a simple linear comparative network: The measured gene patterns in each dataset are expressed mathematically simultaneously as superpositions of a few independent sources that are common to both datasets, as well as sources that are exclusive to one of the datasets or the other [16]. The integrative pseudoinverse projection model approximates any number of datasets from the same organism, e.g., of different types of data such as RNA expression levels and proteins' DNA-binding occupancy levels, as the result of a simple linear integrative network: The measured sample patterns in each dataset are formulated simultaneously as superpositions of one chosen set of measured samples, or of profiles extracted mathematically from these samples, designated the "basis" set [17, 18].

The mathematical variables of these models, i.e., the patterns that these models uncover in the data, represent biological (or experimental) reality. The "eigengenes" uncovered by SVD, the "genelets" uncovered by GSVD, and the pseudoinverse correlations uncovered by pseudoinverse projection, correlate with independent processes, biological or experimental, such as observed genome-wide effects of known regulators or transcription factors, the cellular elements that generate the genome-wide RNA expression signals most commonly measured by DNA microarrays. The corresponding "eigenarrays" uncovered by SVD and "arraylets" uncovered by GSVD, correlate with the corresponding cellular states, such as measured samples in which these regulators or transcription factors are overactive or underactive.

The mathematical operations of these models, e.g., data reconstruction, rotation and classification in subspaces spanned by these patterns, also represent biological (or experimental) reality. Data reconstruction in subspaces of selected eigengenes, genelets, or pseudoinverse correlations, and corresponding eigenarrays, or arraylets, simulates experimental observation of only the processes and cellular states that these patterns represent, respectively. Data rotation in these subspaces simulates the experimental decoupling of the biological programs that these subspaces span. Data classification in these subspaces maps the measured gene- and sample patterns onto the processes and cellular states that these subspaces represent, respectively.

Since these models provide mathematical descriptions of the genetic networks that generate and sense the measured data, where the mathematical variables and operations represent biological (or experimental) reality, these models have the capacity to elucidate the design principles of cellular systems as well as guide the design of synthetic ones. These

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models also have the power to make experimental predictions that might lead to experiments in which the models can be refuted or validated, and to discover previously unknown molecular biological principles. Ultimately, these models might enable the control of biological cellular processes in real time and *in vivo*.

While no mathematical theorem promises that SVD, GSVD and pseudoinverse projection could be used to model DNA microarray data, or genome-scale molecular biological signals in general, these results are not counterintuitive. Similar and related mathematical frameworks have already proven successful in describing the physical world, in such diverse areas as mechanics and perception [19].

First, SVD, GSVD and pseudoinverse projection, interpreted, as they are here, as simple approximations of the networks or systems that generate and sense the processed signals, belong to a class of algorithms called blind source separation (BSS) algorithms. BSS algorithms, such as the linear sparse coding algorithm by Olshausen and Field and the neural network algorithms by Hopfield [20], separate or decompose measured signals into their mathematically defined optimal components. These algorithms have already proven successful in modeling natural signals and mimicking computationally the activity of the brain as it expertly perceives these signals, for example, in face recognition [21].

Second, SVD, GSVD and pseudoinverse projection can be also thought of as generalizations of the eigenvalue decomposition (EVD) and generalized eigenvalue decomposition (GEVD) of symmetric matrices, and inverse projection onto an orthogonal matrix, respectively. In mechanics, the EVD of the symmetric matrix, which tabulates the energy of a system of coupled oscillators, uncovers the eigenmodes and eigenfrequencies of this system, i.e., the normal coordinates, which oscillate independently of one another, and their frequencies of oscillations. One of these eigenmodes represents the center of mass of the system. The GEVD of the symmetric matrices, which tabulate the kinetic and potential energies of the oscillators, compares the distribution of kinetic energy among the eigenmodes with that of the potential energy. The inverse projection onto the orthogonal matrix, which tabulates the eigenmodes of this system, is equivalent to transformation of coordinates to the frame of reference, which is oscillating with the system [22]. SVD, GSVD and pseudoinverse projection are, therefore, generalizations of the frameworks that underlie the mathematical theoretical description of the physical world.

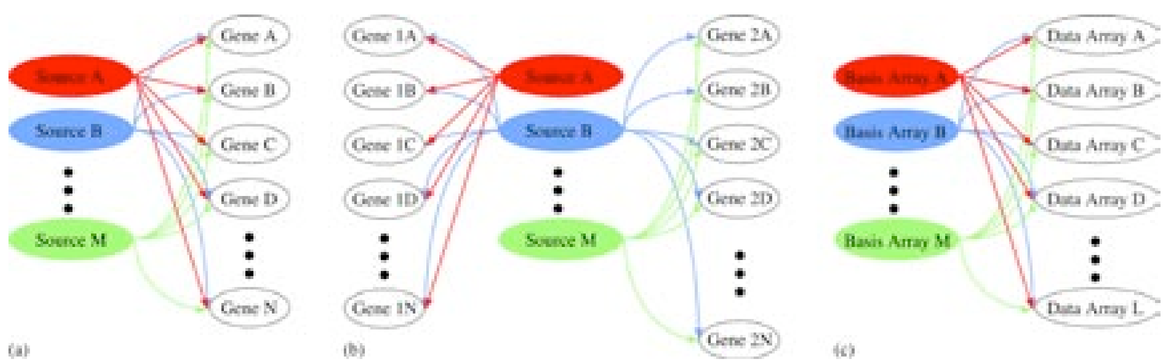

Fig. 1. The first data-driven predictive models for DNA microarray data. (*a*) The SVD model. (*b*) The GSVD comparative model. (*c*) The pseudoinverse projection integrative model.

2. Singular value decomposition (SVD) for modeling DNA microarray data. Let the matrix $\hat{e}$ of size N-genes $\times$ M-arrays tabulate the genome-scale signal, e.g., RNA expression levels, measured in a set of M samples using M DNA microarrays. Singular value decomposition (SVD) is a linear transformation of this DNA microarray dataset from the N-genes $\times$ M-arrays space to the reduced L-eigenarrays $\times$ L-eigengenes space, where $L = \min\{M, N\}$,

$$\hat{e} = \hat{u}\hat{\epsilon}\hat{v}^T. \tag{2.1}$$

In this space, the dataset or matrix $\hat{e}$ is represented by the diagonal nonnegative matrix $\hat{\epsilon}$ of size L-eigenarrays $\times$ L-eigengenes. The diagonality of $\hat{\epsilon}$ means that each eigengene is decoupled of all other eigengenes, and each eigenarray is decoupled of all other eigenarrays, such that each eigengene is expressed only in the corresponding eigenarray. The eigengenes and eigenarrays are orthonormal superpositions of the genes and arrays, such that the transformation matrices $\hat{u}$ and $\hat{v}^T$ are both orthogonal. The eigengenes and eigenarrays are unique up to phase factors of ± 1 for a real data matrix $\hat{e}$, such that each eigengene and eigenarray captures both parallel and antiparallel gene- and array expression patterns, except in degenerate subspaces, defined by subsets of equal eigenexpression levels. SVD is, therefore, data-driven, except in degenerate subspaces.

The fractions of eigenexpression $p_l = \epsilon_l^2 / \sum_{k=1}^{L} \epsilon_k^2$, calculated from the "eigenexpression" levels which are listed in the diagonal of $\hat{\epsilon}$, indicate for each eigengene and eigenarray their significance in the dataset relative to all other eigengenes and eigenarrays in terms of the overall expression information that they capture in the data. Note that each fraction of eigenexpression can be thought of as the probability for any given gene among all genes in the dataset to express the corresponding eigengene, and at the same time, the probability for any given array among all arrays to express the corresponding eigenarray.

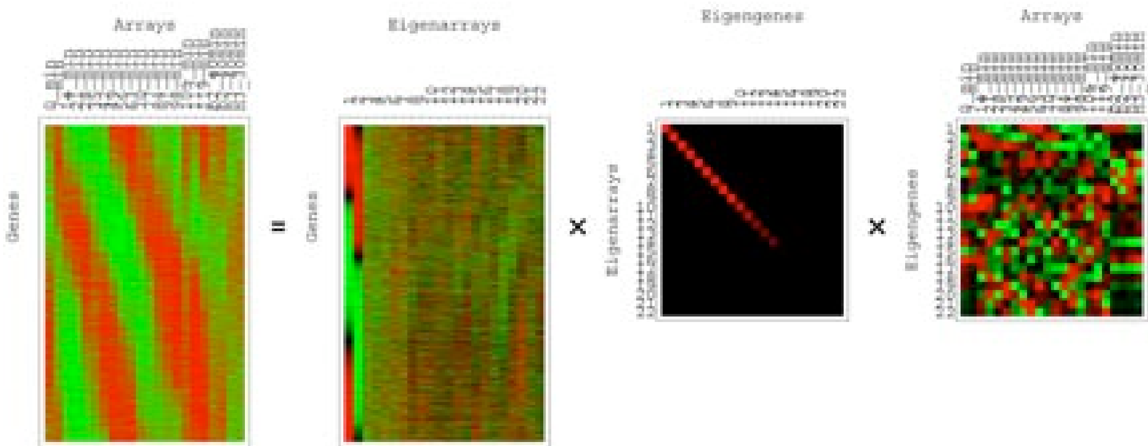

Fig. 2. Raster display of the SVD of a yeast cell cycle RNA expression dataset [4], with overexpression (red), no change in expression (black), and underexpression (green) around the steady state of expression of the 4,579 yeast genes. SVD is a linear transformation of the data from the 4,579-genes × 22-arrays space to the reduced diagonalized 22-eigenarrays × 22-eigengenes space, which is spanned by the 4,579-genes × 22-eigenarrays and 22-eigengenes × 22-arrays bases.

SVD is a BSS algorithm that decomposes the measured signal, i.e., the measured gene- and array patterns of, e.g. RNA expression, into mathematically decorrelated and decoupled patterns, the eigengenes and eigenarrays [13–15]. The correspondence between these mathematical patterns uncovered in the measured signal and the independent biological and experimental processes and cellular states that compose the signal was illustrated, e.g., with an analysis of genome-scale RNA expression data from the yeast *Saccharomyces cerevisiae* during its cell cycle program [Fig. 2]. Significant eigengenes and corresponding eigenarrays were shown to correlate with genome-scale effects of independent sources of expression and their corresponding cellular states. An eigenarray is parallel- and antiparallel associated with the most likely parallel and antiparallel cellular states, or none thereof, according to the annotations of the two groups of n genes each, with largest and smallest levels of signal, e.g., expression, in this eigenarray among all N genes, respectively. A coherent biological theme might be reflected in the annotations of either one of these two groups of genes. The P-value of a given association by annotation is calculated using combinatorics and assuming hypergeometric probability distribution of the K annotations among the N genes, and of the subset of $k \subseteq K$ annotations among the subset of $n \subseteq N$ genes, $P(k; n, N, K) = \binom{N}{n}^{-1} \sum_{l=k}^{n} \binom{K}{l} \binom{N-K}{n-l}$, where $\binom{N}{n} = N! n!^{-1} (N-n)!^{-1}$ is the binomial coefficient [23]. The corresponding eigengene is inferred to represent the corresponding biological process from its pattern of expression.

Mathematical operations with these patterns were shown to simulate biological experiments: (i) the filtering out of eigengenes and eigenarrays, such as these that are associated with experimental artifacts [14, 15], was shown to simulate the experimental suppression of the cellular processes and states that these eigengenes and eigenarrays represent; (ii) rotation in an almost degenerate subspace of eigengenes and eigenarrays was shown to simulate experimental decoupling of the biological programs, such as that the subspace spans; and (iii) the classification of the data according to the eigengenes and eigenarrays was shown to give a global picture of the dynamics of the biological program these represent, e.g., of a picture of cell cycle expression that resembles the traditional understanding of yeast cell cycle regulation [24].

It was shown that the SVD model describes, to first order, the RNA expression of most of the yeast genome during the cell cycle program [25] as being driven by the activities of two periodically oscillating cellular elements or modules, which are orthogonal, i.e., $\pi/2$ out of phase relative to one another. The underlying genetic network or circuit suggested by this model might be parallel in its design to the analog harmonic oscillator. This well-known oscillator design principle is at the foundations of numerous physical oscillators, including (i) the mechanical pendulum; (ii) the LC circuit; and (iii) the chemical Lotka-Volterra irreversible autocatalytic reaction model far from thermodynamic equilibirum [26].

3. Generalized SVD (GSVD) for Comparative Modeling of DNA Microarray Datasets. Let the matrix $\hat{e}_1$ of size N_1-genes $\times$ M_1-arrays tabulate the genome-scale signal, e.g., RNA expression levels, measured in a set of M_1 samples using M_1 DNA microarrays. Let the matrix $\hat{e}_2$ of size N_2-genes $\times$ M_2-arrays tabulate the genome-scale signal, e.g., RNA expression levels, measured in a set of M_2 samples under M_2 experimental conditions, that correspond one-to-one to the M_1 conditions underlying $\hat{e}_1$, such that $M_1 = M_2 \equiv M < \max\{N_1, N_2\}$. This one-to-one correspondence between the two sets of conditions is at the foundation of the GSVD comparative analysis of the two datasets, and should be mapped out carefully. GSVD is a simultaneous linear transformation of the two expression datasets $\hat{e}_1$ and $\hat{e}_2$ from the two N_1-genes $\times$ M-arrays and N_2-genes $\times$ M-arrays spaces to the two reduced M-arraylets $\times$ M-genelets spaces,

$$\begin{aligned} \hat{e}_1 &= \hat{u}_1 \hat{\epsilon}_1 \hat{x}^{-1}, \\ \hat{e}_2 &= \hat{u}_2 \hat{\epsilon}_2 \hat{x}^{-1}. \end{aligned} \tag{3.1}$$

In these spaces the data are represented by the diagonal nonnegative matrices $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$. Their diagonality means that each genelet is decoupled of all other genelets in both datasets simultaneously, such that each genelet is expressed only in the two corresponding arraylets, each of which is associated with one of the two datasets. The genelets are normalized, but not necessarily orthogonal, superpositions of the genes of the first dataset and, at the same time, also the second dataset. The arraylets of the first or the second datasets are orthonormal superpositions of the arrays of the first and second datasets, respectively. The antisymmetric "angular distances" between the datasets $\theta_m = \arctan(\epsilon_{1,m}/\epsilon_{2,m}) - \pi/4$, calculated from the "generalized eigenexpression" levels which are listed in the diagonals of $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$, indicate the relative significance of each genelet, i.e., its significance in the first dataset relative to that in the second dataset, in terms of the ratio of expression information captured by this genelet in the first dataset to that in the second.

GSVD is a comparative BSS algorithm that simultaneously decomposes two measured signals, i.e., the measured gene- and array patterns of, e.g., RNA expression in two organisms, into mathematically decoupled genelets and two sets of arraylets [16]. The correspondence between these mathematical patterns uncovered in the measured signals and the similar and dissimilar among the biological programs that compose each of the two signals was illustrated with a comparative analysis of genome-scale RNA expression data from yeast and human during their cell cycle programs [Fig. 3]: (i) common genelets and corresponding arraylets were shown to span the common yeast and human cell cycle subspace, which is common to both the yeast and human genomes, and is manifested in both datasets; and (ii) exclusive genelets and corresponding arraylets were shown to span the exclusive yeast and human synchronization responses subspaces. Mathematical operations with these patterns were shown to simulate biological experiments: (i) simultaneous reconstruction and classification of the yeast and human data in the common subspace outlines the biological similarity in the regulation of the yeast and human cell cycle programs; whereas (ii) reconstructions and classifications of either dataset in both the common subspace and the corresponding exclusive subspace simulate experimental observation of differential expression of the corresponding genome in the two cellular programs of the cell cycle and the synchronization response, uncovering the pathway- or program-dependent variation in the relations between the expression patterns of, e.g., the yeast genes *KAR4* and *CIK1*, which are known to be are correlated during mating, yet anticorrelated during cell cycle progression [27] (and see also [28]).

It was shown that the GSVD comparative model describes, to first order, the RNA expression of most of the yeast and human [29] genomes during their common cell cycle programs as being driven by the activities of three periodically oscillating cellular elements or modules, which are $\pi/3$ out of phase relative to one another. The underlying eukaryotic genetic network or circuit suggested by this model might be parallel in its design to the digital three-inverters ring oscillator. Elowitz and Leibler [30] recently demonstrated a synthetic genetic circuit analogous to this digital oscillator.

Comparisons of DNA sequence of entire genomes already give new insights into evolutionary, biochemical and genetic pathways. Recent studies showed that the addition of RNA expression data to DNA sequence comparisons improves functional gene annotation and might expand the understanding of how gene expression and diversity evolved. For example, Stuart, Segal, Koller and Kim [31] and independently also Bergmann, Ihmels and Barkai [32] identified pairs of genes for which RNA coexpression is conserved, in addition to their DNA sequences, across several organisms. The evolutionary conservation of the coexpression of these gene pairs confers a selective advantage to the functional relations of these genes. The GSVD comparative model is not limited to genes of conserved DNA sequences, and as such it elucidates universality as well as specialization of molecular biological mechanisms that are truly on genomic scales.

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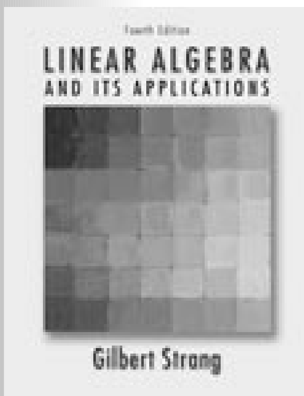

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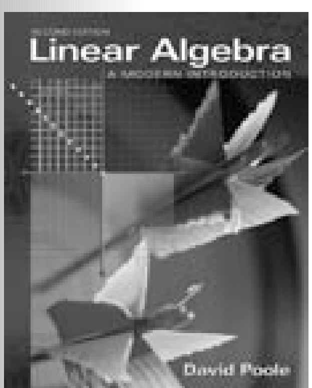

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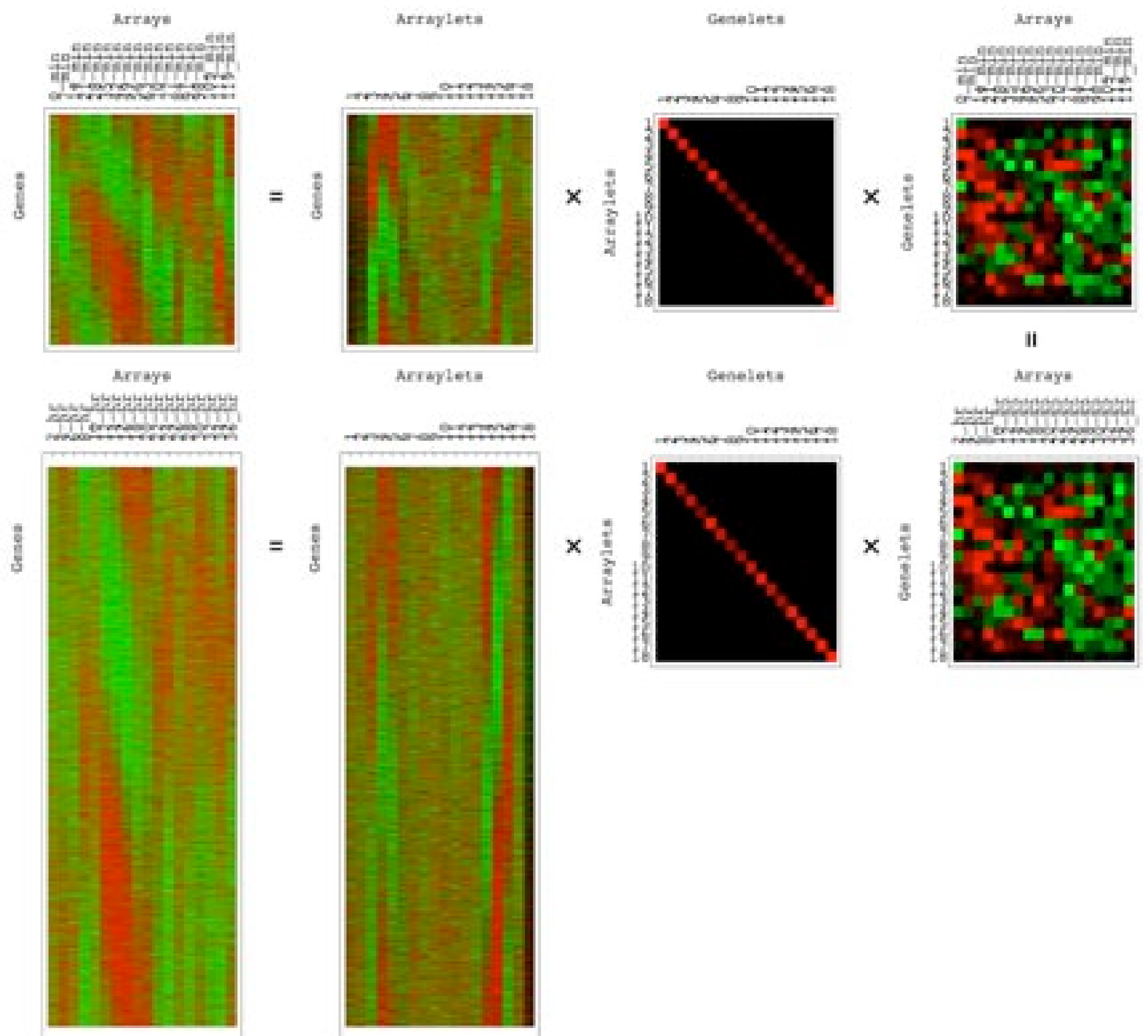

Fig. 3. Raster display of the GSVD of a yeast [4] and a human [5] cell cycles RNA expression datasets, showing a linear transformation of the yeast and human data from the 4,523-yeast and 12,056-human genes × 18-arrays spaces to the reduced diagonalized 18-arraylets × 18-genelets spaces, which are spanned by the 4,523- and 12,056-genes × 18-arraylets bases, respectively, and by the 18-genelets × 18-arrays shared basis.

4. Pseudoinverse projection for integrative modeling of DNA microarray datasets. Let the basis matrix $\hat{b}$ of size N-genomic sites or open reading frames (ORFs) × M-basis profiles tabulate M genome-scale molecular biological profiles of, e.g., RNA expression, measured from a set of M samples or extracted mathematically from a set of M or more measured samples. Let the data matrix $\hat{d}$ of size N-ORFs × L-data samples tabulate L genome-scale molecular biological profiles of, e.g., proteins' DNA-binding, measured for the same ORFs in L samples from the same organism. Moore-Penrose pseudoinverse projection of the data matrix $\hat{d}$ onto the basis matrix $\hat{b}$ is a linear transformation of the data from the N-ORFs × L-data samples space to the M-basis profiles × L-data samples space,

$$\hat{d} \to \hat{b}\hat{c}, \qquad \hat{b}^{\dagger}\hat{d} \equiv \hat{c}, \tag{4.1}$$

where the matrix $\hat{b}^{\dagger}$, i.e., the pseudoinverse of $\hat{b}$, satisfies $\hat{b}\hat{b}^{\dagger}\hat{b} = \hat{b}$, $(\hat{b}\,\hat{b}^{\dagger})^T = \hat{b}\,\hat{b}^{\dagger}$, $\hat{b}^{\dagger}\hat{b}\hat{b}^{\dagger} = \hat{b}^{\dagger}$, and $(\hat{b}^{\dagger}\,\hat{b})^T = \hat{b}^{\dagger}\,\hat{b}$, such that the transformation matrices $\hat{b}\hat{b}^{\dagger}$ and $\hat{b}^{\dagger}\hat{b}$ are orthogonal projection matrices for a real basis matrix $\hat{b}$. In this space the data matrix $\hat{d}$ is represented by the pseudoinverse correlations matrix $\hat{c}$.

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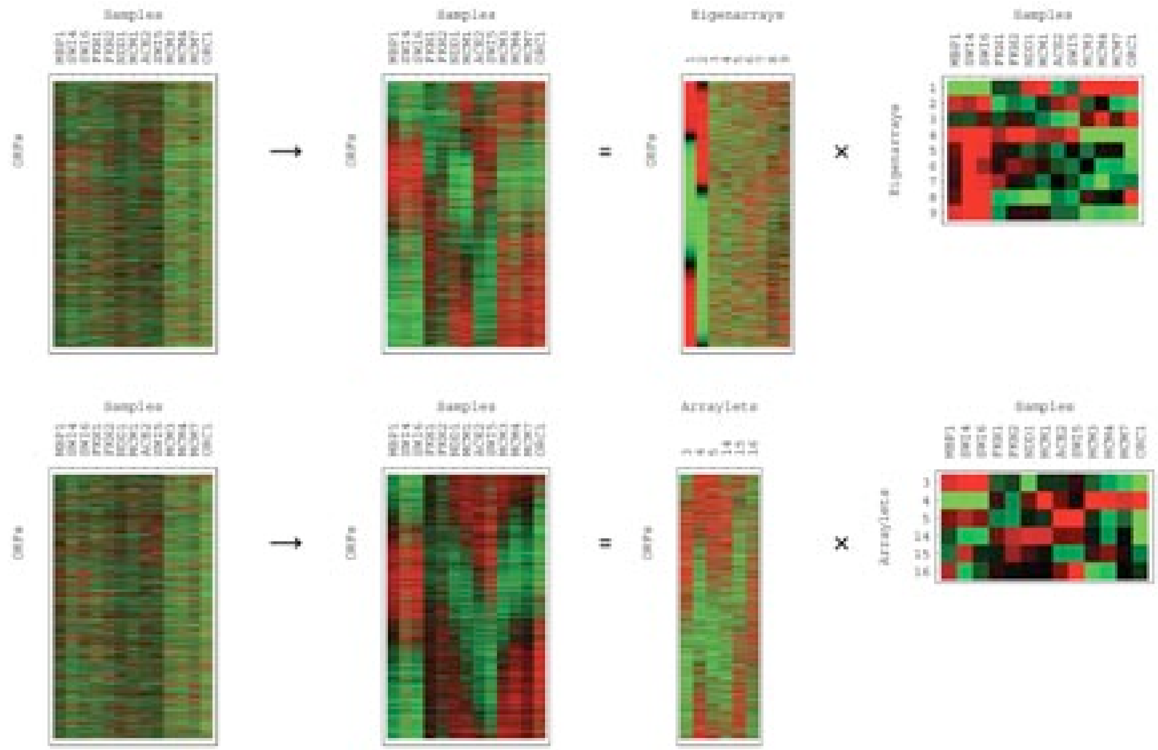

Fig. 4. Raster display of the pseudoinverse projection of the yeast cell cycle transcription factors and replication initiation proteins' DNA-binding data onto the SVD and GSVD cell cycle RNA expression bases, showing a linear transformation of the proteins' binding data from the 2,227 ORFs $\times$ 13-data samples space to the nine eigenarrays of the SVD basis $\times$ 13-data samples space (*upper*), and independently also from the 2,139 ORFs $\times$ 13-data samples space to the six arraylets of the GSVD basis $\times$ 13-data samples space (*lower*).

Pseudoinverse projection is an integrative BSS algorithm that decomposes the measured gene patterns of any given "data" signal of, e.g., proteins' DNA-binding, into mathematically least squares-optimal pseudoinverse correlations with the measured gene patterns of a chosen "basis" signal of, e.g., RNA expression, in a different set of samples from the same organism. The measured array patterns of the data signal are least squares-approximated with a decomposition into the measured array patterns of the basis. The correspondence between these mathematical patterns that are uncovered in the measured signals and the independent activities of cellular elements that compose the signals was illustrated with an integration of yeast genome-scale proteins' DNA-binding occupancy data from cell cycle transcription factors [6] and DNA replication initiation proteins [7] with the cell cycle time course RNA expression data, using as basis sets the eigenarrays and arraylets determined by SVD and GSVD, respectively [Fig. 4]. Pseudoinverse correlations uncovered in the data were shown to correspond to reported functions of, e.g., transcription factors. Mathematical operations with these patterns were shown to simulate biological experiments: (i) pseudoinverse reconstruction of the data in the basis simulates experimental observation of only the cellular states manifest in the data that correspond to those in the basis; and (ii) classification of the basis-reconstructed data samples maps the cellular states of the data onto those of the basis and gives a global picture of the correlations and possibly also causal coordination of these two sets of states.

It was shown that the pseudoinverse projection integrative model correlates for the first time the binding of replication initiation proteins with minima or shutdown of the transcription of adjacent ORFs during the cell cycle stage G_1 [17, 18], under the assumption that the measured cell cycle RNA expression levels are approximately proportional to cell cycle RNA transcription activity. Diffley, Cocker, Dowell and Rowley [33] showed that replication initiation requires binding of these proteins at origins of replications across the yeast genome during G_1. Micklem et al. [34] showed also that these replication initiation proteins are involved with transcriptional silencing at the yeast mating loci. Either one of at least two mechanisms of regulation may be underlying this novel genome-scale correlation between DNA replication initiation and RNA transcription during the yeast cell cycle: The transcription of genes may reduce the binding

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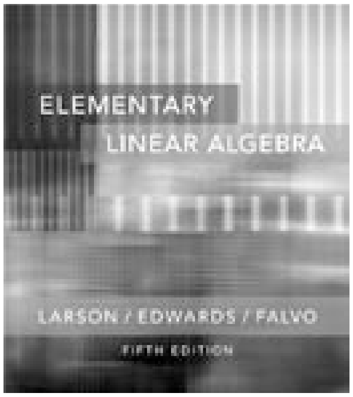

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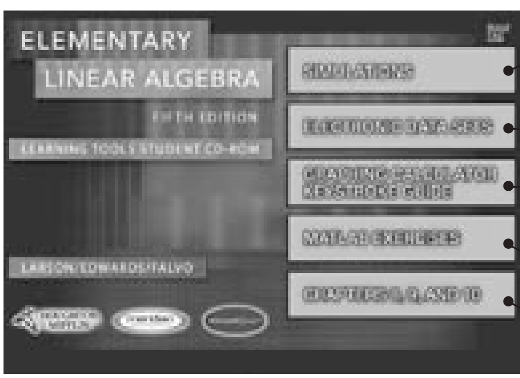

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Alter article cont'd from page 11

efficiency of adjacent origins. Or, the binding of replication initiation proteins to origins of replication may repress, or even shut down, the transcription of adjacent genes.

Integrative analysis of different types of global signals, such as these measured by DNA microarrays, from the same organism, promises to reveal global causal coordination of cellular activities. For example, Bussemaker, Li and Siggia [35] predicted new regulatory motifs by linear regression of profiles of genome-scale RNA expression in yeast vs. profiles of the abundance levels, or counts, of DNA oligomer motifs in the promoter regions of the same yeast genes. Lu, Nakorchevskiy and Marcotte [36] associated the knockout phenotype of individual yeast genes with cell cycle arrest by deconvolution of the RNA expression profiles measured in the corresponding yeast mutants into the RNA expression profiles measured during the cell cycle for all yeast genes that were microarray-classified as cell cycle-regulated.

This is the first time that a data-driven mathematical model, where the mathematical variables and operations represent biological reality, has been used to predict a biological principle that is truly on a genome-scale: The ORFs in either one of the basis or data matrices were selected based on data quality alone, and were not limited to ORFs that are traditionally or microarray-classified as cell cycle-regulated, suggesting that the RNA transcription signatures of yeast cell cycle cellular states may span the whole yeast genome. This novel correlation demonstrates the power of the SVD, GSVD and pseudoinverse projection models to predict previously unknown biological principles.

5. Disscussion: On the Linearity and Orthogonality of Genetic Networks. The SVD model, the GSVD comparative model and the pseudoinverse projection integrative model are all mathematically linear and orthogonal. These models formulate genome-scale molecular biological signals as linear superpositions of mathematical patterns, which correlate with activities of cellular elements, such as regulators or transcription factors, that drive the measured signal, and cellular states where these elements are active. These models associate the independent cellular states with orthogonal, i.e., decorrelated, mathematical profiles, suggesting that the overlap or crosstalk between the genome-scale effects of the corresponding cellular elements or modules is negligible. Recently, Ihmels, Levy and Barkai [37] found evidence for linearity as well as orthogonality in the metabolic network in yeast. Integrating genome-scale RNA expression data with the structural description of this network, they showed that at the network's branchpoints, only distinct branches are coexpressed, and concluded that transcriptional regulation biases the metabolic flow toward linearity. They also showed that individual isozymes, i.e., chemically distinct but functionally similar enzymes, tend to be corregulated separately with distinct processes. They concluded that transcriptional regulation uses isozymes as means for reducing crosstalk between pathways that use a common chemical reaction.

Orthogonality of the cellular states that compose a genetic network suggests an efficient network design: With no redundant functionality in the activities of the independent cellular elements, the number of such elements needed to carry out a given set of biological processes is minimized. An efficient network, however, is fragile. The robustness of biological systems to diverse perturbations, e.g., phenotypic stability despite environmental changes and genetic variation, suggests functional redundancy in the activities of the cellular elements, and therefore also correlations among the corresponding cellular states. Carslon and Doyle [38] introduced the framework of "highly optimized tolerance" to study fundamental aspects of complexity in, among others, biological systems that appear to be naturally selected for efficiency as well as robustness. They showed that trade-offs between efficiency and robustness might explain the behavior of such complex systems, including occurrences of catastrophic failure events.

Linearity of a genetic network may seem counterintuitive in light of the nonlinearity of the chemical processes, which underlie the network. Arkin and Ross showed that enzymatic reaction mechanisms can be thought to compute the mathematically nonlinear functions of logic gates on the molecular level. They also showed that the qualitative logic gate behavior of such a reaction mechanism may not change when situated within a model of the cellular program that uses the reaction: This program functions as a biological switch from one pathway to another in response to chemical signals, and thus computes a nonlinear logic gate function on the cellular scale. Another cellular program that can be thought to compute nonlinear functions is the well-known genetic switch in the bacteriophage λ, the program of decision between lysis and lysogeny [40]. McAdams and Shapiro [41] modeled this program with a circuit of integrated logic components. However, even if the kinetics of biochemical reactions are nonlinear, the mass balance constraints that govern these reactions are linear. Schilling and Palsson [42] showed that the underlying pathway structure of a biochemical network, and therefore also its functional capabilities, can be extracted from the linear set of mass balance constraints corresponding to the set of reactions that compose this network.

That genetic networks might be modeled with linear and orthogonal mathematical frameworks does not necessarily imply that these networks are linear and orthogonal. Dynamical systems, linear and nonlinear, are regularly studied with linear orthogonal transforms. For example, SVD might be used to reconstruct the phase-space description of a dynamical system from a series of observations of the time evolution of the coordinates of the system. In such a reconstruction, the experimental data are mapped onto a subspace spanned by selected patterns that are uncovered in the data by

SVD. The phase-space description of linear systems, for which the time evolution, or "motion," of the coordinates is periodic, such as the analog harmonic oscillator, is the "limit cycle." The phase-space description of nonlinear systems, for which the coordinates' motion is chaotic, such as the chemical Lotka-Volterra irreversible autocatalytic reaction, is the "strange attractor." Broomhead and King [43] were the first to use SVD to reconstruct the strange attractor.

While it is still an open question whether genetic networks are linear and orthogonal, linear and orthogonal mathematical frameworks have already proven successful in describing the physical world, in such diverse areas as mechanics and perception. It may not be surprising, therefore, that linear and orthogonal mathematical models for genome-scale molecular biological signals (i) provide mathematical descriptions of the genetic networks that generate and sense the measured data, where the mathematical variables and operations represent biological (or experimental) reality; (ii) elucidate the design principles of cellular systems as well as guide the design of synthetic ones; and (iii) predict previously unknown biological principles. These models may become the foundation of a future in which biological systems are modeled as physical systems are today.

Acknowledgements. The author thanks the International Linear Algebra Society for selecting her to deliver the 2005 Linear Algebra and its Applications Lecture. The author also thanks D. Botstein and P. O. Brown for introducing her to genomics, G. H. Golub for introducing her to matrix computation and M. van de Rijn for introducing her to translational cancer research. The author also thanks G. M. Church, J. F. X. Diffley, J. Doyle, S. R. Eddy, P. Green, R. R. Klevecz, E. Rivas and J. J. Wyrick for thoughtful and thorough reviews of parts of the work presented in this article. This work was supported by a National Human Genome Research Institute Individual Mentored Research Scientist Development Award in Genomic Research and Analysis (K01 HG00038-05) and by a Sloan Foundation and Department of Energy Postdoctoral Fellowship in Computational Molecular Biology (DE-FG03-99ER62836).

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Professor Richard S. Varga recieves the Hans Schneider Prize
by **Michael Neumann**

Editor's note: Richard S. Varga and Richard A. Brualdi were selected as the 2005 Hans Schneider Prize awardees. Richard Varga received his award at the 2005 ILAS meeting. Richard Brualdi will receive his award at the 2006 ILAS meeting. Below are some excerpts from Michael Neumann's presentation of the award to Richard Varga. Michael expresses his thanks to Ljiljana Cvetković, Volker Mehrmann, and Lothar Reichel for their help in preparing the presentation.

Introduction

With the wide ranging contribution and influence that Professor Richard Varga has had over a period of 50 years on the matrix and linear algebra communities, on the numerical linear algebra community, on the numerical analysis community, on the analysis community, and on the approximation theory community, it is truly hard to choose where to begin. But it is also simple. There are so many statistics about him, that even if the real effect of what lies behind the statistics was not certain, which it is, Richard would merit a prize just for the size and breadth of his contributions. So let's start with Richard's biographical data.

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Hans Schneider Prize cont'd from page 15

Richard Varga's Educational Background

Richard Varga was born in the US 76 years ago. He received a B.S. in Mathematics from the Case Institute of Technology, Ohio, in 1950; an A.M. ("Artium Magister" in Mathematics from Harvard University in 1951; and a PhD in Mathematics from Harvard University in 1954. His dissertation title was: "Properties of a Special Set of Entire Functions and Their Respective Partial Sums". His thesis advisor was J. L. Walsh.

From 1954 until 1960, he worked for Westinghouse Electric Corporation, at the Bettis Atomic Power Laboratory in Pittsburgh. Since 1960, Richard has held many academic posts. He has been a University Professor in the Department of Mathematical Sciences at Kent State University in Ohio since 1970. He was the Director of the Institute for Computational Mathematics at Kent State University from 1980–1998, and he has been its Research Director since then.

Richard Varga has published over 240 papers and he has written 6 books, some of which have been translated into foreign languages. He has organized many international conferences, meetings, and workshops. Among the most celebrated are the Gatlingburg conferences (that have since become the Householder meetings) and the Oberwolfach Meetings on Matrix Theory and Numerical Linear Algebra.

He has had 25 PhD students.

Richard Varga has been an editor of several journals. Among them *Numerische Mathematik* from 1965–present (and its Editor–in-Chief from 1988–2002), *Electronic Transactions on Numerical Analysis* (ETNA) 1983–present, and, of course, for our community, *Linear Algebra and its Applications*, from 1968–present.

Some of Richard Varga's Important and Lasting Contributions

Let me begin with Richard's contributions to linear algebra, matrix theory, and numerical linear algebra. What immediately comes to mind is Richard's notion of a regular splitting of a matrix, for which he proved that an iteration matrix induced by a regular splitting of a matrix A, has a spectral radius less than 1, if and only if, the matrix A is a monotone matrix.

An important example of a monotone matrix is a nonsingular M--matrix, and Varga's comparison theorem yields for such a matrix, a straightforward proof of the well-known Stein-Rosenberg criterion for the comparison of the convergence of the Jacobi and the Gauss-Seidel methods.

The Gauss-Seidel method can be viewed as a special case of the SOR method, when the relaxation parameter equals 1. The moment the relaxation parameter exceeds 1, the iteration matrix loses its nonnegativity and comparison theorems are no longer applicable in coming to decide where in the interval [1,2), the SOR iteration matrix attains its minimal spectral radius.

It is here that Varga's multiple talents and expertise as an analyst, in particular a complex analyst, come to the fore. He generalizes Young's optimal SOR theory for the 2-cyclic case to the general p-cyclic case. His powers as an analyst, as a linear algebraist, and as a numerical analyst continue in the investigation of semi-iterative methods, including Chebyshev's, and their relation to SOR theory. In all the above works what is incredible is Richard's use of both analytical and combinatorial tools to obtain the best possible results.

In the 1970s, Richard Varga published another very influential manuscript. The paper is, of course, "On recurring theorems on diagonal dominance", which was published in the special LAA issue in honor of Olga Taussky-Todd. What is new in the paper is the way Richard brings under one umbrella the topics of M-matrices, H-matrices, diagonal dominance, and Geršgorin circles. No longer does a matrix theorist need to think of them as completely separate entities in matrix theory and numerical linear algebra.

Richard's work on Geršgorin's circles is also a marvelous achievement in bringing and generalizing many results under one roof. He shows how the union of the Geršgorin disks contains the union of Brauer ovals of Cassini, which, in turn, contains the union of the Brualdi lemniscate sets which, in turn, contains the minimal Geršgorin sets which, in turn, contains the spectrum.

We come now to quite a separate area of contribution and interest of Richard's, namely, in analysis. Some areas in analysis that Richard has worked on are: (i) Properties of the exponential function. In part his work on this function has been inspired by his work on numerical methods for the solution of parabolic differential equations. It involves analysis, approximation theory, and it is all beautiful mathematics. (ii) Richard has published a sequence of papers that sharpen inequalities related to the Riemann hypothesis. (iii) He has done numerous works on polynomial and rational approximation in the complex plane.

Some Suggestions Concerning Richard Varga

(a) Don't attempt to physically attack him by day or by night–he is an experienced wrestler, (b) Don't be lulled by his challenge to play table tennis–you are likely to lose, (c) Do go to hear him sing in a choir, (d) When he tells you a joke, don't tell him that he told you that one before, and (e) Let him repair your car–he has kept a Mercedes Benz on the road for over 30 years, repairing it all by himself.

CONGRATULATIONS Richard!

Book Review: *Numerical Linear Algebra* by Xiao-qing Jin and Yi-min Wei, Science Press (Beijing/New York), 2004.
ISBN 7-03-013954-2, vii+174 pages, paperback.

This book was written as a textbook for a course fully dedicated to Numerical Linear Algebra (NLA) at the advanced undergraduate and beginning graduate level. With its emphasis on the rigorous theoretical aspects of the subject, it is suitable for students majoring in mathematics or scientific computing. In its 174 pages, and nine chapters, it covers in substantial detail all the topics covered in modern textbooks on numerical analysis and also includes more advanced topics such as Krylov subspace methods, in particular, conjugate gradients and GMRES, for linear systems. Here is a short description of its contents.

Direct methods for linear systems

Following Chapter 1, which essentially introduces the notation used and motivates the study of NLA, Chapter 2 discusses the solution of linear systems by direct methods, with emphasis on the LU and Cholesky factorizations and pivoting.

Chapter 3, starting with the subject of norms, discusses the concept of stability and of condition numbers, and presents a quite clear and complete treatment of backward error analysis of LU factorization and solution of linear systems by Gaussian elimination.

Chapter 4 concerns the solution of linear least squares problems. Following a brief discussion of the Moore--Penrose generalized inverse, the relevant condition number, and the solution of least squares problems via the QR factorization, the computation of the QR factorization via Householder transformations and Givens rotations is discussed in detail.

Iterative methods for linear systems

Chapter 5 discusses the classical iterative methods for the solution of linear systems. The methods discussed are the Jacobi, Gauss-Seidel, and SOR methods, and their convergence theory for strictly diagonally dominant matrices and hermitian positive definite matrices. This chapter also includes a summary of what is known concerning the optimal relaxation parameter for SOR in the presence of consistently ordered matrices.

Chapter 6 is a nice introduction to the subject of Krylov subspace methods for linear systems. It discusses the steepest descent method, the method of conjugate gradients, and GMRES. It also gives a brief introduction to the topic of preconditioning for conjugate gradients. It provides the known theories of convergence for all three methods in a thorough manner.

Eigenvalue problems

Chapter 7 treats the eigenvalue problem for nonsymmetric matrices. Following a short discussion of the Jordan canonical form and Schur factorization, the power method and its variants are discussed. Most of the rest of the chapter is devoted to the basic QR method (without shifts), and a short discussion of the QR method with shifts is also provided.

Chapter 8 is devoted to the symmetric eigenvalue problem. It treats the symmetric QR method with shifts and the Jacobi method and its variants. The solution of the symmetric tridiagonal eigenvalue problem by the bisection and divide-and-conquer methods is also treated.

Applications

Chapter 9 is unique in that it is concerned with the application of preconditioned GMRES to linear boundary value problems in ordinary differential equations and to delay differential equations.

The reviewer has found this book to be a very useful and pleasant textbook in NLA. The presentation, even of the difficult topics, is concise but to the point and mathematically rigorous. In addition, the treatment is clear and in-depth. In most places, complete proofs are provided; only in a few places is the student/teacher sent to other literature for proofs. The subjects discussed in each chapter are supplemented by exercises that should enhance the knowledge of the student. What is missing is computer exercises. Such exercises could be supplied with the help of MATLAB. The English can be improved somewhat; the authors should have this point in mind if and when they revise or reprint the book.

In conclusion, the reviewer would like to congratulate the authors for their success in squeezing so much useful material in 174 pages in such a pleasant way. He highly recommends *Numerical Linear Algebra* as a textbook for courses in numerical linear algebra.

Reviewed by **Avram Sidi**
asidi@cs.technion.ac.il
Computer Science Department
Technion--Israel Institute of Technology

What is a Comatrix?

In his excellent book *The Professor and the Madman: A Tale of Murder, Insanity, and the Making of the Oxford English Dictionary* Simon Winchester (1998) mentions that a "comatrix" (joint womb) is defined in Henry Cockerham's *The English dictionary* of 1623 as "a maid who make ready and unready her mistress."

So far as I am aware, the word comatrix is not currently employed in matrix theory. This is unfortunate, as in an ideal world it could perhaps be used for the transpose of the matrix of cofactors thus freeing "adjoint" for the transpose of the matrix of complex conjugates as implied by the term "self-adjoint" (or Hermitian).

An interview with the Editor of the Oxford English Dictionary (OED) included at the end of the spoken word

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edition of Winchester's book implies that readers wishing to help with the development of the 20th edition of the OED should consult the relevant pages at `http://www.oed.com`.

Reference

Simon Winchester (1998). *The Professor and the Madman: A Tale of Murder, Insanity, and the Making of the Oxford English Dictionary*, Harper-Collins, New York.

Richard William Farebrother
11 Castle Road, Bayston Hill,
Shrewsbury, England SY3 0NF.

A new book in Linear Algebra
submitted by **Leiba Rodman**

Indefinite Linear Algebra and Applications, Birkhäuser Verlag, Basel, 2005, xii + 357 pages, by Israel Gohberg, Peter Lancaster, and Leiba Rodman.

The following topics of mathematical analysis have been developed in the last fifty years: the theory of linear canonical differential equations with periodic Hamiltonians, the theory of matrix polynomials with selfadjoint coefficients, linear differential and difference equations of higher order with selfadjoint constant coefficients, and algebraic Riccati equations. All of these theories, and others, are based on relatively recent results of linear algebra in spaces with an indefinite inner product, i.e. linear algebra in which the usual positive definite inner product is replaced by an indefinite one. More concisely, we call this subject *indefinite linear algebra.*

This book has the structure of a graduate text in which chapters of advanced linear algebra form the core. The development of the topics follows the lines of a usual linear algebra course. However, chapters giving comprehensive treatments of differential and difference equations, matrix polynomials and Riccati equations are interwoven as the necessary techniques are developed.

The main source of material is our earlier monograph in this field: *Matrices and Indefinite Scalar Product* [GLR3]. The present book differs in objectives and material. Some chapters have been excluded, others have been added, and exercises have been added to all chapters. An appendix is also included which may serve as a summary and refresher on standard results, as well as a source for some less familiar material from linear algebra with a definite inner product.

The theory developed here has become an essential part of linear algebra. This, together with the many significant areas of application, and the accessible style, make this book useful for engineers, scientists and mathematicians alike.

It starts with the theory of subspaces and orthogonalization and then goes on to the theory of matrices, perturbation and stability theory. All of this material is developed in the context of linear spaces with an indefinite inner product. The book also includes applications of the theory to the study of matrix polynomials with selfadjoint constant coefficients, to differential and difference equations (of first and higher order with constant coefficients), and to algebraic Riccati equations.

In the interests of developing a clearer and more comprehensive theory, chapters on orthogonal polynomials, normal matrices, and definite subspaces have been introduced. These changes are all intended to make our subject more accessible

The material has an interesting history. The perturbation and stability results for unitary matrices in a space with zones of stability for canonical differential equation with periodic coefficients were initiated by Krein [K]. The next development in this direction was made by I. M. Gelfand, V. B. Lidskii, and M. G. Neigaus [GelLid]. Further contributions were made by V. M. Starzhinskii and V. A. Yakubovich, W. A. Coppel and A. Howe as well as N. Levinson. The present authors have made contributions to the theory of linear differential and difference matrix equations of higher order with selfadjoint coefficients and to the theory of algebraic Riccati equations.

All of these theories are based on the same material of advanced linear algebra: namely, the theory of matrices acting on spaces with an indefinite inner product. This theory includes canonical forms and their invariants for H-selfadjoint, H-unitary and H-normal matrices, invariant subspaces of different kinds, and different aspects of perturbation theory. This material makes the core of the book and makes up a systematic *Indefinite Linear Algebra*. Immediate applications are made to demonstrate the importance of the theory. These applications are to the solution of time-invariant differential and difference equations with certain symmetries in their coefficients, the solution of algebraic Riccati equations, and to the analysis of matrix polynomials with selfadjoint coefficients.

The material included has been carefully selected to represent the area, to be self-contained and accessible, to follow the lines of a standard linear algebra course, and to emphasize the differences between the definite and indefinite linear algebras.

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[GelLid] I.M. Gelfand and V.B. Lidskii, On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients. *Amer. Math. Soc. Transl.* 8:143-181, 1958.

[GLR3] I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and Indefinite Scalar Products*, volume 8 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1983.

[K] M. G. Krein, Topics in differential and integral equations and operator theory. *Operator Theory: Advances and Applications*, Vol. 7, Birkhäuser Verlag, Basel, 1983. (Translation of 1955 Russian original).

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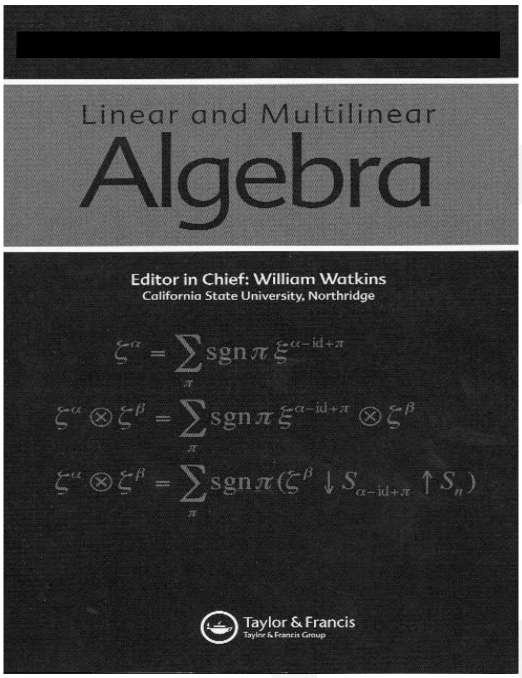

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12th ILAS Conference

report by **Doug Farenick** and **Steve Kirkland**

The 12th Conference of the International Linear Algebra Society was held June 26-29, 2005 in Regina, Canada, making that meeting the first ILAS conference to have been held in Canada. The scientific program for the conference was set by an international organising committee that consisted of Rajendra Bhatia (India), Robert Guralnick (U.S.A.), Daniel Hershkowitz (Israel, ILAS Preseident), Steve Kirkland (Canada, Committee Chair), Volker Mehrmann (Germany), Bit-Shun Tam (Taiwan), Pauline van den Driessche (Canada) and Henry Wolkowicz (Canada).

The conference featured 20 invited speakers, who presented eight one hour and twelve half hour lectures. The invited lectures covered a wide array of research areas, including matrix theory, geometry, algebraic methods, combinatorics and graph theory, numerical methods, operator theory, the teaching of linear algebra, and applications ranging from genomic signal processing to optimization. Among the invited talks, there were four featured lectures: (i) the Hans Schneider Prize Lecture (sponsored by ILAS) by Richard Varga (Kent State); (ii) the LAA Lecture (sponsored by *Linear Algebra and its Applications*) by Orly Alter (Texas); (iii) the LAMA Lecture (sponsored by *Linear and Multilinear Algebra*) by Chi-Kwong Li (William & Mary); and (iv) the ILAS Education Lecture by Anna Sierpinksa (Concordia). In addition, there were five minisymposia, consisting of twenty-eight presentations on a range of fundamental issues of current interest in linear algebra, as well as seventy contributed talks. The journal *Linear Algebra and its Applications* will publish a special issue devoted to papers presented at the conference. The Editors of this volume are Rajendra Bhatia, Robert Guralnick, Steve Kirkland, and Henry Wolkowicz.

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Conference Photo

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The ILAS Conference attracted 150 participants, and ILAS's international character was evident at the meeting, as the list of participants included people from Africa, Asia, Europe, North America and South America. In addition, there were a number of graduate students and postdoctoral fellows at the conference. An ILAS-sponsored lunch on the first day of the meeting introduced the students and postdoctoral fellows by name, and invited all conference participants to welcome these young people in the informal discussions and social activities that normally occur during a mathematics meeting. Further, a joint initiative of ILAS and the Canadian Mathematical Society resulted in a Student Social that was held at a downtown restaurant, attracting 27 participants.

One of the conference's highlights was the presentation of the Hans Schneider Prize, which recognises outstanding contributions to research in linear algebra. At this year's conference, the Hans Schneider Prize was awarded to Richard Varga of Kent State University for his career-long contributions to matrix analysis. Varga's prize lecture was devoted to Geršgorin's Circle Theorem, the subject of his very recent Springer monograph. The prize itself was presented at the conference banquet; those in attendance were treated to an insightful overview of Professor Varga's research given by Michael Neumann (Connecticut) as well as a gracious and good-humoured acceptance speech by Professor Varga. The banquet also featured an after-dinner talk by Chandler Davis (Toronto), who spoke on the development of linear algebra as a field of independent interest, as seen from the perspective of his career and experience.

Financial support for the conference came from a number of sources, including Atlas Conference Services, the Canadian Mathematical Society Student Committee, Elsevier, the Fields Institute, ILAS, the Pacific Institute for the Mathematical Sciences, Taylor & Francis, and the University of Regina. The conference was supported in a variety of technical aspects by the Fields Institute and by University of Regina staff and students. The local organizers (Shaun Fallat, Doug Farenick, Chun-Hua Guo and Steve Kirkland) are pleased that Regina was given the opportunity to be the venue for the 12th ILAS Conference. Like anything that is really worth doing, the planning and hosting of the meeting was both challeging and rewarding.

Graduate students and post-docs honored at ILAS meeting

report by **Jeff Stuart**

Instituting a tradition at our conferences, ILAS introduced all of the students and postdocs at a box lunch in their honor. The twenty six new scholars are listed with their institutions are: Mahmud Akelbek (University of Regina), Oscar Maria Baksalary (Adam Mickiewicz University), Tom Bella (University of Connecticut), Adam Berliner (University of Wisconsin-Madison), Julius Borcea (Stockholm University), Sebastian Cioaba (Queen's University), Louis Deaett (University of Wisconsin-Madison), Sandra Fital (University of Regina), Adja Fosner (University of Maribor), Vicente Galiano (Miguel Hernandez University), Kent Griffin (Washington State University), Victoria Herranz (Miguel Hernandez University), Jong Sam Jeon (Washington State University), Plamen Koev (Massachusetts Institute of Technology), Supranee Lisawadi (University of Regina), Xiaoping Liu (University of Regina), Ricardo Nuno Fonseca de Campos Pereira Mamede (University of Coimbra), Janko Marovt (University of Maribor), Dipra Mitra (University of Regina), Farej Omar (University of Regina), Bhanu Pratap Sharma (Canadian Math Society), Nung-sing Sze (University of Hong Kong), Ryan Tifenbach (University of Regina), Vanesa Cortes Utrillas (University of Zaragoza), Bamdad R. Yahaghi (Institute for Studies in Theoretical Physics and Mathematics) and Fei Zhou (University of Regina).

Graduate students, post-docs and ILAS Executive Committee at Regina meeting

Educational Activities at ILAS 2005

report by **Jane M. Day,** Education Committee Chair

This is a brief snapshot; more details are available from the ILAS Education page (go to http://www.math.technion.ac.il/iic and click on IIC, then Education links).

Anna Sierpinska of Concordia University spoke on "Innovations in teaching linear algebra, why you'll never near the end of it." She said linear algebra is hard to learn and to teach when instructors insist students understand the theory of vector spaces, as they should. This is not surprising since this beautiful theory is the polished result of two centuries of deep applications in which linear spaces and transformations proved essential. Most students have no knowledge of that history, so the ideas seem strange and unmotivated.

Nevertheless, the most practical thing we can offer linear algebra students is help learning to think theoretically, because then they will be able to use their knowledge. No one knows straightforward ways to do this, so dedicated instructors must continue to innovate! She has developed questions which intend to foster theoretical thinking, and using these in weekly quizzes has improved the performance of many students (but not the weakest ones). These exercises are posted on her website: http://alcor.concordia.ca/~sierp/.

Luz DeAlba of Drake Univ. spoke about "Assessment strategies for linear algebra." Drake has mandated development of assessment strategies for all courses, and Luz is doing this for linear algebra. She says it is easy to evaluate the best and worst students but hard to do for the middle ones, who continue to exhibit a mixture of "knowledge and ignorance, care and negligence." However she has identified specific learning outcomes for sophomore linear algebra, a list of references on assessment, and examples of assessment methods and strategies she has developed (which is work in progress). See her slides and handout at http://www.drake.edu/mathcs/dealba/.

Guershon Harel of UC San Diego spoke about "What is mathematics? A pedagogical answer to a philosophical question." He first defined a teacher's knowledge base as content (knowledge of mathematics), cognition (knowledge of how students learn), and pedagogy (knowledge of how to teach), and he defined two categories of knowledge: Way of Understanding and Way of Thinking. The first corresponds to a product of a mental act, and the second to the character of a mental act. (Interpreting, generalizing and modeling are examples of mental acts.) Then he defined mathematics as consisting of two complementary sets: the first is all the institutionalized ways of understanding in mathematics throughout history, and the second is all the ways of thinking that characterize the mental acts whose products comprise the first set. He demonstrated the impact on teaching when considering mathematics according to this definition. Visit http://www.math.sdsu.edu/~hare for this paper and other resources.

Sang-Gu Lee of Sungkyunkwan University described "Korea's e-Campus Vision 2007: a New Learning Environment for Linear Algebra." Lecture rooms are now equipped with projection equipment, Viewcams, tablet PC and internet D-base. This is changing teaching methods: computer demonstrations are easy, students in remote locations can easily participate in classes, and lectures are recorded so students can review them right after class. Sample lectures and Java and Flash tools for linear algebra can be found on the ILAS Education page or http://matrix.skku.ac.kr/newMatrixCal/Test.html, and http://matrix.skku.kr/CLAMC/index.html.

Applied Linear Algebra 2005, in Honor of Richard Varga

report by **Daniel Szyld**

This conference, sponsored by ILAS took place on October 13-15 in the lakeside resort town of Palíc, Serbia and Montenegro. Sixty eight participants from ten countries gathered to honor Richard Varga and his many contributions to the Linear Algebra community.

It was Varga's seventy seventh birthday, which is neither a multiple of five, nor a prime number. Nevertheless it was a fitting celebration on Varga's influence to the field through his work, his mentoring, his tireless editorial work, and many international collaborations, for almost fifty years. During the conference it was announced that *Linear Algebra and its Applications* will have a special issue dedicated to Richard Varga's eightieth birthday, to be published, of course, in three years time.

Several lectures were devoted to the topic of H-matrices, diagonally dominant matrices, and eigenvalue localization. This was natural, one year after the publication of Varga's latest book Geršgorin and his circles. There were of course many other topics discussed exposing the audience to the breadth and health of numerical linear algebra today, with many references to Varga's work.

The conference was the brainchild of Ljiljana Cvetković of the University of Novi Sad, who, together with a wonderful group of local organizers, made sure everyone enjoyed their stay.

There was for example an evening celebration which included a large cake with seventy seven candles. Richard Varga made his secret wish and blew them off in one sweep. This was followed by a dance, accompanied by wonderful live music. The dance lasted for many hours, but everyone recovered by the time of the morning lecture.

Everyone thanked Prof. Cvetković and her colleagues for their warm hospitality which, together with the scientific program,

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made the meeting a great success.

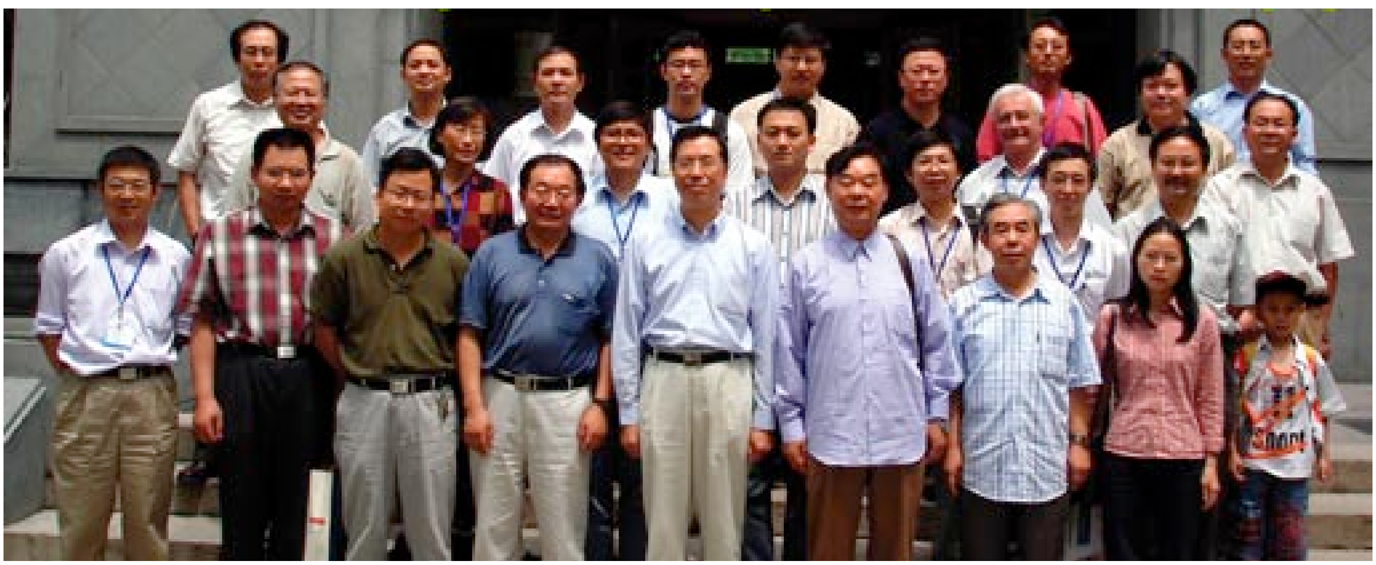
IMAGE readers can get a glimpse of this by looking at the program or the conference pictures at the conference web site: http://ala2005.ns.ac.yu. They can also read selected papers related to the conference in the forthcoming special issue of *Numerical Algorithms*, which is being edited by Michele Benzi, Ljiljana Cvetković, Michel Neumann, and Tomasz Szulc.

1st International Workshop on Matrix Analysis and Applications

report by **Fuzhen Zhang**

The 1st International Workshop on Matrix Analysis and Applications was held June 9-10, 2005 at Beijing Normal University, Beijing, China. Twenty five people from Canada, China, the Czech Republic and the United States participated in the informal two-day workshop and contributed sixteen talks. The Organizing Committee consisted of Tiangang Lei (National Natural Science Foundation of China), Xiuping Zhang (Beijing Normal University) and Fuzhen Zhang (Nova Southeastern University, USA).

The conference was generously supported by the Math Department of Beijing Normal University which provided free delicious meals to all participants for two days, including a banquet dinner. Some participants greatly enjoyed sightseeing of the Great Wall, Forbidden City, Summer Palace, and Shi-Cha-Hai Lake, etc.



Conference Photograph

ILAS-NET

ILAS operates ILAS-NET, an electronic news service that transmits announcements of ILAS activities and circulates other notices of interest to linear algebraists. Announcements for ILAS-NET or requests to be on the mailing list for ILAS-NET, should be sent to Shaun Fallat (sfallat@math.uregina.ca). Subscription to ILAS-NET is independent of membership in ILAS and is free.

Call for Submissions to *IMAGE*

IMAGE welcomes expository articles on emerging applications and topics in Linear Algebra, announcements of upcoming meetings, reports on past conferences, historical essays on linear algebra, book reviews, essays on the development of Linear Algebra in a certain country or region, and letters to the editor or signed columns of opinion. Contributions for *IMAGE* should be sent to either Bryan Shader (bshader@uwyo.edu) or Hans Joachim Werner (hjw.de@uni-bonn.de). The deadlines are October 15 for the fall issue, and April 15 for the spring issue.

Forthcoming Conferences and Workshops in Linear Algebra

Workshop on the Teaching of Linear Algebra

Philadelphia, Pennsylvania

25 March 2006

A workshop on the teaching of Linear Algebra will be held at Drexel University on Saturday March 25, 2006. See

`http://www.drexel.edu/coas/math/workshop/`

for more details and updated information of the workshop.

Invited speakers are: Robert Busby (Drexel University), Peter Lax (New York University), Gilbert Strang (Massachusetts Institute of Technology) and Frank Uhlig (Auburn University).

In addition, there will be a panel discussion with moderator David Lay (University of Maryland) and panel members Jane Day, (San Jose University), Guershon Harel (University of California, San Diego), David Hill (Temple University) and Steven Leon (University of Massachusetts, Dartmouth)

There is a small registration fee of $15 for the workshop. There is an opportunity for contributed talks and posters. Please follow the instructions on the workshop website.

The organizers are: Herman Gollwitzer (Drexel University, hgollwit@math.drexel.edu) and Hugo Woerdeman (Drexel University, hugo@math.drexel.edu).

Graduate Student Combinatorics Conference

Madison, Wisconsin

22-23, April 2006

The 2006 meeting of the Graduate Student Combinatorics Conference will be hosted by the University of Wisconsin–Madison April 22-23, 2006 in Madison, Wisconsin.

The purpose of this conference is to bring together graduate students in combinatorics, let them practice giving talks, learn about new topics, and get to know other graduate students in their field.

The conference will feature contributed 20-minute talks from graduate students, two keynote addresses by Professor Doron Zeilberger of Rutgers, and a social program. More information and an online registration form are available at the conference's website at

`http://www.math.wisc.edu/~gscc06`.

Conference organizers are Adam Berliner (University of Wisconsin), Louis Deaett (University of Wisconsin), and Dimitrije Kostic (Texas A & M University). To contact the organizers with any questions or suggestions, send an email to gscc06@math.wisc.edu.

Western Canada Linear Algebra Meeting

Victoria, British Columbia, Canada

23-24 June 2006

The Western Canada Linear Algebra Meeting (W-CLAM) provides an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While, the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting. Previous W-CLAMs were held in Regina (1993), Lethbridge (1995), Kananaskis (1996), Victoria (1998), Winnipeg (2000) and Regina (2002).

For W-CLAM 2006, invited speakers include Richard Brualdi (Univeristy of Wisconsin-Madison) and Mark Lewis (University of Alberta).

W-CLAM 2006 is partially funded by the Pacific Institute for Mathematical Sciences, and is being held just prior to the SIAM Conference on Discrete Mathematics, which will be in Victoria, June 25-28, 2006.

A website with further information will be available later. If you wish to be included on an email list for further information on this meeting, please contact Dale Olesky (dolesky@cs.uvic.ca)

The 15th International Workshop on Matrices and Statistics

IWMS-2006: Uppsala (Sweden)

13-17 June 2006

The 15th International Workshop on Matrices and Statistics (IWMS-2006) will be held at the University of Uppsala (Uppsala, Sweden) on June 13-17, 2006. This Workshop will be hosted by the Mathematics and Information Technology Centre at Uppsala University. The purpose of this Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory.

Additional emphasis will be put on related numerical linear algebra issues and numerical solution methods, relevant to problems arising in statistics. This Workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in matrix theory and may exchange ideas with researchers from a wide variety of countries.

The Scientific Organizing Committee (SOC) for this Workshop comprises R. William Farebrother (Shrewsbury, England, UK), Augustyn Markiewicz (Poznań, Poland), Simo Puntanen (Tampere, Finland), Dietrich von Rosen (Uppsala, Sweden), George P. H. Styan (Montréal, Québec, Canada), and Hans Joachim Werner (chair; Bonn, Germany, {hjw.de@uni-bonn.de).

The Local Organizing Committee (LOC) consists of Zhanna Adrushchenko (Uppsala), Johannes Forkman (Uppsala), Tatjana Nahtman (Tartu, Estonia), Maya Neytcheva (Uppsala), and Razaw Al Sarraj (Uppsala) and will be chaired by Dietrich von Rosen (Uppsala, Sweden, Dietrich.von.Rosen@bt.slu.se).

This Workshop will include the presentation of both invited and contributed papers on matrices and statistics. Invited Speakers include Theodore W. Anderson (USA), Åke Björck (Sweden), Gene H. Golub (USA), David Harville (USA), Sabine van Huffel (Belgium), Jeffrey J. Hunter (New Zealand), Ingram Olkin (USA), Friedrich Pukelsheim (Germany), Youseff Saad (USA), and Muni Srivastava (Canada). There will also special sessions for papers presented by Ph.D. students. In addition a special session in Honour of Dr. Tarmo Pukkila's 60th Birthday will be organized and chaired by Erkki Liski.

If you wish to present a talk, please submit the title and the abstract to the conference secretariat by February 28, 2006. All abstracts should be no more than 300 words summarizing the paper, include the title of the talk, keywords and the names, affiliation, addresses and email addresses of the authors. Electronic submissions in LaTeX or plain text format are preferred. All abstracts will be refereed by the Organizing Committee.

Several social activities for all participants including a conference dinner and a get-together event will be arranged. Uppsala has a lot to offer, among others, ancient monuments, a cat without a tail, churches, memories of Linnaeus, castles, gardens, etc.

Deadlines are: Abstract Submission (February 28, 2006), Early-bird registration (February 28, 2006), and Registration (April 23, 2006).

Further information about Uppsala, travel and lodging arrangements can be found at the Workshop web page:
http://www.bt.slu.se/iwms2006/iwms06.html

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This Workshop which will be the fifteenth in a series but the first one in Sweden is supported by the Centre of Biostochastics, the Scandinavian Airlines (SAS), the Swedish Research Council, the Swedish Statistical Association, the Swedish Research Council for Environment, Agricultural Sciences & Spatial Planning, and by John Wiley & Sons, Ltd. It is also endorsed by the International Linear Algebra Society.

You might like to combine your participation in IWMS-2006 with the 5th International Conference on Probability and Statistics which will be held at Smolenice Castle, Slovakia (June 5–9, 2006), and/or with the 13th ILAS Conference which will be held in Amsterdam, The Netherlands (July 18–21, 2006).

8th Workshop on Numerical Ranges and Radii

Bremen, Germany
15-16 July 2006

The 8th Workshop on Numerical Ranges and Numerical Radii (WONRA) will take place at Universität Bremen, Germany from July 15 to July 16, 2006. See
http://www.math.uni-bremen.de/aag/wonra06
for more details and updated information of the workshop.The purpose of the workshop is to stimulate research and foster interaction of researchers interested in the subject. The informal workshop atmosphere will guarantee the exchange of ideas from different research areas and, hopefully, the participants will leave informed of the latest developments and newest ideas. One may visit the WONRA website
http://www.math.wm.edu/~ckli/wonra.html
for some background about the subject and previous meetings. The workshop is endorsed and sponsored by ILAS, and Professor Man-Duen Choi will be the ILAS Lecturer.

There will be no registration fees for the workshop. The 8th WONRA will be run in conjunction with the 13th International Linear Algebra Society (ILAS) Conference at Amsterdam (July 18 -21, 2006).

People who are interested in participating or giving a talk at the 8th WONRA should contact the organizers: Chi-Kwong Li (College of William and Mary, Williamsburg, VA, USA, ckli@math.wm.edu), Leiba Rodman (College of William and Mary, Williamsburg, VA, USA, lxrodm@math.wm.edu), and Christiane Tretter (Universität Bremen, ctretter@math.uni-bremen.de). A special issue of *Linear and Multilinear Algebra* will be devoted to the meeting. The organizers of the workshop will be the special editors.

13th ILAS Conference

Amsterdam, The Netherlands
18-21 July, 2006

From July 18 to 21, 2006 the 13th ILAS conference will be held in Amsterdam. This is in the week before the GAMM-SIAM conference on Linear Algebra in Dusseldorf, Germany. The ILAS will also run in conjunction with the 8th WONRA Workshop on Numerical Ranges and Numerical Radii which will take place at Universitat Bremen, Germany, on July 15–16, 2006.

The conference will be organized at the Vrije Universiteit, located in the southern part of the city of Amsterdam, the capital of the Netherlands.

The conference will be mainly structured around a number of themes. For each of those themes an invited lecture will be combined with a mini-symposium. Themes selected so far include:Linear Algebra in Statistics, Numerical Linear Algebra, Matrices in Indefinite Scalar Product Spaces, Structured Matrices, and Positive Linear Algebra.

In addition, there will be the possibility for participants to present their work at the conference, even if it does not fall under one of the special themes.

The registration fee is 140 euros if paid before April 1, 2006, by bank transfer. Otherwise, you pay 160 euros, which then can also be paid cash on arrival. There is no possibility to pay with credit cards!

The conference dinner is 30 euros and an additional 50 euros for each accompanying person. For registration for ILAS 2006 and conference dinner (including bank details) abstract submission, and application for financial support, see the conference website:

http://staff.science.uva.nl/~brandts/ILAS06/

The most important deadlines and dates are: February 12 (Application for financial support) March 1(Decision on financial support applications), April 1 (Title and abstract submission), April 1 (Early registration of 140 euros by bank transfer), July 1 (Late registration of 160 euros by bank transfer), and July 17 (Late registration fee of 160 euros cash).

The city of Amsterdam can be reached easily from around the globe. The main airport of the Netherlands, Schiphol airport, is only a short train ride away from both the city centre and the Vrije Universiteit. In addition, Amsterdam can be reached from the major cities in Europe easily by train. Once inside the city, the excellent network of public transportation will guarantee you an easy trip to your hotel and the conference location.Participants are expected to make their own arrangements for accomodation. This can be done easily online.

The proceedings will appear as a volume of *Linear Algebra and its Applications*. Editors for the volume are Harm Bart, Jan Brandts, Andre Ran and Paul Van Dooren.

GAMM-SIAM Conference on Applied Linear Algebra

Düsseldorf, Germany

24-27 July 2006

The Joint GAMM-SIAM Conference on Applied Linear Algebra organized in cooperation with ILAS will be held at the University of Düsseldorf, Germany, from July 24-27, 2006. This conference takes place the week following the 13th ILAS Conference, Amsterdam, July 18-21.

Linear algebra problems and linear algebra algorithms for their solution are at the very heart of almost all numerical computations and play a prominent role in modern simulation methods in science and engineering.

This conference, which belongs to a series of triannual meetings organized by SIAM in the US, is the premier international conference on applied linear algebra. We expect about 250 participants coming from countries all over the world, working in academia, research labs or industry. The conference is organized jointly by Heinrich-Heine-Universität Düsseldorf (Marlis Hochbruck) and Bergische Universität Wuppertal (Andreas Frommer, Bruno Lang).

Participants will present and discuss their latest results in the area of applied linear algebra, ranging from advances in the theory over the development and analysis of new precise and efficient algorithms to large scale supercomputer applications.

The Program Committee consists of: Michele Benzi, (Atlanta), Zlatko Drmac (Zagreb), Heike Fassbender, (Braunschweig), Sven Hammarling, (Oxford), Daniel Hershkowitz (Haifa), Ilse Ipsen (Raleigh), Bo Kagstrom (Umea), Steve Kirkland (Regina), Rich Lehoucq, (Albuquerque), Volker Mehrmann (Berlin), Julio Moro (Madrid), Jim Nagy (Atlanta), and Paul Van Dooren (Louvain-la-Neuve).

Additional details can be found at the conference website: http://www.ala2006.de/

2nd International Workshop on Matrix Analysis and Applications

Fort Lauderdale, Florida, USA

15-16 December, 2006

This two-day workshop aims to stimulate research and interaction of mathematicians in all aspects of linear algebra and matrix analysis and their applications and to provide an opportunity for researchers to exchange ideas and recent developments on the subjects.

The meeting is sponsored by Nova Southeastern University and the International Linear Algebra Society (ILAS). Richard Brualdi, University of Wisconsin–Madison, will be the ILAS sponsored lecturer.

Online registrations will be accepted until Dec. 8, 2006. A registration fee of $50 will be charged to cover admission tickets and bus transportation for an excursion of tropical Florida. Titles and abstracts should be submitted to Chi-Kwong Li no later than Dec. 8, 2006 by e-mail in LaTeX/TeX . Detailed information about transportation and accommodation will be available via web or e-mail.

The organizing committee consists of Zhong-Zhi Bai (Chinese Academy of Sciences, bzz@lsec.cc.ac.cn), Chi-Kwong Li, (College of William and Mary, ckli@math.wm.edu), Bryan L Shader (University of Wyoming, bshader@uwyo.edu), Hugo Woerdeman (Drexel University, hugo@math.drexel.edu), Fuzhen Zhang (Chair) (Nova Southeastern University, zhang@nova.edu), and Qingling Zhang (Northeastern University, qlzhang@mail.new.edu.cn).

A special issue, Matrix Analysis and Applications, of the *International J. of Control and Information Sciences* will be devoted to the meeting, and the editors are Dennis Bernstein (University of Michigan-Ann Harbor), Fuzhen Zhang (Nova Southeastern University), Hugo Woerdeman (Drexel University) and Qingling Zhang (Northeastern University).

IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 34-1 through 34-10 [IMAGE 34 (Spring 2005), pp. 40 & 39]. Problems 30-3 [IMAGE 30 (April 2003), p. 36] and 32-4 [IMAGE 32 (April 2004), p. 40] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 9 new problems on pp. 44 & 43 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to hjw.de@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \dots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

While we have received one belated solution to this problem which will be considered for possible publication in IMAGE 36, we also look forward to receiving some further solutions to Problem 30-3!

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j}X_1 + u_{2j}X_2 + u_{3j}X_3$, $j = 1, 2, 3$, such that each of the six matrices $W_j^{-1}W_k$, $j \neq k$, has zero trace. Equivalently, we want $(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We still look forward to receiving solutions to Problem 32-4!

Problem 34-1: A Well-Known Matrix Equation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let V and W be given $n \times n$ and $m \times m$ positive semidefinite matrices and let $X,\ y$, and z be given $n \times m,\ n \times 1$ and $m \times 1$ matrices. Then the system of $n + m$ equations in $n + m$ unknowns

$$\begin{pmatrix} V & X \\ X' & W \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}$$

is well-known in Mathematical Statistics when $W = 0$ and $z = 0$ are both null. Readers are invited to identify at least one other well-known instance of this matrix equation.

Solution 34-1.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:*

R.W.Farebrother@Manchester.ac.uk

Consider the problem of estimating $z'\beta$ in the general linear statistical model

$$u = X\beta + \varepsilon, \quad E(\varepsilon) = 0, \quad E(\varepsilon\varepsilon') = \sigma^2 V,$$

where u, z, β and ε are $n \times 1,\ m \times 1,\ m \times 1$ and $n \times 1$ matrices and σ^2 is a positive constant.

Let a be an $n \times 1$ matrix. Then $a'u$ is a linear unbiased estimator of $z'\beta$ with mean $a'X\beta$ and variance $\sigma^2 a'Va$ provided $a'X\beta = z'\beta$ for all values of β. Thus, our problem becomes one of minimizing the scaled variance $a'Va$ subject to $X'a = z$.

Formulating the Lagrangian function for this problem

$$L = \frac{1}{2}a'Va + b'(X'a - z)$$

where b is an $m \times 1$ matrix of Lagrange coefficients, and differentiating L with respect to a and b, we have the first order conditions $Va + Xb = 0$ and $X'a = z$ which may be written in the required form with $W = 0$ and $y = 0$.

Problem 34-2: Eigenvalues of a Class of Tridiagonal Matrices

Proposed by Steven J. LEON, *University of Massachusetts Dartmouth, Dartmouth, Massachusetts, USA:* sleon@umassd.edu

Let α and β be real scalars and let $A_n(\alpha,\beta)$ denote the $n \times n$ tridiagonal matrix whose entries on the main diagonal are all equal to α and whose entries on the other two diagonals are all equal to β except that the $(1,2)$ entry is 2β. For example,

$$A_4(\alpha,\beta) = \begin{pmatrix} \alpha & 2\beta & 0 & 0 \\ \beta & \alpha & \beta & 0 \\ 0 & \beta & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}.$$

Show that the eigenvalues of $A_n(\alpha,\beta)$ are

$$\lambda_j = \alpha + 2\beta\cos\frac{(2j-1)\pi}{2n}, \quad j = 1,\ldots n.$$

Solution 34-2.1 by the Proposer Steven J. LEON, *University of Massachusetts Dartmouth, Dartmouth, Massachusetts, USA:*
sleon@umassd.edu

Since $A_n(\alpha,\beta) = \alpha I + 2\beta A(0,\frac{1}{2})$ it suffices to show that the eigenvalues of $A(0,\frac{1}{2})$ are $\lambda_j = \cos\frac{(2j-1)\pi}{2n}$, $j = 1,\ldots n$. These λ_j's are just the roots of the nth degree Chebyshev polynomial T_n. This follows from the well known property of Chebyshev polynomials that $T_n(\cos\theta) = \cos n\theta$. This property can also be used to show that the Chebyshev polynomials satisfy the recursion relations $T_1(x) = xT_0(x)$ and $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$ (for k ≥ 1). If λ is any root of T_n and we substitute $x = \lambda$ into the first n of the recursion equations, we end up with a system that can be written in matrix form

$$\begin{pmatrix} 0 & 1 & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} T_0(\lambda) \\ T_1(\lambda) \\ \vdots \\ T_{n-2}(\lambda) \\ T_{n-1}(\lambda) \end{pmatrix} = \lambda \begin{pmatrix} T_0(\lambda) \\ T_1(\lambda) \\ \vdots \\ T_{n-2}(\lambda) \\ T_{n-1}(\lambda) \end{pmatrix}.$$

It follows then that the eigenvalues of $A_n(0,\frac{1}{2})$ are the roots of T_n.

Solution 34-2.2 by Andres SAEZ-SCHWEDT, *Departamento de Matematicas Universidad de Leon, Leon, España:*
demass@unileon.es

We want to compute the roots of the polynomial $f_n(x) = \det(A_n(\alpha,\beta) - xI)$, for example:

$$f_4(x) = \det\begin{pmatrix} \alpha - x & 2\beta & 0 & 0 \\ \beta & \alpha - x & \beta & 0 \\ 0 & \beta & \alpha - x & \beta \\ 0 & 0 & \beta & \alpha - x \end{pmatrix}.$$

The first values of $f_n(x)$ are $f_2(x) = (\alpha - x)^2 - 2\beta^2$, $f_3(x) = (\alpha - x)^3 - 3\beta^2(\alpha - x)$, and for $n \geq 4$ we obtain a recurrence relation by expanding the determinant $f_n(x)$ by the last column:

$$f_n(x) = (\alpha - x)f_{n-1}(x) - \beta^2 f_{n-2}(x). \tag{1}$$

For convenience, we also define $f_0(x) = 2$ and $f_1(x) = \alpha - x$, so that the previous recurrence still holds for $n = 2$ and $n = 3$.

On the other hand, for any $\gamma_j = \frac{(2j-1)\pi}{2n}$, $j = 1 \dots n$, one has $\cos(n\gamma_j) = 0$. But $\cos(n\gamma_j) = T_n(\cos(\gamma_j))$, where T_n is the Chebyshev polynomial of degree n given by $T_0(x) = 1, T_1(x) = x$ and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{2}$$

for $n \geq 2$. The elements $\{\cos(\gamma_j)\}$ are exactly the n roots of the equation $T_n(x) = 0$, therefore the n roots of $T_n(\frac{x-\alpha}{2\beta})$ are precisely the elements $\{\lambda_j = \alpha + 2\beta\cos(\gamma_j)\}$. If $\beta = 0$ the proposed problem is trivial. Otherwise, let us prove the following formula:

$$f_n(x) = 2(-\beta)^n T_n(\frac{x-\alpha}{2\beta}). \tag{3}$$

The assertion is certainly true for $n = 0, 1$, since $f_0(x) = 2$ and $f_1(x) = (\alpha - x) = -2\beta(\frac{x-\alpha}{2\beta})$. Now suppose that (3) is valid for $n-2$ and $n-1$. Substituting in the recurrence (1) we obtain:

$$f_n(x) = (\alpha - x)\cdot 2(-\beta)^{n-1}T_{n-1}(\frac{x-\alpha}{2\beta}) - \beta^2\cdot 2(-\beta)^{n-2}T_{n-2}(\frac{x-\alpha}{2\beta}) = 2(-\beta)^n\left(2(\frac{x-\alpha}{2\beta})T_{n-1}(\frac{x-\alpha}{2\beta}) - T_{n-2}(\frac{x-\alpha}{2\beta})\right),$$

which by (2) is equal to $2(-\beta)^n T_n\left(\frac{x-\alpha}{2\beta}\right)$, completing the proof of the claim (3). As $\beta \neq 0$, $f_n(x)$ and $T_n\left(\frac{x-\alpha}{2\beta}\right)$ have the same roots, which are the elements $\{\lambda_j = \alpha + 2\beta\cos(\gamma_j)\}$, as we wanted to prove.

Solution 34-2.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

The following argument is valid for complex α and β.

A nonzero vector $x = [x_1\ x_2\ \cdots\ x_n]^T$ is a λ-eigenvector of $A_n(\alpha, \beta)$ if and only if

$$\beta x_{r-1} + (\alpha - \lambda)x_r + \beta x_{r+1} = 0, \quad 1 \leq r \leq n, \tag{4}$$

with $x_0 = x_2$ and $x_{n+1} = 0$. Let

$$\beta z^2 + (\alpha - \lambda)z + \beta = \beta(z - \zeta)(z - 1/\zeta). \tag{5}$$

The solutions of (4) such that $x_{n+1} = 0$ are of the form

$$x_r = c(\zeta^{r-n-1} - \zeta^{-r+n+1}), \quad 1 \leq r \leq n.$$

The condition $x_0 = x_2$ dictates that $\zeta^{-n-1} - \zeta^{n+1} = \zeta^{-n+1} - \zeta^{n-1}$. With $\zeta = e^{i\theta}$, this is equivalent to $\sin(n+1)\theta = \sin(n-1)\theta$, which in turn is equivalent to $\cos n\theta \sin\theta = 0$. Letting $\theta_j = (2j-1)\pi/2n$ and $c = 1/2i$ yields the eigenvector $x_j = [x_{1j}\ x_{2j}\ \cdots\ x_{nj}]^T$ with

$$x_{rj} = \sin\frac{(2j-1)(r-n-1)\pi}{2n}, \quad 1 \leq r \leq n, \quad 1 \leq j \leq n.$$

From (5), $\lambda = \alpha + \beta(\zeta + 1/\zeta)$. Setting $\zeta = e^{i\theta_j}$ yields the eigenvalue λ_j.

Now let

$$B_4(\alpha,\beta) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & \beta & 0 \\ 0 & \beta & \alpha & \beta \\ 0 & 0 & 2\beta & \alpha \end{pmatrix} \quad \text{and} \quad C_4(\alpha,\beta) = \begin{pmatrix} \alpha & 2\beta & 0 & 0 \\ \beta & \alpha & \beta & 0 \\ 0 & \beta & \alpha & \beta \\ 0 & 0 & 2\beta & \alpha \end{pmatrix},$$

and let $B_n(\alpha,\beta)$ and $C_n(\alpha,\beta)$ be their obvious generalizations. The components of the eigenvectors of these matrices satisfy (4) with $x_0 = 0$, $x_{n-1} = x_{n+1}$ for $B_n(a,b)$ and $x_0 = x_2$, $x_{n+1} = x_{n-1}$ for $C_n(\alpha,\beta)$.

An argument similar to the one used for $A_n(\alpha,\beta)$ shows that $B_n(\alpha,\beta)$ has the same eigenvalues $\lambda_1, \dots, \lambda_n$ as $A_n(\alpha,\beta)$, with associated eigenvectors vector $x_j = [x_{1j}\ x_{2j}\ \cdots\ x_{nj}]^T$ defined by

$$x_{rj} = \sin\frac{(2j-1)r\pi}{2n}, \quad 1 \leq r \leq n, \quad 1 \leq j \leq n.$$

This conclusion can also be obtained by noting that $B_n(\alpha,\beta) = J_n A_n(\alpha,\beta) J_n$, where J_n is the $n \times n$ flip matrix; i.e., $J_n = [\delta_{i,n-j+1}]_{i,j=1}^n$.

By inspection, $\lambda_1 = \alpha + 2\beta$ and $\lambda_n = \alpha - 2\beta$ are eigenvalues of $C_n(\alpha,\beta)$ with associated eigenvectors $x_1 = [x_{11}\, x_{21}\, \cdots\, x_{n1}]^T$ and $x_n = [x_{1n}\, x_{2n}\, \cdots\, x_{nn}]^T$ given by $x_{r1} = 1$ and $x_{rn} = (-1)^r$, $1 \le r \le n$. The components of the other eigenvectors of $C_n(\alpha,\beta)$ are of the form $x_r = c_1\zeta^r + c_2\zeta^{-r}$, where

$$c_1(1-\zeta^2) + c_2(1-\zeta^{-2}) = 0, \quad c_1(1-\zeta^2)\zeta^{n-1} + c_2(1-\zeta^{-2})\zeta^{-n+1} = 0, \tag{6}$$

which has a nontrivial solution with $\zeta \ne \pm 1$ if and only if

$$\zeta = \zeta_j = e^{2i(j-1)\pi/(2n-2)}, \quad 2 \le j \le n-1. \tag{7}$$

Thus, the eigenvalues are

$$\lambda_j = \alpha + 2\beta \cos\frac{2(j-1)\pi}{2(n-1)}, \quad j = 1, \ldots, n.$$

With ζ as in (7), (6) has the solution $c_1 = (1-\zeta_j^{-2})/2i$, $c_2 = -(1-\zeta_j^2)/2i$, which yields the eigenvector $x_j = [x_{1j}\, x_{2j}\, \cdots\, x_{nj}]^T$ with

$$x_{rj} = \sin\frac{2(j-1)r\pi}{2(n-1)} - \sin\frac{2(j-1)(r-2)\pi}{2(n-1)}, \quad 1 \le r \le n, \quad 2 \le j \le n-1.$$

Solutions to Problem 34-2 were also received from Carlos Martins da Fonseca and from Nathan Krislock, Veronica Piccialli & Henry Wolcowicz.

Problem 34-3: On the Spectral Radius

Proposed by Chi-Kwong LI and Sebastian J. SCHREIBER, *The College of William & Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu; sjschr@wm.edu

Suppose $A \in M_m$, $B \in M_n$ and R is an $m \times n$ matrix such that $AR = RB$.

(a) Suppose R has full column rank. Show that the spectrum of B is a subset of that of A, and hence the spectral radius of B is not larger than that of A.

(b) If A and B are nonnegative, and if R has no zero rows or zero columns, show that A and B have the same spectral radius.

Solution 34-3.1 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We first prove a slightly more general result than (a). Then we show that claim (b) is actually incorrect. And finally, we give a corrected version of (b).

THEOREM 1. *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $R \in \mathbb{C}^{m\times n}$ be such that $AR = RB$. If $[\mathcal{R}(B) + \mathcal{N}(B)] \cap \mathcal{N}(R) = \{0\}$, then $\sigma(B) \subseteq \sigma(A)$ and so, in particular, $\rho(B) \le \rho(A)$, with $\mathcal{R}(\cdot)$, $\mathcal{N}(\cdot)$, $\sigma(\cdot)$, and $\rho(\cdot)$ denoting the range (column space), the null space, the spectrum, and the spectral radius, respectively, of the matrix $(\cdot)$.*

PROOF: Let (λ, x) be any eigenpair of B. Then $Bx = \lambda x$ and $x \ne 0$. Needless to say, $x \in \mathcal{N}(B)$ when $\lambda = 0$, and $x \in \mathcal{R}(B)$ when $0 \ne \lambda \in \sigma(B)$. In any case, by means of $[\mathcal{R}(B) + \mathcal{N}(B)] \cap \mathcal{N}(R) = \{0\}$, therefore $Rx \ne 0$. From $ARx = RBx = \lambda Rx$ it now follows that (λ, Rx) is an eigenpair of A. Consequently, $\sigma(B) \subseteq \sigma(A)$ and so, as claimed, $\rho(B) \le \rho(A)$. □

Our result deserves some further comments. First of all, we emphasize that if R is of full column rank or, equivalently, if $\mathcal{N}(R) = \{0\}$, then trivially $[\mathcal{N}(B) + \mathcal{R}(B)] \cap \mathcal{N}(R) = \{0\}$. Since the converse implication, however, does not always hold, it is clear that our theorem is indeed a generalization of claim (a). Secondly, we mention that if our condition is replaced by the less restrictive one $\mathcal{R}(B) \cap \mathcal{N}(R) = \{0\}$, then as in the preceding proof it follows that $(\sigma(B)\backslash\{0\}) \subseteq \sigma(A)$ and so even then $\rho(B) \le \rho(A)$ still holds true.

That claim (b) of Problem 34-3 is incorrect is seen as follows. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \quad B = (1), \quad R = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Although $AR = RB$, $A \geq 0$, $B \geq 0$ and R has no zero rows or zero columns, we have $3 = \rho(A) \neq \rho(B) = 1$, contrary to the assertion. It is now interesting to ask for conditions on A, B and R which are sufficient for $\rho(B) = \rho(A)$ to hold. An answer is given by our concluding theorem.

THEOREM 2. *Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$ be such that $AR = RB$. If A, B and R are nonnegative matrices and RB is a positive matrix, then $\rho(B) = \rho(A)$.*

PROOF: The generalization of Perron's theorem to nonnegative matrices tells us that if $M \geq 0$, then $\rho(M) \in \sigma(M)$, and there exists an associated eigenvector $x \geq 0$ such that $Mx = \rho(M)x$; cf. Meyer (2000, p. 670). It is also well known that $\sigma(M) = \sigma(M')$, with M' denoting the transpose of M; cf. Meyer (2000, p. 503). Since $B \geq 0$ and $A \geq 0$, therefore $Bx = \rho(B)x$ for some $0 \neq x \geq 0$ and likewise $A'y = \rho(A)y$ for some $0 \neq y \geq 0$. Consequently, in virtue of $AR = RB > 0$, $ARx = RBx > 0$. Hence, $ARx = \rho(B)Rx > 0$ and so $\rho(B) > 0$ and $Rx > 0$. Moreover, $y'ARx = \rho(B)y'Rx > 0$. On the other hand, also $y'ARx = \rho(A)y'Rx$. Therefore, $\rho(A)y'Rx = \rho(B)y'Rx$, which in turn can happen only if $\rho(A) = \rho(B)$. □

Reference

C. D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia.

A belated solution to Problem 33-3 was also received from the Proposers Chi-Kwong Li & Sebastian J. Schreiber.

Problem 34-4: A Range Equality for the Commutator with Two Involutory Matrices

Proposed by Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Suppose that A and B are both involutory matrices of the same order, that is, $A^2 = B^2 = I$, where I is the identity matrix. Show that

$$\mathrm{range}(AB - BA) = \mathrm{range}(A - B) \cap \mathrm{range}(A + B).$$

Solution 34-4.1 by Andres SAEZ-SCHWEDT, *Universidad de Leon, Leon, España:* demass@unileon.es

First, it is easy to see that (i) $(A - B)(A + B) = AB - BA = (A + B)(B - A)$ and (ii) $A(AB - BA) = (BA - AB)A = -(AB - BA)A$. From (i), we see that any element in the range of $AB - BA$ is both in the range of $A - B$ and in the range of $A + B$. Conversely, let $v = (A + B)x = (A - B)y$ be a vector in $\mathrm{range}(A - B) \cap \mathrm{range}(A + B)$. By (i), we have that $(A - B)v = (AB - BA)x$ and $(A + B)v = -(AB - BA)y$. Adding both equations we obtain $2Av = (AB - BA)(x - y)$, and hence, using (ii), we see that

$$v = A^2 v = \frac{1}{2}A \cdot 2Av = \frac{1}{2}A(AB - BA)(x - y) = -\frac{1}{2}(AB - BA)A(x - y) = (AB - BA)A\left(\frac{y - x}{2}\right),$$

which lies in the range of $AB - BA$.

Solution 34-4.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

For a complex matrix A, let A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote, as usual, the conjugate transpose, the range (column space), and the null space, respectively, of A. By I_n we denote the identity matrix of order n.

Our solution to Problem 34-4 offers some further insights into the theory of idempotent matrices and makes use of the next three results.

THEOREM 1. *The matrix $R \in \mathbb{C}^{n\times n}$ is an involutory matrix, i.e., $R^2 = I_n$, if and only if there exists an idempotent matrix $P \in \mathbb{C}^{n\times n}$ such that $R = I_n - 2P$.*

PROOF. If R is involutory, then $(I_n - R)^2 = 2(I_n - R)$. For $P := (I_n - R)/2$, thus $P^2 = P$ and $R = I - 2P$, so that the proof of necessity is complete. Conversely, let $R = I_n - 2P$ for some idempotent matrix P. Then $R^2 = (I_n - 2P)^2 = I_n - 4P + 4P = I_n$, i.e., R is indeed involutory, as claimed. □

THEOREM 2. *Let $P, Q \in \mathbb{C}^{n\times n}$ be idempotent matrices. Then:*

(i) $\mathcal{N}(P - Q) = [\mathcal{R}(P) \cap \mathcal{R}(Q)] \oplus [\mathcal{N}(P) \cap \mathcal{N}(Q)]$, $\mathcal{N}(I_n - P - Q) = [\mathcal{N}(P) \cap \mathcal{R}(Q)] \oplus [\mathcal{R}(P) \cap \mathcal{N}(Q)]$, *with '$\oplus$' indicating a direct sum.*

(ii) $\mathcal{N}(P)\cap\mathcal{N}(PQ)=[\mathcal{N}(P)\cap\mathcal{R}(Q)]\oplus[\mathcal{N}(P)\cap\mathcal{N}(Q)]$, $\mathcal{R}(P)\cap\mathcal{N}((I_n-P)Q)=[\mathcal{R}(P)\cap\mathcal{R}(Q)]\oplus[\mathcal{R}(P)\cap\mathcal{N}(Q)]$.

(iii) $\mathcal{N}(PQ-QP)=[\mathcal{R}(Q)\cap\mathcal{N}(P)]\oplus[\mathcal{N}(Q)\cap\mathcal{N}(P)]\oplus[\mathcal{R}(Q)\cap\mathcal{R}(P)]\oplus[\mathcal{N}(Q)\cap\mathcal{R}(P)]$.

PROOF. It is known that if P is idempotent, then I_n-P is idempotent, $\mathcal{R}(I_n-P)=\mathcal{N}(P)$, $\mathcal{N}(I_n-P)=\mathcal{R}(P)$, and $\mathbb{C}^n=\mathcal{R}(P)\oplus\mathcal{N}(P)$; cf. Lancaster (1969, p. 83). Any $x\in\mathbb{C}^n$ can thus be uniquely written as $x=x_1+x_2$, where $x_1\in\mathcal{R}(P)$ and $x_2\in\mathcal{N}(P)$. We then obtain $x=x_1+x_2\in\mathcal{N}(P-Q)\Leftrightarrow x_1=Qx_1+Qx_2\Leftrightarrow(I_n-Q)x_1=Qx_2\Leftrightarrow(I_n-Q)x_1=0$ & $Qx_2=0\Leftrightarrow x_1\in\mathcal{R}(P)\cap\mathcal{R}(Q)$ & $x_2\in\mathcal{N}(P)\cap\mathcal{N}(Q)$, and so the first claim of (i) is established. Since I_n-P is also idempotent, the second result of (i) follows from the first result of (i) just by replacing P by $I-P$. For proving (ii), consider $z\in\mathcal{N}(P)$. In view of $\mathbb{C}^n=\mathcal{R}(Q)\oplus\mathcal{N}(Q)$, there uniquely exist $y_1\in\mathcal{R}(Q)$ and $y_2\in\mathcal{N}(Q)$ such that $z=y_1+y_2$. Hence $z=y_1+y_2\in\mathcal{N}(PQ)\Leftrightarrow PQz=0\Leftrightarrow Py_1=0\Leftrightarrow y_1\in\mathcal{N}(P)\cap\mathcal{R}(Q)$, and so $z\in\mathcal{N}(P)\cap\mathcal{N}(PQ)$ if and only if $z\in[\mathcal{N}(P)\cap\mathcal{R}(Q)]\oplus[\mathcal{N}(P)\cap\mathcal{N}(Q)]$. This is the first claim of (ii). Since I_n-P is idempotent and $\mathcal{N}(I_n-P)=\mathcal{R}(P)$, the second claim of (ii) follows from the first part of (ii) just by replacing P by I_n-P. All that remains now is to prove (iii). For that purpose, let $x\in\mathbb{C}^n$ again be written as $x=x_1+x_2$, where $x_1\in\mathcal{R}(P)$ and $x_2\in\mathcal{N}(P)$. Then $x=x_1+x_2\in\mathcal{N}(PQ-QP)\Leftrightarrow PQ(x_1+x_2)=QP(x_1+x_2)\Leftrightarrow PQx_1+PQx_2=Qx_1\Leftrightarrow(I_n-P)Qx_1=PQx_2\Leftrightarrow(I_n-P)Qx_1=0$ & $PQx_2=0\Leftrightarrow x_1\in[\mathcal{R}(P)\cap\mathcal{R}(Q)]\oplus[\mathcal{R}(P)\cap\mathcal{N}(Q)]$ & $x_2\in[\mathcal{N}(P)\cap\mathcal{R}(Q)]\oplus[\mathcal{N}(P)\cap\mathcal{N}(Q)]$, where the last equivalence follows by virtue of (ii). □

The following interesting observation is an immediate consequence of Theorem 2 (i) & (iii).

THEOREM 3. *Let $P,Q\in\mathbb{C}^{n\times n}$ be idempotent matrices. Then*

$$\mathcal{N}(PQ-QP)=\mathcal{N}(P-Q)\oplus\mathcal{N}(I_n-P-Q). \tag{8}$$

Theorem 3 enables us now to prove the following version of Problem 34-4.

THEOREM 4. *Let $A,B\in\mathbb{C}^{n\times n}$ be involutory matrices. Then*

$$\mathcal{N}(AB-BA)=\mathcal{N}(A-B)\oplus\mathcal{N}(A+B). \tag{9}$$

PROOF. From Theorem 1 we know that there exist idempotent matrices $P,Q\in\mathbb{C}^{n\times n}$ such that $A=I_n-2P$ and $B=I_n-2Q$. Then $AB-BA=(I_n-2P)(I_n-2Q)-(I_n-2Q)(I_n-2P)=4(PQ-QP)$, $A-B=(I_n-2P)-(I_n-2Q)=2(Q-P)$, and $A+B=(I_n-2P)+(I_n-2Q)=2(I_n-P-Q)$. Consequently, $\mathcal{N}(AB-BA)=\mathcal{N}(PQ-QP)$, $\mathcal{N}(A-B)=\mathcal{N}(P-Q)$, and $\mathcal{N}(A+B)=\mathcal{N}(I_n-P-Q)$, and so it is clear that (9) is an immediate consequence of (8). □

That (9) is equivalent to the claimed problem follows by means of the following three well-known facts: (1) A matrix R is involutory if and only if R^* is involutory. (2) For any matrix M, we have $(\mathcal{N}(M^*))^\perp=\mathcal{R}(M)$, with $(\cdot)^\perp$ indicating the orthogonal complement of the linear space $(\cdot)$ with respect to the usual inner product. (3) If $\mathcal{M}$ and $\mathcal{N}$ are linear subspaces of $\mathbb{C}^n$, then $(\mathcal{M}+\mathcal{N})^\perp=\mathcal{M}^\perp\cap\mathcal{N}^\perp$.

Reference

P. Lancaster (1969). *Theory of Matrices.* Academic Press. New York.

Solutions to Problem 34-4 were also received from Johanns de Andrade Bezerra and from the Proposer Yongge Tian.

Problem 34-5: A Rank Equality for Sums of Two Outer Inverses of a Matrix

Proposed by Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

An $m\times n$ matrix X is called an outer inverse of an $n\times m$ matrix A if $XAX=X$. Show that if $a_1\neq 0$, $a_2\neq 0$ and $a_1+a_2\neq 0$, then $\operatorname{rank}(a_1X_1+a_2X_2)=\operatorname{rank}(X_1+X_2)$ for any two outer inverses X_1 and X_2 of A.

Solution 34-5.1 by the Proposer Yongge TIAN, *Shanghai Univ. of Finance & Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Note that rank of a matrix is invariant under elementary block matrix operations. Hence by elementary block matrix operations

$$\operatorname{rank}\begin{pmatrix} X_1 & 0 & a_1X_1 \\ 0 & X_2 & a_2X_2 \\ X_1 & X_2 & 0 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & -a_1X_1-a_2X_2 \end{pmatrix}=\operatorname{rank}(X_1)+\operatorname{rank}(X_2)+\operatorname{rank}(a_1X_1+a_2X_2).$$

On the other hand, note that $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Also by elementary block matrix operations

$$\operatorname{rank}\begin{pmatrix} X_1 & 0 & a_1X_1 \\ 0 & X_2 & a_2X_2 \\ X_1 & X_2 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} X_1 & 0 & a_1X_1 \\ -X_2AX_1 & 0 & a_2X_2 \\ X_1 & X_2 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} (1+a_1a_2^{-1})X_1 & 0 & a_1X_1 \\ 0 & 0 & a_2X_2 \\ X_1 & X_2 & 0 \end{pmatrix} =$$

$$\operatorname{rank}\begin{pmatrix} X_1 & 0 & 0 \\ 0 & 0 & a_2X_2 \\ 0 & X_2 & -a_1(1+a_1a_2^{-1})^{-1}X_1 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} a_1(a_1+a_2)^{-1}a_2X_1 & X_2 \\ X_2 & 0 \end{pmatrix} + \operatorname{rank}(X_1) = \operatorname{rank}\begin{pmatrix} X_1 & X_2 \\ X_2 & 0 \end{pmatrix} + \operatorname{rank}(X_1).$$

Combining the above two rank equalities gives

$$\operatorname{rank}(a_1X_1 + a_2X_2) = \operatorname{rank}\begin{pmatrix} X_1 & X_2 \\ X_2 & 0 \end{pmatrix} - \operatorname{rank}(X_2) = \operatorname{rank}(X_1 + X_2).$$

Problem 34-6: The Schur Complement in an Orthogonal Projector

Proposed by Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Suppose that $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ is an orthogonal projector, that is, $M^2 = M = M^*$. Show that

(a) The Schur complement $D - B^*A^\dagger B$ of A in M satisfies the rank subtractivity condition

$$\operatorname{rank}(D - B^*A^\dagger B) = \operatorname{rank}(D) - \operatorname{rank}(B^*A^\dagger B),$$

where $A^\dagger$ denotes the Moore-Penrose inverse of A.

(b) $\{D^-\} \subseteq \{(B^*A^\dagger B)^-\}$, where $(\cdot)^-$ denotes a g-inverse of a matrix.

(c) $D = B^*A^\dagger B \Leftrightarrow \operatorname{rank}(M) = \operatorname{rank}(A) \Leftrightarrow \operatorname{rank}(D) = \operatorname{rank}(B)$.

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First note that $M^2 = M^* = M$ implies $A = A^*$, $D = D^*$ and

$$A^2 + BB^* = A, \quad AB + BD = B. \tag{10}$$

Define the two Hermitian matrices $K := D - B^*A^\dagger B, L := B^*A^\dagger B$. Then $D = K + L$.

(a): In view of (10) and the properties of the Moore-Penrose inverse, it follows that $LK = (B^*A^\dagger B)(D - B^*A^\dagger B) = B^*A^\dagger BD - B^*A^\dagger BB^*A^\dagger B = B^*A^\dagger(B - AB) - B^*A^\dagger(A - A^2)A^\dagger B = B^*A^\dagger(B - AB) - B^*A^\dagger B - B^*A^\dagger AB = 0$. Since both matrices are Hermitian, we get $(LK)^* = KL = 0$. Therefore, the matrices commute and we conclude that the ranges are orthogonal complements and get the direct sum decomposition

$$\mathcal{R}(K) \oplus \mathcal{R}(L) = \mathcal{R}(K+L) = \mathcal{R}(D), \quad D = P\begin{pmatrix} D_K & 0 \\ 0 & 0 \end{pmatrix}P^* + P\begin{pmatrix} 0 & 0 \\ 0 & D_L \end{pmatrix}P^*, \tag{11}$$

for some unitary P. Therefore, we get the desired rank additivity for K, L (or subtractivity for D, L) result.

(b): Let D^- be a g-inverse of D. From (11), note that both $\mathcal{R}(K), \mathcal{R}(L)$ are D-invariant subspaces. Therefore (easy to show using $D = DD^-D$), these subspaces are invariant for the g-inverse D^- as well. Therefore, D^- is a g-inverse of L. (And, clearly it is a g-inverse for K as well.)

(c): Define

$$Q := \begin{pmatrix} I & 0 \\ -B^*A^\dagger & I \end{pmatrix} \quad \text{and} \quad R := QMQ^* = \begin{pmatrix} A & B - AA^\dagger B \\ B^* - B^*A^\dagger A & K \end{pmatrix}.$$

From $M = M^* = M^2 = M^*M$, we see M is positive semidefinite. Since Q is nonsingular, the congruence (and Sylvester's Theorem) implies that R is positive semidefinite. We now conclude:

$$D = B^*A^\dagger B \quad \Rightarrow \quad K = 0 \Rightarrow B - AA^\dagger B = 0 \quad (\text{since } R \text{ positive semidefinite}) \quad \Rightarrow \quad \operatorname{rank}(M) = \operatorname{rank}(A).$$

Conversely, suppose that $\operatorname{rank}(M) = \operatorname{rank}(A)$. Then the range $\mathcal{R}(B)$ is a subspace of the range $\mathcal{R}(A)$. Since $AA^\dagger$ is the orthogonal projection on $\mathcal{R}(A)$, we conclude that $B - AA^\dagger B = 0$. Therefore, R is block diagonal and, by the nonsingular congruence has the same rank as M and also A. Therefore, $K = 0$, i.e. we have completed the proof of the first equivalence.

Now, suppose that $\operatorname{rank}(M) = \operatorname{rank}(A)$. Then $K = 0$. Since the rank of a product of matrices is not greater than the smallest rank of the matrices, we conclude that $\operatorname{rank} D \leq \operatorname{rank} B$. Now assume that $\operatorname{rank} D < \operatorname{rank} B$. Then $\mathcal{N}(B) \not\subset \mathcal{N}(D)$ and there exists v such that $Dv = 0$ but $Bv \neq 0$. Choose w so that $w^* Bv < 0$. Now consider the positive semidefinite quadratic form

$$0 \leq (\, w^* \quad tv^* \,) \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} w \\ tv \end{pmatrix} = w^* A w + tv^* D tv + 2w^* B tv = w^* A w + 2w^* B tv.$$

i.e. $\mathcal{N}(B) \not\subset \mathcal{N}(D)$ leads to a contradiction when t becomes large (positive). Hence we get $\mathcal{N}(B) \subset \mathcal{N}(D)$. Thus we conclude that $\operatorname{rank} D = \operatorname{rank} B$.

Finally, to conclude the proof we assume that $\operatorname{rank} D = \operatorname{rank} B$ and show that $K = 0$. Since $M \geqslant_{\mathrm{L}} 0$, the principal submatrices $A, D \geqslant_{\mathrm{L}} 0$ and we can form orthogonal diagonalizations $P_A^* A P_A = \Lambda_A$, $P_D^* D P_D = \Lambda_D$. Now define

$$S := \begin{pmatrix} P_A^* & 0 \\ 0 & P_D^* \end{pmatrix} M \begin{pmatrix} P_A & 0 \\ 0 & P_D \end{pmatrix} = \begin{pmatrix} \Lambda_A & P_A^* B P_D \\ P_D B^* P_A^* & \Lambda_D \end{pmatrix}.$$

Then S is an orthogonal projection and we maintain the rank equivalence between Λ_D and $P_A^* B P_D$. Since S is positive semidefinite, $(\Lambda_D)_{ii} = 0$ implies that the rest of the i-th column of S, and so also of $P_A^* B P_D$, is 0. Therefore, we can ignore these columns (and corresponding rows) and so we assume, without loss of generality, that Λ_D is nonsingular. Our rank assumption now implies that $P_A^* B P_D$ is full column rank. We can now assume the same for D and B. And, similarly, we can assume without loss of generality that A is nonsingular. Therefore, $B^* A^\dagger B = B^* A^{-1} B$ is a full rank congruence of a nonsingular matrix and so is of nonsingular. Now by part (b), we conclude that $D^{-1} = (B^* A^{-1} B)^{-1}$. Therefore, $D^{-1} K = D^{-1} D - D^{-1} B^* A^{-1} B = 0$.

Solution 34-6.2 by the Proposer Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Because M is an orthogonal projector, $\operatorname{range}(B) \subseteq \operatorname{range}(A)$ holds. Thus its rank can be written as

$$\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(\, D - B^* A^\dagger B \,).$$

Also from IMAGE Problem 25-4(b) (see Tian (2000)), the rank of M is $\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(D) - \operatorname{rank}(B)$. Thus $\operatorname{rank}(\, D - B^* A^\dagger B \,) = \operatorname{rank}(D) - \operatorname{rank}(B)$. However, $\operatorname{rank}(B^* A^\dagger B) = \operatorname{rank}(A^\dagger B) = \operatorname{rank}(B)$. Thus we have the result in (a). Applying a well-known result:

$$\{A^-\} \subseteq \{B^-\} \Leftrightarrow \operatorname{rank}(A - B) = \operatorname{rank}(A) - \operatorname{rank}(B)$$

to (a) gives (b). The equivalences in (c) are from (a) and (b).

Reference

Y. Tian (2000). Problem 25-4: Two rank equalities associated with blocks of an orthogonal projector. *IMAGE: The Bulletin of the International Linear Algebra Society* **25** (October 2000), 16.

Solution 34-6.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Our solution offers some additional insights into the partitioning of (not necessarily orthogonal) projectors. For example, we reobtain the known result that the Schur complements of the north-west and the south-east block in a symmetrically partitioned orthogonal projector are again orthogonal projectors; see Corollary 2.1 in Baksalary, Baksalary and Szulc (2004). We also show how Problem 34-6 can be extended to cover block-partitioned oblique projectors, i.e., idempotent non-Hermitian matrices. In what follows, for a complex matrix $A \in \mathbb{C}^{n \times n}$, let A^*, $\operatorname{rank}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the rank, the range (column space), and the null space, respectively, of A.

Let $M := \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$. Clearly, M is an orthogonal projector if and only if $M = M^* = M^2$ or, equivalently, if and only if the following six conditions do hold simultaneously:

$$A = A^*,\ D = D^*,\ A(I - A) = BB^*,\ D(I - D) = B^* B,\ B^*(I - A) = DB^*,\ (I - A)B = BD. \tag{12}$$

We note that $\mathcal{R}(BB^*) = \mathcal{R}(B)$. Therefore, by the third condition in (12), $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ or, equivalently, $AA^- B = B$, irrespective of the choice of A^-, with A^- indicating a generalized inverse of A, i.e., any matrix G satisfying $AGA = A$. We note that

$AA^-B = B$ is equivalent to $B^*(A^-)^*A = B^*$, since $A = A^*$. It is also useful to mention that $B^*A^-B = B^*A^+B$. Since A^+ is nonnegative and Hermitian, B^*A^-B is also so. From

$$\begin{pmatrix} I & 0 \\ -B^*(A^-)^* & I \end{pmatrix} M \begin{pmatrix} I & -A^-B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - B^*A^-B \end{pmatrix}$$

it follows that $\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(D - B^*A^-B)$. Consequently, $\operatorname{rank}(M) = \operatorname{rank}(A)$ if and only if $D = B^*A^-B$, which is claim (c) of Problem 34-6. By using the conditions in (12), it is further easy to check that $D - B^*A^-B = (D - B^*A^-B)^* = (D - B^*A^-B)^2$, thus showing that $D - B^*A^-B$, the Schur complement of A in M, is an orthogonal projector. More precisely,

$$D - B^*A^-B = P_{D\mathcal{N}(B)},$$

where $P_{D\mathcal{N}(B)}$ stands for the orthogonal projector that projects onto $D\mathcal{N}(B)$ along $(D\mathcal{N}(B))^\perp = \mathcal{R}(B^*) \oplus \mathcal{N}(D)$, and where '$\oplus$' indicates a direct sum. Therefore, in particular, $(D - B^*A^-B)B^*A^-B = 0$ and so $\mathcal{R}(B^*A^-B) \cap \mathcal{R}(D - B^*A^-B) = \{0\}$. Because B^*A^-B and $D - B^*A^-B$ are (nonnegative definite and) Hermitian matrices it is now clear that $D - B^*A^-B$ is weakly bicomplementary to B^*A^-B. For the sake of clarity, we mention that two matrices, say S and T, are called weakly bicomplementary to each other (or often also disjoint matrices) if and only if $\mathcal{R}(S) \cap \mathcal{R}(T) = \{0\}$ and $\mathcal{R}(S^*) \cap \mathcal{R}(T^*) = \{0\}$; see Werner (1986) or Jain, Mitra and Werner (1996). Since, according to Theorems 2.1 and 3.1 in Werner (1986), see also Theorem 2.3 in Jain, Mitra and Werner (1996), R is weakly bicomplementary to $S \Leftrightarrow \mathcal{R}(S+T) = \mathcal{R}(S) \oplus \mathcal{R}(T) \Leftrightarrow \operatorname{rank}(S+T) = \operatorname{rank}(S) + \operatorname{rank}(T) \Leftrightarrow \{(R+S)^-\} \subseteq \{R^-\} \cap \{T^-\}$, (a) and (b) of Problem 34-6 are obvious.

We conclude with mentioning that Problem 34-6 allows a more general version. Note that the restrictive assumptions in our next Theorem are all redundant if M is an orthogonal projector. The proof of our extension can be left to the reader.

THEOREM. *Let* $M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a symmetrically partitioned idempotent matrix satisfying* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C) \subseteq \mathcal{R}(D)$, $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, *and* $\mathcal{N}(B) \subseteq \mathcal{N}(D)$. *Then:*

(i) $\operatorname{rank}(D) = \operatorname{rank}(D - CA^-B) + \operatorname{rank}(CA^-B)$.

(ii) $\{D^-\} \subseteq \{(D - CA^-B)^-\} \cap \{(CA^-B)^-\}$.

(iii) $D = CA^-B \Leftrightarrow \operatorname{rank}(M) = \operatorname{rank}(A) \Leftrightarrow \operatorname{rank}(D) = \operatorname{rank}(CA^-B)$.

(iv) *The Schur complements of* A *and* D *in* M *are both idempotent but not necessarily Hermitian.*

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A solution to Problem 34-6 was also received from Oskar Maria Baksalary & Xiaoji Liu.

Problem 34-7: A Sufficient Condition for a Matrix to be Normal

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Suppose that $A, R \in \mathbb{R}^{n\times n}$, $R = R^{-1} \neq \pm I_n$, $RAR = A^T$, and $x^T(A^TA - AA^T)y = 0$ whenever $Rx = x$ and $Ry = -y$. Show that A is normal.

Solution 34-7.1 by Johanns DE ANDRADE BEZERRA, *Campina Grande, PB, Brazil:* talita.tao@zipmail.com.br

If $RAR = A^T$ and $R^2 = (R^T)^2 = I$, then $R^TRARR^T = R^TA^TR^T = A$, and hence it follows that $ARR^T = RR^TA$ and $AR^TR = R^TRA$. Consequently, $A^TA = RARR^TA^TR^T = R^2R^TAA^TR^T = R^TAA^TR^T$, and so $A^TA = RAA^TR$. Since $R^2 = I$, it follows that $U \oplus V = \mathbb{R}^{n\times 1}$, where U and V are the eigenspaces of R corresponding to the eigenvalues 1 and -1, respectively.

Next, let $x \in U$ and $y \in V$ be arbitrarily chosen. Because of $x^T(A^TA - AA^T)y = 0$ and $A^TRR^T = RR^TA^T$, clearly $\langle x, A^TAy\rangle - \langle x, AA^Ty\rangle = \langle Ax, Ay\rangle - \langle x, AA^Ty\rangle = \langle R^TA^TR^Tx, R^TA^TR^Ty\rangle - \langle x, AA^Ty\rangle = \langle R^Tx, ARR^TA^TR^Ty\rangle - \langle x, AA^Ty\rangle = \langle R^Tx, AA^TRR^TR^Ty\rangle - \langle x, AA^Ty\rangle = -\langle R^Tx, AA^Ty\rangle - \langle x, AA^Ty\rangle = 0$, and so $\langle x, (RAA^T + AA^T)y\rangle = 0$. Since x is an arbitrary vector from U, it follows that $(RAA^T + AA^T)y \in U^\perp$. But $R(RAA^T + AA^T)y = (RAA^T + AA^T)y$, and so $(RAA^T + AA^T)y \in U$. Hence $(RAA^T + AA^T)y \in U \cap U^\perp = \{0\}$, this yields $RAA^Ty = -AA^Ty$.

On the other hand, $\langle x, A^TAy\rangle - \langle x, AA^Ty\rangle = \langle A^TAx, y\rangle - \langle A^Tx, A^Ty\rangle = \langle A^TAx, y\rangle - \langle RARx, RARy\rangle = \langle A^TAx, y\rangle + \langle RAx, RAy\rangle = \langle A^TAx, y\rangle + \langle A^TR^TRAx, y\rangle = \langle A^TAx, y\rangle + \langle R^TRA^TAx, y\rangle = \langle A^TAx, y\rangle + \langle A^TAx, R^TRy\rangle = \langle A^TAx, y\rangle - \langle A^TAx, R^Ty\rangle = 0$, since $A^TR^TR = R^TRA^T$, and so $\langle (A^TA - RA^TA)x, y\rangle = 0$. Thus, since y is an arbitrary vector from V, it follows that $(A^TA - RA^TA)x \in V^\perp$. But $R(A^TA - RA^TA)x = -(A^TA - RA^TA)x$, and so $(A^TA - RA^TA)x \in V$. Hence $(A^TA - RA^TA)x \in V \cap V^\perp = \{0\}$, this yields $RA^TAx = A^TAx$.

In view of $A^TA = RAA^TR$, clearly $RA^TA = AA^TR$. Thus, $RA^TAx = AA^TRx = AA^Tx = A^TAx$, for any $x \in U$. Similarly, $RA^TAy = AA^TRy = -AA^Ty = RAA^Ty$, then $R(A^TAy - AA^Ty) = 0$, and so $AA^Ty = A^TAy$, for any $y \in V$. Therefore, $AA^T = A^TA$, that is, A is normal.

Solution 34-7.2 by the Proposer William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Let $B = A^TA - AA^T$. We must show that $B = 0$. Since $R^2 = I$ and $RAR = A^T$, it follows that $RA^TR = A$ and

$$RBR = (RA^TR)(RAR) - (RAR)(RA^TR) = AA^T - A^TA = -B. \tag{13}$$

Since the minimal polynomial of R is $x^2 - 1$,

$$R = (\,P \quad Q\,)\begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}(\,P \quad Q\,)^{-1}$$

where $r, s > 0$,

$$P^TP = I_r, \quad \text{and} \quad Q^TQ = I_s. \tag{14}$$

Moreover, we can write

$$B = (\,P \quad Q\,)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}(\,P \quad Q\,)^{-1} \tag{15}$$

with $B_{11} \in \mathbb{R}^{r\times r}$. Then

$$RBR = (\,P \quad Q\,)\begin{pmatrix} B_{11} & -B_{12} \\ -B_{21} & B_{22} \end{pmatrix}(\,P \quad Q\,)^{-1}. \tag{16}$$

From (13), comparing (15) and (16) yields

$$B = (\,P \quad Q\,)\begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}(\,P \quad Q\,)^{-1}.$$

Therefore $PB_{12} = BQ$ and $QB_{21} = BP$, so (14) implies that $B_{12} = P^TBQ$ and $B_{21} = Q^TBP$. Since $RP = P$ and $RQ = -Q$, the last assumption implies that $B_{12} = 0$ and $B_{21} = 0$. Hence, $B = 0$.

Solution 34-7.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

For a complex matrix $A \in \mathbb{C}^{m\times n}$, let A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the range (column space), and the null space, respectively, of A. We recall that a square matrix $R \in \mathbb{C}^{n\times n}$ is called involutory if and only if $R^2 = I_n$, where I_n stands for the identitiy matrix of order n. In what follows, we make extensive use of the following powerful characterization of involutory matrices. An easy proof of that theorem may be found in Werner (2005).

THEOREM 1. *The matrix $R \in \mathbb{C}^{n\times n}$ is an involutory matrix if and only if there exists an idempotent matrix P, i.e., $P^2 = P$, such that $R = I_n - 2P$. In which case, $P = (I_n - R)/2$.*

It is worth mentioning that there is an obvious one-to-one correspondence between the set of involutory matrices and the set of idempotent matrices. Theorem 1 enables us to prove the following more general version of Problem 34-7.

THEOREM 2. *Let $R \in \mathbb{C}^{n\times n}$ be an involutory matrix, i.e., let $R = I_n - 2P$ for some idempotent matrix P. For given $A \in \mathbb{C}^{n\times n}$, let $B := RAR$. The following conditions are then equivalent:*

(i) *B and A commute, i.e., $BA = AB$.*
(ii) $(BA - AB)P = 0$ *and* $(BA - AB)(I_n - P) = 0$.
(iii) $(BA - AB)(I_n - R) = 0$ *and* $(BA - AB)(I_n + R) = 0$.
(iv) $(I_n + R)BA(I_n - R) = 0$ *and* $(I_n - R)BA(I_n + R) = 0$.
(v) $(I_n + R)AB(I_n - R) = 0$ *and* $(I_n - R)AB(I_n + R) = 0$.
(vi) $(I_n - P)BAP = 0$ *and* $PBA(I_n - P) = 0$.
(vii) $(I_n - P)ABP = 0$ *and* $PAB(I_n - P) = 0$.
(viii) *$BAP = PBA$, i.e., BA and P commute.*
(ix) *$ABP = PAB$, i.e., AB and P commute.*
(x) *$ARAP = PARA$, i.e., ARA and P commute.*
(xi) $(I_n + R)^*(BA - AB)(I_n - R) = 0$ *and* $(I_n - R)^*(BA - AB)(I_n + R) = 0$.

PROOF. In view of $P + (I_n - P) = I_n$, $I_n - R = 2P$, and $I_n + R = 2(I_n - P)$, clearly (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), (iv) $\Leftrightarrow$ (vi), and (v) $\Leftrightarrow$ (vii). Trivially, (i) $\Rightarrow$ (xi). By means of $(I_n - R)R = R(I_n - R) = -(I_n - R)$ and $(I_n + R)R = R(I_n + R) = I_n + R$, we obtain $(BA - AB)(I_n - R) = (RARA + ARA)(I_n - R) = (I_n + R)ARA(I_n - R) = (I_n + R)RARA(I_n - R) = (I_n + R)BA(I_n - R) = -(I_n + R)AB(I_n - R)$ and $(BA - AB)(I_n + R) = (RARA - ARAR)(I_n + R) = (RARA - ARA)(I_n + R) = -(I_n - R)ARA(I_n + R) = (I_n - R)RARA(I_n + R) = (I_n - R)BA(I_n + R) = -(I_n - R)AB(I_n + R)$, and hence (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). Because $P^2 = P$, trivially (vi) $\Leftrightarrow$ (viii) and (vii) $\Leftrightarrow$ (ix). By means of $R^2 = I_n$, $R(I_n + R) = I_n + R$ and the definition of B, (ix) $\Leftrightarrow$ (x). We have already seen that $(BA - AB)(I_n - R) = (I_n + R)BA(I_n - R)$ and $(BA - AB)(I_n + R) = (I_n - R)BA(I_n + R)$. If M and N are matrices such that MN is defined, then it is well known that $M^*MN = 0$ if and only if $MN = 0$. With this in mind, it is now clear that (iv) follows from (xi), so that the proof of this theorem is complete. □

From Theorem 2 we single out the following special case.

COROLLARY 3. *Let $R \in \mathbb{C}^{n \times n}$ be an involutory matrix, i.e., let $R = I_n - 2P$ for some idempotent matrix P. Moreover, let $A \in \mathbb{C}^{n \times n}$ be such that $A^* = RAR$. The following conditions are then equivalent:*

(i) *A is normal, i.e., $A^*A = AA^*$.*
(ii) $A^*AP = PA^*A$.
(iii) $AA^*P = PAA^*$.
(iv) *$ARAP = PARA$, i.e., ARA and P commute.*
(v) $(I_n + R)^*(A^*A - AA^*)(I_n - R) = 0$.
(vi) $x^*(A^*A - AA^*)y = 0$ *whenever* $Rx = x$ *and* $Ry = -y$.

PROOF. According to Theorem 2, clearly (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Applying the conjugate transpose operation on both sides of equation $(I_n + R)^*(A^*A - AA^*)(I_n - R) = 0$ results in equation $(I_n - R)^*(A^*A - AA^*)(I_n + R) = 0$, and so we know from Theorem 2 that (v) is equivalent to (i). In virtue of $Rx = x \Leftrightarrow (I_n - R)x = 0 \Leftrightarrow Px = 0 \Leftrightarrow x \in \mathcal{N}(P) \Leftrightarrow x \in \mathcal{R}(I_n - P) \Leftrightarrow x \in \mathcal{R}(I_n + R)$ and $Ry = -y \Leftrightarrow (I_n + R)y = 0 \Leftrightarrow (I_n - P)y = 0 \Leftrightarrow y \in \mathcal{N}(I_n - P) \Leftrightarrow y \in \mathcal{R}(P) \Leftrightarrow y \in \mathcal{R}(I_n - R)$, it is finally also clear that (v) $\Leftrightarrow$ (vi). □

Corollary 3 contains with (i) $\Leftrightarrow$ (vi) the complex analog of the claim of Problem 34-7, and so our solution is complete.

Reference

H. J. Werner (2005). A range equality for the commutator with two involutory matrices. Solution 34-4.2. *IMAGE: The Bulletin of the International Linear Algebra Society* **35** (Fall 2005), 32–33.

A belated solution to Problem 34-7 was also received from Nadya Zharko.

Problem 34-8: A Property for the Sum of a Matrix A and its Moore-Penrose Inverse A^+

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a square complex matrix. Show that the following two statements are equivalent:

(i) $A + A^{+} = 2AA^{+}$.

(ii) $A + A^{+} = AA^{+} + A^{+}A$.

Verify that under (i) or (ii), A must be EP, i.e. the column spaces of A and A^* coincide.

Solution 34-8.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and Xiaoji LIU, *Guangxi University for Nationalities, Nanning, China:* xiaojiliu72@yahoo.com.cn

Let $A \in \mathbb{C}_{n,n}$ of rank r have a singular value decomposition of the form

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{17}$$

where $U, V \in \mathbb{C}_{n,n}$ are unitary and $D \in \mathbb{C}_{r,r}$ is a positive definite diagonal matrix. Clearly, the Moore-Penrose inverse of A is given by

$$A^\dagger = V \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*. \tag{18}$$

If matrices U and V utilized in (17) are partitioned as $U = (U_1 : U_2)$ and $V = (V_1 : V_2)$, where $U_1, V_1 \in \mathbb{C}_{n,r}$ and $U_2, V_2 \in \mathbb{C}_{n,n-r}$, then the product $V^*U \in \mathbb{C}_{n,n}$ can be written as

$$V^*U = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{19}$$

where $W_{ij} = V_i^* U_j$, $i, j = 1, 2$, and thus $W_{11} \in \mathbb{C}_{r,r}$, $W_{12} \in \mathbb{C}_{r,n-r}$, $W_{21} \in \mathbb{C}_{n-r,r}$, and $W_{22} \in \mathbb{C}_{n-r,n-r}$. It is easily seen that matrix V^*U is unitary and from condition $(V^*U)(V^*U)^* = I_n$, where I_n is the identity matrix of indicated order, it follows that

$$W_{11}W_{11}^* + W_{12}W_{12}^* = I_r, \qquad W_{11}W_{21}^* + W_{12}W_{22}^* = 0, \qquad W_{21}W_{21}^* + W_{22}W_{22}^* = I_{n-r}. \tag{20}$$

The solution will be based on the following result related to Theorem 2.2 in Baksalary, Baksalary & Liu (2003).

LEMMA. *Let $A \in \mathbb{C}_{n,n}$ be of rank r and have a singular value decomposition of the form* (17) *with V^*U partitioned as in* (19)*. Then A is an EP matrix if and only of $W_{12} = 0$, $W_{21} = 0$, and W_{11} as well as W_{22} are unitary.*

PROOF. Assume that A is an EP matrix. Then substituting (17) and (18) to condition $AA^\dagger = A^\dagger A$ and premultiplying the obtained equality by V^*, postmultiplying it by U, and adopting notation (19) leads to $W_{12} = 0$, $W_{21} = 0$. Under these conditions, from the first and the last equality in (20) it follows that W_{11} and W_{22} are unitary.

For the converse implication first observe that orthogonal projectors $AA^\dagger$ and $A^\dagger A$ composed of matrices of the forms (17) and (18) are equal $U_1U_1^*$ and $V_1V_1^*$, respectively. The fact that W_{11} is unitary can be expressed as $V_1^*U_1U_1^*V_1 = I_r$ and by premultiplying this condition by V_1, postmultiplying it by V_1^*, and utilizing the observation above, it follows that $A^\dagger AAA^\dagger A^\dagger A = A^\dagger A$. Hence it is seen that product of two orthogonal projectors $AA^\dagger$ and $A^\dagger A$ is idempotent, for which it is necessary and sufficient that the projectors commute, i.e., that $AA^\dagger A^\dagger A = A^\dagger AAA^\dagger$. Taking the above facts into account and denoting the range of a matrix argument by $\mathcal{R}(\cdot)$ it follows that

$$\mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^\dagger AAA^\dagger A^\dagger A) = \mathcal{R}(AA^\dagger A^\dagger A) \subseteq \mathcal{R}(A).$$

Combining the inclusion $\mathcal{R}(A^*) \subseteq \mathcal{R}(A)$ with the observation that $\mathrm{rank}(A) = \mathrm{rank}(A^*)$, leads to equality $\mathcal{R}(A^*) = \mathcal{R}(A)$, which expresses the fact that A is an EP matrix. □

The solution to the problem is given in what follows.

THEOREM. *Let $A \in \mathbb{C}_{n,n}$ and let $A^\dagger$ be its Moore-Penrose inverse. Then the following statements are equivalent:*

$$\text{(i) } A + A^\dagger = 2AA^\dagger, \qquad \text{(ii) } A + A^\dagger = 2A^\dagger A, \qquad \text{(iii) } A + A^\dagger = AA^\dagger + A^\dagger A. \tag{21}$$

Moreover, each of conditions (i)–(iii) *implies $AA^\dagger = A^\dagger A$.*

PROOF. In fact, it is enough to show that each of conditions (i)–(iii) forces A to be an EP matrix. Then the equivalence between conditions (i)–(iii) will be easily seen by rearranging the right-hand sides of the equalities constituting them with the use of equality $AA^\dagger = A^\dagger A$.

Substituting (17) and (18) to condition (i) in (21), premultiplying the obtained equality by V^*, postmultiplying it by U, and adopting notation (19) leads to

$$W_{11}DW_{11} + D^{-1} = 2W_{11}, \qquad W_{11}DW_{12} = 0. \tag{22}$$

Postmultiplying the former of these conditions by D entails $W_{11}D(2I_r - W_{11}D) = I_r$, and thus it is clear that $W_{11}D$ is nonsingular. Consequently, from the latter condition in (22) it follows that $W_{12} = 0$. Substituting this condition to the first equality in (20) shows that W_{11} is unitary, and further from the second equality that $W_{21} = 0$. In consequence, from the third equality in (20) it is seen that also W_{22} is unitary. Thus, on account of Lemma it is clear that condition (i) in (21) implies that A is an EP matrix. The corresponding implication with condition (i) replaced by condition (ii) is established analogously.

Substituting now (17) and (18) to condition (iii) in (21) and premultiplying the obtained equality by V^* and postmultiplying it by V yields $W_{21} = 0$. Substituting this condition to the last equality in (20) shows that W_{22} is unitary, and further from the second equality that $W_{12} = 0$. In consequence, from the first equality in (20) it is seen that also W_{11} is unitary. Thus, on account of Lemma it is clear that also condition (iii) in (21) implies that A is an EP matrix. □

Reference

J. K. Baksalary, O. M. Baksalary & X. Liu (2003). Further properties of the star, left-star, right-star, and minus partial orderings. *Linear Algebra and Its Applications* **375**, 83–94.

Solution 34-8.2 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Following Hartwig and Spindelböck, Corollary 6 (1984), any square matrix $A \in \mathbb{C}^{n\times n}$ can be written in the form

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where U is a unitary matrix, $KK^* + LL^* = I_r$, $\Sigma = \operatorname{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$, $r_1 + r_2 + \ldots + r_t = r = \operatorname{rank}(A)$, and $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$ being the singular values of A. Using this representation, we get

$$A^+ = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*, \quad A^+A = U\begin{pmatrix} K^*K & K^*L \\ L^*K & L^*L \end{pmatrix} U^*, \quad AA^+ = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

Some straightforward calculations then show that each of the conditions (i) and (ii) is equivalent to $L = 0$ and $\Sigma K + K^{-1}\Sigma^{-1} = 2I_r$, which can be rephrased as $L = 0$ and $(\Sigma K - I_r)^2 = 0$.

Observe that the condition $L = 0$ is equivalent for A to be EP (see Corollary 6, Hartwig and Spindelböck, 1984).

Reference

R. E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and A^+ commute. *Linear and Multilinear Algebra* **14**, 241–256.

Solution 34-8.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

The following well-known elementary results open the door to prove this problem. First of all, we recall that if V is any nonnegative definite and Hermitian $n \times n$ matrix and $\mathcal{M}$ is any linear subspace of $\mathbb{C}^n$, then $(V\mathcal{M}) \cap \mathcal{M}^\perp = \{0\}$, where $\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$ with respect to the usual inner product; cf. Werner (2003; Theorem in IMAGE Solution 29-5.4). Secondly, we note that if $M \in \mathbb{C}^{m\times n}$, then $(\mathcal{N}(M^*))^\perp = \mathcal{R}(M)$ and $(\mathcal{R}(M^*))^\perp = \mathcal{N}(M)$, with $(\cdot)^*$, $\mathcal{N}(\cdot)$, and $\mathcal{R}(\cdot)$ denoting the conjugate transpose, the null space, and the range (column space), respectively, of the matrix $(\cdot)$. Thirdly, we mention that MM^+ and M^+M are orthogonal projectors onto $\mathcal{R}(M)$ (along $\mathcal{N}(M^*)$) and onto $\mathcal{R}(M^*)$ (along $\mathcal{N}(M)$), respectively, and so MM^+ and M^+M are, in particular, both nonnegative definite and Hermitian matrices. Fourthly, we note that $\mathcal{N}(M^+) = \mathcal{N}(M^*)$. And finally, we recall to the following characterization: $M \in \mathbb{C}^{n\times n}$ is EP $\Leftrightarrow \mathcal{R}(M) = \mathcal{R}(M^*) \Leftrightarrow \mathcal{N}(M) = \mathcal{N}(M^*) \Leftrightarrow MM^+ = M^+M$. In view of all these observations, it is clear that each of the problem statements (i) and (ii) implies that $A + A^+$ is a nonnegative definite and Hermitian matrix. Therefore, $[(A + A^+)\mathcal{N}(A^*)] \cap \mathcal{R}(A) = \{0\} \Leftrightarrow (A\mathcal{N}(A^*)) \cap \mathcal{R}(A) = \{0\} \Leftrightarrow A\mathcal{N}(A^*) = \{0\} \Leftrightarrow \mathcal{N}(A^*) \subseteq \mathcal{N}(A) \Leftrightarrow \mathcal{N}(A^*) = \mathcal{N}(A) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A) \Leftrightarrow A$ is EP $\Leftrightarrow AA^+ + A^+A = 2AA^+$. This completes our proof.

Reference

H. J. Werner (2003). Product of two Hermitian nonnegative definite matrices. Solution 29-5.4. *IMAGE: The Bulletin of the International Linear Algebra Society* **30** (April 2003), 25.

Problem 34-9: A Sum Property for the Moore-Penrose Inverse of EP Matrices

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a $n \times n$ EP matrix with complex entries whose rows add up to the same sum s. Show that the Moore-Penrose inverse A^+ of A has the property that its rows add up to $1/s$ if $s \neq 0$ and 0 if $s = 0$.

Solution 34-9.1 by Johanns DE ANDRADE BEZERRA, *Campina Grande, PB, Brazil:* talita.tao@zipmail.com.br

Let $\iota \in \mathbb{C}^n$ be the vector with all elements equal to unity. We note that A is EP if and only if $AA^+ = A^+A$. In which case, in particular, $A^+A^+A = A^+$. Therefore, if $A\iota = 0$, then also $A^+\iota = 0$. Next, let $A\iota = s\iota$, where $s \neq 0$. Then $\iota = s^{-1}A\iota$ and so $A^+\iota = s^{-1}A^+A\iota = s^{-1}AA^+\iota = s^{-2}AA^+A\iota = s^{-2}A\iota = s^{-1}\iota$.

Solution 34-9.2 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl

Let $A \in \mathbb{C}_{n,n}$ be an EP (range-Hermitian) matrix, i.e., let $AA^\dagger = A^\dagger A$, where $A^\dagger$ is the Moore-Penrose inverse of A. It is to be shown that if A is such that

$$A1 = s1, \tag{23}$$

where 1 is the vector of n ones and $s \in \mathbb{C}$, then $A^\dagger$ satisfies

$$A^\dagger 1 = s^\dagger 1, \quad \text{where } s^\dagger = \begin{cases} s^{-1}, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0. \end{cases} \tag{24}$$

It is known [see Ben-Israel and Greville (2003, Corollary 3, p. 166)] that A is an EP matrix if and only if $A^\dagger$ is expressible as polynomial in A. For nonzero vector $x \in \mathbb{C}_{n,1}$, consider the class of matrices

$$\mathcal{A}(x) = \{A \in \mathbb{C}_{n,n}\colon Ax = \lambda x \ \text{ for some } \ \lambda \in \mathbb{C}\}.$$

It is easily seen that the identity matrix of order n belongs to $\mathcal{A}(x)$ and, moreover, that for $A_1, A_2 \in \mathcal{A}(x)$ and $\gamma \in \mathbb{C}$, also A_1A_2, $A_1 + A_2$, and $\gamma A_1 \in \mathcal{A}(x)$. Thus, $A \in \mathcal{A}(x)$ implies $A^\dagger \in \mathcal{A}(x)$.

Let $Ax = \lambda x$ and $A^\dagger x = \mu x$. If $\mu = 0$, then $AA^\dagger = A^\dagger A$ along with $AA^\dagger A = A$ imply that $\lambda = 0$. If $\mu \neq 0$, then $AA^\dagger = A^\dagger A$ along with $A^\dagger AA^\dagger = A^\dagger$ entail $\mu(\lambda\mu - 1)x = 0$. Since $x \neq 0$, it follows that $\lambda\mu = 1$. Consequently, if $Ax = \lambda x$, then $A^\dagger x = \lambda^\dagger x$ and the solution to the problem follows by taking x to be the vector of n ones.

Parenthetically notice that the version of Problem 34-9 with $A \in \mathbb{C}_{n,n}$ in (23) replaced by not necessarily EP matrix $A \in \mathbb{R}_{n,n}$ and $A^\dagger$ in (24) replaced by A^D, where A^D denotes the unique Drazin inverse of A, was solved by Schmidt and Trenkler (2001, Section 3). The solution provided therein is closely related to the one presented above, for if A is an EP matrix, then $A^\dagger = A^D$.

References

A. Ben-Israel & T. N. E. Greville (2003). *Generalized Inverses: Theory and Applications* (2nd ed.). Springer, New York.

K. Schmidt & G. Trenkler (2001). The Moore-Penrose inverse of a semi-magic square is semi-magic. *International Journal of Mathematical Education in Science and Technology* **32**, 624–629.

Solution 34-9.3 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

A is an EP matrix if and only if $AA^+ = A^+A$. If $s \neq 0$ and (s, x) is an eigenpair of A, then

$$sx = Ax = AA^+Ax = sAA^+x = sA^+Ax = s^2A^+x,$$

so $A^+x = x/s$. Since $A = AA^+A = A^2A^+$ and $A^+ = A^+AA^+ = (A^+)^2A$, A and A^+ have the same null space. Letting $x = (1, 1, \cdots, 1)'$ yields the stated conclusion.

Solution 34-9.4 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Since A is EP, we have $AA^+ = A^+A$. Let ι denote the $n \times 1$ vector of ones. Now $A\iota = s\iota$ implies $A^+A\iota = sA^+\iota$ and thus $AA^+\iota = sA^+\iota$. Assume $s \neq 0$. Then $\iota \in \mathcal{R}(A)$ and $AA^+\iota = \iota$, since AA^+ is the orthogonal projector on $\mathcal{R}(A)$. Hence we obtain $\iota = sA^+\iota$ or equivalently $A^+\iota = \frac{1}{s}\iota$. When $s = 0$ we have $AA^+\iota = 0$. Premultiplying by A^+ then yields $A^+\iota = 0$.

Problem 34-10: On the Product of Orthogonal Projectors

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be orthogonal projectors with complex entries. Show that PQ is an orthogonal projector if and only if $PQP \leq_L Q$, with $\leq_L$ indicating the Löwner ordering.

Solution 34-10.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl and Xiaoji LIU, *Guangxi University for Nationalities, Nanning, China:* xiaojiliu72@yahoo.com.cn

Let $P, Q \in \mathbb{C}_{n,n}$ be orthogonal projectors and let P be of rank r. Then P and Q can be represented as

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^* \quad \text{and} \quad Q = U \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix} U^*, \tag{25}$$

where $U \in \mathbb{C}_{n,n}$ is unitary, I_r denotes the identity matrix of order r, and $Q_{11} \in \mathbb{C}_{r,r}$, $Q_{22} \in \mathbb{C}_{n-r,n-r}$ are Hermitian. Then condition $PQP \leqslant_{\mathrm{L}} Q$ is equivalent to

$$0 \leqslant_{\mathrm{L}} U \begin{pmatrix} 0 & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix} U^*. \tag{26}$$

In view of Theorem 1 in Albert (1969), condition (26) is fulfilled if and only if $Q_{12} = 0$ and $0 \leqslant_{\mathrm{L}} Q_{22}$. Since Q is an orthogonal projector, the latter condition is always fulfilled while the former one ensures that P and Q defined in (25) satisfy $PQ = QP$, which is a well known necessary and sufficient condition for the product PQ to be an orthogonal projector.

Reference

A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics* **17**, 434-440.

Solution 34-10.2 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Following Baksalary, Kala and Kłaczyński (1983, Theorem 1), by taking $M = Q$ and $B = B^* = P$, the condition $PQP \leq_L Q$ is equivalent to $PQP = QP$.

Since $(PQ)^+$ is idempotent (see Greville, 1974), we may write

$$(PQ)^+ = U \begin{pmatrix} I_r & H \\ 0 & 0 \end{pmatrix} U^*,$$

where U is a unitary matrix (see Hartwig and Loewy, 1992). Hence

$$PQ = (PQ)^{++} = U \begin{pmatrix} E & 0 \\ H^*E & 0 \end{pmatrix} U^*,$$

where $E = (I_r + HH^*)^{-1}$. From this we get

$$PQP = PQQP = U \begin{pmatrix} E^2 & E^2H \\ H^*E^2 & H^*E^2H \end{pmatrix} U^*.$$

The condition $PQP = QP$ then entails $H = 0$, or, equivalently, PQ is an orthogonal projector.

REMARK: According to Corollary 6 in Baksalary, Kala and Kłaczyński (1983), the condition $PQP \leq_L Q$ is equivalent to

$$\operatorname{rank}(Q - PQP) = \operatorname{rank}(Q) - \operatorname{rank}(PQP).$$

References

J. K. Baksalary, R. Kala & K. Kłaczyński (1983). The matrix inequality $M \geq B^*MB$. *Linear Algebra and Its Applications* **54**, 77–86.

T. N. E. Greville (1974). Solutions to the matrix equation $XAX = X$ and relations between oblique and orthogonal projectors. *SIAM Journal on Applied Mathematics* **26**, 828–832.

R. E. Hartwig & R. Loewy (1992). Maximal elements under the three partial orders. *Linear Algebra and Its Applications* **175**, 39–61.

Solution 34-10.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

It is well-known that a complex square matrix M is an orthogonal projector if and only if M is idempotent and Hermitian, i.e., if and only if $M = M^2 = M^*$, where M^* stands for the conjugate transpose of M. Then, evidently, $M = MM^*$, i.e., orthogonal projectors are in particular nonnegative definite and Hermitian. For later use we mention that if M is an orthogonal projector, then $\mathcal{N}(M) = \mathcal{R}(I - M)$, with $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ denoting the null space and the range (column space), respectively, of the matrix $(\cdot)$.

$\underline{\Rightarrow:}$ Let P, Q and PQ be orthogonal projectors, i.e., let P, Q and PQ be idempotent and Hermitian. Then $(I-P)Q = [(I-P)Q]^* = [(I-P)Q]^2$, i.e., $(I-P)Q$ is also an orthogonal projector. Consequently, $0 \leq_{\mathrm{L}} (I-P)Q = Q - PQ = Q - PQP$ and so necessity is established.
$\underline{\Leftarrow:}$ Conversely, let $Q - PQP \geq_{\mathrm{L}} 0$. Since Q and P are orthogonal projectors, $Q \geq_L 0$ and $PQP \geq_L 0$, and so, clearly, $Q - PQP \geq_{\mathrm{L}} 0 \Rightarrow PQP\mathcal{N}(Q) = \{0\} \Leftrightarrow QP\mathcal{N}(Q) = \{0\} \Leftrightarrow QP(I-Q) = 0 \Leftrightarrow QP = QPQ \Leftrightarrow QP = PQ \Leftrightarrow QP = (QP)^* = (QP)^2 \Leftrightarrow QP$ is an orthogonal projector.

IMAGE Problem Corner: More New Problems

Problem 35-6: Spectral Representation of an Arbitrary Diagonalizable Complex Matrix
Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Suppose $A \in \mathbb{C}^{n\times n}$ has minimal polynomial $p(x) = (x - \lambda_1)\cdots(x - \lambda_k)$ with $\lambda_1, \ldots \lambda_k$ distinct. Let $p_i(x) = p(x)/(x - \lambda_i)$ and suppose $Q_i \in \mathbb{C}^{n\times n_i}$ has columns that form an orthonormal basis for the column space of $p_i(A)$, $1 \leq i \leq k$. Show that $AQ_i = \lambda_i Q_i$, $1 \leq i \leq k$, $\sum_{i=1}^{k} n_i = n$, and

$$A = \sum_{i=1}^{k} \lambda_i Q_i Q_i^* \frac{p_i(A)}{p'(\lambda_i)}.$$

Show that A is normal if and only if

$$Q_i^* \frac{p_i(A)}{p'(\lambda_i)} = Q_i^*, \quad 1 \leq i \leq k.$$

Problem 35-7: A Characterization of Oblique Projectors
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a square matrix with complex entries. Show that A is an oblique projector if and only if A is similar to an orthogonal projector.

Problem 35-8: A Characterization of a Particular Class of Square Complex Matrices
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Characterize the class of all square matrices A with complex entries satisfying the identity

$$A^+ + A^\sharp = 2A^+A^\sharp,$$

where A^+ and $A^\sharp$ denote the Moore-Penrose inverse and the group inverse of A, respectively.

Problem 35-9: A Range Equality for Idempotent Hermitian Matrices
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be idempotent and Hermitian matrices of the same order. Show that

$$\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q),$$

where $\mathcal{R}(\cdot)$ denotes the range of a matrix.

Problems 35-1 through 35-5 are on page 44.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to hjw.de@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.

Problem 35-1: An Upper Bound for the Norm of a Matrix Exponential

Proposed by Ken DRIESSEL, *Colorado State University, Fort Collins, Colorado, USA:* driessel@math.colostate.edu
and Wasin SO, *San Jose State University, San Jose, California, USA:* so@math.sjsu.edu

Let N be a strictly upper-triangular matrix with all ones.

1. Prove that $\|e^{-N}\| \leq \sqrt{e}$ where $\|\cdot\|$ is the spectral norm.
2. Is equality possible?

Problem 35-2: Matrix Polynomials as Group Inverses

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ nonzero matrix with complex elements. Establish general conditions under which the matrix polynomial $P(A)$ serves as a group inverse $A^{\#}$ of A:

$$AA^{\#} = A^{\#}A, \quad AA^{\#}A = A, \quad A^{\#}AA^{\#} = A^{\#}.$$

Develop this general result for (i) the case in which $\mathrm{rank}(A) = n-1$, and (ii) the case in which $A^{\#} = A^m$ for some value of $m > 0$.

Problem 35-3: Partitioned Inverse of a Matrix Product

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let $X = (\,X_1 \quad X_2\,)$ and $Y = (\,Y_1 \quad Y_2\,)$ be $n \times k$ matrices of rank k partitioned by their first k_1 columns and the remaining $k_2 := k - k_1$ columns in such a way that $X_1'Y_1$, $X_2'Y_2$ and $X'Y$ are nonsingular. Obtain explicit expressions for the columns of the $n \times k$ matrices $P = (\,P_1 \quad P_2\,)$ and $Q = (\,Q_1 \quad Q_2\,)$ partitioned conformably with $X = (\,X_1 \quad X_2\,)$ and $Y = (\,Y_1 \quad Y_2\,)$ in such a way that $P'Q = (X'Y)^{-1}$.

Problem 35-4: Which Sizes have the Matrices?

Proposed by Alexander KOVAČEC, *University of Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Consider a matrix P, at least three units higher than broad whose lu-diagonal (containing the left upper corner) is filled by 1's; whose rl-diagonal (containing the right lower corner) is filled by -1's; whose diagonal two units above the rl-diagonal is filled by 1's; and whose remaining entries are 0. Let b be a column of height of P of the form $b = (1, 0, \ldots, 0, 1, -1)'$. Which sizes, if any, are possible for P if $Px = b$ is solvable?

Problem 35-5: First and Second Moments Involving a Camouflaged Wishart Matrix

Proposed by Heinz NEUDECKER, *Universiteit van Amsterdam, Amsterdam, The Netherlands:* ericaengels173@hotmail.com

Consider the $m \times (n + m - 1)$ matrix $C' := (A', B')$, where $A \in \mathbb{R}^{m-1,m}$ is constant, $B \in \mathbb{R}^{n,m}$ is such that $B'B$ follows a central Wishart distribution $W(\Omega, n)$, $n > m + 1$, and where $\mathrm{rank}(C) = m$. As usual, let E denote the expected value operator and D denote the variance operator. Prove the following:

(i) $\mathsf{E}(\sum_i |(A', b_i)|^2) = n \cdot \mathrm{tr}(A'A)^*\Omega$, where, for $i = 1, 2, \cdots, n$, b_i denotes the i-th column vector of B', and where $|\cdot|$, $(\cdot)^*$, and $\mathrm{tr}(\cdot)$ stand for the determinant, the adjoint, and the trace, respectively, of $(\cdot)$.

(ii) $\mathsf{D}(\sum_i |(A', b_i)|^2) = n^2 \cdot \{\mathrm{tr}(A'A)^*\Omega\}^2 + 2n \cdot \mathrm{tr}\{(A'A)^*\Omega\}^2$.

Problems 35-6 through 35-9 are on page 43.