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IMAGE

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Forthcoming Conferences and Workshops in Linear Algebra

IMAGE Problem Corner: Old Problems, Most With Solutions

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Cramer's Rule as an Orthogonalisation Procedure

by **Richard William Farebrother**

1. General Orthogonalisation Procedure

In this article it is observed that Cramer's Rule, in addition to being a procedure for solving systems of linear equations, is the basis of a simple stepwise orthogonalisation procedure. Throughout $\mathbb{R}^n$ denotes the vector space of all 1 by n vectors with the standard dot-product.

The orthogonalisation problem at hand is: given a subspace of U of $\mathbb{R}^n$, how does one find an orthogonal basis of the orthogonal complement, $U^\perp$, of U?

First consider the special case that $\dim(U) = n-1$. Let $\mathbf{A}$ be an $(n-1)\times n$ matrix whose rows are a basis for U. Clearly $\mathbf{A}$ has rank $n-1$, and thus there exists an $(n-1)\times(n-1)$ nonsingular submatrix $\mathbf{B}$ of $\mathbf{A}$. Without loss of generality we may assume that $\mathbf{B}$ is formed by the last $n-1$ columns of $\mathbf{A}$. Partition $\mathbf{A}$ as $\mathbf{A} = \begin{bmatrix} \mathbf{b} & \mathbf{B} \end{bmatrix}$, where $\mathbf{b}$ is an $(n-1)\times 1$ vector. Note that the vector $\begin{bmatrix} -1 & \mathbf{x}^T \end{bmatrix}$ is orthogonal to each row of $\mathbf{A}$ if and only if the $(n-1)\times 1$ vector $\mathbf{x}$ satisfies $\mathbf{Bx} = \mathbf{b}$. By Cramer's rule, the solution $\mathbf{x} = [x_i]$ to $\mathbf{Bx} = \mathbf{b}$ is given by $x_i = \det \mathbf{B}_{i\leftarrow \mathbf{b}}/\det \mathbf{B}$, where $\mathbf{B}_{i\leftarrow \mathbf{b}}$ is the matrix obtained from $\mathbf{B}$ by replacing its ith column by $\mathbf{b}$. It follows that

$$(-1, \det \mathbf{B}_{1\leftarrow b}/\det \mathbf{B}, \det \mathbf{B}_{2\leftarrow \mathbf{b}}/\det \mathbf{B}, \ldots, \det \mathbf{B}_{(n-1)\leftarrow \mathbf{b}}/\det \mathbf{B})$$

is a basis for the orthogonal complement of U. Note that $\det \mathbf{B}_{i\leftarrow \mathbf{b}} = (-1)^{i+1} \det \mathbf{A}_i$ where $\mathbf{A}_i$ is obtained from $\mathbf{A}$ by deleting its ith column. Thus the vector

$$(-\det \mathbf{A}_1, \det \mathbf{A}_2, -\det \mathbf{A}_3, \ldots, (-1)^n \det \mathbf{A}_n)$$

of co-factors is also a basis for the orthogonal complement of U. Thus, Cramer's rule is the crux of a procedure for solving the orthogonalisation problem in the case that $\dim(U) = n-1$.

Now consider the general case. We start with a basis

$$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$$

of U, and construct vectors $\mathbf{w}_{p+1}, \mathbf{w}_{p+2}, \ldots, \mathbf{w}_{n-p}$ that are mutually orthogonal and orthogonal to each of $\mathbf{w}_1, \ldots, \mathbf{w}_p$ as follows. Assume that $\mathbf{w}_{p+1}, \ldots, \mathbf{w}_q$ have been constructed and $q < n$. To construct $\mathbf{w}_{q+1}$, form the $q \times n$ matrix whose rows are $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$, and let $\mathbf{A}_q$ be any $q \times (q+1)$ submatrix of $\mathbf{W}_q$ of rank q. By applying Cramer's rule as in the case $p = n-1$, it follows that vector $\mathbf{v}_{q+1}$ of co-factors of $\mathbf{A}_q$ is a nonzero vector which is orthogonal to each row of $\mathbf{A}_q$. Hence the vector $\mathbf{w}_{q+1}$, whose entries in columns corresponding to those of A_q are those of $\mathbf{v}_{q+1}$ and whose other entries are 0, is a nonzero vector which is orthogonal to each of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$. This process is then repeated to find $\mathbf{w}_{q+2}, \ldots, \mathbf{w}_n$.

2. Illustrative Example

We illustrate this procedure in the case when $p = 1$ and U is the subspace spanned by the 1×4 vector $\mathbf{w}_1$ of all 1's.

Taking the first two elements in $\mathbf{w}_1$, and considering the cofactors of m_1 and m_2 in the 2×2 matrix

$$\begin{bmatrix} 1 & 1 \\ m_1 & m_2 \end{bmatrix}$$

we identify the first basis vector of the orthogonal complement of U:

$$\mathbf{w}_2 = [-1\ 1\ 0\ 0].$$

Augmenting $\mathbf{w}_1$ by this row, and taking the first three columns of the augmented matrix $\mathbf{W}_2$ and considering the cofactors of $m_1,\ m_2$ and m_3 in the 3×3 matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

we identify the second basis vector of the orthogonal complement of U:

$$\mathbf{w}_3 = [-1\ -1\ 2\ 0].$$

Again augmenting $\mathbf{W}_2$ by this row and taking the first four columns of the augmented matrix $\mathbf{W}_3$, and considering the cofactors of $m_1,\ m_2,\ m_3$ and m_4 in the 4×4matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ m_1 & m_2 & m_3 & m_4 \end{bmatrix}$$

we identify the third (and last) basis vector of the orthogonal complement of U:

$$\mathbf{w}_4 = \begin{bmatrix} -1 & -1 & -1 & 3 \end{bmatrix},$$

where, for simplicity, the first element of $\mathbf{w}_4$ has been normalised to -1.

The successive basis vectors of the orthogonal complement of $\mathbf{w}_1$ identified in this way are known as Helmert's transformation. Further, if these rows are rescaled to unit length (by dividing them through by $\sqrt{2},\ \sqrt{6},\ \sqrt{12},\ \ldots$), then the corresponding columns has an interesting geometrical interpretation, see Farebrother (2002, p. 41).

3. Concluding Remarks

The orthogonalisation procedure described in Section 1 is numerically expensive and the corresponding solution procedure is rarely mentioned in modern textbooks of computational linear algebra. However, it is familiar in the 2×3 case when it defines the outer product of two 1×3 row vectors. The same technique was also used implicitly by de la

Vallée Poussin's in each iteration of his 1911 minimax absolute residual line fitting procedure, see Farebrother (1999, pp. 41-43) for details.

In principle, the same objection should be lodged against the practical use of Cramer's Rule if n is at all large. I recollect having to advise a colleague in the early 1970s that he could not hope to solve a system of 240 equations in 240 unknowns by Cramer's Rule in much less than 150,000 years as the University of Manchester's CDC 7600 mainframe computer was only able to perform one million multiplications per second.

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Books on Determinants 1841–51

by **Richard William Farebrother**

In his article on the history of determinants, Knobloch (1994, p. 772) notes

> In 1841 Carl Jacobi published three decisive treatises: 'On the formation and the properties of determinants', 'On functional determinants' and 'On alternating functions and their division by the product of the differences of the elements' ...

Knobloch (1994, p.773) continues:

> Around the middle of the nineteenth century, a thorough knowledge of determinant theory was limited largely to Jacobi's pupils. But in 1841 Arthur Cayley published the first English contribution to the new theory. He was the first to delimit the array of elements by two upright lines ...

Indeed, Cayley's 1841 paper served to define the upper limit for the whole of the first (1906) volume of Muir's (1906–1930) five-volume work as is apparent from the title of the first part of Volume I published in 1890, see Farebrother, Jensen and Styan (2002, p. 6).

These statements would seem to necessitate some amendment to the assertion made by Farebrother, Jensen and Styan These statements would seem to necessitate some amendment to the assertion made by Farebrother, Jensen and Styan (2002, p.7):

> The first books on determinants seem to be Jacobi's *'De functionibus alternantibus'* (1841), the 16-page booklet by Cayley (1844) and the 63-page book by Spottiswoode (1851) ...

Now, there are two potential errors in this statement (and in the associated list of books on determinants) which I failed to spot at the proof-reading stage (as my electronic equipment was malfunctioning):

(a) the single entry for Jacobi in the list of books on determinants actually corresponds to a series of three papers originally published as separate articles in Latin in 1841 and subsequently published in German translation in book form in 1896, see Muir (1906, pp. 253, 325, 358) and (1923, pp.63, 189, 257).

However, it is not clear to me why the subsequent publication of Jacobi's work in book form should be sufficient to depose Spottiswoode's 1851 textbook from its primacy of publication, compare Knobloch (1994, p. 773).

(b) Nor does it seem possible to maintain Cayley's claim for priority when it seems more reasonable to regard the simultaneous publication of Cayley's (1844) paper by the Pitt Press in the light of an offprint intended for distribution to his colleagues and students.

I consulted Stephen Stigler on this point and he (personal communication, 14 May 2004) replied:

> I do not know the history of the offprint ... The early separates of Laplace I have are all extracts [cut from bound volumes] ... After 1800 reprints with printed covers start to show up but very slowly – only after 1850 are they moderately common.

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(cont. on p. 5)

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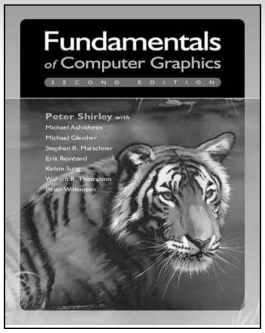

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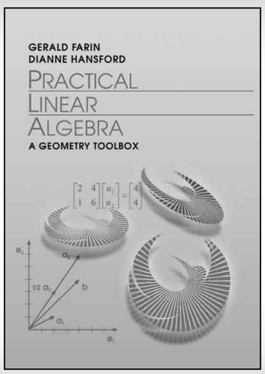

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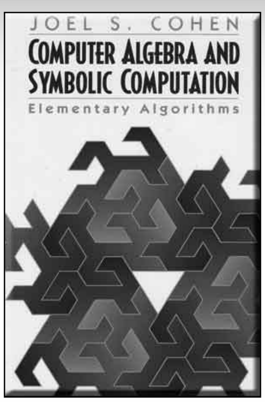

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(Books on Determinants, cont'd from p. 3.)

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ILAS INFORMATION CENTER

The electronic ILAS INFORMATION CENTER (IIC) provides current information on international conferences in linear algebra, other linear algebra activities, linear algebra journals, and ILAS-NET notices. Organizations and individuals are invited to contribute information to this database. Information on how to use IIC can be obtained from Shaun Fallat (sfallat@math.uregina.ca), IIC manager. The primary website can be found at:

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Eugene Tyrtyshnikov elected to Russian Academy of Sciences

Eugene Tyrtyshinikov has been elected as a Corresponding Member of the Russian Academy of Sciences. Founded in 1724 by Peter I the Great, Rossiiskaya Akademiya Nauk (RAS) was opened in 1725 by his widow, Catherine I. It is the highest scientific society and principal coordinating body for research in natural and social sciences, technology, and production in Russia. The highly prestigious life-long membership in the academy is by election, and members can be one of three ranks Academician, Corresponding Member, or Foreign Member. The academy directs the research of other scientific institutions and institutions of higher education, such as Steklov Institute of Mathematics and the Institute of Numerical Mathematics. The Academy includes departments of biology; chemistry and materials; earth sciences; engineering; information science and computer technology; history; mathematics; mechanics, and control processes; philology; physics; and social sciences. Its membership exceeds 1400, with roughly 700 corresponding members, 500 academicians, and 200 foreign members. The membership is a fitting recognition of Eugene's contributions to Numerical Analysis and Linear Algebra in the topics of structured matrices, iterative methods approximation theory, and parallel computations.

New Journal: *Operators and Matrices*

The first issue of a new linear algebra journal, *Operators and Matrices* ('OaM'), will appear in March 2007. OaM's aim is to develop a high standard international journal which will publish top quality research and expository papers in matrix and operator theory and their applications. The journal will publish mainly pure mathematics, but occasionally papers of a more applied nature will be considered. OaM will also publish relevant book reviews. OaM will published quarterly, in March, June, September and December. Articles will be submitted at the Operators and Matrices Web site at http://oam.ele-math.com.

Chi-Kwong Li, Josip Pecaric and Leiba Rodman are its Editors-in-chief, Zlatko Drmac its Managing Editor, and Damir Bakic its Associate Editor. The Editorial board consists of: Daniel Alpay, Tsuyoshi Ando, Hari Bercovici, Rajendra Bhatia, Albrecht Boettcher, Richard A. Brualdi, Raul Curto, Douglas R. Farenick, Takayuki Furuta, Fritz Gesztesy, Moshe Goldberg, Donald Hadwin, Frank Hansen, J. William Helton, Birgit Jacob, Fuad Kittaneh, David R. Larson, Bojan Magajna, Roy Mathias, Scott McCullough, Michael Neumann, Matjaz Omladic, Vladimir V. Peller, Heydar Radjavi, Andre Ran, Peter Šemrl, Ilya M. Spitkovsky, Kresimir Veselic, Ngai-Ching Wong, and Pei-Yuan Wu.

OaM will be published by ELEMENT headquartered in Zagreb, Croatia.

Richard A. Brualdi receives the Hans Schneider Prize in Linear Algebra

by **Michael Neumann**

Editor's note: At the 13th ILAS Conference in Amsterdam, Richard Brualdi was presented the Hans Schneider Prize in Linear Algebra. Richard Varga was co-recipient of the prize and received his prize at the preceding ILAS Conference in Regina. Below are excerpts from the laudaio given by Michael Neumann on behalf of the selection committee. Michael would like to acknowledge the help of Mona Wasow and Bryan Shader.

Introduction

I have a question for the entire audience here: *What is common to the following areas of linear algebra and combinatorics: matrix scalings, eigenvalue-inclusion regions, doubly stochastic matrices, combinatorial matrix theory, algebraic graph theory, qualitative matrix theory, coding theory, and matroid theory?!*

In case you are having difficulties with the question, let me help you with the answer. They are all areas in which Professor Richard Brualdi has made some of his many contributions. I will begin by outlining Richard's body of achievements. You may think that I'm showering him with compliments, but having gone through his record, I can tell to you to the contrary, that this is an underestimation.

- Currently he is the University of Wisconsin Foundation's Beckwith Bascom Professor of Mathematics.
- He has published 203 research papers. Some very influential. Many with beautiful mathematics.
- With the book that came out just now, he has published 3 books and a monograph. I want to mention one of them in particular. It is his 1991 Cambridge University Press book, co-authored with Herb Ryser, whose title is *Combinatorial Matrix Theory*. The reason for this special mention is because later on we shall talk about Richard's dedication to his many PhD students. But Richard showed loyalty and dedication also to his own mentors, in particular to his advisor Herb Ryser. This is a book that he helped finish posthumously out of total dedication.
- He has a Master's and PhD degrees from Syracuse University. His B.S. degree, though, is from the University of Connecticut.
- He has had 32 PhD students who completed their theses and he currently supervises 3 students.
- Throughout his career, he has been on the editorial board of over 20 journals, including every journal that specializes i matrix theory and linear algebra. The most familiar to us is his position as Editor–in–Chief of *Linear Algebra and its Applications* (or LAA) since 1979.
- He has given over 90 invited addresses all over the world, some in very major conferences.
- He has been on the ILAS Board of Directors for a term of two years and President of ILAS for a total of 6 years or two terms.
- He has been the Secretary of the SIAM Activity Group on Linear Algebra for 2 years and he was also the Editor of its Newsletter.
- He has been an organizer of many conferences and workshops with a good proportion of them on linear algebra and under the auspices of either ILAS or SIAM. He also chaired, from 1991 to 1992, the Special Year on Applied Linear Algebra at the Institute of Mathematics and its Applications in Minneapolis.
- He has had continuous federal funding from various agencies for over 35 years.
- He has been a Chair of the Mathematics Department at the University of Wisconsin for 6 years.

You may wonder, could I have ordered Richard's activities according to some criteria and presented them to you in a more systematic way? My answer is no, because in many cases they all occur simultaneously! How can anyone chair a huge math department such as that at the University of Wisconsin, be the Editor–in–Chief of LAA, be the President of ILAS, do research and direct graduate students, while tending to a union of two families? In any case you can see Richard's tremendous qualities of leadership. Indeed, as I was being considered for the headship of the Mathematics Department at the University of Connecticut, I called Richard, knowing that he has had the experience of combining research, teaching, and a heavy administrative load. I heard from him nothing but, go for it if you feel you can make the difference for your colleagues and for your department.

You can see why it wouldn't pay to mess with Richard. It would also be hard to joke with him unless you think that "Combinatorialists do it discretely" is funny. So you may wonder, does this guy have any imperfections? Well, I worked hard to find one and I was told by an inside source that Richard gets the top of the frigidaire confused with his running socks drawer. As you may know, he is an avid runner and one of the best in his age group in Madison. So he has many certificates for winning or participating in running competitions. These he hangs on the top of his frigidaire, while the many math awards that he wins he puts in his running socks drawer.

Brualdi's Scientific Contributions

For over 40 years Richard has been a key figure in the development of several areas in combinatorics and linear algebra, some of which we listed earlier. His contributions are profound. Let us detail some of them.

- Who here has not used Brualdi's 1966 *J. Math. Anal. Appl.* joint paper with Parter and Schneider on the "Diagonal equivalence of a non-negative matrix to a stochastic matrix"**?** This is a seminal paper in the field of matrix scalings.
- In a 1968 LAA Vol. 1 paper, Brualdi and Wielandt give an extremely powerful and useful spectral characterization of the doubly stochastic matrices. In a further sequence of four papers between 1976 and 1977, Brualdi and Gibson offer a deep study of the combinatorial and geometric properties of the polytope of doubly stochastic matrices.
- From the perspective of combinatorics, the class of (0,1)–matrices with a given row and column sum forms an important class of matrices. Brualdi has made major contributions to the understanding of this class, and the more general class of nonnegative matrices with fixed row and column sums. Most notable is his 1981 *European J. of Combinatorics* paper "On Haber's minimum rank term formula", where he adopts a more general approach for observing the combinatorial properties of this class and applies it to derive a very elegant, but deep new proof to an extremely difficult theorem by Haber.
- In his 1982 *Linear and Multilinear Algebra* paper "Matrices, eigenvalues and directed graphs", which is on eigenvalue inclusion regions, Richard uses graph theory to give a sweeping generalization of important results of Levy-Desplanques, Gersgorin, Ostrowski, Taussky, and Varga. His paper showed that matrices should be regarded as both analytical and combinatorial objects. To get a sense of how important this paper is, may I refer you to a recent book by Richard Varga, our co-winner of the Hans Schneider Prize. He calls Richard an expert in the area and he ranks and judges the paper as a seminal one in the subject. Varga's book is simply full of Brualdi.
- In a 1987 LAA paper joint with Hoffman "On the spectral radius of (0,1)-matrices", Brualdi provides very useful tools for relating the spectral radius of a (0,1)-matrix with the combinatorial properties of its underlying digraph.
- The famous Graham-Pollak-Witsenhausen theorem asserts that it requires at least n-1 colors to color the edges of a complete graph on n vertices, so that the edges of each color form a special kind of subgraph called a biclique. In a 1991 *J. of Combinatorial Theory* paper with Alon and Shader, Richard gives an elegant generalization of this theorem.

Now let me turn to some of his recent work. In a recent *J. of Combinatorial Theory* paper with Kirkland entitled "Aztec diamonds and digraphs, and Hankel determinants of Schröder numbers", Richard contributes to the area of Enumerative Combinatorics, specifically, a new method is developed for counting the number of domino tilings of the Aztec diamond. In another recent *Electronic Linear Algebra* work "Vector majorization via Hessenberg matrices", Richard and Hwang significantly strengthened the classic result for vector majorization by showing that the doubly-stochastic matrix involved, can be taken to be permutationally equivalent to a Hessenberg matrix. The linear preserver problem has a long history. It has attracted the attention of great mathematicians such as Frobenius, Dieudonne, and Dynkin. In a 2003 special issue of LAA, Richard establishes a beautiful connection between the theory of linear preservers and graph theory, and he uses this connection and graph theoretic results to establish several new results on this topic.

Other Attributes: Collegiality and Personal Life

As an LAA Associate Editor for 20 years, I have had many correspondences with Richard. You never have to wait long for an answer from him. In all aspects of one's work as an editor, Richard gives such a level of comfort about what you want to recommend or do, that it is simply a pleasure to work with him. His quick response to e-mails, even when he was chair of his department and LAA Editor–in–Chief and ILAS President, has led some to conjecture that Richard has a clone who does part of his work.

As you know, Richard and Mona Wasow, who is here, have been married for over 10 years. Together they have 5 children and 8 grandchildren from previous marriages. Mona speaks of the wonderful way in which Richard has accepted her children. In particular, one of her sons has a debilitating mental illness. That son has come to trust and await the affectionate way that Richard has taken to him.

Richard is a very gracious host when you come to his home. Did you know that Richard loves cooking Italian and vegetarian dishes? He loves gardening. He is a very socially conscientious person. And he is kind and gentle with all people.

We asked earlier how can a person accomplish so much at work, with the family, and at home. It is because he has a very strong work discipline and ethics. For example, his daily routine consists of riding his bike to the office every day, even when it's 30 degrees below and putting in a few hours of work before most of his colleagues even arrive at their office. But it goes well beyond this. We mentioned that he has had 32 PhD students who completed their theses and 3 on the way. Richard's happiest mathematical times are when he is supervising his PhD students or mentoring to his postdoctoral students. He reserves for each student a weekly corner of time. I have heard reports about the enormous length to which Richard goes to help his students and postdocs settle down in Madison. Particularly, if they come from foreign countries.

In conclusion, I submit to you gentle audience, that if ever there was a fitting candidate to receive the Hans Schneider Prize, it is Richard Brualdi. Richard, I congratulate you.

ILAS President/Vice President Annual Report April 2006

1. The following were elected in the ILAS 2005 elections to offices with terms that began on March 1, 2006 and end on February 28, 2009:

Board of Directors: Shmuel Friedland and William Watkins.

The following continue in ILAS offices to which they were previously elected:

President: Daniel Hershkowitz (term ends February 29, 2008)
Vice President: Roger Horn (term ends February 28, 2007)
Secretary/Treasurer Jeff Stuart (term ends February 28, 2009)

Board of Directors:

Ilse Ipsen (term ends February 29, 2008)
Roy Mathias (term ends February 28, 2007)
Joao Filipe Queiro (term ends February 28, 2007)
Reinhard Nabben (term ends February 29, 2008).

Rafael Bru and Hugo Woerdeman completed their three-year terms on the ILAS Board of Directors on February 28, 2006.

We thank the members of the nomination committee (Steve Kirkland - chair, Angelika Bunse-Gerstner, Naomi Shaked-Monderer, Bit-Shun Tam and Goetz Trenkler) for their efforts on behalf of ILAS.

2. Two ILAS-endorsed meetings took place since our last report:

The Householder Meeting on Numerical Linear Algebra, May 23-27, 2005, Campion, USA

Workshop on the Teaching of Linear Algebra, March 25, 2006, Drexel University, Philadelphia, USA.

3. ILAS has endorsed the following conferences of interest to ILAS members:

Fifteenth International Workshop on Matrices and Statistics, June 13-17, 2006, Uppsala, Sweden

Eighth Workshop on Numerical Ranges and Numerical Radii, July 15-16, 2006, Universitat Bremen, Bremen, Germany. Man-Duen Choi is the ILAS Lecturer.

Second International Workshop on Matrix Analysis and Applications, December 15-16, 2006, Nova University, Fort Lauderdale, USA.

POSTA06, The Second Multidisciplinary International Symposium on Positive Systems: Theory and Applications, August 30–September 1, 2006, Grenoble, France.

4. The following ILAS conferences are scheduled:

13th ILAS Conference, Amsterdam, The Netherlands, July 19-22, 2006 (Chairman of the organizing committee is Andre C.M. Ran. Local organizers: Andre Ran, Andre Klein, Peter Spreij and Jan Brandts).

14th ILAS Conference, Shanghai, China, July 16-20, 2007 (Organizing Committee: Richard Brualdi - co-chair, Erxiong Jiang - co-chair, Raymond Chan, Chuanqing Gu, Danny Hershkowitz - ILAS President, Roger Horn, Ilse Ipsen, Julio Moro, Peter Šemrl, Jia-yu Shao and Pei Yuan Wu). Arrival Registration on Sunday, July 15, 2007.

15th ILAS Conference, Cancun, Mexico, June 16-20, 2008 (Chairman of the organizing committee is Luis Verde).

5. ILAS is also a partner in organizing the joint GAMM-SIAM conference on Applied Linear Algebra, Dusseldorf, July 24-27, 2006 (which continues the SIAM series of conferences on Applied Linear Algebra). Ludwig Elsner and Jim Demmel will be ILAS Lecturers at that conference.

6. The *Electronic Journal of Linear Algebra* (ELA) is now in its 15th volume. Its editors-in-chief are Ludwig Elsner and Danny Hershkowitz.

Volume 1, published in 1996, 6 papers.
Volume 2, published in 1997, 2 papers.
Volume 3, the Hans Schneider issue, published in 1998, 13 papers.
Volume 4, published in 1998 as well, 5 papers.
Volume 5, published in 1999, 8 papers.
Volume 6, Proceedings of the Eleventh Haifa Matrix Theory Conference, published in 1999 and 2000, 8 papers.
Volume 7, published in 2000, 14 papers.
Volume 8, published in 2001, 12 papers.
Volume 9, published in 2002, 24 papers.
Volume 10, published in 2003, 25 papers.
Volume 11, published in 2004, 22 papers.
Volume 12, Proceedings of the 2004 Workshop on Non-negative Matrices, Maynooth, 8 papers.
Volume 13, published in 2005, 25 papers.
Volume 14, Proceedings of the 2004 workshop "Directions in Combinatorial Matrix Theory", Banff, 6 papers.
Volume 15 is being published in 2006; as of now it contains 14 papers.

Acceptance percentage in ELA is currently 54%.

ELA's primary site is at the Technion. Mirror sites are located at: Temple University, University of Chemnitz, University of Lisbon, EMIS - The European Mathematical Information Service offered by the European Mathematical Society, and in EMIS's more than 40 Mirror Sites.

7. IMAGE - The Bulletin of ILAS is edited by Bryan Shader and Hans Joachim Werner. ILAS members may elect to receive either a print version of IMAGE or an electronic version.

8. ILAS-NET is managed by Shaun Fallat. As of April 30, 2006, we have circulated 1522 ILAS-NET announcements.

9. The primary site of ILAS INFORMATION CENTER (IIC) is at Regina. Mirror sites are located at: The Technion, Temple University, University of Chemnitz, and University of Lisbon.

Respectfully submitted,

Daniel Hershkowitz, ILAS President,
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Roger Horn, ILAS Vice-President,
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Recently Published Books in Linear Algebra

Combinatorial Matrix Classes by Richard A. Brualdi, (Encylopedia of Mathematics and its Applications, vol. 108, ix + 544 pages, Cambridge University Press, 2006)

This book contains a thorough treatment of many important classes of (0,1)-matrices and nonnegative matrices in general (e.g. doubly stochastic matrices). It is a sequel to *Combinatorial Matrix Theory* by Brualdi and H. J. Ryser.

The Science of Search Engine Rankings by Amy N. Langville and Carl D. Meyer (224 pages, Princeton University Press, 2006).

This is the first book ever about the science of Web page rankings. The book covers the mathematics of Google's PageRank including chapters on: the PageRank problem as a Linear System, methods for accelerating the computation of Pagerank, methods for updating PageRank, amd the sensitivity of PageRank. The book contains several MATLAB codes and links to sample Web data sets.

Differential-Algebraic Equations: Analysis and Numerical Solution by Peter Kunkel and Volker Mehrmann (EMS Textbooks in Mathematics, 2006)

Differential-algebraic equations are a widely accepted tool for the modeling and simulation of constrained dynamical systems in numerous applications, such as mechanical multibody systems, electrical circuit simulation, chemical engineering, control theory, fluid dynamics and many others. This is the first comprehensive textbook that provides a systematic and detailed analysis of initial and boundary value problems for differential-algebraic equations. The analysis is developed from the theory of linear constant coefficient systems via linear variable coefficient systems to general nonlinear systems. Further sections on control problems, generalized inverses of differential-algebraic operators, generalized solutions, and differential equations on manifolds complement the theoretical treatment of initial value problems. Two major classes of numerical methods for differential-algebraic equations (Runge-Kutta and BDF methods) are discussed and analyzed with respect to convergence and order. A chapter is devoted to index reduction methods that allow the numerical treatment of general differential-algebraic equations. The analysis and numerical solution of boundary value problems for differential-algebraic equations is presented, including multiple shooting and collocation methods. A survey of current software packages for differential-algebraic equations completes the text.

The book is addressed to graduate students and researchers in mathematics, engineering and sciences, as well as practitioners in industry. A prerequisite is a standard course on the numerical solution of ordinary differential equations. Numerous examples and exercises make the book suitable as a course textbook or for self-study.

New Editor-in-chief for *IMAGE*

IMAGE welcomes aboard Jane Day as Editor-in-chief. Jane will be replacing Bryan Shader who recently ended his term. Jane will be serving alongside Hans Joachim Werner.

Call for Submission to *IMAGE*

IMAGE welcomes expository articles on emerging applications and topics in Linear Algebra, announcements of upcoming meetings, reports on past conferences, historical essays on linear algebra, book reviews, essays on the development of Linear Algebra in a certain country or region, and letters to the editor or signed columns of opinion. Contributions for *IMAGE* should be sent to either Jane Day (jday@math.sjsu.edu) or Hans Joachim Werner (hjw.de@uni-bonn.de). The deadlines are October 15 for the fall issue, and April 15 for the spring issue.

ILAS 2005 - 2006 Treasurer's Report
March 1, 2005 through February 28, 2006

Net Account Balances on February 28, 2005			
Vanguard (ST Fed. Bond Fund 3661.909 Shares) (10.60% Each: General Fund, Conference Fund and ILAS/LAA Fund, 17.40% Taussky Todd Fund, 7.95% Uhlig Fund, 42.85% Schneider Fund)		$37,864.14	
Checking account		$44,323.09	
Pending checks		$ 2,200.00	
Pending VISA/Mastercard/AMEX		$ 1,350.00	
Outstanding checks payable		($ 685.56)	$85,051.67
General Fund		$34,288.60	
Conference Fund		$10,625.82	
ILAS/LAA Fund		$ 7,882.27	
Olga Taussky Todd/John Todd Fund		$ 9,354.12	
Frank Uhlig Education Fund		$ 3,901.71	
Hans Schneider Prize Fund		$18,999.15	$85,051.67
Income:			
Dues	5380.00		
Corporate Dues	1000.00		
Book Sales	31.00		
General Fund	1405.59		
Conference Fund	108.99		
ILAS\LAA Fund	1088.99		
Taussky-Todd Fund	286.07		
Uhlig Education Fund	86.74		
Schneider Prize Fund	729.72	10,117.10	
Expenses:			
IMAGE (2 issues)	2034.17		
Speakers (4)	3560.00		
ILAS Conference Support	3000.00		
ILAS Web Development	435.00		
Credit Card and Bank Fees	234.87		
License Fees	61.25		
Labor - Mailing & Conference	420.00		
Postage	577.22		
Supplies and Copying	365.86		
Exchange losses, errors	44.94	10,725.10	
Net Account Balances on February 28, 2006			
Vanguard (ST Fed. Bond Fund 3787.048 Shares) (10.60% Each: General Fund, Conference Fund and ILAS/LAA Fund, 17.40% Taussky Todd Fund, 7.95% Uhlig Fund, 42.85% Schneider Fund)		$38,703.63	
Checking account		$43,551.83	
Pending checks		$ 1,910.00	
Pending VISA/Mastercard/AMEX		$ 1,020.00	
Outstanding checks payable		($ 750.00)	$84,435.46
General Fund		$33,431.88	
Conference Fund		$10,734.81	
ILAS/LAA Fund		$ 7,971.26	
Olga Taussky Todd/John Todd Fund		$ 9,640.19	
Frank Uhlig Education Fund		$ 3,988.45	
Hans Schneider Prize Fund		$18,668.87	$84,435.46

Prepared by

Jeffrey L. Stuart, Secretary-Treasurer
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Special Session on Combinatorial Matrix Theory at the Fall Central Section Meeting of the AMS

Report by **Luz M. DeAlba**

A Special Session on Combinatorial Matrix Theory was held October 21-23, 2005 at the Fall Central Section Meeting of the American Mathematical Society, which took place at the University of Nebraska in Lincoln. The goals of this Special Session were to: allow the dissemination of recent results in the area, provide a platform for the exchange of ideas, and encourage graduate students to get involved in Combinatorial Matrix Theory.

The organizers of the session were Leslie Hogben (Iowa State University) and Bryan Shader (University of Wyoming). Twenty invited talks were presented at the meeting, and nearly 30 mathematicians, including one from India and a number of graduate students, participated the three-day event. The session was nicely partitioned into four segments that showcased the areas of Sign Patterns, Minimal Rank and Maximum Multiplicity, Spectra of Graphs, and Completion Problems. The speakers were: Francesco Barioli, Wayne Barrett, Luz DeAlba, J. Ding, Shaun Fallat, Jason Grout, Frank Hall, Leslie Hogben, Brenda Kroschel, Chi-Kwong Li, Zhongshan Li, Judith MacDonald, Sivaram Narayan, Yiu T. Poon, Bhaba Kumar Sarma, Bryan Shader, Wasin So, Jeffrey Stuart, Kevin vander Meulen, and Amy Wangsness. The full program can be found at http://www.ams.org/amsmtgs/2117_program_ss10.html#title.

Two social events at local ethnic restaurants were held. Additional information and conference photos can be found at http://orion.math.iastate.edu/l\hogben/research/AMSLincoln.html or http://orion.math.iastate.edu/hogben/research/AMSLincolnPix.html.

Participants of the AMS Special Session on Combinatorial Matrix Theory

First Workshop on the Teaching of Linear Algebra

Report by **Pavel Grinfeld**

On March 25, 2006, the Drexel University Department of Mathematics hosted its first workshop on the Teaching of Linear Algebra. The workshop featured keynote talks by Peter Lax (NYU), the 2005 Abel Prize Laureate, and Gilbert Strang (MIT) as well as two thirty minute talks by invited speakers Robert Busby (Drexel University) and Frank Uhlig (Auburn University), four fifteen minute talks and three poster presentations. The workshop concluded with a five member panel discussion, moderated by David Lay, entitled "The Direction of Linear Algebra Teaching". The workshop enjoyed the participation of 80 enthusiastic researchers and educators from 35 universities and technical institutions. The workshop provided a forum for educators from around the country to exchange their experiences and ideas.

The question of what to teach in a "second Linear Algebra course" was discussed in several talks and was the primary focus of Peter Lax's talk titled "Linear Algebra! If you know enough". Peter Lax gave a variety of examples of subjects appropriate for an advanced Linear Algebra course.

In his talk, "Linear Algebra - A Happy Chance to Apply Mathematics", Gilbert Strang presented a number short "snippets" of a Linear Algebra course that illustrated how simple topics in Linear Algebra can stimulate the students' ability to do independent research at a very early stage in their career.

The participants of the panel discussion were Jane Day (San Jose University), Guershon Harel (University of California, San Diego), David Hill (Temple University), Steven Leon (University of Massachusetts, Dartmouth). The afternoon session concluded with an extended Q&A session between the workshop participants and the members of the panel.

The workshop was organized by Hugo Woerdeman, Pavel Grinfeld, and Herman Gollwitzer. The workshop was endorsed by ILAS and by SIAM. The resources related to the workshop can be found at http://www.math.drexel.edu.

Participants during the Workshop on the Teaching of Linear Algebra

The Eighth Workshop on Numerical Ranges and Numerical Radii

Report by **Chi-Kwong Li**, **Leiba Rodman**, and **Christiane Tretter**

The Eighth workshop on "Numerical Ranges and Numerical Radii" WONRA 2006 was held on July 15 - July 17, 2006, at the University of Bremen, Germany, in conjunction with the Thirteenth International Linear Algebra Conference, July 18 - July 21, 2006, in Amsterdam, the Netherlands. The workshop took place in the Department of Mathematics building.

WONRA 2006 was sponsored by the University of Bremen, and endorsed by the International Linear Algebra Society. Prof. Man-Duen Choi of the University of Toronto was the ILAS lecturer at WONRA 2006.

The meeting began with opening remarks by Dr. Angelika Bunse-Gerstner, Vice President for Research at the University of Bremen. Thirty two talks by thirty three people were given on various aspects of numerical ranges and radii, their generalizations and applications, including several talks on generalized numerical ranges in quantum computing, control, and information. List of participants, detailed schedule of talks, and abstracts can be found in

http://www.math.uni-bremen.de/aag/wonra06/programme.pdf.

The list of participants is also appended below.

According to a WONRA tradition, there was no registration fee, and as in previous WONRA meetings, participants exchanged many ideas, results and problems on the subject in a very friendly atmosphere. The meeting included dinner at the *(k)ukuk* restaurant and a sightseeing tour of old town Bremen guided by Christiane Tretter.

Workshop pictures can be found in

http://www.math.uni-bremen.de/aag/wonra06/pictures.html.

The proceedings of WONRA 2006 will appear as a special issue of *Linear and Multilinear Algebra*. Chi-Kwong Li, Leiba Rodman, and Christiane Tretter are the editors of the special issue. The deadline for submission of papers for the issue is December 31, 2006.

List of participants:

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Participants of the 8th Workshop on Numerical Ranges and Numerical Radii

The $15^{\rm th}$ International Workshop on Matrices and Statistics

Report by **Hans Joachim Werner**

The $15^{\rm th}$ International Workshop on Matrices and Statistics (IWMS-2006) was held at Uppsala University (Uppsala, Sweden), on June 13–17, 2006. Uppsala University, in Swedish *Uppsala universitet*, founded in 1477, is the oldest University in Scandinavia. This university, which for centuries has been an important place of science, represented by names such as Linnaeus, Celsius and Ångström, and in the last century by several Nobel Laureates such as Svante Arrhenius (Laureate in Chemistry, 1903), Manne Siegbahn (Laureate in Physics, 1924), his son Kai Siegbahn (Laureate in Physics, 1981), and Theodor Svedberg (Laureate in Chemistry, 1926), was a perfect and exciting place to hold our annual meeting this year.

Our Workshop, which was hosted by the Mathematics and Information Technology Centre at Uppsala University, was supported by the Centre of Biostochastics, the Scandinavian Airline (SAS), the Swedish Research Council, the Swedish Statistical Association, the Swedish Research Council for Environment, Agricultural Sciences & Spatial Planning, SPSS Sweden AB, the Linnaeus Centre for Bioinformatics, the MathWorks, the Tampereen Yliopiston Tukisäätiö, and by John Wiley & Sons, Ltd. It was also endorsed by the International Linear Algebra Society (ILAS).

The Scientific Organizing Committee (SOC) for this workshop consisted of Hans Joachim Werner (Bonn, Germany; chair), R. William Farebrother (Shrewsbury, England), Augustyn Markiewicz (Poznań, Poland), Simo Puntanen (Tampere, Finland), George P. H. Styan (Montréal, Québec, Canada), and Dietrich von Rosen (Uppsala, Sweden). The Local Organizing Committee (LOC) comprised Razaw Al Sarraj (Uppsala), Zhanna Andrushchenko (Uppsala), Johannes Forkman (Uppsala), Kristi Kuljus (Uppsala), Tatjana Nahtman (Tartu, Estonia), Maya Neytcheva (Uppsala), and was chaired by Dietrich von Rosen.

The Workshop included the presentation of both invited and contributed papers. The invited speakers were Theodore W. Anderson (USA), Åke Björck (Sweden), Gene H. Golub (USA), David Harville (USA), Sabine van Huffel (Belgium), Jeffrey J. Hunter (New Zealand), Thomas Mathew (USA), João Tiago Mexia (Portugal), Ingram Olkin (USA), Friedrich Pukelsheim (Germany), Youseff Saad (USA), and Muni Srivastava (Canada). In addition there was a special session in Honour of Dr. Tarmo Pukkila's $60^{\rm th}$ Birthday organized and chaired by Erkki Liski.

The purpose of the workshop was to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between matrix theory and statistics. Additional emphasis was put on related numerical linear algebra issues and numerical solution methods, relevant to problems arising in statistics. More than 68 participants from 13 different countries joined this workshop. It is expected that many of the statistically oriented papers presented at this Workshop will be submitted to *Acta et Commentationes Universitatis Tartuences de Mathematica* while a special issue of the journal *Numerical Linear Algebra with Applications (NLA)* will be devoted to those papers dealing with numerical linear algebra issues and numerical solution methods.

On Friday, June 16, there was an Afternoon Outing to Sigtuna and to the Skokloster Castle. Sigtuna is the earliest, still existing, Swedish town. Although less important today, Sigtuna has an important place in Sweden's early history. About 1000 years ago, it was founded, probably by King Eric Segersäll, on what was then the shore of lake Mälaren. Sigtuna operated as a royal and commercial centre for about 250 years. In the $11^{\rm th}$ century Sigtuna became a diocesan town. Although already 100 years later the archbishop moved to Östra Aros (Uppsala), Sigtuna remained an important city for a while. Many church and monastery ruins still stand, and as the old city structure has not been remodelled, it was a real pleasure to visit Sweden's first town which has it all - beauty, history and exciting tales to tell. A guide brought us on an excellent historical journey, with interesting information on history, tales of the many runic stones and the history of the churches. Later in the afternoon we also visited Skokloster Castle which is located at lake Mälaren between Stockholm and Uppsala. This castle was built in baroque style between 1654 and 1676 by the Field Marshal Carl Gustaf Wrangel (after plans by Nicodemus Tessin the Elder) displaying his power and success during Sweden's Age of Greatness. After Wrangel died in 1676 this castle was owned by the families Brahe and von Essen. In 1967 the Swedish government bought this castle and opened it as a museum. There are rooms where the time has stood still. So the large banqueting hall has never been completed and is still in the same condition as the builders have left it in 1676. During a guided tour through Skokloster Castle we could enjoy the castle's unique interiors containing thousands of exhibits such as weapons, tools, furniture, silver, textiles, books and ethnographic rarities, all extremely gorgeous and remarkable. In the evening of the same day, a delicious Workshop Dinner was served in the garden of Skokloster Castle.

As our previous meeting in Auckland (New Zealand) last year, this Workshop in Uppsala again provided an extremely good atmosphere to stimulate contacts and exchange ideas. The Workshop Programme is still downloadable from the Workshop web page

`http://www.bt.slu.se/iwms2006/iwms06.html`

A group photograph can be found on page 19.

Group photo (through the camera of Jeffrey J. Hunter) of the 15th IWMS Conference

13th ILAS Meeting

Report by **Andre Ran**

The 13th ILAS meeting was held in Amsterdam from July 18 to 21. The meeting was well-attended with 224 registered participatns from six different continents. Among the participants were many PhD students, attesting to the fact that the area's future is secure. Despite the unusually hot weather, the participants enjoyed the plenary talks as well as the session talks.

The conference has several highlights, from the start with LAA speaker, Bob Plemmons, via the middle, for instance the LAMA lecturer Steve Kirkland, and to the end with Hans Schneider prize winner Richard Brualdi. But also many of the other plenary and session talks were mentioned by some of the participants as the talk that was for them the high point of the conference. Peter Lancaster, as after dinner speaker, treated the participants to a slide show through the the history of linear algebra, with many pictures from his albums, and his often humorous comments.

The organizers gratefully acknowledge support from several sources. In no particular order these are: the publishing companies Elsevier BV and Taylor and Francis, the two universities in Amsterdam: University of Amsterdam and the Vrije Universiteit Amsterdam, NWO (the Dutch Science Foundation), the Steiltjes Institute, the KNAW (the Royal Academy of Sciences of Netherland), and the Stichting Advancement of Mathematics. The city of Amsterdam welcomed the participants on Tuesday evening in the City Hall with a reception.

Western Canada Linear Algebra Meeting

Report by **Dale Olesky**

The seventh Western Canada Linear Algebra Meeting (WCLAM) was held on June 23-24, 2006 at the University of Victoria, with generous funding provided by the Pacific Institute for the Mathematical Sciences, and the Faculty of Engineering and the Faculty of Science at the University of Victoria. The meeting attracted established researchers, graduate students and postdoctoral fellows with interests primarily in matrix analysis, combinatorial matrix analysis, numerical linear algebra and their applications. There were 45 attendees, which included 18 students and 5 postdoctoral fellows. Thirty of the attendees were from western Canada, eight from Washington state, and the remainder from other parts of the USA, England and India. The meeting honored Pauline van den Driessche on the occasion of her sixty fifth birthday.

Eighteen talks were presented over the two days. The three invited speakers were:

Richard Brualdi (Beckwith Bascom Professor of Mathematics, University of Wisconsin, Madison), The Bruhat Order for (0,1)-Matrices;

Anne Greenbaum (Professor, University of Washington), Characterizations of the Polynomial Numerical Hull; and

Mark Lewis (Canada Research Chair in Mathematical Biology, University of Alberta), The Basic Reproduction Number for Discrete- and Continuous-time Models.

Eight of the fifteen contributed talks were given by graduate students or postdoctoral fellows. Plenty of time was allocated for discussions and questions, and there was a 30 minute session at which open problems were presented and discussed.

Participants of the 7th W-CLAM Meeting

Recent papers published in ELA

Primary website:
http://www.math.technion.ac.il/iic/ela/

Ting-Zhu Huang, Wei Zhang and Shu-Qian Shen, *Regions containing eigenvalues of a matrix*, ELA 15 (2006), 215-224.

Abstract
In this paper, regions containing eigenvalues of a matrix are obtained in terms of partial absolute deleted row sums and column sums. Furthermore, some sufficient and necessary conditions for H-matrices are derived. Finally, an upper bound for the Perron root of nonnegative matrices is presented. The comparison of the new upper bound with the known ones is supplemented with some examples.

Semitransitivity Working Group at LAW'05, Bled, *Semitransitive subspaces of matrices,* ELA 15 (2006), 225-238.

Abstract
A set of matrices S in $M_n(F)$ is said to be semitransitive if for any two nonzero vectors x,y in F^n, there exists a matrix A in S such that either Ax=y or Ay=x. In this paper various properties of semitransitive linear subspaces of $M_n(F)$ are studied. In particular, it is shown that every semitransitive subspace of matrices has a cyclic vector. Moreover, if $|F| \geq n$, it always contains an invertible matrix. It is proved that there are minimal semitransitive matrix spaces without any nontrivial invariant subspace. The structure of minimal semitransitive spaces and triangularizable semitransitive spaces is also studied. Among other results it is shown that every triangularizable semitransitive subspace contains a nonzero nilpotent.

Maria Adam and Michael J. Tsatsomeros, *An eigenvalue inequality and spectrum localization for complex matrices*, ELA 15 (2006), 239-250.

Abstract
Using the notions of the numerical range, Schur complement and unitary equivalence, an eigenvalue inequality is obtained for a general complex matrix, giving rise to a region in the complex plane that contains its spectrum. This region is determined by a curve, generalizing and improving classical eigenvalue bounds obtained by the Hermitian and skew-Hermitian parts, as well as the numerical range of a matrix.

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R. Ben Taher, M. Mouline and Mustapha Rachidi, *Fibonacci-Horner decomposition of the matrix exponential and the fundamental system of solutions*, ELA 15 (2006), 178-190.

Abstract
This paper concerns the Fibonacci-Horner decomposition of the matrix powers A^n and the matrix exponential exp(tA) (A a rxr complex matrix and t real), which is derived from the combinatorial properties of the generalized Fibonacci sequences in the algebra of square matrices. More precisely, exp(tA) is expressed in a natural way in the so-called Fibonacci-Horner basis with the aid of the dynamical solution of the associated ordinary differential equation. Two simple processes for computing the dynamical solution and the fundamental system of solutions are given. The connection to Verde-Star's approach is discussed. Moreover, an extension to the computation of f(A), where f is an analytic function is initiated. Finally, some illustative examples are presented.

Marek Niezgoda, *Upper bounds on certain functionals defined on groups of linear operators,* ELA 15 (2006), 191-200.

Abstract
The problem of estimating certain functionals defined on a group of linear operators generating a group induced cone (GIC) ordering is studied. A result of Berman and Plemmons [Math. Inequal. Appl., 2(1):149–152, 1998] is extended from the sum function to Schur-convex functions. It is shown that the problem has a closed connection with both Schur type inequality and weak group majorization. Some applications are given for matrices.

Rafael Bru, Francisco Pedroche and Daniel B. Szyld, *Subdirect sums of S-strictly diagonally dominant matrices*, ELA 15 (2006), 201-209.

Abstract
Conditions are given which guarantee that the k-subdirect sum of S-strictly diagonally dominant matrices (S-SDD) is also S-SDD. The same situation is analyzed for SDD matrices. The converse is also studied: given an SDD matrix C with the structure of a k-subdirect sum and positive diagonal entries, it is shown that there are two SDD matrices whose subdirect sum is C.

Guoli Ding and Andrei Kotlov, *On minimal rank over finite fields*, ELA 15 (2006), 210-214

Abstract
Let F be a field. Given a simple graph G on n vertices, its minimal rank (with respect to F) is the minimum rank of a symmetric n-by-n F-valued matrix whose off-diagonal zeroes are the same as in the adjacency matrix of G. If F is finite, then for every k, it is shown that the set of graphs of minimal rank at most k is characterized by finitely many forbidden induced subgraphs, each on at most $((|F|^k)/2+1)^2$ vertices. These findings also hold in a more general context.

Forthcoming Conferences and Workshops in Linear Algebra

Robert C. Thompson Matrix Meeting

Auburn University

March 24, 2007

The Robert C. Thompson Matrix Meeting (formerly the Southern California Matrix Meeting) will be held at Auburn University on March 24, 2007. The organizers are Tin Yau Tam, Wasin So and Jane Day. Contributed talks are invited. For details, visit http://www.auburn.edu/~tamtiny/rct2007.html.

In the tradition established by Thompson and Steve Pierce when they began these in the mid 1980's, this will be an informal one-day meeting to encourage the interaction and collaboration of researchers on matricees, including applicaitons, computation and theory. The attendees in 2004 voted to change the name to honor Bob Thompson, deceased in 1995.

2nd International Workshop on Matrix Analysis and Applications

Fort Lauderdale, Florida

December 15-16, 2006

The 2nd International Conference on Matrix Analysis and Applications will be held at Nova Southeastern University, Fort Lauderdale Florida, USA, December 15-16, 2006. The aim of this mathematical meeting is to stimulate research and interaction of researchers interested in all aspects of linear and multilinear algebra, matrix analysis and its applications. The conference is sponsored by the International Linear Algebra Society (ILAS) and Nova Southeastern University.

The Keynote speaker is Richard Brualdi (University of Wisconsin-Madison).

The organizing committee consists of Zhong-Zhi Bai (Chinese Academy of Sciences), Chi-Kwong Li (College of William and Mary), Bryan Shader (University of Wyoming), Hugo Woerdeman (Drexel University), Fuzhen Zhang (Nova Southeastern University) and Qingling Zhang (China Northeastern University).

The registration deadline is November 1, 2006.

A special issue, Matrix Analysis and Applications, of the *International J. of Information & Systems Sciences* will be devoted to the meeting. Paper submissions to the special issue are solicted.

For further information contact Fuzhen Zhang (zhang@nova.eud) or visit the conference website http://undergrad.nova.edu/mst/matrix/.

The 2007 Haifa Matrix Theory Conference

Haifa, Israel

April 16-19, 2007

The 2007 Haifa Matrix Theory Conference will take place at the Technion on April 16-19, 2007 under the auspices of the Technion's Center for Mathematical Sciences.

This will be the fourteenth in a sequence of matrix theory conferences held at the Technion since 1984. The conference will host two special sessions: one to celebrate Hans Schneider's 80th birthday; the other in memory of our colleague Miron Tismenetsy (Timor) who passed away last April.

As in the past all talks will be of 30 minutes duration, and will cover a wide spectrum of theoretical and applied Linear Algebra.

Titles and abstracts of contributed talks should be sent, no later than December 31, 2006, to any of the members of the organizing committee; Abraham Berman (co-chair), Moshe Goldberg (co-chair), Daniel Hershkowitz, Leonid Lerer, and Raphael Loewy.

14th ILAS Conference

Shanghai, P. R. China

July 16–20, 2007

The organizing committee invites participation and presentations on all aspects of linear algebra, matrix theory and its applications.

The deadline for abstracts of contributed talks is March 31, 2007, and early registration is April 30, 2007.

The meeting is sponsored by ILAS and Shanghai University.

Keynote speakers are:

M. Bresar (Slovenia)
M.D. Choi (Canada)

D. Farenick (Canada)
Eric Kaltofen (USA)
Plamen Koev (USA)
Amy Langville (USA)
Wenwei Lin (Taiwan, China)
Michael Ng (Hong Kong, China)
Andre Ran (The Netherlands)
Zhongci Shi (Beijing, China)
Yaokun Wu (Shanghai, China)
Shufang Xu (Beijing, China)

There will be a special issue of *Linear Algebra and its Applications* devoted to the conference. Editors for the issue are Ilse Ipsen, Julio Moro, Peter Šemrl, Jiayu Shao, and Pei Yuan Wu.

The Organizing Committee is:

Richard Brualdi (Madison), co-chair
Erxiong Jiang (Shanghai), co-chair
Raymond Chan, (Hong Kong)
Chuanqing Gu (Shanghai)
Danny Hershkowitz (Haifa), ILAS President
Roger Horn (Salt Lake City)
Ilse Ipsen (Raleigh)
Julio Moro (Madrid)
Peter Šemrl (Ljubljana)
Pei Yuan Wu (Hsinchu)
Jia-yu Shao (Shanghai)

Local Organizers are:
Qinwen Wang
Chuanqin Gu (chair)
Fuping Tan
Xiaomei Jia

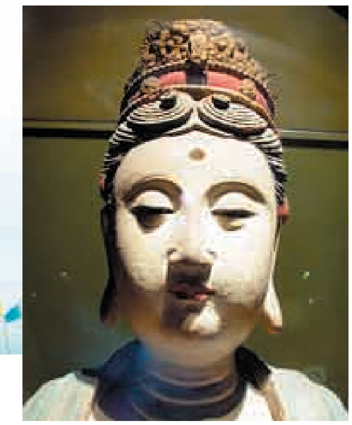
An excursion is being planned for Wednesday. Current thinking is to have meetings from 9 to 11 on Wednesday, an early lunch, and then begin the excursion which would last into the evening. If it can all be arranged the excursion would include:
1) Shanghai Museum
2) YuYuan Gardens with dinner
3) Oriental Pearl Tower (magnificent view of Shanghai)
4) Nanjing Road (an opportunity to walk on this famous street).

Shanghai University has a modern conference center with air-conditioning, and 5 meeting rooms including a very large one; the conference center is a 10 minute walk from the two hotels on the new campus. All of the conference rooms have a computer and a projector mounted on the ceiling. For computer presentations it is best to have a pdf file on a memory stick. Standard overhead projectors will also be available.

It is hoped that there will be an opportunity for those who wish on Monday or Tuesday night to go to a Chinese acrobat/circus show. This would be at an additional cost ($35-45) with a bus provided by the conference organization.

For more information, please visit the conference webpage http://math.shu.edu.cn/ILAS07/.

Oriental Pearl Tower



Statue at Shanghai Museum

YuYuan Gardens

A shop on Nanjing Road

IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problem 30-3 [IMAGE 30 (April 2003), p. 36], IMAGE Problems 35-1 through 35-4 [IMAGE 35 (Fall 2005), p. 44], and IMAGE Problems 35-6 through 35-9 [IMAGE 35 (Fall 2005), p. 43]. Problems 32-4 [IMAGE 32 (April 2004), p. 40] and 35-5 [IMAGE 35 (Fall 2005), p. 44] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 8 new problems on pp. 36 & 35 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to hjw.de@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ . & . & . & . & . & \cdots & . \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

Solution 30-3.1 by Vladimir BOLOTNIKOV and Chi-Kwong LI, *The College of William and Mary, Williamsburg, Virginia, USA:*
vladi@math.wm.edu; ckli@math.wm.edu

Let P be the permutation matrix obtained from I_{n-2} by moving the first two columns to the last two columns, and let N be the lower triangular Jordan block of zero of size $n-4$. Then

$$MP = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \text{ where } M_{11} = xI_{n-4} - C \text{ with } C = -(c_1 N + c_2 N^2 + \cdots c_{n-5} N^{n-5}),$$

$$M_{21} = \begin{pmatrix} c_{n-4} & \cdots & c_1 \\ c_{n-3} & \cdots & c_2 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} c_2 & c_1 \\ \vdots & \vdots \\ c_{n-3} & c_{n-4} \end{pmatrix}, \text{ and } M_{22} = \begin{pmatrix} c_{n-2} & c_{n-3} \\ c_{n-1} & c_{n-2} \end{pmatrix}.$$

Thus, by the assumption on M, for any $x > \|C\|$, we have

$$\begin{aligned} 0 &= \det(MP) \\ &= \det(M_{11}) \det(M_{22} - M_{21} M_{11}^{-1} M_{12}) \\ &= x^{n-4} \det(M_{22} - M_{21}(xI_{n-4} - C)^{-1} M_{12}) \\ &= x^{n-4} \det(M_{22} - M_{21}(x^{-1} \sum_{j=0}^{n-5} (C/x)^j) M_{12}). \end{aligned}$$

Hence, $M_{22} - M_{21}(x^{-1} \sum_{j=0}^{n-5} (C/x)^j) M_{12}$ has rank at most one for all x. Note that

$$M_{22} - M_{21}(x^{-1} \sum_{j=0}^{n-5} (C/x)^j) M_{12} = M_{22} - \sum_{j=1}^{n-4} p_j(1/x) M_{21} N^{j-1} M_{12}$$

for some polynomials $p_j(z)$ of degree j, where $j = 1, \ldots, n-4$. In particular, there are infinitely many x such that $p_j(x) \neq 0$ for $j = 1, \ldots, n-4$. We conclude that all the matrices M_{22} and $M_{21}N^jM_{12}$ for $j = 0, \ldots, n-5$ have the same row space or the same column space. Comparing $M_{21}N^{n-5} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} c_2 & c_2 \end{pmatrix}$ and $M_{21}N^{n-6}M_{12} = \begin{pmatrix} c_2 & c_1 \\ c_3 & c_2 \end{pmatrix} \begin{pmatrix} c_2 & c_1 \\ c_3 & c_2 \end{pmatrix}$, we see that (c_2, c_3) is a multiple of (c_1, c_2). So, the two matrices actually have the same row space as well as the same column space. We can then compare these matrices with $M_{21}N^{n-7}N_{12} = \begin{pmatrix} c_3 & c_2 & c_1 \\ c_4 & c_3 & c_2 \end{pmatrix} \begin{pmatrix} c_2 & c_1 \\ c_3 & c_2 \\ c_4 & c_3 \end{pmatrix}$ and conclude that this matrix also has the same row space and column space as the previous two matrices. Inductively, we see that the rows of M_{12} are multiples of (c_2, c_1) and the columns of M_{21} are multiples of $(c_1, c_2)^t$. Thus, the first two columns and also the first two rows of M are linearly dependent.

Note that our proof shows that the converse of the statement is true as well, and the (if and only if) statement actually holds for M of order 3 or above.

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j}X_1 + u_{2j}X_2 + u_{3j}X_3$, $j = 1, 2, 3$, such that each of the six matrices $W_j^{-1}W_k$, $j \neq k$, has zero trace. Equivalently, we want $(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We still look forward to receiving solutions to Problem 32-4!

Problem 35-1: An Upper Bound for the Norm of a Matrix Exponential

Proposed by Ken DRIESSEL, *Colorado State University, Fort Collins, Colorado, USA:* driessel@math.colostate.edu
and Wasin SO, *San Jose State University, San Jose, California, USA:* so@math.sjsu.edu

Let N be a strictly upper-triangular matrix with all ones.

1. Prove that $\|e^{-N}\| \leq \sqrt{e}$ where $\|\cdot\|$ is the spectral norm.
2. Is equality possible?

Solution 35-1.1 by the Proposers Ken DRIESSEL, *Colorado State University, Fort Collins, Colorado, USA,*
and Wasin SO, *San Jose State University, San Jose, California, USA:*
driessel@math.colostate.edu; so@math.sjsu.edu

1. Let $\lambda_i(\cdot)$ be the i-th largest eigenvalue of a Hermitian matrix. Now
$$\|e^{-N}\|^2 = \lambda_1(e^{-N^*}e^{-N}) \leq \lambda_1(e^{-N^*-N}) = e^{\lambda_1(-N^*-N)} = e^{-\lambda_n(N^*+N)} = e$$
where the inequality follows from Cohen's inequality; see Cohen (1988).
2. Since N is integral and nilpotent of index n, where n is the order of N, the matrix $e^{-N^*}e^{-N}$ is rational. Hence, $\lambda_1(e^{-N^*}e^{-N})$ is an algebraic number, but e is transcendental. Consequently, equality never happens.

Reference

J. E. Cohen (1988). Spectral inequalities for matrix exponentials. *Linear Algebra and Its Applications* **111**, 25–28.

Problem 35-2: Matrix Polynomials as Group Inverses

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ nonzero matrix with complex elements. Establish general conditions under which the matrix polynomial $P(A)$ serves as a group inverse $A^\#$ of A:
$$AA^\# = A^\# A, \quad AA^\# A = A, \quad A^\# AA^\# = A^\#.$$

Develop this general result for (i) the case in which $\text{rank}(A) = n-1$, and (ii) the case in which $A^\# = A^m$ for some value of $m > 0$.

Solution 35-2.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK*, and Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:*
R.W.Farebrother@Manchaster.ac.uk, hjw.de@uni-bonn.de

(i): In what follows, let $A \in \mathbb{C}^{n\times n}$ be an arbitrary but fixed singular matrix with $\text{index}(A) = 1$ (i.e., $\text{rank}(A) = \text{rank}(A^2)$). We recall that A has a group inverse $A^\#$ if and only if $\text{rank}(A) = \text{rank}(A^2)$; in which case $A^\#$ is unique. Let $m(\lambda)$ denote the minimal polynomial of A. This polynomial satisfies $m(A) = 0$ and among all such polynomials it has the smallest degree. Since $\text{index}(A) = 1$, all Jordan blocks belonging to the eigenvalue 0 are of order 1, i.e., 1 is the multiplicity of $\lambda = 0$ as a zero of $m(\lambda)$ or, equivalently, $m(\lambda) = \lambda \cdot p(\lambda)$ for some polynomial p in λ with $p(0) \neq 0$, which in turn implies that $p(\lambda)$ can be rewritten as $p(\lambda) = c \cdot (1 - \lambda \cdot q(\lambda))$ for some polynomial q and some constant c. Consequently, $0 = m(A) = cA(I - Aq(A))$, and hence $0 = A(I - Aq(A))$, i.e., $A = A^2q(A)$. It is easy to check that for the matrix $X := A(q(A))^2$ we obtain $AX = A^2(q(A))^2 = Aq(A) = XA$, $AXA = A^2q(A) = A$, and $XAX = A^3(q(A))^4 = A^2(q(A))^3 = A(q(A))^2 = X$. In other words, the matrix $X = A(q(A))^2$ satisfies the three defining equations of the uniquely defined group inverse $A^\#$ of A. Therefore, $A^\# = A(q(A))^2$. Our preceding lines show that the matrix polynomial $P(A)$ represents the group inverse of A if and only if $P(A) = A(q(A))^2$. It is pertinent to mention that by checking the defining equations of the group inverse, it is also easy to see that $A^\# = A(w(A))^2$ holds true for any polynomial w satisfying $A = A^2w(A)$. Moreover, $A^\#$ exists if and only if $A = A^2w(A)$ for some polynomial w; in which case $A^\# = A(w(A))^2$. We emphasize that we did not make use of the restrictive problem assumption $\text{rank}(A) = n - 1$.

(ii): In virtue of the defining equations of the group inverse, clearly $A^\# = A^m$ for some $m > 0$ if and only if $A^{m+2} = A$. In passing, we mention that if A is idempotent (i.e., $A^2 = A$) or skew-idempotent (i.e., $A^2 = -A$) then it is also tripotent (i.e., $A^3 = A$) with $A^\# = A$. A matrix A with $A^m = A$ for some integer $m \geq 4$ is called multipotent.

COMMENT. We conclude with mentioning that likewise it can be shown that if A is a singular matrix with $\text{index}(A) = k$ (i.e., k is the smallest nonnegative integer with $\text{rank}(A^{k+1}) = \text{rank}(A^k)$), then $A^D = A^k(w(A))^{k+1}$ for some polynomial w if and only if $A^k = A^{k+1}w(A)$, where A^D denotes the uniquely defined Drazin inverse of A. A suitable choice for w is the polynomial q in the representation $m(\lambda) = c \cdot \lambda^k \cdot (1 - \lambda \cdot q(\lambda))$ of the minimal polynomial m of A.

Solution 35-2.2 by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If P is a polynomial then $AP(A) = P(A)A$. Hence, we have only to find P such that $A^2P(A) = A$ and $AP^2(A) = P(A)$.

Let $Q(x) = \sum_{\ell=0}^{k} q_\ell x^\ell$ (with $q_k = 1$) be the minimal polynomial of A. Then A is invertible if and only if $q_0 \neq 0$. If A is invertible then $A^2P(A) = A$ if and only if $AP(A) = I$, which is equivalent to $Q(x) \mid xP(x) - 1$. Therefore $P(x) = (1 + F(x)Q(x))/x$ where F is a polynomial such that $F(0) = -1/q_0$ or, equivalently,

$$P(x) = -\frac{1}{q_0}\sum_{\ell=1}^{k} q_\ell x^{\ell-1} + G(x)Q(x),$$

where $G(x)$ is an arbitrary polynomial. Since $Q(A) = 0$,

$$A^{-1} = -\frac{1}{q_0}\sum_{\ell=1}^{k} q_\ell A^{\ell-1}. \tag{1}$$

Now suppose A is noninvertible. Then $q_0 = 0$ and $Q(x) = xQ_1(x)$, where $Q_1(x) = \sum_{\ell=1}^{k} q_\ell x^{\ell-1}$. We will show that $P(A)$ is a group inverse of A if and only if $q_1 \neq 0$ and

$$xP(x) = 1 + F(x)Q_1(x), \tag{2}$$

where F is a polynomial such that

$$F(0) = -1/q_1 \text{ and } F'(0) = q_2/q_1^2. \tag{3}$$

For necessity, if $A^2P(A) = A$ then $xQ_1(x) \mid x(xP(x) - 1)$, which implies (2) for some polynomial F. From (2), $q_1 = Q_1(0) \neq 0$ and the first equality in (3) holds. If $AP^2(A) = P(A)$ then $xQ_1(x) \mid P(x)(xP(x) - 1)$. Therefore $P(0) = 0$, so (2) implies the second equality in (3). For sufficiency, (3) implies that $P(x)$ in (2) is a polynomial and $P(0) = 0$. From (2),

$$A^2P(A) = A + F(A)Q_1(A)A = A + F(A)Q(A) = A.$$

Therefore, $A^{k+1}P(A) = A^k$ if $k \geq 1$. Since $P(0) = 0$, this implies that $AP^2(A) = P(A)$.

It is straightforward to verify from (2) and (3) that

$$P(x) = \frac{1}{q_1^2} \sum_{\ell=1}^{k-2} (q_2 q_{\ell+1} - q_1 q_{\ell+2}) x^\ell + \frac{q_2}{q_1^2} x^{k-1} + G(x)Q(x),$$

where G is an arbitrary polynomial. Since $Q(A) = 0$, all polynomial group inverses of A reduce to

$$A^\# = \frac{1}{q_1^2} \sum_{\ell=1}^{k-2} (q_2 q_{\ell+1} - q_1 q_{\ell+2}) A^\ell + \frac{q_2}{q_1^2} A^{k-1}. \tag{4}$$

Since we have shown that a noninvertible matrix A has a polynomial group inverse if and only if $Q'(0) \neq 0$, a requirement unrelated to $\text{rank}(A)$, (i) does not seem to warrant special discussion. As for (ii), $P(A) = A^m$ is a group inverse of A if and only if $A(A^{m+1} - I) = 0$, which is equivalent to $Q(x) \mid x(x^{m+1} - 1)$. This holds if and only if A is diagonalizable and its nonzero eigenvalues are $(m+1)$-th roots of unity. In this case A^m can be written as in (1) if A is invertible or as in (4) if A is noninvertible.

Problem 35-3: Partitioned Inverse of a Matrix Product

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let $X = (\,X_1 \quad X_2\,)$ and $Y = (\,Y_1 \quad Y_2\,)$ be $n \times k$ matrices of rank k partitioned by their first k_1 columns and the remaining $k_2 := k - k_1$ columns in such a way that $X_1'Y_1$, $X_2'Y_2$ and $X'Y$ are nonsingular. Obtain explicit expressions for the columns of the $n \times k$ matrices $P = (\,P_1 \quad P_2\,)$ and $Q = (\,Q_1 \quad Q_2\,)$ partitioned conformably with $X = (\,X_1 \quad X_2\,)$ and $Y = (\,Y_1 \quad Y_2\,)$ in such a way that $P'Q = (X'Y)^{-1}$.

Solution 35-3.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:*
R.W.Farebrother@Manchester.ac.uk

Let $M_1 := I_n - Y_1(X_1'Y_1)^{-1}X_1'$ and $M_2 := I_n - Y_2(X_2'Y_2)^{-1}X_2'$. Because $X_1'M_2Y_1$ and $X_2M_1Y_2$ are both nonsingular, we may define $P_1' := (X_1'M_2Y_1)^{-1}X_1'M_2$, $P_2' := (X_2'M_1Y_2)^{-1}X_2'M_1$, $Q_1 := M_2Y_1(X_1'M_2Y_1)^{-1}$, and $Q_2 := M_1Y_2(X_2'M_1Y_2)^{-1}$. Then,

$$\begin{aligned}
P_1'Q_1 &= (X_1'M_2Y_1)^{-1}X_1'M_2M_2Y_1(X_1'M_2Y_1)^{-1} \\
&= (X_1'M_2Y_1)^{-1}, \\
P_1'Q_2 &= (X_1'M_2Y_1)^{-1}X_1'M_2M_1Y_2(X_2'M_1Y_2)^{-1} \\
&= -(X_1'M_2Y_1)^{-1}X_1'M_2Y_1(X_1'Y_1)^{-1}X_1'Y_2(X_2'M_1Y_2)^{-1} \\
&= -(X_1'M_2Y_1)^{-1}X_1'Y_2(X_2'Y_2)^{-1}X_2'M_1Y_2(X_2'M_1Y_2)^{-1}, \\
P_2'Q_1 &= (X_2'M_1Y_2)^{-1}X_2'M_1M_2Y_1(X_1'M_2Y_1)^{-1} \\
&= -(X_2'M_1Y_2)^{-1}X_2'M_1Y_2(X_1'Y_1)^{-1}X_1'Y_2(X_2'M_1Y_2)^{-1} \\
&= -(X_2'M_1Y_2)^{-1}X_2'Y_1(X_1'Y_1)^{-1}X_1'M_2Y_1(X_1'M_2Y_1)^{-1}, \\
P_2'Q_2 &= (X_2'M_1Y_2)^{-1}X_2'M_1M_1Y_2(X_2'M_1Y_2)^{-1} \\
&= (X_2'M_1Y_2)^{-1}.
\end{aligned}$$

Comparing these results with the usual partitioned inversion formulas, we deduce that

$$\begin{pmatrix} P_1'Q_1 & P_1'Q_2 \\ P_2'Q_1 & P_2'Q_2 \end{pmatrix} = \begin{pmatrix} X_1'Y_1 & X_1'Y_2 \\ X_2'Y_1 & X_2'Y_2 \end{pmatrix}^{-1},$$

as required.

Problem 35-4: Which Sizes have the Matrices?

Proposed by Alexander KOVAČEC, *University of Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

Consider a matrix P, at least three units higher than broad whose lu-diagonal (containing the left upper corner) is filled by 1's; whose rl-diagonal (containing the right lower corner) is filled by -1's; whose diagonal two units above the rl-diagonal is filled by 1's;

and whose remaining entries are 0. Let b be a column of height of P of the form $b = (1, 0, \ldots, 0, 1, -1)'$. Which sizes, if any, are possible for P if $Px = b$ is solvable?

Solution 35-4.1 by the Proposer Alexander KOVAČEC, *University of Coimbra, Coimbra, Portugal:* kovacec@mat.uc.pt

By inspection it is evident that necessary for solvability is breadth greater than or equal to 3, and thus height greater than or equal to 6. The solvability of the system $Px = b$ for size of P equal to 6×3 is seen by checking the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

We claim that 6×3 is the only size possible. It will be convenient to assume that P has size $(m+1) \times (m-n+1)$. Then by hypothesis $n = (m+1) - (m-n+1) \geq 3$. Let $a = [a_{m-n}, \ldots, a_1, a_0]^\mathsf{T}$ and form from a the polynomial $a(x) = \sum_{j=0}^{m-n} a_j x^j$. Let $p(x) = x^n + x^2 - 1$, and $b(x) = x^m + x - 1$. Observe that a satisfies $Pa = b$ if and only if $p(x)a(x) = b(x)$. Now

$$b(x)(x+1) - p(x) = (x^{m+1} + x^m + x^2 - 1) - (x^n + x^2 - 1) = x^n(x^{m+1-n} + x^{m-n} - 1) \in \mathbb{Z}[x]p.$$

Since $\gcd(x^n, p) = 1$, we conclude that $b_2(x) := x^{m+1-n} + x^{m-n} - 1 \in \mathbb{Z}[x]p$, that is $p|b_2$. Consequently $m+1-n \geq n$ and every root of p is root of b_2. Now $p(0) = -1$, $p(1) = 1$, and since p as a polynomial function is evidently increasing in $]0, 1[$, it has exactly one root $x_0 \in]0, 1[$. Now supposing $m+1-n > n \geq 3$ implies $x_0^{m+1-n} < x_0^n$, and $x_0^{m-n} < x_0^2$, hence $b_2(x_0) < p(x_0) = 0$, and x_0 cannot be root of b_2. Thus $m+1-n = n$, $m-n = 2$ and so $m = 5, n = 3$. (Indeed $x^5 + x - 1 = (x^3 + x^2 - 1)(x^2 - x + 1)$.)

Problem 35-5: First and Second Moments Involving a Camouflaged Wishart Matrix

Proposed by Heinz NEUDECKER, *Universiteit van Amsterdam, Amsterdam, The Netherlands:* ericaengels173@hotmail.com

Consider the $m \times (n+m-1)$ matrix $C' := (A', B')$, where $A \in \mathbb{R}^{m-1,m}$ is constant, $B \in \mathbb{R}^{n,m}$ is such that $B'B$ follows a central Wishart distribution $W(\Omega, n)$, $n > m+1$, and where $\operatorname{rank}(C) = m$. As usual, let E denote the expected value operator and D denote the variance operator. Prove the following:

(i) $\mathsf{E}(\sum_i |(A', b_i)|^2) = n \cdot \operatorname{tr}(A'A)^*\Omega$, where, for $i = 1, 2, \cdots, n$, b_i denotes the i-th column vector of B', and where $|\cdot|$, $(\cdot)^*$, and $\operatorname{tr}(\cdot)$ stand for the determinant, the adjoint, and the trace, respectively, of $(\cdot)$.

(ii) $\mathsf{D}(\sum_i |(A', b_i)|^2) = n^2 \cdot \{\operatorname{tr}(A'A)^*\Omega\}^2 + 2n \cdot \operatorname{tr}\{(A'A)^*\Omega\}^2$.

We still look forward to receiving solutions to Problem 35-5!

Problem 35-6: Spectral Representation of an Arbitrary Diagonalizable Complex Matrix

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Suppose $A \in \mathbb{C}^{n \times n}$ has minimal polynomial $p(x) = (x - \lambda_1) \cdots (x - \lambda_k)$ with $\lambda_1, \ldots \lambda_k$ distinct. Let $p_i(x) = p(x)/(x - \lambda_i)$ and suppose $Q_i \in \mathbb{C}^{n \times n_i}$ has columns that form an orthonormal basis for the column space of $p_i(A)$, $1 \leq i \leq k$. Show that $AQ_i = \lambda_i Q_i$, $1 \leq i \leq k$, $\sum_{i=1}^k n_i = n$, and

$$A = \sum_{i=1}^k \lambda_i Q_i Q_i^* \frac{p_i(A)}{p'(\lambda_i)}.$$

Show that A is normal if and only if

$$Q_i^* \frac{p_i(A)}{p'(\lambda_i)} = Q_i^*, \quad 1 \leq i \leq k.$$

Solution 35-6.1 by the Proposer William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If

$$q(x) = \sum_{i=1}^{k} \frac{p_i(x)}{p'(\lambda_i)},$$

then $q(\lambda_j) = 1$, $1 \le j \le k$; hence, $q(x) \equiv 1$. Therefore $q(A) = I$, so $\sum_{i=1}^{k} n_i = n$. Since $(A - \lambda_i I)p_i(A) = 0$ and the columns of Q_i are in the column space of $p_i(A)$, it follows that $AQ_i = \lambda_i Q_i$, $1 \le i \le k$. Consequently,

$$A = Q \left(\bigoplus_{i=1}^{k} \lambda_i I_{n_i} \right) Q^{-1},$$

where $Q = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_k \end{pmatrix}$. However,

$$Q^{-1} = \begin{pmatrix} \widehat{Q}_1 \\ \widehat{Q}_2 \\ \vdots \\ \widehat{Q}_k \end{pmatrix}, \quad \text{where} \quad \widehat{Q}_i = Q_i^* \frac{p_i(A)}{p'(\lambda_i)}, \quad 1 \le i \le k.$$

To see this, note that $p'(\lambda_i)Q_i^* Q_i = p'(\lambda_i) I_{n_i}$ and, if $i \ne j$, then $p_i(A)$ contains the factor $A - \lambda_j I$, so $p_i(A)Q_j = 0$. This verifies the stated expansion of A.

If

$$Q_i^* \frac{p_i(A)}{p'(\lambda_i)} = Q_i^*, \quad 1 \le i \le k, \tag{5}$$

then

$$Q_i^* Q_j = \frac{Q_i^* p_i(A) Q_j}{p'(\lambda_i)} = \delta_{ij} I_{n_i}, \quad 1 \le i, j \le n,$$

so A is normal and the stated expansion of A reduces to $A = \sum_{i=1}^{k} \lambda_i Q_i Q_i^*$. Conversely, if A is normal, then

$$\left(Q_i^* - Q_i^* \frac{p_i(A)}{p'(\lambda_i)} \right) Q_j = 0, \quad 1 \le i, j \le k,$$

which implies (5).

Problem 35-7: A Characterization of Oblique Projectors

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a square matrix with complex entries. Show that A is an oblique projector if and only if A is similar to an orthogonal projector.

Solution 35-7.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl and Xiaoji LIU, *Guangxi University for Nationalities, Nanning, China:* xiaojiliu72@yahoo.com.cn

Actually, the solution to Problem 35-7 is known in the literature; see e.g., point 1 of Theorem 4.1 in Zhang (1999). The solution given below refers to different facts than those utilized by Zhang (1999).

Let $A \in \mathbb{C}_{n,n}$ be of rank r. It is well-known that every oblique projector (i.e., idempotent matrix) is similar to a diagonal matrix of its eigenvalues, which are equal to either zero or one. In other words, if $A^2 = A$, then there exists a nonsingular matrix $V \in \mathbb{C}_{n,n}$ such that $A = VDV^{-1}$, where

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

with I_r denoting the identity matrix of order r. Thus, A is similar to an orthogonal projector (i.e., Hermitian idempotent matrix). Conversely, if A is similar to an orthogonal projector, then $A^2 = A$ follows easily.

Reference

F. Zhang (1999). *Matrix Theory – Basic Results and Techniques.* Springer, New York.

Solution 35-7.2 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Sufficiency is trivial. To prove necessity, let A be an oblique projector, that is $A^2 = A$. Define $P_A = AA^+, Q_A = I - P_A$, and $S = A + Q_A$. Then S is nonsingular. For, if $Sx = 0$, then $Ax = -Q_A x$ which implies $Q_A x = 0$. Hence $P_A x = x$ and $Ax = 0$, consequently $x = P_A x = P_A A x = 0$, so that $\ker(S) = \{0\}$.

Letting $P = AS^{-1}$, by $Q_A S^{-1} = Q_A$ and $A = S - Q_A$, we see that $P = P_A$. Finally, $SP_A = P_A$ yields $P_A = SAS^{-1}$. Thus A is similar to the orthogonal projector $P_A = AA^+$.

Solution 35-7.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Let $\mathbb{F}^{n\times m}$ stand for the set of all $n \times m$ matrices over the field $\mathbb{F}$. For a matrix $A \in \mathbb{F}^{n\times m}$, let $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the range (column space), and the null space, respectively, of A. Our solution to this problem follows easily from the following two illuminating observations.

THEOREM 1. *Let $P, Q \in \mathbb{F}^{n\times n}$ be idempotent, i.e., let $P = P^2$ and $Q = Q^2$. If $P + Q - I$ is nonsingular, then P is similar to Q, i.e., there exists a nonsingular matrix, say X, such that $P = X^{-1}QX$. Here I denotes, as usual, the identity matrix of appropriate order.*

PROOF: Clearly, $(P+Q-I)P = QP = Q(P+Q-I)$. Because $P+Q-I$ is nonsingular, we obtain $P = (P+Q-I)^{-1}Q(P+Q-I)$, so that choosing $X = P + Q - I$ completes our proof. □

THEOREM 2. *Let $P, Q \in \mathbb{F}^{n\times n}$ be idempotent matrices such that $\mathcal{R}(P) = \mathcal{R}(Q)$. Then P is similar to Q, i.e., $P \sim Q$.*

PROOF: Note that if $A \in \mathbb{F}^{n\times n}$ is an idempotent matrix, then $I - A$ is idempotent, $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathbb{F}^n$, where $\oplus$ indicates a direct sum, $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, $\mathcal{R}(I - A) = \mathcal{N}(A)$, and $Ax = x$ irrespective of the choice of $x \in \mathcal{R}(A)$. According to Theorem 1, it suffices to show that $P + Q - I$ is nonsingular whenever P and Q are idempotent matrices with $\mathcal{R}(P) = \mathcal{R}(Q)$. Needless to say, $P + Q - I = P - (I - Q)$. Under the assumptions of Theorem 2, we now, by means of the previous observations, easily obtain $\mathcal{N}(P - (I - Q)) = [\mathcal{R}(P) \cap \mathcal{N}(Q)] \oplus [\mathcal{N}(P) \cap \mathcal{R}(Q)] = \{0\}$, thus showing that $P + Q - I$ is, as claimed, necessarily a nonsingular matrix. □

In what follows, let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Moreover, for $A \in \mathbb{F}^{n\times m}$, let A^+ denote the Moore-Penrose inverse of A. We recall that AA^+ is an Hermitian idempotent matrix (i.e., an orthogonal projector) and $\mathcal{R}(AA^+) = \mathcal{R}(A)$. Therefore, if P is a (possibly oblique) projector (i.e., $P^2 = P$), then PP^+ is an orthogonal projector. Because $\mathcal{R}(P) = \mathcal{R}(PP^+)$ it follows by Theorem 2 that $P \sim PP^+$. Conversely, if $P \sim Q$, where Q is an oblique or orthogonal projector, then, trivially, $P^2 = P$, so that the proof of IMAGE Problem 35-7 is complete.

A solution to Problem 35-7 was also received from William F. Trench. He used the same trick as in Solution 35-7.1.

Problem 35-8: A Characterization of a Particular Class of Square Complex Matrices
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Characterize the class of all square matrices A with complex entries satisfying the identity

$$A^+ + A^\sharp = 2A^+A^\sharp,$$

where A^+ and $A^\sharp$ denote the Moore-Penrose inverse and the group inverse of A, respectively.

Solution 35-8.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and Xiaoji LIU, *Guangxi University for Nationalities, Nanning, China:* xiaojiliu72@yahoo.com.cn

For $A \in \mathbb{C}_{n,n}$ of rank r, let $A = FG$, with $F \in \mathbb{C}_{n,r}$, $G \in \mathbb{C}_{r,n}$, be a full-rank factorization of A. Then

$$A^\dagger = G^*(GG^*)^{-1}(F^*F)^{-1}F^* \quad \text{and} \quad A^\# = F(GF)^{-2}G, \tag{6}$$

where the superscript "$*$" denotes a conjugate transpose. Substituting matrices $A^\dagger$ and $A^\#$ given in (6) into $A^\dagger + A^\# = 2A^\dagger A^\#$ yields

$$G^*(GG^*)^{-1}(F^*F)^{-1}F^* + F(GF)^{-2}G = 2G^*(GG^*)^{-1}(GF)^{-2}G. \tag{7}$$

Premultiplying this identity by G and postmultiplying it by F gives

$$GF = I_r, \tag{8}$$

where I_r denotes the identity matrix of order r. (Parenthetically notice that (8) ensures that $A^2 = A = A^{\#}$.) Substituting (8) into (7) and postmultiplying the obtained identity by F leads to $F = G^*(GG^*)^{-1}$. Hence, $A^\dagger = F(F^*F)^{-1}F^*$ and by postmultiplying this condition by $A(= FG)$ it follows that $A^\dagger A = A$. Consequently, A is an orthogonal projector (i.e., $A^2 = A = A^*$). Since this fact ensures that $A = A^\dagger = A^{\#}$, the converse implication is easily established. On account of the observation that the same conclusions are obtained if analogous arguments to those used above are utilized with respect to the identity $A^\dagger + A^{\#} = 2A^{\#}A^\dagger$, we arrive at equivalences

$$A^\dagger + A^{\#} = 2A^\dagger A^{\#} \;\Leftrightarrow\; A^2 = A = A^* \;\Leftrightarrow\; A^\dagger + A^{\#} = 2A^{\#}A^\dagger.$$

Solution 35-8.2 by Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

THEOREM. *Let A be a complex matrix and α a fixed complex number. Then*

$$A^+A^{\#} = \alpha A^+ + (1-\alpha)A^{\#} \tag{9}$$

if and only if a matrix A is Hermitian idempotent (independently of α).

PROOF: Premultiplying and postmultiplying the eqality (9) by A and using the property $AA^{\#} = A^{\#}A$ [see Ben-Israel & Greville (2003, p. 156)] we get $AA^{\#} = A$, which implies that matrix A is an idempotent, and in consequence $A^{\#} = A$. Now, for $\alpha = 1$ the result is obvious. For $\alpha \neq 1$ we obtain it by putting A instead of $A^{\#}$ and premultiplying the equality (9) by A. □

COMMENT 1. *The equality* (9) *with* $\alpha = 0.5$ *coincides with the equation in Problem 35-8.*

COMMENT 2. *For Hermitian idempotent matrices the equality* (9) *with* $\alpha = 0.5$ *coincides with equation (a) in Problem 34-8 proposed by Trenkler (2005). Therefore Hermitian idempotent matrices A belong to the class of solutions of the equation.*

References

A. Ben-Israel & T. N. E. Greville (2003). *Generalized Inverses: Theory and Applications* (2nd ed.). Springer, New York.

G. Trenkler (2005). Problem 34-8: A property for the sum of a matrix A and its Moore-Penrose inverse A^+. *IMAGE: The Bulletin of the International Linear Algebra Society* **34** (Spring 2005), 39.

Solution 35-8.3 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

According to Corollary 6 in Hartwig and Spindelböck (1984), we may write the matrix A in the form

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where U is unitary, $KK^* + LL^* = I_r$, $\Sigma =$ diag $(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t}), r_1 + \ldots + r_t = r =$ rank (A) and $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$. Some straightforward calculations show that

$$A^+ = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*,$$

$$A^\sharp = U\begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix} U^*,$$

and

$$A^+A^\sharp = U\begin{pmatrix} K^*\Sigma^{-1}K^{-1}\Sigma^{-1} & K^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L \\ L^*\Sigma^{-1}K^{-1}\Sigma^{-1} & L^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L \end{pmatrix} \mathbf{U}^*.$$

Note that the existence of the group inverse means that K is nonsingular. Then the condition $A^+ + A^\sharp = 2A^+A^\sharp$ is equivalent to

(i) $K^*\Sigma^{-1} + K^{-1}\Sigma^{-1} = 2K^*\Sigma^{-1}K^{-1}\Sigma^{-1}$, (ii) $K^{-1}\Sigma^{-1}K^{-1}L = 2K^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L$,

(iii) $L^*\Sigma^{-1} = 2L^*\Sigma^{-1}K^{-1}\Sigma^{-1}$, (iv) $L^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L = 0$.

From (i) we get $K^*+K^{-1} = 2K^*\Sigma^{-1}K^{-1}$ and thus $KK^*+I_r = 2KK^*\Sigma^{-1}K^{-1}$. Condition (iii) implies $LL^* = 2LL^*\Sigma^{-1}K^{-1}$. Using $KK^* + LL^* = I_r$ we arrive at $\Sigma^{-1}K^{-1} = I_r$. From (iv) it finally follows that $L^*L = 0$, i.e., $L = 0$. Hence $\Sigma K = I_r$ and $L = 0$, so A can be written as

$$A = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^*$$

such that A is an orthogonal projector. Conversely, if A is an orthogonal projector, we have $A^2 = A = A^+ = A^\sharp$. Hence our class consists of all orthogonal projectors. With the same tools used above one can easily show that the preceding class can also be written as

$$A^+ + A^\sharp = 2A^\sharp A^+.$$

Observe that in general $A^\sharp A^+ \neq A^+A^\sharp$.

Reference

R. E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and A^+ commute. *Linear Algebra and Its Applications* **14**, 241–256.

Solution 35-8.4 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Some elementary (geometric) facts enable us to prove the following characterization

$$A^+ + A^\sharp = 2A^+A^\sharp \;\Leftrightarrow\; A = A^* = A^2 \text{ (i.e., } A \text{ is an orthogonal projector).} \tag{10}$$

It is well-known that $\mathcal{R}(A^\sharp) = \mathcal{R}(A)$ and $\mathcal{R}(A^+) = \mathcal{R}(A^*)$, where $\mathcal{R}(\cdot)$ denotes the range (column space) of the matrix $(\cdot)$ and where A^* stands for the conjugate transpose of A. Rewriting eqn. $A^+ + A^\sharp = 2A^+A^\sharp$ as $A^\sharp = A^+(2A^\sharp - I)$, with I indicating the identity matrix of suitable order, thus tells us that $\mathcal{R}(A) = \mathcal{R}(A^\sharp) \subseteq \mathcal{R}(A^*)$. Because the rank of a matrix always coincides with the rank of its conjugate transpose, we obtain $\mathcal{R}(A) = \mathcal{R}(A^*)$, i.e., A is necessarily an EP matrix. Pre- and postmultiplying eqn. $A^+ + A^\sharp = 2A^+A^\sharp$ by the matrix A and making use of $AA^\sharp A = A = AA^+A$ and $A^\sharp A = AA^\sharp$ readily results in $2A = 2AA^\sharp$. Hence, necessarily $A = A^2$, i.e., A is idempotent. Because an idempotent matrix A is EP if and only if it is Hermitian [see Theorem in Werner (2002)], it is now clear that $A = A^* = A^2$ is a necessary condition for $A^+ + A^\sharp = 2A^+A^\sharp$ to hold. To see that this condition is also sufficient, let A be an orthogonal projector, i.e., let $A = A^* = A^2$. Then $A^+ = A^\sharp = A$, and so trivially $A^+ + A^\sharp = 2A^+A^\sharp$. This completes our proof of characterization (10).

Reference

H. J. Werner (2002). Parital isometry and idempotent matrices. Solution 28-7.5. *IMAGE: The Bulletin of the International Linear Algebra Society* **29** (October 2002), 31–32.

Problem 35-9: A Range Equality for Idempotent Hermitian Matrices
Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let P and Q be idempotent and Hermitian matrices of the same order. Show that

$$\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q),$$

where $\mathcal{R}(\cdot)$ denotes the range of a matrix.

Solution 35-9.1 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

In Groß (1999, Corollary 2) it is stated that for idempotent and Hermitian matrices R and S the following result holds:

$$\mathcal{R}(I - RS) = \mathcal{N}(R) + \mathcal{N}(S),$$

where $\mathcal{N}(\cdot)$ denotes the null space of a matrix. Setting $R = I - P$ and $S = I - Q$, one obtains $\mathcal{R}(I - (I - P)(I - Q)) = \mathcal{N}(I - P) + \mathcal{N}(I - Q)$ which implies $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q)$.

Reference

J. Groß (1999). On the product of orthogonal projectors. *Linear Algebra and Its Applications* **289**, 141–150.

Solution 35-9.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

For a complex matrix A, let A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the range (column space), and the null space, respectively, of A. According to Lemma 3.2 in Werner (1985) [see also (3.1a) in Werner (1987, p. 270)] we have the following fact.

FACT. *Let $V \in \mathbb{C}^{n\times n}$ be an $n \times n$ Hermitian nonnegative definite matrix, and let $X \in \mathbb{C}^{n\times m}$. Then*

$$\mathcal{R}(V, X) = \mathcal{R}(V) + \mathcal{R}(X) = V\mathcal{N}(X^*) \oplus \mathcal{R}(X),$$

where $\oplus$ indicates a direct sum.

Now let $P, Q \in \mathbb{C}^{n\times n}$ be both Hermitian idempotent matrices, i.e., let $P = P^2 = P^*$ and let $Q = Q^2 = Q^*$. Then $P = PP^*$ and $Q = QQ^*$, thus showing that P and Q are Hermitian nonnegative definite matrices. It is further clear that $\mathcal{N}(Q) = \mathcal{N}(Q^*) = \mathcal{R}(I - Q)$, with I denoting the identity matrix of order n. Therefore, according to our above fact,

$$\mathcal{R}(P) + \mathcal{R}(Q) = P\mathcal{N}(Q) \oplus \mathcal{R}(Q) = \mathcal{R}(P(I - Q)) \oplus \mathcal{R}(Q) = \mathcal{R}(P - PQ) + \mathcal{R}(Q). \tag{11}$$

Trivially, $\mathcal{R}(P + Q - PQ) \subseteq \mathcal{R}(P - PQ) + \mathcal{R}(Q)$. To prove the converse inclusion, let y be an arbitrary but fixed vector from $\mathcal{R}(P - PQ) + \mathcal{R}(Q)$. Then $y = (P - PQ)x_1 + Qx_2$ for some (unique) vectors x_1 and x_2. Put $x := (I - Q)x_1 + Qx_2$. Then $(P + Q - PQ)x = [P(I - Q) + Q]x = P(I - Q)x + Qx = P(I - Q)x_1 + Qx_2 = y$, and so $y \in \mathcal{R}(P + Q - PQ)$. Consequently, $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P - PQ) + \mathcal{R}(Q)$. Combining this set equality with (11) immediately results in $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q)$, and our solution is complete.

References

H. J. Werner (1985). More on BLUE estimation in regression models with possibly singular covariances. *Linear Algebra and Its Applications* **67**, 207–214.

H. J. Werner (1987). C. R. Rao's IPM method: a geometric approach. In *New Perspectives in Theoretical and Applied Statistics* (Madan L. Puri et al., eds.), John Wiley & Sons, New York, pp. 367–382.

A solution to Problem 35-9 was also received from Johanns de Andrade Bezerra.

IMAGE Problem Corner: More New Problems

Problem 36-6: Inverses of Some Cayley Matrices

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu.

Find C^{-1} if $C = (r + s \,(\operatorname{mod} n))_{r,s=0}^{n-1}$ with $n \geq 3$.

Problem 36-7: Necessary and Sufficient Conditions for $A + A^*$ to be a Nonnegative Definite Matrix

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a complex square matrix with A^*, A^+, and $A^\#$ denoting the conjugate transpose, the Moore-Penrose inverse, and the group inverse, respectively, of A. Similarly, define the conjugate transpose of A^+ and $A^\#$ as A^{+*} and $A^{\#*}$, respectively. Moreover, let $\geq_{\rm L}$ denote the Löwner ordering. Show that the following statements are equivalent:

(i) $A + A^* \geq_{\rm L} 0$.

(ii) $A^+ + A^{+*} \geq_{\rm L} 0$.

(iii) $A^\# + A^{\#*} \geq_{\rm L} 0$, where the group inverse $A^\#$ of A is assumed to exist.

Problem 36-8: A Characterization of a Particular Class of Square Complex Matrices

Proposed by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Characterize the class of all square matrices A with complex entries satisfying the identity

$$A + A^\sharp = 2A^+A^\sharp,$$

where A^+ and $A^\sharp$ denote the Moore-Penrose inverse and the group inverse of A, respectively.

Problems 36-1 through 36-5 are on page 36.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to hjw.de@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.

Problem 36-1: Integer Matrices with Unit Determinants

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Suppose that, for expository purposes, you wish to construct a 2×2 matrix with a determinant of plus one, and eigenvalues of $f \pm g\sqrt{d}$ corresponding to the column eigenvectors

$$\begin{pmatrix} c \pm \sqrt{d} \\ 1 \end{pmatrix},$$

where c, f, and g are nonzero integers at your disposal and d is a fixed positive integer (though not a square). Explain how such matrices may be identified for certain choices of d!

Problem 36-2: Simple Integer Programming Problems

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Show that magic squares and similar recreational problems involving the insertion of a sequence of positive integers into a rectangular array can be realised as nonlinear integer programming problems (without explicit optimality criteria).

Problem 36-3: Transformations of (Skew-) Idempotent and (Skew-) Involutory Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

In this problem, we generalise Farebrother's (2001, p. 11) definitions of involutory and skew-involutory matrices by replacing the $n \times n$ identity matrix I_n by a given $n \times n$ idempotent matrix P satisfying $P^2 = P$.

(a) Let A be an $n \times n$ idempotent or skew-idempotent matrix satisfying $AP = PA = A$ and $A^2 = mA$ where $m = \pm 1$. Then for $s^2 = \pm 1$, the generalised Householder transformation $B = s(2A - mP)$ is P-involutory (i.e., $B^2 = P$) or skew-P-involutory (i.e., $B^2 = -P$) according as $s^2 = \pm 1$: $B^2 = s^2(4A^2 - 4mPA + m^2P^2) = s^2P$.

(b) Conversely, suppose that B is an $n \times n$ P-involutory or skew-P-involutory matrix satisfying $BP = PB = B$ and $B^2 = s^2P$ where $s^2 = \pm 1$, then the transformation $A = (sB + mP)/2$ is idempotent (i.e., $A^2 = A$) or skew-idempotent (i.e., $A^2 = -A$) according as $m = \pm 1$: $A^2 = (s^2B^2 + 2msBP + m^2P^2)/4 = m(sB + mP)/2 = mA$.

Readers are asked to suggest alternative transformations that maintain the reality of B (or A) when P and A (or B) are real.

Reference

R. W. Farebrother (2001). The naming of parts. *IMAGE: The Bulletin of the International Linear Algebra Society* **27** (October 2001), 11.

Problem 36-4: An Eigenvalue Inequality of 3×3 Complex Hermitian Matrices

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Let A be a 3×3 complex Hermitian matrix. Show that $\lambda_{\min}(A) \le \lambda_{\min}(|A|)$, with equality if and only if A is diagonally similar to $|A|$. Here $|\cdot|$ denotes the entrywise absolute value. Prove or disprove this inequality for complex Hermitian matrices of order ≥ 4.

Problem 36-5: Eigenvalues and Eigenvectors of Some Cayley Matrices

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If r and s are nonnegative integers, let $r = \sum_{i=1}^{\infty} \ell_{ir} 2^{i-1}$ and $s = \sum_{i=1}^{\infty} \ell_{is} 2^{i-1}$ be their binary expansions; i.e., $0 \le \ell_{ir}, \ell_{is} \le 1$, $i \ge 1$. Define

$$r \widehat{+} s = \sum_{i=1}^{\infty} [\ell_{ir} + \ell_{is} \text{ (mod 2)}] 2^{i-1}.$$

Find the eigenvalues and eigenvectors of $C_k = (r \widehat{+} s)_{r,s=0}^{2^k-1}$, $k \ge 1$.

Problems 35-6 through 35-8 are on page 35.