The Bulletin of the International Linear Algebra Society

IMAGE

Serving the International Linear Algebra Community

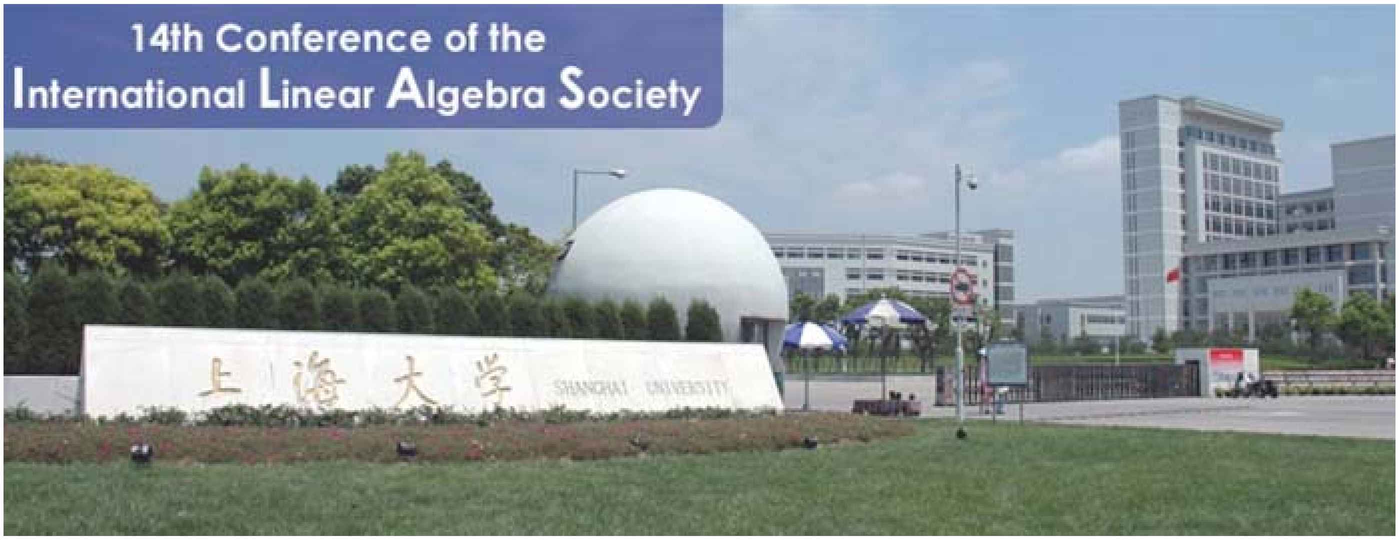
Issue Number 37, pp. 1-32, Fall 2006

Conferences and Workshops

Announcements, Book Review and Reports

IMAGE Problem Corner: Old Problems, Most With Solutions

IMAGE Problem Corner: New Problems



14th ILAS Conference
Shanghai, P. R. China

July 16-20, 2007

The organizing committee invites participation and presentations on all aspects of linear algebra, matrix theory and its applications. The deadline for abstracts of contributed talks is March 31, 2007, and early registration is April 30, 2007. The meeting is sponsored by ILAS and Shanghai University. Keynote speakers are:

M. Bresar (Slovenia)
M.D. Choi (Canada)
D. Farenick (Canada)
Eric Kaltofen (USA)
Plamen Koev (USA)
Amy Langville (USA)
Wenwei Lin (Taiwan,China)
Michael Ng (Hong Kong,China)
Andre Ran (The Netherlands)
Yaokun Wu (Shanghai, China)
Shufang Xu (Beijing, China)

Bryan Shader (University of Wyoming) will be the Olga Taussky/John Todd lecturer.

It is hoped that there will be an opportunity for those who wish to go to a Chinese acrobat/circus show, on Monday or Tuesday night. This would be at an additional cost ($35-45) with a bus provided by the conference organization. For more information, and to register, please visit the conference webpage http://math.shu.edu.cn/ILAS07/.

There will be a special issue of *Linear Algebra and its Applications* devoted to the conference. Editors for the issue are Ilse Ipsen, Julio Moro, Peter Šemrl, Jiayu Shao, and Pei Yuan Wu.

The Organizing Committee is:

Richard Brualdi (Madison), Co-chair
Erxiong Jiang (Shanghai), Co-chair
Raymond Chan (Hong Kong)
Chuanqing Gu (Shanghai)
Danny Hershkowitz (Haifa), ILAS President
Roger Horn (Salt Lake City)
Ilse Ipsen (Raleigh)
Julio Moro (Madrid)
Peter Šemrl (Ljubljana)
Pei Yuan Wu (Hsinchu)
Jia-yu Shao (Shanghai)

Local Organizers are:

Qinwen Wang
Chuanqin Gu (Chair)
Fuping Tan
Xiaomei Jia

An excursion is being planned for Wednesday. Current thinking is to have meetings from 9 to 11 on Wednesday, an early lunch, and then begin the excursion which would last into the evening. If it can all be arranged the excursion would include:

1) Shanghai Museum
2) YuYuan Gardens with dinner
3) Oriental Pearl Tower (magnificent view of Shanghai)
4) Nanjing Road (an opportunity to walk on this famous street).

Shanghai University has a modern conference center with air-conditioning and 5 meeting rooms, including a very large one; the conference center is a 10 minute walk from the two hotels on the new campus. All of the conference rooms have a computer and a projector mounted on the ceiling. For computer presentations it is best to have a pdf file on a memory stick. The standard overhead projectors will also be available.

2007 Haifa Matrix Theory Conference
Haifa, Israel
April 16-19, 2007

The 2007 Haifa Matrix Theory Conference will take place at the Technion on April 16-19, 2007 under the auspices of the Technion's Center for Mathematical Sciences.

This will be the fourteenth in a sequence of matrix theory conferences held at the Technion since 1984. The conference will host two special sessions: one to celebrate Hans Schneider's 80th birthday; the other in memory of our colleague Miron Tismenetsy (Timor) who passed away last April.

As in the past all talks will be of 30 minutes duration, and will cover a wide spectrum of theoretical and applied Linear Algebra.

Titles and abstracts of contributed talks should be sent, no later than December 31, 2006, to any of the members of the organizing committee; Abraham Berman (co-chair), Moshe Goldberg (co-chair), Daniel Hershkowitz, Leonid Lerer, and Raphael Loewy.

2007 Robert C. Thompson Matrix Meeting
Auburn University, Auburn, AL
Saturday, March 24, 2007

The Robert C. Thompson Matrix Meeting (formerly the Southern California Matrix Meeting) is an informal one-day meeting to encourage the interaction and collaboration of researchers on matrices, including applications, computation, and theory. These meetings were originally initiated by Steve Pierce and Robert C. Thompson in the 1980's. The attendees in 2004 voted to change the name to honor Bob Thompson, deceased in 1995.

There will be time slots for 20 minute and 30 minute talks. Send your abstract to one of the organizers, Jane Day (day@math.sjsu.edu), Wasin So (so@math.sjsu.edu), or Tin-Yau Tam (tamtiny@auburn.edu), no later than March 1, 2007.

There is no registration fee. Following the tradition of Thompson and Pierce, a complimentary dinner will be served after the meeting, to which all participants and their guests are cordially invited. RSVP by March 12, 2007 for the dinner, to Tin-Yau Tam (tamtiny@auburn.edu).

For travel and hotel information, visit the conference website: http://www.auburn.edu/~tamtiny/rct2007.html .

UNIVERSITY OF WINDSOR
DEPARTMENT OF MATHEMATICS AND STATISTICS PRESENTS

The 16th International Workshop on Matrices and Statistics

WINDSOR, ONTARIO, CANADA
JUNE 1–3, 2007

IWMS 2007

The purpose of this Workshop is to stimulate research, in an informal setting, and to foster the interaction of researchers in the interface between matrix theory and statistics. The Workshop will include both invited and contributed talks, and a special session with talks and posters by graduate students is planned. A special issue of the journal *Linear Algebra and its Applications* will be published devoted to selected papers presented at the conference.

Registration:

Before May 1, 2007

Faculty: $250
Students/Retirees: $150

Banquet: $50
Day trip: $150
Spouse/Guest Trip $100

After May 1, 2007

Faculty: $350
Students/Retirees: $200

Banquet: $50
Day trip: $150
Spouse/Guest Trip $100

(Please see the website for more information about the workshop venue, registration information)

International Organizing Committee:

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S. Ejaz Ahmed — LOC chair (Windsor)
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FOR MORE INFORMATION, PLEASE VISIT OUR WEBSITE http://www.uwindsor.ca/iwms EMAIL: iwms@uwindsor.ca

Workshop on Operator Theory & Applications College of William & Mary, Williamsburg, VA July 22-26, 2008

This will be a comprehensive, inclusive conference covering all aspects of theoretical and applied operator theory. Tentative list of plenary speakers: J. Agler, A. Boettcher, P. Deift, I. Gohberg, B. Jacob, M. A. Kaashoek, P. Lancaster, B. McCluer, S. A. McCullough, N. Nikolskii, G. Popescu, S. Richter, H. Schneider, and V. Vinnikov.

For details as available, visit http://www.math.wm.edu/~vladi/IWOTA/IOWTA2008.htm. Organizers: J.A. Ball, V. Bolotnikov, J.W. Helton, L. Rodman, and I.M. Spitkovsky.

Special Issue of LAA in Honor of Richard S. Varga

Richard S. Varga will celebrate his 80th birthday on October 9, 2008. He has been one of the leading figures in numerical analysis, approximation theory and linear algebra over the past 50 years, an editor of Linear Algebra and Its Applications for almost 40 years, and is currently one of its distinguished editors.

We are pleased to announce a special issue of LAA in his honor. Papers in pure, applied and numerical linear algebra will be considered. All papers submitted must meet the publication standards of LAA and will be subject to the normal refereeing procedure.

Papers should be sent to any of the special editors, preferably by email in a PDF or PostScript format. Deadline: March 31, 2007.

Guest Editors are: Ljiljana Cvetkovic, Univ. of Novi Sad (lila@im.ns.ac.yu), Andreas Frommer, Univ. Wuppertal (frommer@math.uni-wuppertal.de), Lilia Kolotilina, Steklov Mathematical Institute (liko@pdmi.ras.ru), and Daniel B. Szyld, Temple University (szyld@math.temple.edu).

Georg Heinig Memorial Volume

Vadim Olshevsky, Coordinating Editor

On May 10, 2005, Georg Heinig died unexpectedly at age 57 of a heart attack. He was a leading expert in the field of structured matrices, an irreplaceable colleague, and a good friend. Obituaries can be found at http://www.GeorgHeinig.org.

Researchers are invited to submit papers to “Numerical Methods for Structured Matrices and Applications: Georg Heinig Memorial Volume,” which will be published by Birkhauser Verlag Basel. The topics should be related to his research interests, which include (but are not limited to) structured matrices, fast algorithms, operator theory, applications to system theory, and signal processing. Original research papers are sought. In exceptional cases, the editors will allow survey articles. All papers will be refereed.

Papers must be written in LaTeX2e using the Birkhauser style birkmult.cls. The file and the instructions to the authors can be obtained directly from http://www.birkhauser.ch/math/downloads.

See http:/www.GeorgHeinig.org and http://www.math.uconn.edu/~olshevsky/Heinig/callforpapers.php for more information.

DEADLINES FOR IMAGE

IMAGE welcomes information on linear algebra and related topics, including:

Reports and announcements for workshops and conferences

Feature articles on emerging applications and topics

Historical essays

Book reviews and announcements of new publications

Letters to the editor

New problems and solutions to old problems

Submissions for the Spring 2007 issue are due April 15; for Fall 2007, October 15. Send all material by email: for the Problem Corner to Hans Joachim Werner (hjw.de@uni-bonn.de), everything else to Jane Day (day@math.sjsu.edu).

ILAS INFORMATION CENTER

The electronic ILAS INFORMATION CENTER (IIC) provides current information on international conferences in linear algebra, other linear algebra activities, linear algebra journals, and ILAS-NET notices. Organizations and individuals are invited to contribute information. Contact Shaun Fallat (sfallat@math.uregina.ca), IIC manager, for information on how to use IIC. The primary website is
http://www.ilasic.math.uregina.ca/iic/index1.html
and mirror sites are located at:

htpp://www.math.technion.ac.il/iic/index1.html
htpp://wftp.tu-chemnitz.de/pub/iic/index1.html
htpp://hermite.cii.fc.ul.pt/iic/index1.html
htpp://www.math.temple.edu/iic/index1.html

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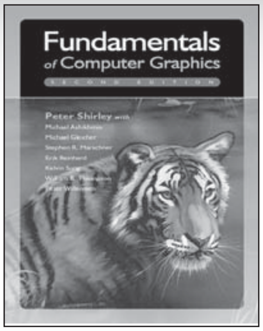


COMPUTER GRAPHICS

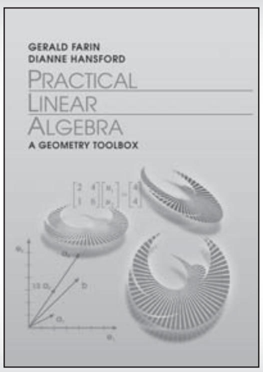
Fundamentals of Computer Graphics

Second Edition

Peter Shirley, Michael Ashikhmin, Michael Gleicher, Stephen R. Marschner, Erik Reinhard, Kelvin Sung, William B. Thompson, Peter Willemsen

$74.00; 1-56881-269-8; Hardcover; 652 pp.

The second edition of this widely adopted text includes a wealth of new material, with new chapters on Signal Processing, Using Graphics Hardware, Building Interactive Graphics Applications, Perception, Curves, Computer Animation, and Tone Reproduction. The authors present the mathematical foundations of computer graphics with a focus on geometric intuition, allowing the programmer to understand and apply those foundations to the development of efficient code.



LINEAR ALGEBRA

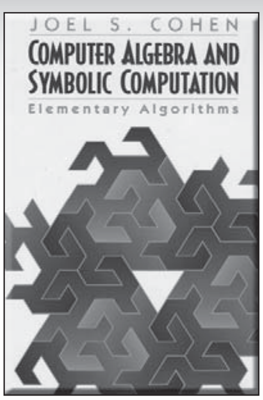
Practical Linear Alegbra

A Geometry Toolbox

Gerald Farin, Dianne Hansford

$67.00; 1-56881-234-5; Hardcover; 394 pp.

Practical Linear Algebra introduces students in mathematics, science, engineering, and computer science to Linear Algebra from an intuitive and geometric viewpoint, creating a level of understanding that goes far beyond mere matrix manipulations. This book covers all the standard linear algebra material for a first-year course; the authors teach by motivation, illustration, and example rather than by using a theorem/proof style.



COMPUTER ALGEBRA

Computer Algebra and Symbolic Computation

Elementary Algorithms

Joel S. Cohen

$55.00; 1-56881-158-6; Hardcover; 323 pp.

This book provides a systematic approach for the algorithmic formulation and implementation of mathematical operations in computer algebra programming languages. The viewpoint is that mathematical expressions, represented by expression trees, are the data objects of computer algebra programs, and by using a few primitive operations that analyze and construct expressions, we can implement many elementary operations from algebra, trigonometry, calculus, and differential equations. With a minimum of prerequisites this book is accessible to and useful for students of mathematics, computer science, and other technical fields.

CD includes full, searchable text and implementations of all algorithms in Maple, Mathematica, and MuPad!

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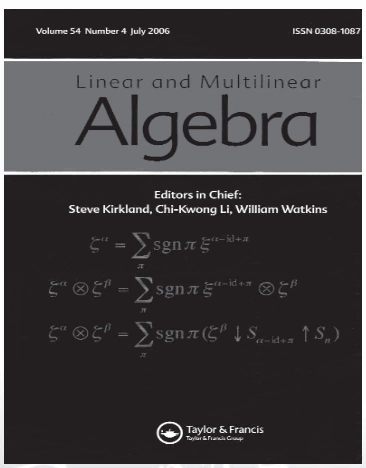

Volume 55, 2007

LAMA publishes original research papers that advance the study of linear and multilinear algebra, or that apply the techniques of linear and multilinear algebra in other branches of mathematics and science. The Journal also publishes research problems, survey articles and book reviews of interest to researchers in linear and multilinear algebra. Appropriate areas include, but are not limited to:

- spaces over fields or rings
- tensor algebras, nonnegative matrices
- inequalities in linear algebra
- combinatorial matrix theory
- numerical linear algebra
- representation theory
- Lie theory
- invariant theory
- operator theory

http://www.tandf.co.uk/journals/titles/03081087.html

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Fiedler Anniversary Honored

Report by Jurjen Duintjer Tebbens and Ctirad Matonoha

On June 12, 2006, an international conference to honor the 80th birthday of Prof. Miroslav Fiedler was held at the Mathematical Institute of the Academy of Sciences of the Czech Republic, in Prague. Many of his colleagues, both foreign experts and new young researchers, met there. It was organized by the Institute of Computer Science AS CR in cooperation with the Mathematical Institute.

During the opening ceremony, Dr. A. Sochor, Director of the Mathematical Institute, emphasized the fundamental contributions of Prof. Fiedler to science, and the President of the Academy of Sciences, Prof. V. Paˇces, awarded him the honorary Academy medal *De scientia et humanitate optime meritis* for his important contributions to science and humanity.

The Director of the Institute of Computer Science, Prof. J. Wiedermann, enriched the conference with a presentation about the meaning of the number 80 in mathematics and history up to the anniversary of Prof. Fiedler. At the end of the opening ceremony Prof. Z. Strakoˇs informed attendees about publication of a special issue of the scientific journal Linear Algebra and its Applications in honor of Prof. Fiedler. The remaining program of the scientific conference contained a variety of interesting lectures by both Czech and foreign experts.

Early in the evening the meeting continued with a social event at Villa Lanna, which was opened by the chairman of the Union of Czech Mathematicians and Physicists, Prof. S. Zajac.

Prof. Miroslav Fiedler was born on the 7th of April 1926 in Prague, where he also graduated from the Charles University. He started to work at the Mathematical Institute in the 1950's and in 1963 he obtained the title DrSc. At the Academy of Sciences he has worked for more than 50(!) years and, although now retired, is still scientifically active. His areas of specialization are mainly linear algebra, numerical linear algebra, graph theory and Euclidean geometry, where he achieved a number of outstanding results. He has been teaching at the Charles University for many years, became a Professor in 1965, and supervised many successful doctoral students. He also worked at several foreign universities, including Cal Tech in Pasadena, Auburn University, and the University of South Carolina.

Many honors and prizes have been awarded to Prof. Fiedler, for instance the National Prize of the Czech Republic (1978), honorary membership of the Union of Czechoslovak Mathematicians and Physicists (1985), the Bolzano Gold Medal (1986), and the Hans Schneider Prize of the International Linear Algebra Society (ILAS) (1993). He was Chairman of the Czechoslovak (later Czech) Committee for Mathematics (1979-1999), Chief Editor of the *Czechoslovak Math. Journal*, Honorary Editor of *Numerische Mathematik*, and Distinguished Editor of *Linear Algebra and Its Applications*. He also served on the editorial boards of *Linear and Multilinear Algebra*, *Mathematica Slovaca*, and the *Electronic Journal of Linear Algebra*, and was a member of the Householder (Gatlinburg) Symposia Steering Committee (1967-1993).

Prof. Fiedler is the author of several monographs and more than 170 scientific papers published in renowned journals.

First Olga Taussky Todd Lecturer at ICIAM: Pauline van den Driessche

The Association for Women in Mathematics and European Women in Mathematics, together with the organizers of the 2007 International Conference on Industrial and Applied Mathematics, have announced that Pauline van den Driessche, University of Victoria, will be the Olga Tausky Todd Lecturer at the 7th ICIAM Congress, to be held in July 2007 at the ETH in Zurich.

This is the first such award, an honor established by AWM and EWM for a woman who has made outstanding contributions in applied mathematics and/or scientific computation. The title pays tribute to the memory of Olga Taussky Todd, whose scientific legacy is in both theoretical and applied mathematics, and whose work exemplifies the qualities to be recognized.

Special consideration was given for this first award to candidates whose work is in one of the areas of Taussky Todd's research: applications of number theory, linear algebra or numerical analysis. The selection process was conducted by the Olga Taussky Todd Prize Committee, chaired by Barbara Lee Keyfitz, president of AWM.

Fiedler Anniversary Conference

Book Review: *Handbook of Linear Algebra*

Roger A. Horn, University of Utah

Handbook of Linear Algebra, Leslie Hogben (Editor-in-Chief); Richard Brualdi (Associate Editor), Anne Greenbaum (Associate Editor), Roy Mathias (Associate Editor), CRC Press, 2006, approx. 1,400 pp. List Price: $119.95 ISBN: 1584885106

This CRC *Handbook* is not the huge flexibly bound brick that we of a certain age lugged around campus (amazingly thin but durable pages, awesome integral tables, physical constants galore, microtype that only youthful eyes could love) back when the 12-inch dangling from our belts didn't need virus protection and TV was still a monochrome experience.

But like its ancestor, the descriptors of this *Handbook* are all big: Four senior editors; 99 distinguished contributors; 1,400 pages; 77 chapters; 40-page glossary; and a 56-page index.

The major divisions of the *Handbook* are: Core linear algebra (26 chapters); Combinatorial matrix theory and graphs (10 chapters); Numerical methods (13 chapters); Applications to optimization, probability and statistics, analysis, physical and biological sciences, computer science, geometry, and algebra (21 chapters); and Computational software (7 chapters).

An enormous amount of material is summarized in the core linear algebra part, yet it contains only a third of the chapters. The breadth of application areas included in this book is unmatched in any text or monograph now available.

Each chapter is divided into sub-topical sections. Sections may begin with Definitions, followed by a list of Facts (these lists are the core of the enterprise), which may then be illustrated by Examples and perhaps by Applications. References to texts or survey papers, and, where necessary, to research papers, are provided for all Facts; there are no proofs. For example, the 15-page chapter on Singular Values contains seven sections: Characterizations; Singular Values of Special Matrices; Unitarily Invariant Norms; Inequalities; Matrix Approximation; Characterizations of Eigenvalues of $C = A+B$ (Hermitian) and Singular Values of $C = AB$ (general); and Miscellaneous Results and Generalizations. The Inequalities section contains two Definitions, 23 Facts, and five Examples.

The editors have made a serious effort to standardize notation and terminology across the contributions of 99 expert authors; definitions, once given, are rarely repeated. Of course, few readers of this *Handbook* will digest it sequentially, and fewer still will remember every one of the more than 1,200 definitions and 450 notations offered up in just the first 49 chapters. For the rest of us, there is a 10-page Notation index, a comprehensive Glossary, and a very detailed Index. For example, the Routh-Hurwitz matrix exhibited in Section 19.2 has entries $S_k(A)$, about which nothing is said. However, the Notation Index tells us that $S_k(A)$ is the "sum of all principal minors of size k of matrix A" and refers us to Section 4.2. The Glossary listing for *principal minor* also refers us to Section 4.2.

This *Handbook* adopts a pragmatic approach to citations that is appropriate to its broad spectrum of intended users outside the linear algebra research community. Readers are often referred to texts or survey articles for proofs and related results rather than to original research articles.

For a complete table of contents (chapter title and contributing author), visit http://www.public.iastate.edu/lhogben/HLA.html, which in due course will also have an errata list.

Comparison of this splendid and comprehensive *Handbook* with an influential earlier work illustrates something noteworthy about our discipline. In 1960, the National Bureau of Standards (NBS) published a "survey of basic material in matrix theory … as a handy reference for research workers and students … useful to the applied scientist and engineer as an accessible source of basic material in matrix theory." The NBS Director assured the reader that "an attempt has been made to include most of the important recent useful results," but he warned that "the field is undergoing rapid development at both a practical and a theoretical level." The comprehensive 1960 NBS survey was prepared by a single author, spanned 24½ pages of text, had a 1½ page index, and could be purchased from the U.S. Government Printing Office for 15 cents [1]. We have come a long way in 46 years.

Bibliography

[1] Marvin Marcus, *Basic Theorems in Matrix Theory*, National Bureau of Standards Applied Mathematics Series No. 57, 1960.

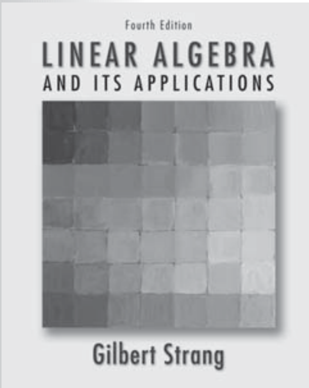
New Journal: *Operators and Matrices*

By Leiba Rodman, for the Editors-in-Chief

This new journal is dedicated to publication of high quality original research papers in matrix and operator theory and their numerous applications, as well as expository papers.

The first issue will appear in March 2007. Additional information can be found at http://www.ele-math.com and the links there. Papers can be submitted electronically to any member of the editorial board, directly to one of editors-in-chief or editorial office, or on line following the instructions at http://www.ele-math.com.

Have you seen the latest Linear Algebra texts from Brooks/Cole?

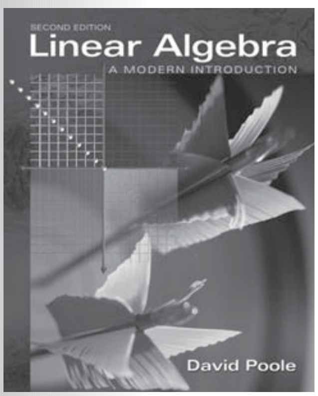


Linear Algebra and Its Applications, Fourth Edition

Gilbert Strang, Massachusetts Institute of Technology
488 pages. 7 3/8 x 9 1/4. 2-color. Casebound. © 2006. ISBN: 0-03-010567-6.
Available Now!

Gilbert Strang demonstrates the beauty of linear algebra and its crucial importance. Strang's emphasis is always on understanding. He explains concepts, rather than concentrating entirely on proofs. The informal and personal style of the text teaches students real mathematics. Throughout the book, the theory is motivated and reinforced by genuine applications, allowing every mathematician to teach both pure and applied mathematics. Applications to physics, engineering, probability and statistics, economics, and biology are thoroughly integrated as part of the mathematics in the text.

- The exercise sets in the book have been greatly expanded and thoroughly updated. They feature many new problems drawn from Professor Strang's extensive teaching experience.
- The Linear Algebra web pages offer review outlines and a full set of video lectures by Gilbert Strang. The sites also include eigenvalue modules with audio (**http://ocw.mit.edu** and **http://web.mit.edu/18.06**).
- An *Instructor's Solutions Manual* (0-03-010568-4) with teaching notes by Gilbert Strang is provided for use with this text. In addition, a *Student Solutions Manual* (0-495-01325-0) with detailed, step-by-step solutions to selected problems will be available.



Linear Algebra: A Modern Introduction, Second Edition

David Poole, Trent University
712 pages. 8 x 10. 2-color. Casebound. © 2006. ISBN: 0-534-99845-3.
Available Now!

Emphasizing vectors and geometric intuition from the start, David Poole's text better prepares students to make the transition from the computational aspects of the course to the theoretical. Poole covers vectors and vector geometry first to enable students to visualize the mathematics while they are doing matrix operations. With a concrete understanding of vector geometry, students are able to visualize and grasp the meaning of the calculations that they will encounter. By seeing the mathematics and comprehending the underlying geometry, students develop mathematical maturity and can think abstractly when they reach vector spaces.

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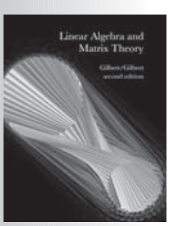
- **Student CD-ROM:** Included with the text, the CD contains data sets for more than 800 problems in MAPLE, MATLAB, and Mathematica, as well as data sets for selected examples. Also contains CAS enhancements to the "Vignettes" and "Explorations," which appear in the text, and manuals for using MAPLE, MATLAB, and Mathematica.

Also Available



Linear Algebra and Matrix Theory, Second Edition

Jimmie Gilbert and **Linda Gilbert**, both of University of South Carolina, Spartanburg
544 pages. Casebound. ©2005.
ISBN: 0-534-40581-9.

Linear Algebra: An Interactive Approach

S. K. Jain, Ohio University
A. D. Gunawardena, Carnegie Mellon University
480 pages. Casebound. ©2004.
ISBN: 0-534-40915-6.

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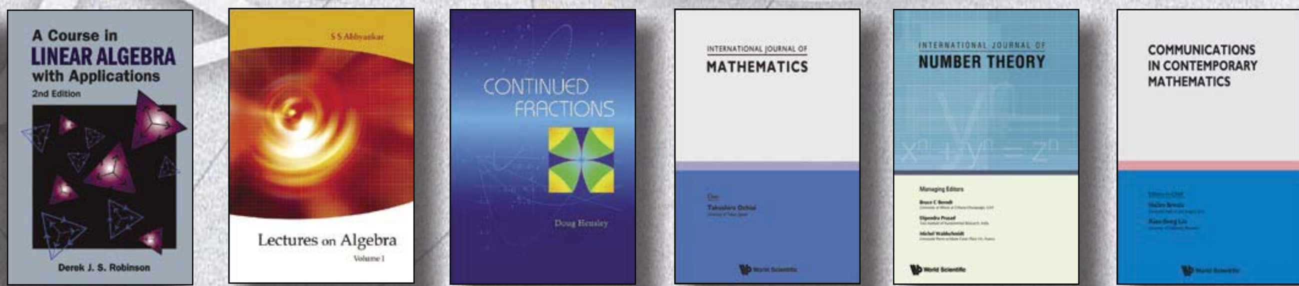

A COURSE IN LINEAR ALGEBRA WITH APPLICATIONS

(2nd Edition)

by **Derek J S Robinson** *(University of Illinois at Urbana-Champaign, USA)*

Review of the first edition:

"... it is very carefully written, both from the point of view of mathematical content and style, and readability ... It should therefore be very suitable as a course book as well as for self-tuition."

Mathematics Abstracts, Germany

This is the second edition of the best-selling introduction to linear algebra. Presupposing no knowledge beyond calculus, it provides a thorough treatment of all the basic concepts, such as vector space, linear transformation and inner product. The concept of a quotient space is introduced and related to solutions of linear system of equations, and a simplified treatment of Jordan normal form is given.

Numerous applications of linear algebra are described, including systems of linear recurrence relations, systems of linear differential equations, Markov processes, and the Method of Least Squares. An entirely new chapter on linear programming introduces the reader to the simplex algorithm with emphasis on understanding the theory behind it.

452pp	**981-270-023-4**	**US$84**	**£48**
Aug 2006	**981-270-024-2(pbk)**	**US$42**	**£24**

LECTURES ON ALGEBRA

Volume I

by **S S Abhyankar** *(Purdue University, USA)*

This book is a timely survey of much of the algebra developed during the last several centuries including its applications to algebraic geometry and its potential use in geometric modeling.

The present volume makes an ideal textbook for an abstract algebra course, while the forthcoming sequel, *Lectures on Algebra II*, will serve as a textbook for a linear algebra course. The author's fondness for algebraic geometry shows up in both volumes, and his recent preoccupation with the applications of group theory to the calculation of Galois groups is evident in the second volume which contains more local rings and more algebraic geometry.

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by **Doug Hensley** *(Texas A&M University, USA)*

The Euclidean algorithm is one of the oldest in mathematics, while the study of continued fractions as tools of approximation goes back at least to Euler and Legendre. Continued fractions have been studied from the perspective of number theory, complex analysis, ergodic theory, dynamic processes, analysis of algorithms, and even theoretical physics, which has further complicated the situation.

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Joint GAMM-SIAM Conference on Applied Linear Algebra July 2006

In July 2006, the SIAM Conference on Applied Linear Algebra took place for the first time in Europe. It was organized jointly with GAMM (Gesellschaft für Angewandte Mathematik und Mechanik), the International Association of Applied Mathematics and Mechanics, and in cooperation with ILAS, and it was held at the University of Düsseldorf in Germany.

This joint conference attracted the attention of a local newspaper, Die Rheinische Post, which featured an article on "How Mathematics Helps to Save Lives." After reminding its readers of the math phobia they may have experienced in school, the article went on to describe the work of Bernd Fischer (University of Lübeck, Germany), who opened the conference with a plenary talk on the use of image registration in medicine. The article ended with a somewhat hesitant admission that mathematics may, possibly, have some practical use after all.

The conference was expertly organized by Marlis Hochbruck from the University of Düsseldorf, and Andreas Frommer and Bruno Lang from the nearby University of Wuppertal. There were 280 attendees from 30 countries, 220 talks, and 12 ever ready student helpers in distinctive T-shirts. ILAS sponsored two speakers: Ludwig Elsner (University of Bielefeld, Germany) and Olga Holtz (University of California at Berkeley). Ludwig Elsner presented an overview of max algebras, results from his recent work with Pauline van den Driesche, and an application to music theory. Unfortunately, Olga Holtz could not attend due to visa problems. She sent Jim Demmel as a substitute, who gave a talk about their joint work on stability of group theoretic algorithms for fast matrix multiplication.

The SIAG Linear Algebra Prize was awarded to Inderjit Dhillon (University of Texas at Austin) and Beresford Parlett (University of California at Berkeley) for their 2004 SIMAX paper Orthogonal Eigenvectors and Relative Gaps. They show how to accurately compute, in floating point arithmetic, all eigenvectors of a large class of $n \times n$ real tridiagonal matrices in n^2 operations. A suitable representation of the tridiagonal matrix as a product of bidiagonal matrices, makes it possible to avoid the time consuming orthogonalization of eigenvectors, which is responsible for the n^3 operation count of conventional methods.

Another highlight of the conference was the banquet speech by Henk van der Vorst (University of Utrecht, Netherlands), the Editor in Chief of SIMAX. In one of Old Town's favorite restaurants, among good food and plenty of Düsseldorfer Alt (the dark beer for which Düsseldorf is known), Henk talked about his professional life where "big progress goes with small steps." Just the previous weekend, Henk had been made Ridder in de Orde de Nederlandse Leeuw (Knight in the Order of the Dutch Lion), a royal distinction bestowed on those who make exceptional contributions to society.

Workshop on Spectra of Families of Matrices Described by Graphs, Digraphs and Sign Patterns Oct. 23-27, 2006

American Institute of Mathematics
Palo Alto, CA

Three problems were investigated: the 2n-conjecture for spectrally arbitrary sign patterns, determining the minimum rank of symmetric matrices described by a graph, and the energy of graphs. There were 34 participants.

AIM workshops differ from typical conferences, in that the focus is on doing research and planning future directions, rather than reporting recent results. Lectures at the workshop, delivered by Shaun Fallat, Pauline van den Driessche, and Dragan Stevanovic, reviewed the background material for the three problems. The entire group then determined specific questions to investigate, and broke into subgroups to work.

Richard Brualdi, Leslie Hogben, and Bryan Shader organized this workshop. Participants included Francesco Barioli, Wayne Barrett, Avi Berman, Richard Brualdi, Steven Butler, Sebastian Cioaba, Dragos Cvetkovic, Jane Day, Louis Deaett, Luz DeAlba, Shaun Fallat, David Farmer, Shmuel Friedland, Chris Godsil, Jason Grout, Willem Haemers, Leslie Hogben, In-Jae Kim, Steve Kirkland, Raphael Loewy, Judith McDonald, Rana Mikkelson, Sivaram Narayan, Olga Pryporova, Uri Rothblum, Irene Sciriha, Bryan Shader, Wasin So, Dragan Stevanovic, Pauline van den Driessche, Hein van der Holst, Kevin Vander Meulen, Amy Wangsness, Amy Yielding. The workshop was supported by AIM and NSF.

A report will be available soon on the AIM website, http://aimath.org/pastworkshops/matrixspectrum.html. Here are some details.

Minimum Rank

Let G be a simple graph and let S_n denote the set of real symmetric $n \times n$ matrices. The *minimum rank* of G is $\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in S_n$ and for $i \neq j$, $a_{ij} \neq 0$ if and only if ij is an edge of $G\}$. The diagonal entries of matrices in this set are arbitary. Minimum rank is easy to compute for trees but otherwise can be difficult to determine. Results included:

- Minimum rank of numerous families of graphs was determined.
- Software is being developed for easy computation of minimum rank for small graphs.
- Possible new upper and lower bounds on minimum rank are under investigation.
- The effect of several graph operations on minimum rank have been obtained.
- Connections between balanced inertia and rank strong vertices are being studied.
- Examples over fields other than the real numbers were produced and several known results were extended to other fields.

These results are being incorporated into an on-line catalog listing graphs, minimum rank and other graph parameters. This catalog will facilitate making and testing conjectures about minimum rank.

2*n* conjecture

A spectrally arbitrary sign pattern (or zero-nonzero pattern), allows every possible spectrum of a real matrix, or, equivalently, allows every monic real polynomial as the characteristic polynomial. The $2n$-conjecture asserts that an $n \times n$ spectrally arbitrary pattern has at least $2n$ nonzero entries. It is known that a spectrally arbitrary $n \times n$ pattern must contain at least $2n$-1 nonzero entries, and numerous examples of spectrally arbitrary n × n sign patterns with 2n nonzero entries are known. The $2n$ conjecture may be valid only for irreducible patterns; possible counterexamples to the general conjecture were constructed by taking the direct sum of an SAP with $2n$ entries with a non-SAP with $2n$-1 entries were investigated. Full patterns that are SAPs were also studied.

Energy

The energy of a graph was defined by chemists to be the sum of the absolute values of the eigenvalues of its adjacency matrix. Certain quantities such as the heat of formation of a hydrocarbon are related to pi-electron energy that can be calculated as the energy of an appropriate ``molecular'' graph. Recently the Laplacian energy of a graph, the analogue of energy for the Laplacian matrix of G, has also been studied. The workshop investigated both.

AIM workshop organizers: Hogben, Brualdi, Shader

Participants in AIM workshop

A working group at AIM workshop (van den Driessche, Loewy, Kirkland, Kim, McDonald)

International Conference on Matrices and Statistics in Memory of Professor Sujit Kumar Mitra

Report by Simo Puntanen

Professor Sujit Kumar Mitra passed away at his home in New Delhi on 18 March 2004. An International Conference on Matrices and Statistics in his memory was held in Hyderabad, India, 6-7 January 2007. The conference was organised by the Statistical Quality Control and Operations Research Unit of the Indian Statistical Institute (ISI), Hyderabad. The Organizing Committee chaired by S.K. Pal (Director, ISI, Calcutta) included R.B. Bapat (ISI, Delhi), P. Bhimasankaram (ISI, Hyderabad), Dibyen Majumdar (University of Illinois at Chicago), Thomas Mathew (University of Maryland Baltimore County), A.L.N. Murthy (ISI, Hyderabad), G.S.R. Murthy (ISI, Hyderabad) and G. Murali Rao (ISI, Hyderabad). The conference was supported by the Department of Science & Technology of Government of India, and the conference venue was the National Geophysical Research Institute, Hyderabad, India.

The conference was inaugurated by B.L.S. Prakasa Rao (University of Hyderabad), and the keynote address was given by J.K. Ghosh (Purdue University). There were 15 participants at the conference: Ravindra B. Bapat (ISI, New Delhi), P. Bhimasankaram (ISI, Hyderabad), Jayanta Kumar Ghosh (Purdue University, USA), K. Krishnamoorthy (University of Louisiana at Lafayette, USA), Dibyen Majumdar (University of Illinois at Chicago, USA), Saroj Malik (Hindu College, Delhi), Thomas Mathew (University of Maryland Baltimore County), A.R. Meenakshi (Annamalai University), Simo Puntanen (University of Tampere, Finland), B.L.S. Prakasa Rao (University of Hyderbad), P.S.S.N.V.P. Rao (ISI, Kolkata), S.B. Rao (ISI, Kolkata), Debasis Sengupta (ISI, Kolkata), Kirti R. Shah (University of Waterloo, Canada), Bikas Kumar Sinha (ISI, Kolkata).

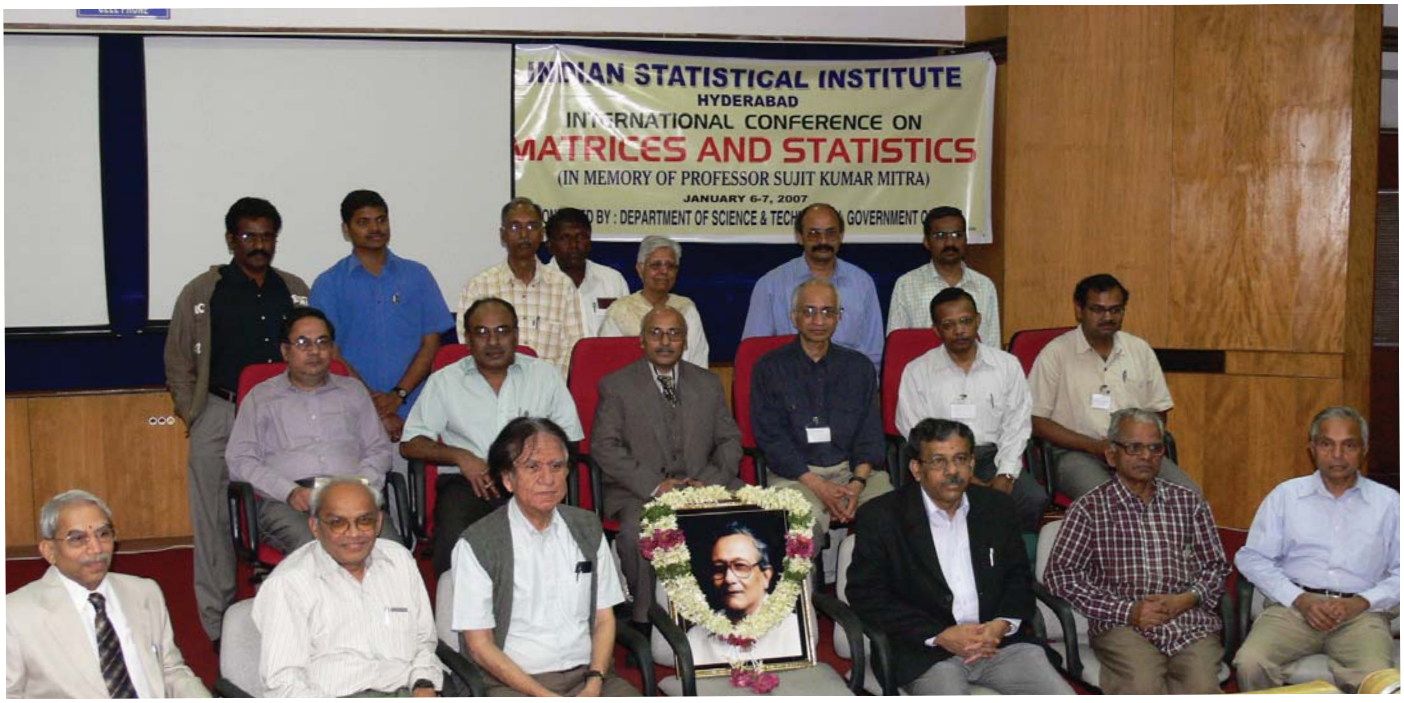
Sujit Kumar Mitra was born on 23 January 1932 in Kolkata, India. He earned his B.Sc. degree in Statistics from Presidency College, Calcutta, in 1949, and his M.Sc. degree in Statistics, from Calcutta University, in 1951. He spent two years in the Department of Statistics at the University of North Carolina at Chapel Hill, completing his Ph.D. thesis, "Contributions to the statistical analysis of categorical data" under the guidance of Samarendra Nath Roy, in 1956.

Dr. Mitra joined the Indian Statistical Institute (ISI), Calcutta, soon after finishing his Masters degree in Calcutta University and was with ISI, Calcutta, till 1971, except for the periods when he was a Ph.D. student at the University of North Carolina and when he spent a short time as a Statistician with Tea Board. He was a Professor of Statistics at ISI, New Delhi, 1974-1991, and Distinguished Scientist there, 1991-1992. He retired in 1992 and continued as Professor Emeritus. Professor Mitra held several visiting professorships in the United States and Japan.

Professor Mitra's main interests were statistical methodology, multivariate analysis, design of experiments, sample surveys and matrix algebra, particularly the theory of generalized inverses of matrices. His classic book with C. Radhakrishna Rao, *Generalized Inverse of Matrices and Its Applications* (Wiley, 1971), has been used with enthusiasm by mathematicians and statisticians all over the world. Very appropriately, Sujit Kumar Mitra was called a "Master of the Row Space and the

Column Space" by Bapat and Hartwig [Linear Algebra and its Applications (1994), vol. 211, pp. 5-14]. Professor Mitra published about 100 research papers and several books; of the papers, 17 appeared in Linear Algebra and its Applications.

Professor Mitra is survived by his wife, Sheila, whom he married in 1958, three daughters, Utsa, Ipshita and Anindita, and a son, Kaustav, and their families.



Participants in S.K. Mitra Conference

Second International Workshop on Matrix Analysis and Applications Nova Southeastern University Fort Lauderdale, FL Dec. 15-17, 2006

by Fuzhen Zhang

The Workshop opened with a warm welcome by Dr. Don Rosenblum, Dean of the Farquhar College of Arts and Sciences, followed by the ILAS invited talk given by Richard Brualdi. Fifty five people from more than ten countries attended. Thirty talks and one poster were presented.

In addition to the talks, the participants had a pleasant tour of the Everglades Holiday Park by airboat, to see Florida alligators, exotic birds and fish. An enjoyable dinner after the excursion highlighted the Workshop with fine wine, beer, food, and good conversation.

The Organizing Committee was Zhong-Zhi Bai, Chinese Academy of Sciences, Beijing, China; Chi-Kwong Li, College of William and Mary, Williamsburg, VA; Bryan Shader, University of Wyoming, Laramie, WY; Hugo Woerdeman, Drexel University, Philadelphia, PA; Fuzhen Zhang (Chair), Nova Southeastern University, Fort Lauderdale, FL; and Qingling Zhang, Northeastern University, Shenyang, China.

The conference was supported by Nova Southeastern University and the International Linear Algebra Society (ILAS). A special issue, Matrix Analysis and Applications, of the International Journal of Information & Systems Sciences will be devoted to the meeting. Editors for the issue include: Dennis Bernstein (Guest editor, University of Michigan-Ann Harbor), Hugo J. Woerdeman (Guest editor), Fuzhen Zhang, and Qingling Zhang.

Participants in 2nd International Workshop on Matrix Analysis and Applications

Summer Research Experiences for Undergraduates at Iowa State

by Leslie Hogben

Each summer more than a dozen excited, eager and enthusiastic undergraduates converge on Ames, Iowa from all over the United States to participate in the Iowa State University Mathematics Department's NSF-funded Research Experience for Undergraduates (REU) program. REUs in general are designed to foster student interest in research and graduate school. Faculty and graduate students work with seven or more research projects each summer. This is one of the largest NSF mathematics REU sites and one of the largest undergraduate summer research programs on the ISU campus. It is specifically designed to exploit the strengths of a large research university (ready access to a large number of faculty and graduate students doing research), while at the same time giving students the kind of individual attention and mentoring that is often found only at smaller colleges.

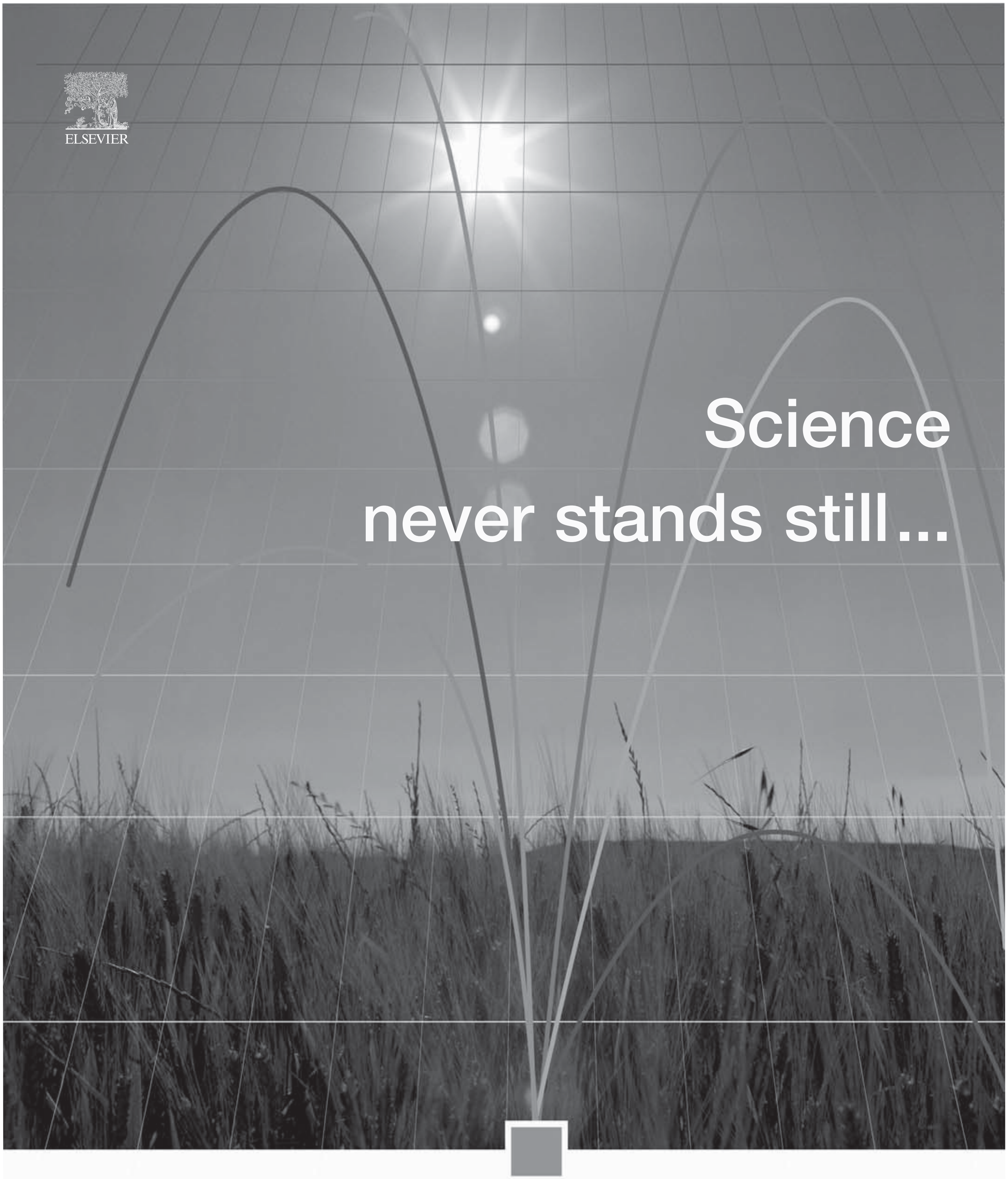
The research topics vary from year to year, depending on which faculty members serve as mentors, although projects have been offered every year in linear algebra, dynamical systems, and mathematical biology.

Leslie Hogben manages the ISU Math REU and mentors one or two matrix theory projects each summer. Her projects have included matrix completion problems, rational realization of eigenvalues of tree sign patterns, minimum rank of symmetric matrices described by a graph, and matrix *D*-stability. During summer 2006, although the projects included such diverse topics as biomolecular modeling and dynamically coupled linear ODEs and Markov chains, the students realized that most of the projects used linear algebra, and that became a theme of the REU.

A typical research group is a team with two undergraduates, one graduate student and one faculty member. Each team meets daily for at least an hour. Initially, faculty teach the necessary background to students; later students report progress and discuss obstacles. Each project team produces a final paper (typically 20 pages) and presents an hour-long report at the symposium held in the final week of the program.

Since 2004, five ISU Math REU papers have appeared in *Linear Algebra and Its Applications*, *Journal of Mathematical Analysis and Applications*, and *Mathematical Biosciences and Engineering*. Four additional papers are under review, and several from summer 2006 are in preparation. Approximately ten students have made research presentations at undergraduate conferences.

In addition to the research projects, there is an extensive program of other activities, both academic and social. During the first two weeks, the students attend several classes about Matlab and LaTeX. Once a week undergraduates, graduate students and some faculty have lunch together, where lot of discussion about life as a graduate student and as a faculty member takes place. Every Tuesday and Thursday afternoon participants gather for "pop and cookies" and conversation; once a week this is followed by a lecture or presentation. Lectures have varied from "The mathematics of secrecy: an introduction to public key cryptography" to "Mathematical modeling of phytoplankton", as well as presentations such as "Graduate school: What is it *really* like?"


ELSEVIER
Science
never stands still...

Social activities have included T-shirt design, a picnic at Big Creek Lake, a trip to an amusement/water park, participation in an intramural soccer team, ice skating, bowling, math movie night (Fermat's Last Tango), a concert by a local folk singer, etc. The undergraduates are housed together in Frederiksen Court student apartments. By living close they develop strong friendships.

The ISU Math REU involves 13-16 undergraduates each summer. Twelve are supported by the NSF REU-site grant; a few others are supported by the Alliance for the Production of African American Ph.D.s in the Mathematical Sciences, which is an NSF-supported partnership between several Iowa universities and historically black colleges and universities; through faculty members' research grants; and through the NASA/Pipelines Sciencebound program.

Each year the students design an REU T-shirt, and in 2006 their shirts supplied the answer to the question, "What do you call a bay eigensheep?" (a lamb, duh)

Recognizing the many benefits of graduate student involvement in the REU, the ISU Mathematics Department funds seven or eight research assistantships each summer for graduate students who participate in the REU. These graduate students do not replace faculty involvement; normally the entire project group meets together, although the graduate student leads the project when the faculty member is away for a short time at a research conference. The graduate students are also encouraged to discuss their graduate school experiences with the undergraduates.

More information about the ISU Math REU can be found on its website, http://orion.math.iastate.edu/reu/homepage.html, and in a paper located at http://orion.math.iastate.edu/reu/ISUREU2.pdf.

Participants in 2006 REU at Iowa State

Positions Available

1. Technische Universitaet Berlin, Germany, open Dec. 1, 2006 until filled:

> Two Postdoctorate positions in Algebra and in Analysis, for applicants with a PhD in pure or applied mathematics. For the algebra position, candidates should have a strong background in commutative algebra, combinatorics, linear and multilinear algebra; for the analysis position, a strong background in real and complex analysis, functional analysis, approximation theory and numerical analysis.
>
> One PhD Studentship in Numerical Analysis/Linear Algebra, for a B.S. graduate with strong background in functional and numerical analysis, linear and abstract algebra and combinatorics.

All these positions are for work on the project "Direct and inverse problems of numerical algebra and analysis" led by Prof. Olga Holtz. To apply, send a CV (including a publication list) to Olga Holtz (holtz@math.tu-berlin.de).

2. University of Birmingham, England:

> Opening for a Lecturer/Senior Lecturer in Computational Mathematics, deadline January 15, 2007.

Applicants in Numerical Linear Algebra and Optimisation are particularly sought, but all areas of computational mathematics and statistics will be considered. At least one additional appointment in optimisation will be made.

Faculty in numerical analysis and optimisation include Roy Mathias, Michal Kocvara (January 2007), Daniel Loghin, Natalia Petrovskya, Joerg Fliege, Peter Butkovic, Sandor Nemeth. Please send inquiries to Roy Mathias (mathias@maths.bham.ac.uk).

3. University of Amsterdam, The Netherlands:

> Opening for one PhD student, deadline December 15, 2006.

Preference for candidates with a solid background in (numerical) analysis of finite element methods or (numerical) linear algebra.

For more information, visit
http://staff.science/uva/n./~brandts/vacancy.html
or contact Jan Brandts (brandts@science.uva.nl).

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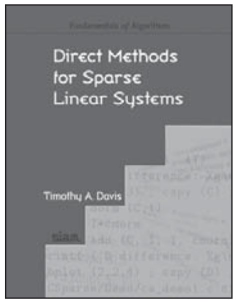

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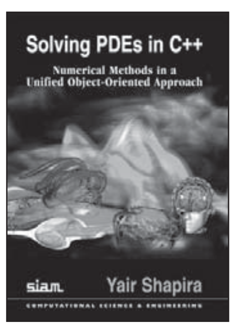
Computational Science and Engineering 1



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Zhangxin Chen, Guanren Huan, and Yuanle Ma
Computational Science and Engineering 2

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IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problem 35-5 [IMAGE 35 (Fall 2005), p. 44], and IMAGE Problems 36-1, 36-2, and 36-4 through 36-8 [IMAGE 36 (Spring 2006), pp. 36 & 35]. Problems 30-3 [IMAGE 30 (April 2003), p. 36], 32-4 [IMAGE 32 (April 2004), p. 40], and 36-3 [IMAGE 36 (Spring 2006), p. 36] are repeated below without solutions; we are still hoping to receive solutions to these problems. We introduce 6 new problems on pp. 32 & 31 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to hjw.de@uni-bonn.de and (b) two paper copies (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 30-3: Singularity of a Toeplitz Matrix

Proposed by Wiland SCHMALE, *Universität Oldenburg, Oldenburg, Germany:* schmale@uni-oldenburg.de
and Pramod K. SHARMA, *Devi Ahilya University, Indore, India:* pksharma1944@yahoo.com

Let $n \geq 5$, $c_1, \ldots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

Editorial Note. We have already published with Solution 30-3.1 an assumed solution to this problem; see IMAGE 36, (Spring 2006), pp. 26–27. In the meantime we have received the following remark. Because of this remark, the problem is reopened.

Remark on Problem 30-3 by Harald WIMMER, *Universität Würzburg, Würzburg, Germany:* wimmer@mathematik.uni-wuerzburg.de

In Solution 30-3.1 (see IMAGE 36 (Spring 2006), pp. 26–27) it was stated that a polynomial matrix $A(t) = \sum M_i t^i \in \mathbb{C}^{2\times 2}[t]$ of rank 1 has the property that the coefficient matrices M_i have the same row space and the same column space. The following example suggests that in general such a claim need not be true.

Consider the matrix

$$A(t) = \begin{pmatrix} 1 \\ t \end{pmatrix} \begin{pmatrix} t & 1 \end{pmatrix} = \begin{pmatrix} t & 1 \\ t^2 & t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t^2 = M_0 + M_1 t + M_2 t^2.$$

We have $\operatorname{rank} A(t) = 1$ for all $t \in \mathbb{C}$, but $\operatorname{rank} M_0 = \operatorname{rank} M_2 = 1$ and $\operatorname{rank} M_1 = 2$.

We still look forward to receiving solutions to Problem 30-3!

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j = u_{1j} X_1 + u_{2j} X_2 + u_{3j} X_3$, $j = 1, 2, 3$, such that each of the six matrices $W_j^{-1} W_k$, $j \neq k$, has zero trace. Equivalently, we want $(W_j^{-1} W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We still look forward to receiving solutions to Problem 32-4!

Problem 35-5: First and Second Moments Involving a Camouflaged Wishart Matrix

Proposed by Heinz NEUDECKER, *Universiteit van Amsterdam, Amsterdam, The Netherlands:* ericaengels173@hotmail.com

Consider the $m \times (n+m-1)$ matrix $C' := (A', B')$, where $A \in \mathbb{R}^{m-1,m}$ is constant, $B \in \mathbb{R}^{n,m}$ is such that $B'B$ follows a central Wishart distribution $W(\Omega, n)$, $n > m+1$, and where $\operatorname{rank}(C) = m$. As usual, let E denote the expected value operator and D denote the variance operator. Prove the following:

(i) $\mathsf{E}(\sum_i |(A', b_i)|^2) = n \cdot \operatorname{tr}(A'A)^*\Omega$, where, for $i = 1, 2, \cdots, n$, b_i denotes the i-th column vector of B', and where $|\cdot|$, $(\cdot)^*$, and $\operatorname{tr}(\cdot)$ stand for the determinant, the adjoint, and the trace, respectively, of $(\cdot)$.

(ii) $\mathsf{D}(\sum_i |(A', b_i)|^2) = n^2 \cdot \{\operatorname{tr}(A'A)^*\Omega\}^2 + 2n \cdot \operatorname{tr}\{(A'A)^*\Omega\}^2$.

Solution 35-5.1 by the Proposer Heinz NEUDECKER, *Universiteit van Amsterdam, Amsterdam, The Netherlands:*
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Clearly, $|(A', b_i)|^2 = |A'A + b_i b_i'|$. Suppose $\operatorname{rank}(A) = m-1$. Use then the Schur decomposition $A'A = T\Lambda T'$ with orthogonal T and diagonal positive semidefinite Λ whose last diagonal element is zero. Let M be the positive definite square submatrix of Λ. Write $T = (S, x)$. Then $A'A + b_i b_i' = T(\Lambda + T' b_i b_i' T)T'$ and

$$\Lambda + T' b_i b_i' T = \begin{pmatrix} M + S' b_i b_i' S & (b_i' x) S' b_i \\ (b_i' x) b_i' S & (b_i' x)^2 \end{pmatrix}.$$

It is easy to see that $|\Lambda + T' b_i b_i' T| = |M| \cdot (b_i' x)^2 = \mu(A'A) \cdot (b_i' x)^2$, where $\mu(A'A)$ is the product of the $m-1$ nonzero (hence positive) eigenvalues of $A'A$ (or AA'). For the determinantal property, in case $b_i' x$ is nonzero, see e.g. Magnus and Neudecker (1999, p. 25, ex. 4). When $b_i' x = 0$ the property will trivially hold. Thus we have shown that $|(A', b_i)|^2 = |A'A + b_i b_i'| = \mu(A'A) \cdot (b_i' x)^2$, when $\operatorname{rank}(A) = m-1$. Hence $\sum_i |(A', b_i)|^2 = \mu(A'A) \cdot x' B' B x$ when $\operatorname{rank}(A) = m-1$. As $\operatorname{rank}(B'B) = m$ by definition, the expression just found is nonzero. It is known that $(A'A)^* = \mu(A'A) \cdot xx'$ when $\operatorname{rank}(A'A) = m-1$, where x is as defined above. But then $\mu(A'A) \cdot x' B' B x = \mu(A'A) \cdot \operatorname{tr}(B' B x x') = \operatorname{tr}[\mu(A'A) \cdot x x' B' B] = \operatorname{tr}[(A'A)^* B' B]$. This leads to the result $\sum_i |(A', b_i)|^2 = \operatorname{tr}[(A'A)^* B' B]$ when $\operatorname{rank}(A) = m-1$.

Consider now the case: $\operatorname{rank}(A) < m-1$. Then $\operatorname{rank}(A', b_i) < m$ and $|(A', b_i)| = 0$. When $\operatorname{rank}(A) < m-1$, then also $\operatorname{rank}(A'A) < m-1$ and further $(A'A)^* = 0$; see Magnus and Neudecker (1999, pp. 40–41) for this and other properties of the adjoint matrix.

We can now state that generally $\sum_i |(A', b_i)|^2 = \operatorname{tr}[(A'A)^* B' B]$. From this we derive

(i) $\mathsf{E}(\sum_i |(A', b_i)|^2) = n \cdot \operatorname{tr}[(A'A)^*\Omega]$.

(ii) $\mathsf{D}(\sum_i |(A', b_i)|^2) = \mathsf{D}\{\operatorname{tr}[(A'A)^* B' B]\} = \mathsf{D}\{(\operatorname{vec}[(A'A)^*])' \operatorname{vec}(B'B)\} = (\operatorname{vec}[(A'A)^*])' \mathsf{D}\{\operatorname{vec}(B'B)\} \operatorname{vec}[(A'A)^*] = n \cdot (\operatorname{vec}[(A'A)^*])'(I + K_{m,m})(\Omega \otimes \Omega)\operatorname{vec}[(A'A)^*] = 2n \cdot (\operatorname{vec}[(A'A)^*])'(\Omega \otimes \Omega)\operatorname{vec}[(A'A)^*] = 2n \cdot \operatorname{tr}\{(A'A)^*\Omega\}^2$.

Unfortunately there was a mistake in the formulation of the second part of the problem.

Reference

J. R. Magnus & H. Neudecker (1999). *Matrix Differential Calculus with Applications in Statistics and Econometrics.* John Wiley & Sons, Chichester, England.

Problem 36-1: Integer Matrices with Unit Determinants

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Suppose that, for expository purposes, you wish to construct a 2×2 matrix with a determinant of plus one, and eigenvalues of $f \pm g\sqrt{d}$ corresponding to the column eigenvectors

$$\begin{pmatrix} c \pm \sqrt{d} \\ 1 \end{pmatrix},$$

where c, f, and g are nonzero integers at your disposal and d is a fixed positive integer (though not a square). Explain how such matrices may be identified for certain choices of d!

Solution 36-1.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:*
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It is readily established that the required 2×2 matrix is that given on the left of the following equation, as it satisfies the conditions:

$$\begin{pmatrix} f+gc & g(d-c^2) \\ g & f-gc \end{pmatrix} \begin{pmatrix} c \pm \sqrt{d} \\ 1 \end{pmatrix} = (f \pm g\sqrt{d}) \begin{pmatrix} c \pm \sqrt{d} \\ 1 \end{pmatrix}.$$

Further, this matrix has a determinant of plus one so that we have to find nonzero integers f and g satisfying the condition

$$f^2 - dg^2 = 1.$$

If it is possible to find a partition of $d = c^2 + k$ such that $c,\ k$ and $g = 2c/k$ are nonzero integers, then we may set $f = \pm(cg+1)$. A range of possible values for c, k, f, and g corresponding to values of $d \leq 10$ are indicated in the following table. For other less obvious solutions to this problem, readers are referred to the discussion of Pell's equation in any textbook on the Theory of Numbers.

c	k	d	g	f
1	1	2	2	3
1	2	3	1	2
-2	-1	3	4	7
2	1	5	4	9
2	2	6	2	5
-3	-2	7	3	8
2	4	8	1	3
-3	-1	8	6	17
3	1	10	6	19

Solution 36-1.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We first note that a square matrix, say $B \in \mathbb{R}^{n\times n}$, is called simple if for each distinct eigenvalue of B the algebraic multiplicity is equal to the geometric multiplicity or, equivalently, if B is similar to a diagonal matrix, i.e., if there exists a nonsingular matrix $X \in \mathbb{R}^{n\times n}$ such that

$$X^{-1}BX = \mathrm{diag}(\lambda_1, \cdots, \lambda_n),$$

in which case the columns of X can be interpreted as n linearly independent right eigenvectors of B with the λ's as corresponding eigenvalues. Square matrices of order n with n distinct eigenvalues are automatically simple; cf. Lancaster (1969, p. 61) or Werner (2004, p. 33).

For convenience, we next put

$$\lambda_1 := f + g\sqrt{d}, \quad \lambda_2 := f - g\sqrt{d}, \quad x_1 := \begin{pmatrix} c+\sqrt{d} \\ 1 \end{pmatrix}, \quad x_2 := \begin{pmatrix} c-\sqrt{d} \\ 1 \end{pmatrix}, \quad D = \mathrm{diag}(\lambda_1, \lambda_2), \quad \text{and} \quad X := (x_1, x_2).$$

Because d and g are nonzero integers, necessarily $\lambda_1 \neq \lambda_2$. By means of the above-mentioned fact, it is thus clear that the desired 2×2 matrix A of this IMAGE problem can be written as

$$A = XDX^{-1}.$$

In view of

$$X^{-1} = \frac{1}{2\sqrt{d}} \begin{pmatrix} 1 & -c+\sqrt{d} \\ -1 & c+\sqrt{d} \end{pmatrix} =: \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

we now easily obtain

$$A = \lambda_1 \cdot x_1 x^1 + \lambda_2 \cdot x_2 x^2 = f \cdot I_2 + g \cdot \begin{pmatrix} c & d - c^2 \\ 1 & -c \end{pmatrix},$$

where I_2 denotes, as usual, the identity matrix of order 2. Since $\det(A) = \lambda_1 \cdot \lambda_2 = f^2 - g^2 d$, A has all required properties if and only if the integers f, g, c, and d satisfy the condition

$$f^2 - g^2 d = 1. \tag{1}$$

Since the latter condition does not depend on c, c can be an arbitrary integer. For a given positive integer d, it thus remains to check, if eqn. (1) is solvable for some nonzero integers f and g, and if so, to determine the complete set of integers f and g solving eqn. (1) in that case. For given integer g between 1 and 19, we list below in Tables 1 & 2 the front parts of the corresponding sequences of pairs of positive integers f and d satisfying eqn. (1).

$g=1$		$g=2$		$g=3$		$g=4$		$g=5$		$g=6$		$g=7$		$g=8$		$g=9$		$g=10$	
f	d	f	d	f	d	f	d	f	d	f	d	f	d	f	d	f	d	f	d
2	3	3	2	8	7	7	3	24	23	17	8	48	47	31	15	80	79	49	24
3	8	5	6	10	11	9	5	26	27	19	10	50	51	33	17	82	83	51	26
4	15	7	12	17	32	15	14	49	96	35	34	97	192	63	62	161	320	99	98
5	24	9	20	19	40	17	18	51	104	37	38	99	200	65	66	163	328	101	102
6	35	11	30	26	75	23	33	74	219	53	78	146	435	95	141	242	723	149	222
7	48	13	42	28	87	25	39	76	231	55	84	148	447	97	147	244	735	151	228
8	63	15	56	35	136	31	60	99	392	71	140	195	776	127	252	323	1288	199	396
9	80	17	72	37	152	33	68	101	408	73	148	197	792	129	260	325	1304	201	404
10	99	19	90	44	215	39	95	124	615	89	220	244	1215	159	395	404	2015	249	620
11	120	21	110	46	235	41	105	126	635	91	230	246	1235	161	405	406	2035	251	630
12	143	23	132	53	312	47	138	149	888	107	318	293	1752	191	570	485	2904	299	894
13	168	25	156	55	336	49	150	151	912	109	330	295	1776	193	582	487	2928	301	906
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 1

$g=11$		$g=12$		$g=13$		$g=14$		$g=15$		$g=16$		$g=17$		$g=18$		$g=19$	
f	d	f	d	f	d	f	d	f	d	f	d	f	d	f	d	f	d
120	119	17	2	168	167	97	48	26	3	127	63	288	287	161	80	360	359
122	123	55	21	170	171	99	50	199	176	129	65	290	291	163	82	362	363
241	480	71	35	337	672	195	194	224	223	255	254	577	1152	323	322	721	1440
243	488	73	37	339	680	197	198	226	227	257	258	579	1160	325	326	723	1448
362	1083	89	55	506	1515	293	438	251	280	383	573	866	2595	485	726	1082	3243
364	1095	127	112	508	1527	295	444	424	799	385	579	868	2607	487	732	1084	3255
483	1928	143	142	675	2696	391	780	449	896	511	1020	1155	4616	647	1292	1443	5768
485	1944	145	146	677	2712	393	788	451	904	513	1028	1157	4632	649	1300	1445	5784
604	3015	161	180	844	4215	489	1220	476	1007	639	1595	1444	7215	809	2020	1804	9015
606	3035	199	275	846	4235	491	1230	649	1872	641	1605	1446	7235	811	2030	1806	9035
725	4344	215	321	1013	6072	587	1758	674	2019	767	2298	1733	10392	971	2910	2165	12984
727	4368	217	327	1015	6096	589	1770	676	2031	769	2310	1735	10416	973	2922	2167	13008
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 2

The results in these tables are not surprising, since for a given integer $g \geq 1$, we always have, for each $n \in \mathbb{N}$, not only

$$(n \cdot g^2 \pm 1)^2 \quad = \quad [n(n \cdot g^2 \pm 2)] \cdot g^2 + 1, \tag{2}$$

but, if g is divisible by 2 or, equivalently, if g^2 is divisible by 4, then, in addition,

$$(n \cdot \frac{g^2}{2} \pm 1)^2 = [n(n \cdot \frac{g^2}{4} \pm 1)] \cdot g^2 + 1. \tag{3}$$

In other words, the square bracket expressions $[\cdots]$ in (2) and (3) then serve for given g as d's. The squares of the corresponding f-values are the left hand expressions in (2) and (3). Most of the entries in Tables 1 & 2 can be obtained in that manner. However, there are some further entries in the columns belonging to $g = 12$ and $g = 15$ in Table 2, which can be explained rather easily as follows. If $g = a \cdot b$ for some positive integers a and b, then $g^2 = a^2b^2$ and, trivially,

$$f^2 = d \cdot (a^2b^2) + 1 \quad \Leftrightarrow \quad f^2 = (da^2) \cdot b^2 + 1. \tag{4}$$

In virtue of (4), it is clear that the d's and f's belonging to $g = 15$ can all be obtained from the column of $g = 5$ or $g = 3$. For example, let's consider the column of $g = 5$. Determine all those pairs (f, d) in that column whose d is divisible by $9 = 3 \cdot 3$. The resulting pairs $(f, \frac{d}{9})$ are the pairs in the column of $g = 15$. In other words, when considering the prime factor factorization of a positive integer g, it is always possible to obtain the entries in the column of that g from the entries of one of any of its prime factors by means of just the illustrated approach.

It is interesting to ask: For which positive integers d is eqn. (1) solvable in $f, g \in \mathbb{N}$? Needless to say, (1) is inconsistent whenever d itself is a square. Are there any other integers d leading to the inconsistency of eqn. (1)? When glancing at our Tables 1 & 2, it is striking that all non-squares between 1 and 18 appear in these two tables except $d = 13$. So one might be tempting to believe that (1) is inconsistent for given $d = 13$. This, however, is erroneous.

Tables 3 through 5 below list for all non-square d's between 1 and 60 the respective smallest positive integers g together with the according integers f. What further latent properties are inherent in these triplets? It would be very interesting to find a closed form formula for the smallest positive integer g just in terms of the given non-square positive integer d. This, however, is left to the interested reader.

d	2	3	5	6	7	8	10	11	12	13	14	15	17	18	19	20	21	22	23	24
g	2	1	4	2	3	1	6	3	2	180	4	1	8	4	39	2	12	42	5	1
f	3	2	9	5	8	3	19	10	7	649	15	4	33	17	170	9	55	197	24	5

Table 3

d	26	27	28	29	30	31	32	33	34	35	37	38	39	40	41	42	43	44
g	10	5	24	1820	2	273	3	4	6	1	12	6	4	3	320	2	531	30
f	51	26	127	9801	11	1520	17	23	35	6	73	37	25	19	2049	13	3482	199

Table 4

d	45	46	47	48	50	51	52	53	54	55	56	57	58	59	60
g	24	3588	7	1	14	7	90	9100	66	12	2	20	2574	69	4
f	161	24335	48	7	99	50	649	66249	485	89	15	151	19603	530	31

Table 5

References

P. Lancaster (1969). *Theory of Matrices.* Academic Press, New York.

H. J. Werner (2004). Invariance of the vector cross product. Solution 32-7.4. *IMAGE: The Bulletin of the International Linear Algebra Society* **33** (Fall 2004), 33–35.

Problem 36-2: Simple Integer Programming Problems

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Show that magic squares and similar recreational problems involving the insertion of a sequence of positive integers into a rectangular array can be realised as nonlinear integer programming problems (without explicit optimality criteria).

Solution 36-2.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

For simplicity, I shall restrict my exposition to two 4×4 problems. In the case of a 4×4 magic square the integers $1, 2, \ldots, 16$ are to be inserted into a 4×4 array A and the problem takes the form:

$$\text{Minimise} \quad \sum_{i=1}^{4} \sum_{j=1}^{4} w_{ij}\, a_{ij} \tag{5}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{4} a_{ij} = 34 \quad \text{for} \quad j = 1, 2, 3, 4, \\
&\sum_{j=1}^{4} a_{ij} = 34 \quad \text{for} \quad i = 1, 2, 3, 4, \\
&\sum_{i=1}^{4} a_{ii} = 34, \\
&\sum_{i=1}^{4} a_{i(5-i)} = 34, \\
&a_{11} + a_{12} + a_{21} + a_{22} = 34, \quad \text{and} && (6) \\
&\sum_{i=1}^{4} \sum_{j=1}^{4} 2^{a_{ij}} = 2^{17} - 2 = 131070, && (7)
\end{aligned}$$

where the optimality criterion (5) is arbitrary, condition (6) is sometimes omitted, and condition (7) ensures that only the numbers $1, 2, \ldots, 16$ feature in the solution to the problem.

In the case of the 4×4 sudoku problem we have:

$$\text{Minimise} \quad \sum_{i=1}^{4} \sum_{j=1}^{4} w_{ij} a_{ij} \tag{8}$$

subject to

$$\sum_{i=1}^{4} 2^{a_{ij}} = 30 \quad \text{for} \quad j = 1, 2, 3, 4, \tag{9}$$

$$\sum_{j=1}^{4} 2^{a_{ij}} = 30 \quad \text{for} \quad i = 1, 2, 3, 4, \quad \text{and} \tag{10}$$

$$2^{a_{11}} + 2^{a_{12}} + 2^{a_{21}} + 2^{a_{22}} = 30, \tag{11}$$

where the optimality criterion (8) is again arbitrary and the conditions (9) – (11) ensure that only the numbers $1, 2, 3, 4$ feature in the solution to the problem.

SUPPLEMENT: For completeness, I note that the 9×9 sudoku problem may be written as:

$$\text{Minimise} \quad \sum_{i=1}^{9} \sum_{j=1}^{9} w_{ij} a_{ij}$$

subject to

$$\sum_{i=1}^{9} 2^{a_{ij}} = 1022 \quad \text{for} \quad j = 1, 2, \cdots, 9,$$
$$\sum_{j=1}^{9} 2^{a_{ij}} = 1022 \quad \text{for} \quad i = 1, 2, \ldots, 9,$$
$$2^{a_{11}} + 2^{a_{12}} + 2^{a_{13}} + 2^{a_{21}} + 2^{a_{22}} + 2^{a_{23}} + 2^{a_{31}} + 2^{a_{32}} + 2^{a_{33}} = 1022,$$
$$2^{a_{41}} + 2^{a_{42}} + 2^{a_{43}} + 2^{a_{51}} + 2^{a_{52}} + 2^{a_{53}} + 2^{a_{61}} + 2^{a_{62}} + 2^{a_{63}} = 1022,$$
$$2^{a_{14}} + 2^{a_{15}} + 2^{a_{16}} + 2^{a_{24}} + 2^{a_{25}} + 2^{a_{26}} + 2^{a_{34}} + 2^{a_{35}} + 2^{a_{36}} = 1022, \quad \text{and}$$
$$2^{a_{44}} + 2^{a_{45}} + 2^{a_{46}} + 2^{a_{54}} + 2^{a_{55}} + 2^{a_{56}} + 2^{a_{64}} + 2^{a_{65}} + 2^{a_{66}} = 1022.$$

Problem 36-3: Transformations of (Skew-) Idempotent and (Skew-) Involutory Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

In this problem, we generalise Farebrother's (2001, p. 11) definitions of involutory and skew-involutory matrices by replacing the $n \times n$ identity matrix I_n by a given $n \times n$ idempotent matrix P satisfying $P^2 = P$.

(a) Let A be an $n \times n$ idempotent or skew-idempotent matrix satisfying $AP = PA = A$ and $A^2 = mA$ where $m = \pm 1$. Then for $s^2 = \pm 1$, the generalised Householder transformation $B = s(2A - mP)$ is P-involutory (i.e., $B^2 = P$) or skew-P-involutory (i.e., $B^2 = -P$) according as $s^2 = \pm 1$: $B^2 = s^2(4A^2 - 4mPA + m^2P^2) = s^2P$.

(b) Conversely, suppose that B is an $n \times n$ P-involutory or skew-P-involutory matrix satisfying $BP = PB = B$ and $B^2 = s^2P$ where $s^2 = \pm 1$, then the transformation $A = (sB + mP)/2$ is idempotent (i.e., $A^2 = A$) or skew-idempotent (i.e., $A^2 = -A$) according as $m = \pm 1$: $A^2 = (s^2B^2 + 2msBP + m^2P^2)/4 = m(sB + mP)/2 = mA$.

Readers are asked to suggest alternative transformations that maintain the reality of B (or A) when P and A (or B) are real.

Reference

R. W. Farebrother (2001). The naming of parts. *IMAGE: The Bulletin of the International Linear Algebra Society* **27** (October 2001), 11.

We still look forward to receiving solutions to Problem 36-3!

Problem 36-4: An Eigenvalue Inequality of 3×3 Complex Hermitian Matrices

Proposed by Roy MATHIAS, *University of Birmingham, Birmingham, United Kingdom:* mathias@maths.bham.ac.uk

Let A be a 3×3 complex Hermitian matrix. Show that $\lambda_{\min}(A) \leq \lambda_{\min}(|A|)$, with equality if and only if A is diagonally similar to $|A|$. Here $|\cdot|$ denotes the entrywise absolute value. Prove or disprove this inequality for complex Hermitian matrices of order ≥ 4.

Solution 36-4.1 by Chi-Kwong LI, *The College of William & Mary, Williamsburg, Virginia, USA:* ckli@math.wm.edu

Let $A = (a_{ij}) \in M_3$ be Hermitian. Suppose A_1 is obtained from A be replacing the diagonal entries by their absolute values. Then $A_1 = A + P$ for a nonnegative diagonal matrix P. So, $\lambda_{\min}(A) = \min\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\} \leq \min\{x^*A_1x : x \in \mathbb{C}^n, x^*x = 1\} = \lambda_{\min}(A_1)$.

Let $f(x) = \det(xI - |A|)$ and $g(x) = \det(xI - A_1)$. Then $f(x)$ and $g(x)$ are real monic polynomials of degree three with real zeros. Note that $f(x) = g(x) - \gamma$ with $\gamma = \det(|A|) - \det(A_1) > 0$. So, the curve $y = f(x)$ is obtained from the curve $y = g(x)$ by lowering it by the value γ. Thus, the smallest real zero of $f(x)$ is on the right side of that of $g(x)$, i.e., $\lambda_{\min}(A_1) \leq \lambda_{\min}(|A|)$.

Combining the above arguments, we have $\lambda_{\min}(A) \leq \lambda_{\min}(|A|)$.

Suppose $A = \begin{pmatrix} 0 & 10 & 2 \\ 10 & 0 & 2 \\ 2 & 2 & -1 \end{pmatrix}$. Then $\lambda_{\min}(A) = \lambda_{\min}(|A|) = -10$, but A is not diagonally similar to $|A|$.

We claim that $\lambda_{\min}(A) = \lambda_{\min}(|A|)$ if and only if

(a) A is diagonally similar to $|A|$, or

(b) A has one negative diagonal entry, say, at the (i,i) position, and there is a diagonal unitary matrix such that (i) D^*AD has nonnegative off-diagonal entries, and (ii) there is a unit vector $x \in \mathbb{C}^3$ with ith entry equal to zero satisfying $D^*ADx = \lambda_{\min}(A)x$.

If (a) holds, then we clearly have $\lambda_{\min}(A) = \lambda_{\min}(|A|)$. Suppose (b) holds. Then $|A| = D^*AD + P$ such that P has only one nonzero entry at the (i,i) position. We have

$$\lambda_{\min}(A) = x^*D^*ADx = x^*D^*ADx + x^*Px = x^*|A|x \geq \lambda_{\min}(|A|) \geq \lambda_{\min}(A).$$

Conversely, suppose $\lambda_{\min}(A) = \lambda_{\min}(|A|)$. From our proof of the inequality, we see that $\lambda_{\min}(A) = \lambda_{\min}(A_1)$ and $\lambda_{\min}(A_1) = \lambda_{\min}(|A|)$. It is also clear from our proof that $\lambda_{\min}(A_1) = \lambda_{\min}(|A|)$ implies that $\det(A_1) = \det(|A|)$. Let D be a diagonal unitary matrix such that D^*A_1D has nonnegative $(1,2)$ and $(1,3)$ entries. Then $\det(|A|) = \det(A_1) = \det(D^*A_1D)$ implies that the $(2,3)$ entry of D^*A_1D is also nonnegative. Hence, $D^*A_1D = |A|$.

Suppose A has nonnegative diagonal entries. Then $A = A_1$ and condition (a) holds. Suppose A has one or more negative diagonal entries, and $D^*AD + P = D^*A_1D = |A|$ for a nonnegative diagonal matrix P. Let $x \in \mathbb{C}^3$ be a unit vector such that $|A|x = \lambda_{\min}(|A|)x$. Then

$$\lambda_{\min}(|A|) = x^*D^*ADx + x^*Px \geq x^*D^*ADx \geq \lambda_{\min}(A) = \lambda_{\min}(|A|).$$

Hence, $x^*Px = 0$ and $x^*D^*ADx = \lambda_{\min}(A)$. If P has one positive entry, i.e., A has one negative diagonal entry, Then we get condition (b). If P has two positive entries, then x^*Ax equals to the nonnegative diagonal entry, which cannot be the smallest eigenvalue because A is not positive semidefinite. If P has three positive entries, then $x = 0$, which is impossible.

The inequality $\lambda_{\min}(A) \leq \lambda_{\min}(|A|)$ is not true for $n \geq 4$. For example, let

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 5 & -5 \\ 0 & 1 & -5 & 4 \end{pmatrix} \oplus 0_{n-4}.$$

Then $\lambda_{\min}(A) = -1.7492$ and $\lambda_{\min}(|A|) = -1.8242$.

Problem 36-5: Eigenvalues and Eigenvectors of Some Cayley Matrices

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If r and s are nonnegative integers, let $r = \sum_{i=1}^{\infty} \ell_{ir} 2^{i-1}$ and $s = \sum_{i=1}^{\infty} \ell_{is} 2^{i-1}$ be their binary expansions; i.e., $0 \leq \ell_{ir}, \ell_{is} \leq 1$, $i \geq 1$. Define

$$r \widehat{+} s = \sum_{i=1}^{\infty} [\ell_{ir} + \ell_{is} \text{ (mod 2)}] 2^{i-1}.$$

Find the eigenvalues and eigenvectors of $C_k = (r \widehat{+} s)_{r,s=0}^{2^k-1}$, $k \geq 1$.

Solution 36-5.1 by the Proposer William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

If $k \geq 1$, then

$$C_{k+1} = \begin{pmatrix} C_k & X_k \\ Y_k & Z_k \end{pmatrix}$$

with

$$X_k = \left(r \widehat{+} (s + 2^k)\right)_{r,s=0}^{2^k-1}, \quad Y_k = \left((r + 2^k) \widehat{+} s\right)_{r,s=0}^{2^k-1},$$

and

$$Z_k = \left((r + 2^k) \widehat{+} (s + 2^k)\right)_{r,s=0}^{2^k-1}.$$

However, if $0 \leq r, s \leq 2^k - 1$, then

$$(r + 2^k) \widehat{+} s = r \widehat{+} (s + 2^k) = (r \widehat{+} s) + 2^k \quad \text{and} \quad (r + 2^k) \widehat{+} (s + 2^k) = r \widehat{+} s.$$

Therefore,

$$C_{k+1} = \begin{pmatrix} C_k & C_k + 2^k U_k \\ C_k + 2^k U_k & C_k \end{pmatrix} = U_1 \otimes C_k + 2^k C_1 \otimes U_k, \quad k \geq 1, \tag{12}$$

where $C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and U_k is the $2^k \times 2^k$ matrix with all entries equal to 1.

Let $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $(1, x_0)$ and $(-1, x_1)$ are eigenpairs of C_1. If $k \geq 2$ and $\mu \in \{0, 1, \ldots, 2^k - 1\}$, let $\mu = \sum_{i=1}^k \ell_{i\mu} 2^{i-1}$ be its binary expansion, and define

$$y_{k\mu} = x_{\ell_{1\mu}} \otimes x_{\ell_{2\mu}} \otimes \cdots \otimes x_{\ell_{k\mu}}.$$

Then $\{y_{k\mu}\}_{\mu=0}^{2^k-1}$ is an orthogonal basis for $\mathbb{R}^{2^k}$ and, since every row of U_k equals y_{k0}^T, $U_k y_{k\mu} = 0$ if $\mu \neq 0$. We will show by induction that if $k \geq 2$, then

$$C_k y_{k\mu} = \lambda_{k\mu} y_{k\mu}, \quad 0 \leq \mu \leq 2^k - 1, \tag{13}$$

with

$$\lambda_{k\mu} = \begin{cases} 2^{k-1}(2^k - 1) & \text{if } \mu = 0, \\ -2^{2k-m-2} & \text{if } \mu = 2^m, \quad 0 \leq m \leq k-1, \\ 0 & \text{if } \mu \notin \{0, 1, 2, \ldots, 2^{k-1}\}. \end{cases} \tag{14}$$

The set $S_k = \{0, 1, \ldots, 2^{k-1}\}$ is a group under $\widehat{+}$ (isomorphic to the k-fold direct product of the additive group with two elements with itself). Hence, every member of S_k appears exactly once in every row of C_k, so every row sum of C_k equals $2^{k-1}(2^k - 1)$. Therefore, since all the components of y_{k0} are ones, (13) and (14) both hold with $\mu = 0$ for all $k \geq 2$. Henceforth we assume that $\mu \neq 0$. Setting $k = 1$ in (12) yields

$$C_2 = U_1 \otimes C_1 + 2C_1 \otimes U_1,$$

so

$$\begin{aligned} C_2 y_{21} &= U_1 x_1 \otimes C_1 x_0 + 2C_1 x_1 \otimes U_1 x_0 = -4y_{21}, \\ C_2 y_{22} &= U_1 x_0 \otimes C_1 x_1 + 2C_1 x_0 \otimes U_1 x_1 = -2y_{22}, \\ C_2 y_{23} &= U_1 x_1 \otimes C_1 x_1 + 2C_1 x_1 \otimes U_1 x_1 = 0, \end{aligned}$$

which verifies (13) and (14) for $k = 2$. Now suppose (13) and (14) hold for some $k \geq 2$. We will use (12) to show that

$$C_{k+1} y_{k+1,\mu} = \lambda_{k+1,\mu} y_{k+1,\mu}, \quad 0 \leq \mu \leq 2^{k+1} - 1,$$

with

$$\lambda_{k+1,\mu} = \begin{cases} -2^{2k-m} & \text{if } \mu = 2^m, \quad 0 \leq m \leq k, \\ 0 & \text{if } \mu \notin \{0, 1, 2, \ldots, 2^k\}, \end{cases}$$

which will complete the induction.

Since $y_{k+1,1} = x_1 \otimes y_{k0}$,

$$\begin{aligned} C_{k+1} y_{k+1,1} &= U_1 x_1 \otimes C_k y_{k0} + 2^k C_1 x_1 \otimes U_k y_{k0} \\ &= 0 - 2^{2k} x_1 \otimes y_{k0} \\ &= -2^{2k} y_{k+1,0}. \end{aligned}$$

If $1 \leq m \leq k$ then $y_{k+1,2^m} = x_0 \otimes y_{k,2^{m-1}}$, so

$$C_{k+1} y_{k+1,2^m} = U_1 x_0 \otimes C_k y_{k,2^{m-1}} + 2^k C_1 x_0 \otimes U_k y_{k,2^{m-1}}. \tag{15}$$

From (13) and (14), $C_k y_{k,2^{m-1}} = -2^{2k-m-1} y_{k,2^{m-1}}$, $1 \leq m \leq k$. Since $U_k y_{k,2^{m-1}} = 0$, $1 \leq m \leq k$ and $U_1 x_0 = 2x_0$, (15) yields

$$C_{k+1} y_{k+1,2^m} = -2^{2k-m} x_0 \otimes y_{k,2^{m-1}} = -2^{2k-m} y_{k+1,2^m}, \quad 1 \leq m \leq k.$$

Finally, if $\mu \notin \{0, 1, 2, \ldots, 2^k\}$, $y_{k+1,\mu} = x_{\ell_{1\mu}} \otimes z_{k\mu}$ with

$$z_{k\mu} = x_{\ell_{2\mu}} \otimes x_{\ell_{3\mu}} \otimes \cdots \otimes x_{\ell_{k+1,\mu}}.$$

Since $\ell_{i\mu} = 1$ for at least one value of $i \in \{2, \dots, k+1\}$, $U_k z_{k\mu} = 0$. Hence,

$$C_{k+1} y_{k+1,\mu} = U_1 x_{\ell_{1\mu}} \otimes C_k z_{k\mu}.$$

If $\ell_{1\mu} = 1$, then $U_1 x_{\ell_{1\mu}} = 0$. If $\ell_{1\mu} = 0$, then $\ell_{i\mu} = 1$ for at least two values of i in $\{2, \dots, k+1\}$, so (13) and (14) imply that $C_k z_{k\mu} = 0$.

Problem 36-6: Inverses of Some Cayley Matrices

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu.

Find C^{-1} if $C = (r + s \,(\mathrm{mod}\, n))_{r,s=0}^{n-1}$ with $n \geq 3$.

Solution 36-6.1 by the Proposer William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

By Theorem 2.1 in Ablow and Brenner (1963), an $n \times n$ matrix A is an anticirculant (i.e., $A = (a_{r+s\,(\mathrm{mod}\, n)})_{r,s=0}^{n-1}$ if and only if $PAP = A$, where P is the circulant with first row $(0\,1\,0 \cdots 0)$. If A is an invertible anticirculant, then $PA^{-1}P = A^{-1}$, so A^{-1} is an anticirculant. Hence, since C is an anticirculant, so is C^{-1}; i.e., $C^{-1} = (b_{r+s\,(\mathrm{mod}\, n)})_{r,s=0}^{n-1}$, where

$$\sum_{s=0}^{n-1} (r + s \,(\mathrm{mod}\, n)) b_s = \delta_{0s}, \quad 0 \leq r \leq n-1. \tag{16}$$

We look for a solution of (16) in the form $b_s = x + \alpha_s$, where x and $\alpha_0, \dots, \alpha_{n-1}$ are to be determined. If $1 \leq r \leq n-1$, then (16) becomes

$$r(x + \alpha_0) + \left(\frac{n(n-1)}{2} - 2r + 1 \right) x + \sum_{s=1}^{n-2} (r + s \,(\mathrm{mod}\, n)) \alpha_s + (r-1)(x + \alpha_{n-1}) = 0,$$

which holds if

$$\alpha_1 = \cdots = \alpha_{n-2} = 0 \quad \text{and} \quad \alpha_{n-1} = -\alpha_0 = \frac{n(n-1)}{2}\, x.$$

Now (16) with $r = 0$ reduces to

$$\frac{n(n-1)}{2}\, x + (n-1)\alpha_{n-1} = 1.$$

Substituting $\alpha_{n-1} = n(n-1)x/2$ in this yields $x = 2/(n^2(n-1))$. Therefore,

$$b_0 = -\frac{n(n-1) - 2}{n^2(n-1)}, \quad b_\ell = \frac{2}{n^2(n-1)}, \quad 1 \leq \ell \leq n-2, \quad b_{n-1} = \frac{n(n-1) + 2}{n^2(n-1)}.$$

Reference

C. M. Ablow & J. L. Brenner (1963). Roots and canonical forms for circulant matrices. *Transactions of the American Mathematical Society* **107**, 360–376.

Problem 36-7: Necessary and Sufficient Conditions for $A + A^*$ to be a Nonnegative Definite Matrix

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a complex square matrix with A^*, A^+, and $A^\#$ denoting the conjugate transpose, the Moore-Penrose inverse, and the group inverse, respectively, of A. Similarly, define the conjugate transpose of A^+ and $A^\#$ as A^{+*} and $A^{\#*}$, respectively. Moreover, let $\geq_{\mathrm{L}}$ denote the Löwner ordering. Show that the following statements are equivalent:

(i) $A + A^* \geq_{\mathrm{L}} 0$.

(ii) $A^+ + A^{+*} \geq_{\mathrm{L}} 0$.

(iii) $A^\# + A^{\#*} \geq_{\mathrm{L}} 0$, where the group inverse $A^\#$ of A is assumed to exist.

Solution 36-7.1 by the Proposer Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

According to Hartwig and Spindelböck (1984, Corollary 6), the matrix A can be written in the form

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where U is unitary, $KK^* + LL^* = I_r$, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$, $r_1 + r_2 + \ldots + r_t = r = \mathrm{rank}(A)$ and $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$ being the singular values of A. Then we get

$$A + A^* = U \begin{pmatrix} \Sigma K + K^*\Sigma & \Sigma L \\ L^*\Sigma & 0 \end{pmatrix} U^*.$$

By Alberts's Theorem (Albert, 1969), the matrix $A + A^*$ is nonnegative definite if and only if $\mathcal{R}(L^*\Sigma) \subset \mathcal{R}(0)$ and $\Sigma K + K^*\Sigma \geq_{\mathrm{L}} 0$, i.e., $L = 0$ and $\Sigma K + K^*\Sigma \geq_{\mathrm{L}} 0$. Observe that when $L = 0$, K becomes unitary. On the other hand we have

$$A^+ = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*$$

which implies

$$A^+ + A^{+*} = U \begin{pmatrix} K^*\Sigma^{-1} + \Sigma^{-1}K & \Sigma^{-1}L \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*.$$

Using again Albert's Theorem, we obtain $A^+ + A^{+*} \geq_{\mathrm{L}} 0$ if and only if $L = 0$ and $K^*\Sigma^{-1} + \Sigma^{-1}K \geq_{\mathrm{L}} 0$. Premultiplying the latter inequality by ΣK and postmultiplying by $K^*\Sigma$, both matrices being nonsingular, yields the equivalent condition: $L = 0$ and $\Sigma K + K^*\Sigma \geq_{\mathrm{L}} 0$. Thus the equivalence of (i) and (ii) is established. For the rest of the proof we note that

$$A^{\#} = U \begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix} U^*$$

so that

$$A^{\#} + A^{\#*} = U \begin{pmatrix} K^{-1}\Sigma^{-1} + \Sigma^{-1}K^{-1*} & K^{-1}\Sigma^{-1}K^{-1}L \\ L^*K^{-1*}\Sigma^{-1}K^{-1*} & 0 \end{pmatrix} U^*.$$

Finally using Albert's Theorem we see that $A^{\#} + A^{\#*} \geq_{\mathrm{L}} 0$ if and only if $L = 0$ and $K^*\Sigma^{-1} + \Sigma^{-1}K \geq_{\mathrm{L}} 0$. Thus the equivalence of (ii) and (iii) is shown.

We note in passing that the condition $L = 0$ says that A is EP, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$, cf. Hartwig and Spindelböck (1984, Corollary 6).

References

A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics* **24**, 434–440.

R. E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and A^+ commute. *Linear Algebra and its Applications* **14**, 241–256.

Solution 36-7.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Let $\mathbb{C}^{n\times n}$ stand for the set of all complex $n \times n$ matrices. For a matrix $B \in \mathbb{C}^{n\times n}$, let $\mathcal{R}(B)$, $\mathcal{N}(B)$, B^*, B^+, and $B^\sharp$ denote the range (column space), the null space, the conjugate transpose, the Moore-Penrose inverse, and the group inverse, respectively, of B. Whereas the Moore-Penrose inverse B^+ of the matrix B does always uniquely exist, the group inverse $B^\sharp$ exists if and only if $\mathcal{R}(B^2) = \mathcal{R}(B)$ or, equivalently, if and only if $\mathrm{rank}(B^2) = \mathrm{rank}(B)$. We recall that if the group inverse of B exists, it is the unique solution $B^\sharp$ to the following three equations

$$BB^\sharp B = B, \quad B^\sharp BB^\sharp = B^\sharp, \quad BB^\sharp = B^\sharp B,$$

in which case $B^{\sharp *} := (B^\sharp)^* = (B^*)^\sharp =: B^{*\sharp}$, $\mathcal{R}(B^\sharp) = \mathcal{R}(B)$, $\mathcal{R}(B^{\sharp *}) = \mathcal{R}(B^*)$ and $BB^\sharp = B^\sharp B$ is the projector onto $\mathcal{R}(B)$ along $\mathcal{N}(B)$. For the sake of clarity, we further recall that the Moore-Penrose inverse of B is defined as the unique matrix B^+ satisfying the four so-called Penrose equations

$$BB^+B = B, \quad B^+BB^+ = B^+, \quad (BB^+)^* = BB^+, \quad (B^+B)^* = B^+B.$$

Consequently, BB^+ is the (orthogonal) projector onto $\mathcal{R}(B)$ along $\mathcal{N}(B^*)$, B^+B is the (orthogonal) projector onto $\mathcal{R}(B^*)$ along $\mathcal{N}(B)$, $\mathcal{R}(B^+) = \mathcal{R}(B^*)$, $(B^+)^+ = B$, $B^{+*} := (B^+)^* = (B^*)^+ =: B^{*+}$, and $\mathcal{R}(B^{+*}) = \mathcal{R}(B)$.

Our solution to this IMAGE problem is based on the following two elementary facts.

FACT 1. *Let $B \in \mathbb{C}^{n\times n}$. If $B + B^* \geq_{\mathrm{L}} 0$, where $\geq_{\mathrm{L}}$ indicates the Löwner ordering, then B is EP, i.e., $\mathcal{N}(B) = \mathcal{N}(B^*)$ or, equivalently, $\mathcal{R}(B) = \mathcal{R}(B^*)$.*

PROOF. Because $B+B^*$ is a Hermitian nonnegative definite matrix, clearly $x^*(B+B^*)x = 0 \Leftrightarrow (B+B^*)x = 0 \Leftrightarrow x \in \mathcal{N}(B+B^*)$. Now, let x be an arbitrary vector such that $Bx = 0$. Then $x^*Bx = 0$. In view of $x^*(B + B^*)x = 2\Re(x^*Bx) = 0$, with $\Re(\cdot)$ denoting the real part of $(\cdot)$, therefore $(B + B^*)x = 0$ and so, in addition, also $B^*x = 0$, i.e., we have $\mathcal{N}(B) \subseteq \mathcal{N}(B^*)$. But then, trivially, $\mathcal{N}(B) = \mathcal{N}(B^*)$, and our proof is complete. □

FACT 2. *Let $B \in \mathbb{C}^{n\times n}$. Then,*

$$B \text{ is EP } \Leftrightarrow B^\sharp \text{ exists and } B^\sharp = B^+ \Leftrightarrow BB^+B^* = B^* \Leftrightarrow B^+BB^{+*} = B^{+*}.$$

PROOF. First, let B be EP. Then $\mathcal{R}(B^2) = \mathcal{R}(BB^*) = \mathcal{R}(B)$, and so $B^\sharp$ does exist. Clearly, $BB^\sharp = B^\sharp B$, where $BB^\sharp$ is now the (orthogonal) projector onto $\mathcal{R}(B)$ along $\mathcal{N}(B^*)$. Therefore, $(BB^\sharp)^* = BB^\sharp = B^\sharp B = (B^\sharp B)^*$, and so it is clear that $B^\sharp$ satisfies the four Penrose equations. Consequently, as claimed, $B^\sharp = B^+$. Conversely, if $B^\sharp$ exists and $B^\sharp = B^+$, then, in virtue of $\mathcal{R}(B^+) = \mathcal{R}(B^*)$ and $\mathcal{R}(B^\sharp) = \mathcal{R}(B)$, directly $\mathcal{R}(B^*) = \mathcal{R}(B)$. This completes the proof of B is EP $\Leftrightarrow$ $B^\sharp$ exists and $B^\sharp = B^+$. Because BB^+ is the (orthogonal) projector onto $\mathcal{R}(B)$ along $\mathcal{N}(B^*)$, B^+B is the (orthogonal) projector onto $\mathcal{R}(B^*)$ along $\mathcal{N}(B)$, and $\mathcal{R}(B^{+*}) = \mathcal{R}(B)$, the remaining characterizations are straightforward. □

The above facts and observations enable us now to prove

$$\text{(i) } A + A^* \geq_{\mathrm{L}} 0 \quad \Leftrightarrow \quad \text{(ii) } A^+ + A^{+*} \geq_{\mathrm{L}} 0 \quad \Leftrightarrow \quad \text{(iii) } A^{\#} + A^{\#*} \geq_{\mathrm{L}} 0.$$

Because we already know that $\mathcal{R}(A^+) = \mathcal{R}(A^*)$, $\mathcal{R}(A^{+*}) = \mathcal{R}(A^{*+}) = \mathcal{R}(A)$, $\mathcal{R}(A^\sharp) = \mathcal{R}(A)$, and $\mathcal{R}(A^{\sharp*}) = \mathcal{R}(A^{*\sharp}) = \mathcal{R}(A^*)$, it follows from Fact 1 that A is necessarily EP if A satisfies any one of the three conditions (i), (ii), and (iii). But then, according to Fact 2, $A^\sharp = A^+$, thus showing that (ii) is indeed identical to (iii). So, (i) $\Leftrightarrow$ (ii) remains to be shown. First, let (i) be satisfied. Multiplying $0 \leq_{\mathrm{L}} A + A^*$ on the left-hand side by A^+ and on the right-hand side by A^{+*} readily results in $0 \leq_{\mathrm{L}} A^+(A + A^*)A^{+*} = A^{+*} + A^+ = A^+ + A^{+*}$, because A is EP and so, in view of Fact 2, $A^+AA^{+*} = A^{+*}$ or, equivalently, $A^+ = (A^+AA^{+*})^* = A^+A^*A^{+*}$. Finally, multiplying $0 \leq_{\mathrm{L}} A^+ + A^{+*}$ on the left-hand side by A and on the right-hand side by A^* results likewise in $0 \leq_{\mathrm{L}} A(A^+ + A^{+*})A^* = A^* + A = A + A^*$, because, again in view of Fact 2, we have $AA^+A^* = A^*$ or, equivalently, $AA^{+*}A^* = A$. This completes the proof of (i) $\Leftrightarrow$ (ii), and our solution is complete.

Problem 36-8: A Characterization of a Particular Class of Square Complex Matrices
Proposed by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Characterize the class of all square matrices A with complex entries satisfying the identity

$$A + A^\sharp = 2A^+A^\sharp,$$

where A^+ and $A^\sharp$ denote the Moore-Penrose inverse and the group inverse of A, respectively.

Solution 36-8.1 by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Using Corollary 6 in Hartwig and Spindelböck (1984), we may write the matrix A in the form

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where U is unitary, $KK^* + LL^* = I_r, \Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$, $r_1 + \ldots + r_t = r = \mathrm{rank}(A)$ and $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$. Then one easily obtains

$$A^+ = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*, \qquad A^\sharp = U \begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix} U^*,$$

$$A^+A^\sharp = U \begin{pmatrix} K^*\Sigma^{-1}K^{-1}\Sigma^{-1} & K^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L \\ L^*\Sigma^{-1}K^{-1}\Sigma^{-1} & L^*\Sigma^{-1}K^{-1}\Sigma^{-1}K^{-1}L \end{pmatrix} U^*,$$

and

$$A + A^\sharp = \begin{pmatrix} \Sigma K + K^{-1}\Sigma^{-1} & \Sigma L + K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix} U^*.$$

Note that $A^\sharp$ exists if and only if K is nonsingular. Then the condition

$$A + A^\sharp = 2A^+A^\sharp$$

is equivalent to $L = 0$ and $\Sigma K + K^{-1}\Sigma^{-1} = 2K^*\Sigma^{-1}K^{-1}\Sigma^{-1}$. Since $KK^* + LL^* = I_r$, we have $KK^* = I_r$ and hence $K^* = K^{-1}$. Furthermore, the condition $L = 0$ is equivalent to the EP-ness of A. It follows that $A + A^\sharp = 2A^+A^\sharp$ if and only if A is EP and $(\Sigma K)^3 + \Sigma K = 2I$, or equivalently $(\Sigma K)^4 + (\Sigma K)^2 = 2\Sigma K$. However, by $L = 0$ we get

$$A = U \begin{pmatrix} \Sigma K & 0 \\ 0 & 0 \end{pmatrix} U^*, \quad \text{so that} \quad A^2 = U \begin{pmatrix} (\Sigma K)^2 & 0 \\ 0 & 0 \end{pmatrix} U^* \quad \text{and} \quad A^4 = U \begin{pmatrix} (\Sigma K)^4 & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

Hence $A + A^\sharp = 2A^+A^\sharp$ if and only if A is EP and $A^4 + A^2 = 2A$.

Reference

R. E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and A^+ commute. *Linear Algebra and Its Applications* **14**, 241–256.

Solution 36-8.2 by the Proposer Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

Some elementary (geometric) facts enable us to prove the following characterization

$$A + A^\sharp = 2A^+A^\sharp \;\Leftrightarrow\; \mathcal{R}(A^*) = \mathcal{R}(A) \quad \text{(i.e., } A \text{ is EP)} \quad \text{and} \quad A^4 + A^2 = 2A. \tag{17}$$

It is well-known that $\mathcal{R}(A^\sharp) = \mathcal{R}(A)$ and $\mathcal{R}(A^+) = \mathcal{R}(A^*) = \mathcal{R}(A^+A)$, where $\mathcal{R}(\cdot)$ denotes the range (column space) of the matrix $(\cdot)$ and where A^* stands for the conjugate transpose of A. From eqn. $A + A^\sharp = 2A^+A^\sharp$ we therefore get $\mathcal{R}(A^*) = \mathcal{R}(A^+) = \mathcal{R}(A^+A^\sharp) = \mathcal{R}(A + A^\sharp) \subseteq \mathcal{R}(A)$. Because the rank of a matrix always coincides with the rank of its conjugate transpose, we obtain $\mathcal{R}(A) = \mathcal{R}(A^*)$, i.e., A is necessarily an EP matrix. Multiplying eqn. $A + A^\sharp = 2A^+A^\sharp$ on the left-hand side by A and on the right-hand side by A^2 and making use of $AA^\sharp A = A = AA^+A$ and $AA^+A^\sharp A = AA^\sharp$, readily results in $A^4 + A^2 = 2A$, and the proof of the "only if" part is complete. To prove the converse implication, let A be an EP matrix such that $A^4 + A^2 = 2A$. Clearly, A is EP if and only if $AA^+ = A^+A$, in which case $A(A^+)^2 = A^+AA^+ = A^+$. Postmultiplying eqn. $A^4 + A^2 = 2A$ by $(A^+)^2A^\sharp$ and making use of $A^\sharp A = A^\sharp A$, $AA^\sharp A = A$, and $A^\sharp AA^\sharp = A^\sharp$ readily results in $A + A^\sharp = 2A^+A^\sharp$. This completes our proof of characterization (17).

We conclude with mentioning some further interesting observations:

(i) Eqn. $A^2 + A^4 = 2A$ is equivalent to $A[\frac{1}{2}(I + A^2)]A = A$, where I stands for the identity matrix of appropriate order. In which case, $A^\sharp = \frac{1}{4}A(I + A^2)^2$.

(i) If $A^2 + A^4 = 2A$, then λ can be an eigenvalue of A only if $\lambda \in \{0, 1, i, -i, \frac{1}{2}(-1 + i\sqrt{7}), -\frac{1}{2}(1 + i\sqrt{7})\}$, where i is the imaginary unit.

(ii) $A + A^\sharp = 2A^+A^\sharp \Leftrightarrow A^+ = A^\sharp$ and $A + A^\sharp = 2(A^\sharp)^2 \Leftrightarrow A^+ = A^\sharp$ and $A^3 + A = 2AA^\sharp$.

IMAGE Problem Corner: One More New Problem

Problem 37-6: Characterization of EP-ness

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Let A be a square matrix with complex entries. Show that A is an EP matrix if and only if

$$AAA^+A^+ = AA^+.$$

Problems 37-1 through 37-5 are on page 32.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to hjw.de@uni-bonn.de, preferably embedded as text, and (b) with two paper copies by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.

Problem 37-1: Another Property for the Sum of a Matrix A and its Moore-Penrose Inverse $A^\dagger$

Proposed by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
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Let $A \in \mathbb{C}_{n,n}$ and let $A^\dagger$ be its Moore-Penrose inverse. Show that the following statements are equivalent:

(i) $A + A^\dagger = AA^\dagger + A^\dagger A$,

(ii) A is an EP matrix (i.e., satisfies $AA^\dagger = A^\dagger A$) such that $Q_A = (I_n - A)^2$, where Q_A is the orthogonal projector onto the orthogonal complement of the column space of A and I_n is the identity matrix of order n.

(Parenthetically note that the present problem is related to the Problem 34-8 [G. Trenkler, A Property for the Sum of a Matrix A and its Moore-Penrose Inverse A^+. *IMAGE: The Bulletin of the International Linear Algebra Society* **34** (Spring 2005), 39].)

Problem 37-2: Rank of a Generalized Projector

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Let $G \in \mathbb{C}_{n,n}$ partitioned as

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

with $G_{11} \in \mathbb{C}_{k,k}$ and $G_{22} \in \mathbb{C}_{l,l}$, be a generalized projector, i.e., satisfy $G^2 = G^*$, where G^* denotes the conjugate transpose of G. Express rank of matrix G in terms of ranks of its submatrices.

Problem 37-3: Rank of a Nonnegative Definite Matrix

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Let $A \in \mathbb{C}_{n,n}$ be a nonnegative definite matrix with Moore-Penrose inverse $A^\dagger$. Show that $\mathrm{trace}(A) \cdot \mathrm{trace}(A^\dagger) \geqslant \mathrm{rank}(A)$. When does equality happen?

Problem 37-4: Do Singular Values Dominate Eigenvalues?

Proposed by David CALLAN, *University of Wisconsin-Madison, Madison, Wisconsin, USA:* callan@stat.wisc.edu

Do singular values dominate eigenvalues? More specifically, suppose A is an $n \times n$ complex matrix with singular values $s_1 \geq s_2 \geq \ldots \geq s_n \geq 0$ and eigenvalues $x_1, x_2, ..., x_n$ ordered by decreasing absolute value. Is it true that $s_1 + s_2 + ... + s_k \geq |x_1| + |x_2| + ... + |x_k|$ for each $k = 1, 2, ..., n$? It is true in the two special cases (i) $k = 1$, (ii) $n = 2$.

Problem 37-5: A Generalised Matrix Transformation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ matrix and let m be an integer in the range $1 \leq m \leq n$. Then we may define the mth row of A by $\{a_{mj}\}$, the mth column of A by $\{a_{im}\}$, the mth primary (or dexter) cyclic-diagonal of A by $\{a_{ij} \,:\, i = j + m (\mathrm{mod}\, n)\}$, and the mth secondary (or sinister) cyclic-diagonal of A by $\{a_{ij} \,:\, i + j = m + 1 (\mathrm{mod}\, n)\}$. In each case, the nth primary or secondary cyclic-diagonal of A may be named the principal primary or secondary diagonal as they are also the familiar non-cyclical diagonals of the matrix.

If n is odd then each row has a single intersection with each column and *vice versa* and each primary diagonal has a single intersection with each secondary diagonal and *vice versa*. For the case $n = 3$, identify a transformation which carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column. Outline at least one possible application of your chosen transformation.

Problem 37-6 is on page 31.