The Bulletin of the International Linear Algebra Society

IMAGE

Serving the International Linear Algebra Community
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IMAGE Problem Corner: New Problems

Report on ILAS07
Shanghai, P.R. China
July 16-20, 2007

By Richard Brualdi

The 14th ILAS Conference was held at Shanghai University in Shanghai, China. The co-chairs of the conference were Richard Brualdi (USA) and Erxiong Jiang (China). It was the largest ever ILAS conference to date with approximately 300 registrants, almost equally divided between people from outside mainland China and those from within.

The invited speakers were Matej Bresar (Slovenia), Man-Duen Choi (Canada), Douglas Farenick (Canada), Erich Kaltofen (USA), Plamen Koev (USA), Amy Langville (USA), Andre Ran (The Netherlands), Yaokun Wu (China), and Shufang Xu (China). In addition, Michael Ng (Hong Kong, China) was the designated SIAM Lecturer chosen and supported by the SIAM Activity Group on Linear Algebra, Bryan Shader (USA) was the Olga Taussky/John Todd ILAS Lecturer, and Hans Schneider was the LAMA Lecturer supported by LAMA publisher Taylor and Francis. There were five minisymposia at the conference, listed with their organizers: (1) Numerical Range (Chi-Kwong Li), (2) Schur Complement and Applications (Fuzhen Zhang), (3) Perron-Frobenius and *M*-matrices (Michael Neumann and Daniel Szyld), (4) Matrices and Graphs (Leslie Hogben and Bryan Shader), (5) Practical and Innovative Ways to Help Students Learn Linear Algebra (Jane Day). The first four were dedicated to Hans Schneider on the occasion of his 80th birthday. There were over 100 contributed talks. Many of the talks were given by linear algebraists from our host country, and they demonstrated quite clearly the vitality of linear algebra in China and the talent and enthusiasm of many young Chinese mathematicians.

The conference was held on the campus of Shanghai University, a new, very large and modern campus less than 10 years old on the outskirts of Shanghai. The facilities were excellent. For lunch and dinner every day we were treated to a Chinese banquet. One evening we were given the opportunity to attend a Chinese Circus where we witnessed a dazzling performance of Chinese acrobats. The conference excursion was held on Wednesday afternoon and evening, and it consisted of four parts: (i) a visit to the Shanghai Museum where we saw magnificent collections of jade, porcelain, bronzes, paintings, calligraphy, and furniture, (ii) a walk on East Nanjing Road, a famous and vibrant commercial street in Shanghai, (iii) Yu Gardens (sometimes described as Shanghai's Chinatown) where we also had dinner in a restaurant, and (iv) the very impressive Oriental Pearl Tower where we had a breathtaking view of modern Shanghai.

The conference banquet--- a super Chinese banquet!---was held on Thursday and was an *80th birthday party for Hans Schneider*.

Opening Session Welcome: Qingwen Wang, Math Dept. Chair, Erxiong Jiang, ILAS07 Co-Chair, and Zhewei Zhou, Vice President, all of Shanghai University; and Richard Brualdi, ILAS07 Co-Chair

Several speakers told of the enormous contributions that Hans has made, and continues to make, to linear algebra and the linear algebra community. An impressive and delicious three-tiered birthday cake was brought out and then there was a singing of *Happy Birthday to Hans*. The enthusiasm of the participants, along with ample beer, wine and music, led to the banquet developing into a disco-party of sorts.

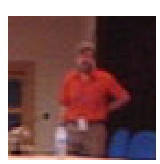
Hans and the Cake

All of the above could not have taken place without the tremendous effort of the local organizing committee of Chuanqing Gu (chair), Fuping Tan, Qinwen Wang, and Xiaomei Jia. I want to especially single out Chuanqing Gu who worked with amazing energy and dedication to make our conference such a great success, and Fuping Tan who managed with skill the program and technical facilities. In addition, many young and delightful Chinese students were available every day to answer questions and give whatever help that was asked for.

During the entire week we experienced the warmth and hospitality of our Chinese hosts. I think everyone agrees that the conference is one that we are going to remember with great fondness for a very long time.

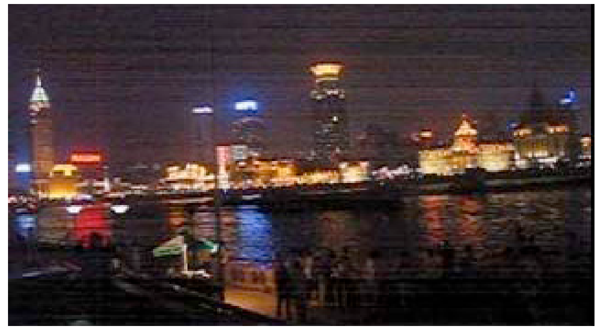
There will be a special issue of LAA devoted to selected papers presented at the conference. Papers should be submitted by November 1, 2007 to one of the special editors:

Ilse Ipsen <ipsen@math.ncsu.edu>,
Julio Moro <jmoro@math.uc3m.es>,
Peter Semrl <Peter.Semrl@fmf.uni-lj.si>,
Jiayu Shao <jyshao@sh163.net>,
Pei Yuan Wu <pywu@math.nctu.edu.tw>



Andre Ran stepped in as opening speaker after Amy Langville's flight was delayed

Closing Session Acknowledgments:
Richard Brualdi; Danny Hershkowitz, ILAS President;
and Peifen Weng, College of Sciences Dean, Shanghai University



View of Shanghai from the Pearl Tower

The Shanghai Museum and some of its beautiful ceramics

The Pearl Tower

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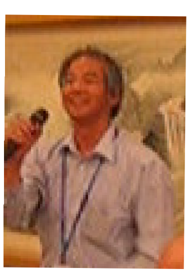
Birthday Celebration and some of the speakers

Richard Brualdi,
Master of Ceremonies,
and Hans

Danny Hershkowitz and Chi-Kwong Li,
Hans' two post docs

Roger Horn



Man-Duen Choi

Hans thanked the speakers and then there was music!
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UPCOMING CONFERENCES AND WORKSHOPS

ILAS08, Cancun, Mexico
June 16-20, 2008

The fifteenth conference of the International Linear Algebra Society (ILAS08) will be held in Cancun, Mexico, June 16-20, 2008. We look forward to meeting you in Cancun.

The confirmed plenary lecturers are Leslie Hogben, Heike Fassbender, Naomi Shaked-Monderer, Juan Manuel Peña, Froilán Dopico, Luca Gemignani, Daniel Kressner, Albrecht Boettcher, Ilya Spitkovsky, Paul Van Dooren, Erxiong Jiang, and James Nagy, the SIAM Lecturer.

There will be sessions for contributed talks as well as special mini-symposia on Max Algebras; Eigenvalues: Theory, Computation and Applications; Matrix Functions and Matrix Equations; Combinatorial Matrix Theory; Nonnegative and Eventually Nonnegative Matrices; and Linear Algebra Education. The talks in the mini-symposia will have a total duration of 25 minutes.

To register and submit an abstract, visit the conference website, http://star.izt.uam.mx/ILAS08. Hotel reservations will soon be possible there also. The deadline for submitting an abstract, and for early registration, is April 1, 2008. The final deadline for online registration is May 14, 2008.

The proceedings will appear as a volume of Linear Algebra and its Applications. The editors for this issue will be Beatrice Meini, Jeff Stuart, Daniel Kressner, Vadim Olshevsky, and Luis Verde-Star.

Send questions to Luis Verde-Star at verde@star.izt.uam.mx.

For information about ILAS and past conferences see the IIC home-page: http://www.ilasic.math.uregina.ca/iic/.

Scientific Committee:
Luis Verde-Star, Chair; Rafael Bru, Luz María de Alba, Daniel Hershkowitz, Andre Klein, Beatrice Meini, Dale Olesky, Vadim Olshevsky, Jeff Stuart, Daniel Szyld.

Local Organizing Committee:
Luis Verde-Star, Chair; María José Arroyo, Rubén Martínez-Avendaño, Martha Takane.

Cancun

Applied Linear Algebra Conference in honor of Ivo Marek
Novi Sad, Serbia, April 28-30, 2008

Following the fashion of the Applied Linear Algebra conference held in honor of Richard Varga in Palić in October 2005, we are pleased to announce a celebration honoring Ivo Marek, another influential figure in applied linear algebra community. This conference will be held April 28-30, 2008, in Novi Sad, Serbia. It is being organized by the Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad. Details can be viewed at http://www.im.ns.ac.yu/events/ala2008/.

The main program will concentrate on computational methods, special classes of matrices, linear systems, eigenvalue problems, and linear algebra applications in industry and science. Selected contributions presented at the conference will be published in a special issue of Numerical Linear Algebra with Applications.

Young researchers are especially encouraged to participate. The Scientific Committee will present awards for the best presentations by young scientists.

5th Linear Algebra Workshop
Kranjska Gora, Slovenia, May 27–June 5, 2008

The Fifth Linear Algebra Workshop will be held at Hotel Kompas and Hotel Larix, Kranjska Gora, Slovenia, May 27–June 5, 2008. The main theme will be the interplay between operator theory and algebra. The workshop will follow our usual format. A few hours of talks will be scheduled for the morning sessions, while afternoons will be reserved for work in smaller groups. In addition there will be two special sections (Real Algebra and its Interactions with Functional Analysis, and Linear Algebra and Discrete Mathematics). Saturday morning will be dedicated to honor Professor Ivan Vidav in celebration of his 90th birthday.

It is our intention to bring together people whose motivation for linear algebra comes primarily from problems in operator theory and algebra, but are willing to learn (or are already familiar with) deeper concepts from either field that may help solve such problems. We believe that we should have more time to think about the problems than to listen to each other's formal talks. So the workshop will be organized in an informal way. After a couple of hours of morning talks as a motivation for later discussions, we will split into smaller groups to concentrate on our problems.

The Organizing Committee Chair is Prof. Matjaž Omladič, Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia. The Scientific Organizing Committee is L. Grunenfelder, I. Klep, T. Košir, M. Omladič, H. Radjavi, P. Rosenthal, L. Rodman, and P. Šemrl.

Those interested in attending should register by January 15, 2008. All information can be found on the web page http://www.law05.si.

Preliminary Announcement
Western Canada Linear Algebra Meeting
Winnipeg, Canada, May 30-31, 2008

A Western Canada Linear Algebra Meeting will be held May 30-31, 2008, at the University of Manitoba in Winnipeg. W-CLAM meetings provide an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the area, and participation is open to anyone who is interested in attending or speaking at the meeting.

Invited speakers include: Michael Gekhtman, Professor of Mathematics, Department of Mathematics, University of Notre Dame; Olga Holtz, Associate Professor of Mathematics, Department of Mathematics, University of California at Berkeley and David Watkins, Professor of Mathematics at Washington State University. For further information, visit http://server.maths.umanitoba.ca/~craigen/WCLAM08/WCLAM.html.

If you wish to be included on an email list for further information on this meeting, please contact the local organizer Robert Craigen <craigenr@cc.umanitoba.ca>. Previous W-CLAM meetings were held in Regina (1993), Lethbridge (1995), Kananaskis (1996), Victoria (1998), Winnipeg (2000), Regina (2002) and Victoria (2006).

IMA Program for Graduate Students
on Linear Algebra and Applications
Ames, Iowa, June 30-July 25, 2008

The Institute for Mathematics and Its Applications (IMA) will sponsor a Participating Institution Summer Program for Graduate Students June 29-July 26, 2008, at Iowa State Univ., on Linear Algebra and Applications. The organizers are Leslie Hogben, Wolfgang Kliemann, and Y. T. Poon.

Morning lecturers will introduce current research problems and background: Bryan Shader, combinatorics; David Watkins, numerical linear algebra, emphasizing eigenvalue calculations; Chi-Kwong Li, matrix inequalities in science and engineering; and Fritz Colonius, applications of linear algebra to dynamical systems. Students will work in groups in the afternoons.

All expenses will be paid for two students from each IMA Participating Institution. Interested faculty who can supply their own funding are welcome to apply. There may be supplemental funding to support students from other US universities. Students at institutions outside the US who can obtain funding from their government or university are also invited to apply. For more information, visit http://www.ima.umn.edu/2007-2008/PISG6.30-7.25.08/.

Ninth Workshop on Numerical Ranges and Numerical Radii
July 19-21, 2008, Williamsburg, Virginia

The 9th Workshop on Numerical Ranges and Numerical Radii (WONRA) will be held at The College of William and Mary on July 19-21, 2008. It will be followed by the International Workshop on Operator Theory and Applications (IWOTA 2008), also to be held at William and Mary. Furthermore, the

18th International Symposium on Mathematical Theory of Networks and Systems (MTNS) will be held at Virginia Tech at Blacksburg, Virginia, July 18-August 1, 2008. Participants may consider coming to one or more of WONRA, IOWTA, and MTNS.

The purpose of the numerical range workshop is to stimulate research and foster interaction of researchers interested in the subject. The informal workshop atmosphere will facilitate the exchange of ideas from different research areas and, hopefully, the participants will leave informed of the latest developments and newest ideas.

Visit the website WONRA (http://www.math.wm.edu/~ckli/wonra.html) to see some background about the subject and previous meetings.

There will be no registration fee. To register, please send the following information to Chi-Kwong Li (ckli@math.wm.edu) by June 30, 2008:

- Name, affiliation, and e-mail.
- Specify whether or not you will give a 25 min. talk. If yes, please attach a texfile of the abstract and title of your talk, using the basic latex template.
- Specify any schedule constraint for your talk.

SIAG/LA-SIMUMAT International Summer School on Numerical Linear Algebra July 21-25, 2008

Initiating a new line of activity, the SIAM Activity Group on Linear Algebra, SIAG/LA, together with SIMUMAT, is organizing its first Summer School on Numerical Linear Algebra at the International Center of Mathematical Meetings in Castro Urdiales, Spain, July 21-25, 2008.

Lecture series will be given on the following subjects:

- Krylov subspace methods for solving linear systems (Michael Eiermann)
- Matrix methods in data mining (Lars Elden)
- Mechanics and linear algebra (Rich Lehoucq)
- Structured eigenvalue problems: modern theory and computational practice (David Watkins)

Depending on funding, some travel support may be granted to selected participants. Further details can be found on the web site www.simumat.es/SIAGLA2008.

Workshop on Operator Theory & Applications Williamsburg, Virginia July 22-26, 2008

This will be a comprehensive, inclusive conference covering all aspects of theoretical and applied operator theory. A tentative list of plenary speakers is: J. Agler, A. Boettcher, P. Deift, I. Gohberg, B. Jacob, M. A. Kaashoek, P. Lancaster, B. McCluer, S. A. McCullough, N. Nikolskii, G. Popescu, S. Richter, H. Schneider, and V. Vinnikov.

The organizers are: J.A. Ball, V. Bolotnikov, J.W. Helton, L. Rodman, and I.M. Spitkovsky. For details, visit: http://www.math.wm.edu/~vladi/IWOTA/IOWTA2008.htm.

The 17th International Workshop on Matrices and Statistics Tomar, Portugal, July 23-26, 2008

IWMS08 will be held in Tomar, Portugal, July 23-26, 2008, to celebrate Professor Theodore Wilbur Anderson's 90th birthday. The conference website is www.ipt.pt/iwms08. For more information, contact:

Tomar Castle

Prof. Francisco Carvalho
fpcarvalho@ipt.pt
+351 249 328 100
Fax: +351 249 328 186.

Preliminary Announcement Robert C. Thompson Matrix Meeting UC Santa Barbara, Goleta, California October 18, 2008

Maribel Bueno will host an RCT Matrix Meeting at UC Santa Barbara on Saturday, October 18, 2008. Details will be announced later.

These RCT meetings are informal one day conferences which were started by Steve Pierce and Robert C. Thompson in the 1980's. They were called Southern California Matrix meetings for a long time, then began to be held in various other locations. In 2004, the attendees agreed to rename these the Robert C. Thompson Matrix Meetings, in memory of Bob Thompson, who died in 1995.

A list of previous ones can be found at http://www.auburn.edu/~tamtiny/rct2007.html.

NEWS

American Institute of Mathematics Groundbreaking for New Research Center

On May 31, 2007, AIM held a gala event to celebrate the groundbreaking for its new facility in Morgan Hill, CA, about 20 miles south of its present location in Palo Alto, CA. Ron Graham of UC San Diego, Member of the AIM Advisory Board, was the keynote speaker at the luncheon. AIM was founded in the early 1990's, in order to facilitate collaborative research among mathematicians on difficult outstanding problems. A workshop on matrix/graph problems was held there in October 2006 (see report in IMAGE 37).

Below is an architect's sketch of the new facility, which will replicate much of Spain's Alhambra Castle. For more information about AIM, see the article later in this issue about Math Institutes, and visit http://www.aimath.org/images/foothill_2.jpg.

Ron Graham, keynote speaker, and Brian Conrey, Executive Director of AIM

In Memoriam

We regret to report that we have lost four ILAS members in 2007:

John Todd died June 21 at the age of 96, at his home in Pasadena, Ca. He was an Emeritus Professor at the California Institute of Technology.

Victor Klee died August 21 at the age of 81. He was an Emeritus Professor at the University of Washington.

Ralph DeMarr died October 10 at the age of 75. He was an Emeritus Professor at the University of Arizona.

Gene Golub died November 16 at the age of 75. He was an Emeritus Professor at Stanford University.

Call for News for *IMAGE*

IMAGE editors welcome information on linear algebra and related topics, including:

- Reports and announcements of workshops, conferences and awards
- Feature articles on emerging applications and topics
- Historical essays
- Book reviews
- Announcements of new books and resources
- Obituaries and death notices
- Letters to the editor
- New problems and solutions to old problems

Send material for the Spring 2008 issue as follows:

Problems and solutions to Hans Joachim Werner by March 1, 2008: hjw.de@uni-bonn.de. It is helpful to receive these as early as possible. All other news to Jane Day by April 1, 2008: day@math.sjsu.edu.

Searchable online files of all issues of IMAGE are now available on its website, http://www.math.technion.ac.il/iic/IMAGE/. Our thanks to Roger Horn, George Styan and Jim Weaver for locating and scanning these files, and to Shaun Fallat for posting them.

First Olga Taussky Todd Lecturer for ICIAM: Pauline van den Driessche

Pauline van den Driessche, University of Victoria and one of the founding members of ILAS, gave the first Olga Taussky Todd Lecture at the International Congress on Industrial and Applied Mathematics held in Zurich in July 2007, on "Matrices in Action for Epidemic Models." This award is jointly sponsored by the Association for Women in Mathematics, European Women in Mathematics, and ICIAM. A news report about her talk appeared in SIAM News (40), October 2007, and was reprinted in the AWM Newsletter (37), November-December 2007.

New Manager for the ILAS Information Center (IIC)

By Daniel Hershkowitz, ILAS President

I am delighted to announce that effective November 15, 2007, Sarah Carnochan Naqvi of the University of Regina will assume the position of ILAS-Net/IIC Manager. We thank Sarah for agreeing to carry on this important responsibility. This position has been held by Shaun Fallat since February 1, 2002, and we thank Shaun for his very dedicated service to ILAS. In addition to managing IIC, he also significantly upgraded our website.

The online IIC provides current information on linear algebra activities such as international conferences, workshops, journals, and ILAS-NET notices. Organizations and individuals are invited to contribute. The primary website for IIC is http://www.ilasic.math.uregina.ca/iic/index1.html. Mirror sites are located at http://www.math.technion.ac.il/iic/index1.html and htpp://wftp.tu-chemnitz.de/pub/iic/index1.html.

Announcements for ILAS-Net should be sent in a text email with no attachments, to owner-ilas-net@math.technion.ac.il. Updates to the IIC webpage should be sent to Sarah Naqvi at ilasic@math.uregina.ca.

Call for Papers: Special Issue of LAA in honor of Thomas J. Laffey

By Volker Mehrmann, LAA Editor-in-Chief

We are pleased to announce a special issue of *Linear Algebra and Its Applications* in honor of Tom Laffey, who will celebrate his 65th birthday in December 2008. He has been an LAA editor for many years and is now one of its senior editors. All papers must meet the publication standards of LAA and will be subject to normal refereeing procedures. The deadline for submission of papers is February 15, 2008 and publication is expected in December 2008. Papers should be sent to one of the special editors of this issue, preferably by email in a PDF or PostScript format. These editors are Rod Gow, rod.gow@ucd.ie; Raphael Loewy, loewy@techunix.technion.ac.il; Joao Filipe Queiro, jfqueiro@mat.uc.pt, and Vladimir Sergeichuk, sergeich@imath.kiev.ua.

ARTICLES AND RESOURCES

Classroom Voting in Linear Algebra

By Kelly Cline, Holly Zullo and Mark Parker. Carroll College, Helena, MT

We have an NSF grant to develop classroom voting questions for linear algebra and differential equations, which are intended to actively involve students in the teaching and learning process. We have written about three hundred multiple-choice and true/false questions for linear algebra and as many for differential equations. We use these questions during class, to stimulate discussion that will clarify concepts. Despite much lively discussion, we cover the same content as when we gave traditional lectures in the same courses. Our students' response to this class style has been overwhelmingly positive. These questions are freely available at http://mathquest.carroll.edu, and we encourage feedback from anyone who uses some of them.

When using classroom voting, a question is posed during class. Students have a few minutes to discuss it in small groups before they decide what they think is the correct answer. Each student uses a handheld "clicker" to vote and the results are instantly tallied so all can see how the votes are spread. Then the instructor leads a Socratic discussion about the problem, asking different students to explain which answer they chose and why. This usually sparks considerable discussion and the correct answer soon becomes clear. More about our experiences can be found in the recent article "Teaching With Classroom Voting," MAA Focus, May/June 2007.

This method was introduced in 1989 by Eric Mazur in physics courses at Harvard, after he gave his students a diagnostic test to assess their true understanding of Newtonian mechanics. He was shocked at the poor results, and began to change his teaching style to what he calls "Peer Instruction." It remains very popular with his students. Other physics departments began using it. More recently mathematics faculty at several colleges and universities have developed similar methods. For example, visit htpp://hermite.cii.fc.ul.pt/iic/index1.html and htpp://www.math.temple.edu/iic/index1.html.

Research with Pre-Mathematicians: REU Programs at The College of William and Mary

By Charles R. Johnson[1]

1. Introduction

We have held an NSF-funded REU summer program at William and Mary since 1989. An earlier history and account of that activity may be found in [1]. Here I want to describe how these work. The purpose of REU programs is to help mathematically talented undergraduates to understand research in mathematics and thus help them make career decisions. An important benefit is the serious, high-level research that can result.

The students who participate are what I describe as "pre-mathematicians." By this I mean young people who have the ability to enter a career in high-level mathematics. Since such a career will require substantial, independent, creative mathematical thought, an honest and serious acquaintance with mathematical research is a valuable complement to conventional course work. Fortunately, a taste of research can help to seduce bright people into mathematics. The appeal of mystery and empowerment given by success in research can be most addictive.

Our concept of summer REU at W&M has always been collaborative research in small groups (at least one pre-mathematician and at least one mentor, no more than 4 total) on serious research problems whose outcome will be of interest to others. The unifying theme has been matrix analysis and applications, often with a considerable combinatorial flavor. Groups work on different problems but it has proven beneficial that the topics are in the same general area, so participants can chat and appreciate each other's work. We often have visitors help as mentors. These range from senior and junior collaborators to postdocs, other junior visitors, and Ph.D. students who are "graduates" of the REU program. For example, in 2005, I had 3 junior colleagues from Poland come for the summer.

About 160 US students apply each summer for NSF-funded REU programs, and a very large percentage of this group will benefit from an REU and be able to do good work. We choose students from this pool, trying to achieve some gender balance which is easier some summers than others. Sometimes a student who seems very strong, but has not had the opportunity for exposure to serious mathematics in their undergraduate experience thus far, may be given extra consideration. This entails some risk, but can also be very beneficial to the student, meeting an REU objective.

1 Adapted from a report prepared for the American Mathematical Society

We also frequently accept a few student participants who are not supported by the NSF; these range from high school students to foreign nationals (from Portugal, Ireland, Korea, etc.), to qualified "walk-ons". The resulting vertical integration (e.g. a high school student, working with an undergraduate, working with a graduate student, working with me) has worked well and been beneficial to all parties. In fact, I strongly feel that carefully selected high school students should be involved in REU, and that, at rather low cost, this would serve all goals of the REU concept. Cooperative agreements with foreign countries would be helpful. Ireland is already implementing an REU program that allows participation by foreign students. It would be of value if more countries sponsored such programs, and if such programs supported some students from abroad.

2. Where Do The Problems Come From?

Perhaps the most important feature of an activity designed to expose pre-mathematicians to research is the availability of appropriate problems. Good problems for this purpose should be (i) accessible, (ii) unsolved, and (iii) important; also (iv) it should be likely that some valuable progress can be made, but (v) the area should be large and open-ended enough that an exhaustively complete solution is unlikely before the end of a summer program. Importance is a judgment but the problem area should relate to other things or be an interesting part of a bigger picture and should be viewed as intriguing by the pre-mathematicians. We have had many large problems progress in parts over many summers. Accessibility depends, to some extent, on participant background.

Our REU activity has been centered around matrix analysis/linear algebra and its applications, broadly defined. A solid beginning linear algebra course is required for admission and a true second course is preferred. Each participant is given a copy of [2] at the beginning of the program, and copies of [3] are readily available. The participants are rapid learners, and learning that helps solve their problem is a very compelling motivation. With on-line and library resources and guidance from someone who knows the area, the topic of matrix analysis has proved ideal as a problem source. In addition, because of its connections with all parts of mathematics and most of its applications, learning about the area serves the participants well.

Most of the problems are spin-offs from my own research. I have very broad interests in matrix analysis and combinatorics and their relationship to other parts of mathematics. Occasionally, good problems are suggested by colleagues or queries I receive. Since prior REU work and ongoing work raise many fresh questions as well, this altogether allows accumulation in a year's time of many "good" problems, many more than I can pursue by myself.

Over the years, we have had many continuing themes for REU problems. These include: (i) matrix completion problems; (ii) the long standing conjecture from statistical physics that the coefficients of $p(t)= Tr[(A + tB)^m]$ are positive whenever A and B are positive definite matrices; (iii) possible multiplicities of the eigenvalues among the Hermitian matrices with a given graph; (iv) minimum rank among positive semidefinite matrices with a given graph; and (v) factorization of matrix and operator functions. Each has been a very rich source of specific problems and nice results, many of which have been published.

3. The Benefits of Undergraduate Research

If the purpose of REU programs in mathematics is to attract strong students to mathematics by giving them a realistic view of mathematical research, that seems to be working well. To be sure, many successful REU students would have gone on to mathematics Ph.D. study anyway, but it is likely that many others would not have gotten "into" mathematics were it not for these summer opportunities. However, I have observed other good consequences.

(i) It was for many years a wonderful tradition in Russia and parts of Eastern Europe that important and established mathematicians would go out of their way to nurture talented pre-mathematicians. This tradition not only improved the discipline and helped new entrants feel a part of the "community," but it also helped establish famous mathematical traditions and establish mathematics as an enduring cultural tradition that transcended politics. Our commitment to REU and research with pre-mathematicians is a modern version of that tradition which helps to accomplish similar objectives here.

(ii) The actual research that results from REU's is one of the most important tangible products. It allows me to pursue interesting questions that I would not have been able to otherwise. Working with REU students provides a nice way to establish pieces of work that can be assembled later into a bigger picture. Some examples are below. In many cases, the results are important and published in very credible journals.

(iii) What may seem to be a failure may also be a success. If a student finds through exposure to research, that it really wasn't for him or her, that is much better and less expensive than finding it out in graduate school. Fortunately, this is an unusual outcome.

(iv) The reinforcement of interest in and commitment to mathematics that comes from a group working together (even if not on the same problem) should not be underestimated. I had not anticipated it, but realized very early that one of the biggest benefits to the students (especially those who were "one-of-a-kind" at their institutions) is being with others of similar interest and outlook. This common experience may result in friendships and acquaintances that persist through a career, In fact I have long term collaborators who were once my REU students, and two of our REU students are now married and professors at Bucknell!

Let me indulge in closing this section with a story of my first experience as a mentor for pre-mathematician research. I was an NRC-NAS postdoc at the National Bureau of Standards (now NIST), which, at the time, had an Eastern European-like tradition of hiring Westinghouse talent search winners in the summer. I worked with one, Tom Leighton, and this resulted in two very nice papers, a classic on possible sign patterns of inverse positive matrices and a very practical one on graph isomorphism and eigenvalues. Tom did well academically and ended up co-founding Akamei Corporation, which survived the bursting of the "techbubble" and is now a very successful company. The value added to the economy by Akamei alone likely dwarfs the low cost of REU!

4. Some Examples of Recent REU Research

I have published about 40 papers with pre-mathematicians (many involving several co-authors), and others are in progress. My colleagues have also published a number of papers with undergraduates. This work has demonstrably advanced the subject. For variety, the following examples do not come from the continuing themes mentioned in Section 3.

4.1 An n-by-n matrix A over a field has an LU factorization if there exist a lower triangular matrix L and an upper triangular matrix U (wlog over the same field) such that A = LU . Such a factorization is important both in theory and in many applications. Though some sufficient and some necessary conditions are known for its existence, until recently no characterization of matrices A was known, for which an LU factorization exists. The sufficient conditions were not necessary, while the necessary conditions were not sufficient. In the REU program one summer, Pavel Okunev took this up with me. I had noticed that it was necessary that the rank of the upper left k-by-k principal submatrix of A plus k should be at least the rank of the first k rows of A plus the rank of the first k columns. Eventually, Pavel came up with a complicated proof of sufficiency which worked for an algebraically closed field. But we felt that the answer should not depend upon the field. Finally, we found a field independent proof of sufficiency, one of the more fundamental results of a summer program and the first ever to reach its original objective before the end (a risk in a narrowly focused problem). But, there is always more to do. The answer raised the question, for a matrix A without an LU factorization, of how many and where entries above the diagonal of L and/or below the diagonal of U might be needed for a factorization. We gave partial answers, and fuller answers have been given since with Maribel Bueno.

4.2 The natural partial order on n-by-n Hermitian matrices A and B is the positive semidefinite partial order: $A \geq B$ if A - B is positive semidefinite. It has long been known that if A and B are positive semidefinite, then $A \geq B$ implies $A^t = B^t$ for all $0 \leq t \leq 1$. In a conversation with a Polish statistician, Czeslaw Stepniak, the question arose of for which pairs of positive semidefinite matrices A and B should we have $A^t = B^t$ for all t > 1? It is clearly necessary that $A \geq B$, and this is sufficient if A and B commute. Another obvious sufficient condition is that all eigenvalues of A are at least any eigenvalue of B. It was not clear what a characterization should be, but in summer 2006, I took this up with undergraduate Becky Hoai and interested colleague, Ilya Spitkovsky, who often advises REU students. Another student also had an interest. After some thought we came up with and proved two characterizations, one very pretty and one of likely practical value. The former is that there exist k matrices $C_1 \geq ... \geq C_k$ such that $A \geq C_1 \geq ... \geq C_k \geq B$, C_i commutes with C_{i+1}, i =1, ..., k – 1, A commutes with C_1 and B commutes with C_k. The number k may have to be as high as n - 1, but not higher. Several related results were given.

4.3 If the zero/nonzero pattern A of an m-by-n matrix over a field is known, but not the values of the nonzero entries, the question of what the rank might be often arises. The maximum possible rank has long been understood, and all ranks between the minimum and maximum occur, but the minimum is quite difficult to characterize. If there is a k-by-k subpattern of A (in general position) that is permutation equivalent to a triangular pattern with nonzero diagonal (call this a "k-triangle") then it is known that min rank $A \geq k$. Thus min rank $A \geq T(A)$, which is defined to be the maximum of such k and is called the "triangle size" of A.

It was known from prior work that min rank A > T (A) can occur, the smallest known example being 7x7. This raises a natural question of for which m, n, r must an m-by-n pattern of minimum rank r (over the real field) have an r-triangle? Rafael Cauto and I had obtained some important partial results about this, and I had suggested the question to many bright pre-mathematicians who had given up after a brief look. In 2005, undergraduate Josh Link took up the question quite seriously, initially with a combination of clever ad hoc arguments and computing. Eventually, he and I found a clever (but still involved) complete solution, which I would never have found without his collaboration. This settled a long-standing natural question, but raises many more. For example, which m, n, r is the "first" instance of an m-by-n pattern A for which min rank A = r = T (A)+2?

4.4 A square matrix is Toeplitz if its entries are constant along diagonals parallel to the main one. As a natural case in a progression of determinantal inequality questions, we raised the following question in summer 2006: which ratios of products of principal minors are bounded among all positive definite Toeplitz matrices? Two pre-mathematicians, Hyo-min Choi from Korea and Alex Porush, a bright high school student from the area, took up this problem. I had addressed such questions before for M-matrices (and inverse M-matrices), totally positive matrices, positive definite matrices and certain structured P-matrices with co-authors Shaun Fallat and Tracy Hall (ex REU). The general positive definite case, especially, is still very unresolved (important partial results) and presents some remarkable difficulties, but a cone theoretic approach evolved from that work. Choi and Porush used this approach but had much creative work to do. Positive semidefinite Toeplitz matrices with special distributions of rank among the principal submatrices had to be constructed (or their existence ruled out) and inequalities suggested by the method had to be proven. They pushed things through n =6, a remarkable and fresh piece of work in a very classical area.

4.5 Ron Smith and I had pioneered linear interpolation problems for special classes of matrices. Example: for which pairs x and y of real n-vectors does there exist a P-matrix (positive principal minors) A such that Ax = y. We had found informative characterizations for many familiar classes.

This raised a natural further question, what about replacing x, y by n-by-k matrices X and Y (wlog of full rank k). This proves to be enormously challenging for virtually every class. I had done the positive definite case, which had a nice answer, but we hadn't done any other. We were concentrating on the P-matrix case and had a natural conjecture, but could not even prove it in the case n =3, k =2. Christian Sykes took up the problem and focused upon the 3,2 case. This is very geometric and analytic, which he liked. Ron and I had been trying to find an elegant 3,2 proof that might generalize. I encouraged Christian not to worry about how many cases he might have to consider, but that resolution of the 3,2 case would affect everyone's thinking. In the end, he found a rather nice algebraic proof of our conjecture via many cases. He has already spoken about this at a meeting in Portugal and won a prize for the talk.

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Your NSF Mathematical Sciences Institutes[1]

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The US NSF Mathematical Sciences Institutes represent a wonderful resource for the mathematics community. In this article we describe the seven US institutes supported by NSF: American Institute of Mathematics (AIM), Institute for Advanced Study (IAS), Institute for Mathematics and its Applications (IMA), Institute for Pure and Applied Mathematics (IPAM), Mathematical Biosciences Institute (MBI), Mathematical Sciences Research Institute (MSRI), and Statistical and Applied Mathematical Sciences Institute (SAMSI), as well as the Banff International Research Station (BIRS) in Canada that is supported by Canada, the US, and Mexico. Links to the pages of all the institutes can be found at http://mathinstitutes.org/. All these institutes provide a valuable and accessible resource for organizing and attending workshops and other programs in linear algebra.

In this article, we emphasize opportunities to organize or participate in week-long research workshops offered by these institutes. Participants are typically provided with support for local expenses and in some cases travel funds are available. Although it is generally expected that one or more of the organizers be US citizens (and in the case of BIRS, one or more be Canadian citizens), participants are usually invited from all over the world. In addition to workshops, many of the institutes have many other programs for researchers, graduate students, undergraduates, industry, and the public; we will only briefly and partially mention such programs in this article. Any of these programs can be highly stimulating to research.

While all the institutes offer workshops, there are substantial differences in suitable topics for proposals. Some of the institutes have a particular institutional focus (for example, SAMSI emphasizes statistics) or a different theme each year with most workshops related to the theme. Also, many of the institutes have some form of university membership and some special programs are restricted to faculty and students at member universities.

The websites for AIM, IMA, IPAM, MSRI, SAMSI and BIRS all make it easy to find out how to propose activities; AIM, IPAM, MSRI and BIRS have links for proposers on their main page; IMA and SAMSI have such links on the "Programs" page.

Many of the institutes have been supportive of linear algebra and offer excellent access to research support. IMA, AIM, and BIRS are of particular interest to ILAS members. IMA hosted a linear algebra year in 1991-1992 organized by R. A. Brualdi, G. Cybenko, A. George, G. Golub, M.B. Luskin, and P. Van Dooren (see http://www.ima.umn.edu/programs/annual/1991-1992.html), sponsored the Topics in Linear Algebra Conference at Iowa State University (ISU) in 2002, and will offer a graduate summer program in linear algebra and applications to be held at ISU in 2008 (see http://www.ima.umn.edu/2007-2008/PISG6.30-7.25.08/). Both AIM and BIRS have hosted workshops in aspects of linear algebra, including the 2006 AIM workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns" (http://aimath.org/pastworkshops/matrixspectrum.html) and the 2004 BIRS 2-day workshop "Directions in Combinatorial Matrix Theory" (http://www.birs.ca/birspages.php?task=displayevent&event_id=04w2525).

We will now briefly discuss each institute separately.

American Institute of Mathematics (AIM) http://www.aimath.org/

AIM was founded in 1994 with support from Fry's Electronics and became an NSF supported institute in 2002 (Fry's continues to be a major sponsor). Since its founding, AIM has emphasized mathematicians working in groups to solve hard problems.

AIM hosts 15-25 five-day workshops per year in all areas of mathematics. The workshops are fully funded (travel as well as local expenses) for up to 32 participants. The deadlines to propose a workshop are Nov. 1 and May 1 each year, and organizers are encouraged to consult with AIM staff when developing a proposal; see http://www.aimath.org/research/workshopproposals.html for more information. Workshops are held throughout the year. Eight participants are selected through an on-line application process (the organizers invite 24 participants). The deadline to apply to participate in a workshop depends on the date of the workshop; see http://www.aimath.org/research/upcoming.html for a list of upcoming workshops.

AIM workshops are quite different from those usually offered by other programs. Far fewer talks are given and the participants spend much of their time doing research in small groups. AIM staff have extensive experience with this unusual format and advise the organizers throughout the process. After the workshop a website of information is created to disseminate results.

[1]The opinions expressed are those of the authors, not those of ISU, NSF, or any of the NSF Mathematical Sciences Institutes.

[2]Full disclosure: The authors have much more experience with some of the institutes than with others. Over the past ten years, as a professor of mathematics at Iowa State University, Leslie Hogben has proposed, organized, and/or participated in programs at IMA (3), AIM (2), and BIRS (2); in fall 2007 she also became the Associate Director for Program Diversity of AIM. Jason Grout participated in four MSRI programs and one AIM workshop while a graduate student at Brigham Young University.

AIM also offers support for small research groups called SQuaREs, hosts long-term projects such as the one that led to the determination of all the representations of E_8 and another that led to the resolution of the perfect graph conjecture, supports post-docs, and provides outreach activities to the local community.

AIM workshops, with their emphasis on doing research in groups during the workshop, are ideally suited to the collaborative style of research practiced by many linear algebraists. AIM is also one of the most accessible of all the institutes, with no annual theme and no membership.

Institute for Advanced Study (IAS) http://www.math.ias.edu/

IAS is the oldest of the institutes and encompasses science as well as mathematics. Unlike the other institutes, it has its own permanent faculty, as well as members, short-term visitors, and post-doctoral fellows, and a wide range of mathematical activities.

One annual activity is the Park City Math Institute (PCMI) http://pcmi.ias.edu/, a summer program on a specific research area. Each PCMI includes researchers, graduate and undergraduate students, and groups which focus on the mathematics which underlies the research area. For example, the 1998 theme was Representation Theory of Lie Groups and the associated undergraduate faculty program focused on linear algebra (http://pcmi.ias.edu/1998/).

Institute for Mathematics and its Applications (IMA) http://www.ima.umn.edu/

Founded in 1982, IMA was one of the first two NSF Mathematical Sciences Institutes and has an emphasis on interdisciplinary research involving mathematics related to challenging problems in science and industry. IMA pioneered the formal involvement of industry throughout its programming, from IMA industrial post-docs to Participating Corporations.

IMA hosts 5-15 workshops each year, mostly related to the annual theme (in 2008-2009 the theme is Mathematics and Chemistry). Workshops related to the annual theme are organized in conjunction with the planning for the theme year; others can be proposed as hot topics. For all workshops, some participants are long-term visitors to the IMA, others are invited for the workshop, and some apply through the on-line application process; see http://www.ima.umn.edu/docs/application.php for more information. As part of the application process, visitors may apply for support (local and/or travel expenses).

IMA has both academic partners (Participating Institutions, called PIs) and industrial partners (Participating Corporations, PCs). Faculty, staff, and students at PIs and PCs enjoy preference for funding for participation in IMA activities, and some programs are restricted to people affiliated with PIs and PCs. IMA also offers a variety of other programs, including long-term memberships for individuals in conjunction with the annual theme and programs specifically for graduate students.

IMA has an applied focus that makes it particularly attractive to those in applied linear algebra. IMA has already sponsored several programs relating to linear algebra and applications. If you have one specific topic in mind, the chance that it will be suitable for an IMA workshop in a specific year is not great, but there are a variety of programs available at any time, and often some have connections to some aspect of linear algebra (this is especially true for IMA programs at Participating Institutions).

Institute for Pure and Applied Mathematics (IPAM) http://www.ipam.ucla.edu/

IPAM emphasizes connections between mathematics and other sciences. Each year IPAM offers two one-semester programs. Each program begins with tutorials, followed by 3-4 workshops, and culminates in a “Oberwolfach-like workshop.” In addition to these workshops associated with the long programs, IPAM hosts workshops on a variety of topics throughout the year.

IPAM accepts workshop proposals unrelated to the long programs, and a workshop with an applied linear algebra focus would be appropriate.

Mathematical Biosciences Institute (MBI) http://www.mbi.osu.edu/

The Mathematical Biosciences Institute at Ohio State University focuses on the mathematical theory, statistical methods, and computational problems in the biosciences. Activities include year-long emphasis programs, week-long workshops, and summer undergraduate programs. Partial support is available for long-term visitors (http://www.mbi.osu.edu/forms/visitorapplication.html). Full and partial support for three-year postdocs is also available (http://www.mbi.osu.edu/postdoctoral/postdoctoral.html).

Mathematical Sciences Research Institute (MSRI) http://www.msri.org/

MSRI was founded in 1982 and is funded primarily by NSF and other government agencies, industry, foundations, and participating institutions. MSRI hosts year-long and semester-long programs (usually more than one at the same time) as well as several shorter workshops (three days to two weeks) and summer graduate-student workshops each year. Proposals for a semester-long or year-long program are generally submitted three years in advance. Proposals for a week-long workshop are generally submitted 1.5-2 years in advance. Additionally, a "Hot Topics" week-long workshop in the spring is selected from proposals submitted by November 1 of the previous year. Information on submitting proposals for any of these programs is at http://www.msri.org/propapps/. A list of upcoming activities is available at http://www.msri.org/activities.

MSRI offers full or partial support to researchers and postdocs for the semester or year programs, but also offers support for a few appointments outside of the themes for the year. Also, for member institutions, MSRI offers full support for three graduate students to attend a summer workshop. These workshops provide a rich educational and research experience for graduate students.

MSRI videotapes many lectures given at the facility and makes them freely available on the web (see http://www.msri.org/communications/vmath). Available recordings include lectures from workshops and other special occasions.

MSRI accepts workshop proposals that are not related to the yearly or semester programs. Many past and future programs at MSRI involve algebra, combinatorics, and computation, so may also be of interest.

Statistical and Mathematical Sciences Institute (SAMSI) http://www.samsi.info/

SAMSI, a partnership of several universities and organizations, emphasizes statistics and is located in North Carolina. Year-long programs and workshops are hosted (see http://www.samsi.info/orograms/ for the current programs). Proposals are generally submitted in September two years before the program starts (see http://www.samsi.info/programs/programssought.shtml). Full or partial support is provided for visiting researchers, postdocs and visiting graduate students. Information on these positions is available at http://www.samsi.info/opportunities/.

A recent program "Random Matrices" (http://www.samsi.info/programs/2006ranmatprogram.shtml) investigated linear algebra. Programs will naturally emphasize applications and connections to statistics.

Banff International Research Station (BIRS) http://www.birs.ca/

BIRS is the newest of the institutes discussed here; it was founded in 2001 and began the operation of its workshops in 2003. Located at the Banff Center, BIRS is modeled on Oberwolfach; the setting, nestled in the mountains, is spectacular.

BIRS is jointly supported by Canada's NSERC, the US NSF, Mexico's

US NSF, Mexico's National Council for Science and Technology, and Alberta's Science Research Authority. Workshops are the main program of BIRS, which operates 44-48 workshops each year for up to 42 participants (in some weeks two 20-person half workshops are offered). BIRS provides housing and all meals for participants. There is no annual theme and the deadline to propose a workshop is Oct. 1 each year (more than a year in advance – Oct. 1, 2008 for 2010); see http://www.birs.ca/applicants/guidelines/ for more information. BIRS hosts workshops in all areas of mathematics throughout the year (except mid-December to mid-January). BIRS does not have member universities, but is affiliated with the Canadian institutes Pacific Institute for the Mathematical Sciences (PIMS) and Mathematics of Information Technology and Complex Systems (MITACS), the U.S. MSRI, and the Mexican Instituto de Matimaticas (IM-UNAM).

BIRS also offers some 2-day workshops and space for smaller groups of researchers (Focused Research Groups and Research in Teams).

BIRS is one of the more accessible institutes since it has no annual theme and operates the largest number of workshops. It is suitable for all areas of linear algebra. Potential organizers should also consider other BIRS programs, such as 2-day workshops and small groups.

What is an Enigmatrician?

According to the solution of Polymath Crossword No.350, published in the Financial Times of London on 7 July 2006, people who compile crosswords for the Manchester Guardian should properly be referred to as "enigmatists." Combining this result with those of [1], this suggests that compilers of problems published in the IMAGE Problem Corner should be known as "enigmatricians."

[1] R. W. Farebrother, *What is a matrician?* IMAGE No.25, October 2000, p. 32.

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IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 37-1 through 37-6 [IMAGE 37 (Fall 2006), pp. 31–32] and a further remark to IMAGE Problem 30-3 [IMAGE 30 (April 2003), p. 36]. Problems 32-4 [IMAGE 32 (April 2004), p. 40] and 36-3 [IMAGE 36 (Spring 2006), p. 36] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 6 new problems on pp. 32 & 31 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material both (a) in macro-free LaTeX by e-mail, preferably embedded as text, to hjw.de@uni-bonn.de and (b) one paper copy (nicely printed please) by classical p-mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany. Please make sure that your name as well as your e-mail and classical p-mail addresses (in full) are included in both (a) and (b)!

Problem 30-3: Singularity of a Toeplitz Matrix

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Let $n \geq 5$, $c_1, \dots, c_{n-1} \in \mathbb{C} \setminus \{0\}$, x an indeterminate over the complex numbers $\mathbb{C}$ and consider the Toeplitz matrix

$$M := \begin{pmatrix} c_2 & c_1 & x & 0 & \cdot & \cdots & 0 \\ c_3 & c_2 & c_1 & x & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & & & & \ddots & \vdots \\ c_{n-3} & c_{n-4} & \cdot & \cdot & \cdot & \cdots & x \\ c_{n-2} & c_{n-3} & \cdot & \cdot & \cdot & \cdots & c_1 \\ c_{n-1} & c_{n-2} & \cdot & \cdot & \cdot & \cdots & c_2 \end{pmatrix}.$$

Prove that if the determinant $\det M = 0$ in $\mathbb{C}[x]$ and $5 \leq n \leq 9$, then the first two columns of M are dependent. [We do not know if the implication is true for $n \geq 10$.]

Editorial Note. We have already published with Solution 30-3.1 an assumed solution to this problem; see IMAGE 36 (Spring 2006), pp. 26–27. Because of a remark by Harald Wimmer (see IMAGE 37 (Fall 2006), p. 19), this problem was reopened. In the meantime we have received a further remark, which is given below.

Further Remark on Problem 30-3 by Alexander KOVACEC, Maria Celeste GOUVEIA, *Universidade de Coimbra, Portugal:* kovacec@mat.uc.pt, mcag@mat.ut.pt and Wiland SCHMALE, *Universität Oldenburg, Germany:* schmale@uni-oldenburg.de

We will give a solution for the 5×5 case, in Hankel matrix language, at the beginning reminiscent of the Li-Bolotnikov approach. Then, using the computer we also show how to prove the $n \times n$ cases for $n = 6, 7$, or with more powerful machines perhaps even higher dimensional cases. Let

$$H_n(x) = \begin{pmatrix} & & & x & c_1 & c_2 \\ & & x & c_1 & c_2 & c_3 \\ & & & & \vdots & \vdots \\ x & c_1 & \dots & \dots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \dots & \dots & c_{n-1} & c_n \\ c_2 & c_3 & \dots & \dots & c_n & c_{n+1} \end{pmatrix}, \text{ so that, in partcular, } H_5(x) = \begin{pmatrix} & & x & c_1 & c_2 \\ & x & c_1 & c_2 & c_3 \\ x & c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}.$$

Throughout, for a natural number n, let GAn, Cn, and HPnC denote the GENERAL ASSUMPTION: "$\det H_n(x) \equiv 0$ and $c_i \neq 0$, $i = 1, \dots, n+1$", the CLAIM: "the last two rows of $H_n(x)$ are dependent", and the HANKEL PENCIL CONJECTURE: "GAn implies Cn (or, equivalently, there exists an $l \in \mathbb{C}$ such that $c_j = l^{j-1} c_1$, for $j = 1, \dots, n+1$)", respectively. Given a partitioned matrix

$$A = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where A is $n \times n$ and E is $k \times k$ invertible, a formula in Brualdi & Schneider (1983, Section 2) says that in such a situation there holds the matrix identity

$$(\det E)(H - GE^{-1}F) = (\det A[1:k, i|1:k, j])_{i,j=k+1,\dots,n}.$$

Using that $\det A = (\det E)\det(H - GE^{-1}F)$, the Sylvester identity follows.

We now discuss HP5C, i.e., HPnC for $n = 5$. So, let $n = 5, k = 3$, and let $A = H_5(x)$. Define the polynomial $m_{ij}(x) = \det H_5(x)[i^c|j^c]$, where $i^c = \{1,2,3,4,5\}\setminus\{i\}$, $j^c = \{1,2,3,4,5\}\setminus\{j\}$. Since $\det E = -x^3$, the hypothesis of HP5C is equivalent to saying

$$\det \begin{pmatrix} m_{55}(x) & m_{45}(x) \\ m_{45}(x) & m_{44}(x) \end{pmatrix} \equiv 0.$$

The 2×2 matrix here obtained is the analogue of the one obtained by Li and Bolotnikov. A computation shows,

$$\begin{pmatrix} m_{55}(x) & m_{45}(x) \\ m_{45}(x) & m_{44}(x) \end{pmatrix} = 1 \begin{pmatrix} c_1^4 & c_1^3c_2 \\ c_1^3c_2 & c_1^2c_2^2 \end{pmatrix} - x \begin{pmatrix} 3c_1^2c_2 & 2c_1c_2^2 + c_1^2c_3 \\ 2c_1c_2^2 + c_1^2c_3 & c_2^3 + 2c_1c_2c_3 \end{pmatrix} + x^2 \begin{pmatrix} c_2^2 + 2c_1c_3 & 2c_2c_3 + c_1c_4 \\ 2c_2c_3 + c_1c_4 & c_3^2 + 2c_2c_4 \end{pmatrix} - x^3 \begin{pmatrix} c_4 & c_5 \\ c_5 & c_6 \end{pmatrix}.$$

Now from knowing that the determinant of this sum is 0 we cannot conclude without further ado, as the proposal in IMAGE 36 (Spring 2006), pp. 26–27, seems to imply, that the determinants of the matrices are 0; see the example by H.K. Wimmer in IMAGE 37 (Fall 2006), p. 19.

At any rate, we can use the matrix polynomial to prove HP5C. From the above we see that GA5 implies that m_{44}, m_{45}, m_{55} are degree 3 polynomials and so their sets of roots, $\mathrm{roots}(m_{ij})$, $i, j \in \{4,5\}$, have cardinalities in $\{1,2,3\}$ and do not contain 0. For polynomials $p = p(x)$, $q = q(x)$, write $p \doteq q$ to mean that there is a $l \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that $p = lq$.

LEMMA. GA5 *implies each of the following facts:*

(F0) $c_6c_4 - c_5^2 = 0$.

(F1) $m_{44} \cdot m_{55} - m_{45}^2 \equiv 0$.

(F2) $m_{45} \doteq m_{55} \Rightarrow$ *C5.*

(F3) *For all* $a, b, g \in \mathbb{C}^*$: $\left.\begin{array}{l} m_{44} = -c_6(x-a)(x-b)^2 \\ m_{45} = -c_5(x-a)(x-b)(x-g) \\ m_{55} = -c_4(x-a)(x-g)^2 \end{array}\right\} \Rightarrow a = b = g.$

Assuming this Lemma valid for a moment, GA5 $\Rightarrow$ C5 is easy to establish by contradiction as follows. Assume GA5 $\&\neg$ C5. Then, by (F2), $m_{45} \not\doteq m_{55}$. From (F1) and the matrix polynomial we infer that for certain $a, b, g \in \mathbb{C}^*$, polynomials m_{44}, m_{55}, m_{45} can be written as in the hypothesis of (F3). But then $a = b = g$, implying $m_{45} \doteq m_{55}$, a contradiction.

Throughout the following proof of our Lemma, we assume that GA5 is satisfied.

PROOF OF (F0): Note that $\mathrm{coeff}(\det H_5(x), x^3) = -c_6c_4 + c_5^2$.

PROOF OF (F1): This is a restatement of $\det \begin{pmatrix} m_{55}(x) & m_{45}(x) \\ m_{45}(x) & m_{44}(x) \end{pmatrix} \equiv 0$.

PROOF OF (F2): Assume $m_{45} \doteq m_{55}$. Then from (F1) we also get $m_{44} = lm_{45}$ for some $l \neq 0$. By looking at the coefficients of x^j, $j = 0,1,2,3$, in both polynomials, we find $c_1^2c_2^2 = lc_1^3c_2$, and so $c_2 = lc_1$. Next $c_2^3 + 2c_1c_2c_3 = l(2c_1c_2^2 + c_1^2c_3)$, so substituting $c_2 = lc_1$ we get $c_3 = l^2c_1$. Next, using $c_3^2 + 2c_2c_4 = l(2c_2c_3 + c_1c_4)$ and writing c_2, c_3 in terms of c_1 we get $c_4 = l^3c_1$. We also get $c_6 = lc_5$ from where together with (F0) we find C5.

PROOF OF (F3): The following table compares the coefficients of the powers x^j, $j = 0,1,2$ of the m_{ij}, $(i,j) = (4,4), (4,5), (5,5)$, in succession, according to the definition, with those obtained by factorization in $\mathbb{C}[x]$. Note that by (F0), $c_5 = lc_4$, $c_6 = l^2c_4$ for some $l \in \mathbb{C}^*$.

m_{44}			m_{45}			m_{55}		
$c_1^2c_2^2$	$\stackrel{1}{=}$	$l^2c_4ab^2$	$c_1^3c_2$	$\stackrel{2}{=}$	lc_4abg	c_1^4	$\stackrel{3}{=}$	c_4ag^2
$c_2^3 + 2c_1c_2c_3$	$\stackrel{1'}{=}$	$l^2c_4b(2a+b)$	$2c_1c_2^2 + c_1^2c_3$	$\stackrel{2'}{=}$	$lc_4(ab+ag+bg)$	$3c_1^2c_2$	$\stackrel{3'}{=}$	$c_4g(2a+g)$
$c_3^2 + 2c_2c_4$	$\stackrel{1''}{=}$	$l^2c_4(a+2b)$	$2c_2c_3 + c_1c_4$	$\stackrel{2''}{=}$	$lc_4(a+b+g)$	$c_2^2 + 2c_1c_3$	$\stackrel{3''}{=}$	$c_4(a+2g)$

From $(3')$ we find by GA5 that $2a+g\neq 0$. From the table we note

$$c_1=\frac{3\mathrm{lhs}(2)}{\mathrm{lhs}(3')}=\frac{3\mathrm{rhs}(2)}{\mathrm{rhs}(3')}=\frac{3abl}{2a+g},\ c_2=\frac{3\mathrm{lhs}(1)}{\mathrm{lhs}(3')}=\frac{3\mathrm{rhs}(1)}{\mathrm{rhs}(3')}=\frac{3ab^2l^2}{g(2a+g)},$$

$$c_4=\frac{\mathrm{rhs}(1)}{l^2ab^2}=\frac{\mathrm{lhs}(1)}{l^2ab^2}=\frac{c_1^2c_2^2}{l^2ab^2}=\frac{81a^3b^4l^4}{g^2(2a+g)^4},\quad c_3=\frac{\mathrm{lhs}(3'')-\mathrm{c}_2^2}{2c_1}=\frac{\mathrm{rhs}(3'')-\mathrm{c}_2^2}{2c_1}=\frac{3ab^3l^3(5a^2+14ag-g^2)}{2(2a+g)^3g^2}.$$

Now equations $1,2,3,3',3''$ are satisfied. From $3\,\mathrm{lhs}(1)=\mathrm{c}_2\mathrm{lhs}(3')$ and $\mathrm{lhs}(1')=\mathrm{c}_2\mathrm{lhs}(3'')$ we obtain $3\,\mathrm{rhs}(1)\mathrm{rhs}(3'')-\mathrm{rhs}(1')\mathrm{rhs}(3')=0$, which means that we must have the first of the equations

$$\begin{aligned} p(a,b,g) &:= (3a^2+4ag-g^2)b-(4a^2g+2ag^2) &= 0,\\ q(a,b,g) &:= (-g^3+3ag^2+42a^2g+10a^3)b-(9ag^3+27a^2g^2+18a^3g) &= 0, \end{aligned}$$

while the second equation follows from requiring $0=\mathrm{lhs}(2'')-\mathrm{rhs}(2'')$, since q emerges as factor that has to be 0. Since $g\neq 0$, and p,q are homogeneous, we may assume $g=1$. Then we infer

$$0=\mathrm{coeff}(q(a,b,1),b)\cdot p-\mathrm{coeff}(p(a,b,1),b)\cdot q=7(-1+a)^3a(1+2a).$$

Therefore $a\in\{1,-1/2\}$. But $a=-1/2$ implies via $p(-1/2,b,1)=-9/4b$, that $b=0$, which we excluded. Therefore $a=1$. But now $0=p(1,b,1)$ yields $b=1$. Thus $a=b=g=1$, as we wished to prove. □

We next SKETCH A COMPUTER ASSISTED PROOF FOR THE $n\times n$ CASES WITH $n=3,4,5,6,7$. Quite generally, upon scaling, one may assume in $H_n(x)$ that $c_1=1$. The assumption GAn is of course equivalent to demanding $c_2,\ldots,c_n\neq 0$ and the following system of algebraic equations in $c_2,c_3,\ldots,c_{n+1}$:

$$cx_j(c_2,\ldots,c_{n+1})=\mathrm{coeff}(\det H_n(x),x^j)=0,\ j=0,1,2,...,n-2.$$

We show in detail the 7×7 case, since analogous investigations of the $n\times n$ cases with $n=3,...,6$ exhibit quite similar phenomena. We also put $c_2=a$, since it plays a special role. Execute the Mathematica lines c2=a; cxlist={cx0,cx1,cx2,cx3,cx4,cx5} and gb=GroebnerBasis[cxlist,{c8,c7,c6,c5,c4,c3,c2}]. Equipped with Mathematica version 2.2, a 486-PC, model 1993, generates after about 80 minutes an ordered list of 42 polynomials; we call these simply g1, g2, ..., g42. One finds Factor[g1]=(-a^2 + c3)^16. So we execute the condition c3=a^2; necessary for solution. Then the next few polynomials in gb yield 0, but one finds Factor[g9]=142686402687062742698729 98*(-a^3 + c4)^8. So execute c4=a^3. Again many polynomials now vanish. But Factor[g25]= 35*(-a^4 + c5)^4. Thus execute c5=a^4. Then the next nonzero polynomial is Factor[g36]=1400*(-a^5 + c6)^3. Hence put c6=a^5. Then Factor[g41]=(-a^6 + c7)^2. So c7=a^6. Then g42=-a^7 + c8. So c8=a^7 is the last necessary condition for the solution of the system. With this the proof is done: with this (and only this) setting (i.e. $c_j=a^{j-1}$) all the polynomials cx_j (or those in the equivalent Groebner Basis) vanish. □

Other elimination orders of variables, or letting c_1 be variable instead of putting it equal to 1, prohibited the treatment of even the case $n=6$, on the mentioned computer at least. Interestingly, the further theoretically admissible reduction to $c_1=1$ & $c_6=c_7=c_8$ (purporting to prove $c_2=c_3=\ldots=1$) incurred equally prohibitive memory or time costs. The proof shows furthermore that to obtain the conclusion it seems to be sufficient to assume $c_1\neq 0$.

We conclude with mentioning that meanwhile we have found new general results that lead to considerable conceptual simplifications of the above arguments; these allow us to check the validity of the conjecture for $H_n(x)$ for all $n\leq 8$ on the above mentioned computer within minutes. Two of us submitted recently a paper on the subject to *Linear Algebra and Its Applications*

Reference

R. Brualdi & H. Schneider (1983). Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplase, Muir, and Caley. *Linear Algebra and Its Applications* **52–53**, 769–791.

Problem 32-4: A Property in $\mathbb{R}^{3\times 3}$

Proposed by J. M. F. TEN BERGE, *University of Groningen, Groningen, The Netherlands:* j.m.f.ten.berge@ppsw.rug.nl

We have real nonsingular matrices X_1, X_2, and X_3 of order 3×3. We want a real nonsingular 3×3 matrix U defining $W_j=u_{1j}X_1+u_{2j}X_2+u_{3j}X_3$, $j=1,2,3$, such that each of the six matrices $W_j^{-1}W_k$, $j\neq k$, has zero trace. Equivalently, we want

$(W_j^{-1}W_k)^3 = (a_{jk})^3 I_3$, for certain real scalars a_{jk}. Conceivably, a matrix U as desired does not in general exist, but even a proof of just that would already be much appreciated.

We still look forward to receiving solutions to Problem 32-4!

Problem 36-3: Transformations of (Skew-) Idempotent and (Skew-) Involutory Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

In this problem, we generalise Farebrother's (2001, p. 11) definitions of involutory and skew-involutory matrices by replacing the $n \times n$ identity matrix I_n by a given $n \times n$ idempotent matrix P satisfying $P^2 = P$.

(a) Let A be an $n \times n$ idempotent or skew-idempotent matrix satisfying $AP = PA = A$ and $A^2 = mA$ where $m = \pm 1$. Then for $s^2 = \pm 1$, the generalised Householder transformation $B = s(2A - mP)$ is P-involutory (i.e., $B^2 = P$) or skew-P-involutory (i.e., $B^2 = -P$) according as $s^2 = \pm 1$: $B^2 = s^2(4A^2 - 4mPA + m^2P^2) = s^2P$.

(b) Conversely, suppose that B is an $n \times n$ P-involutory or skew-P-involutory matrix satisfying $BP = PB = B$ and $B^2 = s^2P$ where $s^2 = \pm 1$, then the transformation $A = (sB + mP)/2$ is idempotent (i.e., $A^2 = A$) or skew-idempotent (i.e., $A^2 = -A$) according as $m = \pm 1$: $A^2 = (s^2B^2 + 2msBP + m^2P^2)/4 = m(sB + mP)/2 = mA$.

Readers are asked to suggest alternative transformations that maintain the reality of B (or A) when P and A (or B) are real.

Reference

R. W. Farebrother (2001). The naming of parts. *IMAGE: The Bulletin of the International Linear Algebra Society* **27** (October 2001), 11.

We still look forward to receiving solutions to Problem 36-3!

Problem 37-1: Another Property for the Sum of a Matrix A and its Moore-Penrose Inverse $A^\dagger$

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and Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $A \in \mathbb{C}_{n,n}$ and let $A^\dagger$ be its Moore-Penrose inverse. Show that the following statements are equivalent:

(i) $A + A^\dagger = AA^\dagger + A^\dagger A$,

(ii) A is an EP matrix (i.e., satisfies $AA^\dagger = A^\dagger A$) such that $Q_A = (I_n - A)^2$, where Q_A is the orthogonal projector onto the orthogonal complement of the column space of A and I_n is the identity matrix of order n.

(Parenthetically note that the present problem is related to the Problem 34-8 [G. Trenkler, A Property for the Sum of a Matrix A and its Moore-Penrose Inverse A^+. *IMAGE: The Bulletin of the International Linear Algebra Society* **34** (Spring 2005), 39].)

Solution 37-1.1 by the Proposers Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
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On account of Corollary 6 in Hartwig & Spindelböck (1984), any matrix $A \in \mathbb{C}_{n,n}$ of rank r can be represented as

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{1}$$

where $U \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, ..., \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > ... > \sigma_t > 0$, $r_1 + r_2 + ... + r_t = r = \mathrm{rank}(A)$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy $KK^* + LL^* = I_r$. From (1) it follows that

$$A^\dagger = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*, \tag{2}$$

and hence

$$AA^\dagger = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^* \quad \text{and} \quad A^\dagger A = U \begin{pmatrix} K^*K & K^*L \\ L^*K & L^*L \end{pmatrix} U^*. \tag{3}$$

Furthermore, from (3) it is easily seen that A satisfies $AA^\dagger = A^\dagger A$ (i.e., A is EP) if and only if $L = 0$ (note that this condition is equivalent to $K^* = K^{-1}$).

Since the orthogonal projector onto the orthogonal complement of the column space of A is of the form $Q_A = I_n - AA^\dagger$, direct calculations with the use of (1) and the left-hand side formula in (3) show that condition $Q_A = (I_n - A)^2$, constituting a part of statement (ii) of the problem, holds if and only if $(I_r - \Sigma K)^2 = 0$ and $K\Sigma L = 2L$. Combining these conditions with the requirement that matrix A is EP, leads to the conclusion that statement (ii) is equivalent to the conjunction

$$L = 0 \quad \text{and} \quad (I_r - \Sigma K)^2 = 0. \tag{4}$$

Therefore, the solution is established, for from (1), (2), and (3) it straightforwardly follows that also statement (i) of the problem is equivalent to conjunction (4).

Reference

R.E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and $A^\dagger$ commute. *Linear and Multilinear Algebra* **14**, 241-256.

Solution 37-1.2 by Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

FACT 1. *Let $A \in \mathbb{C}_{n,n}$. Then $Q_A = (I_n - A)^2$ is the orthogonal projector onto the orthogonal complement of the column space of A only if $(I_n - A)^2 A = 0$.*

FACT 2. *Let $A \in \mathbb{C}_{n,n}$. Then $A + A^+ \geq_{\mathrm{L}} 0$ $\;(i.e.,\ is\ nonnegative\ definite)$ if and only if there exists an unitary matrix U and a nonsingular matrix P such that*

$$A = U \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} U^* \tag{5}$$

and $P + P^{-1} \geq_{\mathrm{L}} 0$ (for nonsingular matrix A we omit zero matrices in the decomposition (5)).

PROOF. Using Corollary 6 of Hartwig and Spindelböck (1984) and Albert's Theorem (Albert, 1969) we get (5). □

THEOREM. *Let $A \in \mathbb{C}_{n,n}$. Then*

$$A + A^+ = AA^+ + A^+A \tag{6}$$

if and only if

$$AA^+ = A^+A \tag{7}$$

and

$$(I_n - A)^2 A = 0. \tag{8}$$

PROOF. Premultiplying and postmultiplying the equality (6) by A and using (5) we get (7) and (8). Premultiplying (8) by $(A^+)^2$ and using (7) we get (6). □

References

A. Albert (1969). Condition for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics* **24**, 434-440.

R. E. Hartwig & K. Spindelböck (1969). Matrices for which $\mathbf{A}^*$ and $\mathbf{A}^+$ commute. *Linear Algebra and its Applications* **14**, 241–256.

Solution 37-1.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

First, we recall that if V is any nonnegative definite and Hermitian $n \times n$ matrix and $\mathcal{M}$ is any linear subspace of $\mathbb{C}^n$, then $(V\mathcal{M}) \cap \mathcal{M}^\perp = \{0\}$, where $\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$ with respect to the usual inner product; cf. Theorem in Werner (2003). Secondly, we note that if $M \in \mathbb{C}^{m\times n}$, then $(\mathcal{N}(M^*))^\perp = \mathcal{R}(M)$ and $(\mathcal{R}(M^*))^\perp = \mathcal{N}(M)$, with $(\cdot)^*$, $\mathcal{N}(\cdot)$, and $\mathcal{R}(\cdot)$ denoting the conjugate transpose, the null space, and the range (column space), respectively, of the matrix $(\cdot)$. Thirdly, we mention that $MM^\dagger$ and $M^\dagger M$ are the orthogonal projectors onto $\mathcal{R}(M)$ (along $\mathcal{N}(M^*)$) and onto $\mathcal{R}(M^*)$ (along $\mathcal{N}(M)$), respectively, and so $MM^\dagger$ and $M^\dagger M$ are, in particular, both nonnegative definite and Hermitian matrices. Fourthly, we note that $\mathcal{N}(M^\dagger) = \mathcal{N}(M^*)$. And finally, we recall to the following characterization: $M \in \mathbb{C}^{n\times n}$ is EP $\Leftrightarrow \mathcal{R}(M) = \mathcal{R}(M^*)$ $\Leftrightarrow \mathcal{N}(M) = \mathcal{N}(M^*) \Leftrightarrow MM^\dagger = M^\dagger M$. In view of these observations, is is clear that problem condition (i) implies that $A + A^\dagger$ is nonnegative definite and Hermitian. In which case, $[(A + A^\dagger)\mathcal{N}(A^*)] \cap \mathcal{R}(A) = \{0\} \Leftrightarrow (A\mathcal{N}(A^*)) \cap \mathcal{R}(A) = \{0\} \Leftrightarrow$ $A\mathcal{N}(A^*) = \{0\} \Leftrightarrow \mathcal{N}(A^*) \subseteq \mathcal{N}(A) \Leftrightarrow \mathcal{N}(A^*) = \mathcal{N}(A) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A) \Leftrightarrow A$ is EP; see also Werner (2005). Consequently, $A + A^\dagger = AA^\dagger + A^\dagger A \Leftrightarrow A + A^\dagger = 2AA^\dagger \Leftrightarrow A^2 + A^\dagger A = 2A$ (notice that $AA^\dagger A = A$, and that $A^2A^\dagger = A^\dagger$ if A is EP) $\Leftrightarrow$ $A^\dagger A = 2A - A^2$ (or, equivalently, $(2I - A)A$ represents the orthogonal projector onto $\mathcal{R}(A^*)$ along $\mathcal{N}(A)$) $\Leftrightarrow I - A^\dagger A = (I - A)^2$

$\Leftrightarrow (I - A)^2$ is the orthogonal projector onto $\mathcal{N}(A)$ along $\mathcal{R}(A^*)$. Because each of the previous equivalent conditions implies that A is EP, our solution is complete.

References

H. J. Werner (2003). Product of two Hermitian nonnegative definite matrices. Solution 29-5.4. *IMAGE: The Bulletin of the International Linear Algebra Society* **30** (April 2003), 25.

H. J. Werner (2005). A property for the sum of a matrix A and its Moore-Penrose inverse A^+. Solution 34.8-3. *IMAGE: The Bulletin of the International Linear Algebra Society* **35** (Fall 2005), 40.

Problem 37-2: Rank of a Generalized Projector

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Let $G \in \mathbb{C}_{n,n}$ partitioned as

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

with $G_{11} \in \mathbb{C}_{k,k}$ and $G_{22} \in \mathbb{C}_{l,l}$, be a generalized projector, i.e., satisfy $G^2 = G^*$, where G^* denotes the conjugate transpose of G. Express the rank of matrix G in terms of ranks of its submatrices.

Solution 37-2.1 by the Proposers Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
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An essential role in our solution is played by Theorem in Puntanen, Styan & Tian (2001), according to which the rank of any $A \in \mathbb{C}_{n,n}$ can be expressed as

$$\operatorname{rank}(A) = \operatorname{rank}(X^*A) + \operatorname{rank}(Y^*A) - \operatorname{rank}(X^*P_AY), \tag{9}$$

where $X \in \mathbb{C}_{n,k}$, $Y \in \mathbb{C}_{n,l}$ are such that the columnwise partitioned matrix $(\, X : Y\,)$ is of full row rank, $X^*Y = 0$, and P_A denotes the orthogonal projector onto the column space of A, i.e., $P_A = AA^\dagger$, where $A^\dagger$ is the Moore-Penrose inverse of A.

Straightforward calculations show that matrix G defined in the problem satisfies $G^2 = G^*$ if and only if

$$G_{11}^2 + G_{12}G_{21} = G_{11}^*,\ G_{11}G_{12} + G_{12}G_{22} = G_{21}^*,\ G_{21}G_{11} + G_{22}G_{21} = G_{12}^*,\ G_{21}G_{12} + G_{22}^2 = G_{22}^*. \tag{10}$$

Another key observation is that the class of generalized projectors is included in the class of hypergeneralized projectors, which is composed of matrices $H \in \mathbb{C}_{n,n}$ satisfying $H^2 = H^\dagger$; see Corollary in Groß & Trenkler (1997). Thus,

$$P_G = G^3. \tag{11}$$

Assume that $X \in \mathbb{C}_{n,k}$ and $Y \in \mathbb{C}_{n,l}$ are of the forms $X^* = (\, I_k : 0\,)$ and $Y^* = (\, 0 : I_l\,)$, where subscripted I denotes the identity matrix of indicated order. Then, in view of (11), it follows that

$$X^*P_GY = (\, G_{11} : G_{12}\,) \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix}, \tag{12}$$

and hence, utilizing the first and second condition in (10), we obtain

$$X^*P_GY = (\, G_{11}^*G_{12} + G_{21}^*G_{22}\,).$$

(Parenthetically note that, with the use of the second and fourth condition in (10), equality (12) yields

$$X^*P_GY = (\, G_{11}G_{21}^* + G_{12}G_{22}^*\,).)$$

In consequence, from (9) we get

$$\operatorname{rank}(G) = \operatorname{rank}((\, G_{11} : G_{12}\,)) + \operatorname{rank}((\, G_{21} : G_{22}\,)) - \operatorname{rank}((\, G_{11}^*G_{12} + G_{21}^*G_{22}\,)).$$

References

J. Groß & G. Trenkler (1997). Generalized and hypergeneralized projectors. *Linear Algebra and Its Applications* **264**, 463-474.

S. Puntanen, G.P.H. Styan & Y. Tian (2001). Two rank equalities associated with blocks of an orthogonal projector. Solution 25-4.3. *IMAGE: The Bulletin of the International Linear Algebra Society* **26** (April 2001), 8-9.

Solution 37-2.2 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

We begin with the following observations.

LEMMA. *Let $G \in \mathbb{C}^{n\times n}$ of rank r be a generalized projector, i.e., let G be such that $G^2 = G^*$. Then,*

(i) $G^3 = G^*G = GG^*$ *is nonnegative definite and Hermitian;*

(ii) $G^4 = (G^*)^2 = (G^2)^* = G$*;*

(iii) $GG^*G = G$ *or, equivalently,* $G^\dagger = G^*$*, with* $G^\dagger$ *denoting the Moore-Penrose inverse of matrix* G*;*

(iv) G *is EP, i.e.,* $\mathcal{R}(G) = \mathcal{R}(G^*)$*, with* $\mathcal{R}(\cdot)$ *denoting the range (column space) of matrix* $(\cdot)$*;*

(v) $G = UTU^*$ *for some column-unitary matrix* $U \in \mathbb{C}^{n\times r}$ *(i.e.,* $U^*U = I_r$*, where* I_r *denotes the identity matrix of order* r*) and some (unitary) matrix* $T \in \mathbb{C}^{r\times r}$ *with* $T^2 = T^*$ *and* $T^*T = I_r$*.*

We notice that condition (v) *is actually not only necessary but also sufficient for* G *to be a generalized projector.*

PROOF. The verifications of the first four results are straightforward. In order to prove (v), let G be a generalized projector of rank r and let $G = UV^*$ be a full-rank factorization of G, where without loss of generality, U can be assumed to be column-unitary. Since, according to (iv), G is EP, V can be written as $V = UT^*$ for some nonsingular matrix T. Hence, $G = UTU^*$. It follows from (i)–(iv) that G^*G is the orthogonal projector onto $\mathcal{R}(G^*) = \mathcal{R}(G) = \mathcal{R}(U)$. Consequently, $G^*G = UT^*TU^* = UU^*$ or, equivalently, $T^*T = I_r$, i.e., T is a unitary nonsingular matrix. Moreover, in view of $G^2 = G^*$, clearly $UT^2U^* = UT^*U^*$ or, equivalently, $T^2 = T^*$. This completes the proof of (v). Conversely, if $G = UTU^*$ for some column-unitary matrix U and some unitary matrix T with $T^2 = T^*$, then $G^2 = G^*$, so that G is, as claimed, a generalized projector. □

Next, let $G \in \mathbb{C}^{n\times n}$ be a generalized projector. Let $r = \operatorname{rank}(G)$. Then, in view of the above Lemma, there exist matrices $U \in \mathbb{C}^{n\times r}$ and $T \in \mathbb{C}^{r\times r}$ such that

$$G = UTU^*,\ U^*U = I_r,\ T^*T = I_r,\ T^2 = T^*.$$

In what follows, let

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

be a symmetrical partitioning of G, i.e., let G_{11} and G_{22} be square matrices of order $s \times s$ and $(n-s)\times(n-s)$, respectively. Clearly, when partitioning $U = (\,U_1^* \quad U_2^*\,)$ in accordance with this block partitioning of G, then G can be rewritten as

$$G = \begin{pmatrix} U_1TU_1^* & U_1TU_2^* \\ U_2TU_1^* & U_2TU_2^* \end{pmatrix},$$

i.e., $G_{ij} = U_iTU_j^*$, for $i, j = 1, 2$. Then,

$$GG^* = UU^* = \begin{pmatrix} U_1U_1^* & U_1U_2^* \\ U_2U_1^* & U_2U_2^* \end{pmatrix}, \tag{13}$$

and $\operatorname{rank}(G) = \operatorname{rank}(GG^*)$. Our above Lemma tells us that GG^* is an idempotent Hermitian matrix. By Theorem (i) in Werner (2005), we know that

$$\operatorname{rank}(U_2U_2^* - U_2U_1^\dagger U_1U_2^*) = \operatorname{rank}(U_2U_2^*) - \operatorname{rank}(U_2U_1^\dagger U_1U_2^*). \tag{14}$$

In passing, we mention that $U_2U_2^* - U_2U_1^\dagger U_1U_2^*$ is the pseudo-Schur complement of $U_1U_1^*$ in matrix GG^*. Letting

$$M = \begin{pmatrix} I_s & -(U_1U_1^*)^\dagger U_1U_2^* \\ 0 & I_{n-s} \end{pmatrix},$$

we have

$$GG^*M = \begin{pmatrix} U_1U_1^* & 0 \\ U_2U_1^* & U_2(I_r - U_1^\dagger U_1)U_2^* \end{pmatrix}. \tag{15}$$

Since M is nonsingular,

$$\operatorname{rank}(GG^*) = \operatorname{rank}(GG^*M) = \operatorname{rank}(U_1U_1^*) + \operatorname{rank}(U_2(I_r - U_1^\dagger U_1)U_2^*). \tag{16}$$

Because $U_1^\dagger U_1$ is a nonnegative definite and Hermitian matrix, by the same token as in IMAGE Solution 37-1.3, $(U_1^\dagger U_1 \mathcal{R}(U_2^*)) \cap \mathcal{N}(U_2) = \{0\}$. Therefore, $\operatorname{rank}(U_2U_1^\dagger U_1U_2^*) = \operatorname{rank}(U_1U_2^*)$. By means of this observation, it now follows from (14) and (16) that

$$\operatorname{rank}(G) = \operatorname{rank}(U_1U_1^*) + \operatorname{rank}(U_2U_2^*) - \operatorname{rank}(U_1U_2^*) \tag{17}$$

or, equivalently, in terms of the blocks in matrix G,

$$\operatorname{rank}(G) = \operatorname{rank}((\, G_{11} \quad G_{12}\,)) + \operatorname{rank}((\, G_{21} \quad G_{22}\,)) - \operatorname{rank}(G_{11}G_{21}^* + G_{12}G_{22}^*).$$

The beautiful rank formula (17) tells us that the rank of matrix G is the sum of the ranks of the two main diagonal blocks minus the rank of one of the off-diagonal blocks in the partitioning (13) of matrix GG^*.

Reference

H. J. Werner (2005). The Schur complement in an orthogonal projector. Solution 34.6-3. *IMAGE: The Bulletin of the International Linear Algebra Society* **35** (Fall 2005), 35–36.

Problem 37-3: Rank of a Nonnegative Definite Matrix

Proposed by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let $A \in \mathbb{C}_{n,n}$ be a nonnegative definite matrix with Moore-Penrose inverse $A^\dagger$. Show that $\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) \geqslant \operatorname{rank}(A)$. When does equality happen?

Solution 37-3.1 by the Proposers Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Equality happens if and only if $\operatorname{rank}(A) \leqslant 1$. The case $A = 0$ is trivial. Let now $\operatorname{rank}(A) = 1$. Then $A = \lambda aa^*$ for some real $\lambda > 0$ and some $a \in \mathbb{C}_{n,1}$ such that $a^*a = 1$. Then $A^\dagger = \lambda^{-1}aa^*$, $\operatorname{trace}(A) = \lambda$, $\operatorname{trace}(A^\dagger) = \lambda^{-1}$, and, consequently,

$$\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) = \lambda \cdot \lambda^{-1} = 1 = \operatorname{rank}(A).$$

Now let $\operatorname{rank}(A) > 1$. Since A is nonnegative definite, it can be expressed in the form

$$A = U\Lambda U^*,$$

where $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_n)$, $\lambda_j \geqslant 0$, $j = 1, ..., n$, and U is unitary. Hence,

$$\operatorname{trace}(A) = \sum_{j \in J} \lambda_j \quad \text{and} \quad \operatorname{trace}(A^\dagger) = \sum_{j \in J} \lambda_j^{-1},$$

where $J = \{j \colon j \in \{1, ..., n\}$ and λ_j is a positive eigenvalue of $A\}$. Since $\operatorname{rank}(A) \geqslant 2$ (i.e., at least two λ_js are greater than zero), $\operatorname{rank}(A) = |J| > 1$ and

$$\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) = (\sum_{j \in J} \lambda_j)(\sum_{i \in J} \lambda_i^{-1}) = \operatorname{rank}(A) + \sum_{j \in J} \sum_{\substack{i \in J \\ i \neq j}} \frac{\lambda_j}{\lambda_i} > \operatorname{rank}(A).$$

Solution 37-3.2 by Johanns de Andrade BEZERRA, *Campina Grande, PB, Brazil:* talita.tao@zipmail.com.br

For given nonnegative definite matrix $A \in \mathbb{C}^{n \times n}$, let $r = \operatorname{rank}(A)$. Then A can be written as $A = U\Lambda U^*$, where U is a unitary matrix of order n and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$ contains the eigenvalues of A in decreasing order. Since A of rank r is nonnegative definite, clearly $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Notice that the Moore-Penrose inverse $A^\dagger$ can be written as $A^\dagger = U\Lambda^\dagger U^*$, where $\Lambda^\dagger = \operatorname{diag}(\lambda_1^{-1}, \ldots \lambda_r^{-1}, 0, \ldots, 0)$. Consequently, $\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) = (\lambda_1 + \cdots + \lambda_r) \cdot (\lambda_1^{-1} + \cdots + \lambda_r^{-1}) = r + a \geq r$

for suitable $a \geq 0$. From this representation it directly follows that $\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) = r$ or, equivalently, $a = 0$ holds if and only if $r = \operatorname{rank}(A) \leq 1$.

Solution 37-3.3 by Sachindranath JAYARAMAN, *Indian Institute of Technology, Madras, India:* sachinj@smail.iitm.ac.in

We begin by recalling the following four well-known facts:

(1) If $B, C \in \mathbb{C}^{n\times n}$ are nonnegative definite and Hermitian, then $\operatorname{trace}(BC) \leq \operatorname{trace}(B) \cdot \operatorname{trace}(C)$.

(2) If A is nonnegative definite and Hermitian, then its Moore-Penrose inverse $A^\dagger$ is also so.

(3) For any complex matrix A, $\operatorname{trace}(AA^\dagger) = \operatorname{rank}(A)$.

(4) For any complex matrix A, $(AA^\dagger)^2 = AA^\dagger = \left(AA^\dagger\right)^* = \left(AA^\dagger\right)^\dagger$.

Now let A be a nonnegative definite and Hermitian matrix. Then, in view of facts (1)–(3), as claimed

$$\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) \geq \operatorname{rank}(A). \tag{18}$$

Next, put $B = C = AA^\dagger$. Then, according to facts (1)–(4), $\operatorname{trace}(AA^\dagger) \leq \left(\operatorname{trace}(AA^\dagger)\right)^2$ or, equivalently, $\operatorname{rank}(A) \leq (\operatorname{rank}(A))^2$. This in turn shows that equality can generally happen in (18) only if $\operatorname{rank}(A) \leq 1$. For proving the converse, let A be a nonnegative definite and Hermitian matrix of rank less than or equal to 1, i.e., let A be of the form $A = \lambda uu^*$, where $u^*u = 1$ and $\lambda \geq 0$. Then $A^\dagger = \lambda^\dagger uu^*$, where $\lambda^\dagger = 0$ if $\lambda = 0$, and where $\lambda^\dagger = \lambda^{-1}$ if $\lambda > 0$. Because $\operatorname{trace}(A) = \lambda$ and $\operatorname{trace}(A^\dagger) = \lambda^\dagger$, it is easy to see that in this case we have $\operatorname{trace}(A) \cdot \operatorname{trace}(A^\dagger) = \operatorname{rank}(A)$.

Problem 37-4: Do Singular Values Dominate Eigenvalues?

Proposed by David CALLAN, *University of Wisconsin-Madison, Madison, Wisconsin, USA:* callan@stat.wisc.edu

Do singular values dominate eigenvalues? More specifically, suppose A is an $n \times n$ complex matrix with singular values $s_1 \geq s_2 \geq \ldots \geq s_n \geq 0$ and eigenvalues $x_1, x_2, ..., x_n$ ordered by decreasing absolute value. Is it true that $s_1 + s_2 + ... + s_k \geq |x_1| + |x_2| + ... + |x_k|$ for each $k = 1, 2, ..., n$? It is true in the two special cases (i) $k = 1$, (ii) $n = 2$.

Editorial Note. We received emails from Roger A. Horn, David London, and Joao Filipe Queiro that the question addressed in Problem 37-4 has been solved a long time ago; see, for instance, Horn & Johnson (1991, Theorem 3.3.13(a)).

Reference

R. A. Horn & Ch. R. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press, Cambridge.

Problem 37-5: A Generalised Matrix Transformation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ matrix and let m be an integer in the range $1 \leq m \leq n$. Then we may define the mth row of A by $\{a_{mj}\}$, the mth column of A by $\{a_{im}\}$, the mth primary (or dexter) cyclic-diagonal of A by $\{a_{ij} \,:\, i = j + m (\operatorname{mod} n)\}$, and the mth secondary (or sinister) cyclic-diagonal of A by $\{a_{ij} \,:\, i + j = m + 1 (\operatorname{mod} n)\}$. In each case, the nth primary or secondary cyclic-diagonal of A may be named the principal primary or secondary diagonal as they are also the familiar non-cyclical diagonals of the matrix.

If n is odd then each row has a single intersection with each column and *vice versa* and each primary diagonal has a single intersection with each secondary diagonal and *vice versa*. For the case $n = 3$, identify a transformation which carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column. Outline at least one possible application of your chosen transformation.

Solution 37-5.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Consider the 3×3 matrix:

$$A = \begin{pmatrix} (1,2) & (3,3) & (2,1) \\ (2,3) & (1,1) & (3,2) \\ (3,1) & (2,2) & (1,3) \end{pmatrix}.$$

The transformation represented by this matrix carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column, as required. Further, this transformation clearly exhibits a fixed point at (3, 1) combined with an eight period cycle which carries (1, 1) into (1, 2), (1, 2) into (3, 3), (3, 3) into (1, 3), (1, 3) into (2, 1), (2, 1) into (2, 3), (2, 3) into (3, 2), (3, 2) into (2, 2), (2, 2) into (1, 1), and so on.

Collecting the results relating to second successors, we obtain the matrix:

$$\begin{pmatrix} (3,3) & (1,3) & (2,3) \\ (3,2) & (1,2) & (2,2) \\ (3,1) & (1,1) & (2,1) \end{pmatrix}$$

from which we may deduce that two applications of this transformation carries rows 1, 2 and 3 into columns 3, 1 and 2, and columns 1, 2 and 3 into rows 3, 2 and 1. Similarly, on collecting the results relating to fourth successors, we have the matrix:

$$\begin{pmatrix} (2,1) & (2,3) & (2,2) \\ (1,1) & (1,3) & (1,2) \\ (3,1) & (3,3) & (3,2) \end{pmatrix}$$

from which we may deduce that four applications of this transformation carries rows 1, 2 and 3 into rows 2, 1 and 3, and columns 1, 2 and 3 into columns 1, 3 and 2; and hence that eight applications of this transformation restores every element to its original position.

This technique may readily be generalised to matrices of higher order provided that n is odd. A generalization to matrices of even order is more problematical as each primary diagonal has two intersections with each of $n/2$ secondary diagonals and *vice versa* when n is even.

APPLICATION: Let $e_1, e_2, ..., e_9$ be a set of nine distinct elements, then a minor generalization of the standard 9×9 *sudoku* problem concerns the possibility of inserting nine copies of each of these elements into a 9×9 array C in such a way that each row contains all nine elements, each column contains all nine elements, and each of nine contiguous 3×3 submatrices contains all nine elements. In the present application, we are concerned with a further generalisation of the standard problem that also requires that the principal primary and secondary diagonals of the 9×9 matrices C, P_*C and P_*^2C each contains all nine elements where P is the 3×3 rotation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which carries the first row of a $3 \times m$ matrix into its third row, its second row into its first row, and its third row into its second row, and P_* is the 9×9 Kronecker product $P_* = P \otimes I_3$ that performs a similar service for blocks of three rows in a $9 \times m$ matrix.

Given any arrangement of the nine elements $e_1, e_2, ..., e_9$, we define the 3×3 matrix B:

$$B = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}.$$

In this context, the values of i and j in a typical element (i, j) from our transformation serve to define the row and column rotations required to bring the $h = 3(i-1)+j$th element of B into the upper left corner of the transformed matrix $M_h = P^{i-1}BQ^{j-1}$ where $Q = P' = P^{-1} = P^2$. Further, on substituting the 3×3 matrix M_h for the symbol (i, j) in our selective transformation, we have the 9×9 matrix:

$$\begin{pmatrix} M_2 & M_9 & M_4 \\ M_6 & M_1 & M_8 \\ M_7 & M_5 & M_3 \end{pmatrix}$$

or

2	3	1	9	7	8	4	5	6
5	6	4	3	1	2	7	8	9
8	9	7	6	4	5	1	2	3
6	4	5	1	2	3	8	9	7
9	7	8	4	5	6	2	3	1
3	1	2	7	8	9	5	6	4
7	8	9	5	6	4	3	1	2
1	2	3	8	9	7	6	4	5
4	5	6	2	3	1	9	7	8

where, for simplicity, we have omitted the es. This 9×9 matrix clearly solves the generalised *sudoku* problem defined above. Additional solutions may be obtained by making an alternative choice of a 3×3 transformation matrix A or by allocating the nine given elements to the matrix B in a different order. In particular, the substitution of $AQ,\ AQ^2,\ PA,\ PAQ,\ PAQ^2,\ P^2A,\ P^2AQ$ or P^2AQ^2 for A results in a similar rotation of the block rows and block columns of the 9×9 matrix C.

It is not clear whether the technique described here will serve to identify all possible solutions in the 9×9 case. However, it is certainly sufficient in the 4×4 case with $c_{11},\ c_{12},\ c_{21}$ and c_{22} given, as this problem has only two solutions corresponding to the choice of $c_{33} = c_{12}$ or $c_{33} = c_{21}$.

APPENDIX. From each starting value, the chosen transformation yields the following sequences:

(1, 1)	(1, 2)	(3, 3)	(1, 3)	(2, 1)	(2, 3)	(3, 2)	(2, 2)	(1, 1)
(1, 2)	(3, 3)	(1, 3)	(2, 1)	(2, 3)	(3, 2)	(2, 2)	(1, 1)	(1, 2)
(1, 3)	(2, 1)	(2, 3)	(3, 2)	(2, 2)	(1, 1)	(1, 2)	(3, 3)	(1, 3)
(2, 1)	(2, 3)	(3, 2)	(2, 2)	(1, 1)	(1, 2)	(3, 3)	(1, 3)	(2, 1)
(2, 2)	(1, 1)	(1, 2)	(3, 3)	(1, 3)	(2, 1)	(2, 3)	(3, 2)	(2, 2)
(2, 3)	(3, 2)	(2, 2)	(1, 1)	(1, 2)	(3, 3)	(1, 3)	(2, 1)	(2, 3)
(3, 1)	(3, 1)	(3, 1)	(3, 1)	(3, 1)	(3, 1)	(3, 1)	(3, 1)	(3, 1)
(3, 2)	(2, 2)	(1, 1)	(1, 2)	(3, 3)	(1, 3)	(2, 1)	(2, 3)	(3, 2)
(3, 3)	(1, 3)	(2, 1)	(2, 3)	(3, 2)	(2, 2)	(1, 1)	(1, 2)	(3, 3)

Problem 37-6: Characterization of EP-ness

Proposed by Götz TRENKLER, *Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Let A be a square matrix with complex entries. Show that A is an EP matrix if and only if

$$AAA^{+}A^{+} = AA^{+}.$$

Solution 37-6.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl
and the Proposer Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Similarly as Solution 37-1.1, also the present one is based on Corollary 6 in Hartwig & Spindelböck (1984). Using matrices (1) and (2) given above, condition $AAA^{+}A^{+} = AA^{+}$ can be equivalently expressed as $\Sigma KK^{*}\Sigma^{-1} = I_r$, or, in other words, $KK^{*} = I_r$. In view of $KK^{*} + LL^{*} = I_r$, the latter condition is equivalent to $L = 0$, which, as pointed out in Solution 37-1.1, holds if and only if A is EP.

Reference

R.E. Hartwig & K. Spindelböck (1984). Matrices for which A^{*} and $A^{\dagger}$ commute. *Linear and Multilinear Algebra* **14**, 241-256.

Solution 37-6.2 by Hans Joachim WERNER, *University of Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

For a matrix $A \in \mathbb{C}^{m\times n}$, let A^*, $A^\dagger$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the Moore-Penrose inverse, the range (column space), and the null space, respectively, of A. Below we will make use of the following well-known facts: (i) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, (ii) $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, (iii) $AA^\dagger A = A$, and (iv) $A \in \mathbb{C}^{n\times n}$ is EP $\Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A) \Leftrightarrow AA^\dagger = A^\dagger A$.

Now, let A be EP. By means of (iii) and characterization (iv), we then obtain $AAA^\dagger A^\dagger = AA^\dagger AA^\dagger = AA^\dagger$, thus showing that

$$AAA^\dagger A^\dagger = AA^\dagger \tag{19}$$

is indeed a necessary condition for A to be an EP matrix. For proving the converse, let $A \in \mathbb{C}^{n\times n}$ be such that equation (19) or, equivalently, equation $A(I - AA^\dagger)A^\dagger = 0$ is satisfied. We recall that $I - AA^\dagger$ is the (orthogonal) projector onto $\mathcal{N}(A^*)$ along $\mathcal{R}(A)$ in $\mathbb{C}^n$ and that $[\mathcal{R}(A) + \mathcal{R}(A^*)] \oplus [\mathcal{N}(A^*) \cap \mathcal{N}(A)] = \mathbb{C}^n$, with $\oplus$ indicating a direct sum. Therefore, in particular,

$$[\mathcal{R}(A) + \mathcal{R}(A^*)] \cap [\mathcal{N}(A^*) \cap \mathcal{N}(A)] = \{0\}.$$

By means of this observation plus the facts (i)–(iii), clearly

$$A(I - AA^\dagger)A^\dagger = 0 \Leftrightarrow \mathcal{R}(A^*) \subseteq \mathcal{R}(A) + [\mathcal{N}(A^*) \cap \mathcal{N}(A)],$$

which can happen only if $\mathcal{R}(A^*) \subseteq \mathcal{R}(A)$ or, equivalently, only if $\mathcal{R}(A^*) = \mathcal{R}(A)$, and so our solution to this problem is complete.

A Solution to Problem 37-6 was also received from Johanns de Andrade Bezerra.

IMAGE Problem Corner: Three More New Problems

Problem 39-4: On (α, β)-normal Operators in Hilbert Spaces
Proposed by Mohammad Sal MOSLEHIAN, *Ferdowsi University, Mashhad, Iran:* moslehian@ferdowsi.um.ac.ir

An operator T acting on a (separable) Hilbert space is called (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ if

$$\alpha^2 T^*T \le TT^* \le \beta^2 T^*T.$$

For fixed $\alpha > 0$ and $\beta \neq 1$,

(i) give an example of an (α, β)-normal operator which is neither normal nor hyponormal;
(ii) is there any "nice" relation between norm, numerical radius and spectral radius of an (α, β)-normal operator?

Problem 39-5: Two Commutativity Equalities for the Regularized Tikhonov Inverse
Proposed by Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

The regularized Tikhonov inverse of a complex matrix A with respect to a positive number μ is defined to be

$$A^\# = (\,\mu I + A^*A\,)^{-1}A^*,$$

where A^* is the conjugate transpose of A. Suppose A is a square matrix of order n. Show that

(i) $AA^\# = A^\# A \Leftrightarrow AA^* = A^*A.$
(ii) $A^*A^\# = A^\# A^* \Leftrightarrow AA^*A^2 = A^2A^*A.$

Problem 39-6: Two Equalities for the Moore-Penrose Inverse of a Row Block Matrix
Proposed by Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Suppose A and B are two $m \times n$ and $m \times p$ matrices, respectively, and let $(\cdot)^*$ and $(\cdot)^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse of a complex matrix, respectively. Show that

(i) $(\,A \quad B\,)(\,A \quad B\,)^\dagger = \frac{1}{2}(AA^\dagger + BB^\dagger) \Leftrightarrow \mathrm{range}(A) = \mathrm{range}(B).$
(ii) $(\,A \quad B\,)^\dagger = \frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} \Leftrightarrow AA^* = BB^*.$

Problems 39-1 through 39-3 are on page 32.

IMAGE Problem Corner: New Problems

Please submit solutions, as well as new problems, both (a) in macro-free LaTeX by e-mail to hjw.de@uni-bonn.de, preferably embedded as text, and (b) with one paper copy by regular mail to Hans Joachim Werner, IMAGE Editor-in-Chief, Department of Statistics, Faculty of Economics, University of Bonn, Adenauerallee 24-42, D-53113 Bonn, Germany.

Problem 39-1: A Simple Procedure for a Generalised Matrix Transformation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ matrix and let m be an integer in the range $1 \leq m \leq n$. Then we may define the mth row of A by $\{a_{mj}\}$, the mth column of A by $\{a_{im}\}$, the mth primary (or dexter) cyclic-diagonal of A by $\{a_{ij} \,:\, i = j + m (\operatorname{mod} n)\}$, and the mth secondary (or sinister) cyclic-diagonal of A by $\{a_{ij} \,:\, i + j = m + 1 (\operatorname{mod} n)\}$. In each case, the nth primary or secondary cyclic-diagonal of A may be named the principal primary or secondary diagonal as they are also the familiar non-cyclical diagonals of the matrix. [That is in the context of IMAGE Problem 37-5.]

(i) For the case n odd, identify a simple procedure for generating a transformation which carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column.

(ii) Illustrate your procedure in the case $n = 5$.

Problem 39-2: On the Range of A^+A^*

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:*, Shuangzhe.Liu@canberra.edu.au,
and Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

For an $n \times n$ matrix A with complex entries, let $\Phi(A) = A^+A^*$, where A^+ is the Moore-Penrose inverse and A^* is the conjugate transpose of A, respectively. Assume that A is EP, i.e., A and A^+ have identical column spaces. Show that the following statements are equivalent:

(i) $\Phi(A)$ is a partial isometry.

(ii) A is normal.

(iii) $A\Phi(A) = \Phi(A)A$.

REMARK: The present problem generalizes Proposition 1 in C. R. DePrima & C. R. Johnson, The range of $A^{-1}A^*$ in $GL(n, \mathbb{C})$, *Linear Algebra and Its Applications* (1974) **9**, 209-222.

Problem 39-3: An Inequality Involving the Khatri-Rao Sum of Positive Definite Matrices

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:*, Shuangzhe.Liu@canberra.edu.au,
Götz TRENKLER, *University of Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de,
and Zhongpeng YANG, *Putian University, Putian China:* yangzhongpeng@126.com

For two $m \times m$ positive definite matrices A and B, the Khatri-Rao sum is $A \diamond B = A * I_m + I_m * B$, where $*$ denotes the Khatri-Rao product and I_m is the $m \times m$ identity matrix. Let $n \times n$ matrix $S = A \diamond B^{-1} + A^{-1} \diamond B$, where A, A^{-1}, B and B^{-1} are compatibly partitioned. Prove the following inequality in the Löwner sense

$$4I_n \leq S \leq \frac{2(\lambda + \mu)}{\sqrt{\lambda\mu}} I_n,$$

where I_n is the $n \times n$ identity matrix, λ and μ are positive scalars such that all the eigenvalues of A, A^{-1}, B and B^{-1} are contained in the interval [λ, μ].

REMARK: For the notations, see e.g. S. Liu and G. Trenkler, Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal for Information & Systems Sciences* (2008) **4**, 160–177 in which our inequality is collected, and Z.P. Yang, X. Zhang and C.G. Cao, Inequalities involving Khatri-Rao products of Hermitian matrices, *Korean Journal of Computational & Applied Mathematics* (2002) **9**, 125–133.

Problems 39-4 through 39-6 are on page 31.