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IMAGE

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UPCOMING CONFERENCES AND WORKSHOPS

Workshop on Spectral Graph Theory With Applications to Computer Science, Combinatorial Optimization and Chemistry Military Institute of Engineering Rio de Janeiro, Brazil, December 1-4, 2008

The Scientific Committee is:

- B. Mohar, Simon Fraser University, Canada;
- Chris Godsil, University of Waterloo, Canada;
- Domingos M. Cardoso, University of Aveiro, Portugal;
- Dragos Cvetkovic, University of Belgrade, Serbia (Chairman);
- E. Hancock, University of York, England, UK;
- Ivan Gutman, University of Kragujevac, Serbia;
- Jayme Luiz Szwarcfiter, Federal University of Rio de Janeiro, Brazil;
- Nair Abreu, Federal University of Rio de Janeiro, Brazil;
- Peter Rowlinson, University of Stirling, Scotland;
- Pierre Hansen, University of Montreal, Canada;
- Richard A. Brualdi, University of Wisconsin - Madison, USA;
- S. K. Simić, University of Belgrade, Serbia;
- Steve Kirkland, University of Regina, Canada;
- Willem H. Haemers, Tilburg University, The Netherlands.

Visit http://www.sgt.pep.ufrj.br/~tegrio/ for details.

Conference on Numerical Analysis and Scientific Computation with Applications Agadir, Morocco, May 18-22, 2009

Submitted by Hassane Sadok

The main topics will be large linear systems and preconditioning; eigenvalue problems; high-performance and parallel computation; linear algebra and control; model reduction; multigrid and multilevel methods; numerical methods for PDEs': Finite element, Finite volume, Meshless methods; approximation; radial basis functions; scattered data approximation; splines; optimization; applications: image processing, financial computation, learning machines.

The plenary speakers will be M. Buhmann, Germany; C. Brezinski, France; B. Datta, USA; B. Philippe, France; L. Reichel, USA; Y. Saad, USA; M. Seaid, UK; D. Silvester, UK; D. Szyld, USA; and C. Vasquez, Spain.

Important Dates:
January 15, 2009 - Deadline for titles and abstracts
March 1, 2009 - Deadline for early registration
October 1, 2009 - Full paper submission.

The proceedings will be a special volume of Applied Numerical Mathematics, subject to regular refereeing. For details, visit the web page: http://www-lmpa.univ-littoral.fr/NASCA09/ or contact us at nasca09@lmpa.univ-littoral.fr.

MAT-TRIAD 2009 -3rd International Workshop on Matrix Theory and Its Applications
Będlewo, Poland, March 23-27, 2009

The MAT-TRIAD 2009 will be held at the Mathematical Research and Conference Center of the Polish Academy of Sciences, in Będlewo near Poznań, Poland. The purpose of the workshop is to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications in other parts of mathematics and offer them a possibility to discuss current developments in these subjects. Presentations of both invited and contributed papers will be included. The list of invited speakers will be opened by four winners of Young Scientists Awards of MAT-TRIAD 2007.

Two lecture series will be given on Iterative Methods for Large Scale Linear Feasibility Problems, by Andrzej Cegielski, and Special Classes of Matrices, by Charles R. Johnson. For more details, visit http://mtriad09.amu.edu.pl/.

3rd International Conference on Mathematics and Statistics
The Mathematics & Statistics Research Unit of ATINER
June 15-18, 2009, Athens, Greece

Papers from all areas of Mathematics, Statistics, Mathematics and Engineering, and Mathematics and Education are welcome. Send an abstract of about 300 words, via email to Dr. Vladimir Akis, Head, Mathematics & Statistics Research Unit, Athens Institute for Education and Research (ATINER), atiner@atiner.gr. Dr. Akis is also a Professor of Mathematics and Computer Science at UCLA.

To help evaluate papers, contribute to editing, or chair a session, email Dr. Gregory T. Papanikos (gtp@atiner.gr), Director, ATINER. For details, including registration, preliminary program and excursions, visit http://www.atiner.gr/.

15th Haifa Matrix Theory Conference
Technion Center for Mathematical Sciences
Haifa, Israel, May 18-21, 2009

This will be the fifteenth in a sequence of matrix theory conferences held at the Technion since 1984. As in the past, all talks will be of 30 minutes duration, and the topics will cover a wide spectrum of theoretical and applied linear algebra. This conference will also host a special session on the teaching of linear algebra.

Send titles and abstracts of contributed talks by December 31, 2008, to any member of the Organizing Committee:

Abraham Berman (berman@tx.technion.ac.il),
Daniel Hershkowitz, Co-chair
(hershkow@techunix.technion.ac.il)
Raphael Loewy, Co-chair (loewy@techunix.technion.ac.il).

3rd International Workshop on Matrix Analysis and Applications
Hangzhou, China, July 9-13, 2009

The Third International Workshop on Matrix Analysis and Applications will be held in Hangzhou (Lin'An), China, July 9-13, 2009, under the auspices of Zhejiang Forestry University, Zhejiang Province, China. The purpose of this conference is to stimulate research, providing an opportunity for researchers to present their newest results and to meet for informal discussions. The previous meetings of the workshop series have taken place in Beijing, China and Fort Lauderdale, FL, USA.

The Keynote Speaker will be Chi-Kwong Li, Ferguson Professor of Mathematics at The College of William and Mary, USA. The Electronic Journal of Linear Algebra (ELA) will devote a special issue to the conference, and abstracts of the talks will be published by World Academic Press. This meeting is sponsored by the National Natural Science Foundation of China and Zhejiang Forestry University, and is endorsed by the International Linear Algebra Society.

Chairs of the organizing committee are Changqing Xu (Zhejiang Forestry University, China cqxurichard@163.com), Guanghui Xu (Zhejiang Forestry University, China) and Fuzhen Zhang (Co-chair, Nova Southeastern University, Ft Lauderdale, USA. zhang@nova.edu). For more information and updates, visit www.nova.edu/~zhang/09MatrixWorkshop.html.

3rd International Symposium on Positive Systems (POSTA09)
Valencia, Spain, September 2-4, 2009

Rafael Bru, Universidad Politecnica de Valencia,
Chairman of Organizing Committee

POSTA09 aims to be a multidisciplinary forum where leading researchers may report on recent advances and developments in different fields related to positive systems such as Compartmental Systems, Markov Models, Biological Models, Max-Plus Algebra, 1D and 2D Systems, Nonnegative Matrices, among others.

We invite you to submit a paper and/or to propose an invited session. All accepted papers will appear in the conference proceedings published in a special volume of Lecture Notes in Control and Information Sciences.

Deadline for Invited Sessions: November 22, 2008
Deadline for papers: January 20, 2009

At least one author for each paper should be registered by April 15, 2009. For further details visit the conference web page http://posta09.webs.upv.es/.

Other Conferences of Interest

January 5-8, 2009: AMS/MAA Joint Meetings, Special Session on Spectra of Matrix Patterns and Applications to Dynamical Systems, Washington, DC, USA.
July 6-8, 2009: SIAM Conference on Control and Its Applications, Denver, CO, USA.
July 6-10, 2009: SIAM Annual Meeting, Denver, CO, USA.
October 26-29, 2009: SIAM Conference on Applied Linear Algebra, Seaside, CA, USA. ILAS speakers: Shaun Fallat and Christian Mehl.
June 21-25, 2010: ILAS, Pisa, Italy. Contact: Dario Bini.
July 27-31, 2010: LinStat 2010, Tomar, Portugal
June or August, 2011: ILAS, Braunschweig, Germany. Contact: Heike Fassbender.

REPORTS ON CONFERENCES AND WORKSHOPS

Report on ILAS 15
June 16-20, 2008, Cancun, Mexico

The ILAS Conference in Cancun was very successful. There were 268 participants, 13 plenary speakers, 90 contributed talks, and another 85 talks in the eight mini-symposia. An ILAS business meeting was also held, chaired by new President Steve Kirkland. We extend warm appreciation to Luis Verde-Star, Chair of the Organizing Committee, and to all who served on that committee as well as those who helped with arrangements.

Former and current ILAS Presidents in Cancun: Hans Schneider, Richard Brualdi, Danny Hershkowitz, and Steve Kirkland

The LAMA Lecture, by Juan Manuel Peña, was "From Total Positivity to Positivity: Related Classes of Matrices," and the SIAM Linear Algebra Lecture, by James Nagy, was "Kronecker Products in Imaging Sciences."

The Hans Schneider Prize winners for 2008, Cleve Moler and Beresford Parlett, were not able to attend this meeting. It is expected that the formal presentation of these awards will be made at the upcoming ILAS Conference in Pisa, Italy, in June 2010.

The weather was warm and beautiful. Social events included an excursion to the Mayan archeological site of Tulúm and a closing banquet. More conference photos can be found on the conference website http://star.izt.uam.mx/ILAS08/.

Participants in ILAS 15, Cancun, Mexico, June 16-20, 2008

A Conference Celebrating Richard Varga's Eightieth Birthday

By Daniel Szyld

About a hundred scientists and students from fifteen European countries, Canada, Israel, Lebanon, and the U.S., gathered in Kalamata, in the Southwestern Peloponnese, Greece, during the first week of September 2008 to honor Richard S. Varga on the occasion of his eightieth birthday. The five-day conference was called ``Numerical Analysis 2008: Recent approaches to Numerical Analysis, Theory, Methods, and Applications." This followed two earlier regional conferences of similar theme, one of which was in the same location. The organizers were Georgios Akrivis, Stratis Gallopoulos, Apostolos Hadjidimos, Ilias Kotsireas, Dimitrios Noutsos, and Michael Vrahatis.

There were sixteen plenary speakers and eleven contributed sessions, covering diverse topics in numerical analysis. As is to be expected in a meeting honoring Richard Varga, many talks were on areas of applied and numerical linear algebra. Topics included methods for inverse problems, image reconstruction, regularization, computerized tomography, data mining, matrix functions, matrix polynomials, approximations, linear systems of special structures, and of course (apropos of Varga's latest book on this theme), Geršgorin circles, eigenvalue estimates, and generalizations of H-matrices.

First row: Richard Varga (center) with his wife (right), daughter and grandchildren (left). On the far right is the main organizer Michael Vrahatis and his wife.

There were lively scientific discussions during coffee breaks. It was particularly rewarding to see groups of young researchers from several countries coalescing, in what one hopes will be long-term contacts. This nice outcome may have been prompted by the blessing the participants received from the Bishop of the Messinia Diocese during the opening ceremony. As another testament to the collaborative climate of the meeting, it was uplifting to notice that one afternoon session began with a speaker from Russia, and ended with a speaker from Georgia. With no parallel sessions, the schedule was packed from 9:00 until 20:00 most days. However, many participants were seen going to the beach just across the street from the hotel for an early swim before breakfast, and walking in groups along the street bordering the beaches before the morning heat or after the sessions on the way to some Greek tavern.

There was an excursion to Ancient Messini, which has been excavated and restored over the last thirty years. Our guides were people who currently work on the excavations and restorations. This cultural visit was followed by a dinner which overlooked Messini and the rows of olive trees in nearby mountains. We enjoyed the first of two cakes and singing ``Happy Birthday Richard." The second celebration was at the banquet the next night, and it was noted that the celebration would continue at another conference to be held in October at Kent State University. Before the banquet, an exquisite classical music concert was performed by a trio: piano, cello, and a soprano. Most of their pieces were by Greek composers of the twentieth century, including Skalkotas, Konstantinides and Hadjidakis, and some ``Greek dances" by Maurice Ravel.

A discussion session with no preconceived agenda closed the meeting, and this gave rise to diverse comments on teaching methodologies for numerical analysis courses, and experiments with project-based teaching in various countries. Everyone seemed happy with the quality of the conference, and many praised the organizers effusively, saying that the high organizational standards encountered in Kalamata would be very hard to match in other conferences. For a complete list of talks and a photo gallery, visit the conference web site http://www.math.upatras.gr/numan2008. The organizers plan to have international meetings in Greece like this one approximately every two years. Many attendees will be happy to return.

A Second Conference Celebrating Richard Varga's Eightieth Birthday

By Daniel Szyld

Since 1970, people have been making a pilgrimage to Kent State University to visit and work with Richard S. Varga. On October 17-18, 2008, the week after his eightieth birthday, this milestone was commemorated when about sixty mathematicians from the U.S. and Europe made this pilgrimage once more for a two-day conference. Many in the audience had attended the conferences for Varga's 60th and 70th birthdays, but some of the young participants had not even finished primary school twenty years ago.

This conference was fully supported by Kent State University, and featured 36 talks representing the many diverse areas of mathematics in which Varga has worked during his distinguished career, including approximation theory, complex analysis, numerical analysis, numerical linear algebra, iterative methods, and the numerical solution of differential equations. For the speakers and titles, visit the conference web site: http://math.kent.edu/~li/RSV80.

Lothar Reichel organized this event, with the help of Laura Dykes, Paul Farrell, Jing Li, Arden Ruttan, and Laura Smithies, all from Kent State University. As has been the case for other conferences at Kent, they fully succeeded in making sure all participants enjoyed the event.

There was a great piano concert, performed by Per Enflo, and a banquet with some tributes to Varga from his local colleagues. Daniela Calvetti, Chair of the Department of Mathematics at nearby Case Western Research University, presented Varga with a commemorative plaque. He obtained his undergraduate degree from Case Western in 1950.

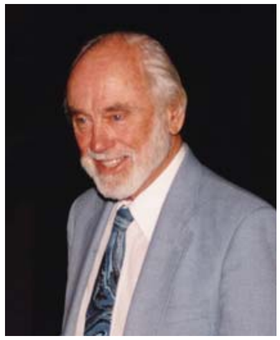
One of the "pilgrims" to Kent was Volker Mehrmann, one of the 25 Ph.D. students who finished under Varga's guidance. Mehrmann regaled the audience with an after-dinner speech showing photos of Varga covering many decades, and prepared "quizzes" for the audience, to identify people in some of the pictures. Mehrmann also reported that a special issue of Linear Algebra and its Applications dedicated to Varga on the occasion of his 80th birthday had just been published. This is volume 249, issue 10, 2008, and is available online.

Taken together, conferences and special issues are fitting tributes to Richard Varga. He is a leader in our profession, an ILAS Hans Schneider Prize recipient, a generous man, and a role model for many friends and colleagues.

Participants, Kent State University Conference in Honor of Richard Varga's 80th Birthday

17th International Workshop on Matrices and Statistics (IWMS) in Celebration of T.W. Anderson's 90th Birthday: Tomar, Portugal, 23–26 July 2008

Francisco Carvalho, Simo Puntanen & George P. H. Styan



The 17th International Workshop on Matrices and Statistics (IWMS'08) http://www.iwms08.ipt.pt/ was held in Tomar, Portugal, 23–26 July 2008. This Workshop was dedicated to Professor Theodore Wilbur Anderson, who celebrated his 90th birthday on 5 June 2008. Ten years ago, the 7th International Workshop on Matrices and Statistics was held in Fort Lauderdale, Florida, USA, 11–14 December 1998, in celebration of T. W. Anderson's 80th Birthday.

T. W. Anderson was born on June 5, 1918, in Minneapolis, Minnesota. He received a B.S. in mathematics from Northwestern University in 1939, and a Ph.D. in 1945 in mathematics from Princeton University, where his supervisor was Samuel S. Wilks. He was with the Cowles Commission at the University of Chicago, and then in 1946 he became a faculty member at Columbia University. Since 1967 he has been in his present position as a Professor of Statistics and Economics at Stanford University. T. W. Anderson has received many honors: a Guggenheim Fellow, Editor of the *The Annals of Mathematical Statistics*, President of the Institute of Mathematical Statistics, Vice-President of the American Statistical Association, an honorary degree from Sweden and one from Northwestern University. He is a Fellow of the American Academy of Arts and Sciences and a member of the National Academy of Sciences.

The International Organizing Committee (IOC) for IWMS'08 comprised George P. H. Styan (Montréal, Canada) – Honorary Chair, Simo Puntanen (Tampere, Finland) – Chair, Augustyn Markiewicz (Poznań, Poland) – Vice-Chair, S. Ejaz Ahmed (Windsor, Canada), Jeffrey J. Hunter (Auckland, New Zealand), Götz Trenkler (Dortmund, Germany), Dietrich von Rosen (Uppsala, Sweden), and Hans Joachim Werner (Bonn, Germany). The members of the Local Organizing Committee (LOC) were João T. Mexia (Lisbon, Portugal) – Chair, Carlos A. Coelho (Lisbon, Portugal) – Vice-Chair, Katarzyna Filipiak (Poznań, Poland), Ricardo Covas (Tomar, Portugal), Miguel Fonseca (Lisbon, Portugal), and Francisco Carvalho (Tomar, Portugal).

The organizing institutes were the New University of Lisbon and the Polytechnical Institute of Tomar, with sponsors the Faculty of Sciences and Technology (New University of Lisbon), Center of Mathematics and Applications (New University of Lisbon), National Institute of Statistics – INE, SPSS, Delta, Hotel Dos Templários (Tomar), Luso-American Foundation, the International Linear Algebra Society, Science and Technology Foundation, UNICER, and UNILEVER.

The 190-page *Book of Abstracts* for IWMS'08 was edited by Katarzyna Filipiak and Francisco Carvalho, and includes not only the abstracts of the talks and the full program, but also the following articles:

- "A short history of the International Workshop on Matrices and Statistics (IWMS)", by Simo Puntanen & George P. H. Styan, http://mtl.uta.fi/iwms/
- "Ted W. Anderson, The Multivariate Man" by Yadolah Dodge, *Student*, 3 (1999), 133–140,
- "A Conversation with T. W. Anderson" by Morris H. DeGroot, *Statistical Science*, 1 (1986), 97–105.

Invited talks were given by S. Ejaz Ahmed (Canada), T. W. Anderson (USA), Ravindra B. Bapat (India) – the ILAS Speaker, Adi Ben-Israel (USA), Carlos A. Braumann (Portugal), Tadeusz Caliński (Poland), Charles R. Johnson (USA), Lynn Roy LaMotte (USA), Augustyn Markiewicz (Poland), Stanisław Mejza (Poland), Michael D. Perlman (USA), Dinis Pestana (Portugal), Dietrich von Rosen (Sweden), Muni S. Srivastava (Canada), George P. H. Styan (Canada), Götz Trenkler (Germany), Júlia Volaufová (USA), Douglas P. Wiens (Canada), and Roman Zmyślony (Poland). The Polish research journal *Discussiones Mathematicae* will promote a special issue for selected papers presented at IWMS'08.

We were delighted that the local organizers had arranged, through the Portuguese Post Office (CTT), a special postage stamp showing George in his top hat from the Tampere honorary doctorate promotion in 2000. In the background is the polygon rotunda (charola) of the Convent of Christ (Castelo Templário and Convento da Ordem de Cristo de Tomar). The Convent of Christ is one of Portugal's most important historical and artistic monuments and has been in the World Heritage list of UNESCO since 1983. It is a combination of a castle and a convent, used by the Templar Knights. Another Portuguese stamp which depicts part of the Convent of Christ was issued in 1973 (*Scott* #1129) and shows the famous Window of the Chapter House (Janela do Capítulo) with typical Manueline motifs.

The 18th International Workshop on Matrices and Statistics will be held in Smolenice Castle, Slovakia, 23–27 June 2009. Local Chair: Viktor Witkovský, IOC Chair: Júlia Volaufová. http://www.um.sav.sk/en/iwms2009.html

17th International Workshop on Matrices and Statistics (IWMS)
in Celebration of T.W. Anderson's 90th Birthday: Tomar, Portugal, July 2008 [Photo: Simo Puntanen]

By Katarzyna Filipiak

The conference LINSTAT'2008 was held at the Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences in Będlewo, Poland, on April 21–25, 2008. Its purpose was to bring together researchers sharing an interest in statistics and its applications, to celebrate the 80th birthday of Prof. Tadeusz Caliński. A special session was dedicated to highlight his numerous contributions to the development of statistics and biometry and discuss current developments in these areas, and to note his wide collaboration with many scientists from different countries.

LINSTAT'2008 was attended by 70 participants from 15 countries. The keynote speakers were Anthony Atkinson, Hannu Oja, Hans-Peter Piepho, and Bimal K. Sinha. Fifteen invited talks were presented by Hilmar Drygas, Jörg Kaufmann, Mirosław Krzyśko, Lynn R. LaMotte, Jean-Pierre Masson, Stanisław Mejza, João T. Mexia, Simo Puntanen, Gavin Ross, Mirella Sari-Gorla, George P.H. Styan, Götz Trenkler, Rob Verdooren, Júlia Volaufová, and Roman Zmyślony. Moreover, 36 contributed talks and 8 posters were presented.

The topics were mainly estimation, prediction and testing in linear models, robustness of relevant statistical methods, estimation of variance components appearing in linear models, generalizations to nonlinear models, design and analysis of experiments, including optimality and comparison of linear experiments as well as solutions to problems of mathematical statistics with the use of matrix algebra. There was much friendly and stimulating discussion. Selected papers based on the talks will be published in a special issue of *Statistical Papers*.

The Scientific Committee awarded prizes for the best talks presented by Ph.D. students and young researchers. The winners were: First Prize – Wojciech Rejchel (Poland); Second Prize – Thomas Rusch (Austria); and Third Prize – Klaus Nordhausen (Finland). The prize for the best poster went to Alin Aylin (Turkey).

The meeting included an excursion to the Museum of Agriculture, where Professor Caliński gave a talk entitled "On the influence of agricultural experimentation on the development of mathematical statistics."

The conference was organized and supported by the Stefan Banach International Mathematical Center, Committee of Mathematics of the Polish Academy of Sciences, Faculty of Mathematics and Computer Science and Institute of Socio-Economic Geography and Spatial Management of the Adam Mickiewicz University, Department of Mathematical and Statistical Methods of the Poznań University of Life Sciences, Institute of Plant Genetics of the Polish Academy of Sciences, Faculty of Mathematics, Computer Science and Econometrics of the University of Zielona Góra, Polish Biometric Society, and Polish Mathematical Society, and sponsored by GlaxoSmithKline Pharmaceuticals.

The International Scientific Committee was Augustyn Markiewcz (Poland; chair), João T. Mexia (Portugal), Simo Puntanen (Finland), Götz Trenkler (Germany), Dietrich von Rosen (Sweden), and Roman Zmyślony (Poland). The Local Organizing Committee was chaired by Katarzyna Filipiak, and included Tomasz Kossowski, Augustyn Markiewicz (vice chair), Marcin Przystalski, Agnieszka Suchocka, and Waldemar Wolynski. The participants agreed that the Conference was extremely fruitful and well organized. The list of participants of LINSTAT'2008, abstracts of talks and posters, and a photo-gallery can be found at http://linstat08.up.poznan.pl.

The next meeting in the LINSTAT series will be held in Tomar, Portugal, 27-31 July 2010. The conference chairs are João T. Mexia (Scientific Committee) and Francisco Carvalho (Organizing Committee) and the official web page is http://www.linstat2010.ipt.pt.

Report on the 2008 Western Canadian Linear Algebra Meeting

By Shaun Fallat

The 2008 Western Canadian Linear Algebra Meeting (WCLAM), held at the University of Manitoba May 30-31, 2008, attracted over 40 participants from throughout North America. It featured three fully supported invited speakers that delivered insightful and elegant lectures: David Watkins, Washington State University, "Understanding the QR Algorithm, Part IX"; Michael Gekhtman, University of Notre Dame, "Biorthogonal Cauchy Polynomials, Total Positivity and Matrix Models"; and Olga Holtz, University of California - Berkeley and Technische Universität Berlin, "Matrix Methods in Stability Theory."

There were 19 other presentations on a wide range of topics including sign patterns, matrix polynomials, magic squares, matrix equations, Hadamard matrices, and combinatorial matrix theory. Following WCLAM tradition, these included several lectures from graduate students and postdoctoral fellows. A new component of this meeting was a poster session, and three students presented posters which could be viewed throughout the meeting.

The scientific organizers for this meeting were: Shaun Fallat (Regina), Steve Kirkland (Regina), Hadi Kharaghani (Lethbridge), Peter Lancaster (Calgary), Michael Tsatsomeros (Washington State), and Pauline van den Driessche (Victoria). The local organizer was Robert Craigen of the University of Manitoba. We gratefully acknowledge the financial support from the University of Manitoba and the Pacific Institute for the Mathematical Sciences. The next WCLAM is planned for May or June 2010 in Calgary, AB.

Participants, WCLAM, University of Manitoba, May 2008

Good Vibrations in Dubrovnik

By Beresford Parlett

This International Workshop on the Accurate Solution to Eigenvalue Problems (IWASEP), was held June 9-12, 2008, in the picturesque city of Dubrovnik, Croatia. This was the seventh in a series devoted to eigenvalue calculations. The term "accurate," about how to achieve high relative accuracy, has become just another topic, and the workshops now cover the whole spectrum of eigensolution, including applications.

Many attendees of the first meetings in 1996, 1998 and 2000 were not present. Only Ivan Slapničar, the local organizer for the third time, Sanja Singer and Beresford Parlett have attended all seven. This time it could be that the Householder matrix algebra meeting held near Berlin the week before and the ILAS meeting held in Cancun, Mexico the week after, claimed some of them (Demmel, Dhillon, Ipsen,...). Although we lost our regular Russian delegation (Godunov, Malyshev), we gained a Spanish contingent (Dopico, Molera, Roman), and Japanese participants (Sakurai, Asakura). Pete Stewart attended for the second time, and Olga Holtz for the first time. Could we be seeing the attraction of Dubrovnik, a town made for perambulation and the jewel of the Adriatic Sea?

Before going into technicalities I mention the format. Although the handling of posters had been very successful in previous meetings there were none in Dubrovnik. There was a place for anyone who wanted to speak. We all stayed in adjacent rooms in the attractive ”Dormitory” on the top floor of the majestic 1901 building of the University of Zagreb (yes, Zagreb – the University of Dubrovnik came later). We took most meals in the attractive restaurant Sesame close by. All sessions were plenary and there were almost as many talks of 50 minutes as of 25 but, as usual, there was plenty of discussion during presentations. Monday and Tuesday were full but Wednesday and Thursday sessions stopped at lunch time.

The Wednesday excursion was by bus, through the most southerly part of Croatia right next to Montenegro. One of the villages is the home of the Veselić family and, to our merriment, Krešimir Veselić spotted a nephew as we drove through it. There was a two hour break for swimming and we reached our mountain-top restaurant at 8:30 p.m. Some of the older participants were ready to leave at 11 p.m., only to learn that the owner had a special dessert prepared and would be gravely insulted if we left before midnight. As a mitigating factor, local celebration of European football championship matches kept some of us awake until 1 a.m. anyway.

Now let me turn to the program. Several themes from early meetings (relative perturbation theory, sophisticated versions of Jacobi, and the ”holy grail”) have been brought to successful fruition and were not at the forefront. The focus now is on exploiting structure for specific and challenging applications. Volker Mehrmann pointed out many of the difficulties in acoustic field computations in car interiors. Mark Embree talked about computing the ”bands” in the spectrum of periodic potentials in the ID Schrodinger operator. Incidentally Mark was a welcome young first timer at IWASEP and maintained Jim Demmel's tradition (or is it the Berkeley tradition?) of insightful questions during, as well as after, each presentation.

Peter Arbenz described the challenges in computations arising in opto-electronics. Here we have coupled Schrodinger equations whose discretization leads to large, sparse, generalized Hermitian eigenproblems which are often indefinite. Spectrum folding (via a quadratic function of the matrix) is used for locating crucial band gaps.

Continuing the list of special applications was Danny Sorensen who described a problem given him by Roland Glowinski. Here the matrices are 3×3 but depend on parameters. The goal is to minimize the smallest eigenvalue under quadratic constraints on the eigenvalues. The task arises in the numerical solution of boundary value PDE's of Monge-Ampere type. Here we are into the modern geometric theory for PDE's.

An excellent tradition in IWASEP is to include Louis Komzsik, who has headed the NASTRAN package for over 20 years and who keeps reminding us of the needs of users and the current industrial challenges. This time he described how structural problems, such as airplane wings, have now evolved into dynamic simulations of wing bending and car crashes. The key phrase, new to me, is "product life cycle management" (PLM), and this turns symmetric matrices into unsymmetric ones.

Michiel Hochstenbach, in a welcome return to IWASEP, talked about delay differential equations and computing critical delays. This leads to quadratic two-parameter eigenvalue problems in which one of the eigenvalues must be pure imaginary. There is no fully satisfactory method at present.

Not all talks were on applications. Zlatko Drmač, and his student Zvonimir Bujanović, have finally proved the convergence of Block Kogbetliantz methods for the SVD. The difficulty has been to find the analogue for blocks of the conditions for the element case, which were found by Forsythe and Henrici in 1960 and relaxed by Fernando in 1989. Another theoretical achievement was Froilan Dopico's demonstration that standard Jacobi, applied implicitly (i.e. one sided) to the factored form XDX^T, D diagonal and invertible, delivers high relative accuracy in all eigenvalues provided only that X is well conditioned for inversion. Of course, such a factorization is not, in general, available.

Olga Holtz presented theoretical connections between complex rational functions, Hankel and Hurwitz minors, resultants, discriminants and continued fractions. In particular, stability and hyperbolicity of polynomials is related to total nonnegativity of certain infinite Hurwitz matrices. Krešimir Veselić exhibited some "distorted Cassinis" which bound eigenfrequencies of damped systems. Josip Matejaš and Vjeran Hari presented improved perturbation bounds for eigenvalues of different kinds of scaled diagonally dominant matrices.

Other talks did present new methods. Pete Stewart revisited the problem of explicit orthogonalization with respect to a given positive definite matrix B. We need realistic error bounds to know how much reorthogonalization is needed. Beresford Parlett showed that the computation of orthogonal eigenvectors, for tridiagonals and isolated very tight clusters of eigenvalues, becomes easy if all the eigenpairs are found at the same time instead of computing them sequentially. The trick is to find a distinguished sparse basis for the invariant subspace.

Plamen Koev presented a fast and accurate method to compute eigenvalue distributions of certain random matrix classes, like Wishart matrices. The method is based on clever evaluation of Schur polynomials and has connection with representation theory. Carla Ferreira presented an elegant but sophisticated iteration that takes 3 dqds steps at once to keep arithmetic real while finding complex conjugate eigenvalues of an unsymmetric tridiagonal matrix in Jacobi form.

Tetsuya Sakurai presented a method for finding interior eigenvalues of definite pencils that exploits parallel architectures. In order to avoid the standard inner/outer iteration bottleneck he uses a simple but accurate quadrature formula, on a circle in the complex plane, to avoid solving the linear system arising in inverse iteration.

Beresford Parlett is a professor emeritus in the Department of Mathematics at the University of California, Berkeley.

Participants, IWASEP, Dubrovnik, June 2008

IMA Summer Program for Graduate Students: Linear Algebra and Applications
June 30-July 25, 2008
Department of Mathematics, Iowa State University

Report by Sue Ellen Tuttle

Iowa State University hosted this Institute for Mathematics and Its Applications (IMA) 2008 Participating Institutions Summer Program for Graduate Students. Close to 40 students, representing nearly two dozen institutions from around the world, participated. The principal speakers were Fritz Colonius, University of Augsberg, Chi-Kwong Li, The College of William and Mary, Bryan Shader, University of Wyoming, and David Watkins, University of Washington.

Five of the students talked about their experiences after the end of the third week. In her research at Washington State University, graduate student Elizabeth Bodine's work centers on spectrally arbitrary zero-nonzero patterns over finite fields. So when Shader opened the Summer Program by looking at the minimum rank of skew-symmetric matrices, Bodine was thrilled. "During our small research group time, Dr. Shader had us looking at minimum rank problems in a different context than they have been studied in the past," Bodine said. "It was different from my own research, but close enough to be useful."

Bodine's experience is precisely in line with IMA's desire to expand and strengthen the talent base engaged in mathematical research. It was a little over a year ago that the 2008 program organizers—ISU colleagues Leslie Hogben, Wolfgang Kliemann, Yiu Tung Poon and Jason Grout—learned that their proposal for a summer program for graduate students in linear algebra had been chosen by IMA for implementation in summer 2008. Bodine deeply appreciates their efforts. "This has been a fantastic opportunity to work with some really talented researchers and see how they approach problems, and what they try," she said. And, while the first week was the closest to her own research, Bodine believes it has been good to explore other areas and become more well-rounded.

First year graduate student Ryan Walker, whose advisor at the University of Kentucky nominated him to participate, was also pleased by the diversity of ideas afforded in this setting. From the perspective of a member in the early graduate school experience, Walker observed, "It is a great opportunity to see how the research process works." In addition to allowing him to explore a number of areas within linear algebra, Walker found it exciting and energizing to hear professors discuss areas which they are passionate about. "I am impressed with how well organized it is," he said. "If they were to change anything, I think it would be nice if to include a mechanism that formally allowed more advanced students an opportunity to share what they are working on with newer students."

Raphael Del Valle-Vega, from the University of Puerto Rico at Rio Piedras, agrees that there is great value in interacting with other students around their research. "Even a small detail can make a difference in your own research," he explained. He plans to do research looking at quantum coding theory, and found Dr. Chi-Kwong Li's background explanation of this area particularly helpful. "It has been well put together," he said of his second experience in an IMA summer program. Del Valle-Vega, who attended last year's program at Texas A&M, returned this year because the topic so closely fit his proposed research.

So far in her masters' program, Violeta Kovacev-Nikolic of Western Michigan University has taken numerical linear algebra and matrix algebra. She thought a workshop on linear algebra would be a great fit to follow those courses. "The lectures have been interesting," nodded Kovacev-Nikolic. With a smile, she added, "I have learned that there is so much more out there, and I am beginning to understand how much more I need to learn." "Dr. David Watkins' session on numerical linear algebra has been closest to my own experience so far," Kovacev-Nikolic said. "Getting information about the other areas will be a huge help in the future." If anything, this experience has made Kovacev-Nikolic thirst for more. "I would like to take a whole course in each area," she acknowledged. "While it's great to get a taste of each area, I actually prefer a slower pace with more details."

Yanfei Jing traveled the furthest to participate. Last year, Jing's advisor at the University of Electronic Science and Technology of China invited Dr. Leslie Hogben to Chengdu. During Hogben's time there, Jing learned about the upcoming Linear Algebra workshop, and decided to make his first visit to the United States. It has been a worthwhile trip. "This is a good chance to meet so many good students and elegant professors," he shared enthusiastically. "It is a good international communications experience. And there are so many interesting topics." "I am looking at solutions of linear equations," Jing explained. "Dr. Watkins was looking at the solutions of eigenvalues, very similar." And, Jing continued, "When he heard about my research, he encouraged me to share it with the group." Jing was nervous, but Watkins encouraged him, noting that he had the ability and technique to give representations in English. "It really helped my confidence," Jing said. "Members of the group asked so many questions that it was very interesting for me. It was good to be able to communicate on the same level. And the students asked me to share my code."

Jing has enjoyed his stay outside the classroom as well. “The weather has been beautiful,” he said. “It has been so simple to sleep here. I knew I must change my sleep habits because the hours are different. I love the environment around Frederiksen Court.” Evenings and weekends have allowed Yangei and others from the workshop to also explore the community and make new friends. In the final week of the ISU-hosted workshop, students looked at applications of linear algebra to dynamical systems with Dr. Fritz Colonius.

The Institute for Mathematics and its Applications was established in 1982 by the National Science Foundation, as a result of a national competition. The primary mission of the IMA is to increase the impact of mathematics by fostering research of a truly interdisciplinary nature, linking mathematics of the highest caliber and important scientific and technological problems from other disciplines and industry. Allied with this mission, the IMA also aims to expand and strengthen the talent base engaged in mathematical research applied to or relevant to such problems.

IMA Program Participants, Iowa State University, Summer 2008

The Ninth Workshop on Numerical Ranges and Numerical Radii
The College of William and Mary
July 19-21, 2008

Report by Chi-Kwong Li

The Ninth Workshop on Numerical Ranges and Numerical Radii, sponsored by the College of William and Mary and endorsed by the International Linear Algebra Society, was held on July 19-21, 2008, in conjunction with the19th International Workshop on Operator Theory and its Applications, July 22-26, 2008. The workshop took place on campus, at the Mathematics Department in Jones Hall.

Participants in WONRA, The College of William and Mary, July 2009

Twenty two talks on different aspects of numerical ranges and radii were given. See the program and abstracts at http://www.math.wm.edu/~ckli/08abs.pdf for details. There were about three dozen participants from more than ten different countries. Similar to previous meetings on numerical range, participants exchanged ideas, results and problems on the subject in a friendly atmosphere. A workshop dinner took place on July 20 at the Wok ‘n Roll Restaurant.

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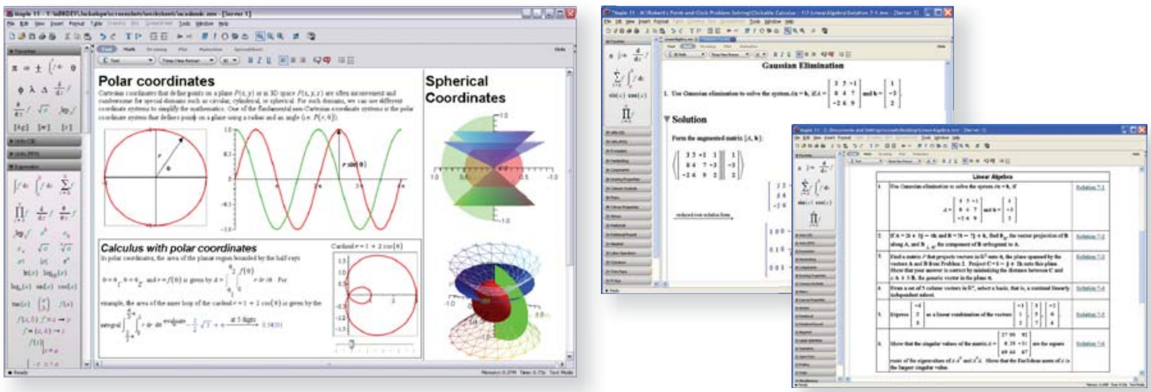

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Robert C. Thompson Matrix Meeting
University of California, Santa Barbara, October 18, 2008

Report by Jane Day

Dean Bruce Tiffney welcomed the participants and recalled Bob Thompson's career at UCSB and his role in initiating these informal one-day matrix theory conferences. Charles Johnson spoke about various ways to define "eventually true" for matrix properties and raised a number of interesting questions, giving answers for some. Visit http://www.math.ucsb.edu/~mbueno/RCthompson/rctm08.htm to view the program. Some participants made a noontime walk to the beach, and the conference closed with a dinner. Maribel Bueno, Fuzhen Zhang, Wasin So and Charles Johnson formed the Program Committee. Prof. Bueno and Roberto Costas Santos were the local hosts.

NEWS AND ANNOUNCEMENTS

Nick Higham Receives Fröhlich Prize

The London Mathematical Society has awarded its 2008 Fröhlich Prize to Nicholas Higham, School of Mathematics, University of Manchester, in recognition of his leading contributions to numerical linear algebra and numerical stability analysis. This prize is awarded in alternate years for original and extremely innovative work in any branch of mathematics, in memory of Albrecht Fröhlich.

The award says Higham's work is "characterized by identifying a fundamental computational problem; analysing the algorithms that have been proposed already for it; finding a key improvement that leads to a better algorithm, often surprising experts who thought the problem was already solved; proving theoretically that the new algorithm works; implementing it in software; and publishing a definitive paper on the subject that is a model of scholarship and clarity." Higham is Director of Research within the School of Mathematics, Director of the Manchester Institute for Mathematical Sciences (MIMS) and Head of the Numerical Analysis Group. He held a Royal Society Wolfson Research Merit Award (2003-2008).

Research Position, University of Lisbon

The Centro de Estruturas Lineares e Combinatórias (CELC), University of Lisbon, Portugal, invites applications for two five-year research positions in Linear Systems Theory and Control, or Combinatorics, Algebraic Combinatorics, and Combinatorial, Probabilistic, and Statistical Aspects of Algebra. Candidates are expected to exhibit a promise for excellence in research and must have a PhD in Mathematics; minimum three years of research experience after PhD; planning and organizational skills; communication skills; proven ability to write reports and high quality publications; fluency in written and spoken English. There are several duties associated with the position: write reports and research papers, assist with the Mathematics PhD and MSc programs, and do minor administrative duties.

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The Banach Journal of Mathematical Analysis (BJMA) is an international and peer-reviewed electronic journal presenting papers on functional analysis and its applications. The first volume appeared in 2007.

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ARTICLES AND BOOKS

The Development of Linear Algebra in Czechia

By Miroslav Fiedler, Prague


Miroslav Fiedler

Czechia is a certain geographical substitute for the Czech part of Czechoslovakia. It seems that the first mathematicians in the area who were knowledgable of matrix theory were Ludvík Kraus (1857-1885) and Eduard Weyr (1852-1903). Apparently, Kraus, who died prematurely, attracted Eduard Weyr's interest in the subject. (Let us mention that Eduard's elder brother Emil Weyr, 1848-1894, was also an important mathematician. Together they published in Czech the textbook *Basic Higher Geometry* in three volumes – 1871, 1874, 1878.)

As is now well known, Eduard Weyr developed a theory on the eigenvalue structure of matrices ([13], [14]) which is in a sense alternative to the theory of elementary divisors. He also contributed to the theory of quaternions and presented, independently of Frobenius, a proof of the Cayley-Hamilton theorem. He also proved the important result on convergence of matrix powers.

The first half of the twentieth century was full of events: two world wars, creation of Czechoslovakia, and its destruction and recovery. In spite of these disruptions, the relatively good education system and activities of the Union of Czech Mathematicians and Physicists led to general improvement of the level of mathematics in the area. In particular, the fields of geometry, topology and real analysis developed.

There was also a strong group of mathematicians at the German part of Charles University in Prague. (A historical remark: this university split in 1882 into the Czech and German parts.) I would emphasize the important role of Charles Loewner, then Karl Löwner (1893-1968). Born not far from Prague, he studied in Prague, supervised by G. Pick. In the 1930's, he became a professor in Prague and published an important paper [7] on matrix monotone functions. In 1939, he emigrated to the United States.

During the second world war, all Czech universities were closed after student protests in 1939. After 1945, there was a boom in the renewed Czechoslovakia in all kinds of sciences, including mathematics. Even during the difficult times after the Communists' coup in 1948, mathematics maintained access to the scientific literature and contacts with world scientists.

The Czechoslovak Academy of Sciences was created in 1952 after preparation in a few institutes, including the Central Institute of Mathematics headed by the famous mathematician Eduard Čech. It was due to Čech's merit and insight that he created a group of about 10 postgraduate students and arranged that the best Czech professors held advanced courses and seminars for them. After receiving the degree of CSc. (a title introduced in 1955, equivalent to Ph.D.), they were to become teachers and researchers to fill the gap after the war. It is interesting that, although most of these students were not directly inclined to linear algebra, five of us sooner or later published papers on topics in this area: I. Babuška, M. Fiedler, J. Mařík, O. Pokorná and V. Pták.

Although Pták was more interested in functional analysis and I in geometry, we soon found in the Mathematics Institute of the Academy a topic of common interest, matrix theory and its applications.

We first proved, independently of R. S. Varga, the monotonicity property of convergence of generalized Seidel methods [5]. Then we published a paper on M-matrices [6] (which we called matrices of class K in honor of the late Russian matematician Kotelyanskij) that was originally meant as a survey paper. There we introduced Z-matrices and P-matrices. This paper became one of the most cited papers in the Czechoslovak Mathematical Journal. During a period of about 40 years, we published 25 joint papers.

V. Pták worked, of course, on a number of topics in linear algebra by himself. He introduced and studied the notion of the critical exponent and the notion of the infinite companion matrix, as well as a general approach to the convergence of iterative processes. Among Pták‘s contributions to linear agebra, I should not forget his beautiful coordinate-free proof of the Jordan normal form theorem which was republished in English in [10].

My contributions were inspired in large part by connections with combinatorics (mostly graph theory) and in particular, Euclidean n-dimensional geometry. The paper [3], which used the Laplacian matrix for defining and studying connectivity properties of an undirected finite graph, and the paper [2], which contains some best possible conditions for biorthogonal systems of vectors in a Euclidean space, were apparently the most important of these.

A list of all papers of V. Pták and myself appeared in the article [12].

A number of excellent mathematicians started their research in numerical and functional analysis. Their best representative is I. Marek who became famous by studying the splitting properties and the related convergence properties of matrices, as well as the aggregation and desaggregation process in matrices [8], [9].

A more liberal political climate in the early 1960‘s made it possible for us to attend international meetings. In 1962, the year of the 100th anniversary of the Union of Czech (then Czechoslovak) Mathematicians and Physicists, almost 20 Czech and Slovak mathematicians attended the International Congress of Mathematicians in Stockholm. We were able to present our results and discuss them with A. S. Householder, R. S. Varga and others. In particular, the theory of generalized norms of matrices that Pták and I were working on attracted interest of A. Householder and his colleagues and led to invitations to meetings and even stays in the U. S. Thus V. Pták spent a year in Seattle and Florida, and I spent half-a-year in Pasadena with J. Todd and Olga Taussky-Todd.

An even more liberal climate in Slovakia, and the high position of the prominent Slovak mathematician Štefan Schwarz in the Academy of Science, allowed an international meeting on graph theory to be organized in 1963, in the Smolenice castle in Slovakia. It was essentially the first truly international meeting on this subject and a number of prominent specialists from both sides of the Iron Curtain participated (F. Harary, G. Dirac, P. Erdös, T. Gallai, G. Ringel, J. W. Moon, and A. A. Zykov). I was the secretary of the organizing committee and presented there, independently of S. Parter, the interplay between graph theory and the zero-nonzero structure of the coefficients of linear systems during the elimination process [1].

After my return from Pasadena we organized a summer school in Slovakia on numerical methods and related fields. There again people from the west and east attended (A. S. Householder, F. L. Bauer, R. S. Varga, V. N. Kublanovskaya) and this was the first such international gathering on this subject. After that, I. Babuška and his colleagues organized a series of conferences Liblice I-IV, where G. Golub, J. Wilkinson, W. Givens and A. N. Tichonov and others from abroad participated.

The improvement in contacts with the international mathematical community was interrupted in 1968 by the invasion of the Soviets and their vassals. There was still a year, until early 1970, of opportunity to go abroad, even with families, which had been impossible before the 1960‘s. Pták went to England, I. Marek visited R. S. Varga at Kent State, I. Babuška went to College Park, Maryland, and I spent the year 1969-70 in Auburn, Alabama, invited by Emilie V. Haynsworth.

After the hardliners among the communist leaders returned to power, times were difficult for Czechoslovak mathematicians (but even much harder for the historians and philosophers). Some first class people refused to return from their stays abroad, e. g. I. Babuška, V. Dlab, and J. Mařík. Nevertheless, Pták organized his summer schools in functional analysis every year, and I had seminars in Prague and Bratislava. We had some excellent students, including T. Bušinská, P. Butkovič, J. Rohn, Z. Vavřín, A. Vrba, and J. Zemánek.

About ten years later we were able to resume our personal contacts with western scientists, although going abroad was possible only if your family stayed at home. Whereas J. Zemánek went to Warszaw and P. Butkovič to Birmingham, Z. Vavřín and V. Pták strengthened their contacts with Karla Rost in Chemnitz (then Karl-Marx Stadt). I was happy to work with T. L. Markham from Columbia, South Carolina; we have published 22 joint papers. Among them, let me mention a series of papers on generalized totally positive matrices and related classes, and the paper [4] introducing new ideas on the Moore-Penrose inverse.

In 1999, Mila Pták died. He certainly was an accomplished and dynamic individual with a very broad spectrum of mathematical knowledge, and a leading personality in our scientific group. J. Rohn works on linear-algebraic problems with inexact coefficients (in particular, coefficients in intervals). His beautiful result [11] states that if an interval square matrix is completely nonsingular (i. e., every matrix with entries in the given interval is nonsingular), then for any two matrices A and B within these intervals, the product of A and inverse of B is a P-matrix.

In more recent decades, research in Czechia seems to be concentrated on numerical problems. In addition to I. Marek, certainly Z. Strakoš, M. Tůma and M. Rozložník are the most active and prospective persons in this direction.

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Ed. Note: We are honored that Professor Fiedler agreed to write this short history for IMAGE. His profound contributions to matrix theory include the seminal paper [3] above, which made clear the significance of the eigenvalues of the Laplacian matrix of a graph for measuring its algebraic connectivity. An eigenvector corresponding to the second smallest such eigenvalue is now called the Fiedler vector of a graph and is useful in a variety of applications.

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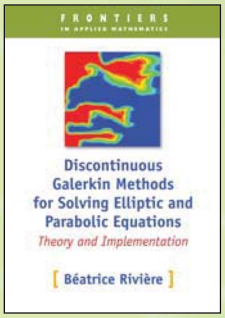

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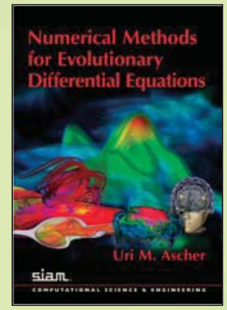

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Double Sudoku Graeco-Latin Squares

Richard William Farebrother & George P. H. Styan

1. Basic 16-Card Problem. Given a standard pack of playing cards, a centuries-old problem asks whether it is possible to lay out the Ace, King, Queen and Jack of Clubs, Diamonds, Hearts and Spades in a 4×4 array in such a way that each row, each column, and both principal diagonals contain one card of each denomination and one card of each suit without duplicating any combination in the entire array.

In our approach here, we convert this traditional problem into a double sudoku problem by noting that a similar condition also applies to the elements in the upper left corner matrix. Now, the row, column and diagonal conditions require that the cards in the (1, 1), (1, 2) and (2, 2) cells have distinct denominations and distinct suits. In principle, the cards in the (1, 2) and (2, 1) cells might share a single characteristic. But, if they do, then the row and column constraints imply that none of the cells on the principal secondary diagonal {(1, 4), (2, 3), (3, 2), (4, 1)} can share the same characteristic, contrary to requirements. In other words, the four elements of the upper left 2×2 matrix, and hence all four 2×2 corner matrices, must contain one card of each denomination and one card of each suit.

Thus, since the problem of interest takes the form of a diagonally-restricted double sudoku problem, we are able to make use of results by Farebrother [5] on 4×4 sudoku matrices. Examining the twelve typical sudoku matrices identified in this paper, we find that all but two have elements duplicated on their principal primary diagonal and deduce that any solution to our problem must be based on two variants $\mathbf{F}_1, \mathbf{F}_2$ of the first matrix given by Farebrother [5], where

$$\mathbf{F}_1 = \begin{bmatrix} w & x & y & z \\ y & z & w & x \\ x & w & z & y \\ z & y & x & w \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} \diamondsuit & \heartsuit & \spadesuit & \clubsuit \\ \spadesuit & \clubsuit & \diamondsuit & \heartsuit \\ \clubsuit & \spadesuit & \heartsuit & \diamondsuit \\ \heartsuit & \diamondsuit & \clubsuit & \spadesuit \end{bmatrix}.$$

Matrices of the types $\mathbf{F}_1, \mathbf{F}_2$ were called "magic Latin squares" by Federer [7, 1955, p. 478], who observes that "The analyses for these designs were developed by G. M. Cox[1], Univ. of North Carolina, unpublished results"; see also Dagnelie [2, 3], Martin [10]. Recently, Styan [12] defined Latin squares like $\mathbf{F}_1$ and $\mathbf{F}_2$ to be "in Sudoku form" and noted that of the 24 standard-form 4×4 Latin squares precisely half, i.e., 12, are in this Sudoku form.

Interchanging the last two rows of $\mathbf{F}_1$ and replacing w, x, y, z by $\diamondsuit, \heartsuit, \spadesuit, \clubsuit$, respectively, yields $\mathbf{F}_2$. Similarly, interchanging the last two columns of $\mathbf{F}_1$ and replacing w, x, y, z by $J =$ Jack, $A =$ Ace, $Q =$ Queen, $K =$ King, respectively, yields:

[1] Gertrude Mary Cox (1900–1978).

$$\mathbf{F}_3 = \begin{bmatrix} J & A & K & Q \\ Q & K & A & J \\ A & J & Q & K \\ K & Q & J & A \end{bmatrix}.$$

Finally, superimposing $\mathbf{F}_3$ on $\mathbf{F}_2$, yields

$$\mathbf{G} = \begin{bmatrix} J\diamondsuit & A\heartsuit & K\spadesuit & Q\clubsuit \\ Q\spadesuit & K\clubsuit & A\diamondsuit & J\heartsuit \\ A\clubsuit & J\spadesuit & Q\heartsuit & K\diamondsuit \\ K\heartsuit & Q\diamondsuit & J\clubsuit & A\spadesuit \end{bmatrix}$$

and, on substituting K for $\clubsuit$, J for $\diamondsuit$, Q for $\heartsuit$ and A for $\spadesuit$ and *vice versa*, yields the transpose $\mathbf{G}'$.

Of course, we may readily obtain $(4!)^2 = 576$ distinct solutions from each of these two fundamental solutions by permuting the orders of the w, x, y, z assignments, or 1152 solutions in all as observed, e.g., by Gardner [8, p. 175].

2. Geometrical Properties. A close inspection of these typical matrices reveals that the elements in the four central cells {(2, 2), (2, 3), (3, 2) (3, 3)}, the four corner cells {(1, 1), (1, 4), (4, 1) (4, 4)}, the middle cells in the first and last rows {(1, 2), (1, 3), (4, 2) (4, 3)}, the middle cells in the first and last columns {(2, 1), (3, 1), (2, 4) (3, 4)}, and the cells in the cyclic diagonals {(1, 3), (2, 4), (3, 1) (4, 2)} and {(1, 2), (2, 1), (3, 4) (4, 3)} also contain one card of each denomination and one card of each suit. We therefore deduce that each of the 1152 matrices identified in Section 1 will be transformed into another member of the same set by reversing the order of the rows and/or the columns; transposing the matrix about its principal primary diagonal and/or about its principal secondary diagonal; rotating the entire matrix through one, two or three right angles; moving the first row to the last position or *vice versa* at the same time as moving the first column to the last position or *vice versa*; interchanging the first two rows with the last two rows and/or interchanging the first two columns with the last two columns.

Indeed, on dividing the number of distinct solutions by eight in respect of any three of the first four transformations, we find that there are 144 (geometrically) fundamental solutions to our variant of the problem, as confirmed by the following quotation from Gardner [8]:

> ... Counting the number of different solutions is not trivial.

W. W. Rouse Ball, in his classic *Mathematical Recreations and Essays,* said there are 72 fundamental solutions, not counting rotations and reflections. This is a mistake that persisted through the book's eleventh edition. but was dropped from later editions revised by H. S. M. Coxeter.

Dame Kathleen Ollerenshaw, a noted British mathematician who was once Lord Mayor of Manchester, found there are twice as many fundamental solutions, 144, making the number of solutions including rotations and reflections $8 \times 144 = 1152$. She recently described a simple procedure for generating all 1152 patterns in an article written for the blind. ...

3. 81-Card Problems. A natural generalisation of the problem of Section 1 supposes that we have a pack of eighty-one cards with nine denominations in nine suits (or nine colours or nine additive multiples of ten) and asks whether it is possible to construct a 9×9 array with one card of each denomination and one card of each suit in each row, in each column, in each of nine contiguous 3×3 submatrices, and in both principal diagonals. Somewhat surprisingly, the answer to this question is already implicit in the 9×9 matrix given in Farebrother [6, 2007, p. 30]. For, restoring the e_is (which were omitted there for simplicity), and setting $e_1 = 1, e_2 = 5, e_3 = 9,\ e_4 = 6, e_5 = 7, e_6 = 2, e_7 = 8, e_8 = 3, e_9 = 4$, we obtain a diagonally restricted sudoku matrix, which when expressed in alphabetic form and superimposed on its transpose in numerical form yields the following solution:

$E5$	$I7$	$A3$	$D2$	$H4$	$C9$	$F8$	$G1$	$B6$
$G9$	$B2$	$F4$	$I6$	$A8$	$E1$	$H3$	$C5$	$D7$
$C1$	$D6$	$H8$	$B7$	$F3$	$G5$	$A4$	$E9$	$I2$
$B4$	$F9$	$G2$	$A1$	$E6$	$I8$	$C7$	$D3$	$H5$
$D8$	$H1$	$C6$	$F5$	$G7$	$B3$	$E2$	$I4$	$A9$
$I3$	$A5$	$E7$	$H9$	$C2$	$D4$	$G6$	$B8$	$F1$
$H6$	$C8$	$D1$	$G3$	$B5$	$F7$	$I9$	$A2$	$E4$
$A7$	$E3$	$I5$	$C4$	$D9$	$H2$	$B1$	$F6$	$G8$
$F2$	$G4$	$B9$	$E8$	$I1$	$A6$	$D5$	$H7$	$C3$

Further, by interchanging the first three rows of the alphabetic portion of this matrix with its last three rows and the first three columns of the numerical portion of this matrix with its last three columns, we obtain a second solution formed by superimposing a diagonally restricted sudoku matrix in alphabetic form on its transpose in numerical form as follows:

$H8$	$C1$	$D6$	$G2$	$B4$	$F9$	$I5$	$A7$	$E3$
$A3$	$E5$	$I7$	$C6$	$D8$	$H1$	$B9$	$F2$	$G4$
$F4$	$G9$	$B2$	$E7$	$I3$	$A5$	$D1$	$H6$	$C8$
$B7$	$F3$	$G5$	$A1$	$E6$	$I8$	$C4$	$D9$	$H2$
$D2$	$H4$	$C9$	$F5$	$G7$	$B3$	$E8$	$I1$	$A6$
$I6$	$A8$	$E1$	$H9$	$C2$	$D4$	$G3$	$B5$	$F7$
$E9$	$I2$	$A4$	$D3$	$H5$	$C7$	$F6$	$G8$	$B1$
$G1$	$B6$	$F8$	$I4$	$A9$	$E2$	$H7$	$C3$	$D5$
$C5$	$D7$	$H3$	$B8$	$F1$	$G6$	$A2$	$E4$	$I9$

This second solution is particularly interesting as the numerical portion of the matrix is identical with the matrix identified by Bailey, Cameron & Connelly [1, p.13] as typical of one of two equivalence classes having one element of each denomination in each of nine rows, in each of nine columns, in each of nine 'broken rows', in each of nine 'broken columns', in each of nine contiguous 3×3 submatrices, and in each of nine locations within these nine 3×3 submatrices, where the mask for a broken row (column) is defined by the Kronecker product of a 3×3 matrix with a single nonzero column (row) and a 3×3 matrix with a single nonzero row (column).

4. Graeco-Latin Squares. If one set of symbols is formally taken from the Roman (= Latin) alphabet and the other from the Greek alphabet then we obtain a Graeco-Latin square of the type employed in experimental design. Now, any $m \times m$ diagonal Graeco-Latin Square may be converted into a standard magic square by assigning the additive values $0, m, 2m, ..., (m-1)m$ to one set of symbols and the values $0, 1, ..., m-1$ to the other before adding unity to each entry. Conversely, we may readily check whether a conventional $m \times m$ magic square (with entries between 1 and m^2) corresponds to a Graeco-Latin Square by subtracting unity from each entry before expressing the result as the sum of two terms from the above sequences, that is, as two-digit numbers in the base m.

It is immediately apparent that every diagonal Graeco-Latin square yields a pair of magic squares but the converse is not true, see, e.g., Hodges [9, p. 51]. In particular, the magic square featured in Dürer's *Melencolia I* (see, e.g., Trenkler & Trenkler [13]) does not yield a Graeco-Latin square[2]. On the other hand, our analysis establishes that there are 1152 Graeco-Latin squares of order 4×4 satisfying numerical variants of the constraints outlined in Section 1.

[2] In a recent talk, Styan [11] observed that of the 880 magic squares of order 4×4 just 144 yield a diagonal Graeco-Latin square.

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Blood Relations

Richard William Farebrother

Suppose that we wish to establish the family relationship between any two persons: we first identify the 'keystones' in the set of their common ancestors. Each keystone will have two or more children; some will be ancestors of one of the selected individuals and others will be ancestors of the second individual but none will be ancestors of both.

Each of these keystones will be the basis of a distinct blood relationship. Consider any pair of descents from any one keystone to the selected individuals. If one individual is the (m-1)-times-great grandchild of the common ancestor ($m \geq 0$) whilst the other is his or her (n-1)-times-great grandchild ($n \geq 0$) and if $m \leq n$, then the first individual is the m^{th} cousin of a (n-m)$^{\text{th}}$ (ancestor of the second and the selected individuals may be said to be 'm^{th} cousins removed by (n-m) generations'.

For example, in the edition of 'Who do you think you are?' screened on BBC TV on 20 and 21 August 2008, it was established that the new Mayor of London Boris Johnson's six-times-great grandmother, the then Princess Royal, was the sister of Queen Elizabeth II of England's four-times-great grandfather King George III. Adding another 'great' to identify George III's parents, the then Prince and Princess of Wales, we find that Boris Johnson is an eighth cousin of Prince William of Wales and a sixth cousin twice removed to Queen Elizabeth II.

Further, if $n = m$ and the pair of m^{th} cousins have p of their q^2 $m+1$ (m-1)-times-great grandparents in common then they may be said to be 'of the p/q-blood'. For example, the children of two sisters by the same husband are both half-siblings or brothers and sisters of the half blood from their father and first cousins of the whole blood from their mothers. However, it should be noted that the concept of half and whole blood is usually restricted to the case $m = 0$. Moreover, 0^{th} cousins are usually known as siblings or brothers and sisters when removed zero generations and as (r-1)-times-great aunts and uncles (or nieces and nephews) when removed backwards (or forwards) r generations.

Positive Definite Matrices by Rajendra Bhatia

Princeton Series in Applied Mathematics Princeton University Press, 2007. 254 pages.

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Rajendra Bhatia's most recent book, *Positive Definite Matrices*, is in many ways a sequel to his earlier—and, in my view, quite successful—book *Matrix Analysis* (Springer GTM, Volume 169). Whereas *Matrix Analysis* is a carefully detailed exposition of the theoretical aspects of the spectral structure of normal matrices, *Positive Definite Matrices* is more directed toward applications of the theory to other parts of mathematics.

Bhatia's new book begins with a short review of the basic facts concerning positive matrices, where here the adjective "positive" refers to positive definite and semidefinite complex matrices. The reader of this book surely knows all about the spectral theorem and so forth, yet I learned new things (such as the pretty and short proof that $A^{1/2} \geq B^{1/2}$ if $A \geq B$) from this introductory chapter. The list of introductory topics in Chapter 1 also includes the min-max variational principle and matrix monotone functions.

Positive Definite Matrices is devoted to the following topics: (i) positive-preserving linear maps on M_n (the algebra of $n \times n$ complex matrices), (ii) various averaging operations for matrices, (iii) positive definite functions, and (iv) the manifold of positive definite matrices. In other words, Bhatia's book touches upon concepts from operator algebras, quantum information theory, harmonic analysis, and differential geometry.

In each of the four settings above, Bhatia aims to emphasise the linear algebraic foundations of the applications in question. One danger of this sort of goal is that important conceptual understanding is exchanged for un-enlightened brute force calculation. However, Bhatia achieves the opposite effect: his linear algebraic approach beautifully clarifies much of the phenomena in question. As examples, I mention the very elegant proofs of the Lieb–Ruskai theorem on the strong subadditivity of entropy (Chapter 4) and the fact that the (Riemannian) manifold of strictly positive definite matrices has nonpositive curvature (Chapter 6). These are significant theorems, with nontrivial proofs already in the literature. It is not to say that Bhatia has simplified the results—rather, he has revealed the matrix-theoretic content in these results. It is a point of view, really; and for people who love to work with matrices, it is a point of view that enlightens the topics at hand and is worth pursuing further.

Chapters 2 and 3 treat positive and completely positive linear maps on M_n. Many results are taken from the literature on operator spaces, but specialised here to M_n. Chapter 4 covers means of matrices (the geometric mean, the harmonic mean, et cetera), and Chapter 5 touches upon positive definite functions. In this latter chapter there are some nice measure/integral theorems (Herglotz's theorem, Bochner's theorem), and applications to norm inequalities.

Chapter 6 concerns the set P_n of strictly positive definite $n \times n$ matrices as a differentiable manifold. This chapter also contains a very nice interpretation of the geometric mean of $A, B \in P_n$ as the midpoint of the geodesic in P_n connecting A with B. Throughout the book, exercises appear regularly as the material is being developed, and the motivation for the topics at hand is generally left to a literature discussion at the end of each chapter.

Readers of *Positive Definite Matrices* will vary from core linear algebraists to researchers in related fields. For example, the field of quantum information theory is now very much in vogue, and there is a great deal in Chapters 2–4 that is relevant to researchers in this area, especially graduate students. At the same time, the treatment stops short of a fuller discourse on the linear analysis required for quantum information theory.

Two topics that are not covered, but are of use to quantum information theorists and mathematical physicists, are one-parameter semigroups of completely positive linear maps on M_n and the Kraus decomposition of normal completely positive linear maps on the infinite-dimensional operator algebra $B(H)$. These topics are within reach, but an author must draw the line somewhere, and Bhatia has elected to focus on a selection of topics rather than develop any single topic extensively. Nevertheless, *Positive Definite Matrices* is a coherent work.

There is no obvious competitor for Bhatia's book, due in part to its focus, but also because it contains some very recent material drawn from research articles. Beautifully written and intelligently organised, *Positive Definite Matrices* is a welcome addition to the literature. Readers who admired his *Matrix Analysis* will no doubt appreciate this latest book of Rajendra Bhatia.

The Argand Brothers of Geneva

By Richard William Farebrother

Readers of Image will perhaps be surprised to learn that there are two entries under `Argand' in the Cambridge Biographical Encyclopaedia [2, p.40], namely those for the brothers Aimé Argand (1750--1803) and Jean-Robert Argand (1768--1822) both born in Geneva, Switzerland.

The well-known contribution of the younger brother is described by Lewis [3, p.724] in the following terms: "In 1797, at the age of 52, the Norwegian surveyor Caspar Wessel presented a paper to the Royal Academy of Denmark in which he demonstrated essentially the same geometric representation of complex numbers used today. Then, in 1806, the 38-year-old Jean Argand, Swiss and apparently self educated, while working in Paris as a bookseller published a graphical representation of complex numbers as *lignes en direction*, the first such to be printed."

For a brief notice of the more substantial work of the elder brother we refer to Jenny Uglow [4, p.375] who notes that: "... the improved oil lamp invented by the Genevan Aimé Argand with a tubularwick and glass chimney which would cut out the smoke and smell. [See Woolfe [5]]. When Argand came to London in 1783 to find skilled workers, he met [André] de Luc, who introduced him to the Court where he thrilled the Royal Family with demonstrations of balloons."

Although Aimé Argand's lamps proved a commercial failure at the time, they were subsequently employed with great success in the lighthouses constructed by the immediate relatives of Robert Louis Stevenson (1850-1894); see Bathurst [1].

Finally, I note that a permanent record of a balloon ascent made at 11.40 a.m. on 17 May 1785 is embodied in the postal address of the headquarters of the [British] Co-operative Bank at 1 Balloon Street, Manchester.

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ILAS NEWS

Correction to ILAS Ballot

Each current ILAS member should have received a paper ballot by regular post around the end of November 2008. This election is for a new ILAS Secretary/Treasurer and for two new members for the ILAS Board of Directors.

The instruction should have said "Ballot must be received by January 20, 2009" (not 2008).

Invitation to Join ILAS

IMAGE is supported by the members of ILAS through their dues. Non-members are invited to join ILAS to help support IMAGE, as well as enjoy all the other benefits of membership.

To learn about ILAS, its structure, conferences and publications, visit the ILAS homepage (http://www.ilasic.math.uregina.ca/iic/). A membership application form is there, as well as PDF files of all past issues of IMAGE, which can be searched and downloaded.

ILAS Members can choose to receive a printed copy of each issue of IMAGE, or to read it online. The online version features color photographs. The printed version is printed in black and white, to reduce cost.

Highlights from the ILAS Board Meeting at Cancun, June 19, 2008

President Kirkland expressed thanks to Luis Verde-Star and the organizing committee for the wonderful setting and smooth functioning of the 2008 ILAS meeting. He also thanked retiring President Danny Hershkowitz; the two Board members whose terms are expiring, Ilse Ipsen and Reinhard Nabben; Jeff Stuart for his years of good work as ILAS Secretary/Treasurer; and Shaun Fallat for his work as Manager of ILAS-NET. Sarah Carnochan Naqvi has now assumed that duty.

Kirkland called attention to changes in the guidelines for ILAS Lecturers. The due date for applications will now be September 30 each year. Visit http://www.ilasic.math.uregina.ca/iic/misc/lecturers.html for details.

He also announced that a full scale ILAS Program Review will soon begin, and the Review Committee will be Harm Bart (Chair), Shaun Fallat, Francoise Tisseur, and Xingzhi Zhan. This Committee is charged to review the past and present ILAS program and make recommendations for the removal or strengthening of existing programs and establishment of new programs. The Committee is to consider financial implications in their recommendations and solicit input from ILAS members. The Committee will report its findings to the ILAS Executive Committee by February 29, 2009, and then to the Board.

ILAS Secretary/Treasurer Jeff Stuart reported that ILAS is financially healthy. The bank balance has been growing, primarily because the cost of mailing IMAGE has decreased substantially as most subscribers have opted to read the electronic version, and because we have several corporate sponsors. ILAS membership has also been growing, especially from those countries where ILAS conferences recently took place. Membership fees have been waived for some new members.

ILAS Outreach Director James Weaver reported that the current ILAS corporate sponsors are A.K. Peters LTD, Elsevier, Maplesoft, SIAM, Taylor & Francis, Wiley-Blackwell and World Scientific. This sponsorship entitles each to a one page advertisement in IMAGE. Elsevier and Taylor & Francis both make substantial contributions to support ILAS conferences, in addition to their corporate dues. As we work with SIAM we are able to advertise in SIAM News.

The ILAS Institutional Membership Committee is Rafael Bru, Wasin So, James Weaver (chair) and Steve Kirkland (ex-officio). Weaver reported that this Committee is looking for people to help recruit more corporate sponsors.

Weaver reported that he is beginning to archive ILAS documents at both the University of West Florida and the University of Texas at Austin, including the articles of incorporation for ILAS. (Ed. note: Weaver was instrumental in obtaining official incorporation of ILAS as a 501(c)3 (nonprofit) organization in 1989. See the article “How ILAS Began”, IMAGE 40, p. 14-17.)

Weaver also reminded those present that all issues of IMAGE are now posted on the ILAS website, where they can be searched and downloaded (http://www.ilasic.math.uregina.ca/iic/publications/).

Roger Horn, new Chair of the ILAS Journals Committee, reported that submissions to the Electronic Journal of Linear Agebra (ELA) continue to increase, and as a consequence the Editors-in-Chief, Ludwig Elsner and Daniel Hershkowitz, intend to expand the ELA Editorial Board to handle the increased work load.

The Journals Committee recently approved a proposal for functional reorganization of the IMAGE Editorial Board that is intended to make the newsletter more interesting and useful for ILAS members.

The ILAS Education Committee is Luz De Alba, Guershon Harel, David Lay, Sang-Gu Lee and Steve Leon (Chair). Leon reported on their recent meeting. They intend to support linear algebra education events at meetings organized by other organizations such as SIAM, AMS, and MAA, as well as at ILAS meetings.

This Committee also plans to conduct two surveys: one to request information about second courses in linear algebra and a worldwide survey on curriculum similar to that conducted by the Linear Algebra Curriculum Study Group, which was published in 1993. (Ed. note: The first of those is on page 31 in this issue.)

David Lay reported that the MAA has offered 60 free copies of its book “Resources for Teaching Linear Algebra”, to be distributed as appropriate. This book has valuable information but may not be well known outside the United States and Canada. The Committee will send these to appropriate people in other countries.

The Education Committee will also continue to maintain an ILAS Linear Algebra Education web page (http://www.ilasic.math.uregina.ca/iic/education/).

Appreciation for H.J. Werner

By Steve Kirkland, President of ILAS

With this issue of IMAGE, Hans Joachim Werner finishes his editorial service to IMAGE.

Professor Werner began this work as a Senior Associate Editor of IMAGE in the Fall of 1997. He became an Editor in Chief in the Fall of 2000, and served in that capacity for eight years, working alongside fellow Editors in Chief George Styan (2000-2003), Bryan Shader (2003-2006) and Jane Day (2006-2008).

Throughout this time, Professor Werner held primary responsibility for the IMAGE Problems Corner. A quick tally shows that he handled literally thousands of problems and solutions over his years with IMAGE -- a remarkable feat indeed.

On behalf of ILAS, I extend my appreciation to Professor Werner for his long service and contributions to IMAGE, and wish him the best in his future endeavours.

New Editorial Structure for IMAGE

By Jane Day, Editor in Chief

All past and present editors of IMAGE also thank Professor Werner for his long and faithful work on IMAGE.

We welcome Fuzhen Zhang as a new Contributing Editor, who will be in charge of the Problems Corner beginning with Issue 42. In fact, each of the Contributing Editors has generously agreed to focus on a specific area of interest:

- Problems and solutions: Fuzhen Zhang (zhang@nova.edu)
- History of linear algebra: Peter Šemrl (peter.semrl@fmf.uni-lj.si)
- Book reviews: Oskar Maria Baksalary (baxx@amu.edu.pl)
- Linear algebra education: Steve Leon (sleon@umassd.edu)
- Advertisements: Jim Weaver (jweaver@uwf.edu)
- All other news: Jane Day (day@math.sjsu.edu)

Call for News for IMAGE Issue 42, Spring 2009

The Editors' goal is to make IMAGE as thorough and interesting as possible to the linear algebra community. Your ideas and contributions will be welcome. Send them to the appropriate Editors, as listed above.

Topics of interest include:

Conference announcements and reports
News about journals and special issues
Problems and solutions
Articles on history and other special topics
Books, websites, funding sources
Honors and awards
Transitions: new appointments, responsibilities, deaths
Linear algebra education
Possible corporate sponsors
Employment opportunities

Guidelines for authors:
Send material in plain text, Word, or in both PDF and Latex (Article.sty with no manual formatting is preferred). Photos should be in JPG format.

Deadlines for submissions to Issue 42, Spring 2009:

- March 1, 2009: Problems and solutions to F. Zhang. It is helpful for him to receive these even earlier when possible.

- April 1, 2009: All other submissions to the appropriate Editors. If you wish to submit something after April 1, it may be possible; communicate as soon as you can with Editor Jane Day.

The publication dates for IMAGE are June 1 for spring issues, December 1 for fall issues.

List of Books of Interest to Linear Algebraists Now Posted on ILAS Website

There exist many fine books about linear algebra and its applications. George Styan began compiling a list of these some years ago, and Oskar Baksalary has agreed to maintain that list now, in addition to soliciting reviews of selected titles.

The current list is now posted on the ILAS-IIC website (http://www.ilasic.math.uregina.ca/iic/). Please have a look at it, and tell Editor Baksalary (baxx@amu.edu.pl) when you find other titles or informative reviews which could be included.

Call for Announcements for ILAS-Net

ILAS-Net/IIC is the online bulletin board for ILAS, to provide current information on linear algebra activities such as international conferences, workshops, journals, and ILAS notices. Organizations and individuals are invited to contribute.

Announcements for ILAS-Net should be sent in a text email with no attachments, to owner-ilas-net@math.technion.ac.il. Updates to the IIC webpage should be sent to the IIC Manager, Sarah Naqvi, at ilasic@math.uregina.ca.

The primary website for IIC is http://www.ilasic.math.uregina.ca/iic/index1.html. Mirror sites are located at http://www.math.technion.ac.il/iic/index1.html and htpp://wftp.tu-chemnitz.de/pub/iic/index1.html.

TWO SURVEYS

Survey of the State of the "Challenges in Matrix Theory" Project, a Decade Later

Please Respond

By Frank Uhlig, Challenges Coordinator

The journal Linear Algebra and Its Applications (LAA) plans to publish a report on the current state of the "Challenges in Matrix Theory Project". For this I kindly ask IMAGE readers to inform me of the state of solutions, and if - as I hope - there have been positive developments, to send me an account of this progress.

I know of two problems that have found partial or full answers. Are there any more? Contributions should be relatively short, 1-2 pages of explanation of the new results, with quotes and so forth. Or more simply, just send a reference and a short explanation. I will collate this information into the survey.

If you intend to contribute to this report, please send submissions by January 1, 2009 or let me know when you will send such.

Address submissions to Frank Uhlig, Challenges Coordinator, Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849-5310, USA; Telephone: 334 844-6584; email: uhligfd@auburn.edu.

This "Challenges in Matrix Theory" project was proposed in LAA 233 (1996), 1-3. Ten problems in pure and applied linear algebra were published between 1998 and 2002. We thank the authors, and now it seems time for the linear algebra community to collect and record its collective knowledge about solutions or partial answers that have been found for these ten challenging problems.

Here are the references.

1998 LAA 278:

1. Alberto Borobia and Julio Moro, On the boundary of the set of spectra of nonnegative matrices, p. 287-293.

2. Georg Heinig, Matrices with higher order displacement structure, p. 295-301.

3. Erxiong Jiang, Challenging eigenvalue perturbation problems, p. 303-307.

4. A. C. Antoulas, Approximation of linear operators in the 2-norm, p. 309-316.

5. Reng-Cang Li, Spectral variations and Hadamard products: some problems, p. 317-326.

6. Jean Jacques Loiseau, Sabine Mondie, Ion Zaballa and Petr Zagalak, Assigning the Kronecker of a matrix pencil by row or column completions, p. 327-336.

2000 LAA 304:

7. Bruno Codenotti, Matrix rigidity, p. 181-192.

8. Bruno Codenotti, Gianna Del Corso and Giovanni Manzini, Matrix rank and communication complexity, p. 193-200.

2001 LAA 332-334:

9. Frank Uhlig, Constructive ways for generating (generalized) real orthogonal matrices as products of (generalized) symmetries, p. 459-467.

2002 LAA 345:

10. Olga Holtz and Hans Schneider, Open problems on GKK tau-matrices, p. 263-267.

Survey of Second Courses in Linear Algebra

Please Respond

By Steven Leon, Chair, ILAS Education Committee

If your department offers a second course in linear algebra, we would appreciate your taking a few minutes to complete the survey below. The results of your responses will be summarized in a future issue of IMAGE. If your department offers more than one second linear algebra course, please describe each one. Thanks in advance for your cooperation.

This survey is a followup to one of the recommendations made by the Linear Algebra Curriculum Study Group in 1990. Those were that mathematics departments should offer a matrix-oriented first course and also a second course (e.g., numerical, applied, theoretical, matrix analysis, etc.).

Although such first courses now appear to be pretty standard, it is not clear how many second courses are presently offered. The ILAS Education Committee discussed this issue recently, and agreed that efforts are still needed to promote this matter. As a first step, we ask that IMAGE readers help us assess the courses currently being offered.

Survey Regarding Second Courses in Linear Algebra

Please return the following survey by email or on paper to Steven Leon, sleon@umassd.edu or Mathematics Department, University of Massachusetts Dartmouth, Dartmouth, MA 02747-2300; telephone: 508-999-8320. We seek the following information about each second linear algebra course offered by your university.

1. Your name, university and email address?

2. What are the main topics covered in your second linear algebra course?

3. What level (sophomore, junior, senior) is the course?

4. Students taking the course are principally from what disciplines?

5. About how many students take the course each year?

6. What software (if any) do you use or plan to use in the course?

7. What textbooks do you use or recommend?

8. Provide any other comments that you feel are relevant, such as how long your department has been offering this course, has it led to development of more advanced courses or theses, etc.

IMAGE Problem Corner: Old Problems, Most With Solutions

We present solutions to IMAGE Problems 39-1 through 39-3, 39-5, and 39-6 [IMAGE 39 (Fall 2007), pp. 31–32]. Problems 36-3 [IMAGE 36 (Spring 2006), p. 36] and 39-4 [IMAGE 39 (Fall 2007), p. 32] are repeated below without solution; we are still hoping to receive solutions to these problems. We introduce 14 new problems on pp. 42-44 and invite readers to submit solutions to these problems as well as new problems for publication in IMAGE. Please submit all material in macro-free LaTeX along with the PDF file by e-mail to zhang@nova.edu. Fuzhen Zhang will be the new IMAGE Problem Corner Editor from IMAGE 42 onwards.

Problem 36-3: Transformations of (Skew-) Idempotent and (Skew-) Involutory Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

In this problem, we generalise Farebrother's (2001, p. 11) definitions of involutory and skew-involutory matrices by replacing the $n \times n$ identity matrix I_n by a given $n \times n$ idempotent matrix P satisfying $P^2 = P$.

(a) Let A be an $n \times n$ idempotent or skew-idempotent matrix satisfying $AP = PA = A$ and $A^2 = mA$ where $m = \pm 1$. Then for $s^2 = \pm 1$, the generalised Householder transformation $B = s(2A - mP)$ is P-involutory (i.e., $B^2 = P$) or skew-P-involutory (i.e., $B^2 = -P$) according as $s^2 = \pm 1$: $B^2 = s^2(4A^2 - 4mPA + m^2P^2) = s^2P$.

(b) Conversely, suppose that B is an $n \times n$ P-involutory or skew-P-involutory matrix satisfying $BP = PB = B$ and $B^2 = s^2P$ where $s^2 = \pm 1$, then the transformation $A = (sB + mP)/2$ is idempotent (i.e., $A^2 = A$) or skew-idempotent (i.e., $A^2 = -A$) according as $m = \pm 1$: $A^2 = (s^2B^2 + 2msBP + m^2P^2)/4 = m(sB + mP)/2 = mA$.

Readers are asked to suggest alternative transformations that maintain the reality of B (or A) when P and A (or B) are real.

Reference

R. W. Farebrother (2001). The naming of parts. *IMAGE: The Bulletin of the International Linear Algebra Society* **27**, 11.

We still look forward to receiving solutions to Problem 36-3!

Problem 39-1: A Simple Procedure for a Generalised Matrix Transformation

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be an $n \times n$ matrix and let m be an integer in the range $1 \leq m \leq n$. Then we may define the mth row of A by $\{a_{mj}\}$, the mth column of A by $\{a_{im}\}$, the mth primary (or dexter) cyclic-diagonal of A by $\{a_{ij} : i = j + m \pmod{n}\}$, and the mth secondary (or sinister) cyclic-diagonal of A by $\{a_{ij} : i + j = m + 1 \pmod{n}\}$. In each case, the nth primary or secondary cyclic-diagonal of A may be named the principal primary or secondary diagonal as they are also the familiar non-cyclical diagonals of the matrix. [That is in the context of IMAGE Problem 37-5.]

(i) For the case n odd, identify a simple procedure for generating a transformation which carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column.

(ii) Illustrate your procedure in the case $n = 5$.

Solution 39-1.1 by the Proposer Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:*
R.W.Farebrother@Manchester.ac.uk

Given n odd and an empty $n \times n$ matrix, we suppose that the transformation carries a typical element (associated with the quadruplet $(ijkl)$) into the ith row, the jth column, the kth primary diagonal, and the lth secondary diagonal, where $k = j - i \pmod{n}$ and $l = i + j \pmod{n}$.

A specific transformation may now be obtained by attaching consistent values to any three rows, columns, primary diagonals or secondary diagonals that intersect in two or three cells. In particular, we note the following simple, if somewhat inefficient, procedure.

Attaching any one of n values to the fourth element of all quadruplets in the first row and any one of n values to the third element of all quadruplets in the first column, then the intersection of the first row with the first column at $(1, 1)$ identifies the value of the first element of all quadruplets in the principal primary diagonal which passes through the points $(1, 1)$, $(2, 2)$ and (n, n). Attaching any one of $n - 1$ values to the fourth element of all quadruplets in the last row, then the intersection between this row and the principal primary diagonal at (n, n) identifies the value of the third element of all quadruplets in the last column.

Now the intersection between the nth row and the first column at $(n, 1)$ identifies the value of the first element of all quadruplets in the primary diagonal that passes through this point and through $(1, 2)$, $(2, 3)$, and $(n-1, n)$ whilst the intersection between the first row and the nth column at $(1, n-1)$ identifies the value of all quadruplets in the primary diagonal that passes through this point and through $(2, 1)$, $(3, 2)$, and $(n, n-1)$. The intersections between these two primary diagonals and the first and last rows and columns at the points $(1, 2)$, $(n-1, n)$, $(2, 1)$, and $(n, n-1)$ identifies the values to be associated with the second and $(n-1)$th rows and columns.

Missing the opportunity of defining four more primary diagonals at this stage, we simply note that the intersections between the two cyclic diagonals immediately above and below the principal primary diagonal and the second and $(n-1)$th rows and columns define the values to be associated with the third and $(n-2)$th rows and columns; and so on to completion.

APPLICATION: Setting $n = 5$, and arbitrarily setting the fourth element of all quadruplets in the first row of the empty 5×5 matrix equal to 4, and the third element of all quadruplets in the first column equal to 2, we find that the primary principal diagonal is associated with a value of 1 = (4-2)/2 and set the first element of all quadruplets in this diagonal equal to 1.

Again, arbitrarily setting the fourth element of all quadruplets in the fifth row equal to 5, we find that the last column is associated with a value of 3 = 5-2, and we have to set the third element of all quadruplets in this column equal to 3. In this way we obtain the matrix:

$$\begin{pmatrix} (1*24) & (***4) & (***4) & (***4) & (**34) \\ (**2*) & (1***) & (****) & (****) & (**3*) \\ (**2*) & (****) & (1***) & (****) & (**3*) \\ (**2*) & (****) & (****) & (1***) & (**3*) \\ (**25) & (***5) & (***5) & (***5) & (1*35) \end{pmatrix}$$

From this matrix we may establish the values associated with the first primary diagonals immediately above and below the principal primary diagonal, and hence the values associated with the second and fourth rows and the second and fourth columns, we thus have:

$$\begin{pmatrix} (1*24) & (4*14) & (**44) & (***4) & (3*34) \\ (3*23) & (1*13) & (4**3) & (**43) & (**33) \\ (**2*) & (3*1*) & (1***) & (4*4*) & (**3*) \\ (**21) & (**11) & (3**1) & (1*41) & (4*31) \\ (4*25) & (**15) & (***5) & (3*45) & (1*35) \end{pmatrix}$$

From which it is easy to obtain the full matrix:

$$\begin{pmatrix} (1324) & (4514) & (2254) & (5444) & (3134) \\ (3523) & (1213) & (4453) & (2143) & (5333) \\ (5222) & (3412) & (1152) & (4342) & (2532) \\ (2421) & (5111) & (3351) & (1541) & (4231) \\ (4125) & (2315) & (5555) & (3245) & (1435) \end{pmatrix}$$

Now, the first two elements suffice to identify the cells of this matrix, and we delete the third and fourth elements to obtain:

$$\begin{pmatrix} (1,3) & (4,5) & (2,2) & (5,4) & (3,1) \\ (3,5) & (1,2) & (4,4) & (2,1) & (5,3) \\ (5,2) & (3,4) & (1,1) & (4,3) & (2,5) \\ (2,4) & (5,1) & (3,3) & (1,5) & (4,2) \\ (4,1) & (2,3) & (5,5) & (3,2) & (1,4) \end{pmatrix}$$

Clearly, the transformation represented by this matrix carries each column into a primary diagonal, each primary diagonal into a row, each row into a secondary diagonal, and each secondary diagonal into a column.

This transformation clearly exhibits a twenty-period cycle that carries (1, 1) into (1, 3), (1, 3) into (2, 2), (2, 2) into (1, 2), (1, 2) into (4, 5), (4, 5) into (4, 2), (4, 2) into (5, 1), (5, 1) into (4, 1), (4, 1) into (2, 4), (2, 4) into (2, 1), (2, 1) into (3, 5), (3, 5) into (2, 5), (2, 5) into (5, 3), (5, 3) into (5, 5), (5, 5) into (1, 4), (1, 4) into (5, 4), (5, 4) into (3, 2), (3, 2) into (3, 4), (3, 4) into (4, 3), (4, 3) into (3, 3), (3, 3) into (1, 1). It also exhibits a five period cycle that carries (4, 4) into (1, 5), (1, 5) into (3, 1), (3, 1) into (5, 2), (5, 2) into

(2, 3), (2, 3) into (4, 4). However, whatever the starting value, it is clear that twenty applications of this transformation carries every element back to its original position.

Further, on examining fourth successors in this sequence, we find that four applications of this transformation carries rows 1, 2, 3, 4 and 5 into rows 4, 5, 1, 2 and 3 and columns 1, 2, 3, 4 and 5 into column 5, 1, 2, 3 and 4 in a five period block cycle.

Problem 39-2: On the Range of A^+A^*

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au,
and Götz TRENKLER, *Technische Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

For an $n \times n$ matrix A with complex entries, let $\Phi(A) = A^+A^*$, where A^+ is the Moore-Penrose inverse and A^* is the conjugate transpose of A, respectively. Assume that A is EP, i.e., A and A^+ have identical column spaces. Show that the following statements are equivalent:

(i) $\Phi(A)$ is a partial isometry.

(ii) A is normal.

(iii) $A\Phi(A) = \Phi(A)A$.

REMARK: The present problem generalizes Proposition 1 in C. R. DePrima & C. R. Johnson, The range of $A^{-1}A^*$ in $GL(n, \mathbb{C})$, *Linear Algebra and Its Applications* (1974) **9**, 209-222.

Solution 39-2.1 by Jan HAUKE, *Adam Mickiewicz University, Poznań, Poland:* jhauke@amu.edu.pl

FACT 1: *Let $A \in \mathbb{C}_{n,n}$ be EP. Then (see Meyer (2000, Chapter 5.11)) there exists an unitary matrix U and a nonsingular matrix P such that*

$$A = U \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

FACT 2: *Let $P \in \mathbb{C}_{n,n}$ be nonsingular. Then*

$$(P^{-1}P^*)^{-1} = (P^{-1}P^*)^* \iff PP^* = P^*P. \tag{1}$$

PROOF. Premultiplying the left side of (1) by $P^{-1}P^*$ and postmultiplying it by P^* we get the desired result. □

FACT 3: *Let $A \in \mathbb{C}_{n,n}$ be EP and $\Phi(A) = A^+A^*$. Then:*

(a) $\Phi(A)$ *is partial isometry if and only if* $(P^{-1}P^*)^{-1} = (P^{-1}P^*)^*$.

(b) A *is normal if and only if* $PP^* = P^*P$.

(c) $A\Phi(A) = \Phi(A)A$ *if and only if* $P^* = P^{-1}P^*P$.

So, the equivalence of (a), (b), and (c) implies the equivalence (i) ⇔ (ii) ⇔ (iii).

Reference

C.D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

Solution 39-2.2 by the Proposers Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au,
and Götz TRENKLER, *Technische Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

Following Corollary 6 in Hartwig and Spindelböck (1984), we may write the matrix A in the form

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where U is unitary, $KK^* + LL^* = I_r$, $\Sigma = \operatorname{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$, $r_1 + \ldots + r_t = r = \operatorname{rank}(A)$ and $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$. Since A is EP, we can conclude that $L = 0$. It follows that $K^* = K^{-1}$,

$$A^* = U \begin{pmatrix} K^*\Sigma & 0 \\ 0 & 0 \end{pmatrix} U^*, \quad A^+ = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*, \quad \text{and} \quad \Phi(A) = A^+A^* = U \begin{pmatrix} K^*\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

(i) $\Rightarrow$ (ii): Let $\Phi(A)$ be a partial isometry. Then $\Phi(A)[\Phi(A)]^*\Phi(A) = \Phi(A)$ which gives $K^*\Sigma^{-1}K^*\Sigma^2 K\Sigma^{-1}KK^*\Sigma^{-1}K^*\Sigma = K^*\Sigma^{-1}K^*\Sigma$ or, equivalently, $K^*\Sigma^2 K = \Sigma^2$. This implies $K^*\Sigma K = \Sigma$ and thus $\Sigma K = K\Sigma$ which describes the normality of A.

(ii) $\Rightarrow$ (iii): Let A be normal, i.e., $K\Sigma = \Sigma K$. Now we have

$$A\Phi(A) = U\begin{pmatrix} \Sigma KK^*\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{pmatrix}U^* \quad \text{and} \quad \Phi(A)A = U\begin{pmatrix} K^*\Sigma^{-1}K^*\Sigma\Sigma K & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

Since $K\Sigma = \Sigma K$, we obtain

$$A\Phi(A) = \Phi(A)A = U\begin{pmatrix} K^*\Sigma & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

(iii) $\Rightarrow$ (i): Assume that $A\Phi(A) = \Phi(A)A$. It follows that $K^*\Sigma = K^*\Sigma^{-1}K^*\Sigma^2 K$, and consequently $K^*\Sigma^{-1}K^*\Sigma^2 K\Sigma^{-1}KK^*\Sigma^{-1}K^*\Sigma = K^*\Sigma^{-1}K^*\Sigma$. Hence $\Phi(A)$ is a partial isometry. $\square$

Reference

R.E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and $A^\dagger$ commute. *Linear and Multilinear Algebra* **14**, 241-256.

Solution 39-2.3 by Hans Joachim WERNER, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

First, we note that if $M \in \mathbb{C}^{m\times n}$, then $(\mathcal{N}(M^*))^\perp = \mathcal{R}(M)$ and $(\mathcal{R}(M^*))^\perp = \mathcal{N}(M)$, with $(\cdot)^*$, $\mathcal{N}(\cdot)$, and $\mathcal{R}(\cdot)$ denoting the conjugate transpose, the null space, and the range (column space), respectively, of the matrix $(\cdot)$, and with $(\cdot)^\perp$ indicating the orthogonal complement of linear space $(\cdot)$ with respect to the usual inner product. Secondly, we mention that MM^+ and M^+M are the orthogonal projectors onto $\mathcal{R}(M)$ (along $\mathcal{N}(M^*)$) and onto $\mathcal{R}(M^*)$ (along $\mathcal{N}(M)$), respectively. Thirdly, we note that $\mathcal{R}(M^+) = \mathcal{R}(M^*)$, $\mathcal{N}(M^+) = \mathcal{N}(M^*)$, $(M^+)^* = (M^*)^+$, and $(MM^*)^+ = (M^+)^*M^+$. And finally, we recall the following well-known characterizations: (a) $A \in \mathbb{C}^{n\times n}$ is EP $\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^*) \Leftrightarrow \mathcal{N}(A) = \mathcal{N}(A^*) \Leftrightarrow AA^+ = A^+A$, and (b) $A \in \mathbb{C}^{n\times n}$ is a partial isometry $\Leftrightarrow AA^*A = A \Leftrightarrow A^+ = A^*$. In what follows, we assume throughout that A is EP. Hence, $AA^+ = AA^+$.

We next wish to show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). First, let $A\Phi(A) = \Phi(A)A$ or, equivalently, let $AA^+A^* = A^+A^*A$. Then, in view of $AA^+A^* = A^*$, clearly $A^* = A^+A^*A$. Premultiplying this equation by A, in view of $AA^+A^* = A^*$, gives $AA^* = AA^+A^*A = A^*A$, i.e., A is normal. This completes the proof of (iii) $\Rightarrow$ (ii). Next, let A be normal, i.e., let $AA^* = A^*A$. By means of $(A^+)^*A^+ = (AA^*)^+$ and $\mathcal{R}(A^*A) = \mathcal{R}(A^*)$, we then obtain $\Phi(A)(\Phi(A))^*\Phi(A) = A^+A^*(A^+A^*)^*A^+A^* = A^+A^*A(A^+)^*A^+A^* = A^+A^*A(AA^*)^+A^* = A^+A^*A(A^*A)^+A^* = A^+A^* = \Phi(A)$. This shows that $\Phi(A)$ is a partial isometry and so the proof of (ii) $\Rightarrow$ (i) is complete. Finally, let $\Phi(A)$ be a partial isometry. Because A is EP, we then obtain $\mathcal{R}(\Phi(A)) = \mathcal{R}(A^+A^*) = A^+\mathcal{R}(A^*) = A^+\mathcal{R}(A) = \mathcal{R}(A^+A) = \mathcal{R}(A^*) = \mathcal{R}(A)$ and $\mathcal{R}((\Phi(A))^*) = \mathcal{R}((A^+A^*)^*) = \mathcal{R}(A(A^*)^+) = A\mathcal{R}(A) = A\mathcal{R}(A^*) = \mathcal{R}(AA^*) = \mathcal{R}(A) = \mathcal{R}(A^*)$. Consequently, $\mathcal{R}(\Phi(A)) = \mathcal{R}((\Phi(A))^*) = \mathcal{R}(A) = \mathcal{R}(A^*)$ and so $\Phi(A)(\Phi(A))^* = (\Phi(A))^*\Phi(A) = AA^+ = A^+A$. Premultiplying equation $A^+A = \Phi(A)(\Phi(A))^*$ by A yields $A = AA^+A = A\Phi(A)(\Phi(A))^* = AA^+A^*(A^+A^*)^* = A^*(A^+A^*)^*$, i.e., $A^*(A^+A^*)^* = A$ or, equivalently, $A^+A^*A = A^*$. Hence, $\Phi(A)A = A^*$. Combining this observation with $A\Phi(A) = AA^+A^* = A^*$ directly results in $A\Phi(A) = \Phi(A)A$, thus completing our proof of (i) $\Rightarrow$ (iii).

Problem 39-3: An Inequality Involving the Khatri-Rao Sum of Positive Definite Matrices

Proposed by Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au,
Götz TRENKLER, *Technische Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de,
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For two $m\times m$ positive definite matrices A and B, the Khatri-Rao sum is $A \diamond B = A * I_m + I_m * B$, where $*$ denotes the Khatri-Rao product and I_m is the $m \times m$ identity matrix. Let $n \times n$ matrix $S = A \diamond B^{-1} + A^{-1} \diamond B$, where A, A^{-1}, B and B^{-1} are compatibly partitioned. Prove the following inequality in the Löwner sense

$$4I_n \le S \le \frac{2(\lambda+\mu)}{\sqrt{\lambda\mu}}I_n,$$

where I_n is the $n\times n$ identity matrix, λ and μ are positive scalars such that all the eigenvalues of A, A^{-1}, B and B^{-1} are contained in the interval $[\lambda, \mu]$.

REMARK: For the notations, see e.g., S. Liu and G. Trenkler (2008), Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information & Systems Sciences* **4**, 160–177 in which our inequality is collected, and Z.P. Yang, X. Zhang

and C.G. Cao (2002), Inequalities involving Khatri-Rao products of Hermitian matrices, *The Korean Journal of Computational & Applied Mathematics* **9**, 125–133.

Solution 39-3.1 by the Proposers Shuangzhe LIU, *University of Canberra, Canberra, Australia:* Shuangzhe.Liu@canberra.edu.au, Götz TRENKLER, *Technische Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de, and Zhongpeng YANG, *Putian University, Putian China:* yangzhongpeng@126.com

It is known that

$$(X'HX)^{-1} \leq X'H^{-1}X \leq \frac{(\lambda+\mu)^2}{4\lambda\mu}(X'HX)^{-1}, \tag{2}$$

where $H > 0$ is a positive definite matrix with eigenvalues contained in the interval $[\lambda, \mu]$ and $X'X = I_n$; see e.g., Zhang (1999, page 206, Problem 11). Let's use $H = \mathrm{blockdiag}(A \bowtie I_m, A^{-1} \bowtie I_m, I_m \bowtie B, I_m \bowtie B^{-1})$ and $X = (Z', Z', Z', Z')'/2$, where $\bowtie$ denotes the Tracy-Singh product, Z is the selection matrix such that $Z'Z = I_n$, $Z'(A \bowtie I_m)Z = A * I_m$ and $X'X = I_n$; see e.g., Liu and Trenkler (2008) and Yang, Zhang and Cao (2002). Hence

$$\begin{aligned} X'HX &= (Z', Z', Z', Z')/2 \\ &\quad \mathrm{blockdiag}(A \bowtie I_m, A^{-1} \bowtie I_m, I_m \bowtie B, I_m \bowtie B^{-1})(Z', Z', Z', Z')'/2 \\ &= (A * I_m + A^{-1} * I_m + I_m * B + I_m * B^{-1})/4 \\ &= S/4, \end{aligned} \tag{3}$$

$$\begin{aligned} X'H^{-1}X &= (Z', Z', Z', Z')/2 \\ &\quad \mathrm{blockdiag}(A \bowtie I_m, A^{-1} \bowtie I_m, I_m \bowtie B, I_m \bowtie B^{-1})^{-1}(Z', Z', Z', Z')'/2 \\ &= (A * I_m + A^{-1} * I_m + I_m * B + I_m * B^{-1})/4 \\ &= S/4. \end{aligned} \tag{4}$$

Note that S is positive definite, because the Khatri-Rao products of positive definite matrices remain positive definite. Inserting (3) and (4) into (2) gives

$$(S/4)^{-1} \leq S/4 \leq \frac{(\lambda+\mu)^2}{4\lambda\mu}(S/4)^{-1}. \tag{5}$$

Pre- and post-multiplying (5) by $(S/4)^{1/2}$, we get

$$I_n \leq (S/4)^2 \leq \frac{(\lambda+\mu)^2}{4\lambda\mu} I_n.$$

This implies that

$$I_n \leq S/4 \leq \frac{(\lambda+\mu)}{2\sqrt{\lambda\mu}} I_n,$$

as in general $P \geq Q \geq 0$ implies $P^{1/2} \geq Q^{1/2}$, see Zhang (1999, page 169, Theorem 6.9). So

$$4I_n \leq S \leq \frac{2(\lambda+\mu)}{\sqrt{\lambda\mu}} I_n.$$

REMARK: A special case is when the Khatri-Rao product becomes the Hadamard product and the Tracy-Singh product becomes the Kronecker product.

References

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F.-Z. Zhang (1999). *Matrix Theory. Basic Results and Techniques.* Springer, New York.

Problem 39-4: On (α, β)-normal Operators in Hilbert Spaces

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An operator T acting on a (separable) Hilbert space is called (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ if

$$\alpha^2 T^*T \le TT^* \le \beta^2 T^*T.$$

For fixed $\alpha > 0$ and $\beta \neq 1$,

(i) give an example of an (α, β)-normal operator which is neither normal nor hyponormal;

(ii) is there any "nice" relation between norm, numerical radius and spectral radius of an (α, β)-normal operator?

Editorial Note. A partial solution to this problem is given in S.S. Dragomir & M.S. Moslehian. Some inequalities for (α, β)-normal operators in Hilbert spaces. *Facta Universitatis, Series: Mathematics and Informatics* (to appear).

We still look forward to receiving complementary solutions to Problem 39-4!

Problem 39-5: Two Commutativity Equalities for the Regularized Tikhonov Inverse

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The regularized Tikhonov inverse of a complex matrix A with respect to a positive number μ is defined to be

$$A^{\#} = (\,\mu I + A^*A\,)^{-1}A^*,$$

where A^* is the conjugate transpose of A. Suppose A is a square matrix of order n. Show that

(i) $AA^{\#} = A^{\#}A \Leftrightarrow AA^* = A^*A$.

(ii) $A^*A^{\#} = A^{\#}A^* \Leftrightarrow AA^*A^2 = A^2A^*A$.

Solution 39-5.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland:* baxx@amu.edu.pl, and Götz TRENKLER, *Technische Universität Dortmund, Dortmund, Germany:* trenkler@statistik.uni-dortmund.de

We restate Problem 39-5 as the proposition below, with matrices A and A^* interchanged.

PROPOSITION. *Let $A \in \mathbb{C}_{n,n}$ and let $\mu \in \mathbb{R}$ be positive. Moreover, let*

$$A^{\#} = (\mu I_n + AA^*)^{-1}A. \tag{6}$$

Then:

(i) $A^*A^{\#} = A^{\#}A^* \;\Leftrightarrow\; AA^* = A^*A$,

(ii) $AA^{\#} = A^{\#}A \;\Leftrightarrow\; A^*A(A^*)^2 = (A^*)^2AA^*$.

PROOF. On account of Corollary 6 in Hartwig & Spindelböck (1984), any matrix $A \in \mathbb{C}_{n,n}$ of rank r can be represented as

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{7}$$

where $U \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, ..., \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > ... > \sigma_t > 0$, $r_1 + r_2 + ... + r_t = r$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy

$$KK^* + LL^* = I_r. \tag{8}$$

From (7) it follows that matrix $A^{\#}$ defined in (6) can be rewritten as

$$A^{\#} = U\begin{pmatrix} (\mu I_r + \Sigma^2)^{-1}\Sigma K & (\mu I_r + \Sigma^2)^{-1}\Sigma L \\ 0 & 0 \end{pmatrix} U^*.$$

In consequence, the four products involved in equivalence (i) take the forms

$$A^*A^\# = U\begin{pmatrix} K^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma K & K^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma L \\ L^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma K & L^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma L \end{pmatrix}U^*, \tag{9}$$

$$A^\# A^* = U\begin{pmatrix} (\mu I_r+\Sigma^2)^{-1}\Sigma^2 & 0 \\ 0 & 0 \end{pmatrix}U^*, \tag{10}$$

$$AA^* = U\begin{pmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{pmatrix}U^*, \quad \text{and} \quad A^*A = U\begin{pmatrix} K^*\Sigma^2 K & K^*\Sigma^2 L \\ L^*\Sigma^2 K & L^*\Sigma^2 L \end{pmatrix}U^*, \tag{11}$$

whereas the four products involved in equivalence (ii) take the forms

$$AA^\# = U\begin{pmatrix} \Sigma K(\mu I_r+\Sigma^2)^{-1}\Sigma K & \Sigma K(\mu I_r+\Sigma^2)^{-1}\Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{12}$$

$$A^\# A = U\begin{pmatrix} (\mu I_r+\Sigma^2)^{-1}\Sigma K\Sigma K & (\mu I_r+\Sigma^2)^{-1}\Sigma K\Sigma \\ 0 & 0 \end{pmatrix}U^*, \tag{13}$$

$$A^*A(A^*)^2 = U\begin{pmatrix} K^*\Sigma^3 K^*\Sigma & 0 \\ L^*\Sigma^3 K^*\Sigma & 0 \end{pmatrix}U^*, \quad \text{and} \quad (A^*)^2AA^* = U\begin{pmatrix} K^*\Sigma K^*\Sigma^3 & 0 \\ L^*\Sigma K^*\Sigma^3 & 0 \end{pmatrix}U^*. \tag{14}$$

On account of the nonsingularity of Σ, it is seen that $L^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma L = 0 \Leftrightarrow L = 0$, with the condition on the right-hand side being, in view of (8), equivalent to $K^* = K^{-1}$. Hence, from formulae (9) and (10) it is obvious that $A^*A^\# = A^\# A^*$ if and only if $L = 0$ and $K^*\Sigma(\mu I_r+\Sigma^2)^{-1}\Sigma K = (\mu I_r+\Sigma^2)^{-1}\Sigma^2$. Clearly, the matrices Σ and $(\mu I_r+\Sigma^2)^{-1}$ commute and, thus, the latter condition can be rewritten as $(\mu I_r+\Sigma^2)^{-1}\Sigma^2 K = K\Sigma^2(\mu I_r+\Sigma^2)^{-1}$, which, by premultiplying and postmultiplying both sides by $\mu I_r+\Sigma^2$, can be reduced to $\Sigma^2 K = K\Sigma^2$. Denoting the entries of $K \in \mathbb{C}_{r,r}$ by k_{ij}, $i,j = 1,...,r$, it is seen that $\Sigma^2 K = K\Sigma^2$ can be equivalently expressed as the set of $r(r-1)$ scalar equations $(\sigma_i^2 - \sigma_j^2)k_{ij} = 0$, $i \neq j$, $i,j = 1,...,r$, each of which is satisfied if and only if either $\sigma_i = \sigma_j$ or $k_{ij} = 0$. Hence, we arrive at $\Sigma^2 K = K\Sigma^2 \Leftrightarrow \Sigma K = K\Sigma$. Concluding, the left-hand side condition in equivalence (i) holds if and only if $L = 0$ and $\Sigma K = K\Sigma$. On the other hand, the fact that this conjunction constitutes also necessary and sufficient conditions for the normality of A (i.e., $AA^* = A^*A$) is easily seen from formulae (11).

For the proof of equivalence (ii), first observe that from formulae (12) and (13) it follows that $AA^\# = A^\# A$ if and only if $\Sigma K(\mu I_r+\Sigma^2)^{-1}\Sigma K = (\mu I_r+\Sigma^2)^{-1}\Sigma K\Sigma K$ and $\Sigma K(\mu I_r+\Sigma^2)^{-1}\Sigma L = (\mu I_r+\Sigma^2)^{-1}\Sigma K\Sigma L$. On account of (8) and the nonsingularity of Σ, combining the former of these conditions postmultiplied by K^* with the latter one postmultiplied by L^*, leads to $\Sigma K(\mu I_r+\Sigma^2)^{-1} = (\mu I_r+\Sigma^2)^{-1}\Sigma K$. Hence, by premultiplying and postmultiplying both sides by $\mu I_r+\Sigma^2$ and, once again, referring to the nonsingularity of Σ, we get $\Sigma^2 K = K\Sigma^2$, i.e., $\Sigma K = K\Sigma$. Conversely, from (14) it is seen that $A^*A(A^*)^2 = (A^*)^2AA^*$ is equivalent to $K^*\Sigma^3K^*\Sigma = K^*\Sigma K^*\Sigma^3$ and $L^*\Sigma^3K^*\Sigma = L^*\Sigma K^*\Sigma^3$. Combining the former of these conditions premultiplied by K with the latter one premultiplied by L, on account of (8) and the nonsingularity of Σ, gives $\Sigma^2K^* = K^*\Sigma^2$, or, equivalently, $\Sigma K = K\Sigma$. □

In a comment to Proposition it is noteworthy that the two conditions occurring in its proof attribute matrix A to the known classes of matrices. Namely, A having representation (7) is an EP matrix (i.e., A and its Moore–Penrose inverse commute) if and only if $L = 0$, and A is a star-dagger matrix (i.e., the conjugate transpose and the Moore–Penrose inverse of A commute) if and only if $\Sigma K = K\Sigma$. Thus, the proof corresponding to equivalence (i) reestablished the known fact that a matrix is normal if and only if it is simultaneously EP and star-dagger, whereas the proof corresponding to equivalence (ii) the fact that AA^*A and A commute if and only if A is star-dagger; see e.g., Hartwig & Spindelböck (1984).

Reference

R.E. Hartwig & K. Spindelböck (1984). Matrices for which A^* and $A^\dagger$ commute. *Linear and Multilinear Algebra* **14**, 241-256.

Solution 39-5.2 by Alicja SMOKTUNOWICZ, *Warsaw University of Technology, Poland:* smok@mini.pw.edu.pl

In order to prove (i), let $A(n\times n)$, $\mu > 0$ and define $B = (\mu I + A^*A)^{-1}$. Then $A^\# = BA^*$ and $A^*A = B^{-1} - \mu I$. Notice that the condition $AA^\# = A^\# A$ is equivalent to $ABA^* = BA^*A$. First we verify the implication $AA^\# = A^\# A \Longrightarrow AA^* = A^*A$. Postmultiplying the equation $ABA^* = BA^*A$ by A we get $AB(A^*A) = B(A^*A)A$. Now put $A^*A = B^{-1} - \mu I$. Then $AB(B^{-1} - \mu I) = B(B^{-1} - \mu I)A$, so $A - \mu AB = A - \mu BA$, thus $AB = BA$. Then we rewrite $ABA^* = BA^*A$ as $ABA^* = BAA^* = BA^*A$ and by the nonsingularity of B we get the desired equality $AA^* = A^*A$. Now assume that $AA^* = A^*A$. Premultiplying this equation by A we get $AA^*A = A^*AA$, i.e., $A(A^*A) = (A^*A)A$. Therefore, $A(B^{-1} - \mu I) = (B^{-1} - \mu I)A$.

This leads to $AB^{-1} - \mu A = B^{-1}A - \mu A$. We see that $AB^{-1} = B^{-1}A$, so $BA = AB$. From this and the assumption $AA^* = A^*A$ we have $ABA^* = BAA^* = B(AA^*) = B(A^*A)$, which completes the proof of (i).

To prove (ii), let $A = U\Sigma V^*$ be the SVD of $A(n \times n)$ of rank r. Here U and V are unitary matrices, $\Sigma = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0)$ is diagonal and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ (so called the singular values of A). We have the following decompositions: $A^* = V\Sigma U^*$, $AA^* = U\Sigma^2 U^*$, $A^*A = V\Sigma^2 V^*$ and $A^{\#} = V(\mu I + \Sigma^2)^{-1}\Sigma U^*$. Notice that $A^*A^{\#} = A^{\#}A^*$ is equivalent to $(A^{\#})^*A = A(A^{\#})^*$, which can be rewritten as

$$\Sigma(\mu I + \Sigma^2)^{-1}(V^*U)\Sigma = \Sigma(V^*U)\Sigma(\mu I + \Sigma^2)^{-1}. \tag{15}$$

Both Σ and $(\mu I + \Sigma^2)^{-1}$ are diagonal, so they commute. Therefore, (15) is equivalent to

$$(\mu I + \Sigma^2)^{-1}\Sigma(V^*U)\Sigma = \Sigma(V^*U)\Sigma(\mu I + \Sigma^2)^{-1}. \tag{16}$$

It is clear that equality in (16) holds if and only if $\Sigma(V^*U)\Sigma(\mu I + \Sigma^2) = (\mu I + \Sigma^2)\Sigma(V^*U)\Sigma$, which is equivalent to

$$\Sigma(V^*U)\Sigma^3 = \Sigma^3(V^*U)\Sigma. \tag{17}$$

Clearly, (17) holds if and only if $A^2A^*A = AA^*A^2$. This completes the proof of (ii).

Solution 39-5.3 by the Proposer Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:* yongge@mail.shufe.edu.cn

Note that

$$\begin{aligned} AA^{\#} - A^{\#}A &= A(\mu I + A^*A)^{-1}A^* - (\mu I + A^*A)^{-1}A^*A \\ &= (\mu I + A^*A)^{-1}[(\mu I + A^*A)A(\mu I + A^*A)^{-1}A^* - A^*A]. \end{aligned}$$

Hence,

$$\mathrm{rank}(AA^{\#} - A^{\#}A) = \mathrm{rank}((\mu A + A^*A^2)(\mu I + A^*A)^{-1}A^* - A^*A) = \mathrm{rank}(\begin{pmatrix} \mu I + A^*A & A^* \\ \mu A + A^*A^2 & A^*A \end{pmatrix}) - n$$
$$= \mathrm{rank}(\begin{pmatrix} \mu I & A^* \\ \mu A & A^*A \end{pmatrix}) - n = \mathrm{rank}(\begin{pmatrix} I & 0 \\ 0 & A^*A - AA^* \end{pmatrix}) - n = \mathrm{rank}(AA^* - A^*A).$$

The equivalence in (i) is a direct consequence of the rank equality. Next, note that

$$\begin{aligned} A^*A^{\#} - A^{\#}A^* &= A^*(\mu I + A^*A)^{-1}A^* - (\mu I + A^*A)^{-1}(A^*)^2 \\ &= (\mu I + A^*A)^{-1}[(\mu I + A^*A)A^*(\mu I + A^*A)^{-1}A^* - (A^*)^2]. \end{aligned}$$

Consequently,

$$\mathrm{rank}(A^*A^{\#} - A^{\#}A^*) = \mathrm{rank}(A(\mu I + A^*A)^{-1}A(\mu I + A^*A) - A^2) = \mathrm{rank}\begin{pmatrix} \mu I + A^*A & \mu A + AA^*A \\ A & A^2 \end{pmatrix} - n$$
$$= \mathrm{rank}\begin{pmatrix} \mu I & AA^*A - A^*A^2 \\ A & 0 \end{pmatrix} - n = \mathrm{rank}\begin{pmatrix} I & 0 \\ 0 & A^2A^*A - AA^*A^2 \end{pmatrix} - n = \mathrm{rank}(AA^*A^2 - A^2A^*A).$$

The equivalence in (ii) is a direct consequence of the rank equality.

Problem 39-6: Two Equalities for the Moore-Penrose Inverse of a Row Block Matrix
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Suppose A and B are two $m \times n$ and $m \times p$ matrices, respectively, and let $(\cdot)^*$ and $(\cdot)^{\dagger}$ denote the conjugate transpose and the Moore-Penrose inverse of a complex matrix, respectively. Show that

(i) $(\,A \quad B\,)(\,A \quad B\,)^{\dagger} = \frac{1}{2}(AA^{\dagger} + BB^{\dagger}) \Leftrightarrow \mathrm{range}(A) = \mathrm{range}(B)$.

(ii) $(\,A \quad B\,)^{\dagger} = \frac{1}{2}\begin{pmatrix} A^{\dagger} \\ B^{\dagger} \end{pmatrix} \Leftrightarrow AA^* = BB^*$.

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We utilize a different notation than the one proposed in the problem, denoting by $\mathcal{R}(.)$ the range of a matrix, and by $(.\ :\ .)$ the columnwise partitioned matrix. Recall that the Moore–Penrose inverse of $K \in \mathbb{C}_{m,n}$ is the unique matrix $K^\dagger \in \mathbb{C}_{n,m}$ satisfying the four equations

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A. \tag{18}$$

It is known that $P_A = AA^\dagger$ and $P_{A^*} = A^\dagger A$ represent the orthogonal projectors onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

For the proof of equivalence (i), first observe that, on account of the first condition in (18), the left-hand side equality in (i) implies $BB^\dagger A = A$ and $AA^\dagger B = B$, or, equivalently, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Thus, $\mathcal{R}(A) = \mathcal{R}(B)$, establishing the necessity. To show the reverse implication, we utilize the well-known formula $P_{(A\,:\,B)} = P_A + (I_m - P_A)P_B$, where I_m denotes the identity matrix of order m. Under $\mathcal{R}(A) = \mathcal{R}(B)$, this formula reduces to $P_{(A\,:\,B)} = P_A$, or, in other words, $(A\ :\ B)(A\ :\ B)^\dagger = AA^\dagger$. Hence, in view of the fact that $\mathcal{R}(A) = \mathcal{R}(B)$ can be equivalently expressed as $AA^\dagger = BB^\dagger$, the proof of equivalence (i) is complete.

To proof equivalence (ii), first observe that direct verifications of definition (18) show that $\frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}$ is the Moore–Penrose inverse of $(A\ :\ B)$ if and only if $\mathcal{R}(B) = \mathcal{R}(A)$ and $B^*(A^*)^\dagger = B^\dagger A$. Premultiplying and postmultiplying the latter condition by B and A^*, respectively, on account of the former condition, leads to $AA^* = BB^*$. Since, $\mathcal{R}(KK^*) = \mathcal{R}(K)$ for any matrix K, the necessity is established. In the proof of the reverse implication, we utilize Theorem 3.1 in Horn and Olkin (1996), according to which $AA^* = BB^*$ if and only if $A = BU$ for some $U \in \mathbb{C}_{p,n}$ with orthonormal rows. Hence, $A^\dagger = U^*B^\dagger$ and, in consequence,

$$(A\ :\ B) = B(U\ :\ I_p) \quad \text{and} \quad \begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = \begin{pmatrix} U^* \\ I_p \end{pmatrix} B^\dagger. \tag{19}$$

Substituting representations (19) into definition (18) shows that $\frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}$ is indeed the Moore–Penrose inverse of $(A\ :\ B)$.

Reference

R.A. Horn & I. Olkin (1996). When does $A^*A = B^*B$ and why does one want to know? *American Mathematical Monthly* **103**, 470-482.

Solution 39-6.2 by the Proposer Yongge TIAN, *Shanghai University of Finance and Economics, Shanghai, China:*
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"$\Leftarrow$ of (i)": Recall that the orthogonal projector onto the range of a matrix M can be written as $MM^\dagger$. Also note that if $\mathrm{range}(A) = \mathrm{range}(B)$, then $\mathrm{range}((\,A \quad B\,)) = \mathrm{range}(A) = \mathrm{range}(B)$. Hence the orthogonal projectors onto the ranges of $(\,A \quad B\,)$, A and B are the same, that is, $(\,A \quad B\,)(\,A \quad B\,)^\dagger = AA^\dagger = BB^\dagger$ hold.

"$\Rightarrow$ of (i)": Premultiplying $2\,(\,A \quad B\,)(\,A \quad B\,)^\dagger = AA^\dagger + BB^\dagger$ by $(\,A \quad B\,)^*$ yields the following equality

$$2\,(\,A \quad B\,)^* = (\,A \quad B\,)^*\,(AA^\dagger + BB^\dagger) = \begin{pmatrix} A^* + A^*BB^\dagger \\ B^* + B^*AA^\dagger \end{pmatrix}.$$

Comparing both sides leads to $BB^\dagger A = A$ and $AA^\dagger B = B$, which imply $\mathrm{range}(A) = \mathrm{range}(B)$.

"$\Leftarrow$ of (ii)": Pre- and post-multiplying $AA^* = BB^*$ by $A^\dagger$ and $(B^\dagger)^*$ gives

$$A^*(B^\dagger)^* = A^\dagger B. \tag{20}$$

The equality $AA^* = BB^*$ also implies that $\mathrm{range}(A) = \mathrm{range}(B)$, so that

$$AA^\dagger = BB^\dagger. \tag{21}$$

Under (20) and (21),

$$\frac{1}{2}(A,\, B)\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = \frac{1}{2}(AA^\dagger + BB^\dagger) = AA^\dagger = BB^\dagger, \tag{22}$$

$$\frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}(A,\, B) = \frac{1}{2}\begin{pmatrix} A^\dagger A & A^\dagger B \\ B^\dagger A & B^\dagger B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A^\dagger A & (B^\dagger A)^* \\ B^\dagger A & B^\dagger B \end{pmatrix}, \tag{23}$$

so that both (22) and (23) are Hermitian. It can also be derived from (22) that

$$\frac{1}{2}(\,A \quad B\,)\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}(\,A \quad B\,) = (\,A \quad B\,), \quad \frac{1}{4}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}(\,A \quad B\,)\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}. \tag{24}$$

By the definition of the Moore-Penrose inverse, (22), (23) and (24) imply that $\frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}$ is the Moore-Penrose inverse of $(A,\, B)$.

"$\Rightarrow$ of (ii)": Recall that any $M^\dagger$ can be written as $M^\dagger = M^*(MM^*)^\dagger$. Hence $(A,\, B)^\dagger$ can be written as $(A,\, B)^\dagger = \begin{pmatrix} A^*(AA^*+BB^*)^\dagger \\ B^*(AA^*+BB^*)^\dagger \end{pmatrix}$. Thus $(A,\, B)^\dagger = \frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}$ implies

$$2A^*(AA^*+BB^*)^\dagger = A^\dagger \ \text{ and } \ 2B^*(AA^*+BB^*)^\dagger = B^\dagger. \tag{25}$$

Pre- and post-multiplying the two equalities in (25) by AA^*A, BB^*B and AA^*+BB^*, and simplifying yield

$$2AA^*AA^* = AA^*(AA^*+BB^*), \quad 2BB^*BB^* = BB^*(AA^*+BB^*),$$

that is, $AA^*AA^* = AA^*BB^*$ and $BB^*BB^* = BB^*AA^*$. Adding both equalities leads to $AA^*AA^* + BB^*BB^* = AA^*BB^* + BB^*AA^*$, that is, $(AA^*-BB^*)(AA^*-BB^*)^* = 0$, which implies $AA^*-BB^* = 0$.

Solution 39-6.3 by Hans Joachim Werner, *Universität Bonn, Bonn, Germany:* hjw.de@uni-bonn.de

First, we note that if $M \in \mathbb{C}^{m\times n}$, then $(\mathcal{N}(M^*))^\perp = \mathcal{R}(M)$ and $(\mathcal{R}(M^*))^\perp = \mathcal{N}(M)$, with $(\cdot)^*$, $\mathcal{N}(\cdot)$, and $\mathcal{R}(\cdot)$ denoting the conjugate transpose, the null space, and the range (column space), respectively, of the matrix $(\cdot)$, and with $(\cdot)^\perp$ indicating the orthogonal complement of linear space $(\cdot)$ with respect to the usual inner product. Secondly, we mention that $MM^\dagger$ and $M^\dagger M$ are the orthogonal projectors $P_{\mathcal{R}(M)}$ onto $\mathcal{R}(M)$ (along $\mathcal{N}(M^*)$) and $P_{\mathcal{R}(M^*)}$ onto $\mathcal{R}(M^*)$ (along $\mathcal{N}(M)$), respectively. Thirdly, we note that $\mathcal{R}(M^\dagger) = \mathcal{R}(M^*)$, and $\mathcal{N}(M^\dagger) = \mathcal{N}(M^*)$. And finally, we recall the following well-known characterization: Given $M \in \mathbb{C}^{m\times n}$, then $G = M^\dagger$ if and only if $MG = P_{\mathcal{R}(M)}$, $GM = P_{\mathcal{R}(M^*)}$ and $\operatorname{rank}(M) = \operatorname{rank}(G)$ hold simultaneously.

Proof of (i): We prove the following version of (i):

$$P_{\mathcal{R}(A)+\mathcal{R}(B)} = \frac{1}{2}(P_{\mathcal{R}(A)} + P_{\mathcal{R}(B)}) \quad \Longleftrightarrow \quad \mathcal{R}(A) = \mathcal{R}(B).$$

First, let $\mathcal{R}(A) = \mathcal{R}(B)$. Then $\mathcal{R}(\,A \quad B\,) = \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) = \mathcal{R}(B)$ and so $P_{\mathcal{R}(A)} = P_{\mathcal{R}(B)} = P_{\mathcal{R}(A)+\mathcal{R}(B)}$ or, equivalently, $P_{\mathcal{R}(A)+\mathcal{R}(B)} = \frac{1}{2}(P_{\mathcal{R}(A)} + P_{\mathcal{R}(B)})$, as claimed. Conversely, let $P_{\mathcal{R}(A)+\mathcal{R}(B)} = \frac{1}{2}(P_{\mathcal{R}(A)} + P_{\mathcal{R}(B)})$. Then $(\,A \quad B\,) = P_{\mathcal{R}(A)+\mathcal{R}(B)}(\,A \quad B\,) = \frac{1}{2}(P_{\mathcal{R}(A)} + P_{\mathcal{R}(B)})(\,A \quad B\,)$ or, equivalently, $P_{\mathcal{R}(B)}A = A$ and $P_{\mathcal{R}(A)}B = B$, which in turn happens if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(B) \subseteq \mathcal{R}(A)$, i.e., if and only if $\mathcal{R}(A) = \mathcal{R}(B)$. This completes the proof of (i).

Proof of (ii): For convenience, put

$$M := \frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}. \tag{26}$$

First, let

$$(\,A \quad B\,)^\dagger = M. \tag{27}$$

Then $P_{\mathcal{R}(A)+\mathcal{R}(B)} = \frac{1}{2}(P_{\mathcal{R}(A)} + P_{\mathcal{R}(B)})$ and so, according to (i), $\mathcal{R}(A) = \mathcal{R}(B)$. From (27) it further follows that

$$P_{\mathcal{R}\begin{pmatrix} A^* \\ B^* \end{pmatrix}} = \frac{1}{2}\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}(\,A \quad B\,) = \frac{1}{2}\begin{pmatrix} A^\dagger A & A^\dagger B \\ B^\dagger A & B^\dagger B \end{pmatrix}. \tag{28}$$

Relationships (28) particularly tell us that

$$\begin{pmatrix} A^* \\ B^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A^\dagger A & A^\dagger B \\ B^\dagger A & B^\dagger B \end{pmatrix}\begin{pmatrix} A^* \\ B^* \end{pmatrix} \tag{29}$$

which happens if and only if

$$A^* = A^\dagger BB^*, \quad B^* = B^\dagger AA^*. \tag{30}$$

Premultiplying equation $A^* = A^\dagger BB^*$ by A or equation $B^* = B^\dagger AA^*$ by B, in view of $\mathcal{R}(A) = \mathcal{R}(B)$, readily results in $AA^* = BB^*$, as claimed. Conversely, let

$$AA^* = BB^*. \tag{31}$$

Clearly, (31) implies (30), and because we already do know that (30) is equivalent to (29), we have (29). Observe next that the relationship (31) implies $\mathcal{R}(A) = \mathcal{R}(B)$ or, equivalently, $\mathcal{N}(A^*) = \mathcal{N}(B^*)$. Therefore, according to (i),

$$(\, A \quad B\,)(\, A \quad B\,)^\dagger = \frac{1}{2}(AA^\dagger + BB^\dagger) = \frac{1}{2}(\, A \quad B\,)\begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix}. \tag{32}$$

Furthermore, from the definition of M in (26), we now additionally obtain

$$\mathcal{N}(M) = \mathcal{N}(A^\dagger) \cap \mathcal{N}(B^\dagger) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*) = \mathcal{N}(A^*) = \mathcal{N}(B^*).$$

Hence,

$$\mathrm{rank}(M) = \mathrm{rank}(\, A \quad B\,). \tag{33}$$

Because $M = (\, A \quad B\,)^\dagger$ if and only if all the equations in (33), (32) and (29) do hold simultaneously, our proof of claim (ii) is complete.

A Solution to Problem 39-6 was also received from Johanns de Andrade Bezerra.

IMAGE Problem Corner: New Problems

Problem 41-1: A Lower Bound for the Rank of $A + A^*$

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Let A^* be the conjugate transpose of $A \in \mathbb{C}_{n,n}$. Provide a nontrivial lower bound for rk$(A + A^*)$, where rk$(.)$ denotes the rank of a matrix argument.

Problem 41-2: An Inequality Involving a Semi-Inner Product

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Let $\mathcal{X}$ be a vector space over $\mathbb{C}$, equipped with a semi-inner product $< \cdot, \cdot >$. For $a, b, c, d \in \mathcal{X}$, show that

$$2\,| < a, c >< b, d > | \leqslant < a, a >< b, b > + < c, c >< d, d > .$$

Problem 41-3: A Property of the Range of Generalized and Hypergeneralized Projectors

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Let A^* and $A^\dagger$ denote the conjugate transpose and the Moore–Penrose inverse of $A \in \mathbb{C}_{n,n}$, respectively.

(i) Prove or disprove that if A is a hypergeneralized projector, i.e., satisfies $A^2 = A^\dagger$, then

$$\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*), \tag{34}$$

where $\mathcal{R}(.)$ is the range (column space) of a matrix argument.

(ii) Is relationship (34) fulfilled if A is a generalized projector, i.e., satisfies $A^2 = A^*$?

REMARK. The notions of generalized and hypergeneralized projectors were introduced by J. Groß & G. Trenkler (1997). Generalized and hypergeneralized projectors. *Linear Algebra and its Applications* **264**, 463-474.

Problem 41-4: Integer Matrices with Unit Determinants II

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

(i) For positive integer values of d, f and g the equation $(f^2 - dg^2)^2 + d(2fg)^2 = (f^2 + dg^2)^2$ supplies sets of three integers satisfying the generalised Pythagorean equation $x^2 + dy^2 = z^2$. Outline a matrix proof of this result.

(ii) In IMAGE Problem 36-1 [see IMAGE **37** (Fall 2006), pp. 20-23 for the problem and its solutions by Farebrother and by Werner] we were concerned with the construction of a 2×2 matrix with a determinant of plus one, and thus with the solution of Pell's Equation $f^2 - dg^2 = +1$ where f and g are nonzero integers and d is a fixed positive integer (though not a square). If instead we assume that the matrix has a determinant of minus one, then we are concerned with the solution of a variant of Pell's Equation (which, for obvious reasons, we shall name Nell's Equation) $f^2 - dg^2 = -1$. Identify integer solutions to this equation for certain choices of d. Use the result established in Part (i) to obtain the corresponding solutions to Pell's Equation.

Problem 41-5: Square Root of a Real Symmetric 3×3 Matrix

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Given a real symmetric positive definite 3×3 matrix A, outline a direct procedure not involving the singular values or eigenvalues of A for computing a real symmetric positive definite 3×3 matrix B satisfying $B^2 = A$.

Problem 41-6: Square Root of a 2×2 P-Matrix

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Given a 2×2 matrix A with positive diagonal elements and a positive determinant, identify a 2×2 matrix B with the same properties and satisfying $B^2 = A$.

Problem 41-7: Simplification of a 3×3 Matrix

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England, UK:* R.W.Farebrother@Manchester.ac.uk

Let A be a 3×3 matrix with nonzero elements. Show that A can be expressed in the form $A = D + fg'$ where D is a 3×3 diagonal matrix and f and g are 3×1 column matrices if and only if the off-diagonal elements of A satisfy the condition: $a_{12}a_{23}a_{31} = a_{13}a_{21}a_{32}$.

Problem 41-8: On the Moore-Penrose Inverse of a Particular Block Partitioned Matrix

Proposed by William F. TRENCH, *Trinity University, San Antonio, Texas, USA:* wtrench@trinity.edu

Prove the following theorem.

THEOREM. *Let $c_0, \dots, c_{k-1}$, $d_0, \dots, d_{k-1}$ be positive integers with $k \geq 2$ and let μ, $p_0, \dots, p_{k-1}$, and $q_0, \dots, q_{k-1}$ be integers such that the set $\mathcal{T} := \{(r, s) \in \mathbb{Z}_k \times \mathbb{Z}_k \,|\, p_r + q_s \equiv \mu \pmod{k}\}$ is nonempty. Suppose that $C = [C_{rs}]_{r,s=0}^{k-1}$ with $C_{rs} \in \mathbb{C}^{c_r \times d_s}$ and $C_{rs} = 0$ if $(r, s) \notin \mathcal{T}$. Then $C^\dagger = [D_{rs}]_{r,s=0}^{k-1}$ with $D_{rs} \in \mathbb{C}^{d_r \times c_s}$ and $D_{rs} = 0$ if $(s, r) \notin \mathcal{T}$. Moreover, if $r \neq r'$ and $s \neq s'$ whenever (r, s) and (r', s') are distinct pairs in $\mathcal{T}$, then $D_{rs} = C_{sr}^\dagger$, $(s, r) \in \mathcal{T}$, where $(\cdot)^\dagger$ indicates the Moore-Penrose inverse of matrix $(\cdot)$.*

Problem 41-9: Conjugate Annihilating Normal Matrices

Proposed by Geoffrey Goodson, *Towson University, Towson, USA,* goodson@towson.edu
and Roger A. Horn, *University of Utah, Salt Lake City, USA*, rhorn@math.utah.edu

Let A be a normal matrix with complex conjugate $\bar{A}$ and transpose A^T. Show that $A\bar{A} = 0$ if and only if $AA^T = A^TA = 0$.

Problem 41-10: Difference of Two Symmetric Matrices

Proposed by Heinz Neudecker, *University of Amsterdam, Amsterdam, The Netherlands*
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany*, trenkler@statistik.uni-dortmund.de

Let $\mathbf{Y}$ be an $n \times n$ real symmetric matrix with rank $n-1$. Furthermore suppose that there exists a nonsingular symmetric matrix $\mathbf{T}$ satisfying $\mathbf{YTY} = \mathbf{Y}$. Show that $\mathbf{Y} - \mathbf{T}^{-1}$ is either nonnegative or nonpositive definite of rank one.

Problem 41-11: Positive Semidefinite Matrices and Tensor Product

Proposed by Chi-Kwong Li, *College of William and Mary, Williamsburg, USA,* ckli@math.wm.edu

Let $A,\ B$ be $m \times m$ and $C,\ D$ be $n \times n$ positive definite matrices, respectively. For any $r \in (0,1)$ and $a,\ b > 0$, show that

$$(aA + bB)^r \otimes (aC + bD)^{(1-r)} \geq a(A^r \otimes C^{(1-r)}) + b(B^r \otimes D^{(1-r)}),$$

where $X \otimes Y$ denotes the tensor product of matrices X and Y and $P \geq Q$ means $P - Q$ is positive semidefinite. Deduce that the result also holds if the tensor product is replaced by the Hadamard product.

Problem 41-12: An Inequality Involving a Product of Two Orthogonal Projectors

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technical Universität Dortmund, Dortmund, Germany*, trenkler@statistik.uni-dortmund.de

Let P and Q be $n \times n$ complex orthogonal projectors (i.e., Hermitian idempotent matrices of order n). Show that

$$[\mathrm{trace}(PQ)]^2 \leqslant \mathrm{rank}(PQ)\,\mathrm{trace}(PQPQ),$$

with equality if and only if λPQ is an oblique projector (i.e., idempotent matrix) for some scalar $\lambda > 0$.

Problem 41-13: Range Additivity of A and A^*

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
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Let A be a square complex matrix and let A^* denote its conjugate transpose. An eigenvector x of A with associated eigenvalue λ is a *normal eigenvector* if both $Ax = \lambda x$ and $A^*x = \bar{\lambda}x$. Consider the range additivity condition

$$\mathrm{range}(A + A^*) = \mathrm{range}A + \mathrm{range}A^*. \tag{35}$$

(a) Suppose A^2 is normal. Show that A satisfies (35) if and only if for every normal eigenvector of A the associated eigenvalue is either zero or not pure imaginary. In particular, if A is normal then it satisfies (35) if and only if no nonzero eigenvalue of A is pure imaginary.

(b) Suppose $A^2 = \lambda A$ for some $\lambda \in \mathbb{C}$. Show that A satisfies (35) if and only if for every normal eigenvector of A the associated eigenvalue is either zero or not pure imaginary. The special cases $\lambda = 0$ (A is self-annihilating) and $\lambda = 1$ (A is a projection) are noteworthy.

(c) Suppose $A^k = A^*$ for some $k \in \{2, 3, \ldots\}$. If k is even, show that A satisfies (35); if k is odd, show that A satisfies (35) if and only if no nonzero eigenvalue of A is pure imaginary.

Examples: Consider $A_1 = [i]$, $A_2 = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2i & 1 \\ 0 & 0 \end{bmatrix}$.

The matrix A_1 does not satisfy the range additivity property and has a nonzero pure imaginary eigenvalue associated with a normal eigenvector; moreover, $A_1^3 = A^*$. The matrix A_2 satisfies the range additivity property and has nonzero pure imaginary eigenvalues; A_2^2 is normal but A_2 has no normal eigenvectors. The matrix A_3 satisfies the range additivity property and has a nonzero pure imaginary eigenvalue; $A_3^2 = 2iA_3$ but A_3 has no normal eigenvectors.

Problem 41-14: Minimum Eigenvalue of a Special Matrix

Proposed by Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, USA*, zhang@nova.edu

(a). Let $x_1, \ldots, x_n \in [0,1]$ and $A = (x_i + x_j)$. Find the smallest eigenvalue of A in terms of $x_1, \ldots, x_n$.
(b). Let Ω_n be the set of all $n \times n$ matrices as defined in (a). Find the best lower and upper bounds for the eigenvalues of all these matrices; that is, find the largest number α and the smallest number β so that $\lambda \in [\alpha, \beta]$ for all eigenvalues λ of any matrix A in Ω_n.