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# IMAGE

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**Upcoming Conferences and Workshops**



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**IMAGE Problem Corner: Old Problems With Solutions**



**IMAGE Problem Corner: New Problems**

# UPCOMING CONFERENCES AND WORKSHOPS

## 3rd International Conference on Mathematics and Statistics

**Athens, Greece**
**June 15-18, 2009**

The Mathematics & Statistics Research Unit of the Athens Institute for Education and Research (ATINER) will hold its 3rd International Conference in Athens, Greece, June 15-18, 2009. Prof. Vladimir Akis is the Chair, and the conference website is www.atiner.gr/docs/Mathematics.htm.

All areas of Mathematics, Statistics, Mathematics & Engineering and Mathematics & Education will be included, and selected papers will be published in a Special Volume of the Conference Proceedings.

The Athens Institute for Education and Research (ATINER) was established in 1995 as an independent academic organization with the mission to become a forum where academics and researchers from all over the world could meet in Athens and exchange ideas on their research and discuss the future developments of their discipline. It is organized in four research divisions and nineteen research units. Each research unit organizes at least one annual conference and undertakes various small and large research projects.

## SIAM Annual Meeting and Conference on Control and Its Applications

**Denver, Colorado, USA**
**July 6-10, 2009**

The 2009 SIAM Annual Meeting will be held in Denver, Colorado, USA, on July 6-10, 2009. It aims to include all areas of applied mathematics. Visit http://www.siam.org/meetings/an09/ for details.

Jointly with it, there will be a SIAM Conference on Control and Its Applications, July 6-8. This meeting will showcase a wide range of topics in control and systems theory, including real-time optimization and data assimilation, cellular and biological regulation, control techniques for financial mathematics, cooperative control for unmanned autonomous vehicles, biomedical control, risk sensitive control and filtering, control of smart systems, flow control and quantum control. Visit http://www.siam.org/meetings/ct09/ for details.

## 18th International Workshop on Matrices and Statistics 2009

**Smolenice Castle, Slovakia**
**June 23-27, 2009**

The 18th International Workshop on Matrices and Statistics will be held in Smolenice Castle, Slovakia, June 23-27, 2009. The purpose is to stimulate exchanges of ideas and research at the interfaces of matrix theory, statistics, and stochastic processes. The Workshop includes invited and contributed talks and posters. It has been endorsed by the International Linear Algebra Society.

The invited speakers are Miroslav Fiedler (Czech Republic), Karl Gustafson (USA), Mike Kenward (Great Britain), Tonu Kollo (Estonia), Lubomír Kubacek (Czech Republic), Yonghui Liu (China), Ivo Marek (Czech Republic), Ingram Olkin (USA), Andrej Pazman (Slovakia), Friedrich Pukelsheim (Germany), Alastair J. Scott (New Zealand), and Roman Zmyslony (Poland).

The Organizing Committee is Julia Volaufova - Chair (USA), Simo Puntanen - Vice-Chair (Finland), George P. H. Styan - Honorary Chair (Canada), Augustyn Markiewicz (Poland), Jeffrey J. Hunter (New Zealand), S.Ejaz Ahmed (Canada), Goetz Trenkler (Germany), Dietrich von Rosen (Sweden), and Hans Joachim Werner(Germany). For more information, visit http://www.um.sav.sk/en/iwms2009-12.html.

## Workshop on Industrial Mathematical and Statistical Modeling

**Raleigh, North Carolina, USA**
**July 20-28, 2009**

This workshop will be held at North Carolina State University in Raleigh, NC. Its purpose is to expose graduate students in mathematics, engineering, and statistics to exciting real-world problems from industry and government. Besides giving students experience in the team approach to problem solving, the workshop can help them to decide what kind of professional career they want. The organizers are Ilse Ipsen, Pierre Gremaud, and Ralph Smith.

Local expenses and travel expenses will be covered for students at US institutions. For more information, visit http://www.ncsu.edu/crsc/events/imsm09/; questions can be directed to imsm_09@ncsu.edu.

## 3rd International Workshop on Matrix Analysis and Applications
**Hangzhou (Lin'An), China**
**July 9-13, 2009**

This workshop aims to stimulate research and interaction of mathematicians and scientists in all aspects of linear algebra and matrix analysis and their applications, and to provide an opportunity to exchange ideas and recent developments on the subjects. The two previous meetings in this workshop series took place in Beijing, China in 2007 and in Fort Lauderdale, Florida, USA in 2003. The conference website is http://www.nova.edu/~zhang/09MatrixWorkshop.html.

The Invited Keynote Speaker is Chi-Kwong Li, Ferguson Professor of Mathematics at The College of William and Mary, USA.

Other speakers include Z.-J. Bai, University of California - Davis, USA; N. Bebiano (University of Coimbra, Portugal), M.-Z. Ding (University of Florida, USA), C.-Q. Gu (Shanghai University, USA), L.-Z. Ji (University of Michigan, USA), A.H. Li (Montclair State University, USA), R.-C. Li (University of Texas-Arlington, USA), Z.-S. Li (Georgia State University, USA), J.-Z. Liu (Xiangtan University, China), Y.-T. Poon (Iowa State University, USA), L.-F. Qian (Florida Atlantic University, USA, T.-Y. Tam (Auburn University, USA), Y.-G. Tian (Central University of Finance and Economics, China), Q.-W. Wang (Shanghai University, China), Y.-M. Wei (Fudan University, China), P.-Y. Wu (Taiwan National Chiao Tung University), R. Nung-Sing Sze (University of Connecticut, USA), and S.-F. Xu (Peking University, China), T.-Z. Huang (University of Electronic Science & Technology, China).

This meeting is sponsored by the National Natural Science Foundation of China and Zhejiang Forestry University, and is endorsed by ILAS. Chairs of the organizing committee are Changqing Xu (Zhejiang Forestry University, China, cqxurichard@163.com), Guanghui Xu (Zhejiang Forestry University, China) and Fuzhen Zhang (Nova Southeastern University, Ft Lauderdale, Florida, USA, zhang@nova.edu). For details about the conference, visit www.nova.edu/~zhang/09MatrixWorkshop.html.

Hangzhou is well-known as one of the most renowned and prosperous cities of China for much of the last 1000 years, as well as for its beautiful natural scenery. It is located in the Yangtze River Delta, 180 kilometres southwest of Shanghai. It is at the eastern foot of a scenic range of hills, the Tianmu ("Eye of Heaven") Mountains, and on the shore of the famous Xi (West) Lake, celebrated in Chineses poetry and paintings for its beauty and a favourite imperial retreat.

West Lake, Hangzhou

## Positivity VI Conference
**El Escorial, Madrid, Spain**
**July 20-24, 2009**

The International Conference on Positivity and Its Applications will be held in the town of El Escorial, about 41 kilometers northwest of Madrid's center, in a palace and monastery complex also called El Escorial. At this conference there will be four advanced courses, ten invited lectures and a number of short talks.

The advanced course speakers will be Ch. D. Aliprantis, Purdue University, USA; W. Arendt, Ulm University, Germany; B. De Pagter, Delft Technical University, Holland; and N.J. Kalton, Missouri-Columbia University, USA. The conference will also include a celebration of the 80th birthday of Professor W.A.J. Luxemburg.

The organizing committee is Francisco L. Hernández, César Ruiz, Víctor M. Sánchez, Antonio Suárez, Pedro Tradacete (Madrid Complutense University) and Julio Flores (Rey Juan Carlos University). For more information, visit http://www.mat.ucm.es/~cpositi6.

El Escorial Palace

## Linear and Numerical Linear Algebra: Theory, Methods and Applications

**DeKalb, Illinois, USA**
**August 12-14, 2009**

This conference, to be held at Northern Illinois University, USA, on August 12-14, 2009, aims to strengthen communication between theoretical, applied, and computational areas of linear algebra. It will bring together researchers in linear and numerical linear algebra and targeted application areas, such as control and systems theory, signal and image processing, and optimization, for exchange of ideas and discussions of recent developments and future directions of research.

Besides plenary and other talks, two special sessions are being organized: one for talks by graduate students and researchers at the early stages of their careers, and another on the role of linear and numerical linear algebra in CSE Education.

The Department of Mathematical Sciences of Northern Illinois University will host the conference, with partial support from the Institute for Mathematics and Its Applications (IMA). It is co-sponsored by the IMA Institutes at Northern Illinois University, Kent State University, the University of Wyoming, and the University of Kentucky.

The organizing committee is Biswa Datta (Chair), NIU (dattab@math.niu.edu); Greg Ammar, NIU; Karabi Datta, NIU; Sien Deng, NIU; Yoopyo Hong, NIU; Lothar Reichel, Kent State University; Vadim Olshevsky, University of Connecticut; Bryan Shader, University of Wyoming; Vadim Sokolov, Argonne National Laboratory; and Qian Ye, University of Kentucky. For details, visit www.math.niu.edu/LA09.

## 3rd Annual Conference on Positive Systems: Theory and Applications (POSTA09)

**Valencia, Spain**
**September 2-4, 2009**

POSTA09 will be held at the Universidad Politécnica de Valencia in Valencia, Spain. The program includes plenary talks, invited sessions on Optimal Control of Positive Systems, Stability and Control of Positive Systems (I), Stability and Control of Positive Systems (II), and Total Positivity, as well as contributed talks. Invited speakers are H. L. Smith, Arizona State University; M. E. Valcher, Università di Padova; and Paul Van Dooren, Université Catholique de Louvain. Rafael Bru is the Conference Chairman (rbru@mat.upv.es).

The conference proceedings will be published in a special volume of the Lecture Notes in Control and Information Sciences. The Editors for this issue are Rafael Bru and Sergio Romero.

The City of Arts and Sciences, Valencia

For details of the conference, visit http://posta09.webs.upv.es. There are wireless facilities on the campus; for more information about the university, visit www.upv.es.

## International Conference of Numerical Analysis and Applied Mathematics 2009

**Crete, Greece**
**September 18-22, 2009**

ICNAAM 2009 will be held in Greece, on the island of Crete, September 18-22, 2009. The conference topics will include all research areas of Numerical Analysis and Computational Mathematics as well as Applied and Industrial Mathematics. There will be a large number of invited speakers, and the conference will include a celebration of the 60th birthday of Professor Dr. Ernst Hairer. The Proceedings will be published in the American Institute of Physics Conference Proceedings (http://www.icnaam.org/proceeding.htm).

A harbor on Crete

The Conference Chair and Organizer is Dr. T.E. Simos, President of the European Society of Computational Methods in Sciences and Engineering. For details, visit http://www.icnaam.org.

## SIAM Conference on Applied Linear Algebra
**Seaside, California, USA**
**October 26-29, 2009**

The SIAM Conference on Linear Algebra will be held October 26-29, 2009, in Seaside, a town on Monterey Bay and adjacent to the town of Monterey. This conference is organized every three years by the SIAM Activity Group on Linear Algebra (SIAG/LA). Its goal is to bring together researchers and practitioners from academia, research laboratories, and industries all over the world to discuss current work.

The SIAM Activity Group on Linear Algebra Prize will be awarded at this conference. This distinguished prize is for the author(s) of the most outstanding paper, as determined by the prize committee, on a topic in applicable linear algebra published in English in a peer-reviewed journal in the three calendar years preceding the year of the award. The papers must contain significant research contributions to the field of linear algebra, with direct or potential applications.

Members of the prize selection committee are: Michele Benzi (Chair), Emory University; Inderjit Dhillon, University of Texas at Austin; Peter Lancaster, University of Calgary, Canada; Julio Moro, University Carlos III, Madrid, Spain; and Henk van der Vorst, University of Utrecht, The Netherlands.

The Conference Organizing Committee is Angelika Bunse-Gerstner, Universität Bremen, Germany; Raymond Chan, Chinese University of Hong Kong, Hong Kong; Inderjit Dhillon, University of Texas at Austin, USA; Mark Embree, Rice University, USA; Andreas Frommer, University of Wuppertal, Germany; Anne Greenbaum, University of Washington, USA; Chen Greif, University of British Columbia, Canada; Misha Kilmer, Tufts University, USA; Michael Mahoney, Stanford University, USA; James Nagy, Emory University, USA; Esmond Ng (Chair), Lawrence Berkeley National Laboratory, USA; Valeria Simoncini, Universita di Bologna, Italy; and Daniel Szyld, Temple University, USA.

For more information, visit http://www.siam.org/meetings/la.

Jellyfish at Monterey Bay Aquarium


This is a call for minisymposia proposals and contributed talks for the 16th Conference of ILAS, which will be held in Pisa, Italy, on June 21-25, 2010, at the Palazzo dei Congressi. The Scientific Committee is Avi Berman, Michele Benzi, Dario A. Bini, Luca Gemignani, Leslie Hogben, Steve Kirkland, Julio Moro, Ilia Spitkovsky, Francoise Tisseur, and Eugene Tyrtyshnikov. The Committee invites proposals for minisymposia and contributed talks.

The deadline for proposals for minisymposia is September 27, 2009 and notification of the final selection of minisymposia will be given no later than October 31. A minisymposium consists of about 10-15 talks of 25 minutes each. Its aim is to highlight the recent developments of a specific area of linear algebra and its applications. A minisymposium proposal consists of a title, a list of about 10-15 speakers (or potential speakers), a brief abstract, and full contact information for the organizer(s). Email these proposals to the Scientific Committee at bini@dm.unipi.it, subject ILAS-2010.

The deadline for proposals for contributed talks is October 31, 2009, and notification of the final selection of contributed talks will be given no later than December 5. A proposal for a contributed talk consists of an abstract with title, a brief overview of the proposed talk, and full contact information for the corresponding author. Information on how to submit an abstract will be available in the web page of the conference. A contributed talk will be 25 minutes.

The Local Organizing Committee is Dario A. Bini (Chair), Gianna Del Corso, Bruno Iannazzo, Beatrice Meini, Ornella Menchi, Federico Poloni. The conference website (http://www.dm.unipi.it/~ilas2010) will soon be available.

## 0-1 Matrix Theory and Related Topics

**Coimbra, Portugal**
**June 17-19, 2010**

Matrices with entries consisting only of zeros and ones, whose entry sums of rows and columns are constrained, play an active role in modern mathematics and its applications, extending far beyond their natural context of Matrix Theory, Combinatorics or Graph Theory.

The purpose of this meeting is to bring together mathematicians from different areas with a view to exploring a number of new properties on the set A(R,S), whose insertion tableau has a previously-fixed shape, and identifying fruitful avenues for further research. In spite of their extremely demanding nature, recent developments and procedures have evidenced a remarkable elegance and beauty, strengthening the interdisciplinary approach of the issue.

Our intention is that this meeting will attract more mathematicians to this exciting and important area and foster collaborations with other scientific users. For more information, including a list of participants to date, visit http://www.mat.uc.pt/~cmf/01MatrixTheory, the conference website.

This meeting is endorsed by the International Linear Algebra Society. It is sponsored by the Center for Mathematics and the Department of Mathematics at the University of Coimbra, and by the American Mathematical Society.

The Scientific Committee is Richard A. Brualdi, University of Wisconsin, USA; Brendan McKay, Australian National University, Australia; Christian Krattenthaler, University of Vienna, Austria; José Dias da Silva, University of Lisbon, Portugal; Natália Bebiano, University of Coimbra, Portugal; and Carlos Fonseca, University of Coimbra, Portugal.

The Organizing Committee is Ricardo Mamede, University of Coimbra, Portugal; Alessandro Conflitti, University of Coimbra, Portugal; Milica Anđelić, University of Aveiro, Portugal; and Ana Nata, Polytechnic Institute of Tomar, Portugal.

There will be a peer-refereed special issue of Linear Algebra and its Applications devoted to the papers presented at the conference. The submitted papers will go through the usual editorial/referee process. The submission deadline is to be announced. The special editors of the issue are: C.M. da Fonseca, J.A. Dias da Silva, Natália Bebiano, and Geir Dahl.

Coimbra rests on hills overlooking the Mondego River.

## The International Conference on Trends and Perspectives in Linear Statistical Inference

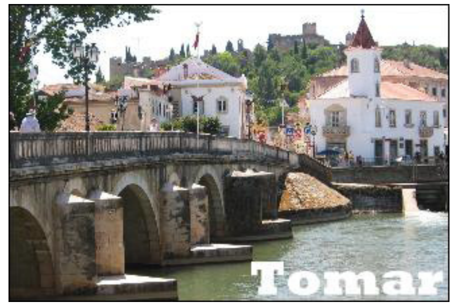
**Tomar, Portugal**

**July 27-31, 2010**

LINSTAT2010 will be held in Tomar, Portugal, on July 27-31, 2010. The purpose of this conference is to bring together researchers to discuss current developments in a variety of aspects of statistics and its applications, and also to encourage young scientists.

The best poster as well as the best talk will be chosen. Young Scientists awarded at Linstat´08 will be Invited Speakers at this next conference.

The conference organizers are Joao T. Mexia (Scientific Committee) and Francisco Carvalho (Organizing Committee). For more information, visit http://www.linstat2010.ipt.pt.




## International Conference on Recent Trends in Graph Theory and Combinatorics

**Cochin, India**

**August 12-15, 2010**

From Ambat Vijayakumar

This Conference on Recent Trends in Graph Theory and Combinatorics (ICRTGC-2010) will be held August 12-15, 2010, in Cochin, India. The academic program will consist of plenary and invited talks by eminent researchers in graph theory, combinatorics and related topics, contributed presentations and mini symposia on specific themes.

The International Scientific Committee is G.E. Andrews – USA; S. Arumugam-India; R.Balakrishnan – India; R.B. Bapat – India; W. Goddard – USA; M.C Golumbic – Israel; G. Gutin – U.K ; P. Hell – Canada; G. O. H. Katona – Hungary; S. Klavzar – Slovenia; J. Nesetril – Czech Republic; S. B. Rao – India; A. Raspaud - France; A. Rosa – Canada; S. S. Sane – India; V.T. Sos – Hungary; X. Zhu – Taiwan.

Chinese Fishing Nets, Cochin

For details, send email to the Conference Convener, Ambat Vijayakumar (vambat@gmail.com) or to (icrtgc2010@gmail.com).

Cochin is a beautiful city, known as the "Queen of the Arabian Sea." It is the commercial and industrial capital of the state of Kerala, which translates as "The God's Own Country." In Cochin one sees an excellent blend of traditions with modernity, a perfect reflection of the cosmopolitan society of Kerala. It is a spectacular city brimming with history, legends and beauty. Since time immemorial, Arabs, Chinese, Dutch, British and Portuguese have had trade relations with Cochin.

Cochin has an international airport and Hyderabad is just ninety minutes by flight.

Notice our conference in Cochin will be held just before ICM-2010, the International Congress of Mathematicians that is to take place at Hyderabad, India on August 19-27, 2010. We extend a cordial invitation to you to come to Cochin for the ICRTGC-2010 and then on to Hyderabad for ICM-2010.

## MAT-TRIAD 2009, Będlewo, Poland, March 23-27, 2009

Report by Katarzyna Filipiak

MAT-TRIAD 2009, the third MAT-TRIAD Conference, was held at the Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences in Będlewo, Poland, on March 23-27, 2009. The conference was organized and supported by the Stefan Banach International Mathematical Center, Faculty of Mathematics and Computer Science, the Institute of Socio-Economic Geography and Spatial Management of the Adam Mickiewicz University, the Department of Mathematical and Statistical Methods of the Poznań University of Life Sciences, and the Polish Mathematical Society.

The International Scientific Committee members were Ljiljana Cvetković (University of Novi Sad, Serbia), Heike Fassbender (Technical University Braunschweig, Germany), Simo Puntanen (University of Tampere, Finland) and Tomasz Szulc (Adam Mickiewicz University, Poznań, Poland; chair). The Local Organizing Committee was chaired by Augustyn Markiewicz, and included Katarzyna Filipiak, Jan Hauke, Aneta Sawikowska, Anna Szczepańska, and Waldemar Wołyński. The purpose of MAT-TRIAD Workshops is to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications and offer them a possibility to discuss current developments in these subjects.

There were 56 participants from 13 different countries. Two lecture series were given: "Iterative methods for large scale linear feasibility problems" by Andrzej Cegielski and "Special classes of matrices" by Charles R. Johnson. The list of invited speakers was opened by winners of Young Scientist Awards of MAT-TRIAD 2007: Vladimir Kostić, Miguel Fonseca, Dorota Kubalińska, and Iwona Wrobel. Moreover, during the five days, there were five invited talks by S. Ejaz Ahmed, Charles R. Johnson, Joao T. Mexia, Juan M. Pena, and Dietrich von Rosen, 26 contributed talks and one poster. The talks were mainly devoted to matrix analysis, methods of linear algebra, algorithms of numerical linear algebra, and applications of mathematical statistics with the use of matrix algebra. The atmosphere was friendly and stimulating, and most talks were extensively discussed.

The International Scientific Committee selected the best talks presented by Ph.D. students and young scientists in the areas of application of the methods of matrix algebra with a special emphasis on statistics. The winners were Ricardo Covas (Portugal) and Aneta Sawikowska (Poland).

Young Scientist Awardees
R. Covas and A. Sawikowska

The short courses, including Charlie's impressive use of chalk and blackboard, were warmly welcomed by the participants. The attendees shared the common opinion that the workshop was extremely fruitful and well organized. They strongly supported the idea to continue the MAT-TRIAD series in 2011. The list of participants, abstracts of talks and posters, and the photo gallery can be found at http://www.mtriad09.amu.edu.pl.

MAT TRIAD Participants

# Workshop on Nonnegative Matrix Theory: Generalizations and Applications

**Palo Alto, California, USA**

**December 1-5, 2008**

Report by Judi McDonald

This workshop was sponsored by and held at the American Institute of Mathematics (AIM) in Palo Alto, California.

It was organized by Judi McDonald, Washington State University, Hans Schneider, University of Wisconsin, and Michael Tsatsomeros, Washington State University. It brought together mathematicians with a common interest in nonnegativity as it arises in linear algebra, operator theory and max algebra.

The goals included making progress on important open problems, identifying new directions and challenges, developing global themes that tie notions of nonnegativity together, as well as exchanging and comparing ideas in the study of nonnegativity of linear and non-linear maps.

The activities included morning presentations that focused on reviewing classical results and bringing the participants up to date on the status of open problems and related research. The afternoons were mostly devoted to group work, reporting of efforts and results to all participants, as well as informal discussions.

Several dynamic groups formed during the workshop to work on specific problems. The groups regularly reported their progress, questions and arising challenges.

A summary of that activity can be found at the website www.aimath.org/pastworkshops/nonnegmatrixrep.pdf.

On Friday, December 6, we also had the opportunity to surprise Tom Laffey with cake and wishes for a happy birthday!

As was announced during the workshop, there will be a special volume of the Electronic Journal of Linear Algebra (ELA) on the occasion of the workshop. Papers in nonnegative matrix theory inspired by the themes of the workshop are invited. These include, but are not limited to, spectral properties of nonnegative matrices and operators, inverse eigenvalue problems for nonnegative matrices, properties and patterns of eventually nonnegative matrices, cone and exponential nonnegativity, as well as matrices in max algebra.

Everyone is welcome to contribute to this volume. No formal deadline has been set for submission of papers. Publication of papers will be immediate, following review and acceptance according to the standard policies of ELA.

Papers should be prepared according to the journal guidelines found in www.math.technion.ac.il/iic/ela, and can be submitted to any one of the special editors: Judith McDonald (jmcdonald@math.wsu.edu, Hans Schneider (hans@math.wisc.edu), or Michael Tsatsomeros (tsat@wsu.edu). See page 26 for more details about this volume.

Participants, Workshop on Nonnegative Matrix Theory

## New England Numerical Analysis Day

**Kingston, Rhode Island, USA**

**April 4, 2009**

Report by Tom Bella

The first New England Numerical Analysis Day was held on April 4, 2009 at the University of Rhode Island. The event, organized by Jim Baglama, Li Wu, and Tom Bella, was an informal meeting dedicated to numerical analysis and related areas, and will hopefully be the first in an annual series of such meetings. The idea for this was proposed by Steve Leon at the Gene Golub Memorial Conference held at the University of Massachusetts Dartmouth in 2008, and follows the idea of the informal Bay Area conferences on numerical linear algebra that Gene used to organize.

About 40 attendees, including many students, came to hear the talks given by Vladimir Druskin, Alan Edelman, Misha Kilmer, Steve Leon, Vadim Olshevsky, Upendra Prasad, Lothar Reichel, Gilbert Strang, Michael Zavlavsky, and Pavel Zhlobich.

The conference was sponsored by the URI Department of Mathematics, URI Foundation, URI Visiting Scholars Program, and the Rhode Island Alpha Pi Mu Epsilon Chapter.

Attendees generally expressed their satisfaction with the conference and much interest in having a second New England Numerical Analysis Day, so hopefully this time next year a similar report will be appearing. The program and abstracts, as well as some photos from the conference, can be found on the conference webpage http://www.math.uri.edu/~tombella/nenad .

# *ILAS NEWS*

## New ILAS Officers

From Steve Kirkland, ILAS President

In the Fall 2009 ILAS elections, Leslie Hogben of Iowa State University, USA, has been elected to the position of Secretary/Treasurer, for a three-year term beginning on March 1, 2009. She has taken over from Jeff Stuart, who held this office for 9 years. The Secretary/Treasurer is responsible for the organization's financial records and records of official meetings, as well as the membership and mailing lists. Also, Dario Bini, University of Pisa, Italy, and Shaun Fallat, University of Regina, Canada, were elected as members of the ILAS Board, for three-year terms beginning on March 1, 2009.

On behalf of ILAS, I take this opportunity to thank Jeff Stuart for his long and faithful service to ILAS, the members of the Nominating Committee - Rajendra Bhatia, Jane Day, Christian Mehl, Michael Neumann (chair), and Michael Tsatsomeros - for their efforts on behalf of ILAS, and also to thank all of the nominees for their participation in the election. Finally, I extend thanks to Arcelia Bettencourt and Tammy Prentice, who helped count the ballots.

Professor Hogben served as Assistant Secretary-Treasurer for several years prior to becoming Secretary-Treasurer, so the transition should be smooth. Henceforth all ILAS correspondence and dues payments should be sent to her: Professor Leslie Hogben, Department of Mathematics, Iowa State University, Ames, IA 50011 USA (lhogben@iastate.edu).

## Proposals for ILAS Lectures at Non-ILAS Conferences

From Steve Kirkland, ILAS President

This is a reminder and call for proposals for sponsorship of ILAS Lectures at non-ILAS conferences.

As part of ILAS's commitment to supporting activities in linear algebra, the Society maintains a program of ILAS Lectureships at non-ILAS conferences. Typically ILAS will sponsor up to two such speakers per year.

All proposals for ILAS Lectureships at non-ILAS conferences will be reviewed once per year, with an annual deadline of September 30 for the receipt of proposals. Details regarding the guidelines and format for proposals can be found at http://www.ilasic.math.uregina.ca/iic/misc/non_ilas_guidelines.html. Organizers of non-ILAS conferences are encouraged to consider submitting proposals through this program.

**ILAS 2008 - 2009 Treasurer's Report**
March 1, 2008 through February 28, 2009

| | | | |
|---|---|---|---|
| **Net Account Balances on February 29, 2008** | | | |
| Vanguard (ST Fed. Bond Fund 6639.103 Shares) | | $71,370.36 | |
| Checking account | | $26,255.12 | |
| Cash | | $ 100.00 | |
| Outstanding checks payable | | ($ 2,089.63) | |
| Pending Deposits | | $ 1,900.00 | **$97,535.85** |
| General Fund | | $37,483.90 | |
| Conference Fund | | $11,755.01 | |
| ILAS/LAA Fund | | $12,415.34 | |
| Olga Taussky Todd/John Todd Fund | | $ 9,911.64 | |
| Frank Uhlig Education Fund | | $ 4,531.40 | |
| Hans Schneider Prize Fund | | $21,438.56 | **$97,535.85** |
| **Income:** | | | |
| Dues | 8,035.00 | | |
| Corporate Dues | 1,200.00 | | |
| General Fund | 3,000.00 | | |
| Conference Fund | 201.00 | | |
| LAMA for speaker | 1,500.00 | | |
| Taussky-Todd Fund | 251.00 | | |
| Uhlig Education Fund | 111.00 | | |
| Schneider Prize Fund | 346.00 | | |
| Interest | 3,340.57 | $17,984.57 | |
| **Expenses:** | | | |
| ILAS Speakers (3) | 1,950.00 | | |
| ILAS Conference Support (Net) | 220.00 | | |
| ILAS Exec Support | 2,000.00 | | |
| LAMA Speaker | 1,500.00 | | |
| IMAGE (2 issues) | 1,967.41 | | |
| Dues Mailing | 353.69 | | |
| Credit Card and Bank Fees | 515.64 | | |
| License Fees | 122.50 | | |
| Labor - Conference | 300.00 | | |
| Supplies, Copying, Postage | 91.73 | | |
| Errors, currency exchange, other | 93.65 | $9,114.61 | |
| **Net Account Balances on February 28, 2009** | | | |
| Vanguard (ST Fed. Bond Fund 6893.256 Shares) | | $74,653.96 | |
| Checking account | | $24,186.85 | |
| Pending credit card amounts | | $ 5,200.00 | |
| Pending deposits | | $ 2,365.00 | **$106,405.81** |
| General Fund | | $43,464.18 | |
| Conference Fund | | $11,956.01 | |
| ILAS/LAA Fund | | $12,840.57 | |
| Olga Taussky Todd/John Todd Fund | | $10,592.14 | |
| Frank Uhlig Education Fund | | $ 4,844.34 | |
| Hans Schneider Prize Fund | | $22,708.57 | **$106,405.81** |

Jeffrey L. Stuart, ILAS Secretary-Treasurer

## Danny Hershkowitz New Minister of Science and Technology for Israel

Danny Hershkowitz, a founding member of ILAS, ILAS President 2002-2008, and one of the Chief Editors of the ILAS Electronic Journal of Linear Algebra (ELA) from its inception in 1996, has recently been elected to the 18th Knesset, Israel's Parliament.

He has also been appointed the new Minister of Science and Technology for the State of Israel. In an interview with Haim Watzman on May 17, 2009 (http://southjerusalem.com/2009/05/interview-with-the-science-minister/#more-1201), Danny said free and well-funded basic research is vital to the development of Israel's human resources and the country needs to make a long-term commitment to funding basic research in science.

Even though Danny is well known to most ILAS members, it is appropriate to record here a few more of his remarkable achievements. He earned a D.Sc. in Mathematics at the Israel Institute of Technology (Technion) in 1982, with Avi Berman as thesis advisor. Hans Schneider invited him to a postdoctoral position at the University of Wisconsin and that was the beginning of a long and fruitful collaboration.

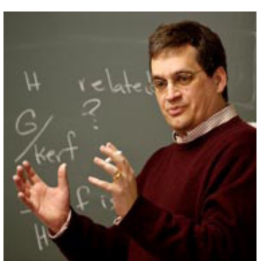
Danny has a distinguished research record – his web page lists 80 research papers, dozens of invited talks, and 11 Ph.D. students through 2006. He has earned many honors, including the Landau Research Prize in Mathematics, the New England Academic Award for Excellence in Research, the Technion's Award for Excellence in Teaching, and the Henri Gutwirth Award for Promotion of Research.

He is a Senior Editor for Linear Algebra and Its Applications and has been Editor of several special issues. He has helped organize numerous mathematics conferences, including chairing the Organizing Committee for the 9th ILAS Meeting, which was held in Haifa in 2001.

He served as Head of the Technion's Division of Continuing Education and External Studies for four years, later was Dean of the Faculty of Mathematics another four years, then became Head of the Technion Faculty Association.

From its origins in the late 1980's, he was an enthusiastic participant in helping ILAS become strong, serving as Managing Editor of ELA as well as Chief Editor, as Vice President before becoming President, and also serving on the Journals Committee and the Hans Schneider Prize Committee. He is married, the father of five children, and rabbi for the Ahuza neighborhood in Haifa.

## Bryan Shader Receives Distinguished Faculty Award at the University of Wyoming



Professor Bryan L. Shader is this year's recipient of the George Duke Humphrey Distinguished Faculty Award at the University of Wyoming. This award, which honors Wyoming's 13th president, recognizes teaching effectiveness, distinction in scholarly work and distinguished service to the university and state.

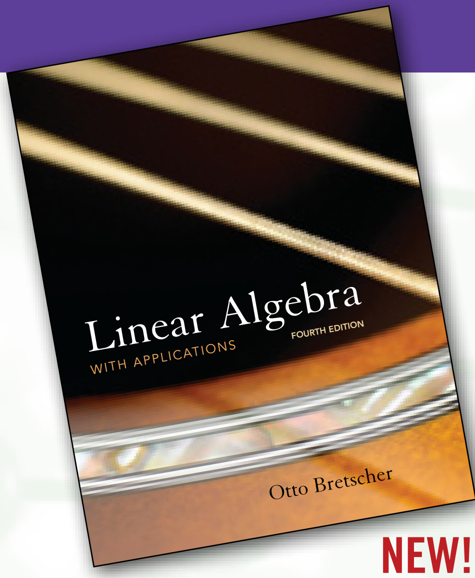
Numerous colleagues and students at Wyoming and elsewhere laud Shader's accomplishments as a researcher in combinatorial matrix theory, Department Chairman, and superb teacher. Many mention his generous nature, always finding time to encourage and promote his colleagues and students.

He is an Associate Editor of Linear Algebra and Its Applications, Advisory Editor for the Electronic Journal of Linear Algebra, and on the Editorial Board of Linear and Multilinear Algebra. He has been Editor of special issues for both LAA and ELA. He is currently on the ILAS Board of Directors and was Editor-in-Chief of ILAS IMAGE from 2003-2006.

Shader received his B.S. degree from the University of Wyoming in 1984 and his Ph.D. from the University of Wisconsin in 1990, where his thesis advisor was Richard Brualdi. He has directed 8 Ph.D. students at Wyoming, plus projects for numerous undergraduate and Masters degree students.

He is co-author with Richard Brualdi of the book *Matrices of Sign-solvable Systems*, Cambridge Press, 1995. He has authored more than 70 research articles and presented over a dozen plenary talks at mathematics conferences. He has also organized many mathematics conferences and workshops, and been named a "Top Prof" at Wyoming several times. More details about his Humphrey Award can be found at http://www.uwyo.edu/news/showrelease.asp?id=30883.

# STAY AT THE FOREFRONT OF LINEAR ALGEBRA WITH PEARSON!

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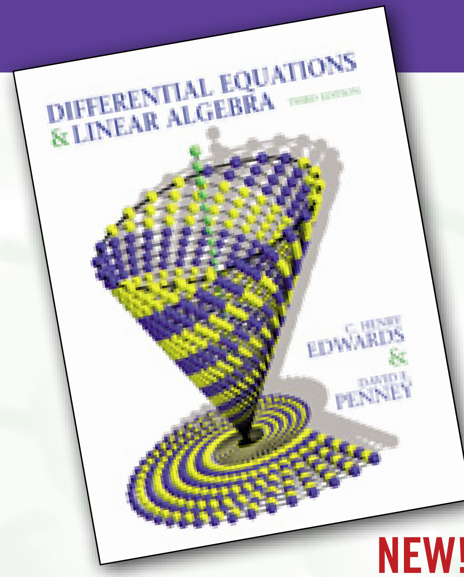


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## ILAS Members Elected to SIAM Offices

Two members of ILAS have been elected to three-year terms on the SIAM Board of Trustees: Iain Duff, University of Strathcylde, UK, and Nicholas Higham, University of Manchester, UK. Professor Duff continues as Chair of this Board.

Two other ILAS members, Michele Benzi, Emory University, USA, and Henry Wolkowicz, University of Waterloo, Canada, have been elected to serve three-year terms on the SIAM Council.

## IMAGE Problem Corner
## Editorial Guidelines/Policies

From Fuzhen Zhang, Editor, Problem Corner

1. New Problems

a. The Problem Corner aims to provide a platform for sharing new, interesting and elegant problems in teaching and research;

b. New problems should be of interest to a reasonable fraction of IMAGE readers;

c. New problems must be new and the solution submitted must be correct;

d. IMAGE usually does not publish problems without provided solutions. Exceptions may be made if a problem is very interesting;

e. Submission material must be in Macro-free LaTex and PDF, and sent via e-mail to zhang@nova.edu;

f. Each issue of IMAGE will contain 5 to 10 new problems, depending on their lengths.

2. Solutions

a. Among similar solutions, we prefer those from less-frequent solvers;

b. We generally do not use proposers' solutions unless no others are suitable;

c. We usually publish one or two solutions for each problem.

3. The Editor will collaborate closely with the Problem Corner Working Team, seriously valuing and considering their suggestions and comments.

## Call for News for IMAGE Issue 43, Fall 2009

All news of interest to the linear algebra community is welcome. The deadlines for IMAGE issue 43 are as follows: September 1, 2009 for submissions to the Problem Corner, and October 1, 2009 for all other material.

Later submissions such as conference reports will be accepted when possible; please alert Editor Jane Day early about when to expect your material. Send your contributions and suggestions to the appropriate Editor, as follows.

- History: Peter Šemrl (peter.semrl@fmf.uni-lj.si)

- Books: Oskar Baksalary (baxx@amu.edu.pl))

- Education: Steve Leon (sleon@umassd.edu)

- Advertisements: Jim Weaver (jweaver@uwf.edu)

- Problems and solutions: Fuzhen Zhang (zhang@nova.edu)

- All other news, including conference announcements and reports, information about honors, transitions, funding sources, employment opportunities, short notes, etc.: Jane Day (day@math.sjsu.edu)

## It is Easier Now to Search IMAGE Online

A new option on the webpage where IMAGE issues are archived (http://www.ilasic.math.uregina.ca/iic/IMAGE/) allows a user to view the Table of Contents before downloading the entire newsletter. The first 12 issues did not include a list of contents, but we will soon extend the new search option to those issues.

Also, larger issues have been compressed so they download much more quickly now.

We thank our webmistress Sarah Carnochan Naqvi for making these improvements.

# ***ARTICLES***

Editorial note: Our series on the development of linear algebra in various parts of the world continues in this issue. We thank the distinguished authors Rajendra Bhatia, for writing about its rich history in India, and M. Radjabalipour and H. Radjavi, for writing about how interest in linear algebra took root and flourished in Iran.

We also welcome Rafael Bru's news about the widespread and growing research in linear algebra in Spain and the new ALAMA network which facilitates communication among groups of mathematicians.

# Linear Algebra in India

By Rajendra Bhatia, Indian Statistical Institute
New Delhi, India

Rajendra Bhatia

The best known book by an Indian with the word "linear" in its title is C. R. Rao's *Linear Statistical Inference and Its Applications*, first published in 1965. Chapter 1 of this classic is titled *Algebra of vectors and matrices* and, like the rest of the book, is a masterly concise introduction to its subject. This is quite apt since Rao carries in his veins the blood of Arthur Cayley, the very father of matrices. According to the Mathematics Genealogy Project, Rao's family tree goes via Ronald Fisher, James Jeans, Edmund Whittaker, Andrew Forsyth to Arthur Cayley.

Before he wrote his book Rao had made major contributions to several areas of statistics, some of them close to linear algebra. One of these became an independent topic of research as the theory of generalised inverses. Rao's first paper on this topic seems to have been [R1]. Its successors [R2] and [R3] indicated that he was working towards a general theory around the subject.

Rao worked at the Indian Statistical Institute (ISI) in Calcutta (now called Kolkata) where there was an extraordinary concentration of talent and creativity. S. K. Mitra, having learnt matrix theory from Rao, was giving his students a set of notes comprising a series of exercises on matrices, with the promise that anyone who does all of them would have learnt statistics. He also taught this at summer schools that ISI regularly conducted at that time.

In 1968 Mitra wrote two papers on generalised inverses [M1], [M2] based on his lectures in the summer schools. Rao, who was planning to write a book on generalised inverses to be published by the Statistical Publishing Society in Calcutta, invited Mitra to be a coauthor. This led to the book *Generalized Inverse of Matrices and Its Applications* by C. R. Rao and S. K. Mitra published by Wiley in 1971. It was very nearly the first full length book on the subject - two other books appeared in the same year, one by Pringle and Rayner [PR] and another by Boullion and Odell [BO]. The "Concluding Remarks" in the book [PR] contain the revealing sentence "Indeed, the research of the Indian School appearing in *Sankhyā* (copies of which reach us several months after the nominal publication date) continues unabated."

For Rao matrix theory was one of his several interests but for Mitra it became his main area of research. In the early seventies he shifted to the newly created Delhi Centre of ISI and remained there till his retirement in the nineties. He attracted several visitors from around the world (A. Ben Israel, R. Hartwig, S. K. Jain, S. Puntanen, M. L. Puri, G. H. Styan, H. Werner) who came to work with him. The Indian statistician C. G. Khatri from Gujarat University in Ahmedabad often collaborated with Rao and Mitra and made many contributions to matrix theory, some of them inspired by problems in statistics.

Several other leading lights at ISI worked on multivariate statistical analysis where matrices play a major role. The most prominent were P. C. Mahalanobis, S. N. Roy and R. C. Bose. The latter had taught C. R. Rao at Calcutta University. His unpublished lecture notes on analysis of variance at Calcutta and later (1959) at the University of North Carolina at Chapel Hill contain several uses of "conditional inverse" of matrices. Bose's famous work on designs and codes has had a strong influence on the work of several linear algebraists.

Almost contemporaneous with this work by Indian statisticians is the work of some physicists interested in quantum theory. A representative of this would perhaps be A. Ramakrishnan in Madras (now Chennai) who published several papers on matrices and wrote a book *L-matrix Theory or the Grammar of Dirac Matrices* (Tata McGraw Hill, 1972). Better known to linear algebraists is the book *Elements of Matrix Theory* (Hindustan Publishing Corporation, 1977) by another physicist M. L. Mehta, the author also of *Random Matrices and the Statistical Theory of Energy Levels* (Academic Press, 1967). Mehta, an Indian, spent most of his working life in France.

Since 1980 most of the work on linear algebra in India has been done at the Delhi Centre of the ISI. There are two notable exceptions that I mention first. B. V. Limaye at the Indian Institute of Technology in Bombay (now Mumbai) has written several interesting papers on numerical approximation of eigenvalues and eigenvectors. Some of his Ph.D. students have continued to work on these problems. H. L. Vasudeva after doing a Ph.D. with W. F. Donoghue has been at the Panjab University in Chandigarh, where he has worked on matrix functions and matrix inequalities, sometimes in collaboration with his students.

R. B. Bapat was one of the first students in the Master of Statistics program at ISI, Delhi. After completing this degree in 1976 he went for a Ph.D. to the University of Illinois, USA, where his advisor was T. E. S. Raghavan, who had earlier obtained his Ph.D. at ISI, Calcutta. Bapat joined the ISI faculty in 1983 and has become very well known for his high quality work on nonnegative matrices, permanents, matrix inequalities, graph theory, max algebras, and generalised inverses. On each of these topics he has written very influential papers. Between 1983 and 1985 one of his coworkers at ISI was V. S. Sunder, who was attracted to matrix theory between his major work on operator theory and von Neumann algebras. Matrices continue to lurk behind Sunder's work on the latter topic.

Bapat's book *Linear Algebra and Linear Models* was the first one in the series *Texts and Readings in Mathematics*, Hindustan Book Agency, 1993. It was later published by Springer. With Raghavan as coauthor he has written the authoritative monograph *Nonnegative Matrices and Applications* in the series *Encyclopedia of Mathematics*, Cambridge University Press, 1997.

S. R. Mohan and T. Parthasarathy, both at ISI Delhi wrote several important papers on the linear complementarity problem. Mohan's major interest was in operations research and linear programming while T. Parthasarathy has made several contributions to the theory of games including matrix games. Both obtained their Ph.D.'s from ISI at Calcutta.

Finally I have to say something about my own work, most of it with others. In 1976 my Ph.D. advisor K. R. Parthasarathy (a renowned probabilist) gave a course based on T. Kato's famous book and proposed for study the general problem of continuity of spectral objects. Not quite knowing what to do I turned for advice to the only other mathematician around - K. K. Mukherjea, a topologist. Following his suggestions I made some progress and was then sent for further direction to C. S. Seshadri, the algebraic geometer at the Tata Institute in Bombay. This made for a somewhat unusual initiation into matrix theory.

After finishing my Ph.D. thesis, *Estimation of Spectral Variation*, I spent a year at Berkeley and during my stay in the U.S. met for the first time real experts in matrix theory - B. N. Parlett, Chandler Davis, S. Friedland. Davis was an external examiner for my thesis and from him I learnt about unitarily invariant norms, Mirsky's conjecture (proposing a generalisation to normal matrices of the Weyl-Lidskii-Mirsky inequalities for Hermitian matrices) and the Davis-Kahan sin theta theorem on perturbation of eigenvectors. I worked on these problems for the next few years.

For the last 25 years I have been on the faculty of ISI, Delhi. Some of the most outstanding matrix analysts have visited here several times. These include T. Ando, M.-D. Choi, C. Davis, L. Elsner, J. Holbrook, F. Kittaneh, P. Šemrl, J. A. Dias da Silva and X. Zhan, with all of whom I have collaborated here and at their home institutions. Though their major interests are elsewhere, my colleagues K. R. Parthasarathy and K. B. Sinha have written very significant papers on matrix analysis as well.

The major themes of this work have been spectral variation problems, unitarily invariant norms, computation of derivatives of matrix functions, operator inequalities, matrix monotone and convex functions, Fourier analysis and differential geometry of matrix manifolds, positivity, approximations, quantum entropy and information. Some of this work has been reported in my three books *Perturbation Bounds for Matrix Eigenvalues* (Longman), *Matrix Analysis* (Springer), and *Positive Definite Matrices* (Princeton University Press). It is a pure coincidence that these appeared at 10 year intervals in 1987, 1997 and 2007, and the first one was reprinted by SIAM, also in 2007.

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# Linear Algebra in Iran

By Mehdi Radjabalipour
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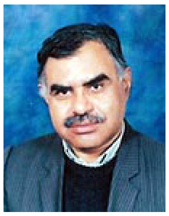
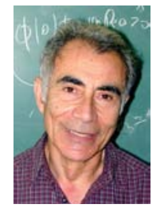

M. Radjabalipour

Like Molière's Monsier Jourdain, who had spoken in prose for forty years without even knowing it, most of us have been doing linear algebra since we were in elementary school. One of the authors fondly remembers his sixth grade teacher in Tabriz, Iran, whose favourite word problem was a variation on this one: If 2 sheep and 5 cows together cost 56 tomans, and if 3 sheep and 4 cows together cost 49 tomans, what is the price of a sheep and a cow each? The solution he presented sounded like magic to us, especially when he got more ambitious one day and totally impressed and mystified us by adding a few horses and another equation to the mix. A couple of years had to pass before we learned, in junior high school, that what our teacher had been doing was inductively eliminating various animals from the system.

Classical mathematics had always played a part in the education of Iranian elite, but modern mathematics came to Iran mainly after the Constitutional Revolution of 1906. A few decades later, Western-style schools had multiplied and a University had been founded in the capital city. In the Thirties, the autocratic old Shah of Iran presided over the newly created Academy, one of whose main duties was finding Persian equivalents for foreign words in modern science and technology and then having them approved by His Majesty for exclusive use by all.

As far as we know, the word "matrix" did not come up, either in the deliberations of the Academy or in the early Persian textbooks in mathematics. In our first-year college courses in analysis, the authors learned all about determinants without once encountering the word "matrix" and graduated without knowing what "linear algebra" meant as a field of study, although we took courses in which it would have been quite natural to invoke the tools of linear algebra. Not entirely in jest, we now speculate that our erstwhile Royal Academicians in the late Thirties looking for a Persian equivalent for "matrix" must have searched for the origin of the word in their French and English dictionaries, found something like "female animal employed for breeding," and given up, fearing the monarch's disapproval! (In fact, two of our professors who, many years later, when there was no academy for language, ventured to define a matrix in advanced books, did call it "zehdaan" and "bookan"-- both Persian words for womb. However, these two terms never gained enough popularity.)

Until the late Fifties, only one university in Iran offered a B.S. degree in mathematics, and no advanced degrees. The program was strong in classical analysis, but topology and modern algebra were considered graduate-school material, and thus almost totally absent. The first references to linear algebra we have found were in the textbooks published after the late Forties. A book in complex analysis [1], published in 1949 by one of our professors, has a short account of real vector spaces to accommodate the geometric representation of complex numbers. Another book in geometry [2], the first we have found that mentions groups and contains a good chunk treating translations, rotations, and other linear transformations, was published in 1952. Other books, including [5, 6, 9, 11], published in the Fifties, discuss systems of linear equations in several unknowns and the corresponding determinants.


H. Radjavi

The first textbook in Persian systematically dealing with linear algebra and topology [4] was published in 1963. Then came the first book fully devoted to matrix theory [10], and its title is "Matrix Calculus," but, as the author clearly states, it is meant for engineering, and not for mathematics, students. Similarly, there is a 1967 textbook [7] titled "Matrix Algebra" meant for applications in statistics. At last, in 1975 a book was published in Persian with the title, "Linear Algebra and Matrices" [8] explicitly for students of mathematics. It is followed by other titles, including [3].

By the end of the Sixties, a few graduate programs had been established in Iranian universities. Independent courses in linear algebra, among other subjects, were being taught in several universities. Some universities used English and French books, but many textbooks by well-known authors were also translated into Persian and published by university presses. "Maatris" was now the accepted term; nobody would call a matrix a womb again.

In 1970, the first Iranian mathematics conference was held in Shiraz, which soon led to the founding of the Iranian Mathematical Society, whose regular annual meetings have been instrumental in developing mathematical research in the country. Its most recent annual meeting, held last August, was its 39th. Lectures in linear algebra and operator theory have been regularly given at these meetings, but the Society has also sponsored specialized conferences and seminars in analysis, algebra, applications, etc. Starting in 1996, they have also had conferences totally devoted to linear algebra and its applications, which have met, roughly, once every three years.

It seems that the first research papers in mathematical journals by an author with an Iranian address were published in the Sixties. Since we are only concerned here with linear algebra research in Iran, we naturally exclude the work done by Iranian mathematicians abroad, unless the work was done in collaboration with researchers residing in Iran. With this self-imposed restriction, and with the warning to the reader that we have no claim for completeness, we start to look back and count.

It is not surprising that the real starting point of linear algebra research in Iran coincides with the establishment of Ph.D. programs in Iranian universities around 1986. We have counted two linear algebra papers published in international journals before 1991. There are 17 in the Nineties. The dramatic increase came with the new century: there are 37 between 2000 and 2005. When we stopped our search at the end of 2008, the number of papers for the three year period 2006-2008 was 57. Quite a few of these are in the Journal of Algebra, Linear and Multilinear Algebra, and other journals that publish material related to linear algebra or operator theory. In order to give the reader a flavour of the kind of problems the Iranian linear algebraists are looking at, without exceeding the space allotted to us, we have decided to list only those papers which were published in one journal: Linear Algebra and its Applications. We have also made a rough prediction, based on our data, about a lower bound for the number of linear algebra papers to be written by Iranians residing in Iran during the next decade: 350.

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## The Spanish Network ALAMA

From Rafael Bru

ALAMA is a research thematic network formed by researchers and professors of many Spanish universities working on different areas from Linear Algebra over rings and fields to Matrix Analysis. These subjects are considered in all aspects – theoretical, numerical and computational - in addition to possible applications in Control Theory, Code Theory, Differential Equations, Orthogonal Polynomials and other topics.

This research network is supported by the Ministry of Education and Science of the Spanish Government. **ALAMA** stands for "**Á**lgebra **L**ineal, **A**nálisis **M**atricial y **A**plicaciones".

The main objective of the network is to gather all Spanish people working in the above mentioned fields for the following purposes:

(i) To stimulate bilateral contacts among all members.
(ii) To organize biannual meetings in Spain about the topics of the network.
(iii) To create visibility for Linear Algebra at Spanish Applied Mathematics events such as the CEDYA meeting.
(iv) To organize summer schools and short courses.
(v) To involve young mathematicians and scientists on postdoctoral programs who are working in fields related to the network.
(vi) To distribute information, mainly through a web page.

Although the goals are primarily about activities inside Spain, the international connection is always present as many ALAMA members are also members of ILAS and of the SIAM Activity Group on Linear Algebra.

The network is managed by a Board Committee, whose members are listed on the web page (http://alama.webs.upv.es), which is in Spanish.

In its first year, ALAMA was chaired by Juan-Miguel Gracia, with Inmaculada de Hoyos as Secretary. In September 2008, at the second Business Meeting, Rafael Bru was elected to chair the network for a two year term. Presently, José Mas serves as Secretary, and there are more than one hundred members in the network.

Activities of the network so far include organizing the ALAMA meeting in Vitoria that was held September, 25-26, 2008, and constructing the ALAMA website. In addition, members of the network organized a Summer School on Numerical Linear Algebra that was held in Santander, July 21-25, 2008 and are currently organizing the POSTA09 meeting to be held in Valencia, September 2-3, 2009.

A long and rich history about ALAMA members has been written by Juan-Miguel Gracia and you can read that on the ALAMA web page. Here is a brief summary.

In Vitoria, at the University of the Basque Country, a group has been working on Linear Algebra since 1981. Actually, it can be considered the first Spanish group on these topics, and in the beginning was supported by Graciano de Oliveira of the University of Coimbra.

In 1985 a group from the Polytechnic University of Valencia started a joint collaboration with José Vitória (University of Coimbra) and Michael Neumann (University of Connecticut).

A third Linear Algebra group in Spain, at the Polytechnic University of Catalunya, mainly in Barcelona, has a strong collaboration with the Vitoria group.

At the University of Zaragoza some members are working on totally positive matrices and orthogonal polynomials.

In Madrid, there are three groups on Linear Algebra. One is in the Carlos III University and another at UNED (a kind of Open University), and there are relationships among members of these groups. The third group is in the Polytechnic University of Madrid.

Other important groups are in the University of Alacant, University of Valladolid, University of Lleida and University of Leon. Currently, new groups are starting at the Miguel Hernández University and at Castilla la Mancha University. We hope the membership will continue to increase in the future and maintain strong ties with ILAS.

# NOTES ABOUT BOOKS

## Handbook of Linear Algebra

Reviews about this monumental reference appeared in Choice Reviews Online, on January 2, 2009 (http://www.cro2.org/). Here is a brief quote from one of the reviews: “This is a Herculean labor of love on the editor’s part, a successful effort that should be appreciated and applauded by anyone working and/or teaching in this important area of mathematics.” (Henry Ricardo). The editors of the Handbook are Leslie Hogben with Richard Brualdi, Anne Greenbaum, and Roy Mathias. It was published by Chapman & Hall, 2007, and is distributed by CRC Press.

## A Combinatorial Approach to Matrix Theory and Its Applications

Richard A. Brualdi and Dragos Cvetkovic have co-authored this new book. It was published by Chapman & Hall in 2009, and is also distributed by CRC Press. Most researchers in and users of matrix theory know the importance of graphs in linear algebra, but typically the graph is a tangential tool. Here graphs are used to explain and illuminate basic matrix constructions, formulas, computations, ideas, and results, and this approach clarifies many matrix ideas. The authors believe this should be of interest to mathematicians, electrical engineers, chemists, physicists and specialists in other sciences. Each author has written a previous book related to this one.

## Linear Algebra: Challenging Problems for Students:

The 2nd edition of this resource book by Fuzhen Zhang was published by The Johns Hopkins University Press in May 2009. The author reports that this fully updated and revised edition is a good study tool for students in advanced linear algebra or those who just want to brush up on the subject. It defines the discipline’s main terms, explains its key theorems, and provides over 425 example problems ranging from elementary to quite difficult. Vital concepts are highlighted at the beginning of each chapter and a final section contains hints and solutions.

## Introductory Combinatorics

The 5th edition of this classic text by Richard A. Brualdi has been published by Prentice Hall. A discussion of changes made since the 4th edition can be found in ILAS-Net Message 1681, 2008 (http://www.ilasic.math.uregina.ca/iic/).

## Introduction to Linear Algebra

The 4th edition of this classic book by Gilbert Strang has been published by SIAM. The text now has challenge problems in addition to review problems. This text accompanies Strang’s video lectures, which are available on the MIT OpenCourseWare site http://ocw.mit.edu. More than a million viewers have seen him explain the beauty of linear algebra and the importance of its applications in these videos.

## Introduction to Real Analysis

The author of this text, William F. Trench, reports that it is now available for download free of charge from http://ramanujan.math.trinity.edu/wtrench/index.shtml. The book was previously published by Pearson Education, 2003.

## List of books on the ILAS-IIC website

There is a list of books about linear algebra and its applications on the ILAS-IIC website (http://www.ilasic.math.uregina.ca/iic/IMAGE/). George Styan began collecting such titles and the list seems useful so we continue this project. It is now maintained by Oskar Baksalary, the Book Editor for IMAGE. If you find new titles or informative reviews that would be appropriate to include, please send those suggestions to Professor Baksalary (baxx @amu.edu.pl).

# JOURNAL NEWS

## Recent Issue of LAA
## Devoted to the 2008 (14th) ILAS Conference

Volume 430 of Linear Algebra and Its Applications, Issues 5-6, which appeared in March 2009, is devoted to the 14th ILAS Conference, held in Cancun Mexico in June 2008. The Editors for this volume are Ilse Ipsen, Julio Moro, Peter Šemrl, Jia-Yu Shao and Pei Yuan Wu.

## Call for Papers for Special Issue of LAA In Honor of Heinrich Voss

Papers are invited for a special issue of Linear Algebra and Its Applications in honor of Heinrich Voss, who will celebrate his 65th birthday June 16, 2009. Heinrich Voss has made many major contributions in numerical linear algebra and has been one of the leading figures in the recent developments in the analysis and numerical solution of nonlinear eigenvalue problems.

All papers submitted must meet the publication standards of Linear Algebra and Its Applications and will be subject to normal refereeing procedure.

The responsible Editor-in-Chief of the special issue is:
Volker Mehrmann, Institut f. Mathematik, MA 4-5, TU Berlin, D-10623 Berlin, Germany (mehrmann@math.tu-berlin.de).

The deadline for submission of papers is November 15, 2009, and publication is expected in the fall of 2010. Papers can be sent to any one of the following Editors of this special issue, preferably by email in PDF or PostScript format:

Timo Betcke, Department of Mathematics, University of Reading, Whiteknights, PO Box 220, Berkshire, RG6 6AX, UK (t.betcke@reading.ac.uk).

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Siegfried M. Rump, Institute for Reliable Computing, Hamburg University of Technology, Schwarzenbergstr. 95, 21071 Hamburg, Germany (rump@tu-harburg.de).

## Call for Papers for Special Issue of LAA in Honor of G. W. (Pete) Stewart

Papers are invited for a special issue of Linear Algebra and Its Applications in honor of G. W. (Pete) Stewart, who will celebrate his 70th birthday in October 2010. Pete has been an editor of Linear Algebra and Its Applications for many years and is one of its distinguished editors.

All papers submitted must meet the publication standards of Linear Algebra and Its Applications and will be subject to normal refereeing procedures.

The responsible Editor-in-Chief of the special issue is:
Volker Mehrmann, Institut f. Mathematik, MA 4-5, TU Berlin, D-10623 Berlin, Germany (mehrmann@math.tu-berlin.de).

The deadline for submission of papers is July 15, 2009, and publication is expected in May 2010. Papers can be sent to any one of the following Editors of this special issue, preferably by email in PDF or PostScript format:

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Misha Kilmer, Department of Mathematics, Tufts University, Medford, MA 02155, USA (misha.kilmer@tufts.edu).

## Call for Papers for Special Issue of ELA on Nonnegative Matrix Theory

Papers are invited for a special issue of the Electronic Journal of Linear Algebra on nonnegative matrix theory, inspired by the themes of the Workshop on Nonnegative Matrix Theory: Generalizations and Applications, which was held at the American Institute of Mathematics in Palo Alto, CA, USA, in December 2008.

These themes include, but are limited to, spectral properties of nonnegative matrices and operators, inverse eigenvalue problems for nonnegative matrices, properties and patterns of eventually nonnegative matrices, cone and exponential nonnegativity, as well as matrices in max algebra.

Publication of papers will be immediate, following review and acceptance according to the standard policies of ELA. Papers should be prepared according to the journal guidelines found in http://www.math.technion.ac.il/iic/ela/ and can be submitted to any one of the special editors Judith McDonald (jmcdonald@math.usu.edu), Hans Schneider (hans@math.wisc.edu), and Michael Tsatsomeros (tsat@wsu.edu).

## Call for Papers Operators and Matrices

We are pleased to announce that the third volume of Operators and Matrices is in production. The first issue was published in March 2007. Operators and Matrices is a quarterly journal dedicated to publication of high quality research papers in matrix and operator theory and their applications, as well as expository papers and book reviews. The editors welcome submissions in these areas.

The journal Operators and Matrices is covered by Mathematical Reviews, Zentralblatt, and Referativnyi Zhurnal. It is also on the Thomson Reuters' SCIE and CC lists. The publisher is Element, Zagreb, Croatia. The Editors-in-Chief are Chi-Kwong Li, Josip Pecaric, and Leiba Rodman.

The journal's Editorial Board, submission and subscription information, and full text of previous issues and forthcoming papers can be found at http://www.ele-math.com.

## Call for Papers Banach Journal of Mathematical Analysis

The Banach Journal of Mathematical Analysis (BJMA) is an international and peer-reviewed electronic journal. Papers on Functional Analysis, Operator Theory and Matrix Analysis (in particular, ones involving norm and operator inequalities) are welcomed. The journal website is http://www.math-analysis.org/.

BJMA is reviewed in Zentralblatt Mathematica and Mathematical Reviews, and has recently been selected for coverage in SCOPUS.

It is also a partner in Project Euclid (http://projecteuclid.org). This is an online service whose mission is to advance scholarly communication in the field of theoretical and applied mathematics and statistics. Through a collaborative partnership arrangement, it provides full-text searching, reference linking, interoperability through the Open Archives Initiative, and long-term retention of data for low-cost independent and society journals.

## Call for Papers From Pushpa Publishing House

The Pushpa Publishing House (http://www.pphmj.com) in Allahabad, India, requests original research papers and critical survey articles are invited for the following journals. All published articles are reviewed in Mathematical Reviews, MathSciNet and Zentralblatt für Mathematik. Contents of published articles are available on http://pphmj.com/journals/jpanta.htm.

(1) The JP Journal of Algebra, Number Theory and Applications (JPANTA) aims to provide a common forum for significant research both on theoretical and on applied aspects of current interest in algebra and number theory and their innovative links between themselves and the various fields of applications. It is a bimonthly journal that began publication in 2001.

(2) The Advances and Applications in Discrete Mathematics (AADM) aims to provide a common forum for significant research in different areas of current interest covering all branches and aspects of Discrete Mathematics and Combinatorics, including graph theory, networks, block designs, coding theory, matroid theory and combinatorial and discrete geometry. This is a quarterly journal, first published in 2008.

## Call for Papers for New Journal - Algebra, Number Theory: Advances and Applications

The Scientific Advances Publishers in Allahabad, India, announce a new publication in 2009, the Journal of Algebra, Number Theory: Advances and Applications. This a fully refereed international journal, which publishes original research papers and survey articles in all areas of algebra, number theory and their applications. For more information, visit http://scientificadvances.org/journals4P1.htm.

# IMAGE Problem Corner: Old Problems With Solutions

Note: Six new problems are on the back cover, and we invite readers to submit solutions of these. Also, we still look forward to receiving solutions for unsolved problems from past issues: Problems 36-3 [IMAGE 36 (Spring 2006), p. 36], 39-4 [IMAGE 39 (Fall 2007), p. 32], and 41-4 [IMAGE 41 (Fall 2008), p. 43].

SOLUTIONS TO PROBLEMS 41-1 THROUGH 41-14, EXCEPT 41-4 IS REPEATED WITHOUT A SOLUTION:

**Problem 41-1: A Lower Bound for the Rank of $A + A^*$**
Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

Let $A^*$ be the conjugate transpose of $A \in \mathbb{C}_{n,n}$. Provide a nontrivial lower bound for rk$(A + A^*)$, where rk$(.)$ denotes the rank of a matrix argument.

**Solution 41-1** by the proposers Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

On account of Corollary 6 in Hartwig & Spindelböck (1984), any matrix $A \in \mathbb{C}_{n,n}$ of rank $r$ can be represented as

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

where $U \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of $A$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy

$$KK^* + LL^* = I_r. \tag{1}$$

With the use of this representation, several useful characterizations of matrix $A$ can be established. For instance, by noting that

$$A^* = U \begin{pmatrix} K^*\Sigma & 0 \\ L^*\Sigma & 0 \end{pmatrix} U^* \quad \text{and} \quad A^\dagger = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*,$$

where $A^\dagger$ denotes the Moore–Penrose inverse of $A$, it is seen that $A$ is normal, i.e., $AA^* = A^*A$, if and only if $L = 0$ and $\Sigma K = K\Sigma$, whereas $A$ is EP, i.e., $AA^\dagger = A^\dagger A$, if and only if $L = 0$.

Let

$$A + A^* = U \begin{pmatrix} G & H \\ H^* & 0 \end{pmatrix} U^*,$$

with $G = \Sigma K + K^*\Sigma$ and $H = \Sigma L$. From the Hermicity of $A + A^*$ it follows that $(A + A^*)^2$, being of the form

$$(A + A^*)^2 = U \begin{pmatrix} G^2 + HH^* & GH \\ H^*G & H^*H \end{pmatrix} U^*,$$

is nonnegative definite. Thus, since rk$[(A + A^*)^2]$ = rk$(A + A^*)$, combining Theorem 1 in Albert (1969) with Corollary 19.1 in Marsaglia & Styan (1974) leads to

$$\mathrm{rk}(A + A^*) = \mathrm{rk}(H) + \mathrm{rk}[G^2 + HH^* - GH(H^*H)^\dagger H^*G].$$

Whence, utilizing the fact that $H^\dagger = (H^*H)^\dagger H^*$, we arrive at

$$\mathrm{rk}(A + A^*) = \mathrm{rk}(H) + \mathrm{rk}(HH^* + G\widetilde{P}_H G), \tag{2}$$

where $\widetilde{P}_H$ denotes the orthogonal projector onto the orthogonal complement of the column space of $H$, i.e., $\widetilde{P}_H = I_r - P_H$, with $P_H = HH^\dagger$ being the orthogonal projector onto the column space of $H$. Taking into account that the nonsingularity of $\Sigma$ ensures that $\text{rk}(H) = \text{rk}(L)$, from (2) we get

$$\text{rk}(A + A^*) \geqslant \text{rk}(L) \geqslant 0,$$

leading to the conclusion that $\text{rk}(A + A^*) > 0$ when $A$ is not EP. Observing that, on account of (1),

$$A\widetilde{P}_A = U \begin{pmatrix} 0 & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,$$

it is seen that another consequence of the nonsingularity of $\Sigma$ is that $\text{rk}(A\widetilde{P}_A) = \text{rk}(L)$. Thus, $\text{rk}(A + A^*) \geqslant \text{rk}(A\widetilde{P}_A)$, or, equivalently, $\text{rk}(A + A^*) \geqslant \text{rk}(P_{A^*}\widetilde{P}_A)$.

*References*

A. Albert (1969). Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM Journal on Applied Mathematics* **17**, 434-440.

R.E. Hartwig & K. Spindelböck (1984). Matrices for which $A^*$ and $A^\dagger$ commute. *Linear and Multilinear Algebra* **14**, 241-256.

G. Marsaglia & G.P.H. Styan (1974). Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra* **2**, 269-292.

**Problem 41-2: An Inequality Involving a Semi-Inner Product**

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

Let $\mathcal{X}$ be a vector space over $\mathbb{C}$, equipped with a semi-inner product $< \cdot, \cdot >$. For $a, b, c, d \in \mathcal{X}$, show that

$$2\,|< a, c >< b, d >| \leqslant < a, a >< b, b > + < c, c >< d, d > .$$

**Solution 41-2** by the proposers Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

The result will be established by combining relationships originating from the two well-known inequalities. Namely, the inequality linking arithmetic and geometric means, which ensures that

$$\sqrt{< a, a >< c, c >< b, b >< d, d >} \leqslant \frac{1}{2}(< a, a >< b, b > + < c, c >< d, d >), \tag{3}$$

and the Cauchy–Bunyakovsky–Schwarz inequality, which entails

$$|< a, c >|\,|< b, d >| \leqslant \sqrt{< a, a >< c, c >< b, b >< d, d >}. \tag{4}$$

It is clearly seen from (3) and (4) that the asserted inequality necessarily holds.

**Problem 41-3: A Property of the Range of Generalized and Hypergeneralized Projectors**

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

Let $A^*$ and $A^\dagger$ denote the conjugate transpose and the Moore–Penrose inverse of $A \in \mathbb{C}_{n,n}$, respectively.

(i) Prove or disprove that if $A$ is a hypergeneralized projector, i.e., satisfies $A^2 = A^\dagger$, then

$$\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*), \tag{5}$$

where $\mathcal{R}(.)$ is the range (column space) of a matrix argument.

(ii) Is relationship (5) fulfilled if $A$ is a generalized projector, i.e., satisfies $A^2 = A^*$?

*Remark.* The notions of generalized and hypergeneralized projectors were introduced by J. Groß & G. Trenkler (1997). Generalized and hypergeneralized projectors. *Linear Algebra and its Applications* **264**, 463-474.

**Solution 41-3** by Hans Joachim Werner, *University of Bonn, Bonn, Germany,* hjw.de@uni-bonn.de

Our solution to this problem is based on the following characterization of generalized projectors; see Werner (2007).

THEOREM: *A matrix $A \in \mathbb{C}^{n\times n}$ of rank $r$ is a generalized projector (i.e., $A^2 = A^*$) if and only if $A = UTU^*$ for some column-unitary matrix $U \in \mathbb{C}^{n\times r}$ (i.e., $U^*U = I_r$, where $I_r$ denotes the identity matrix of order $r$) and some (unitary) matrix $T \in \mathbb{C}^{r\times r}$ with $T^2 = T^*$ and $T^*T = I_r$.*

By means of this characterization we obtain the following Corollary.

COROLLARY: *If $A$ is a generalized projector, then $\mathcal{R}(A + A^*) = \mathcal{R}(A) = \mathcal{R}(A^*)$, which in turn immediately implies that*

$$\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*). \tag{6}$$

PROOF: Let $A \in \mathbb{C}^{n\times n}$ be a generalized projector. If $A = 0$, the claim is trivial. If $A \neq 0$, then $r := \operatorname{rank}(A) \geq 1$ and, according to our above Theorem, $A = UTU^*$ for some column-unitary matrix $U \in \mathbb{C}^{n\times r}$ and some (unitary) matrix $T \in \mathbb{C}^{r\times r}$ with $T^2 = T^*$ and $T^*T = I_r$. From $\frac{1}{2}(T + T^*)(T + T^* - I_r) = I_r$ it follows that with $T$ also $T + T^*$ is nonsingular. Consequently, $\mathcal{R}(A) = \mathcal{R}(U) = \mathcal{R}(A^*) = \mathcal{R}(A + A^*)$ and so $\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*)$. □

According to the previous Corollary, relationship (6) thus holds for any generalized projector. One might be tempted to believe that for each hypergeneralized projector (6) is also correct. This, however, is erroneous as the following simple example illustrates. Consider the matrix

$$A = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2}i & 1 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & -1 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix} = A^{-1} = A^\dagger \qquad \text{and} \qquad A + A^* = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Whereas $A$ and $A^*$ are nonsingular $2\times 2$ matrices, the matrix $A + A^*$ is a singular $2\times 2$ matrix of rank 1. Consequently, $\mathcal{R}(A + A^*)$ is a proper subset of $\mathbb{C}^n = \mathcal{R}(A) = \mathcal{R}(A^*)$. This completes our solution to this problem.

*Reference*

H.J. Werner (2007). Rank of a Generalized Projector. Solution 37-2.2. *IMAGE: The Bulletin of the International Linear Algebra Society* **39** (Fall 2007), 26.

*Also solved by the proposers.*

**Problem 41-4: Integer Matrices with Unit Determinants II**

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England, UK,* R.W.Farebrother@Manchester.ac.uk

(i) For positive integer values of $d$, $f$ and $g$ the equation $(f^2 - dg^2)^2 + d(2fg)^2 = (f^2 + dg^2)^2$ supplies sets of three integers satisfying the generalised Pythagorean equation $x^2 + dy^2 = z^2$. Outline a matrix proof of this result.

(ii) In IMAGE Problem 36-1 [see IMAGE **37** (Fall 2006), pp. 20-23 for the problem and its solutions by Farebrother and by Werner] we were concerned with the construction of a $2 \times 2$ matrix with a determinant of plus one, and thus with the solution of Pell's Equation $f^2 - dg^2 = +1$ where $f$ and $g$ are nonzero integers and $d$ is a fixed positive integer (though not a square). If instead we assume that the matrix has a determinant of minus one, then we are concerned with the solution of a variant of Pell's Equation (which, for obvious reasons, we shall name Nell's Equation) $f^2 - dg^2 = -1$. Identify integer solutions to this equation for certain choices of $d$. Use the result established in Part (i) to obtain the corresponding solutions to Pell's Equation.

*We are looking forward to receiving a solution to this problem.*

**Problem 41-5: Square Root of a Real Symmetric $3 \times 3$ Matrix**

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England, UK,* R.W.Farebrother@Manchester.ac.uk

Given a real symmetric positive definite $3\times 3$ matrix $A$, outline a direct procedure not involving the singular values or eigenvalues of $A$ for computing a real symmetric positive definite $3 \times 3$ matrix $B$ satisfying $B^2 = A$.

**Solution 41-5** by Richard William Farebrother, *Bayston Hill, Shrewsbury, England, UK,* R.W.Farebrother@Manchester.ac.uk

If $A$ is an $n \times n$ real symmetric positive definite matrix then it may be written as $A = UMU'$, where $U$ is an $n \times n$ orthogonal matrix of eigenvectors and $M$ is the corresponding $n \times n$ diagonal matrix with positive eigenvalues on its diagonal. Let $B = U\Lambda U'$, where $\Lambda$ is an $n \times n$ diagonal matrix with $jj$th element $\lambda_j = \sqrt{\mu_j}$. Then $B$ is the unique real symmetric positive definite square root of $A$. However, we are asked to find this matrix without recourse to the eigenvalues and eigenvectors of $A$.

If $n = 3$ then the characteristic equation $\det(B - \lambda I) = 0$ of $B$ may be written as $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$ and the characteristic equation $\det(A - \mu I) = 0$ of $A$ as $\mu^3 - d_1\mu^2 + d_2\mu - d_3 = 0$, where, for $j = 1, 2, 3, c_j$ and $d_j$ are the (strictly positive) $j$th spurs of $B$ and $A$ respectively.

Setting $\mu = \lambda^2$ and $A = B^2$, we find that these characteristic equations are related by:

$$\det(A - \mu I) = \det(B - \lambda I) \cdot \det(B + \lambda I)$$

so that the $c$-coefficients may be obtained by equating terms in:

$$[\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3][\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3] = \lambda^6 - d_1\lambda^4 + d_2\lambda^2 - d_3,$$

whence

$$d_1 = c_1^2 + 2c_2, \quad d_2 = c_2^2 - c_1c_3, \quad d_3 = c_3^2.$$

Thus $C_3$ is clearly the positive square root of $D_3$ and $c_1$ and $c_2$ are the real positive roots (supposed unique) of the quartic equations:

$$4(d_2 + c_1c_3) = (d_1 - c_1^2)^2$$

and

$$(d_2 - c_2^2)^2 = d_3(d_1 - 2c_2)$$

which may readily be solved by Ferrari's procedure as the coefficients of the terms in $c_1^3$ and $C_2^3$ are zero.

Further, by the Cayley-Hamilton Theorem, $B$ satisfies its own characteristic equation $B^3 - c_1B^2 + c_2B - c_3I = 0$, so that the required value of $B$ may be obtained from $[A + c_2I]B = [c_1A + c_3I]$ by Gaussian Elimination. Moreover, since $A = UMU'$ is a real symmetric positive definite matrix with $j$th eigenvalue $\mu_j > 0$, we find that $B = U\Lambda U'$ is also a real symmetric positive definite matrix with $j$th eigenvalue $\lambda_j = [c_1\mu_j + c_3]/[\mu_j + c_2] > 0$. Thus, as required, $B$ is the unique positive definite square root of $A$.

In principle, the technique outlined here may be applied for all values of $n$. However, its use in practice would seem to be restricted to cases in which $n$ is relatively small and fixed, as in the spherical data problems discussed by Mardia & Gadzden (1977) and Jupp & Mardia (1980). It would seem to have no role if the value of $n$ is large and variable, as in econometric problems involving the set of BLUS residuals.

*References*

P.E. Jupp & K.V. Mardia (1980). A General Correlation Coefficient for Directional Data and Related Regression Problems. *Biometrika*, **67**:163–173.

K.V. Mardia & R.J. Gadsden (1977). A Small Circle of Closest Fit for Spherical Data and areas of vulcanism. *Applied Statistics* **26**:238–245.

*Appendix*

*Case n = 2:*

Noting that the $2\times 2$ matrix $B$ satisfies its own characteristic equation, we find that $B^2 - c_1B + c_2I = 0$ and thus $B = \frac{1}{c_1}(A + c_2I)$, where the strictly positive values of $c_1$ and $c_2$ may be obtained by equating terms in:

$$[\lambda^2 - c_1\lambda + c_2][\lambda^2 + c_1\lambda + c_2] \quad = \lambda^4 - d_1\lambda^2 + d_2$$

In this context, we have $d_1 = c_1^2 - 2c_2$ and $d_2 = c_2^2$, so that $c_2$ is the positive square root of $d_2$ and $c_1$ is the positive square root of $d_1 + 2c_2$.

*Case n = 4:*

In a similar way, if $n = 4$, then the characteristic equation $det(B - \lambda I) = 0$ of $B$ may be written as

$$\lambda^4 - c_1\lambda^3 + c_2\lambda^2 - c_3\lambda + c_4 = 0$$

and the characteristic equation $\det(A - \mu I) = 0$ of $A$ as

$$\mu^4 - d_1\mu^3 + d_2\mu^2 - d_3\mu + d_4 = 0$$

where, for $j = 1, 2, \ldots, 4,\quad c_j$ and $d_j$ are the (strictly positive) $j$th spurs of $B$ and $A$ respectively.

Setting $\mu = \lambda^2$ and $A = B^2$, we find that these characteristic equations are related by:

$$\det(A - \mu I) = \det(B - \lambda I) \cdot \det(B + \lambda I)$$

where the $c$-coefficients are obtained by equating terms in

$$\begin{aligned} &[\lambda^4 - c_1\lambda^3 + c_2\lambda^2 - c_3\lambda + c_4] \\ [\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4] &= [\lambda^4 + c_2\lambda^2 + c_4]^2 - [c_1\lambda^3 + c_3\lambda]^2 \\ &= \lambda^8 - d_1\lambda^6 + d_2\lambda^4 - d_3\lambda^2 + d_4 \end{aligned}$$

or:

$$d_1 = c_1^2 - 2c_2 d_2 = 2c_4 - 2c_1c_3 + c_2^2 d_3 = c_3^2 - 2c_2c_4 d_4 = c_4^2$$

Thus, $C_4$ is clearly the positive square root of $D_4$ and $C_2$ is the strictly positive real solution (supposed unique) of the quartic equation:

$$[-d_2 + 2c_4 + c_2^2]^2 = 4[d_1 + 2c_2][d_3 + 2c_2c_4]$$

which may readily be solved by Ferrari's procedure as the coefficient of the term in $C_2^3$ is zero.

Further, by the Cayley-Hamilton Theorem, $B$ satisfies its own characteristic equation $B^4 - c_1B^3 + c_2B^2 - c_3B + c_4I = 0$, so that the required value of $B$ may be obtained from $[c_1A + c_3I]B = [A^2 + c_2A + c_4I]$ by Gaussian Elimination. Moreover, since $A = UMU'$ is + a real symmetric positive definite matrix with $j$th eigenvalue $\mu_j > 0$, we find that $B = U\Lambda U'$ is also a real symmetric positive definite matrix with $j$th eigenvalue $\lambda_j = [c_1\mu_j + c_3]^{-1}[\mu_j^2 + c_2\mu_j + c_4] > 0$. Thus, as required, $B$ is the unique positive definite square root of $A$.

**Problem 41-6: Square Root of a $2 \times 2$ P-Matrix**

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England, UK,* R.W.Farebrother@Manchester.ac.uk

Given a $2 \times 2$ matrix $A$ with positive diagonal elements and a positive determinant, identify a $2 \times 2$ matrix $B$ with the same properties and satisfying $B^2 = A$.

**Solution 41-6** by Ewa Pawelec, *Warsaw University of Technology, Poland,* epawelec@mini.pw.edu.pl and Alicja Smoktunowicz, *Warsaw University of Technology, Poland,* smok@mini.pw.edu.pl

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a given matrix with positive diagonal elements $a, d$ and a positive determinant $\det(A) = ad - bc$. We would like to find a matrix $B = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ with the same properties such that $B^2 = A$. From this it follows that

$$x^2 + yz = a, \tag{7}$$

$$y(x + t) = b, \tag{8}$$

$$z(x + t) = c, \tag{9}$$

$$t^2 + yz = d, \tag{10}$$

and additionally

$$xt - yz = \det(B) = \sqrt{\det(A)}. \tag{11}$$

Multiplying (11) by 2 and adding this result to (7) and (10) gives $(x + t)^2 = r^2$, where

$$r = \sqrt{a + d + 2\sqrt{\det(A)}}. \tag{12}$$

Since $x, t > 0$ then using (8)-(9) we have $x + t = r,\ y = b/r,\ \ z = c/r$. From (7) and (10), we obtain $x^2 = (ar^2 - bc)/r^2,\ t^2 = (dr^2 - bc)/r^2$. We see that $x = (a + \sqrt{\det(A)})/r$ and $t = (d + \sqrt{\det(A)})/r$, so $B = \frac{1}{r}(A + \sqrt{\det(A)}\ I)$, where $I$ is $2 \times 2$ identity matrix and $r$ is defined by (12). It is easy to verify that $B$ is a solution of our problem.

*Also solved by the proposer.*

**Problem 41-7: Simplification of a $3 \times 3$ Matrix**

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England, UK,* R.W.Farebrother@Manchester.ac.uk

Let $A$ be a $3 \times 3$ matrix with nonzero elements. Show that $A$ can be expressed in the form $A = D + fg'$, where $D$ is a $3 \times 3$ diagonal matrix and $f$ and $g$ are $3 \times 1$ column matrices if and only if the off-diagonal elements of $A$ satisfy the condition: $a_{12}a_{23}a_{31} = a_{13}a_{21}a_{32}$.

**Solution 41-7** by Nir Cohen, *University of Campinas, Brazil,* nir@ime.unicamp.br

Proof of necessity: If $A = D + fg^t$ with $D$ diagonal then direct calculation gives

$$a_{12}a_{23}a_{31} = (f_1g_2)(f_2g_3)(f_3g_1) = (f_2g_1)(f_3g_2)(f_1g_3) = a_{21}a_{32}a_{13}.$$

Proof of sufficiency: Let $A = (a_{ij})$ be a given matrix with non-zero entries. Define by $\Delta_{ij}$ the $2 \times 2$ determinant obtained from $A$ by removing row $i$ and column $j$. Set

$$D = diag\{\Delta_{23}/a_{32}, -\Delta_{13}/a_{31}, \Delta_{12}/a_{21}\},$$

$$f = a_{21}[a_{13}/a_{23}, 1, a_{31}/a_{21}]^t, \qquad g = [1, a_{32}/a_{31}, a_{23}/a_{21}]^t.$$

(These are well defined since entries were assumed non-zero). Direct calculation shows that

$$D + fg^t = \begin{pmatrix} s & t & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Moreover, assuming the identity $a_{12}a_{23}a_{31} = a_{21}a_{32}a_{13}$ we get

$$s = a_{11} + \left\{ -\frac{a_{12}a_{31}}{a_{32}} + \frac{a_{21}a_{13}}{a_{23}} \right\} = a_{11}, \qquad t = \frac{a_{21}a_{13}a_{32}}{a_{23}a_{31}} = a_{12}$$

implying that $D + fg^t = A$.

*Remark.* Problem 41-7 has appeared as an exercise in [1] page 171, where the context was the determination of the minimal rank of $A - D$ over all $D$ diagonal. It was commented there without proof that the minimal rank is 1 under the identity in question, and is 2 otherwise.

*Reference*

N. Cohen, C.R. Johnson, L. Rodman, H.J. Woerdeman (1989). Ranks of completions of partial matrices. *Operator Theory: Advances and Applications* **40**:165-185.

*Also solved by the proposer.*

**Problem 41-8: On the Moore-Penrose Inverse of a Particular Block Partitioned Matrix**

Proposed by William F. Trench, *Trinity University, San Antonio, Texas, USA,* wtrench@trinity.edu

Prove the following theorem.

THEOREM. *Let $c_0, \ldots, c_{k-1}$, $d_0, \ldots, d_{k-1}$ be positive integers with $k \geq 2$ and let $\mu$, $p_0, \ldots, p_{k-1}$, and $q_0, \ldots, q_{k-1}$ be integers such that the set $\mathcal{T} := \{(r,s) \in \mathbb{Z}_k \times \mathbb{Z}_k \,|\, p_r + q_s \equiv \mu (\bmod\ k)\}$ is nonempty. Suppose that $C = [C_{rs}]_{r,s=0}^{k-1}$ with $C_{rs} \in \mathbb{C}^{c_r \times d_s}$ and $C_{rs} = 0$ if $(r,s) \notin \mathcal{T}$. Then $C^\dagger = [D_{rs}]_{r,s=0}^{k-1}$ with $D_{rs} \in \mathbb{C}^{d_r \times c_s}$ and $D_{rs} = 0$ if $(s,r) \notin \mathcal{T}$. Moreover, if $r \neq r'$ and $s \neq s'$ whenever $(r,s)$ and $(r',s')$ are distinct pairs in $\mathcal{T}$, then $D_{rs} = C_{sr}^\dagger$, $(s,r) \in \mathcal{T}$, where $(\cdot)^\dagger$ indicates the Moore-Penrose inverse of matrix $(\cdot)$.*

**Solution 41-8** by the proposer William F. Trench, *Trinity University, San Antonio, Texas, USA,* wtrench@trinity.edu

We label the equations respectively as follows:

$$\mathcal{T} = \{(r,s) \in \mathbb{Z}_k \times \mathbb{Z}_k \,|\, p_r + q_s \equiv \mu \pmod{k}\}; \tag{13}$$

$$C_{rs} = 0 \quad \text{if} \quad (r,s) \notin \mathcal{T}; \tag{14}$$

$$C^\dagger = [D_{rs}]_{r,s=0}^{k-1} \quad \text{with} \quad D_{rs} \in \mathbb{C}^{d_r \times c_s}; \tag{15}$$

$$D_{rs} = 0 \quad \text{if} \quad (s,r) \notin \mathcal{T}; \tag{16}$$

$$D_{rs} = C_{sr}^\dagger, \quad (s,r) \in \mathcal{T}. \tag{17}$$

In any case, $C^\dagger$ can be written as in (15). Let $\zeta = e^{-2\pi i/k}$,

$$R = \zeta^{p_0} I_{c_0} \,\oplus\, \zeta^{p_1} I_{c_1} \,\oplus \cdots \oplus\, \zeta^{p_{k-1}} I_{c_{k-1}},$$

and

$$S = \zeta^{-q_0} I_{d_0} \,\oplus\, \zeta^{-q_1} I_{d_1} \,\oplus \cdots \oplus\, \zeta^{-q_{k-1}} I_{d_{k-1}}.$$

Then

$$RCS^* = \left[\zeta^{p_r+q_s} C_{rs}\right]_{r,s=0}^{k-1} = \zeta^\mu C,$$

where (13) and(14) imply the second equality; hence $C = \zeta^{-\mu} RCS^*$. Now let

$$D = \zeta^\mu S C^\dagger R^*. \tag{18}$$

We will show that $C$ and $D$ satisfy the Penrose conditions and that (15) and (16) hold.

Since $R$ and $S$ are unitary,

$$CD = RCC^\dagger R^* = (CD)^*, \quad DC = SC^\dagger CS^* = (DC)^*,$$

$$CDC = \zeta^{-\mu} RCC^\dagger CS^* = \zeta^{-\mu} RCS^* = C,$$

and

$$DCD = \zeta^\mu SC^\dagger CC^\dagger R^* = \zeta^\mu SC^\dagger R^* = D.$$

Hence $D = C^\dagger$, which implies (15). Now (18) implies that $D = \zeta^\mu SDR^*$, so

$$D_{rs} = \zeta^{\mu - q_r - p_s} D_{rs}, \quad 0 \le r, s \le k-1.$$

This and (13) imply (16).

If the second assumption holds, there is a permutation $\{r_0, r_1, \ldots, r_{k-1}\}$ of $\mathbb{Z}_k$ such that

$$\mathcal{T} \subset \{(r_0, 0), (r_1, 1), \quad (r_{k-1}, k-1)\}.$$

Since $C_{rs}^\dagger = 0_{sr}$ if $C_{rs} = 0_{rs}$, (14) implies that

$$C = U\left(C_{r_0,0} \oplus C_{r_1,1} \oplus \cdots \oplus C_{r_{k-1},k-1}\right),$$

where $U$ is a permutation matrix. Hence, our earlier result implies that $C^\dagger$ is of the form

$$C^\dagger = \left(D_{r_0,0} \oplus D_{r_1,1} \oplus \cdots \oplus D_{r_{k-1},k-1}\right) U^T.$$

Given that $C$ and $C^\dagger$ satisfy the Penrose conditions, it is straightforward to verify that the pairs $(C_{r_0,0}, D_{r_0,0})$ $\ldots (C_{r_{k-1},k-1}, D_{r_{k-1},k-1})$ separately satisfy them. This implies (17) and completes the proof.

**Problem 41-9: Conjugate Annihilating Normal Matrices**

Proposed by Geoffrey Goodson, *Towson University, Towson, USA,* goodson@towson.edu
and Roger A. Horn, *University of Utah, Salt Lake City, USA*, rhorn@math.utah.edu

Let $A$ be a normal matrix with complex conjugate $\bar{A}$ and transpose $A^T$. Show that $A\bar{A} = 0$ if and only if $AA^T = A^TA = 0$.

**Solution 41-9.1** by C.M. da Fonseca, *University of Coimbra, Portugal,* cmf@mat.uc.pt
and Milica Anđelić, *University of Aveiro, Portugal,* milica@matf.bg.ac.yu

Let $A = U^*\Lambda U$ be a Schur decomposition of $A$, where $U$ is a unitary matrix and $\Lambda$ is a diagonal matrix, all of the same size. Then $\bar{A} = U^T\bar{\Lambda}\bar{U}$ and $A^T = U^T\Lambda\bar{U}$ . Therefore,

$$A\bar{A} = 0 \Leftrightarrow (U^*\Lambda U)(U^T\bar{\Lambda}\bar{U}) = 0 \Leftrightarrow U^*(\Lambda UU^T\bar{\Lambda})\bar{U} = 0 \Leftrightarrow \Lambda UU^T\bar{\Lambda} = 0 \Leftrightarrow \Lambda UU^T\Lambda = 0 \Leftrightarrow U^*(\Lambda UU^T\Lambda)\bar{U} = 0 \Leftrightarrow AA^T = 0\,.$$

Since $A\bar{A} = 0$ is equivalent to $\bar{A}A = 0$, the other equivalence is analogous, i.e., $A\bar{A} = 0$ if and only if $A^TA = 0$.

**Solution 41-9.2** by Fuzhen Zhang, *Nova Southeastern University, USA,* zhang@nova.edu

$\Rightarrow$: $A\bar{A} = 0 \Rightarrow A^*A\bar{A} = 0 \Rightarrow AA^*\bar{A} = 0 \Rightarrow \bar{A}A^TA = 0$. So $(A^TA)^*(A^TA) = A^*\bar{A}A^TA = 0$ and $A^TA = 0$. Similarly, $A\bar{A} = 0 \Rightarrow \bar{A}A = 0 \Rightarrow \bar{A}AA^* = 0 \Rightarrow \bar{A}A^*A = 0 \Rightarrow AA^T\bar{A} = 0$. So $(AA^T)(AA^T)^* = AA^T\bar{A}A^* = 0$ and $AA^T = 0$.
$\Leftarrow$: $(A\bar{A})^*(A\bar{A}) = A^TA^*A\bar{A} = A^TAA^*\bar{A} = 0$. So $A\bar{A} = 0$.

*Also solved by Johanns de Andrade Bezerra, Naomi Shaked-Monderer, Ewa Pawelec and Alicja Smoktunowicz, Hans Joachim Werner, and the proposers.*

**Problem 41-10: Difference of Two Symmetric Matrices**
Proposed by Heinz Neudecker, *University of Amsterdam, Amsterdam, The Netherlands*
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany*, trenkler@statistik.uni-dortmund.de

Let $Y$ be an $n \times n$ real symmetric matrix with rank $n-1$. Furthermore suppose that there exists a nonsingular symmetric matrix $T$ satisfying $YTY = Y$. Show that $Y - T^{-1}$ is either nonnegative or nonpositive definite of rank one.

**Solution 41-10.1** by Johanns de Andrade Bezerra, *Campina Grande, Brazil,* pav.animal@hotmail.com

The identity $YTY = Y$ implies $(TY)^2 = TY$, hence $U := TY$ is idempotent and $rank(U) = rank(Y) = n-1$, and so $V := I - U$ is idempotent of rank 1.

The matrix $Y - T^{-1}$ is symmetric as a difference of two symmetric matrices. In addition, $Y - T^{-1} = -T^{-1}(I - TY) = -T^{-1}V$, hence $rank(Y - T^{-1}) = rank(V) = 1$.

**Solution 41-10.2** by Jan Hauke, *Adam Mickiewicz University, Poznań, Poland*, jhauke@amu.edu.pl

Let $\mathbf{Y}$ and $\mathbf{T}$ be hermitian matrices, such that

$$rank(\mathbf{T}) = rank(\mathbf{Y}) + 1, range(\mathbf{Y}) \subseteq range(\mathbf{T}), \text{ and } \mathbf{YTY} = \mathbf{Y}.$$

Then [1]:
FACT 1. $\mathbf{YT}$ is an idempotent.
FACT 2. $\mathbf{T}^+\mathbf{T} - \mathbf{YT}$ is an idempotent ($\mathbf{T}^+$ denotes the Moore- Penrose inverse of $\mathbf{T}$).
FACT 3. $rank(\mathbf{Y} - \mathbf{T}^+) = rank(\mathbf{T}^+ - \mathbf{Y}) = rank(\mathbf{T}^+\mathbf{T} - \mathbf{YT}) = trace(\mathbf{T}^+\mathbf{T} - \mathbf{YT}) = rank(\mathbf{T}^+\mathbf{T}) - rank(\mathbf{YT}) = 1$.
FACT 4. $\mathbf{Y} - \mathbf{T}^+ = \alpha\mathbf{x}\mathbf{x}^*$, for some real scalar $\alpha$ and some vector $\mathbf{x}$ ($\mathbf{x}^*$ is the conjugate transpose of $\mathbf{x}$).
It follows from FACTS 1-4 that $\mathbf{Y} - \mathbf{T}^+$ is either nonnegative or nonpositive definite of rank one.

*Reference*

[1] C.D. Meyer (2000). *Matrix Analysis and Applied Linear Algebra.* Society for Industrial and Applied Mathematics Philadelphia, PA, USA.

*Also solved by Nir Cohen, Naomi Shaked-Monderer, Hans Joachim Werner, and the proposers.*

**Problem 41-11: Positive Semidefinite Matrices and Tensor Product**
Proposed by Chi-Kwong Li, *College of William and Mary, Williamsburg, USA,* ckli@math.wm.edu

Let $A,\ B$ be $m \times m$ and $C,\ D$ be $n \times n$ positive definite matrices, respectively. For any $r \in (0,1)$ and $a,\ b > 0$, show that

$$(aA + bB)^r \otimes (aC + bD)^{(1-r)} \geq a(A^r \otimes C^{(1-r)}) + b(B^r \otimes D^{(1-r)}),$$

where $X \otimes Y$ denotes the tensor product of matrices $X$ and $Y$ and $P \geq Q$ means $P - Q$ is positive semidefinite. Deduce that the result also holds if the tensor product is replaced by the Hadamard product.

**Solution 41-11** by the proposer Chi-Kwong Li, *College of William and Mary, Williamsburg, USA*, ckli@wm.edu

Note that there is an invertible $S$ such that $SAS^*$ and $SBS^*$ are diagonal matrices. Similarly, there is an invertible $T$ such that $TCT^*$ and $TDT^*$ are diagonal matrices. So, the problem reduces to the scalar cases. The inequality then follows readily from the standard Hölder inequality.

Since Hadamard products of the matrices are submatrices of the tensor products of the matrices, the result follows.

**Problem 41-12: An Inequality Involving a Product of Two Orthogonal Projectors**
Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany*, trenkler@statistik.uni-dortmund.de

Let $P$ and $Q$ be $n \times n$ complex orthogonal projectors (i.e., Hermitian idempotent matrices of order $n$). Show that

$$[\mathrm{trace}(PQ)]^2 \leqslant \mathrm{rank}(PQ)\,\mathrm{trace}(PQPQ),$$

with equality if and only if $\lambda PQ$ is an oblique projector (i.e., idempotent matrix) for some scalar $\lambda > 0$.

**Solution 41-12.1** by Nir Cohen, *University of Campinas, Brazil* nir@ime.unicamp.br

We start with a slightly more general result.

Claim 1. *Let $A$ be a matrix with real eigenvalues. Then*

$$tr^2(A) \leq rank(A) \cdot tr(A^2). \tag{19}$$

*Moreover, if* (19) *holds with equality then the non-zero eigenvalues of $A$ are equal and the zero eigenvalues are all* **semi-***simple.*

*Proof.* Let $J = D + U$ be the Jordan form of $A$, where $D$ is real diagonal and $U$ is strictly upper triangular. Let $k$ be the number of non-zero eigenvalues of $A$, i.e. $k = rank(D)$. Let $d \in \mathbb{R}^k$ be the vector consisting of the $k$ diagonal entries. The following relations are evident:

$$(I) \quad |tr(A)| = |tr(D)| \leq \|d\|_1, \qquad (II) \quad tr(A^2) = tr(D^2) = \|d\|_2^2,$$

$$(III) \quad k \leq rank(A), \qquad (IV) \quad \|d\|_1^2 \leq k\|d\|_2^2.$$

(Item (IV) is a well known norm bound and follows from the Cauchy-Schwarz inequality). Altogether we have $tr^2(A) \leq \|d\|_1^2 \leq k\|d\|_2^2 \leq rank(A)tr(A^2)$ proving (19).

Next we discuss equality, where we discard the trivial case where $A$ is nilpotent. Equality in (19) implies equality in each of (I,III,IV); and it is easy to check that equality in (I,IV) implies that $d = (c, \cdots, c)$. We claim that (III) holds with equality iff all the zero eigenvalues of $A$ are simple. Indeed, $rank(A)$ equals the sum of ranks of its Jordan blocks, and $k = rank(D)$ equals the sum of ranks of the non-nilpotent blocks. Hence $rank(A) = rank(D)$ implies that nilpotent Jordan blocks should have zero rank, which implies simplicity of the zero eigenvalue. □

Much of the applicability of Claim 1, including the case treated here of a product of two orthogonal projections, is due to the following result:

Claim 2. *Let $A$ be a product of two Hermitian matrices $H, P$ with $P$ p.s.d. Then the eigenvalues of $A$ are real, and are nonnegative if also $H$ is p.s.d.*

First assume that $P$ is p.d. Then this is Wigner's result (see [1] Theorem 7.6.3 followed by relevant exercises, or [2]) where it is also shown that $A$ and $H$ have the same inertia, hence $A$ has nonnegative eigenvalues if $H$ is p.s.d. If $P$ is only p.s.d, the same follows by considering $P_n = P + (1/n)I$ with $n \to \infty$. □

*Example 3. If $A$ is a product of two positive definite matrices and satisfies (19) with equality then $A = cI$ for some $c > 0$.*

Indeed, $A$ is diagonalizable (Wigner, [2]) and by Claim 2 has positive eigenvalues, so in the proof of Claim 1 we may take $U = 0$, so $A$ and $D$ are similar. But equality in (I,IV) implies that $D = cI$, hence $A = D$. □

We can now prove the required property, slightly generalized.

Claim 4. *Let $A = PQ$ where $P, Q$ are Hermitian and $P^2 = P$.*

*(i) The non-zero eigenvalues of $A$ are simple; the corresponding eigenvectors are orthogonal.*

*(ii) If* (19) *holds with equality then $A$ is an oblique projection.*

*Proof.* (i) Observe that the Hermitian positive semidefinite matrix $B := PQP$ satisfies (i). Let $\lambda \neq 0$ be an eigenvalue of $A$ and consider the associated spectral equations

$$Av_1 = \lambda v_1, \qquad Av_2 = \lambda v_2 + v_1, \qquad \cdots$$

Multiplying from the left by $P$ we get $\lambda P v_1 = PAv_1 = Av_1 = \lambda v_1$, or $Pv_1 = v_1$ and similarly $Pv_k = v_k$; this shows that $Bv_k = APv_k = Av_k$ holds for all $k$, implying the same spectral equations for $B$:

$$Bv_1 = \lambda v_1, \qquad Bv_2 = \lambda v_2 + v_1, \qquad \cdots.$$

Thus eigenvectors must be pairwise orthogonal, and $v_k$ $(k \geq 2)$ do not exist, since $B$ satisfies (i).

(ii) By part (i), non-zero eigenvalues are simple. If, in addition, $A$ satisfies (19) with equality, by Claim 1 the zero eigenvalue is also simple. Thus, $A$ is diagonalizable, and the proof proceeds as in Example 3. Namely, $A = SDS^{-1}$ with $D$ real, and by Claim 1 $d = (c, \cdots, c)$ with $c \in \mathbb{R}$ non-zero ($c \geq 0$ if $Q$ is p.s.d), hence $c^{-1}D$ is a projection and so $c^{-1}A$ is an oblique projection. □

*References* R.A. Horn, C.R. Johnson. *Matrix Analysis*. Cambridge 1985.

E.P. Wigner, On weakly-positive matrices. *Canadian J. Math* **15**:313-317, 1963.

*Editorial note: For a similar result as Claim 1, see F. Zhang, Matrix Theory, p.217, Springer, 1999.*

**Solution 41-12.2** by Hans Joachim Werner, *University of Bonn, Bonn, Germany,* hjw.de@uni-bonn.de

Let $P, Q \in \mathbb{C}^{n\times n}$ be arbitrary but fixed ORTHOGONAL projectors, i.e., let $P = P^2 = P^*$ and $Q = Q^2 = Q^*$, where $(\cdot)^*$ indicates the conjugate transpose of $(\cdot)$. Then $PQ = PQ^2$ and $PQ = P^2Q$. Because any product of conformable matrices can be permuted CYCLICALLY without altering the trace of the product, we therefore obtain $\mathrm{trace}(PQ) = \mathrm{trace}(PQP) = \mathrm{trace}(QPQ)$. We note that the matrices $PQP$ and $QPQ$ are nonnegative definite and Hermitian. Furthermore, it should be emphasized that the matrices $QP$, $PQ$, $PQP$ and $QPQ$ all have the same eigenvalues of according (algebraic and geometric) multiplicities. Theses eigenvalues are all nonnegative. Let $r$ be the number of (not necessarily distinct) positive eigenvalues, and let $\lambda_1, \ldots, \lambda_r$ denote these positive eigenvalues. Because the trace of a square matrix coincides with the sum of its eigenvalues, we have $\mathrm{trace}(PQ) = \mathrm{trace}(PQP) = \mathrm{trace}(QPQ) = \mathrm{trace}(QP) = \sum_{i=1}^{r} \lambda_i$. Next, we note that $\mathrm{rank}(PQP) = \mathrm{rank}(QP) = \mathrm{rank}(PQ) = \mathrm{rank}(PQP) = r$. Finally, we recall that the eigenvalues of $(PQ)^2 = PQPQ$ are obtained as the squares of the eigenvalues of $PQ$. With all these observations in mind, it is clear that

$$(\mathrm{trace}(PQ))^2 \leq \mathrm{rank}(PQ) \cdot \mathrm{trace}(PQPQ) \iff (\sum_{i=1}^{r} \lambda_i)^2 \leq r \cdot \sum_{i=1}^{r} \lambda_i^2 \iff (\iota^*\lambda)^2 \leq \iota^*\iota \cdot \lambda^*\lambda,$$

where $\iota = (1, 1, \ldots, 1)^* \in \mathbb{C}^r$ and $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)^*$. Because the last inequality is the famous CAUCHY-BUNYAKOVSKII-SCHWARZ (CBS) INEQUALITY that relates the inner product $\iota^*\lambda$ to the norms of the vectors, our solution to this problem is nearly complete. It suffices to mention that equality holds in the CBS-Inequality if and only if $\lambda = \alpha\iota$ for some positive scalar $\alpha$ (in case that $r \geq 1$, i.e., $PQ \neq 0$). The latter is equivalent to $PQP \cdot PQP = \alpha \cdot PQP$ (i.e., $(\alpha I - PQ)PQP = 0$) or, equivalently, to $(\alpha I - PQ)PQ = 0$ (i.e., $\alpha PQ = PQPQ$)).

*Also solved by the proposers.*

**Problem 41-13: Range Additivity of $A$ and $A^*$**

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl,
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Let $A$ be a square complex matrix and let $A^*$ denote its conjugate transpose. An eigenvector $x$ of $A$ with associated eigenvalue $\lambda$ is a *normal eigenvector* if both $Ax = \lambda x$ and $A^*x = \bar{\lambda}x$. Consider the range additivity condition

$$\mathrm{range}(A + A^*) = \mathrm{range}A + \mathrm{range}A^*. \tag{20}$$

(a) Suppose $A^2$ is normal. Show that $A$ satisfies (20) if and only if for every normal eigenvector of $A$ the associated eigenvalue is either zero or not pure imaginary. In particular, if $A$ is normal then it satisfies (20) if and only if no nonzero eigenvalue of $A$ is pure imaginary.

(b) Suppose $A^2 = \lambda A$ for some $\lambda \in \mathbb{C}$. Show that $A$ satisfies (20) if and only if for every normal eigenvector of $A$ the associated eigenvalue is either zero or not pure imaginary. The special cases $\lambda = 0$ ($A$ is self-annihilating) and $\lambda = 1$ ($A$ is a projection) are noteworthy.

(c) Suppose $A^k = A^*$ for some $k \in \{2, 3, \ldots\}$. If $k$ is even, show that $A$ satisfies (20); if $k$ is odd, show that $A$ satisfies (20) if and only if no nonzero eigenvalue of $A$ is pure imaginary.

Examples: Consider $A_1 = [i]$, $A_2 = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2i & 1 \\ 0 & 0 \end{bmatrix}$.

The matrix $A_1$ does not satisfy the range additivity property and has a nonzero pure imaginary eigenvalue associated with a normal eigenvector; moreover, $A_1^3 = A^*$. The matrix $A_2$ satisfies the range additivity property and has nonzero pure imaginary eigenvalues; $A_2^2$ is normal but $A_2$ has no normal eigenvectors. The matrix $A_3$ satisfies the range additivity property and has a nonzero pure imaginary eigenvalue; $A_3^2 = 2iA_3$ but $A_3$ has no normal eigenvectors.

**Solution 41-13** by the proposers Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl, Roger A. Horn, *University of Utah, Salt Lake City, USA*, rhorn@math.utah.edu and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

All of the assertions are invariant under unitary similarity, so in each case it suffices to consider a suitable canonical form under unitary similarity and verify the assertion for the canonical blocks.
(a) If $A^2$ is normal, then there is a unitary $U$ such that $U^*AU$ is a direct sum of blocks, each of which is

$$[\lambda] \quad \text{or} \quad \tau \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}, \ \tau \in \mathbb{R},\ \lambda, \mu \in \mathbb{C},\ \tau > 0, \text{ and } |\mu| < 1;$$

see [2, Theorem 7.2] or [1, Theorem 1]. We have $\operatorname{range}([\lambda] + [\lambda]^*) = \operatorname{range}([\lambda + \bar{\lambda}]) = \operatorname{range}[\lambda] + \operatorname{range}[\bar{\lambda}]$ if and only if either $\lambda = 0$ or $\lambda \neq 0$ and $\lambda + \bar{\lambda} \neq 0$. We always have

$$\operatorname{range}\left(\begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}^*\right) = \operatorname{range}\begin{bmatrix} 0 & 1+\bar{\mu} \\ 1+\mu & 0 \end{bmatrix} = \operatorname{range}\begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} + \operatorname{range}\begin{bmatrix} 0 & \bar{\mu} \\ 1 & 0 \end{bmatrix} = \mathbb{C}^2$$

since $\mu \neq -1$. If $A$ is normal, then every eigenvector is a normal eigenvector; in this case, the assertion also follows directly from the spectral theorem (a canonical form for normal matrices under unitary similarity).
(b) If $A^2 = \lambda A$ for some $\lambda \in \mathbb{C}$, then there is a unitary $U$ such that $U^*AU$ is a direct sum of blocks, each of which is

$$[\lambda] \quad \text{or} \quad \begin{bmatrix} \lambda & \tau \\ 0 & 0 \end{bmatrix}, \ \tau \in \mathbb{R},\ \tau > 0;$$

see [2, Theorem 8.14]. The analysis for the 1-by-1 blocks is the same as in (a), and we always have

$$\operatorname{range}\left(\begin{bmatrix} \lambda & \tau \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda & \tau \\ 0 & 0 \end{bmatrix}^*\right) = \operatorname{range}\begin{bmatrix} \lambda+\bar{\lambda} & \tau \\ \tau & 0 \end{bmatrix} = \operatorname{range}\begin{bmatrix} \lambda & \tau \\ 0 & 0 \end{bmatrix} + \operatorname{range}\begin{bmatrix} \bar{\lambda} & 0 \\ \tau & 0 \end{bmatrix} = \mathbb{C}^2.$$

(c) If $A^k = A^*$ for some positive integer $k$, then $A^*$ (a polynomial in $A$) commutes with $A$, so $A$ is normal and there is a unitary $U$ such that $U^*AU$ is a direct sum of blocks, each of which is $[\lambda]$ for some $\lambda \in \mathbb{C}$. Let $\omega = e^{2\pi i/(k+1)}$. We have $[\lambda^k] = [\lambda]^k = [\lambda]^* = [\bar{\lambda}]$, so $\lambda \in \{0, 1, \omega, \omega^2, \ldots, \omega^k\}$, which contains no nonzero pure imaginary element if $k$ is even. The assertion follows from (a).

*Remark.* The case $k = 2$ in part (c) gives an alternative solution to part (ii) of IMAGE Problem 41-3.

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[2] R. Horn and V. Sergeichuk. Canonical forms for unitary congruence and *congruence. *Linear Multilinear Algebra* (to appear). http://arxiv.org/pdf/0710.1530

*Also solved by Johanns de Andrade Bezerra.*

**Problem 41-14: Minimum Eigenvalue of a Special Matrix**

Proposed by Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, USA*, zhang@nova.edu

(a). Let $A = (x_i + x_j)$ with $x_1, \ldots, x_n \in [0, 1]$. Find the smallest eigenvalue of $A$ in terms of $x_1, \ldots, x_n$.
(b). Let $\Omega_n$ be the set of all $n \times n$ matrices in (a). Find the best lower and upper bounds for the eigenvalues of all matrices in $\Omega_n$; that is, find the largest number $\alpha$ and the smallest number $\beta$ so that $\lambda \in [\alpha, \beta]$ for all eigenvalues $\lambda$ of any matrix $A$ in $\Omega_n$.

**Solution 41-14.1** by Nir Cohen, *University of Campinas, Brazil* nir@ime.unicamp.br

We assume that $n \geq 2$ and use the following notation:

$$A = xe^t + ex^t, \qquad x = [x_1, \cdots, x_n]^t \quad e = [1, \cdots, 1]^t.$$

Also, we say that $x$ is $k$-extremal $(0 \leq k \leq n)$ if $k$ of its entries have unit value and the remaining entries are zero.

CLAIM. *Let $\Omega_n$ be the set of symmetric $n\times n$ matrices of the form $A = xe^t + ex^t$ with $x_i \in [0,1]$. Denote by $\lambda^*(n)$ and $\lambda_*(n)$ the largest and smallest eigenvalues attained on $\Omega_n$.*

*(i) $\lambda^*(n) = 2n$, attained only when $x = e$.*

*(ii) $\lambda_*(n)$ is bounded below by $-n/4$, and is attained by an extremal vector. The equality $\lambda_*(n) = -n/4$ is valid only when $4|n$.*

*(iii) More precisely, assume that $n = 4k + m$ $(-2 \le m \le 1)$. Then $\lambda_*(n) = k - \sqrt{nk}$, attained precisely on the $k$-extremal vectors.*

*Proof.* Ignoring the trivial case $x = ce$ $(0 \le c \le 1)$, we shall assume that $x, e$ are linearly independent. Consider the eigenvalue equation $(A - \lambda I)v = 0$. If $\lambda \neq 0$ then $v \in \mathrm{R}(A) = span\{x, e\}$. But $x$ is not an eigenvector, hence up to a constant multiple we may write $v = e + \mu x$ $(\mu \in \mathbb{C})$. Now the eigenvalue equation becomes

$$nx + se + \mu sx + \mu t^2 e = \lambda e + \mu\lambda x, \qquad i.e., \qquad \begin{cases} s + \mu t^2 = \lambda \\ n + \mu s = \mu\lambda \end{cases}$$

The solution is $\mu = \pm\sqrt{n}/t$, obtaining the non-zero eigenvalues of $A$ as

$$\lambda = s \pm \sqrt{n}t.$$

(i) To obtain the upper bound $\lambda^*(n)$, we observe that in $\Omega_n$ we have $s \le n$, $t \le \sqrt{n}$, and both $s, t$ are maximized exactly when $x = e$, in which case $s + \sqrt{n}t = 2n$.

(ii) The quest for the lower bound reduces to minimization of the continuous function $f(x) = s - \sqrt{n}t = \|x\|_1 - \sqrt{n}\|x\|_2$ over the compact set $\{x \in \mathbb{R}^n : \ 0 \le x_i \le 1\}$. By the extreme value theorem, the minimum is attained at some point, say $x_* = [x_{*1}, \cdots x_{*n}]^t$.

$f(x)$ is strictly concave in each variable (since $s$ is linear and $t$ is strictly convex). Therefore, $x_*$ is necessarily extremal, namely, $x_{*i} \in \{0,1\}$ for all $i$. More precisely, if $x$ is $k$-extremal we get

$$t_*^2 = s_* = k, \qquad f(x_*) = F(k) := k - \sqrt{nk}.$$

It only remains to minimize $F(k)$ over $k \in [0,n]$ an integer. Note that $F(k)$ is strictly convex for $k > 0$ (since $k$ is linear and $\sqrt{k}$ is strictly concave). Moreover, $F(k)$ is differentiable and $F(0) = F(n) = 0$, hence its only minimizer $k_* \in (0,n)$ satisfies

$$0 = F'(k) = 1 - \sqrt{n/4k_*}, \qquad or \qquad k_* = n/4.$$

Thus, $\lambda_*(n) = f(x_*) \ge F(n/4) = -n/4$, confirming the claim. Clearly, the bound is attained only if $k_*$ is an integer, i.e. $4|n$.

(iii) We assume now that $n \not\equiv 0(mod 4)$. Convexity of $F(k)$ implies that $k_0$ is an integer minimizer for $F(k)$ iff $k_0$ is the smallest integer for which $F(k_0) < F(k_0+1)$. After some algebra this reduces to $k_0 = \lceil (n-1)^2/4n \rceil$.

But then it can be checked directly that $k_0 = k$ whenever $n = 4k + m$ with $m = 1, -1, -2$. Indeed, in each of these cases one verifies the strict inequalities $k - 1 < (n-1)^2/4n < k$. This completes the proof of (iii). □

**Solution 41-14.2** by Naomi Shaked-Monderer, *Emek Yezreel College, Yezreel, Israel,* nomi@tx.technion.ac.il

(a) Let $e$ be the vector of all ones, $x = (x_1, \ldots, x_n)^T$, and $x_0 = x^T e$. $\|\cdot\|$ denotes here the $l_2$-norm on $\mathbb{R}^n$. With these notations, $A = xe^T + ex^T$. Hence, if $x = 0$, $A = 0$ and its only eigenvalue is zero. If $x \neq 0$ is a scalar multiple of $e$, say $x = \lambda e$, then $A = 2\lambda ee^T$ is the matrix all of whose entries are equal to $2\lambda$, its only non-zero eigenvalue is $2n\lambda$, and its smallest eigenvalue is $0$. So consider the case that $x$ is not a scalar multiple of $e$. Then the column space of $A$ is spanned by $\{x, e\}$, and $A$ has exactly two nonzero eigenvalues. Since the sum of these two eigenvalues is trace$(A) = 2x^Te = 2x_0$, these eigenvalues are $x_0 \pm t$ for some $t \ge 0$. If $v$ is an eigenvector corresponding to the largest eigenvalue, then $v = \mu x + \nu e$. Note that $Ax = x_0 x + \|x\|^2 e$ is not a scalar multiple of $x$, and $Ae = \|e\|^2 x + x_0 e$ is not a scalar multiple of $e$, hence both $\mu$ and $\nu$ are nonzero. $Av = (x_0+t)v$ means that $(xe^T + ex^T)(\mu x + \nu e) = (x_0 + t)(\mu x + \nu e)$, so

$$(\mu x_0 + \nu\|e\|^2)x + (\mu\|x\|^2 + \nu x_0)e = \mu(x_0+t)x + \nu(x_0+t)e.$$

Since $\{x, e\}$ is linearly independent, this implies that

$$\nu\|e\|^2 = \mu t \quad \text{and} \quad \mu\|x\|^2 = \nu t.$$

Hence

$$\frac{t}{\|x\|^2} = \frac{\mu}{\nu} = \frac{\|e\|^2}{t},$$

and therefore $t = \|e\|\|x\|$. The smallest eigenvalue of $A$ is therefore $x^T e - \|e\|\|x\| = \sum_{i=1}^n x_i - \sqrt{n}\sqrt{\sum_{i=1}^n x_i^2}$ (and the largest is $x^T e + \|e\|\|x\|$). These formulas turn out to hold also for any $x \geq 0$, which is a scalar multiple of $e$.

(b) By the above, $\alpha = \min\{x^T e - \|e\|\|x\| \,|\, 0 \leq x \leq e\}$. Since the function $f(x) = x^T e - \|e\|\|x\|$ is a concave function on the convex set $\Gamma = \{x \,|\, 0 \leq x \leq e\}$, it attains its minimum value at an extreme point of $\Gamma$. Each extreme point of $\Gamma$ is of the form $x_\sigma = \sum_{i\in\sigma} e_i$, where $\sigma \subseteq \{1, \ldots, n\}$ and $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. If the number of elements in $\sigma$ is $k$, then $f(x_\sigma) = k - \sqrt{n}\sqrt{k}$. The minimum of $\lambda - \sqrt{n}\sqrt{\lambda}$ on the interval $[0, n]$ is attained at $\lambda = \frac{n}{4}$, hence $f$ attains its minimum on $\Gamma$ at $x_\sigma$, where the number of elements in $\sigma$ is either $\lfloor \frac{n}{4} \rfloor$ (when $n \equiv 0$ or $1 (\bmod 4)$) or at $\lceil \frac{n}{4} \rceil$ (when $n \equiv 2$ or $3 (\bmod 4)$). That is,

$$\alpha = \begin{cases} \lfloor \frac{n}{4} \rfloor - \sqrt{n}\sqrt{\lfloor \frac{n}{4} \rfloor}, & n \equiv 0 \text{ or } 1 (\bmod 4) \\ \lceil \frac{n}{4} \rceil - \sqrt{n}\sqrt{\lceil \frac{n}{4} \rceil}, & n \equiv 2 \text{ or } 3 (\bmod 4) \end{cases}$$

Similarly, $\beta = \min\{x^T e + \|e\|\|x\| \,|\, 0 \leq x \leq e\}$. Let $g(x) = x^T e + \|e\|\|x\|$. By Cauchy-Schwarz, for every $x \in \Gamma$,

$$g(x) \leq 2\|e\|\|x\| \leq 2\|e\|^2 = 2n.$$

And since $g(e) = 2\|e\|^2$, this is the maximum value of $g$ on $\Gamma$, i.e., $\beta = 2n$.

*Also solved by C.-K. Li, E. Poon, Hans Joachim Werner, and F. Zhang.*

# IMAGE Problem Corner: New Problems

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in IMAGE. Meanwhile we still look forward to receiving solutions to unsolved problems in the past issues of IMAGE: Problems 36-3 [IMAGE 36 (Spring 2006), p. 36], 39-4 [IMAGE 39 (Fall 2007), p. 32], and 41-4 [IMAGE 41 (Fall 2008), p. 43].

Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to IMAGE Problem Corner editor Fuzhen Zhang (zhang@nova.edu). The working team of the Problem Corner consists of Dennis S. Bernstein, Nir Cohen, Shaun Fallat, Dennis Merino, Edward Poon, Peter Šemrl, Wasin So, Nung-Sing Sze, and Xingzhi Zhan.

**Problem 42-1: Square Root of a Product of Two Orthogonal Projectors**

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.uni-dortmund.de

Let $P$, $Q$ be $n \times n$ orthogonal projectors (i.e., Hermitian idempotent matrices) such that $PQ$ is nonzero. Find a square root of $PQ$.

**Problem 42-2: Properties of a Certain Matrix Product**

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England,* R.W.Farebrother@Manchester.ac.uk

Given an $n \times n$ nonzero matrix $A$ with complex elements. Suppose that $B$ is a second $n \times n$ matrix with complex elements satisfying $AB = BA$ and $ABA = A$. Investigate the properties of the $n \times n$ matrix $C = BAB$.

**Problem 42-3: Commutators and Nonderogatory Matrices**

Proposed by Geoffrey Goodson, *Towson University, Towson, USA,* goodson@towson.edu

Let $A, B$ and $D$ be $n$-by-$n$ matrices. It is known that if $AB - BA = D$, where $AD = DA$, then $D$ is nilpotent; and if $A$ is diagonalizable, then $D = 0$. If instead $AB - BA^T = D$ and $AD = DA^T$, then $D$ need not be nilpotent, but it is also known that if $A$ is diagonalizable then $D = 0$.

More generally, show that if $AB - BA^T = D$ and $AD = DA^T$, where $A$ is nonderogatory, then $D$ is symmetric, singular and the eigenvalue $\lambda = 0$ of $D$ has a geometric multiplicity greater than or equal to the number of distinct eigenvalues of $A$.

**Problem 42-4: Eigenvalue Interlacing for Normal Matrices**

Proposed by Roger A. Horn, *Department of Mathematics, University of Utah, Salt Lake City, Utah, USA,* rhorn@math.utah.edu

Prove the following generalizations of eigenvalue interlacing for bordered or additively perturbed Hermitian matrices.

1. Let $n \geq 3$, $A \in M_{n-1}$, $x, y \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{C}$, and $k \in \{2, \ldots n-1\}$ be given and let

$$C = \begin{bmatrix} A & y \\ x^* & \beta \end{bmatrix} \in M_n.$$

Suppose that $A$ and $C$ are normal and let $\mathcal{D} \subset \mathbb{C}$ be a given closed disk that is not a point. If there are $k$ eigenvalues of $A$ in $\mathcal{D}$, then there are at least $k-1$ eigenvalues of $C$ in $\mathcal{D}$. If there are $p$ eigenvalues of $C$ in $\mathcal{D}$, then there are at least $p-2$ eigenvalues of $A$ in $\mathcal{D}$.

2. Let $A \in M_n$ and $x, y \in \mathbb{C}^n$ be given and let $C = A + yx^*$. Suppose that $A$ and $C$ are normal and let $\mathcal{D} \subset \mathbb{C}$ be a given closed disk that is not a point. Then the numbers of eigenvalues of $A$ and $C$ in $\mathcal{D}$ differ by no more than 1.

**Problem 42-5: Obtuse Basis and Acute Dual Basis for Euclidean Space**

Proposed by Tin-Yau Tam, *Auburn University, Auburn, USA,* tamtiny@auburn.edu

If $\{e_1, \ldots, e_n\}$ is an obtuse basis of the Euclidean space $E$ (i.e., $(e_i, e_j) \leq 0$ for all $i \neq j$), prove that the dual basis $\{e_1^*, \ldots, e_n^*\}$ (i.e., $(e_i^*, e_j) = \delta_{ij}$) is acute (i.e., $(e_i^*, e_j^*) \geq 0$ for all $i \neq j$).

**Problem 42-6: Positive Semidefinite Matrices**

Proposed by Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, USA,* zhang@nova.edu

Let $a$ and $b$ be (positive) real numbers such that $a^3 + b^3 = 2$. Show that $a + b \leq 2$. Discuss the analog of this for matrices: If $A$ and $B$ are $n \times n$ positive semidefinite matrices such that $A^3 + B^3 = 2I_n$, does it necessarily follow that $A + B \leq 2I_n$?