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IMAGE

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About IMAGE

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UPCOMING CONFERENCES AND WORKSHOPS

Western Canada Linear Algebra Meeting (WCLAM)
University of Lethbridge, Alberta, Canada, May 12-13, 2012

The 2012 Western Canada Linear Algebra Meeting (WCLAM) will be held at the University of Lethbridge in Alberta, Canada, on May 12-13, 2012. This is the 11th in a series of biennial linear algebra meetings that have been held at various sites throughout Western Canada. While the meeting has a regional base, it also attracts researchers from elsewhere. Postdocs and graduate students are encouraged to participate and may be eligible for financial assistance.

There will be three invited speakers: Ioana Dumitriu, University of Washington, Chi-Kwong Li, College of William & Mary (ILAS speaker), and Heydar Radjavi, University of Waterloo.

Other participants are encouraged to give a 25 minute presentation or poster. Please submit title and abstract by February 28, 2012 to Hadi Kharaghani (kharaghani@uleth.ca). The organizers are: Shaun Fallat, University of Regina, Canada; Hadi Kharaghani, University of Lethbridge, Canada; Steve Kirkland, National University of Ireland, Maynooth, Ireland; Peter Lancaster, University of Calgary, Canada; Michael Tstatsomeros, Washington State University, USA and Pauline van den Driessche, University of Victoria, Canada.

For registration, deadlines and other details, please visit http://www.cs.uleth.ca/~holzmann/WCLAM/.

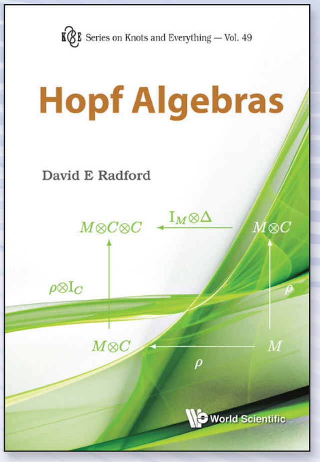
SIAM Conference on Applied Linear Algebra
Universitat Politècnica de València, València, Spain, June 18-22, 2012
Submitted by José Mas Marí

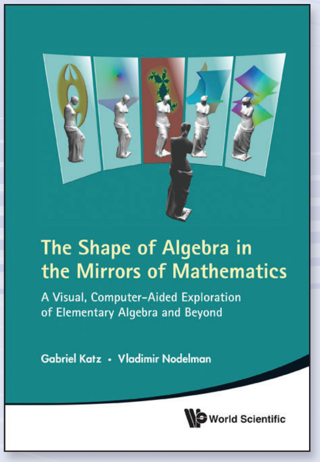
The 2012 SIAM Conference on Applied Linear Algebra will be held in València, Spain on June 18-22, 2012. ILAS members are encouraged to participate. The plenary speakers include two sponsored by ILAS: Françoise Tisseur, University of Manchester, England, and Michael Tsatsomeros, Washington State University, USA.

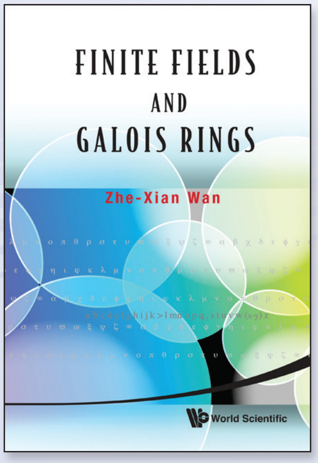
This SIAM conference, held every three years, is the premier international conference on applied linear algebra. It brings together diverse researchers and practitioners from academia, research laboratories, and industries all over the world to present and discuss their latest work. This will be the second time the conference is held outside the United States.

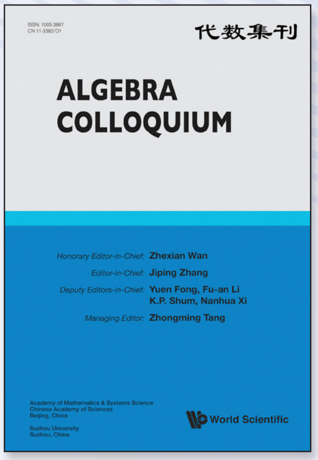
The deadline for submission of papers is January 15, 2012 and for early registration is April 28, 2012. Minisymposia proposals were due November 15, 2011. For more information, visit http://siamla2012.webs.upv.es/.

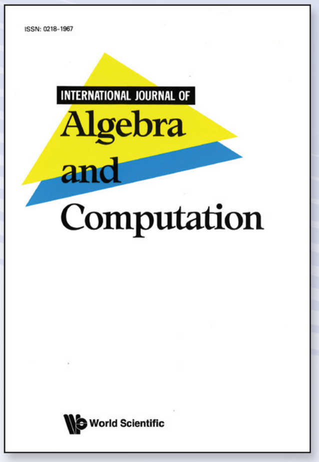
The SIAM Activity Group on Linear Algebra Prize, which is given every three years, will be awarded at this conference. It goes to the most outstanding paper on a topic in applicable linear algebra published in English in a peer-reviewed journal within the three calendar years preceding the year of the award. The period for the 2012 award is January 1, 2009 - December 31, 2011.

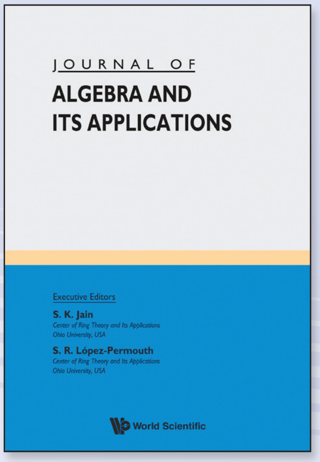












Conference on Applications of Graph Spectra in Computer Science

Centre de Recerca Mathematica (CRM), Bellaterra, Barcelona, Spain, July 16-20, 2012

Spectra of matrices associated with a graph or network are widely used in combinatorial, computer science and complex networks. The aim of this conference is to bring together the large and diverse group of researchers from graph theory, computer science and complex networks who are interested in theory or applications of graph spectra, to discuss current trends and future directions in this area. The conference is expected to raise awareness of disparate groups of researchers, provide an exchange of ideas and foster an increase in collaborations between theory- and application-oriented researchers. For young researchers in particular, this will be an excellent opportunity to see the full scope of the subject and many interesting directions they can explore.

The preliminary list of plenary speakers is Mark E.J. Newman, University of Michigan, USA; Fan Chung, University of California at San Diego, USA; Daniel Spielman, Yale University, USA; Van H. Vu, Rutgers University, USA; Jurgen Jost, Max Planck Institute for Mathematics in Sciences, Leipzig, Germany; Miguel Angel Fiol, UPC Barcelona, Spain; Piet Van Mieghem, Delft University of Technology, Netherlands; Edwin Hancock, University of York, UK; Dragos Cvetkovic, Serbian Academy of Sciences and Arts, Serbia; and Anirban Banerjee, Max Planck Institute of Molecular Genetics, Berlin, Germany.

The Scientific Committee is Dragan Stevanović (Chair), University of Primorska, Koper, Slovenia and University of Nis, Serbia; Robert Elsässer, University of Paderborn, Institute for Computer Science, Germany; Francesc Comellas, UPC, Barcelona, Spain; Vladimir Nikiforov, University of Memphis, Department of Mathematical Sciences, USA; Nair Maria Maia de Abreu, Federal University of Rio de Janeiro, Graduate School and Research in Engineering, Brazil; Fan Chung Graham, University of California at San Diego, USA; Piet Van Mieghem, Delft University of Technology, Netherlands; Dragoš Cvetković, Mathematical Institute, Serbian Academy of Science and Arts, Serbia; Miguel Angel Fiol, UPC, Barcelona, Spain; Maria José Serna, UPC, Barcelona, Spain; Dieter Mitsche, ETH, Zurich, Switzerland; and Steve Kirkland, Hamilton Institute, NUI Maynooth, Ireland.

For more information, check the website http://www.crm.es/Activitats/Activitats/2011-2012/GraphSpectra/web-graphspectra/ or contact secreteria@crm.cat.

Gaudi Dragon in Park Güell, Barcelona

Discussion Group on Teaching Linear Algebra at ICME 2012

Seoul, Korea, July 10 and 13, 2012

One of the Discussion Groups at the International Congress on Mathematical Education (ICME) in July 2012 will be DG 3: "Issues Surrounding Teaching Linear Algebra" (http://icme12.org/sub/dg/dg03.html). The discussion leaders for this are Abraham Berman (Israel) and Sang-Gu Lee (Korea). It will meet in two sessions: Tuesday July 10 and Friday July 13, both 5:00-6:30 pm. Berman and Lee are Co-Chairs of the Organizing Committee, and the other committee members are Steven Leon (USA), Sepideh Stewart (New Zealand), David Strong (USA), and K. Ravi Subramaniam (India), the ICME Liaison IPC Member. The topics will include: what is the meaning of understanding linear algebra; how to improve students' conceptual understanding and ability to think formally; skills to foster and how exams can test those skills; visualization and application of concepts; effective use of technology; and how to establish second courses in linear algebra.

ICME 2012 will be held July 8-15, 2012 in Seoul, Korea (http://icme12.org/), under the auspices of the International Commission on Mathematical Instruction (ICMI). The program will be rich and varied, with 8 Plenary Lectures, 78 regular invited lectures, 37 Topical Study Groups, 17 Discussion Groups on a variety of issues about teaching, and a number of other major subprograms. The Discussion Groups are to be venues where interested participants can meet and discuss, in a genuinely interactive way, certain challenging, controversial or emerging issues and dilemmas about teaching that are of concern to an international or regional audience.

COEX, Seoul

Gene Golub SIAM Summer School on Simulation and Supercomputing in the Geosciences
Monterey, California, July 29-August 10, 2012

Natural hazards caused by storm surges, earthquakes and tsunamis have exposed a large risk to modern societies in recent years. In order to mitigate the effects of such events, advanced simulation and computational techniques are needed for data analysis, early warning and planning purposes. This 2012 Gene Golub Summer School (G2S3) will introduce cutting edge simulation techniques for rapid assessment and accurate process studies of geoscientific problems involving a large range of relevant scales.

There will be three courses: Tsunami and Storm-surge Simulations, Numerical Methods for Wave Propagation, and Supercomputing: From Multi-core to Many-core Platforms. The primary lecturers will be Michael Bader, Technische Universität München, Germany, Jörn Behrens, Universität of Hamburg, Germany, Francis X. Giraldo, Naval Postgraduate School, USA, and Randall J. LeVeque, University of Washington, USA.

Applications are invited. Registration is free for those selected. At least partial funding for local accommodation and meal expenses will also be available for all participants. Limited travel funds may be available. Visit http://www.siam.org/students/g2s3/ for details. The deadline for applications is February 1, 2012.

Proposals for G2S3 2013 are also being sought. For more information, visit http://www.siam.org/students/g2s3/summer_call.php.

Monterey coast

Haifa Matrix Theory Conference
The Technion, Haifa, Israel, November 12-15, 2012
Preliminary Announcement

View of Haifa

The 2012 Haifa Matrix Theory Conference will take place at the Technion, on November 12-15, 2012, under the auspices of the Technion's Center for Mathematical Sciences. This will be the sixteenth in a sequence of matrix theory conferences held at the Technion since 1984. The ILAS lecturer in the conference will be Prof. Thomas Laffey.

As in the past, all talks will be of 30 minutes duration, and will cover a wide spectrum of theoretical and applied linear algebra, as well as the teaching of linear algebra. In particular, the conference will host a special session in memory of Prof. Michael (Miki) Neumann.

Titles and abstracts of contributed talks should be sent, no later than May 31, 2012, to any member of the organizing committee: Abraham Berman (berman@technion.ac.il), Raphael Loewy (loewy@technion.ac.il) and Naomi Shaked-Monderer (nomi@technion.ac.il).

International Conference on Trends and Perspectives in Linear Statistical Inference, LinStat 2012 and 21st International Workshop on Matrices and Statistics, IWMS 2012
Będlewo, Poland, July 16-20, 2012

The International Conference on Trends and Perspectives in Linear Statistical Inference, LinStat 2012, and the 21st International Workshop on Matrices and Statistics, IWMS 2012, will be held July 16-20 at Będlewo near Poznań, Poland. Previous LinStat conferences were held in Będlewo in 2008 and in Tomar, Portugal in 2010. The purpose of these conferences is to bring together researchers in statistics, its applications, and applications of matrix analysis to statistics. Researchers and graduate students in the areas of linear algebra, statistical models and computation are particularly encouraged to attend the workshop.

The Scientific Committee will award the best presentation and best poster of the young scientists, and they will be invited speakers at the next LinStat or IWMS. The Chair of the Scientific Committee for LinStat 2012 is Augustyn Markiewicz (Poland). Simo Puntanen (Finland) is Chair of the Scientific Committee for IWMS 2012. The Chair of the Local Organizing Committee is Katarzyna Filipiak (Poland). For more information, please visit http://linstat2012.au.poznan.pl/.

Call for Proposals: 3rd GAMM Applied and Numerical Linear Algebra Workshop, in 2012

GAMM, the International Association of Applied Mathematics and Mechanics, is soliciting proposals for the third workshop Applied and Numerical Linear Algebra (ANLA), to be held in 2012.

The deadline for proposals is December 23, 2011. Guidelines can be found at http://www.math.temple.edu/~siagla/summer.html.

Email proposals to Michele Benzi, SIAG/LA Chair and ISSNLA Steering Committee Chair, at benzi@mathcs.emory.edu. For information about the previous ANLA workshops, visit http://www.math.temple.edu/~siagla/summer.html.

Call for Proposals: 3rd International Summer School on Numerical Linear Algebra (ISSNLA), in 2013

The SIAM Activity Group on Linear Algebra is soliciting proposals for the third ISSNLA, to take place during the summer of 2013. See Guidelines for Proposals at http://www.math.temple.edu/~siagla/summer.html.

The deadline for submissions is December 23, 2011. Proposals should be emailed to Michele Benzi, SIAG/LA ISSNLA Steering Committee Chair, benzi@mathcs.emory.edu.

The first ISSNLA took place in Castro Urdiales, Spain, in 2008 and was held in cooperation with SIMUMAT (Mathematical Modeling and Numerical Simulation in Science and Technology). The second ISSNLA was held in Selva di Fasano, Italy, in 2010, and was also the first Gene Golub SIAM Summer School. Now the ISSNLA and Gene Golub summer schools are separate events. For additional information, see http://math.temple.edu/~siagla/summer.html.

Other Upcoming Conferences

Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses, Maripal, India, January 2-11, 2012

18th International Symposium on Mathematical Methods Applied to Sciences, San José, Costa Rica, February 21-24, 2012

International Conference on Preconditioning Techniques, Oxford, England, June 19-21, 2013

Householder Symposium XIX on Numerical Linear Algebra, Spa, Belgium, June 8-13, 2014

CONFERENCE REPORTS

6th Linear Algebra Workshop
Kranjska Gora, Slovenia, May 25-June 1, 2011

The main theme of the workshop was the interplay between operator theory and algebra. After a few hours of talks in the morning, afternoons were used for work in smaller groups. The plenary speakers were J. Holbrook, T. Laffey, C.-K. Li, R. Meshulam, P. Moravec, J. Okniński, H. Radjavi, L. Rodman, and P. Šemrl. There were also 37 other talks.

Saturday morning May 28 was dedicated to Professor Matjaž Omladič on the occasion of his 60th birthday. The workshop was supported by the Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia; Faculty of Mathematics and Physics, University of Ljubljana; and the Slovenian Research Agency. There were about 80 participants. To see a list of those, a complete list of abstracts and some photos, please visit http://conferences.imfm.si/conferenceDisplay.py?confId=24.

Directions in Matrix Theory
Coimbra, Portugal, July 9-10, 2011
Report by Carlos Fonseca

This workshop was a forum of discussion of frontline areas of matrix theory and its applications, bringing together renowned researchers and recognized experts of diverse fields fostering the exchange of experiences and insights from different perspectives. More than 80 researchers from 18 countries participated. The plenary lecturers and their titles were Béla Bollobás (Univ. of Cambridge, UK) - The cut norm and the giant component; Shmuel Friedland (Univ. of Illinois at Chicago, USA) - Some open problems in matchings in graphs; Willem H. Haemers (Tilburg Univ., The Netherlands) - The maximum order of adjacency matrices with a given rank; Natália Bebiano (Univ. de Coimbra, Portugal) - An inverse eigenvalue problem for pseudo periodic symmetric matrices; Steve Kirkland (National Univ. of Ireland Maynooth, Ireland) - Sign patterns for eigenmatrices of nonnegative matrices; Juan Manuel Peña (Univ. de Zaragoza, Spain) - Structured matrices with bidiagonal decompositions; Volker Mehrmann (Technische Universität Berlin, Germany) - Self-adjoint linear differential-algebraic operators; and Thomas J. Laffey (Univ. College Dublin, Ireland) - Identifying the spectra of nonnegative matrices. There were also 36 contributed talks.

There will be a special issue of the Electronic Journal of Linear Algebra for the contributed and invited talks. The guest editors are Oskar Maria Baksalary, C.M. da Fonseca, and Shmuel Friedland. The submission deadline was November 30, 2011.

The Scientific Committee was Shmuel Friedland (Univ. of Illinois at Chicago, USA), Steve Kirkland (National Univ. of Ireland Maynooth, Ireland), Natália Bebiano (Univ. of Coimbra, Portugal), and Carlos Fonseca (Univ. of Coimbra, Portugal, Chair). The Local Organizing Committee was Ana Nata (Polytechnic Institute of Tomar, Portugal) and Rute Andrade (secretary of the Centre for Mathematics, Univ. of Coimbra). The meeting was endorsed by ILAS and also sponsored by the Centre for Mathematics, Univ. of Coimbra - CMUC and the Department of Mathematics, Univ. of Coimbra - DMUC. For abstracts and other details, visit http://www.mat.uc.pt/~cmf/Directions2011/DMT2011.htm.

Workshop on Operator Inequalities
Mashhad, Iran, June 6, 2011
Report by Sal Moslehian

A one-day workshop entitled “Operator Inequalities” was held on June 6, 2011 at Ferdowsi University of Mashhad, Iran. The main speaker was Professor Tsuyoshi Ando from Hokkaido University, Sapporo, Japan. About 30 faculty and students participated and actively exchanged ideas.

The talks were Numerical Range Inequalities (T. Ando), Operator Dunkl-Williams Inequality (F. Dadipour), Norm Inequalities (M. Erfanian Omidvar), Operator Monotone Functions (H. Najafi), Preserving Linear Maps (S. Hejazian), and Normal Operators (S.M.S. Nabavi Sales).

Professor Mohammad Sal Moslehian organized the workshop and it was sponsored by the Center of Excellence in Analysis on Algebraic Structures (CEAAS) of Mashhad University in cooperation with the Tusi Mathematics Research Group. CEAAS supported

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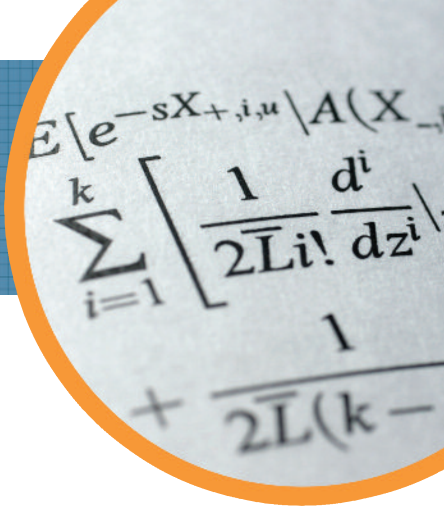
Topics in Tensors: A Summer School by Shmuel Friedland
Coimbra, Portugal, July 6-8, 2011
Report by Carlos Fonseca

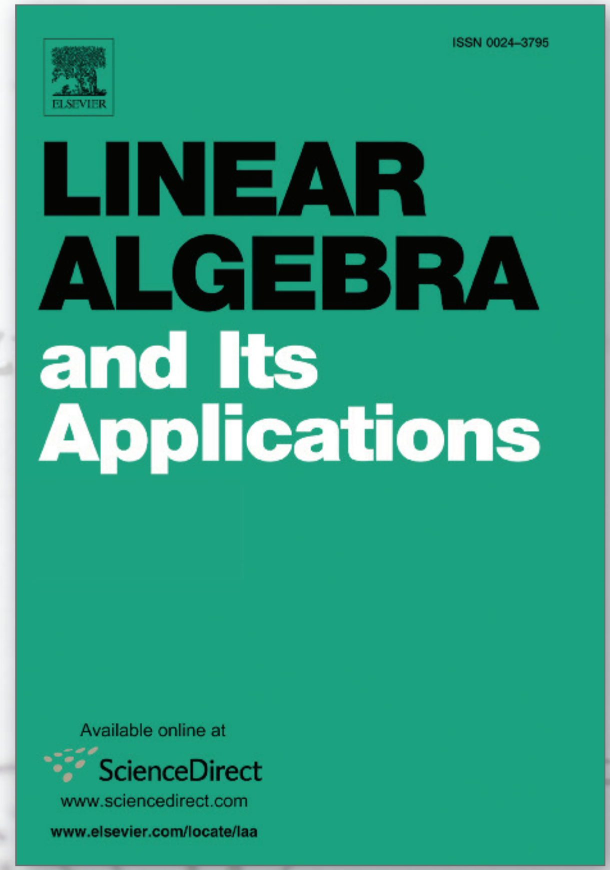
A Summer School led by Shmuel Friedland with the title “Topics in Tensors” was held at the Department of Mathematics of the University of Coimbra, Portugal, July 6-8, 2011. Tensors have emerged as a new hot topic of research in the first decade of 21st century. Motivated by applications in computer science, data processing, bio-statistics and engineering, it blends the tools of algebraic geometry and linear and multilinear algebra to create a vast array of new problems, methods and results. The aim of this summer school was to present several major aspects in this area which reflect the state of the art.

Each day there were morning and afternoon lecture sessions by Prof. Friedland. The theme of the first day was “Ranks of 3-tensors and a solution of the salmon problem”. The next day was dedicated to “Nonnegative tensors: eigenvalues, singular values, and scaling”, and the last day to “Approximation of tensors by low rank tensors”. There was also a special session “Secant varieties of Segre varieties and tensor decomposition”, presented by Anthony Geramita, on the first day. The slides of all lectures are available at http://www.mat.uc.pt/~cmf/SummerSchool/TopicsInTensors.htm.

The school provided a great opportunity to get to know other people working in different areas of tensors, to meet distinguished scholars, and to establish contacts that may lead to research collaborations in the future.

The organizers were Natália Bebiano and Carlos Fonseca, and the summer school was supported by CMUC, the Centre for Mathematics, University of Coimbra. There were 28 participants from Belgium, France, Iran, Ireland, Italy, Portugal, Spain, and Switzerland.





17th ILAS Conference, Pure and Applied Linear Algebra: The New Generation
Braunschweig, Germany, August 22-26, 2011

The 17th ILAS Conference took place at the Technical University of Braunschweig, Germany, August 22-26, 2011. There were 240 researchers from 34 countries and at least 5 continents. Following each plenary talk, parallel sessions were held for the minisymposia and contributed talks. Despite the very full schedule, there was also ample time to mingle, meet old friends and make new ones. An excursion, conference dinner, two receptions and other activities complemented the scientific program. Beautiful warm and sunny weather also helped to make the conference a success.

This conference had a special emphasis on young researchers, which was reflected by predominantly young plenary speakers and seven invited minisymposia featuring young researchers.

The plenary speakers were: Rajesh Pereira, Univ. of Guelph, "Matrix Methods in Analytic Theory of Polynomials"; Zlatko Drmač, Univ. of Zagreb, "Accurate and Stable Numerical Linear Algebra in Control Theory"; Melina Freitag, "Tikhonov Regularization for Large Scale Inverse Problems"; Michiel Hochstenbach, TU Eindhoven, "Recent Progress in the Solution of Discrete Ill-posed Problems"; Joseph M. Landsberg, Texas A&M Univ., Multilinear Algebra and Geometry"; Roland Hildebrand, "Linear Group Representation in the Service of Conic Optimization"; Diederich Hinrichsen, Univ. Bremem, "Interconnected Systems With Uncertain Couplings: Stability Radii and Sharp Inclusion Theorems"; and Daniel Potts, "Parameter Estimation for Exponential Sums". These talks were well received and highlighted many aspects of modern pure and applied linear algebra and their interplay.

Each of the young researchers' minisymposia was proposed by two organizers from different institutions, each holding a PhD for no longer than six years and not yet a tenured professor. The topics and organizers were: "Max-plus Linearity and its Applications in Computer Science and Scheduling", Rob M.P. Goverde and Sergei Sergeev; "The Theory of Orbits in Numerical Linear Algebra and Control Theory", Fernando De Terán and Marta Peña; "Matrix Means: Theory and Computation", Miklos Palfia and Bruno Iannazzo; "Modern Methods for PDE Eigenvalue Problems", Joscha Gedicke and Agnieszka Międlar, "Combinatorial Matrix Theory", Minerva Catral and Amy Wangsness Wehe; "Numerical Methods for the Solution of Algebraic Riccati Equations", Federico Poloni and Timo Reis; and "Parallel Computing in Numerical Linear Algebra", Alfredo Remón and Jens Saak. There were 38 talks in these, all very high quality and well attended.

The topics and organizers of the other invited minisymposia were "Total Positivity: Recent Advances in Theory and Applications", Plamen Koev and Juan Manuel Peña; "Quasi- and Semiseparable Matrices", Raf Vandebril and Vadim Olshevsky; "Compressed

Sensing and Sparse Approximation Algorithms", Holger Rauhut and Gitta Kutyniok; "Matrix Polynomials and Their Eigenproblems", Françoise Tisseur and Ion Zaballa; and "Tensor Decompositions", Eugene Tyrtyshnikov and Lars Grasedyck.

A welcome reception the evening before the opening session gave participants an early opportunity to mingle. The next day a student lunch was offered by ILAS and a reception for all participants was held that evening at the "Dornse", the great hall in the gothic Old Town Hall of Braunschweig. Peter Benner welcomed the participants and introduced them to the history and highlights of the ancient city. Both receptions were made possible by generous support from prudsys, a data analytics company, and Elsevier.

Reception in Old Town Hall

On Wednesday afternoon, several buses took participants on an excursion to Goslar and the old mines of Rammelsberg, both UNESCO world heritage sites. Goslar is a historic village with more than 1500 timber-framed buildings of various epochs. The conference dinner, with free beer, was held at a local brewery, concluding a long and interesting day.

Besides Elsevier and prudys, the conference was also sponsored by DFG, a federal agency supporting research, TU Braunschweig, Taylor & Francis, NICONET, a working group developing control theory software, GAMM, the International Association of Applied Mathematics and Mechanics, SIAG/LA, SIAM's special interest group for linear algebra, Springer Publishing, Oxford University Press, IOP Publishing and Duke University Press. The full program, abstracts and conference photos can be found at http://www.ilas2011.de/. The local arrangements committee, Heike Fassbender, Matthias Bollhofer and Peter Benner, are to be thanked for their excellent planning. As usual for ILAS meetings, the proceedings will appear as a volume of LAA. All papers will be subject to the usual refereeing procedure.

The next ILAS conference is scheduled to take place in Providence, RI, USA, June 3-7, 2013. In 2012, ILAS members are encouraged to participate in the SIAM Applied Linear Algebra meeting to be held June 18-22, 2012 in Valencia, Spain.

Hugo as Gauss,
leading tour of Braunschweig.
"Hugo" is stage name of Jürgen Köpke.

Watching the Glockenspiel
in Goslar's central plaza

Goslar, "half timbered" house

Peter Benner, telling some
history of Braunschweig

Opening Speaker
Rajesh Pereira

GAMM Speaker
Melina Freitag

Great student helpers
with local organizers
Bollhofer, Benner and Fassbender

Conference Dinner at Brauhaus Goslar

Summary School on Numerical Linear Algebra for Dynamical and High-Dimensional Problems

Trogir, Croatia — October 10–15, 2011

Report by Bart Vandereycken (bart.vandereycken@epfl.ch)

Catching the final weeks of sunny weather, this summer school was held at Hotel Medena in Trogir, Croatia. There were introductory lectures in the morning and hands-on training sessions in the afternoons.

On the first day, Prof. Zlatko Drmač of the University of Zagreb explained the importance of backward stable and numerically accurate linear algebra routines in his lecture "Numerical Algorithms in Control". Following the NLA tradition of focusing on orthogonal matrix decompositions, he showed how they improve the numerical computation of control-theoretic decompositions. As a striking case study, a subtle bug in the pivoted QR decomposition of LAPACK was detailed, showing how even the tiniest quantities should be computed as accurately as possible.

The summer school continued on Tuesday when Prof. Peter Benner of MPI Magdeburg explained how to perform "Model Reduction for Linear Dynamical Systems". This course recalled some of the most prominent methods used for linear systems: interpolatory methods by rational interpolation and balanced truncation by solving Lyapunov equations. Both methods were compared from a system theoretical point of view and with respect to numerical implementation. In particular, recent developments in numerical linear algebra were shown to widen the applicability of model reduction and make it implementable for real-world examples from several disciplines.

Encouraging the spirit of collaboration, the participating PhD students presented their own projects on Wednesday morning. That afternoon, the conference excursion to Split featured a local guide taking the participants to see the remainders of the Diocletian's Palace, named after a fourth century Roman emperor.

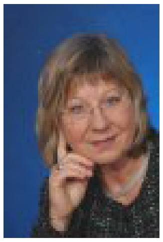
Next up was "Pseudospectra and Behavior of Dynamical Systems" by Prof. Mark Embree of Rice University. In the same spirit as the by now classic book "Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators", the lectures showed how eigenvalues can be misleading for understanding the behavior of dynamical systems and that pseudospectra offer a valuable alternative. After introducing their fundamental properties and the MATLAB® package EigTool [Thomas G. Wright. EigTool. `http://www.comlab.ox.ac.uk/pseudospectra/eigtool/`, 2002], the theory was put to the test by explaining and improving on the behavior of numerical algorithms involving nonnormal matrices, like GMRES and the low-rank ADI method, and computation of the matrix exponential.

The last day featured "Low-Rank Tensor Techniques for High-Dimensional Problems" by Prof. Daniel Kressner of the École Polytechnique Fédérale de Lausanne. His focus was linear algebra for coping efficiently with high-dimensional problems arising in fields as diverse as quantum physics to parameter-dependent and stochastic PDEs. Much attention was given to taming the notation by introducing compact and easy-to-use concepts like tensor networks. In the afternoon, the MATLAB toolbox htucker [`http://www.sam.math.ethz.ch/NLAgroup/htucker_toolbox.html`] indeed showed how low-rank tensors can be efficient building blocks for truly high-dimensional problems.

About 40 participants officially ended the conference with a conference dinner on Friday evening. The summer school was organized by Peter Benner, Zlatko Drmač, Daniel Kressner, and Ninoslav Truhar. It was supported by GAMM (the International Association of Applied Mathematics and Mechanics), and endorsed by the GAMM activity group "Applied and Numerical Linear Algebra". Slides of the lectures and the accompanying exercises are available [`http://www.mpi-magdeburg.mpg.de/mpcsc/events/trogir/index`].

GAMM Workshop on Applied and Numerical Linear Algebra: Model Reduction

Bremen, Germany, September 22-23, 2011



Angelika Bunse-Gerstner

The special topic for this annual ANLA workshop was Model Reduction. The plenary speakers and topics were Paul Van Dooren (UC Louvain-la-Neuve, Belgium), Updating Indefinite Matrix Approximations; Thanos Antoulas (Jacobs University, Bremen and Rice Univeristy, Houston), Parametric Model Reduction; and Caroline Böss (Allianz Global Corporate & Specialty (AGCS)), Model Order Reduction for discrete unstable systems. On Thursday night, there was a conference dinner. The workshop ended at noon on Friday and was followed by a Festkolloquium that afternoon honoring Angelika Bunse-Gerstner on the occasion of her 60th birthday.

Workshop on Linear Algebra and Its Applications
Maynooth, Ireland, October 17, 2011
Report by Steve Kirkland and Robert Shorten

In order to mark the retirement of Abraham Berman, and to acknowledge his important contributions to linear algebra and combinatorics, a one-day Workshop on Linear Algebra and its Applications was held at the Hamilton Institute, National University of Ireland Maynoothon October 17, 2011. The workshop also served as a way of thanking Professor Berman (Avi, to his many friends) for his support for the Hamilton Institute. Indeed, Avi has been a great friend to the Hamilton. He has not only been actively involved in the Institute's research and educational programmes, but has also promoted and fostered the Hamilton Institute through his role as a member of the Institute's Scientific Advisory Board.

A cake for Avi

About 40 people attended the workshop. It featured lectures by Patrizio Colaneri (Politecnico di Milano), Karl-Heinz Förster (TU Berlin), Shmuel Friedland (University of Illinois), Steve Kirkland (NUI Maynooth), Thomas Laffey (University College Dublin), Raphael Loewy (Technion), and Helena Šmigoc (University College Dublin), as well as Avi himself. The talks covered a range of topics including operators, tensors, cones, completely positive matrices, and of course nonnegative matrices. Not surprisingly, the atmosphere was relaxed, with plenty of time for informal discussion. Videos for all of the talks are available online at the following address: http://www.hamilton.ie/linalgworkshop2011/videos.htm.

The workshop provided an excellent opportunity for Avi Berman's friends, colleagues, and admirers to express their appreciation for his numerous scientific accomplishments, and to wish him the best in retirement.

ARTICLES

A Modern Pulveriser

By Richard William Farebrother
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"Pulverizer" refers to the Jayadeva-Bhaskaracharya technique for solving Pell's equation $61x^2 + 1 = y^2$ for minimum integers x and y (see [1] and [4]). Here we consider a generalization of this problem and technique. Suppose that we are given integer values for p, q, r, s and that we can find an integer value for n such that $(pn + q)^2 + (rn + s)^2 = g^2$ for some integer value of g. Then we may define $d = p^2 + r^2$, $e = pq + rs$, and $h = |ps - qr|$ and deduce that $f = |dn + e|$ and g satisfy the equation $f^2 + h^2 = dg^2$ in integers.

This was the technique employed with $h = 1$ in the third part of my unpublished solution to [2], and it yields a table similar to the final table of [3]. In particular, when $p = 6$, $q = 5$, $r = 1$, $s = 1$ then we have $d = 61$, $e = 11$, $h = 1$ and find that $n = 487$ identifies the solution $f = 29718$, $g = 3805$ to Nell's equation $f^2 + 1 = 61g^2$. The smallest solution to Pell's equation $61x^2 + 1 = y^2$ given in [1] then follows by applying the generalised Pythagorean formula of [2] to obtain $x = 2fg$, $y = f^2 + 61g^2$.

References

[1] Richard William Farebrother, Pulverisers Ancient and Modern, *Image* 40, Spring 2008, p. 23.

[2] Richard William Farebrother, Problem 41-4: Integer Matrices with Unit Determinants II, *Image* 41, Fall 2008, p. 43.

[3] Hans-Joachim Werner, Problem 41-4: Integer Matrices with Unit Determinants II, Solution 41-4, *Image* 43, Fall 2009, p. 35.

[4] George Gheverghese Joseph (1990), *The Crest of the Peacock: Non-European Roots of Mathematics*, Penguin Books, Harmondsworth. Second edition, Princeton University Press, Princeton, N.J., 2000.

One Horse Racing Story in Ancient China, Two Card Games, and Three Matrix Theorems

by Chi-Kwong Li,
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1 A horse racing story

Here is a famous Chinese story set about 2300 years ago (see for example [6, 7]). General Tian Ji was a high official in the country Qi. He had a good horse trainer and very good horses in his stable. The king of Qi, Qi Wei Wang, also had very good horses and proposed a match. They planned to have three rounds of races and each side would send a horse to compete in each round; no horse was allowed to race twice. The winner of two or more races would be the ultimate winner. General Tian was very anxious to win both the monetary reward as well as the glory. He did some research about the performance of the three best horses of King Qi and found out that they had similar qualities to his horses. Actually, the King's three horses were slightly better. So he consulted Sun Bin, a very famous expert in war tactics, for a good strategy. Sun appeared to be a very "mathematical" person. He asked all the rules and given conditions carefully before proposing a strategy. In particular, he asked whether King Qi had already assigned his horses for the three matches. Receiving an affirmative answer, he told General Tian that he could win using the following strategy:

use Tian's weakest horse to race against King Qi's best horse;
use Tian's best horse to race against King Qi's median horse;
and use Tian's median horse to race against with King Qi's weakest horse.

So General Tian won the second and the third races and became the overall winner.

2 Two kinds of card games

One can imagine that if King Qi and General Tian conducted the horse race again, King Qi would be smart enough not to disclose his assignment in advance. To have a fair game, each party would just bring the horse to the racing track when each race was about to start. Assuming that the horses can perform consistently, they do not even need to bring their horses to the racing track. They can hire a trustworthy referee and test their horses and record the speeds of the n best horses in each stable. Then the outcome of a race between two horses is determined by simply comparing their speeds, i.e. the one with higher speed wins. Mathematically, the n races can be represented by the following game.

Card Game 1: Suppose A has n cards with numbers $a_1 \geq \cdots \geq a_n$, and B has n cards with numbers $b_1 \geq \cdots \geq b_n$. They can then present their cards in n rounds, say, in their favorite order of

$$a_{i_1}, \ldots, a_{i_n} \qquad \text{and} \qquad b_{j_1}, \ldots, b_{j_n},$$

where $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ are permutations of $(1, \ldots, n)$. Here we assume that the opposite party has no knowledge of the permutation of the opponent. Then for $k = 1, \ldots, n$, A may win, lose, or tie the kth round depending on whether $a_{i_k} > b_{j_k}$, $a_{i_k} < b_{j_k}$, or $a_{i_k} = b_{j_k}$, respectively.

For example, suppose A and B each have three cards with numbers 30, 25, 20. Then the different outcome vectors (W, L, T) recording the number of wins, losses and ties for A would be $(2,1,0), (1,2,0), (0,0,3), (1,1,1)$. It is interesting to note that the outcome vectors $(3,0,0)$ and $(0,3,0)$ are impossible with the given data.

In general, let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. The possible games played by A and B can be generated by $\mathbf{a}P - \mathbf{b}Q$ for different permutation matrices P and Q. So there are as many as $(n!)^2$ different games if each of the vectors $\mathbf{a}$ and $\mathbf{b}$ have no repeated entries. To determine the possible outcome vectors (W, L, T), it suffices to consider

$$(a_1, \ldots, a_n) - (b_1, \ldots, b_n)Q$$

for different permutation matrices Q. Although there are $n!$ choices for the permutation matrix Q, the number of possible outcome vectors (W, L, T) is much less than $n!$. Let

$$S = \{(x, y, z) : x,\ y,\ z \text{ are non-negative integers satisfying } x + y + z = n\}.$$

Then S has $\frac{1}{2}(n+1)(n+2)$ elements and any possible (W, L, T) must lie in S. Using modern technology, one can write a simple computer program to determine which $(x, y, z) \in S$ is a possible outcome vector (W, L, T) for a pair

$\mathbf{a}, \mathbf{b}$. Note that the winner of the game is determined by the pair (W, L) since $T = n - W - L$. In the next section, we will describe a simple criterion to determine all possible (W, L) vectors, and will henceforth refer to these as outcomes.

The outcome of Card Game 1 depends not only on the strength of the card but the arrangement of the cards. For example, suppose A has 3 cards $\mathbf{a} = (1, 3, 5)$ and B has $\mathbf{b} = (2, 4, 6)$. Then clearly, B is stronger than A but A can still win the game in the matching $(3, 5, 1) - (2, 4, 6)$. To eliminate such "unfair" outcome happening, we introduce the following game.

Card Game 2: Players A and B start with n cards. In each round of this game, each side presents a card, say, a_{i_1} and b_{j_1}.

(a) If $a_{i_1} = b_{j_1}$, then the players tie the round and start a new round if there are still cards in their hands.

(b) If $b_{j_1} < a_{i_1}$, then B may choose to present another card b_{j_2} with $a_{i_1} \leq b_{j_2}$. This shows that B has a card which is at least as strong as a_{i_1}. Then, A can present another card a_{i_2} with $a_{i_2} \geq b_{j_2}$. Such moves continue until one of the sides concedes for the lack of higher card or for some strategic reason. Now, if we get a sequence

$$b_{j_1} < a_{i_1} \leq b_{j_2} \leq a_{i_2} \leq \cdots \leq b_{j_k} \leq a_{j_k}, \tag{1}$$

then A will be the winner of that particular round and the cards $b_{j_1}, a_{i_1}, \ldots, b_{j_k}, a_{j_k}$ will be set aside. If we get a sequence

$$b_{j_1} < a_{i_1} \leq b_{j_2} \leq a_{i_2} \leq \cdots \leq b_{j_k} \leq a_{j_k} \leq b_{j_{k+1}}, \tag{2}$$

then B wins the round if $b_{j_{r+1}} > a_{i_r}$, for some $1 \leq r \leq k$; otherwise the round is a tie. In both cases, the cards $a_{i_1}, b_{j_2}, \ldots, a_{j_k}, b_{j_{k+1}}$ will be set aside. B keeps b_{j_1} and can use it for the following rounds.

(c) If $a_{i_1} < b_{j_1}$, then we can apply the rule in (b) by interchanging the roles of A and B.

At the end of the game, we only record the pair (p, q), where p is the number of rounds won by A and q is the number of rounds won by B. The one who wins more rounds will be the ultimate winner. For any possible outcome (W, L, T) of Card Game 1, (W, L) is also a possible outcome of Card Game 2 but the converse does not hold. For example, if $\mathbf{a} = (1, 3, 5)$ and $\mathbf{b} = (2, 4, 6)$, the match can end in one round

$$1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6$$

and have $(p, q) = (0, 1)$ for this match, whereas $(0, 1, 2)$ cannot be an outcome of Card Game 1.

To illustrate the complexity in determining all possible outcomes (p, q) for given $\mathbf{a}$, $\mathbf{b}$, consider the following example from [3, Example 2.6].

Example 1: Let $\mathbf{a} = (6, 6, 4, 3, 3, 2, 1)$ and $\mathbf{b} = (5, 4, 3, 3, 1, 1, 1)$. We will use $\underline{x}$ to denote x in $\mathbf{b}$. Then the matches

$$\underline{1} < 6, \quad \underline{1} < 3 = \underline{3} < 6, \quad 4 < \underline{5}, \quad 3 < \underline{4}, \quad 2 < \underline{3}, \quad 1 = \underline{1}$$

and

$$\underline{1} < 6, \quad \underline{1} < 6, \quad \underline{1} < 4, \quad 3 < \underline{5}, \quad 3 < \underline{4}, \quad 2 < \underline{3}, \quad 1 < \underline{3}$$

show that $(2, 3)$ and $(3, 4)$ are possible outcomes in Card Game 2. It is not so easy to see that $(2, 4)$ is **not** a possible outcome. We will describe how to check this in the next section.

3 Three matrix theorems

First, let us describe how to check whether a (p, q) vector can occur as the outcome of Card Game 2. It turns out that the basic strategy of Sun Bin is very useful. If A wants to lose only q rounds and keep his best cards for the remaining rounds, the best way is to let his q lowest cards compete with the q highest cards of B. Hopefully, A's remaining cards will dominate those of B and win at least p rounds. Thus, if A and B have cards

$$a_1 \geq \cdots \geq a_n \qquad \text{and} \qquad b_1 \geq \cdots \geq b_n,$$

A would like to see that

(C1) $(a_1, \ldots, a_{n-q}) - (b_{q+1}, \ldots, b_n)$ is a non-negative vector with at least p positive entries.

Of course, B would have the same desire and wants to see that

(C2) $(b_1, \ldots, b_{n-p}) - (a_{p+1}, \ldots, a_n)$ is a non-negative vector with at least q positive entries.

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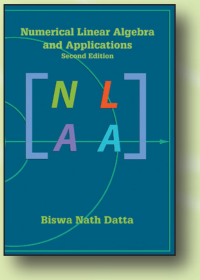

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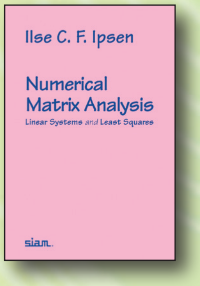

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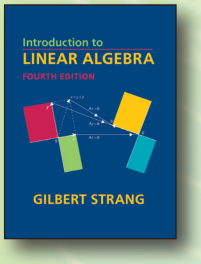

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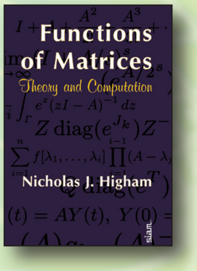

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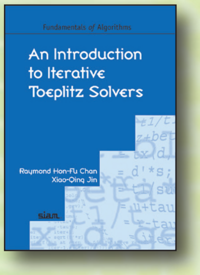

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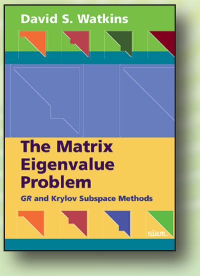

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It turns out that these two conditions are necessary and sufficient for (p, q) to be a possible outcome. For instance, in Example 1, in order to show that $(p, q) = (2, 4)$ is not possible, we observe that

$$(b_1, \ldots, b_{7-2}) - (a_{2+1}, \ldots, a_7) = (5, 4, 3, 3, 1) - (4, 3, 3, 2, 1) = (1, 1, 0, 1, 0)$$

has only 3 positive entries. Thus (C2) is not satisfied.

Moreover, these (p, q) pairs correspond to the inertia values $(p, q, n-p-q)$ of a complex Hermitian (or real orthogonal) matrix of the form $A - B$, where A and B have eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Here a Hermitian matrix M has inertia (p, q, r) if M has p (q and r, respectively) positive (negative and zero, respectively) eigenvalues.

Theorem 1 *Let $D_a = \mathrm{diag}\,(a_1, \ldots, a_n)$ and $D_b = \mathrm{diag}\,(b_1, \ldots, b_n)$ with $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. Then the following are equivalent:*

(a) *Conditions* (C1) *and* (C2) *hold.*

(b) *There are complex unitary (or real orthogonal) matrices U and V such that*

$$U^* D_a U - V^* D_b V$$

has inertia $(p, q, n - p - q)$.

(c) *Either $p = q = 0$ ($\Rightarrow \mathbf{a} = \mathbf{b}$) or the diagonal entries of D_a and D_b can be divided into $p + q$ sequences so that p of the sequences have the form*

$$b_{j_1} \leq a_{i_1} \leq \cdots \leq b_{j_k} \leq a_{i_k} \quad \textit{with at least one inequality } b_{j_t} < a_{i_t}, \tag{3}$$

and q of the sequences have the form

$$a_{i_1} \leq b_{j_1} \leq \cdots \leq a_{i_k} \leq b_{j_k} \quad \textit{with at least one inequality } a_{i_t} < b_{j_t}. \tag{4}$$

Evidently, sequence (3) corresponds to a winning round of A, and sequence (4) corresponds to a winning round of B. So condition (c) of Theorem 1 corresponds to an outcome of Game 2 in which A wins p rounds and loses q rounds.

This theorem was proved in [3, Theorem 2.1]. Actually, it was shown that if (3) holds, then one can find unitary matrices X and Y to construct a rank one positive semidefinite matrix of the form

$$X^* \mathrm{diag}\,(a_{i_1}, \ldots, a_{i_k}) X - Y^* \mathrm{diag}\,(b_{j_1}, \ldots, b_{j_k}) Y.$$

Similarly, one can construct a rank one negative semidefinite matrix using the numbers satisfying (4).

In addition to testing whether an individual vector (p, q) can be the outcome of Card Game 2, we would like to describe all the possible (p, q) outcomes for some given $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ with $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \ldots \geq b_n$. There are several constraints on (p, q):

(i) $p + q \leq n$.

(ii) a lower bound on p is

$$p_0 = \begin{cases} n & \text{if } b_1 < a_n, \\ \min\{p : (b_1, \ldots, b_{n-p}) - (a_{p+1}, \ldots, a_n) \text{ is non-negative}\} & \text{otherwise.} \end{cases}$$

(iii) a lower bound on q is

$$q_0 = \begin{cases} n & \text{if } a_1 < b_n, \\ \min\{q : (a_1, \ldots, a_{n-q}) - (b_{q+1}, \ldots, b_n) \text{ is non-negative}\} & \text{otherwise.} \end{cases}$$

So all the possible outcomes will lie in the triangle

$$\Delta = \{(p, q) : p \geq p_0, q \geq q_0, p + q \leq n\}.$$

By Theorem 1, (p_0, q_0) is a possible outcome. Of course, each player would like to minimize the number of games won by the opponent. It is interesting that A and B can both win this minimum number of rounds in a competition. This pair is attained when both A and B follow the

minimal strategy: *at every turn, use the card with the minimum value, if possible.*

It is not difficult to show that the above strategy is optimal in the sense that if one chooses the minimal strategy, the opponent has no strategy to achieve a better result (of maximizing difference of the numbers of winning and losing rounds) than using the minimal strategy.

Of course, if Q_0 is the number of positive entries in the vector $(b_1, \ldots, b_{n-p_0}) - (a_{p_0+1}, \ldots, a_n)$, then each (p_0, q) pair with $q_0 \leq q \leq Q_0$ is a possible outcome. Similarly, for any possible p such that $(b_1, \ldots, b_{n-p}) - (a_{p+1}, \ldots, a_n)$ is non-negative with Q_p entries, and if (p, q) is an outcome, then $q \leq Q_p$.

Similarly, if P_0 is the number of positive entries in the vector $(a_1, \ldots, a_{n-q_0}) - (b_{q_0+1}, \ldots, b_n)$, then each (p, q) pair with $p_0 \leq p \leq P_0$ is a possible outcome. Again, for a fixed possible q, one can decide when (p, q) is a possible outcome.

In any event, one can follow the above procedure to construct the region of possible outcome pairs (p, q). In particular, it was proved in [3] that the region can be described as the grid points lying inside a convex polygon, whose vertices are possible outcomes (p, q). To avoid the technical details, we just present the geometrical aspect in the following theorem. Interested readers can see [3, Theorem 3.3] for details.

Theorem 2 *Suppose* $D_a = \operatorname{diag}(a_1, \ldots, a_n)$ *and* $D_b = \operatorname{diag}(b_1, \ldots, b_n)$ *are real diagonal matrices with diagonal entries arranged in descending order. Let* $\mathbf{In}(\mathbf{a}, \mathbf{b})$ *be the set of ordered pairs* (p, q) *such that* $(p, q, n-p-q)$ *is the inertia of some* $n \times n$ *Hermitian matrix of the form* $U^* D_a U - V^* D_b V$ *for some unitary matrices* U *and* V*. Then* $\mathbf{In}(\mathbf{a}, \mathbf{b})$ *consists of all the integer pairs enclosed by a convex polygon determined with vertices in* $\mathbf{In}(\mathbf{a}, \mathbf{b})$*. In particular,* $(p_0, q_0, n-p_0-q_0) \in \mathbf{In}(\mathbf{a}, \mathbf{b}) \subseteq \Delta$.

Applying Theorem 1 to Example 1, where $\mathbf{a} = (6, 6, 4, 3, 3, 2, 1)$ and $\mathbf{b} = (5, 4, 3, 3, 1, 1, 1)$, we see that $(p, q) = (2, 0)$ is possible as

$$(a_1, \ldots, a_{7-0}) - (b_{1+0}, \ldots, b_7) = (6, 6, 4, 3, 3, 2, 1) - (5, 4, 3, 3, 1, 1, 1) = (1, 2, 1, 0, 2, 1, 0)$$

and

$$(b_1, \ldots, b_{7-2}) - (a_{2+1}, \ldots, a_7) = (5, 4, 3, 3, 1) - (4, 3, 3, 2, 1) = (1, 1, 0, 1, 0).$$

In fact, if $A = \operatorname{diag}(6, 4, 6, 2, 3, 3, 1)$ and $B = B_1 \oplus B_2$ with

$$B_1 = \begin{pmatrix} 7/2 & \sqrt{15}/2 \\ \sqrt{15}/2 & 5/2 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 7/2 & \sqrt{5}/2 \\ \sqrt{5}/2 & 3/2 \end{pmatrix} \oplus \operatorname{diag}(3, 3, 1),$$

then A and B are 7×7 Hermitian matrices with vectors of eigenvalues $\mathbf{a}$ and $\mathbf{b}$ such that

$$A - B = \begin{pmatrix} 5/2 & -\sqrt{15}/2 \\ -\sqrt{15}/2 & 3/2 \end{pmatrix} \oplus \left[\begin{pmatrix} 5/2 & -\sqrt{5}/2 \\ -\sqrt{5}/2 & 1/2 \end{pmatrix} \oplus \operatorname{diag}(0, 0, 0) \right]$$

is a 7×7 Hermitian matrix with inertia vector $(2, 0, 5)$. We can also test every (p, q) pair of nonnegative integers with $p + q \leq 7$ and depict possible outcomes as points in $\mathbb{R}^2$ as in Figure 1.

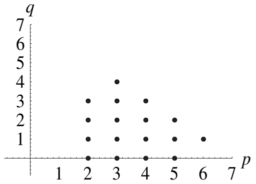

Figure 1: Possible outcomes (p, q) for the game in Example 1.

Next, we turn to Card Game 1. If $(p, q, n-p-q)$ is a possible outcome of Card Game 1, then (p, q) is a possible outcome of Card Game 2. But the converse does not hold in general. For instance, typically, $(p_0, q_0, n-p_0-q_0)$ is not a possible outcome of Card Game 1. Note that if $(p, q, n-p-q)$ is an outcome of Card Game 1, then there are $n-p-q$ tie rounds in the competition. So $\mathbf{a}$ and $\mathbf{b}$ should have $n-p-q$ pairs of common entries, i.e. $a_{i_t} = b_{j_t}$ for $t = 1, \ldots, n-p-q$. Thus we have the following.

(C3) If r_0 is the maximum number of pairs (a_i, b_j) with $a_i = b_j$, then $n - p - q \leq r_0$.

It turns out that this is the only obstacle. We have the following result; see [3, Theorem 4.1].

Theorem 3 *Suppose* $D_a = \operatorname{diag}(a_1, \ldots, a_n)$ *and* $D_b = \operatorname{diag}(b_1, \ldots, b_n)$ *are real matrices. Then the following are equivalent.*

(a) *Conditions* (C1), (C2) *and* (C3) *hold.*

(b) *There are permutation matrices* P *and* Q *such that* $P^t D_a P - Q^t D_b Q$ *has inertia* $(p, q, n-p-q)$.

(c) $(p, q, n-p-q)$ *is a possible outcome of Card Game 1.*

Applying this theorem to the previous example, we see that except for the pair $(2, 0)$ each pair (p, q) in Figure 1 will be the inertia $(p, q, 7-p-q)$ of a matrix of the form $P^t D_a P - Q^t D_b Q$ for some permutation matrices P and Q.

4 Coda

As mentioned in [3], the study of the inertia of the sum of Hermitian matrices with prescribed eigenvalues has connections to many other subjects. There are many directions worthy of further investigation.

For example, one can study the probabilities of the possible outcomes of Card Game 1. Suppose D_a and D_b are diagonal matrices, where the diagonal entries are the card values of the two players. Clearly, there is a correspondence for the probabilities of the different inertia vectors of a matrix of the form $P^t D_a P - Q^t D_b Q$, where P and Q are permutation matrices. The formulation and solution of the probability question for the possible outcomes of Card Game 2 is not so obvious. It is even more difficult to study the probabilities of the different inertia vectors of a matrix of the form $U^* D_a U - V^* D_b V$, where U and V are unitary (or real symmetric) matrices. For instance, if D_a and D_b have one common diagonal entry, then theoretically $U^* D_a U - V^* D_b V$ may be singular. However, the probability of the matrix being singular is zero if we choose U and V "arbitrarily" (in any reasonable sense).

The study of matrices of the form $U^* D_a U - V^* D_b V$ can be viewed as the study of matrices $X - Y$ with X, Y chosen respectively from the unitary orbits

$$\mathcal{U}(D_a) = \{U^* D_a U : U \text{ unitary}\} \quad \text{and} \quad \mathcal{U}(D_b) = \{V^* D_b V : V \text{ unitary}\}.$$

One may study other algebraic and analytic properties, such as the ranks, determinants, and norms of such matrices; see [2, 4, 5]. In fact, one may study the (complete or partial) set of eigenvalues of the matrices [1]. One may apply the results to matrices of the form $U^* D_a U + V^* D_c V$ by letting $D_c = -D_b$. It is then natural to consider matrices of the form

$$\sum_{j=1}^{m} U_j^* D_j U_j, \qquad U_1, \ldots, U_m \text{ are unitary}, \tag{5}$$

for given real diagonal matrices $D_1, \ldots, D_m$. However, there is no easy formulation of the problem as horse racing when $m > 2$. As pointed out in [3], one can use the results in [1] to determine the possible inertia for matrices of the form in (5). But the statements of results are more complicated than those involving only two matrices, and it is unclear how one can deduce our results on two matrices using this alternative approach.

Acknowledgement Research of the first author is supported by an NSF grant and an HK RGC grant.

References

[1] H. Bercovici, W. S. Li, D. Timotin, The Horn conjecture for compact selfadjoint operators, *American Journal of Mathematics* 131(2009), 1543-1567.

[2] M.D. Choi and C.K. Li, The ultimate estimate of the upper norm bound for the summation of operators, *Journal of Functional Analysis* 232 (2006), 455-476.

[3] C.K. Li and Y.T. Poon, Sum of Hermitian matrices with given eigenvalues: inertia, rank, and multiple eigenvalues, *Canadian Journal of Mathematics* 62 (2010), 109-132.

[4] C.K. Li, Y.T. Poon and N.S. Sze, Ranks and determinants of the sum of matrices from unitary orbits, *Linear and Multilinear Algebra* 56 (2008), 105-130.

[5] C.K. Li, Y.T. Poon and N.S. Sze, Eigenvalues of the sum of matrices from unitary similarity orbits, *SIAM J. Matrix Analysis Appl.* 30 (2008), 560-581.

[6] Chinese story online: Tian Ji's horse race, http://www.chinesestoryonline.com/smart-story/215-tian-ji-sai-ma.html

[7] Chinese folktale: Tian Ji winning the horse race, http://www.sino-impression.com/History2008416110427.html

BOOKS

Matrix Theory -- From Generalized Inverses to Jordan Form

By Robert Piziak and P. L. Odell, Chapman & Hall/CRC, 2007. 548 pages. ISBN: 978-1-58488-625-9

Reviewed by K. C. Sivakumar, Indian Institute of Technology, kcskumar@iitm.ac.in

The text book has been designed for a second level course in linear algebra and matrix theory taught at the senior undergraduate and beginning graduate level. Seemingly, the motivation has come from the recommendations of a National Science Foundation meeting on undergraduate linear algebra curriculum. The book has evolved from the notes of the authors, who have been teaching linear algebra for many years.

The book contains twelve chapters. The first considers the central problem in linear algebra, viz., solving systems of linear equations. Three points of view are provided: the geometric, vector and matrix view. A measured treatment of ordinary invertible matrices is presented in this chapter together with the formulae of Henderson-Searle and Sherman-Morrison-Woodbury for the inverse of a sum of matrices. The second chapter begins with briefly reviewing Gaussian elimination, moves on to discuss the minimal polynomial and presents the LU and the LDU factorizations. The Frame algorithm for finding the inverse of a square nonsingular matrix is studied and the Cayley-Hamilton theorem is proved as a corollary.

In the third chapter, after reviewing the notions of rank and nullity, the Sylvester's rank formula is derived along with many of the important corollaries that go with it. The relationship between direct sum decompositions of vector spaces and idempotent matrices comes next. The notion of the index of a matrix is then introduced and the core-nilpotent factorization is proved. In what is agreeably the most demanding part of the book, an in-depth study of the structure of nilpotent matrices is presented. The chapter concludes with a brief discussion on left and right inverses.

In Chapter 4, the existence of a full-rank factorization for any matrix is proved after a quick review of echelon forms, including the Hermite normal form. The central object of study of the book, namely the Moore-Penrose inverse, is then introduced through the Penrose equations and its existence is proved using a full-rank factorization. This paves the way for a natural discussion on the other classes of generalized inverses, including the group inverse and the Drazin inverse, in Chapter 5. This is followed by a very short chapter 6 on norms of vectors and matrices.

Chapter 7 deals with inner product spaces, Gram-Schmidt orthonormalization and QR-factorization, including Kung's algorithm for computing it. Then there is a discussion of the four famous fundamental subspaces associated with a matrix, as treated by G. Strang [Amer. Math. Monthly, 100, (1993), 848-855]. The optimality properties associated with the Moore-Penrose inverse are discussed in the last part of this chapter.

In Chapter 8, the authors develop properties of projections, which will mainly be used in the subsequent chapter, about their relationship with the Moore-Penrose inverse and the spectral theorem. This material has been drawn from the papers by Piziak, Odell and Hahn [*Comput. Math. App.*, 37, (1999), 67-74] and by Piziak and Odell [Comput. Math. App., 48, (2004), 177-190]. The spectral theorem is proved in Chapter 9. In Chapter 10, the primary decomposition theorem and Schur's triangularization theorem are proved, followed by the singular value decomposition theorem. Interestingly, the statement and hence the proof employ the Moore-Penrose inverse. The Jordan canonical form is studied in detail in Chapter 11. Chapter 12 gives an introduction to multilinear algebra.

Each chapter ends with a list of references for further reading. Undoubtedly, these will be useful for anyone who wishes to pursue the topics deeper. MATLAB is now a widely used language for dealing with matrices and the book has many MATLAB examples and problems presented at appropriate places.

The reviewer feels that the book will become a widely used classroom text for a second course on linear algebra. It can be used profitably by graduate and advanced level undergraduate students. It can also serve as an intermediate course for more advanced texts in matrix theory. This is a lucidly written book by two authors who have made many contributions to linear and multilinear algebra.

Perturbation Bounds for Matrix Eigenvalues

By Rajendra Bhatia, SIAM (2007) 191 pages, ISBN 978-0-898716-31-3

Reviewed by Chandler Davis, University of Toronto, davis@math.toronto.edu

The original version of this survey by Prof. Bhatia, with the same title, was published by Longman in 1987. It was an archetypical survey: a review, by one centrally involved, of a burst of research activity motivated by a few salient problems. The field was explained from the author's viewpoint, with generous citation of many contributors. That survey was complete only according to the scope of the explanations: Bhatia ordinarily did not mention byways that he did not explain, and sources he did not explain are mostly not in the bibliography. A marvelously readable and illuminating text resulted— but not a compendium.

The 2007 edition reviewed here consists of a reproduction of the Longman text (without the original preface, despite the mention thereof on the Table of Contents), followed by "Supplements 1986-2006", which includes text (pp. 133-184), supplementary bibliography (pp. 185-190), and a few errata. There is still no author index. The notes on sources have the same brisk, informative style as in the original. Not much has been lost in coherence, either: Bhatia_2007 plainly is on excellent terms with Bhatia_1987, and the augmented book can be read with the same pleasure as the former one.

The core problem in these surveys is this: Given two $n \times n$ matrices A, B, with respective eigenvalue sets $\mathrm{Eig}(A) = \alpha_1, ..., \alpha_n, \mathrm{Eig}(B) = \beta_1, ..., \beta_n$ (counting multiplicities), can we bound the distance between the sets in terms of $||A - B||$? Here the distance is to be the *optimal matching distance* $d(\mathrm{Eig}(A), \mathrm{Eig}(B))$, defined as the minimum of $\max_i |\alpha_i - \beta_{\sigma(i)}|$ over all permutations σ of the indices $1, ..., n$; the norm is the *operator norm* defined by

$$||C|| = \max_{||x||=1} ||Cx||.$$

The optimal matching distance is often much bigger than the Hausdorff distance, but it seems to be the proper focus of attention in this investigation, as we see from the founding result (H. Weyl, 1912): If A and B are both Hermitian then $d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \leqq ||A - B||$. The point here is that the eigenvalues of each matrix may naturally be ordered, and then d is just $\max_i |\alpha_i - \beta_i|$. With different hypotheses upon A and B, it may be harder to know what permutation of the eigenvalues is optimal in this sense, and one fears the conclusion may turn out to be weaker.

The leading conjecture for many of us was that the same conclusion might hold under the hypothesis of normality of A and B. Hardly had the Bhatia survey appeared, setting forth various special cases where this could be proved, than John Holbrook found a 3×3 counterexample to the general conjecture. The subject retained its importance, and gained in mystery, but its focus was shattered. Bhatia pursued the remaining paths, and continued to be an exemplary spokesperson for this active area, but the exposition inevitably has less of a unifying theme in the "Supplements" than in the original survey.

The other big change in the subject between the original book and the reissue concerns Alfred Horn's conjecture. Here we revert to Hermitian A and B. The much-discussed Lidskiĭ-Wielandt Theorem gives many inequalities bounding sums of some k-tuples of eigenvalues of $A + B$ in terms of sums of suitably chosen k-tuples of $\mathrm{Eig}(A)$ and of $\mathrm{Eig}(B)$. It is simple, but it is not conclusive. A succession of inequalities of the same description were found by A.R. Amir-Moez and others— inequalities not derivable from Lidskiĭ-Wielandt by inequality manipulations alone, but provable using the matrix context. The elusive goal was a full listing of all such inequalities; Alfred Horn defined the goal clearly in 1962. The dramatic success of the program came in the 1990s, and is surely a watershed for the field. This culmination of the story is told only briefly here. Bhatia prefers to refer readers to other expositions, one by W. Fulton and one by himself.

From the viewpoint of applications, many other definitions of the difference between two matrices are useful beside $||A - B||$. Most accessible to the matrix-theorist would be $|||A - B|||$, where $||| \cdot |||$ is some other unitarily-invariant norm— that is, one such that $|||UCV||| = |||C|||$ for all unitary U and V. This book gives a self-contained treatment of the von Neumann theory of these norms and goes on to explain the relation between the various results on spectral variation involving them and other measures. Similarly, alternative notions of distance between eigenvalue sets are explored.

It might be expected that the numerical analysts' contributions to the subject would diverge from those of the mere theorem-provers, to concentrate only on results allowing tight error bounds. But the relation of the theory of spectral variation to the derivation of useful error bounds is more complicated than that, and some numerical analysts' theorems are not limited to improving numerical estimates (while some non-numerical-analysts do improve estimates, of course). The good fellowship between number-crunchers and theorem-provers is exemplified by some of the surveys of these questions standing alongside Bhatia's, like those of Ilse Ipsen, of G. Golub and C.F. Van Loan, and of G.W. Stewart and J.-G. Sun. This is an area of mathematics which withstands any attempt to split it into applied and theoretical camps, and one which has a long life ahead of it.

Before Sudoku — The World of Magic Squares

By Seymour S. Block and Santiago A. Tavares

Oxford University Press (2009), 239 pages, ISBN-13: 978-0195367904

Reviewed by Peter D. Loly, University of Manitoba, loly@cc.umanitoba.ca

The authors, both chemical engineers, have written a concise survey of a fascinating family of integer matrices. Readers of *IMAGE* may have a special interest in the n^{th} order Latin squares with elements 1...n in every row and column (so that all rows and columns have the same line sum). Ninth order Sudoku solutions have the additional constraint that each number appears only once in each of the nine adjoining 3-by-3 subsquares. By contrast, order n natural magic squares have elements 1...n^2 with constant line sum in the rows and columns, as well as the diagonals. Latin squares and magic squares are doubly a.ne integer matrices which often exhibit more elegant results than typical matrices over $\mathbb{R}$. Since the widespread reappearance of Sudoku puzzles in 2005, these have been used as a pivot to the study of magic squares, for example by A. van den Essen ["Magische vierkanten: van Lo-Shu tot sudoku," 2008] and also by Block and Tavares in the present book (hereafter B&T).

The early history of magic squares dates back at least two and a half millenia in China. The squares were thought to have been passed along and extended through India and Persia by itinerant scholars until they reached the Mediterranean area by the 9th century. In the 10th century, Islamic scholars wrote lengthy texts about magic squares up to order nine, with another recorded in a Chinese text in 1275 C.E. A Latin square of the same order is found in a 12/13th century manuscript by Albuni [Descombes, "Les Carr's Magiques," Vuibert, 2000]. From the 14th century, magic squares attracted the attention of Manuel Moschopoulos, Agrippa of Nettlesheim, Albrecht Dürer, and many others. The past few centuries have seen a growing literature of archived works and self-published material. Now recreational mathematicians interact via email with the authors of archived articles. So it is helpful to have B&T added to a number of recent books by other scholars.

From the excellent website of H. Heinz [http://www.magic-squares.net], one can see a measure of the amateur activity. B&T also includes some results from Heinz's prolific colleague J. Hendricks. The latter encouraged the present reviewer in his first venture into these waters with a non-magic pandiagonal square [Loly, "A purely pandiagonal 4 × 4 square and the Myers-Briggs Type Table," *Journal of Recreational Mathematics* 31 (2000/2001) 29–31]. B&T consists of a rather personal selection of topics, with substantial sections not covered in previous treatments. It includes extensive discussion of magic squares and art in Chapter 10 where earlier connected line patterns have been extended considerably. There is also a welcome discussion of musical renderings of magic squares.

B&T references contributions up to 2007, but omits progress around 2006 on Franklin bent diagonal squares (see van den Essen above). Also missing is the landmark work of Dame Kathleen Ollerenshaw and David Brée (1997) on most-perfect pandiagonal magic squares, as well as the epic statistical population estimates of magic squares for orders 6 through 10 by Walter Trump [http://www.trump.de/ magic-squares/howmany.html]. There is also no mention of compound or composite ones [Chan and Loly, "Iterative Compounding of Square Matrices to Generate Large-Order Magic Squares," *Mathematics Today* 38 (2002) 113–118]. The interpretation of Latin and magic squares as matrices opens up many interesting mathematical explorations, much of which can be found via [Loly, Cameron, Trump and Schindel, "Magic square spectra, "*Linear Algebra and Its Applications* 430 (2009) 2659–2680]. While Latin squares have well–known design applications, B&T also discusses uses. Also see two expository articles on philatelic uses of Latin squares [Styan and Loly, *Chance*, 23 (2010) Vol. 1, 57–62 and Vol. 2, 57–62]. Incidentally, an order 4 square attributed to Franklin (Figure 7.9) can be found in work by Dürer from 1514 (Figure 5.8), a tad before Franklin's birth!

I will keep B&T in my library and I recommend it as a good starting point for newcomers to the field for the variety of magical squares introduced; however B&T contains no significant linear algebra.

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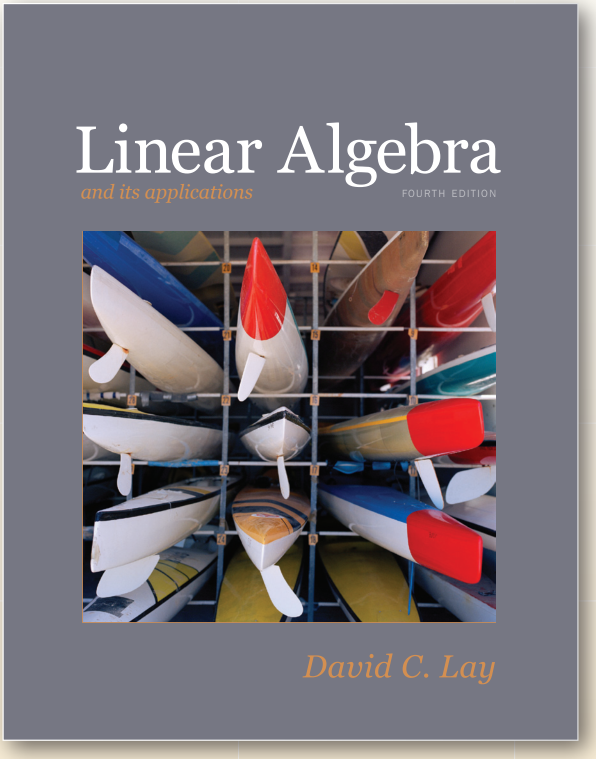

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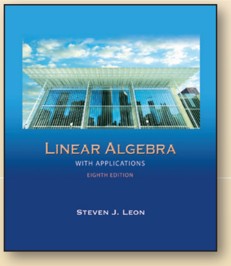

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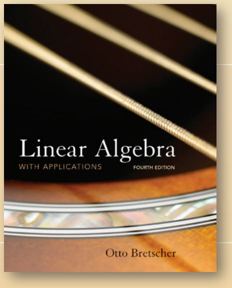

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New book: Matrix Completions, Moments and Sums of Hermitian Squares, by Mihály Bakonyi and Hugo J. Woerdeman. Princeton Press, 2010. 536 p. ISBN 978-0691128894.

This book provides a comprehensive account of a quickly developing area of mathematics and applications, with complete proofs of many recently solved problems and many exercises. It is ideal for a special topics course for graduate or advanced undergraduate students in mathematics or engineering, requiring only a basic background in undergraduate mathematics, primarily linear algebra and some complex analysis. With an extensive discussion of the literature, it will also be a valuable resource for researchers.

New book: The Crest of the Peacock: Non-European Roots of Mathematics, 3rd ed., by George Gheverghese Joseph. Princeton Press, 2010. 610 p. ISBN 978-0691135267.

This book is a multicultural tour of the history of non-European mathematics, including the influence Egyptians and Babylonians had on the Greeks, the Arabs' major creative contributions, and the wide contributions of India and China. This new edition also includes recent findings and emphasizes the dialogue between civilizations.

New book: Graph Algorithms in the Language of Linear Algebra, edited by Jeremy Kepner and John Gilbert. SIAM, 2011, 357 p. ISBN 978-0898719-901.

A linear algebraic approach is widely accessible and can improve algorithmic complexity, ease of implementation, and improved performance. Suitable as text on linear algebraic graph algorithms.

New Book: Matrix Theory – Basic Results and Techniques, 2nd ed., by Fuzhen Zhang. Springer, 2011. 400 p. ISBN 9781461410980.

This book aims to concisely present fundamental ideas, results and techniques in linear algebra, emphasizing matrix theory. It can be used as a textbook or supplement for senior undergraduate or graduate students, and as a reference for researchers. Major changes in the 2nd edition include expansion of topics such as matrix functions, nonnegative matrices and matrix norms and updated material on numerical range, Kronecker and Hadamard products, compound matrices and eigenvalue and singular value inequalities.

List of Books of Interest to Linear Algebraists

Oskar Baksalary, Book Consulting Editor for IMAGE, maintains a list of books about linear algebra and its applications on the ILAS website (http://www.ilasic.math.uregina.ca/iic/). He includes references to reviews of selected titles. Please tell him when you find additional titles or informative reviews that should be included (obaksalary@gmail.com).

CALL FOR PAPERS FROM JOURNALS

LAA moves to Web Submission

From Editors Richard Brualdi, Volker Mehrmann, Hans Schneider, and Peter Šemrl

After much planning and work, LAA moved to Web Submission using Elsevier's electronic editorial system EES on October 3, 2011. Henceforth all LAA submissions and subsequent editorial work will go to the web address http://ees.elsevier.com/laa. We have worked with an Elsevier web trainer to try to make this new system as user friendly and simple as possible for editors, referees, and authors. Contrary to our past practice, now all submissions will go through one of the LAA Editors in Chief, with authors allowed to suggest a handling editor different from the Editor in Chief selected.

You can find information about EES at http://ees.elsevier.com/laa, including interactive tutorials for authors and referees. Technical help is always available at support@elsevier.com for authors and referees. And we will help as necessary if difficulties arise. As always, we appreciate your LAA contributions as authors and referees and we hope that this transition to EES goes well for all.

LAA Special Issue for Papers Presented at the 2011 ILAS Conference

There will be a special issue of Linear Algebra and Its Applications devoted to papers presented at the 17th Conference of ILAS, which was held in Braunschweig, Germany, August 22-26, 2011. The Guest Editors for this issue are Ravi Bapat, Matthias Bollhöfer, Froilán M. Dopico, and Heike Faßbender. The responsible Editor-in-Chief of the special issue is Peter Šemrl.

The deadline for submissions is March 31, 2012. All papers submitted for the conference proceedings will be subject to the usual LAA refereeing process, and should be submitted via the Elsevier Editorial System (EES) at http://ees.elsevier.com/laa/.

When selecting Article Type, please choose "Special Issue 17th ILAS Conference" and when requesting an editor, select P. Šemrl. You will then have an opportunity to suggest one of the four guest editors that you would like to handle your paper.

LAA Special Issue in Honor of Harm Bart

Linear Algebra and its Applications is pleased to announce a special issue in honor of Professor Harm Bart in recognition of his many important contributions to matrix and operator theory and applications, and on the occasion of his 70th birthday in 2012. We solicit papers for the special issue within the research interests of Harm Bart or the entire scope of LAA.

The deadline for submissions is June 1, 2012. Submissions will be subject to normal refereeing procedures, according to the usual standards of LAA. They should be submitted via the Elsevier Editorial System, http://ees.elsevier.com/laa/. Choose the special issue "In Honor of Harm Bart" and the responsible Editor in Chief is Richard Brualdi.

Authors will have the opportunity to suggest one of the following special editors for this issue to handle their submission: Albrecht Boettcher, Technical University Chemnitz, Germany; Harry Dym, Weizmann Institute, Israel; Rien Kaashoek, Vrije Universiteit, The Netherlands; and Andre Ran, Vrije Universiteit, The Netherlands.

LAA Special Issue on Sparse Approximate Solution of Linear Systems

This issue is devoted to the mathematical aspects of sparsity in underdetermined linear systems. Its goals are to highlight recent developments, open problems, and the many facets, techniques, and results of this field. Significant new results as well as survey papers and research problems are encouraged.

Topics of interest include Compressed sensing; Sparse representations; Sparse approximation; Sparse and low rank models (e.g. block sparsity, joint sparsity etc.); Recovery and approximation algorithms, including l-1 minimization and greedy algorithms; Random matrices in connection with sparsity, in particular structured random matrices; Low rank recovery, algorithms and analysis; Applications of sparse recovery/sparse approximation/low rank recovery; Sublinear algorithms for sparse and low rank recovery; and Sparsity and widths (Gelfand widths, Kolmogorov widths etc.).

The deadline for submissions is July 31, 2012, and the special issue is expected to appear in 2013. Submit papers to the responsible Editor in Chief Volker Mehrmann, choosing special issue "Sparse Approximate Solution of Linear Systems", via the electronic submission system at http://ees.elsevier.com/laa. Papers must meet the publication standards of LAA and will be refereed in the usual way. More details can be found at http://www.math.wisc.edu/~hans/speciss.html and at http://www.math.wisc.edu/~hans/SASLS_Call.pdf.

ILAS AND OTHER LINEAR ALGEBRA NEWS

2011 ILAS Election and Electronic Voting

From Stephen Kirkland, ILAS President

The nominations for the ILAS 2011 election are:

For a three-year term to begin March 1, 2012 as Secretary-Treasurer – Leslie Hogben, Ames, Iowa, USA.
For two open three-year terms to begin March 1, 2012 as "at large" members of the ILAS Board of Directors – Dale Olesky, Victoria, Canada; Peter Šemrl, Ljubljana, Slovenia; Valeria Simoncini, Bologna, Italy; Ion Zaballa, Bilbao, Spain.

The Nominating Committee was Wayne Barrett, Froilan Dopico, Helena Šmigoc, Jeff Stuart (Chair), and Bit-Shum Tam. Many thanks to that committee for their important service and to the nominees for agreeing to stand for election.

The ILAS Board of Directors recently amended the Society's bylaws in order to allow ILAS to implement electronic voting in its elections. The amended bylaws have been posted on the ILAS web site. ILAS members who have questions or concerns regarding the amended bylaws should feel free to contact any member of the ILAS Executive Board.

Voting will be open from November 18, 2011 until January 22, 2012. Members with no e-mail on file were mailed paper ballots. If you did not receive an e-mail or paper ballot and wish to vote, please verify that you are a 2011 member by consulting the website http://www.ilasic.math.uregina.ca/iic/misc/membership.html. If you are a member, you can obtain your username and password by contacting Leslie Hogben (hogben@aimath.org), ILAS Secretary-Treasurer.

Nominations Sought for Hans Schneider Prize

The Hans Schneider Prize in Linear Algebra is awarded by The International Linear Algebra Society for research contributions and achievements at the highest level of Linear Algebra. The Prize may be awarded for either outstanding scientific achievement or for a lifetime contribution. The Prize is awarded every three years, and the winner(s) are invited to give a lecture at an appropriate ILAS conference.

The committee charged with selection of the next Hans Schneider Prize winner(s) has been appointed by the ILAS President upon the advice of the ILAS Executive Board. The members are:

S. Friedland, friedlan@uic.edu,
S. Kirkland, Stephen.Kirkland@nuim.ie (ex officio - ILAS president),
P. Šemrl, peter.semrl@fmf.uni-lj.si,
L. Rodman, lxrodm@math.wm.edu (chair),
B. Shader, bshader@uwyo.edu, and
V. Simoncini, valeria@dm.unibo.it.

Nominations, of distinguished individuals judged worthy of consideration for the Prize, are now being invited from members of ILAS and scientific community in general. In nominating an individual, the nominator should include:

(1) a brief biographical sketch of the nominee, and
(2) a statement explaining why the nominee is considered worthy of the prize, including references to publications or other contributions of the nominee which are considered most significant in making this assessment.

The prize guidelines can be found at http://www.ilasic.math.uregina.ca/iic/misc/hsguidelines.html, and the list of all Hans Schneider Prize winners at http://www.ilasic.math.uregina.ca/iic/misc/hsall.html. Nominations are open until December 31, 2011, and should be sent to the committee chair, Leiba Rodman (lxrodm@math.wm.edu).

New Editors for IMAGE

By Jane Day

There will be some new staff for IMAGE, starting three year terms in 2012. Kevin Vander Meulen will be the new Editor in Chief. He teaches at Redeemer University College in Ancaster, Ontario, Canada. He's looking forward to being EIC and even offered to help edit this present issue, which was a great help. There will also be four new Consulting Editors: Carlos Fonseca, Univ. of Coimbra, Portugal - Interviews of Senior Linear Algebraists; Amy Wehe, Fitchburg State University, USA - Advertisements; Minerva Catral, Xavier University, USA - Announcements and Reports about Conferences and Journals; and Bojan Kuzma, University of Ljubljana, Slovenia - Problems Corner.

Three of our Consulting Editors will continue their outstanding work for IMAGE: Peter Šemrl, University of Ljubljana, Slovenia - History of Linear Algebra; Oskar Baksalary, Adam Mickiewicz University, Poland - Books; and Steve Leon, University of Massachusetts at Dartmouth, USA - Linear Algebra Education

Fuzhen Zhang, who has been an excellent Consulting Editor of the Problems Corner since 2009, asked to be replaced in 2012. Jim Weaver retired in 2010 from being the Consulting Editor for Advertisements. (See article about all of Jim's service to ILAS in IMAGE issue 44, Fall 2010, p. 34.)

I have enjoyed working on IMAGE and thank ILAS for this opportunity. I warmly thank all the Consulting Editors who have served with me, for their wonderful support. All of us are grateful for ongoing support from the ILAS Journals Committee, and I especially thank Roger Horn, who chaired that committee for most of my tenure as EIC and provided much wise advice for a novice editor. I'm also very happy about all the enthusiastic new volunteers who are joining the staff of IMAGE.

Send News for IMAGE Issue 48 by April 1, 2012

IMAGE issue 48 is due to appear online June 1, 2012. Send your news for this issue to the appropriate editor by April 1, 2012:
* Problems and Solutions to Bojan Kuzma (bojan.kuzma@upr.si)
* History of Linear Algebra to Peter Šemrl (peter.semrl@fmf.uni-lj.si)
* Book Reviews to Oskar Baksalary (obaksalary@gmail.com)
* Linear Algebra Education news to Steve Leon (sleon@umassd.edu)
* Announcements and Reports of Conferences, Workshops and Journals to Minerva Catral (catralm@xavier.edu)
* Advertisements to Amy Wehe (awehe@fitchburgstate.edu)
* Interviews of Senior Linear Algebraists to Carlos Fonseca (cmf@mat.uc.pt)

* All other news and material that would be of interest to the linear algebra community to Kevin Vander Meulen, Editor in Chief (kvanderm@redeemer.ca). This includes:
    * Feature articles such as surveys, interesting applications, new developments
    * Short items like cartoons, short proofs and notes
    * Honors and awards, job openings, funding opportunities for faculty and students
    * Announcements of new books, software, websites
    * Brief references to articles appearing elsewhere that might interest the linear algebra community
    * Transitions (new positions, obituaries)
    * Letters to the Editor
    * Suggestions for IMAGE

Report of the ILAS Education Committee

Submitted by Abraham Berman

The members of the ILAS Education committee are Avi Berman, Sang-Gu Lee, Steven Leon (chair), and David Strong. Our activities have focused mainly on organizing activities relating to linear algebra education at various mathematics conferences. David Strong has been an organizer of MAA sessions on linear algebra education at recent Joint Meetings of the AMS and MAA. David Strong, Pepperdine University, USA, David Lay, University of Maryland, USA and Gil Strang, Massachusetts Institute of Technology, USA, have organized an MAA Special Session "Innovative and Effective Ways to Teach Linear Algebra", to be held at the annual 2012 Joint Mathematics Meetings, January 4-7, 2012 in Boston, Massachusetts, USA. There will be two sessions: Wednesday, January 4, 2012, 2:15-6:00 pm, in Room 311, Hynes; and Friday, January 6, 2012, 7:20-11:55 am, in Rhythms I, Sheraton Hotel. They organized similar discussions at the Joint Meetings in January 2011, which were highly successful.

The Education Committee is sponsoring a Discussion Group session titled "Issues Surrounding Teaching Linear Algebra" at the International Conference of Mathematics Education (ICME12), which will be held in Seoul, Korea, July 8-15, 2012. (See details on page 5 of this issue). Professors Abraham Berman, Sang-Gu Lee and Kazuyoshi Okubo will co-chair the discussion. For more information please contact Sang-Gu Lee (sglee@skku.edu) or Avi Berman (berman@tx.technion.ac.il).

The Education Committee is also planning a session about teaching linear algebra for the next Haifa Matrix Theory Conference, which will be in Haifa, Israel on November 12-15, 2012. (See page 6 of this issue for more about this conference.)

The ILAS Education Website (http://www.ilasic.math.uregina.ca/iic/education/) is a valuable resource for information on linear algebra education and tools for teaching linear algebra. It is maintained by Sang-Gu Lee.

14th Householder Award in Numerical Linear Algebra

The 14th Alston S. Householder Award for the best PhD dissertation in numerical linear algebra submitted in 2008-2010 was awarded at the 18th triennial Householder Symposium on Numerical Linear Algebra, held June 12-17, 2011, at Granlibakken Conference Center in Tahoe City, California, USA.

There were two co-winners, chosen from a record number of 29 nominations: Bart Vandereycken (KU Leuven): "Riemannian and multilevel optimization for rank-constrained matrix problems" and Paul Willems (Bergische University of Wuppertal): "On MR3-type algorithms for the tridiagonal symmetric eigenproblem and the bidiagonal SVD".

The awards committee also recognized a short list of David Amsallem (Stanford University, US), Stephan Guttel (Technical University of Freiberg, Germany), Mark Hoemmen (University of California at Berkeley, US) and Federico Poloni (University of Pisa, Italy).

The committee members were Michele Benzi, James Demmel (chair), Howard Elman, Volker Mehrmann, Sabine Van Huffel, and Steven Vavasis.

OPPORTUNITIES

Graduate Fellowships

US Department of Energy Science Graduate Fellowships, 3-year awards for US citizens: partial tuition, stipend and research allowance for full-time graduate study and thesis/dissertation research at U.S. academic institutions. http://scgf.orau.gov/. Deadline January 3, 2012.

Temple University Department of Mathematics Postdoctoral Research Assistant Professor Positions: non-tenure track, for new or recent PhDs. See details at http://math.temple.edu/research. Apply through mathjobs.org by December 15, 2011.

University College Dublin, Mathematical Sciences, PhD Studentship: for EU students only, this student will work under Dr. Helena Šmigoc on nonnegative matrices; 18,000 Euros, tuition and research travel annually. Applications to Helena.Smigoc@ucd.ie.

Other Sources for Mathematics Positions

Online listings of university and other mathematics positions are maintained at AMS MathJobs.Org (https://www.mathjobs.org/jobs) and the SIAM Job Board (http://jobs.siam.org/home/index.cfm?site_id=686 or http://twitter.com/SIAMJobBoard). The website Academic Keys http://sciences.academickeys.com/ lists open positions in all academic fields.

IMAGE Problem Corner: Old Problems With Solutions

Problem 46-1: Space Decomposition in Terms of Column and Null Spaces

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* baxx@amu.edu.pl
and Götz Trenkler, *Technische Universität Dortmund, Dortmund, Germany,* trenkler@statistik.tu-dortmund.de

Let $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote column and null spaces of a matrix argument, respectively. For given $K \in \mathbb{C}_{n,n}$, find a decomposition $U \oplus V = \mathbb{C}_{n,1}$ such that $U, V \subseteq \mathbb{C}_{n,1}$ depend exactly on the four subspaces $\mathcal{R}(K)$, $\mathcal{R}(K^*)$, $\mathcal{N}(K)$, and $\mathcal{N}(K^*)$, where K^* stands for the conjugate transpose of K.

Solution 46-1.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA,* eaherman@gmail.com

Let

$$U = \mathcal{N}(K) \cap \mathcal{N}(K^*), \qquad V = \mathcal{R}(K) + \mathcal{R}(K^*)$$

If $u \in U$, $v \in \mathcal{R}(K)$, and $w \in \mathcal{R}(K^*)$, then

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle = 0 + 0 = 0$$

Hence U and V are orthogonal subspaces of $\mathbb{C}_{n,1}$. Furthermore, if $u \in \mathbb{C}_{n,1}$ is orthogonal to V, it is orthogonal to both $\mathcal{R}(K)$ and $\mathcal{R}(K^*)$ and hence belongs to, respectively, $\mathcal{N}(K^*)$ and $\mathcal{N}(K)$. Thus, $u \in U$. Therefore U and V are orthogonal complements in $\mathbb{C}_{n,1}$, and so $U \oplus V = \mathbb{C}_{n,1}$.

Solution 46-1.2 by Hans Joachim Werner, *University of Bonn, Bonn, Germany,* hjw.de@uni-bonn.de

For every $A \in \mathbb{C}_{m,n}$, according to the Orthogonal Decomposition Theorem,

$$\mathbb{C}_{m,1} = \mathcal{R}(A) \oplus \mathcal{N}(A^*) \qquad \text{and} \qquad \mathbb{C}_{n,1} = \mathcal{N}(A) \oplus \mathcal{R}(A^*);$$

see, e.g., [Carl D. Meyer (2000), *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, p. 405]. We note that $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$ and $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$, where $(\cdot)^\perp$ indicates the orthogonal complement of the space argument with respect to the usual standard inner product. For every square matrix $K \in \mathbb{C}_{n,n}$, one can directly state the following four additional decompositions

$$\begin{aligned}
\mathbb{C}_{n,1} &= [\mathcal{R}(K) \cap \mathcal{R}(K^*)] \oplus [\mathcal{N}(K^*) + \mathcal{N}(K)],\\
\mathbb{C}_{n,1} &= [\mathcal{N}(K) \cap \mathcal{N}(K^*)] \oplus [\mathcal{R}(K^*) + \mathcal{R}(K)],\\
\mathbb{C}_{n,1} &= [\mathcal{R}(K) \cap \mathcal{N}(K)] \oplus [\mathcal{N}(K^*) + \mathcal{R}(K^*)],\\
\mathbb{C}_{n,1} &= [\mathcal{R}(K^*) \cap \mathcal{N}(K^*)] \oplus [\mathcal{N}(K) + \mathcal{R}(K)];
\end{aligned}$$

observe that if $\mathcal{M}$ and $\mathcal{N}$ are two linear subspaces of $\mathbb{C}_{n,1}$, then $[\mathcal{M} \cap \mathcal{N}]^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$ and $(\mathcal{M}^\perp)^\perp = \mathcal{M}$.

Also solved by the proposers.

Problem 46-2: Commutator

Proposed by Gerald Bourgeois, *University of French Polynesia, Tahiti*, bourgeois.gerald@gmail.com

If both matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ commute with their commutator $C = AB - BA$, it is known that A and B are simultaneously triangularizable; this is the "Little McCoy" theorem (see N. McCoy, *On quasi-commutative matrices*, Trans. Amer. Math. Soc.36 (1934) 327–340). If only A commutes with C, show that A and B must be simultaneously triangularizable if $n = 2$, but they need not be simultaneously triangularizable if $n \geq 3$.

Solution 46-2.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA,* eaherman@gmail.com

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are 2×2 matrices and that A commutes with $C = AB - BA$. By Schur's theorem, we may assume that A is upper-triangular. Upon calculation, we have

$$\begin{aligned}
C = AB - BA &= \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\\
&= \begin{bmatrix} a_{12}b_{21} & a_{12}(b_{22} - b_{11}) + (a_{11} - a_{22})b_{12} \\ (a_{22} - a_{11})b_{21} & -a_{12}b_{21} \end{bmatrix}
\end{aligned}$$

A calculation reveals that

$$AC - CA = \begin{bmatrix} a_{12}(a_{22} - a_{11})b_{21} & a_{12}(b_{22} - b_{11})(a_{11} - a_{22}) + (a_{11} - a_{22})^2 b_{12} - 2a_{12}^2 b_{21} \\ (a_{22} - a_{11})^2 b_{21} & a_{12}(a_{22} - a_{11})b_{21} \end{bmatrix}$$

Since A and C commute, all entries of $AC-CA$ are zero. We may assume that $b_{21} \neq 0$, since otherwise the matrix that triangularized A also triangularized B. Hence, from the $(2,1)$-entry of $AC - CA$, $a_{22} = a_{11}$; and so, from the $(1,2)$-entry, $a_{12} = 0$. That is, A is a multiple of the identity, which implies that A and B are simultaneously diagonalizable.

For $n \geq 3$, let A be the $n \times n$ matrix whose $(1,n)$-entry is 1 and all other entries are 0, and let B be the $n \times n$ matrix whose diagonal and subdiagonal entries are 1 and all other entries are 0. Then the $(1, n-1)$-entry of $AB - BA$ is 1, the $(2,n)$-entry is -1, and all other entries are 0. Hence A commutes with $AB - BA$, as both products are zero. We will show that A and B are not simultaneously diagonalizable by using [R. Horn and C. Johnson, *Matrix Analysis*, 1985, Theorem 2.4.15, page 94]:

> Let $A, B \in \mathcal{M}_n$, with $\sigma(A) = \{\alpha_1, \ldots, \alpha_n\}$ and $\sigma(B) = \{\beta_1, \ldots, \beta_n\}$, including multiplicities. There is a nonsingular $S \in \mathcal{M}_n$ such that both $S^{-1}AS$ and $S^{-1}BS$ are upper triangular if and only if there is a permutation $i_1, \ldots, i_n$ of the indices $1, 2, \ldots, n$ such that $\sigma(p(A,B)) = \{p(\alpha_j, \beta_{i_j}) : j = 1, \ldots, n\}$ for all polynomials $p(t,s)$ with complex coefficients in two (noncommuting) variables.

Note that $\alpha_j = 0$ and $\beta_j = 1$ for all $j = 1, \ldots, n$. Let $p(x,y) = xy^{n-1}$, and so $p(\alpha_j, \beta_{i_j}) = p(0,1) = 0$ for all $j = 1, \ldots, n$. We will show that $\sigma(AB^{n-1}) = \{1, 0, \ldots, 0\}$, which implies, by the quoted theorem, that A and B are not simultaneously diagonalizable. Since rows 2 through n of A are all zeros, the same is true of AB^{n-1}; hence 0 is an eigenvalue of AB^{n-1} with multiplicity at least $n-1$. If $1 \leq k \leq n-1$, an induction argument shows that the $(1, n-k)$-entry of AB^k is 1 and all entries to the left of this entry are 0. In particular, the $(1,1)$-entry of AB^{n-1} is 1, and so 1 is an eigenvalue of AB^{n-1}.

Solution 46-2.2 by Leo Livshits, *Colby College, Waterville, Maine, USA*, llivshi@colby.edu

1. $n = 2$.
   (a) The validity of the claims
   $$\text{“}A \text{ commutes with } AB - BA\text{”} \tag{1}$$
   and
   $$\text{“}A \text{ and } B \text{ are simultaneously triangularizable”} \tag{2}$$
   remains unchanged if A is replaced by $\alpha A - \delta I$ ($\alpha, \delta \in \mathbb{C}$). It follows that in demonstrating that $(1) \Longrightarrow (2)$ in $\mathcal{M}_2(\mathbb{C})$ we can assume without loss of generality that A is a singular matrix and its spectrum is a subset of $\{0,1\}$.
   Hence we can assume that the Jordan form of A is one of the following: (i). $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, (ii). $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.
   Since the validity of claims (1) and (2) is not affected by a simultaneous similarity applied to A and B, we can assume that A is one of the two Jordan form matrices listed above, and $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.
   (b) Note that condition (1) is equivalent to
   $$A^2B + BA^2 - 2ABA = 0. \tag{3}$$
   A direct calculation shows that (3) entails $c = 0$ for each of the two listed Jordan forms of A. Hence in either case A and B are simultaneously triangularizable.

2. $n \geq 3$.
   (a) Let $A = N \oplus 0_{n-2 \times n-2}$ where $n \geq 3$ and $N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $A^2 = 0$ and so $A^2B + BA^2 - 2ABA = 0$ for every $B \in \mathcal{M}_n(\mathbb{C})$ such that $B(2,1) = 0$. In other words, for every such B, A commutes with $AB - BA$.
   (b) Let B be the matrix defined by
   $$B(i,j) = \begin{cases} 0, & \text{if } (i,j) = (2,1) \\ 1, & \text{otherwise} \end{cases}$$
   Then A commutes with $AB - BA$, but A and B are not simultaneously triangularizable. Here is why: if A and B were simultaneously triangularizable then so would be A and B^2, and since A is nilpotent, we would have that AB^2 is also nilpotent, and hence has zero trace. Yet $Tr(AB^2) = B^2(2,1) = n - 2 > 0$.

Editorial note: The problem is also solved by Johanns de Andrade Bezerra. Johanns de Andrade Bezerra points out that for the case of $n = 2$, by Jacobson's lemma [R.A. Horn and C.R. Johnson, *Matrix Analysis*, 1985, Problem 12, p. 98], if $AB - BA = C$ and $AC = CA$, then C is a nilpotent matrix. By [V.V. Prasolov, *Problems and Theorems in Linear Algebra*, Translation of Mathematical Monographs, Volume 134, 1994, Theorem 40.5, p. 175], if $rank(C) \leq 1$, then A and B are simultaneously triangularizable. If $n = 2$, clearly $rank(C) \leq 1$. Minghua Lin and Henry Wolkowicz note that a counterexample for $n \geq 3$ is available in [G. Bourgeois, *What about* a, b *if* $ab - ba$ *and* a *commute*, arXiv:1103.4200v1 [math.RA]].

Problem 46-3: Lower Bound for Probability
Proposed by Christopher Hillar, *Mathematical Sciences Research Institute, Berkeley, California, USA*, chillar@msri.org

Find a good lower bound for the probability $p(m, n)$ that m random binary column vectors in $\{0, 1\}^n$ are linearly independent.

Solution 46-3 by Hans Joachim Werner, *University of Bonn, Bonn, Germany,* hjw.de@uni-bonn.de
Let $\mathbb{F} := \{0, 1\}$ and consider the Galois field $(\mathbb{F}, +, \cdot)$ with characteristic 2, where the binary field operations '$+$' and '$\cdot$' are defined clockwise, i.e.,

$$0 + 0 = 0, \quad 1 + 0 = 0 + 1 = 1, \quad 1 + 1 = 0$$

and

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

Moreover, for each $n \in \mathbb{N}$, let $\mathbb{F}^n$ stand for the set of all n-tuples (vectors) with entries from $\mathbb{F}$. Clearly, $(\mathbb{F}^n, +, \cdot)$, with '$+$' and '$\cdot$' indicating the usual (elementwise) vector addition and scalar multiplication, respectively, represents a linear space over the field $\mathbb{F}$. The cardinality of $\mathbb{F}^n$ is 2^n, i.e., $\mathbb{F}^n$ consists of exactly 2^n different vectors. In what follows, let X be a random vector that attains equally likely each of the 2^n possible vectors from $\mathbb{F}^n$. Under this assumption, which nowadays is also known as Laplace's principle of equally likely elementary outcomes, the desired probabilities can easily be determined. To that end, for each $m \in \mathbb{N}$, let $X_1, X_2, \ldots, X_m$ be independent identically distributed random vectors, all of which are distributed as X, i.e., let $X_1, X_2, \ldots, X_m$ be a mathematical sample of size m. Then we are interested in

$$p(m, n) := \mathsf{P}(A_m) = \frac{\text{number of outcomes } x_1, \ldots, x_m \text{ being favorable for } A_m}{\text{totally number of equally likely outcomes}}$$

with $A_m := (X_1, X_2, \ldots, X_m$ are linearly independent) and $\mathsf{P}(A_m)$ denoting the probability of event A_m. Clearly, for each $2 \leq j \leq m$, a concrete sample $x_1, x_2, \ldots, x_j$ of size j of vectors from $\mathbb{F}^n$ is linearly independent if, and only if, (a) $x_1, x_2, \ldots, x_{j-1}$ is linearly independent, and (b) x_j is not a linear combination of $x_1, x_2, \ldots, x_{j-1}$. Observe that if $x_1, x_2, \ldots, x_{j-1}$ are linearly independent, then there are exactly 2^{j-1} different vectors in $\mathbb{F}^n$ that are linear combinations of these vectors $x_1, x_2, \ldots, x_{j-1}$. Consequently, there exist $2^n - 2^{j-1}$ different vectors x in $\mathbb{F}^n$ such that the vectors $x_1, x_2, \ldots, x_{j-1}, x$ are linearly independent. Moreover, x_1 is linearly independent if and only if $x_1 \neq 0$, and so there are exactly $2^n - 1 = 2^n - 2^0$ such vectors. With these observations in mind, it is now clear that for each natural number m we have

$$p(m, n) = \begin{cases} \frac{\prod_{j=1}^{m}(2^n - 2^{j-1})}{2^{mn}} & \text{if } 1 \leq m \leq n, \\ \\ 0 & \text{else.} \end{cases}$$

Editorial note: The proposer provided an answer $p(m, n) \geq 1 - 2^{m-n}$ with the submission of the problem.

Problem 46-4: Hermitian Matrices with Zero Product
Proposed by Roger Horn, *University of Utah, Salt Lake City, USA,* rhorn@math.utah.edu

Let A and B be $n \times n$ Hermitian matrices with rank$(A) = r$, rank$(B) = s$. Let $p_A(t) = t^n + a_{n-1}t^{n-1} + \cdots$ be the characteristic polynomial of A, and similarly for the characteristic polynomials $p_B(t)$ (coeffs are b_i) and $p_{A+B}(t)$ (coeffs are c_i). (a) Suppose that $Im(A) + Im(B) = Im(A + B)$. If $a_{n-r}b_{n-s} = c_{n-r-s}$, show that $AB = 0$. (b) If $t^n p_{A+B}(t) = p_A(t)p_B(t)$, show that $AB = 0$.

Solution 46-4.1 by Minghua Lin, *University of Waterloo, Ontario, Canada,* mlin87@ymail.com
and Henry Wolkowicz, *University of Waterloo, Ontario, Canada,* hwolkowicz@uwaterloo.ca

(a) $rank(A) = r$ and $rank(B) = s$ imply $a_{n-r} \neq 0$ and $b_{n-s} \neq 0$. Now $a_{n-r}b_{n-s} = c_{n-r-s}$ shows $c_{n-r-s} \neq 0$ and thus $rank(A + B) = r + s = rank(A) + rank(B)$. By the dimension formula

$$\dim(Im(A)) + \dim(Im(B)) = \dim(Im(A + B)) + \dim(Im(A) \cap Im(B)),$$

$Im(A+B) = Im(A) + Im(B)$ and $rank(A+B) = rank(A) + rank(B)$, we have $\dim(Im(A) \cap Im(B)) = 0$, i.e., $Im(A) \cap Im(B) = \{0\}$ and so $Im(B) \subseteq (Im(A))^{\perp} = Ker(A^*) = Ker(A)$, i.e., $AB = 0$.

(b) We note that this is a stronger version of Craig-Sakamoto theorem. It first appears in [J. Ogawa, I. Olkin, *A tale of two countries: The Craig-Sakamoto-Matusita theorem*, J. Statist. Plann. Inference 138 (11) (2008) 3419-3428], and for a simple proof, we refer to [H. Carrieu, *Close to the Craig-Sakamoto theorem,* Linear Algebra Appl., 432 (2010) 777-779].

Solution 46-4.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA,* eaherman@gmail.com

Both (a) and (b) follow from

Theorem. Let A and B be $n \times n$ Hermitian matrices. Let $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_s$, and $\gamma_1, \dots, \gamma_q$ denote the nonzero eigenvalues of A, B, and $A+B$, respectively. If $q = r + s$ and

$$\left(\prod_{i=1}^{r} \alpha_i\right)\left(\prod_{j=1}^{s} \beta_j\right) = \prod_{k=1}^{r+s} \gamma_k$$

then $AB = 0$.

Proof. First we reduce the problem to the case when $A+B$ is nonsingular. Since $\text{Im}(A+B) \subseteq \text{Im}(A) + \text{Im}(B)$,

$$q = \dim(\text{Im}(A+B)) \le \dim(\text{Im}(A) + \text{Im}(B)) \le \dim(\text{Im}(A)) + \dim(\text{Im}(B)) = r + s = q$$

Hence

$$\text{Im}(A+B) = \text{Im}(A) + \text{Im}(B) \tag{4}$$

We may assume $A+B$ is diagonal:

$$A + B = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} + B$$

where $\gamma = \text{diag}(\gamma_1, \dots, \gamma_q)$ is nonsingular. Hence, by (4), for all $x \in \mathbb{C}^q$ and $y \in \mathbb{C}^{n-q}$ there exist $u \in \mathbb{C}^q$ and $v \in \mathbb{C}^{n-q}$ such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = (A+B)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \gamma u \\ 0 \end{bmatrix}$$

So $A_{12}^* x + A_{22} y = 0$, which implies $A_{12} = 0$ and $A_{22} = 0$. Thus

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \gamma - A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

Also, A_{11} and $\gamma - A_{11}$ have the same nonzero eigenvalues as A and B, respectively. Therefore, if the Theorem holds when $A+B$ is nonsingular, we have $A_{11}(\gamma - A_{11}) = 0$ and so $AB = 0$.

In the rest of the proof, we assume that $A+B$ is nonsingular. Now we assume A is diagonal rather than $A+B$:

$$A + B = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$$

where $\alpha = \text{diag}(\alpha_1, \dots, \alpha_r)$ is nonsingular. Then

$$\left(\prod_{i=1}^{r} \alpha_i\right)\left(\prod_{j=1}^{s} \beta_j\right) = \prod_{k=1}^{r+s} \gamma_k = \det(A+B) = \det\begin{bmatrix} \alpha + B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} = \left(\prod_{i=1}^{r} \alpha_i\right) \det B_{22} \tag{5}$$

The final equality in (5) holds for the following reason. Since rank$(B) = s$, every $k \times k$ submatrix of B with $k > s$ has determinant equal to 0. Let $\langle \alpha_{i_1}, \ldots, \alpha_{i_m} \rangle$ be any subsequence of $\langle \alpha_1, \ldots, \alpha_r \rangle$ with $0 \leq m < r$. Form the $(n-m) \times (n-m)$ submatrix B' of B in which rows and columns $i_1, \ldots, i_m$ are deleted. Since $n - m > n - r = s$, det$(B') = 0$. In the expansion of det $\left[\begin{smallmatrix} \alpha + B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{smallmatrix}\right]$, select all terms in which $\prod_{k=1}^m \alpha_{i_k}$ occurs (and no other α_j appears). When $\prod_{k=1}^m \alpha_{i_k}$ is factored out of the sum of these terms, what remains is det(B').

Equation (5) implies det$(B_{22}) = \prod_{j=1}^s \beta_j$. In particular, B_{22} is nonsingular, and so

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} - B_{12}B_{22}^{-1}B_{12}^* & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ B_{22}^{-1}B_{12}^* & I \end{bmatrix}$$

Since B_{22} has the same rank as B, there exists a $s \times r$ matrix X such that

$$B_{12}^* = B_{22}X \quad \text{and} \quad B_{11} = B_{12}B_{22}^{-1}B_{12}^* = X^*B_{22}X$$

Thus

$$B = \begin{bmatrix} X^*B_{22}X & X^*B_{22} \\ B_{22}X & B_{22} \end{bmatrix} = \begin{bmatrix} X^* \\ I \end{bmatrix} B_{22}[X\ I]$$

It remains only to show that $X = 0$. There is a unitary matrix U such that $U^*BU = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \beta \end{smallmatrix}\right]$, where $\beta = \text{diag}(\beta_1, \ldots, \beta_s)$ is nonsingular. Let $C = [C_1\ C_2] = [X\ I]U$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} = U^* \begin{bmatrix} X^* \\ I \end{bmatrix} B_{22}[X\ I]U = \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} B_{22}[C_1\ C_2]$$

Hence $C_1^*B_{22}[C_1\ C_2] = 0$, which implies $C_1^* = 0$ since rank$([C_1\ C_2]) = $ rank$([X\ I]) = s$; and $\beta = C_2^*B_{22}C_2$, which implies det$(\beta) = |\text{det}(C_2)|^2\,\text{det}(B)$ and so $|\text{det}(C_2)| = 1$. Let $U = \left[\begin{smallmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{smallmatrix}\right]$. Then

$$[0\ C_2] = C = [X\ I] \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = [XU_{11} + U_{21}\ XU_{12} + U_{22}]$$

which implies

$$U_{21} = -XU_{11} \quad \text{and} \quad U_{22} = C_2 - XU_{12} \tag{6}$$

From $UU^* = I$, we have

$$U_{11}U_{11}^* + U_{12}U_{12}^* = I,\ \ U_{21}U_{11}^* + U_{22}U_{12}^* = 0,\ \ U_{21}U_{21}^* + U_{22}U_{22}^* = I \tag{7}$$

Substituting both of equations (6) into the second of equations (7), we have $-XU_{11}U_{11}^* + C_2U_{12}^* - XU_{12}U_{12}^* = 0$, and so $X = C_2U_{12}^*$ by the first of equations (7). Substituting both of equations (6) into the third of equations (7), we have $XU_{11}U_{11}^*X^* + C_2C_2^* - XU_{12}C_2^* - C_2U_{12}^*X^* + XU_{12}U_{12}^*X^* = I$, which, by using the first of equations (7) and $X = C_2U_{12}^*$, simplifies to $C_2C_2^* = XX^* + I$. This implies $\|C_2^*u\| \geq \|u\|$ for all $u \in \mathbb{C}^s$ which, together with the fact that $|\text{det}(C_2)| = 1$, shows that C_2 is unitary. Therefore, since $I = C_2C_2^* = XX^* + I$, we conclude that $X = 0$. □

(a) The assumption Im(A) + Im(B) = Im$(A + B)$ will not be used. Let $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$, and $\gamma_1, \ldots, \gamma_q$ denote the nonzero eigenvalues of A, B, and $A + B$, respectively. Then

$$p_A(t) = (t - \alpha_1) \cdots (t - \alpha_r)t^{n-r}$$

and similarly for $p_B(t)$ and $p_{A+B}((t)$. Hence

$$a_{n-r} = (-1)^r \prod_{i=1}^{r} \alpha_i, \quad b_{n-s} = (-1)^s \prod_{j=1}^{s} \beta_j, \quad c_{n-q} = (-1)^q \prod_{k=1}^{q} \gamma_k$$

Also, in each polynomial, all coefficients of smaller powers of t are zero. Since $c_{n-r-s} = a_{n-r}\, b_{n-s} \neq 0$, it follows that $n-r-s \geq n-q$ and so $q \geq r+s$. Since $q = \text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B) = r+s$, we conclude that $q = r+s$ and so

$$(-1)^{r+s} \left(\prod_{i=1}^{r} \alpha_i\right) \left(\prod_{j=1}^{s} \beta_j\right) = a_{n-r}\, b_{n-s} = c_{n-r-s} = (-1)^{r+s} \prod_{k=1}^{r+s} \gamma_k$$

The Theorem therefore implies that $AB = 0$.

(b) By factoring both sides of the equation $t^n p_{A+B}(t) = p_A(t) p_B(t)$ into linear factors and canceling all occurrences of the factor t, we see that there is a one-one correspondence between the nonzero eigenvalues of $A+B$ and the nonzero eigenvalues of A and B. In particular, the hypotheses of the Theorem hold, and so $AB = 0$.

Also solved by the proposer and by Hans Joachim Werner.

Problem 46-5: A Trace Inequality

Proposed by Minghua Lin, *University of Waterloo, Ontario, Canada,* mlin87@ymail.com

Let A and B be $m \times n$ strictly contractive complex matrices (i.e., their spectral norms are less than 1). Show that

$$\left|Tr(I - B^*A)^{-1}\right|^2 \leq Tr(I - A^*A)^{-1}\, Tr(I - B^*B)^{-1}.$$

Solution 46-5.1 by Suvrit Sra, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

We use lower-case letters for matrices and tr(x) for trace of matrix x, proving that the desired inequality is equivalent to

$$\begin{pmatrix} \text{tr}(I - a^*a)^{-1} & \text{tr}(I - a^*b)^{-1} \\ \text{tr}(I - b^*a)^{-1} & \text{tr}(I - b^*b)^{-1} \end{pmatrix} \geq 0.$$

Since a is strictly contractive, we can write

$$(I - a^*a)^{-1} = \sum\nolimits_{k \geq 0} (a^*a)^k;$$

similarly we can expand $(I - b^*b)^{-1}$ and $(I - a^*b)^{-1}$, etc.

Since sums of positive definite matrices are positive definite, it suffices to prove that the matrix

$$\begin{pmatrix} \text{tr}(a^*a)^k & \text{tr}(a^*b)^k \\ \text{tr}(b^*a)^k & \text{tr}(b^*b)^k \end{pmatrix},$$

is positive-semidefinite. Equivalently, we must prove that

$$|\text{tr}(a^*b)^k|^2 \leq \text{tr}(a^*a)^k \text{tr}(b^*b)^k. \tag{8}$$

Using the notation $|a| = (a^*a)^{1/2}$, recall by Weyl's majorant relation, $|\text{tr}\ x| \leq \text{tr}|x|$. Thus, we have

$$|\text{tr}(a^*b)^k|^2 \leq (\text{tr}|a^*b|^k)^2 = \|\ |a^*b|^{k/2}\ \|_F^4, \tag{9}$$

where as usual we have the *Frobenius norm* $\|x\|_F^2 := \text{tr}|x|^2$. Now, by the known inequality [R. Horn and C. Johnson, *Topics in Matrix Analysis,* Cambridge University Press, 1991, p. 212]

$$\|\ |xy|^r\ \|^2 \leq \|\ |x|^{2r}\ \|\|\ |y|^{2r}\ \|, \quad r > 0, \tag{10}$$

for any unitarily invariant norm. Using inequality (10) on (9), we get

$$|\mathrm{tr}(a^*b)^k|^2 \le \| \, |a^*b|^{k/2} \, \|_F^4 \le \| \, |a|^k \, \|_F^2 \| \, |b|^k \, \|_F^2 = (\mathrm{tr}|a|^{2k})(\mathrm{tr}|b|^{2k}),$$

which proves (8).

Solution 46-5.2 by the proposer Minghua Lin, *University of Waterloo, Ontario, Canada,* mlin87@ymail.com

Note that

$$H := \begin{pmatrix} Tr(I-A^*A)^{-1} & Tr(I-B^*A)^{-1} \\ Tr(I-A^*B)^{-1} & Tr(I-B^*B)^{-1} \end{pmatrix} = \begin{pmatrix} Tr\sum\limits_{j=0}^{\infty}(A^*A)^j & Tr\sum\limits_{j=0}^{\infty}(B^*A)^j \\ Tr\sum\limits_{j=0}^{\infty}(A^*B)^j & Tr\sum\limits_{j=0}^{\infty}(B^*B)^j \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} Tr(A^*A)^j & Tr(B^*A)^j \\ Tr(A^*B)^j & Tr(B^*B)^j \end{pmatrix}$$

and

$$|Tr(B^*A)^j|^2 \le \big(Tr|(B^*A)^j|\big)^2 \le Tr(A^*A)^j Tr(B^*B)^j,$$

where the second inequality is a consequence of [Corollary 4.27, X. Zhan, *Matrix Inequalities*, Springer, 2002] by taking the trace norm. So H is positive semidefinite, the desired inequality follows.

Problem 46-6: Singular Matrix as a Product of n Idempotent Matrices
Proposed by Éric Pité, *Paris, France,* eric.pite@telecom-paristech.org

Show that any singular $n \times n$ matrix (over any field) is a product of n idempotent matrices.

Solution 46-6 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA,* eaherman@gmail.com

We show that any singular $n \times n$ matrix of rank r is a product of $r+1$ idempotent matrices.
Lemma. If A is a nonsingular $n \times n$ matrix, there exist matrices $X_1, \ldots, X_n$ of rank at most 1 such that $A = \prod_{k=1}^{n}(I + X_k)$.
Proof. The result is trivial when $n = 1$. Suppose $n \ge 2$, and let A be a nonsingular $n \times n$ matrix over a field $\mathbb{F}$. Assume the Lemma is true for all nonsingular matrices of smaller size. Choose vectors $v_1, \ldots, v_n \in \mathbb{F}^n$ such that both $\{v_1, v_2, \ldots, v_n\}$ and $\{Av_1, v_2, \ldots, v_n\}$ are linearly independent. Let X be the $n \times n$ matrix such that $Xv_1 = Av_1 - v_1$ and $Xv_j = 0$ for $j = 2, \ldots, n$. Then X has rank at most 1, and $I + X$ is nonsingular since its range includes the linearly independent vectors $Av_1, v_2, \ldots, v_n$. Hence $(I+X)^{-1}Av_1 = v_1$. Let U be the subspace of vectors $u \in \mathbb{F}^n$ such that $(I+X)^{-1}Au = u$. Hence $k = \dim(U) \ge 1$. Let $\{u_1, \ldots, u_n\}$ be a basis of $\mathbb{F}^n$ such that $\{u_1, \ldots, u_k\}$ is a basis of U. The matrix of $(I+X)^{-1}A$ relative to this basis has the form $B = \left[\begin{smallmatrix} I_k & B_1 \\ 0 & B_2 \end{smallmatrix}\right]$, where B_2 is nonsingular and does not have 1 as an eigenvalue. Also

$$S = \begin{bmatrix} I_k & Y \\ 0 & I_{n-k} \end{bmatrix} \Rightarrow SBS^{-1} = \begin{bmatrix} I_k & Y \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} I_k & B_1 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I_k & -Y \\ 0 & I_{n-k} \end{bmatrix} = \begin{bmatrix} I_k & B_1 + Y(B_2 - I_{n-k}) \\ 0 & B_2 \end{bmatrix}$$

Hence, if $Y = -B_1(B_2 - I_{n-k})^{-1}$, the above result shows that $(I+X)^{-1}A$ is similar to $\left[\begin{smallmatrix} I_k & 0 \\ 0 & B_2 \end{smallmatrix}\right]$. By the induction assumption applied to B_2, there exist matrices $X_1, \ldots, X_{n-k}$ of rank at most 1 such that $B_2 = \prod_{j=1}^{n-k}(I_{n-k} + X_j)$. Therefore $(I+X)^{-1}A$ is similar to

$$\begin{bmatrix} I_k & 0 \\ 0 & \prod_{j=1}^{n-k}(I_{n-k} + X_j) \end{bmatrix} = \prod_{j=1}^{n-k} \begin{bmatrix} I_k & 0 \\ 0 & (I_{n-k} + X_j) \end{bmatrix} = \prod_{j=1}^{n-k} \left(I + \begin{bmatrix} 0 & 0 \\ 0 & X_j \end{bmatrix} \right)$$

Since any matrix similar to the sum of the identity matrix and a matrix of rank at most 1 has that same form, A is a product of $n + 1 - k$ such matrices, where $n + 1 - k \le n$. □

We prove the main result also by induction on n. Again the result is trivial when $n = 1$. Suppose $n \ge 2$, and let A be a singular $n \times n$ matrix. Assume the main result is true for all singular matrices of smaller size. Let r be the rank of A. Then A is similar to a matrix of the form $B = \left[\begin{smallmatrix} A_1 & 0 \\ A_2 & 0 \end{smallmatrix}\right]$ where A_1 is $r \times r$. Suppose A_1 is singular. Then

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ A_2 & 0 \end{bmatrix}$$

By the induction assumption applied to A_1, there exist idempotent matrices $P_1, \dots, P_r$ such that $A_1 = \prod_{k=1}^r P_k$. Therefore

$$B = \left(\prod_{k=1}^{r} \begin{bmatrix} P_k & 0 \\ 0 & I_{n-r} \end{bmatrix} \right) \begin{bmatrix} I_r & 0 \\ A_2 & 0 \end{bmatrix}$$

which is a product of $r+1$ idempotent matrices. Finally, suppose A_1 is nonsingular. Then

$$S = \begin{bmatrix} A_1 & 0 \\ A_2 & I_{n-r} \end{bmatrix} \Rightarrow S^{-1}BS = \begin{bmatrix} A_1^{-1} & 0 \\ -A_2A_1^{-1} & I_{n-r} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & I_{n-r} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

By the Lemma, there exist matrices $X_1, \dots, X_r$ of rank at most 1 such that $A_1 = \prod_{k=1}^r (I_r + X_k)$. Next, factor each X_k as $X_k = Y_k Z_k$, where Y_k has a single column and Z_k has a single row. Further, define $1 \times r$ matrices $W_1, \dots, W_{r+1}$ by $W_1 = 0$ and $W_{k+1} = W_k + Z_k$ for $k = 1, 2, \dots, r$. Thus

$$A_1 = \prod_{k=1}^{r} (I_r + Y_k(W_{k+1} - W_k)) = \prod_{k=1}^{r} [I_r - Y_kW_k \;\; Y_k] \begin{bmatrix} I_r \\ W_{k+1} \end{bmatrix}$$

Therefore, letting $Y_{r+1} = 0$, we conclude that

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \left(\prod_{k=1}^{r} [I_r - Y_kW_k \;\; Y_k] \begin{bmatrix} I_r \\ W_{k+1} \end{bmatrix} \right) [I_r \;\; 0] = \prod_{k=1}^{r+1} \begin{bmatrix} I_r \\ W_k \end{bmatrix} [I_r - Y_kW_k \;\; Y_k]$$

which is a product of $r+1$ idempotent matrices.

Editorial note: Hans Joachim Werner points out that in [J.A. Erdos: *On Products of Idempotent Matrices,* Glasgow Math. J. **8**: 118–122 (1967)], it is shown that every singular square matrix can be written as a product of a finite number of idempotent matrices. In [C.S. Ballantine: *Products of Idempotent Matrices,* Linear Algebra Appl. **19**: 81–86 (1978)], this result has been quantified to a certain extent. In particular, Ballantine shows that each singular $n \times n$ matrix (over any field) can be expressed as a product of n idempotent matrices. The proposer provides another reference [J. Araujo and J.D. Mitchell, *An Elementary Proof That Every Singular Matrix Is a Product of Idempotent Matrices*, Amer. Math. Monthly 112 (2005) 641–645].

Problem 46-7: Trace Inequality of a Positive Semidefinite Matrix with Exponential Function

Proposed by Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, USA,* zhang@nova.edu

Let A be an $n \times n$ positive semidefinite (Hermitian) matrix. Show that

$$\operatorname{tr}(e^A) \leq e^{\operatorname{tr}(A)} + (n-1)$$

with equality if and only if rank$(A) \leq 1$.

Solution 46-7.1 by Felicja Okulicka-Dłużewska, *Warsaw University of Technology, Poland,* F.Okulicka-Dluzewska@mini.pw.edu.pl and Alicja Smoktunowicz, *Warsaw University of Technology, Poland,* A.Smoktunowicz@mini.pw.edu.pl

First note that as A is $n \times n$ and positive semidefinite, we have $A = UDU^*$, where U is unitary, $D = \operatorname{diag}(d_i)$ with $d_i \geq 0$ for $i = 1, 2, \dots, n$. From this we get

$$\operatorname{tr}(e^A) = \operatorname{tr}(e^D) = \operatorname{tr}(\operatorname{diag}(e^{d_i})) = \sum_{i=1}^{n} e^{d_i}. \tag{11}$$

On the other hand,

$$e^{\operatorname{tr}(A)} = e^{\operatorname{tr}(D)} = e^{\sum_{i=1}^n d_i}. \tag{12}$$

Now we apply the inequality for nonnegative numbers $d_1, d_2, \dots, d_n$:

$$\sum_{i=1}^{n} e^{d_i} \leq e^{\sum_{i=1}^{n} d_i} + (n-1). \tag{13}$$

To prove (13), first observe that for two arbitrary nonnegative numbers x and y we have the inequality $(e^x - 1)(e^y - 1) \geq 0$, which can be rewritten as $e^x + e^y \leq e^{x+y} + 1$. Equality holds if and only if $x = 0$ or $y = 0$. Now (13) follows immediately by a simple induction on n. From (11)–(13) we get the desired conclusion.

Solution 46-7.2 by João R. Cardoso, *Coimbra Institute of Engineering, Coimbra, Portugal,* jocar@isec.pt

Assuming that A is an $n \times n$ positive semidefinite (Hermitian) matrix, its eigenvalues, which will be denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$, are real nonnegative. By the multinomial theorem, the inequality

$$\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k \leq (\lambda_1 + \lambda_2 + \cdots + \lambda_n)^k .$$

holds for any positive integer k. Recalling that the trace of a matrix is the sum of its eigenvalues and that the eigenvalues of e^A are $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}$, we have

$$\begin{aligned} \mathrm{tr}(e^A) &= e^{\lambda_1} + \cdots + e^{\lambda_n} = \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} + \cdots + \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \\ &= n + \sum_{k=1}^{\infty} \frac{\lambda_1^k}{k!} + \cdots + \sum_{k=1}^{\infty} \frac{\lambda_n^k}{k!} \leq n + \sum_{k=1}^{\infty} \frac{(\lambda_1 + \cdots + \lambda_n)^k}{k!} = n - 1 + e^{\lambda_1 + \cdots + \lambda_n}, \end{aligned}$$

which shows that

$$\mathrm{tr}(e^A) \leq e^{\mathrm{tr}(A)} + n - 1. \tag{14}$$

That equality in (14) occurs if and only if $(A) \leq 1$, is a consequence of the following facts:

(i) $(A) \leq 1$ if and only if the spectrum of A is of the form $\{\lambda, 0, \dots, 0\}$, with $\lambda \geq 0$, that is, there is at most one simple eigenvalue of A that is non zero.

(ii) For any integer $k \geq 2$ and for all nonnegative real numbers $a_1, a_2, \dots, a_\ell$, with $a_i \neq a_j$,

$$(a_1 + a_2 + \cdots + a_\ell)^k = a_1^k + a_2^k + \cdots + a_\ell^k$$

if and only if there is at most one a_i that is non zero.

Remark. It is worthwhile noting that (14) can be used to derive the following interesting inequality involving sums and products of positive integers $k_1, k_2, \dots, k_n$:

$$k_1 + k_2 + \cdots + k_n \leq k_1 k_2 \cdots k_n + n - 1.$$

Solution 46-7.3 by Suvrit Sra, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

We prove that for a semidefinite matrix X, we have $\mathrm{tr}(e^X) \leq e^{\mathrm{tr}(X)} + n - 1$, with equality if and only if $rank(X) \leq 1$.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of X, and let $u_i = e^{\lambda_i} - 1$. The the above inequality reduces to proving

$$1 + (u_1 + \cdots + u_n) \leq \prod\nolimits_i (1 + u_i),$$

which is trivially true, because the right-hand side above equals

$$1 + (u_1 + \cdots + u_n) + \text{cross-terms},$$

where all the cross-terms are nonnegative.

Now, if the matrix has rank 0, then all the $u_i = 0$, if it has rank 1 then suppose $u_1 > 0$. In both cases, equality trivially holds above. If two or more $u_i \neq 0$, then there will be a positive cross-term, and the inequality will be strict. Conversely, by looking at the cross terms we again see that equality happens only when all the cross-terms are zero, which implies that the matrix must be of rank less than or equal to 1.

Editorial note: The problem is also solved by Eugene A. Herman, Minghua Lin and Henry Wolkowicz, Omran Kouba, Leo Livshits, and Hans Joachim Werner. The proposer notes that this is immediate from an application of the following theorem to the function $e^x - 1$. Theorem. If $f : \mathbb{R}_+ \mapsto \mathbb{R}$ is a strictly convex function with $f(0) \leq 0$, then $\sum_{i=1}^n f(x_i) < f\big(\sum_{i=1}^n x_i\big)$ for nonnegative numbers $x_1, \dots, x_n$, with at least two x_i nonzero. See [F. Zhang, *Matrix Theory*, 2nd edition, Springer 2011, p. 340.]

IMAGE Problem Corner: New Problems

Problems: We introduce 7 new problems in this issue and invite readers to submit solutions for publication in IMAGE. Solutions: We present solutions to all problems in the previous issue [IMAGE 46 (Spring 2011), p. 48]. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to IMAGE Problem Corner editor Bojan Kuzma (kuzma@fmf.uni-lj.si). Fuzhen Zhang has finished his term as IMAGE Problem Corner editor. He thanks everyone involved in this column for his/her support and help.

NEW PROBLEMS:

Problem 47-1: Idempotent Matrices

Proposed by Johanns de Andrade Bezerra, *Jardim Paulistano, Campina Grande - PB - Brasil*, pav.animal@hotmail.com

Let A and B be $n \times n$ idempotent matrices, I_n be the $n \times n$ identity matrix, and $X = Im(A) + Im(B)$. Show that

$$dim\big(Ker(I - AB)\big) = dim(ImA \cap ImB) + dim\big(X \cap Ker(A) \cap Ker(B)\big).$$

Problem 47-2: Another Characterization of Normality

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland,* obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany,* trenkler@statistik.tu-dortmund.de

Let A be a square matrix with complex entries and let $A^\dagger$ and A^* denote the Moore–Penrose inverse and the conjugate transpose of A, respectively. Show that A is normal if and only if the column space of A^* is contained in that of A, and $A^\dagger A^*$ is a contraction.

Problem 47-3: Matrices That Commute With Their Conjugate And Transpose

Proposed by Geoffrey Goodson, *Towson University, MD, USA*, ggoodson@towson.edu

It is known that if $A \in M_n$ is normal ($AA^* = A^*A$), then $A\bar{A} = \bar{A}A$ if and only if $AA^T = A^TA$. This leads to the question: do both $A\bar{A} = \bar{A}A$ and $AA^T = A^TA$ imply that A is normal? One can give an example to show that this is false when $n = 4$. In fact any matrix of the form $\left[\begin{smallmatrix} I_{ab} & I_{cd} \\ 0 & I_{ab} \end{smallmatrix}\right]$, where $I_{ab} = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$, $a, b, c, d \in \mathbb{C}$, $c^2 + d^2 = 0$, c and d not both zero, has this property. Show that if $A \in M_n$, $n = 2$ or $n = 3$, then $A\bar{A} = \bar{A}A$ and $AA^T = A^TA$ do imply that A is normal.

Problem 47-4: Completely Invertible Matrices

Proposed by Christopher Hillar, *Mathematical Sciences Research Institute, Berkeley, California, USA,* chillar@msri.org

Given an $n \times n$ complex matrix A, its *field of values* $\mathcal{F}(A)$ is the set (containing the eigenvalues of A) in the complex plane:

$$\mathcal{F}(A) := \{x^*Ax : x^*x = 1\} \subseteq \mathbb{C}.$$

(Here, x^* for a column vector x is its conjugate transpose $\bar{x}^\top$). Show that $0 \notin \mathcal{F}(A)$ if and only if $0 \notin \mathcal{F}(A^{-1})$.

Problem 47-5: Diagonalizable Matrix With Real Eigenvalues

Proposed by Roger Horn, *University of Utah, Salt Lake City, USA,* rhorn@math.utah.edu

Let $A_1, \ldots, A_N \in M_n(\mathbb{C})$ be Hermitian and positive definite and let $B = (\sum_{i=1}^N A_i)(\sum_{i=1}^N A_i^{-1})$. Show that B is diagonalizable and has real eigenvalues, and the smallest eigenvalue of B is not less than N^2.

Problem 47-6: An Exponential Order

Proposed by Minghua Lin, *University of Waterloo, Ontario, Canada,* mlin87@ymail.com

Let A and B be positive invertible operators on a Hilbert space H and let M and m be constants such that $M \geq A \geq m > 0$. If $\log A - \log B$ is a positive operator, show that $e^{\frac{M}{m}A} \geq e^B$.

Problem 47-7: Singular Value Inequalities

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Let $\lambda_i(X)$, $i = 1, 2, \ldots, n$, denote the ith largest eigenvalue of an $n \times n$ complex matrix X whose eigenvalues are all real. If A and B are $n \times n$ complex matrices, show that

$$\frac{1}{2}\lambda_i(A^*A + B^*B + A^*B + B^*A) \leq \lambda_i(AA^* + BB^*), \quad i = 1, 2, \ldots, n.$$

Solutions to Problems 46-1 through 46-7 start on page 31.