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FEATURE INTERVIEW

"Don't Be Afraid to Tackle New Problems"

Richard S. Varga Interviewed by Volker Mehrmann[1]

Richard Varga

V.M. - Dear Richard, I am very happy that you give me this interview. Let me begin with asking you: "how did you get started in linear algebra?"

R.V. - I was a graduate student in mathematics at Harvard University from the Fall of 1950 to the Spring of 1954, and I wrote a thesis in function theory, about the zeros of the partial sums of an entire function, such as $exp(z)$, which had **nothing** to do with linear algebra! But, in the early Spring of 1954, Dr. Henry Garabedian, who had retired from the Department of Mathematics at Northwestern University and had taken a job at the Westinghouse Bettis Atomic Power Laboratory in Pittsburgh, PA, came to Harvard to **recruit** young mathematicians to work on the huge mathematical problems arising in the design of nuclear reactors, which would power aircraft carriers, battleships, submarines, and land-based power plants. It was a totally new area which arose just when electronic computers made great forward strides. My situation was this. I was 1A in the draft, having been deferred in college and graduate school, and with a war going on in Korea, I was expecting to be immediately drafted into the U.S. Army on receiving my Ph.D. degree in June, 1954 from Harvard. But Dr. Garabedian assured me that I would be deferred to work on nuclear reactor designs, which would be more beneficial to the United States then my being a foot-soldier in the Army. (I agreed!)

Working on those nuclear reactors at Bettis was clearly fantastic, and everything was **new** and **exiting**. There were physicists, chemists, and engineers who needed their problems solved! Basically, most of these problems were huge eigenvalue problems, up to the size eight million by eight million. This is how I got started in this area.

My thesis, written under Professor J.L. Walsh, was in function theory, but I did have course work in linear algebra, from that famous book, "Survey of Modern Algebra," by Birkhoff and MacLane, but not much more! Actually, there was not that much out there in linear algebra then.

V.M. - Can you tell me what events, ideas or colleagues propelled your research and influenced the development of your main ideas?

R.V. - I had the good fortune of sharing my office, as a graduate student, with Dr. David M. Young, in the summer of 1951, who had just finished his Ph.D. thesis in 1950, under Professor Garrett Birkhoff, and this is how I learned about the SOR (successive over-relaxation method) for iteratively solving elliptic difference equations. I had no idea that it would be a very powerful and useful tool for solving many large reactor problems.

At Bettis in 1954, I suggested that Garrett Birkhoff be asked to be a consultant at Bettis, because of Garrett's very wide research contributions in many fields of mathematics. This was put into play, and we worked together at Bettis on many interesting problems. Two widely known papers of ours were "Reactor criticality and nonnegative matrices," in the *SIAM Journal* in 1958, and "Implicit alternating directions" in the *Transactions of the AMS* in 1959.

In a short time after arriving at Bettis, I was assigned the task of being the "lead-man" on applying SOR to reactor problems, based on my prior knowledge of this method. The hardest part was how to estimate a good relaxation constant for the method. This resulted in the two well-known papers with Gene H. Golub, "Chebyshev semi-iterative methods," which appeared in *Numerische Mathematik* in 1961. I published later a paper on "p-cyclic matrices, a generalization of the Young-Frankel successive overrelaxation scheme," which appeared in the in the *Pacific Journal of Mathematics* in 1959 and showed that I understood David Young's thesis!

One of the really interesting things about working at Bettis was sharing an office with Ely M. Gelbard, who joined Bettis just at the time I did. He too was a new Ph.D. in physics from the University of Chicago in 1954. Sad to say, Ely knew more **practical mathematics** than I did, and fortunately, I learned much from him in my Bettis days. (Ely won the Ernest Orlandi Lawrence Award in Physics, as well as the Eugene Wigner Award; he died in 2002.)

Others who have helped me in my research were Olga Taussky-Todd and her husband, John Todd, from the California Institute of Technology. I remember well discussing mathematics problems with Olga, while we visited Los Alamos National Laboratory in New Mexico. She was always very polite and encouraging, but I could never practice my German

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on her. Also, Alston Householder and I shared an office in Munich in 1965, having been invited by Fritz Bauer. Alston and I became very good friends. We ended up writing a paper, together with James H. Wilkinson, in *Numerische Mathematik* in 1970 on Geršgorin's Theorem, correcting a result in this area.

V.M. - What do you consider your main contributions in linear algebra?

R.V. - My older contributions have been in the linear algebra of iterative methods for solving large matrix problems, which includes SOR and ADI methods. Also, special topics like M-matrix theory, and generalized ultrametric matrices (jointly with Reinhard Nabben) were lots of fun. More recently, my attention has been on the Geršgorin-like estimates of eigenvalue problems, including Brauer ovals of Cassini and Brualdi lemniscates, which have appeared in my book [*Geršgorin and His Circles*, Springer-Verlag, 2004]. More recently, I have been working with Ljiljana Cvetković and Vladimir Kostić on estimating eigenvalues of generalized eigenvalue problems, $det(A - zB) = 0$, where A and B are $n \times n$ complex matrices with B singular. Also, I have been working with D. Noutsos on the generalization of the Perron-Frobenius theory of nonnegative matrices to general complex matrices. Again, lots of fun!

V.M. - What do you see are important topics that need to be addressed in linear algebra?

R.V. - In linear algebra, I believe that the area of **tensors** will be a very fruitful area of research, and I look forward to this, as others do also. The current results in this area are rather fragmented, but that is the normal story of advancing mathematics!

I have heard that Tom Laffey and others have made great strides recently on the Inverse Eigenvalue Problem for Nonnegative Primitive Matrices, which I have always found to be a fascinating open problem, arising from the beautiful work of Boyle and Handelman in 1991. I don't see here any waiting applications for this theory, but it is a beautiful area of research, one that I am following.

Another fascinating open problem, which also may not now have any applications, is the Riemann Hypothesis, an area which still interests me, and this is an area where I **have** contributed research results, mostly with George Csordas. I just can't put this down either!

V.M. - Can you tell us some anecdotes that would be interesting to our audience?

R.V. - The following is a true story. I was a first-year graduate student in mathematics at Harvard University, and Ted Rivlin, also a graduate student at Harvard in mathematics but two years ahead of me, lived in the same graduate dorm that I did. One day he came by and showed his calculations on the zeros of the first **four** partial sums of $exp(z)$, i.e, zeros of $1 + z, (1 + z + z^2)/2$, etc., and they all did lie in the open left-half plane! He then conjectured that **all** the zeros of all partial sums of $exp(z)$ lie in the open left half-plane! I worked on this problem and quickly showed that all these zeros stay away from the semi-infinite strip $\{z = x + iy : \ x \geq 0 \text{ and } \ |y| \leq 1\}$, which was not quite what Ted had conjectured. But then, Ted found out from Prof. Walsh, (his and my Ph.D. advisor), that G. Szegö had shown in 1924 that the zeros of the partial sums of $exp(z)$ definitely have zeros in the right hand plane, and, in fact, have a positive density in the right half-plane! But what I had was new, and it turned out to be my first published paper, in 1952. The moral here is "Don't be afraid to tackle new problems."

V.M. - What is your favorite linear algebra result?

R.V. - My favorite linear algebra result is a result of Olga Taussky-Todd from 1949. It states that if A is an $n \times n$ irreducible matrix, and if λ is an eigenvalue of A which does not lie in the interior of **any** Geršgorin disk of A, then the boundary of **each** Geršgorin disk passes through λ. It is just beautiful!

V.M. - Do you have any advice for young linear algebraists?

R.V. - Yes. Be willing to work on any problem that you think is interesting mathematics, whether it be in design of nuclear reactors, or in breast-cancer research.

V.M. - Thank you very much for the interview.

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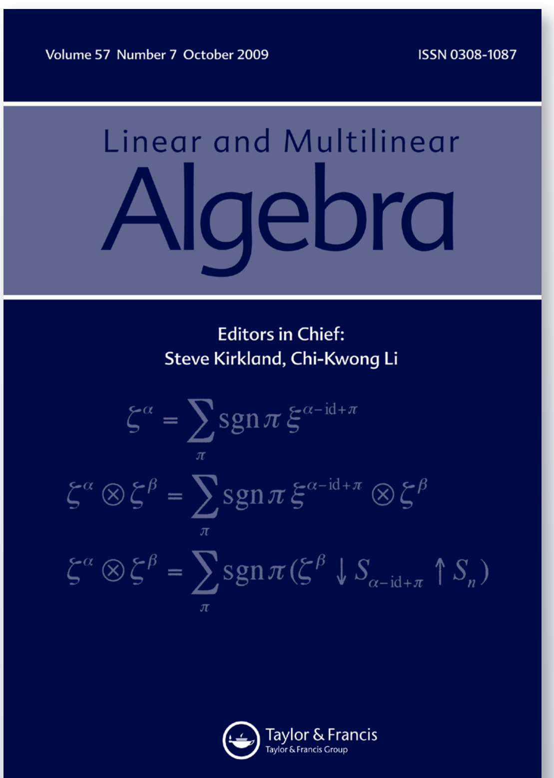

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ARTICLES

Linear Algebra in Croatia: *Spiritus Movens* and Curious Events

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Early history. Modern teaching of mathematics in Croatia started in 1876 with the appointment of Czech professor Karel Zahradník, from Prague, to the Royal University of Franz Joseph I in Zagreb.[1] He was the first professor of mathematics in Croatia, an autonomous province of Austro-Hungarian Empire at the time. In the academic year 1876–1877, Zahradník taught the courses *Algebarska analiza* and *O determinantih.* In the summer of 1878, he published a 39 page book *O determinantih drugoga i trećega stupnja.*[2] It is interesting that this book [78], written in Croatian,[3] is cited in Muir's book [58] on the history of determinants. In 1879, Zahradník published a Czech translation of the book.[4] The Croatian original of the book is cited in 1879 in the section *Publications récentes* in the *Nouvelles Annales Mathématiques, Journal des Candidats aux Ecoles Polytechnique et Normale* (Vol. 18). Another small detail, an illustration of the Zeitgeist, is to be found in the January issue of the political newspaper *Slovenec* (Ljubljana, 1879). After the news and discussions on the aftermath of the Berlin congress, placing Bosnia-Herzegovina under Austro-Hungarian administration, and other hot political themes of those turbulent times, Zahradník's little book on determinants is mentioned in the section of new publications in Croatia (other titles include historical and political themes, and e.g. the translation of the Jules Verne's *Une Fantaisie du Docteur Ox*).

In the winter semester of the academic year 1897-1898, Zahradník published the lecture notes *O determinantih* (112 pages). In 1882, he was elected to the *South Slavic Academy of Sciences and Arts* in Zagreb.[5] In 1899, he took his next mission and became the first rector of the Brno Czech Technical University. For more details on the life of this great teacher see [32], [7], and [8].

Modern teaching of linear algebra. The first modern linear algebra course at Croatian universities, known as AGLA (*Analytic Geometry with Linear Algebra*) was introduced by Stanko Bilinski [9] and Rajko Draščić in the Department of Mathematics of the University of Zagreb in the 1960s. In Zagreb, AGLA evolved into the two courses, *Linear Algebra*, and *Euclidean Spaces*, but AGLA is still taught at some Croatian universities with its original name and concept. The true *spiritus movens*[6] of the development of linear algebra as well as functional and numerical analysis, was Svetozar Kurepa. His 1967 book [41] has influenced many generations of students of mathematics and physics. Based on the course *Selected Topics in Mathematical Physics* (an obligatory course for students of mathematics, applied mathematics, and mathematical physics at the University of Zagreb), the almost-800-page book contains the theory of finite dimensional vector spaces, decompositions of matrices (operators), operator functions, unitary spaces, spectral theory, analytical perturbation theory, introduction to group representation theory, multilinear algebra, tensors, linear programming, and elements of numerical linear algebra. Krešo Horvatić and Mirko Polonijo were also teachers who left deep footprints teaching the theory of linear algebra and Euclidean spaces to many generations of students.

Research. Of course, it is not possible in this short memoir to give a truly detailed review of all current research activities and the complete historical perspective of linear algebra in Croatia. In the following, we just touch some of the themes of linear algebra and its applications that have been the focus of Croatian researchers. The list of references is by no means complete.

Modern research started in the 1960s and several groups emerged with independent developments in several directions of linear algebra and its applications. Some of the first results include the simple algebraic proof of sufficient conditions for simultaneous diagonalization of two Hermitian bilinear functionals by Kraljević's [38], the development of eigenvalue and eigenvector analytic perturbation theory [39], the work on condition number and column-wise distance to singularity [40], and various new results in numerical linear algebra. At the Institute of Mathematics of the University of Zagreb in the 1960s, numerical linear algebra focused on iterative methods for linear systems and generalized inverses. Those were also the pioneering days of implementations of numerical algorithms on electronic computers.

Sanjo Zlobec, now at the McGill University, published his first result on generalized inverses [79] in 1967 as a master student in Zagreb. In the same year he defended his master's thesis *Matrix Iterative Analysis in Finite Dimensional Spaces and Applications* under the supervision of S. Kurepa. In 1968, following a recommendation by Ljubomir Martić (who started teaching *Linear Programming* as a graduate course in the Mathematics Department in 1963), Zlobec moved to Northwestern University, and in 1970 completed his Ph.D. thesis [80] under the supervision of Adi Ben-Israel. He continued working on generalized inverses (e.g. [81]), mathematical programming, sensitivity and data envelopment

[1] The founding of the University was ratified by the Emperor in 1874.
[2] *On determinants of second and third order.*
[3] Zahradník published his work in Croatian, Czech, German and French.
[4] *Prvé počátky nauky o determinantech. Pro vyšší střední školy.* Praha, 1879.
[5] *Academia Scientiarum Et Artium Slavorum Meridionalium.*, now *Croatian Academy of Sciences and Arts.*
[6] driving force.

analysis. Subsequently, Zlobec and Luka Neralić from the Faculty of Economics in Zagreb (who spent some time at the Center for Cybernetic Studies in Austin) collaborated with Abraham Charnes.

The research in the area of numerical linear algebra very soon focused on matrix diagonalization methods. The beginning of this work is connected to a curious sequence of events. In the Mathematics Department in Zagreb in 1969, Krešimir Veselić was appointed to teach *Numerical Analysis*, a very new topic at that time. Given he was a functional analyst and mathematical physicist, he had no experience with numerical analysis. But a tall student fellow in the last row of the classroom already had a hands-on experience with numerical methods. He was Egon Zakrajšek from the University of Ljubljana who, for some formal reasons, had to take this course even though Egon was teaching a similar course in Ljubljana. For the final exam, Egon was allowed to pick his own topic and present it to the class. It was a brilliant presentation of the classical Jacobi method for computing eigenvalues of symmetric matrices, then completely unknown to Croatian researchers.[7]

Veselić[8] and Vjeran Hari, caught by the simplicity and elegance of the Jacobi idea,[9] started to apply the so called Jacobi-like methods to some classes of non-symmetric matrices – with mediocre success. Early works of this type are [73] and [29]. An important feature was the use of both trigonometric and hyperbolic plane transformations (rotations). In particular, Hari [28, 30] developed global and asymptotic convergence theory for general Jacobi diagonalization methods, and later [19], for general block transformations. Then there was a breakthrough showing the high relative accuracy of Jacobi SVD (see [16] and then [17, 15]). A final step was done in [20, 21] in which the high relative accuracy was coupled with a competitive efficiency. Both papers were awarded the 2009 SIAM/SIAG best paper prize and the Jacobi SVD software was included in the LAPACK software package. Similar but weaker results were obtained for the Jacobi eigenreduction of general indefinite symmetric Hermitian matrices in [74] and these were numerically analyzed in [65, 66]. All this was accompanied by a bulk of related lateral works concerning the convergence theory, perturbations and sensitivity under numerical perturbations [76]. Numerical linear algebra in spaces with indefinite inner products is particularly challenging; the main contributors have been Sanja and Saša Singer, [64, 31]. A series of workshops on accurate numerical linear algebra (IWASEP meetings) was initiated by the groups in Croatia, Germany (K. Veselić at FernUniversität in Hagen, and later Volker Mehrmann at Technische Universität in Berlin) and the United States (Jesse Barlow at Penn State and Jim Demmel at Berkeley). The series of workshops in Split 1996, Penn State 1998, Hagen 2000, Split 2002, Dubrovnik 2008,[10] Berlin 2010, and Napa Valley 2012 continues with the next meeting in Dubrovnik June 2-5, 2014.

The analytic perturbations of eigenvalues of matrix functions and variational principles for eigenvalues of pencils of Hermitian matrices are important topics that have been intensively studied, with early results in an operator setting [72], and important contributions by Branko Najman in the 1990s. A major contribution in Najman's research of the first topic was the use of Newton diagrams, which lead to a series of important papers [59, 42, 43, 44], and these developed techniques were later used to treat singular perturbations of linear and quadratic pencils [45, 60]. The most important paper from the line of investigation in the second topic [10] is a culmination of the research that spanned several papers. In the paper, the authors gave a complete variational characterization of eigenvalues of indefinite Hermitian pencils. This was later used by Ivica Nakić and K. Veselić [61, 62] to analyze perturbations of indefinite Hermitian matrix pairs. A series of a posteriori estimates based on Rayleigh quotients can be found in [18, 11, 25].

At the *Josip Juraj Strossmayer University of Osijek*, an active group of scientists led by Ninoslav Truhar, works on numerical linear algebra in control theory [5, 70], matrix perturbation theory [26, 47, 46, 68], linear matrix equations [6, 69], and linear vibrating systems [67]. Applications of linear algebra in mechanics is of particular interest in the graduate program. Lecture notes of a graduate course on oscillations of linear systems, taught by K. Veselić, have recently been published [75]. Other research activities in Osijek include the theory and practice of least squares approximations; the main contributions have been by Rudolf Scitovski, Dragan Jukić and Kristian Sabo. In cooperation with Peter Benner (Max-Planck-Institut, Magdeburg), Mark Embree (Rice University), Daniel Kressner (EPF Lausanne) and Z. Drmač (Zagreb), the Osijek group organized the *Summer School on Numerical Linear Algebra for Dynamical and High-Dimensional Problems* in Trogir, Croatia 2011.

An active research group at the Department of Mathematics in Zagreb works on various aspects of the theory of vertex algebras and infinite-dimensional Lie algebras. Dražen Adamović, Antun Milas (now at SUNY-Albany) and Ozren Perše have investigated representation theory of vertex algebras by applying algebraic and linear-algebraic methods. New rational vertex algebras were constructed in [1], a family of vertex algebras appearing in logarithmic conformal field theory in physics were investigated in [2, 3], and classification results for certain vertex algebras were presented in [4]. Construction of combinatorial bases and related combinatorial identities were discovered in papers by Mirko Primc, Tomislav Šikić, Miroslav Jerković and Goran Trupčević [37, 48, 77, 71]. Representation theory has a long tradition, starting with the initiative of Kurepa in the 1960s. Some main contributions have been developed by Marko Tadić,

[7]The Jacobi method, originally published in 1846, had been rediscovered by H. Goldstine, F. Murray and Von Neumann in 1951.

[8]In 1976, Veselić moved to the University of Dortmund and in 1979, to the FernUniversität in Hagen, but since then he engaged in lively correspondence and collaboration with his students and co-workers in Croatia. Eight Croatian students obtained their doctorate in matrix analysis and numerical linear algebra under his supervision in Hagen and then returned to Zagreb.

[9]For a long time, Zakrajšek was very skeptical of Jacobi methods, being fascinated by the efficiency of the QR family of algorithms.

[10]See the article *Good vibrations in Dubrovnik*, by Beresford Parlett in *IMAGE* 41 (Fall 2008). For more articles on IWASEP by Parlett, see the *SIAM News* issues from October 1996, March 1999, January 2001, December 2002, and November 2004.

Goran Muić, Dragan Miličić and Gordan Savin (D.M. and G.S. now at the University of Utah). A series of conferences on representation theory in the *Inter-University Centre* in Dubrovnik continues with *Representation Theory XIII*, June 19–30, 2013.

Linear preserver problems were studied in [23] within the scope of matrix theory, and in [36, 22] for operator theory. More precisely, Maja Fošner (University of Maribor), Dijana Ilišević and Chi-Kwong Li (College of William and Mary, Williamsburg) [23] initiated the study of generalized bicircular projections; Ajda Fošner (University of Primorska, Koper) and D. Ilišević [33, 34, 22] also developed this line of investigation. Auxiliary results in [22] are related to rank preserving linear maps and surjective linear isometries on the spaces of symmetric and antisymmetric operators. Furthermore, D. Ilišević and Aleksej Turnšek (University of Ljubljana) [36] studied orthogonality-preserving and approximately-orthogonality-preserving maps in the setting of inner product C^*-modules. D. Ilišević and Néstor Thome (Universitat Politècnica de València) [35] characterized all complex matrices A such that the Hermitian part, $H(A)$, respectively its skew-Hermitian part $S(A)$, is a potent matrix.

In the Department of Mathematics at the University of Rijeka, Dean Crnković is leading a group working in design and combinatorial matrix theory. The main topic of their research is the construction and analysis of combinatorial designs and related structures, such as Hadamard matrices, strongly regular graphs, and linear codes (see [12, 14]). They also study finite groups that act as automorphism groups of these structures (see [13]). This research usually involves the construction and analysis of certain $(0, 1)$ matrices.

The group led by Josip Pečarić is working on operator inequalities and operator means. Frank Hansen (Copenhagen University), J. Pečarić and Ivan Perić [27] presented a general formulation of Jensen's operator inequality for a bounded continuous field of self-adjoint operators, for a unital field of positive linear mappings, and for every operator convex function. The converses of Jensen's inequality are given in [56], and in [52] for real valued convex functions. Jensen's operator inequality for real valued continuous convex functions is presented in [50, 49] with conditions on the spectra of the operators. A refinement of Jensen's inequality and an extension are given in [54] and [53]. Some of the results on inequalities among operator means are in [24, 51]. Recently, by using [55, 63], an order among quasi-arithmetic means is introduced in [57].

In 2006, Pečarić co-founded a new journal on matrices and operators. Together with Leiba Rodman and Chi–Kwong Li, he was one of three founding Editors-in-Chief of *Operators and Matrices* (OaM), with the first issue published in March 2007 by *Element*, Zagreb.

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Linear Algebra in Italy: Before and After Computers

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After the advent of computers at the midpoint of the 20^{th} century, the interest for the computational aspect of mathematics enjoyed tremendous growth. In linear algebra this change has been very evident, fostered by the ubiquitous role of linear algebra in applications. In fact, after the invention of computers, research interests were quite focused on the design and analysis of linear algebra tools for the efficient solution of complex computational problems in applications, scientific computing, modeling, and simulations. Therefore, it seems natural to divide this short history of linear algebra in Italy into two parts: before and after the advent of computers.

Before the advent of computers. Since the 19^{th} century, several Italian mathematicians have published papers with contents related to linear algebra. At that time, specialization was not as strong as nowadays, and mathematicians were more eclectic in their research. Linear algebra appeared mainly as a tool to investigate more general problems and not as a discipline in its own. In the context of encouraging his students to become familiar with bilinear forms, in 1873 E. Beltrami provided the first published derivation of the singular value decomposition of a matrix, albeit for square nonsingular matrices [4, 58].[1] In 1887, V. Volterra [61] formalized the concepts of the derivative and integral of a matrix, and then generalized a Cauchy theorem to matrices [62]. In 1888, G. Peano [49] discussed the matrix function e^X as well as the Taylor series of a matrix. Then in 1894, G. Peano [50] provided a formalization of the abstract concepts of linear space.

Relevant contributions to linear algebra were given at the beginning of the 20^{th} century by A. Capelli who devoted an entire chapter of his book [16] to the theory of determinants, matrices and linear systems. His name is also bound to the Rouché-Capelli theorem and to the Capelli's identity related to the representation theory of Lie algebras. Capelli gave results on the rank of Sylvester matrices associated with two polynomials $a(x), b(x)$ and the degree of $\gcd(a(x), b(x))$ [17]. An extension to Bezout matrices was later provided by O. Nicoletti in 1909 [47]. It is exciting to read these papers today and find out how intriguing they appear.

From the beginning of the 20^{th} century, papers with a clear emphasis on linear algebra became more frequent. We provide a necessarily partial description of those contributions which, in our opinion, focus on linear algebra. Some results touch on topics that nowadays enjoy a wide interest and, even though dated at the beginning of the last century, are very modern in their contents, notations, and methodology. We are aware that we are neglecting other mathematicians who, in their more general studies, have skillfully used linear algebra tools to treat problems of a different nature.

One particular topic explores the problem of extending the concept of function from the scalar to the matrix case. For instance, the electrical engineer Giorgi, who invented the Giorgi system of measurement which is the precursor to the International System (SI), introduced the definition of matrix function based on the Jordan canonical form in a couple of papers dated 1928 [39, 40]. In the same year, L. Fantappiè [38] described the conditions that an analytic function of a matrix must satisfy. An important contribution was given by M. Cipolla [36]. In 1933, C.C. MacDuffee [44, p. 101] writes

> "*What seems to be the most satisfactory definition so far proposed for a multiple-valued analytic function of a matrix is the one given by Cipolla. It is an extension of the definition of Giorgi and differs from it only in the respect that different determinations for f may be used in the matrices $f(A_1), \ldots, f(A_k)$, and P must range over all matrices such that $A = P^I(A_1 \dotplus \cdots \dotplus A_k)P$ holds.*"

Other contributions of Cipolla involve extensions and simplifications of Hadamard bounds on determinants (1912), on the discriminant of algebraic equations (1917), and on matrix canonical forms (1921).

Explicit results in some special cases have been obtained by Botasso in 1913 for $f(X) = X^n$ when the minimum polynomial is quadratic [14], and by Martis in 1928 for matrices of order 2 [45] according to the definition of Fantappiè [38], and later on for the exponential function of matrices of order 2 [46] according to the definition of Giorgi [39, 40]. In the same period, Porcu-Tortrini [51] gave $f(X)$ explicitly for matrices of order 2 according to the definition of Giorgi, while Amante and Amato contributed to the analysis of matrix equations [1, 2, 3]. In 1909, Cecioni [22] discussed linear matrix equations of the form $AX = XB$. He discovered, independently of Frobenius, that the number of solutions of the equation $AX = XB$ is $\sum e_{i,j}$ where $e_{i,j}$ is the degree of the greatest common divisor of the invariant factor a_i of $\lambda I - A$ and the invariant factor b_j of $\lambda I - B$. Matrix equations were also analyzed in 1927 by L. Toscano [59].

Another subject of interest was the study of Riemann matrices, where G. Scorza [54, 55], C. Rosati [52], N. Spampinato [56, 57], and A. Lo Voi [42, 43] gave relevant contributions since 1916. Scorza also studied properties of Pfaffians [53]. An algorithm for the numerical solution of linear systems was devised by G. Cimmino [35] in 1938. This method, known

[1] Thanks to Steven Leon for bringing this event to our attention.

as Cimmino's method, is still used in certain applications. Several years later, L. Fantappiè [37] and S. Cherubino [23] reviewed the concept of matrix function.

Cherubino spent most of his academic life at the University of Pisa. He had a significant role in linear algebra because of his many original ideas, his great work on the foundations of linear algebra, and his prolific output of about 200 papers (as well as some monographs) during his long career (from 1911 to 1955). His monograph [34] on the theory of matrices is of particular importance. He provided a matrix canonical form and gave results on the form of commuting matrices, on the Sylvester-Hadamard theorem, on the holomorphic matrix functions, on the Volterra integrals with applications to systems of linear differential equations, on logarithms and roots of matrices, on Riemann matrices and more. In the last part of his career, he applied the theory of matrices to mathematical models in economics. We just cite a few papers concerning linear algebra [23]–[34]. Among the students of Cherubino, we cite in particular M. Passaquindici [48] and L. Cantoni [15].

An extensive bibliography concerning contributions in linear algebra until the 1930s, including many Italian mathematicians, can be found in the books by J.H.M. Wedderburn [63] and by C.C. MacDuffee [44].

Below we list the cited mathematicians with the universities where they taught:

Salvatore Amante (Roma, Palermo); Vincenzo Amato (1881-1963, Catania, Messina); Lionello Cantoni (Bologna); Alfredo Capelli (1855-1910, Pavia, Palermo, Napoli); Luigi Caprioli (1913-2007); Francesco Cecioni (1884-1968, Pisa); Salvatore Cherubino (1885-1970, Napoli, Siena, Padova, Messina, Pisa); Gianfranco Cimmino (1908-1989, Napoli, Cagliari, Bologna); Michele Umberto Leone Cipolla (1880-1947, Catania, Palermo); Luigi Fantappiè (1901-1956, Pisa, San Paulo, Roma); Giovanni Giorgi (1871-1950, Roma); Antonino Lo Voi (1900-1972, Palermo); Silvia Martis (1896-1959, Roma, Cagliari); Onorato Nicoletti (1875-1929, Pisa, Modena, Pisa); Maria Passaquindici (Pisa); Giuseppe Peano (1858-1932, Torino); E. Porcu-Tortrini; Carlo Rosati (1876-1929, Parma, Pisa); Gaetano Scorza (1876-1933, Pisa, Cagliari, Parma, Catania, Roma); Nicolò Spampinato (1892-1971, Palermo); Letterio Toscano (1905-?, Messina); Vito Volterra (1860-1940, Pisa, Torino, Roma).

After the advent of computers. With the advent of computers, a great impetus was given to research into the computational aspects of mathematics and, in particular, to research in the field of numerical linear algebra. Here we mainly focus our attention on this area which has a unified origin, and do not treat the field of core linear algebra where scientific production is more scattered.

In Pisa in the 1950s, with a financial support of the nearby provinces, work was begun on the first Italian electronic computer. The project, proposed by the physicist E. Fermi, was directed by the mathematician A. Faedo and the physicist M. Conversi with the sponsorship of the company Olivetti. In 1957, the CEP (Calcolatrice Elettronica Pisana) was built.

A group of engineers, physicists and mathematicians coalesced around this project with the goal of developing mathematical software for solving applied problems. In this framework, the interest for the computational aspects of linear algebra started to grow. M. Capovani was one of the main actors who developed this part of linear algebra. In the late 1960s he asserted the principle that problems of the real world, once modeled in terms of linear algebra, reveal matrix structures which reflect the specificity of the physical problem. His belief was that, only through the analysis of these structures, it is possible to design effective and fast algorithms for solving complex problems rooted in applications. One of the first problems he attacked involved investigating the structures of the inverses of tridiagonal matrices. Consequently, he published the pioneering papers [18], [19]. Later, he extended these results to block banded matrices [6]. Contributions of linear algebra to applications were given in [20] and [21].

Capovani then faced the problem of studying the complexity of matrix multiplication and tried to improve Strassen's algorithm. The underlying idea was still the same: to first analyze the structures behind the problem, in this case the tensor of matrix multiplication, and then to design algorithms. Capovani formed and directed a group made up by D. Bini, G. Lotti and F. Romani. The group improved Strassen's algorithm by introducing the new concepts of border rank and of approximate complexity [9]. The concept of border rank was a break-through and remains the basic ingredient of all the known fast algorithms for matrix multiplication (see Chapter 47 of [41]). The same concept was investigated for classes of structured matrices like the Toeplitz and band-Toeplitz class [8, 7].

Near the end of the 1960s, the time was ripe for creating a university degree in Computer Science (Scienze dell'Informazione) in Italy. This happened in Pisa; Capovani was one of its supporters. Later on, other universities joined. A fundamental course in the curriculum was Numerical Analysis. The course was characterized by a strong emphasis in numerical linear algebra. In this framework, the book *Metodi Numerici per l'Algebra Lineare* [10], played an important role in introducing young scholars to numerical methods in linear algebra.

The attention addressed to computational and algorithmic issues started to grow significantly. Other groups working on numerical linear algebra were formed in other universities and in research institutes like the CNR (the National Research Council). The group founded by Capovani proliferated and generated several branches in other universities. It is worthwhile to list some people, originating with the Capovani group, who gave an important impetus to the research in linear algebra in Italy: besides D. Bini (Pisa), G. Lotti (Parma) and F. Romani (Pisa), we mention R. Bevilacqua and O. Menchi (Pisa), P. Zellini (Roma 2); later on G. Del Corso (Pisa), F. Di Benedetto (Genova), C. Di Fiore (Roma 2), D. Fasino (Udine), G. Fiorentino (Livorno), L. Gemignani (Pisa), B. Meini (Pisa), S. Serra-Capizzano (Como) and Paolo Tilli (Torino).

Another branch, formed years later at the CNR in Pisa and, in part related to the Capovani group, included M. Arioli (now at Rutherford Appleton Laboratory, UK), B. Codenotti, P. Favati, A. Laratta, M. Leoncini, G. Resta. Even though their research was not entirely focused on linear algebra, they gave interesting contributions. The newest generation includes many young scholars, including A. Aricò (Napoli 2), P. Boito (Limoges), M. Donatelli (Como), C. Estatico (Genova), B. Iannazzo (Perugia), and F. Poloni (Pisa).

The research interests, developed inside this group, are characterized by structured matrix analysis, and touched many areas from theory to applications. The main topics of interest concerned Toeplitz and Hankel matrices, matrix algebras and preconditioners, band matrices and their inverses, tensor rank, complexity, spectral and computational properties, semiseparable and quasiseparable matrices, matrix and polynomial computations, matrix equations, matrix polynomials and matrix functions. The focus of much of the applied work research involved image restoration, Markov chains and queuing models. Many results have been given on the asymptotic spectrum of Toeplitz matrices, and applications to multigrid methods and conjugate gradient preconditioning.

The geography of today. Besides the branches of the Capovani groups that settled in Como, Udine, Genova, Pisa, Roma and Torino, other groups and scholars are involved more or less closely with linear algebra. Many of them treat linear algebra as a side topic of their interests in numerical analysis. Other groups perform their research on central topics in numerical linear algebra.

A geographical overview of the groups with research interests in numerical linear algebra follows. For space reasons we just give some pointers and we do not report bibliographic references which can easily be obtained by surfing on the Web. The list of names is far from being complete and we apologize to people we missed.

Important contributions to the research activity concerning numerical linear algebra and its applications have been given by the group from the University of Bologna. In particular, V. Simoncini developed results on Krylov subspaces methods, F. Sgallari and S. Morigi developed extensive applications of numerical linear algebra to image processing, and to the biomedical environment. Contributions have also been given by L. Montefusco, E. Loli Piccolomini and collaborators.

Among the Italian scholars who graduated in mathematics at the University of Bologna, and who spent the most part of their carrier abroad, we must mention M. Benzi (Emory University) and D. Calvetti (Case Western Reserve University). Both of them got their Ph.D. at the North Carolina State University. Benzi is one of the most authoritative scholars in numerical linear algebra and numerical analysis while Calvetti specializes in imaging and inverse problems.

The analysis of linear algebra tools for solving problems from engineering is the main research topic of the group working at the Faculty of Engineering of the University of Padova. In particular, many contributions in this direction were given by L. Bergamaschi and collaborators. At the same faculty, E. Fornasini, E. Valcher and collaborators have applied linear algebra to the analysis of dynamical systems. Also at the Faculty of Sciences of the same university, Michela Redivo-Zaglia has applied linear algebra to extrapolation.

Significant contributions to linear algebra from a numerical point of view were given at the University of Bari. In particular, D. Trigiante and the group that continues his work (including P. Amodio, L. Brugnano, F. Iavernaro, and F. Mazzia) provided applications to difference and differential equations, polynomial root finding, and stability of algorithms. Other results concerning applications of linear algebra to differential problems have been given by R. Peluso, L. Lopez and collaborators. One of their collaborators is L. Dieci, an Italian scholar now at the Georgia Institute of Technology. N. Mastronardi contributed to numerical linear algebra, in particular, focusing on structured matrix analysis with applications. After some time spent at the KU-Leuven where he has got his Ph.D., Mastronardi moved to the CNR in Bari at the Istituto per le Applicazioni del Calcolo. At the same institute, F. Diele and T. Laudadio work on applications of linear algebra, and I. Sgura, now at the University of Lecce, works on inverse problems.

Many different Italian universities have had groups working in linear algebra. The University of Cagliari has employed S. Seatzu, C. van der Mee and G. Rodriguez, with focused work on operator theory, infinite (multi-index) Toeplitz matrices, spectral factorization, least-squares problems, and preconditioners. At the University of Genova, the group headed by M. Bertero includes P. Brianzi and P. Boccacci. They worked on applications of linear algebra to inverse problems and image restoration. At the same university, C. Fassino applies linear algebra tools to problems in computer algebra. At the universities of Roma, several people are doing work in linear algebra and its applications addressing different facets. Among them, we cite D. Bertaccini, S. Fanelli, C. Manni, at Roma 2, and L. Pasquini and S. Noschese, at Roma 1. While the research at the University of Firenze mainly covers numerical analysis, it also contains contributions in numerical linear algebra. Some of the scholars involved are S. Bellavia, C. Conti, M.G. Gasparo, R. Morandi, B. Morini, and A. Papini. In Bergamo, the group headed by E. Spedicato specializes entirely on the ABS method. At the universities of Ferrara and Modena, S. Bonettini, V. Ruggiero, L. Zanni, and R. Zanella work on optimization and image deblurring. At the University of Calabria, A. Eisenberg treated the case of generalized Vandermonde matrices with some colleagues.

The group started by A. Bellen (Trieste), has specialized on delay differential equations. This group includes N. Guglielmi (L'Aquila), I. Moret (Trieste), P. Novati (Padova) and M. Zennaro (Trieste). They developed contributions on the properties of families of matrices, in particular on the joint spectral radius and related issues.

In evidence of the linear algebra activity of the Italian groups, many scholars have provided editorial service for important international journals. Here we list some of them together with the names of the journals: M. Benzi (*SIMAX*, *SINUM*, *SISC*, *Numer. Algorithms*, *NLAA*, *ETNA*, *Adv. Numer. Anal.*, *CAIM*, *IJCSM*), D. Bertaccini (*ETNA*), D. Bini (*SIMAX*, *Calcolo*), L. Brugnano (*JCAM*, *Appl. Math. Comp.*), B. Codenotti and P. Favati (*Calcolo*), N. Mastronardi (*SIMAX*), B. Meini (*LAA*, *Stochastic Models*), M. Redivo-Zaglia (*Int. J. Appl. Math. Eng. Sci.*, *J. Appl. Math.*, *Numer. Algorithms*), S. Seatzu (*Calcolo*), S. Serra-Capizzano (*Numer. Algorithms*), V. Simoncini (*SIMAX*, *SINUM*, *NLAA*, *ETNA*), E. Valcher (*Automatica*, *Multidim. Syst. Signal Process.*, *SICON*).

Several books on topics related to linear algebra and its applications are co-authored by Italian scholars: Mastronardi [60] has two volumes on semiseparable matrices; Bini, Iannazzo and Meini [11] wrote about algebraic Riccati equations; Bini, Latouche and Meini [12] published on numerical solution of Markov chains; and Bini and Pan [13] have a book on matrix and polynomial computations. Two chapters have been contributed to the *Handbook of Linear Algebra* [41], one on matrix multiplication and the other on Markov chains.

Many scholarly meetings focus on linear algebra topics. Several research projects concerning structured matrix analysis and applications have been granted by the MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca) in the years 1997–2012. A series of international conferences on topics of structured matrix analysis and applications have been organized in Cortona under the support of Istituto Nazionale di Alta Matematica (INDAM) in the years 1996, 2000, 2004, and 2008. In 2010, the 16$^{\text{th}}$ ILAS Conference was organized in Pisa. Almost every year, a two-day workshop on linear algebra is organized in different universities to trace the most current linear algebra research in Italy.

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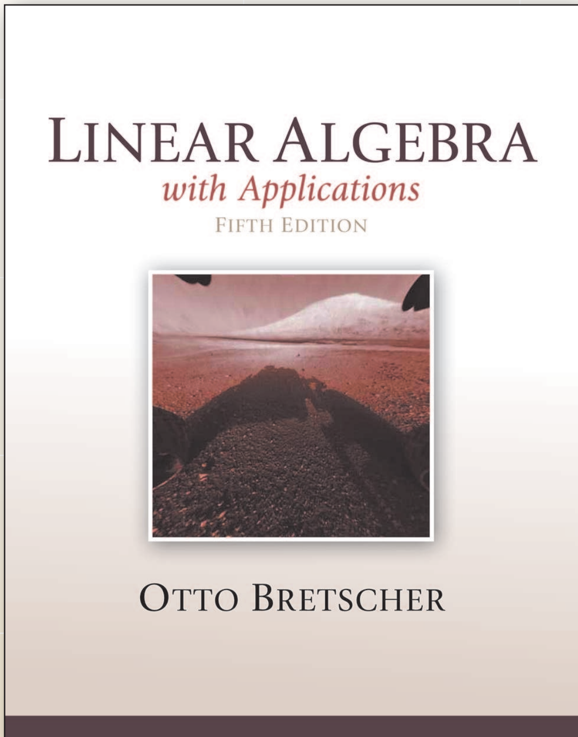

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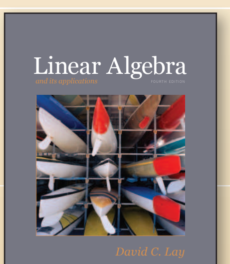

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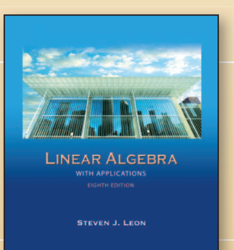

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JPEG Compression: The Big Picture

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Introduction: The transmission and storage of very large amounts of digital data require the compression of those data to meet bandwidth and memory space limitations. This is particularly true today given the preponderance and ubiquity of image, audio, and video files acquired by and shared among users of digital cameras, smartphones, tablets, and personal computers. Often some loss in data information is deemed acceptable in exchange for faster data transfer. Blocking artifacts in streaming videos and pixelations in images due to low resolution are common occurrences on the internet, where a greater premium is frequently placed on faster navigation than data fidelity. This paper discusses some of the mathematics underlying one particular image compression method (JPEG) that leads to significant reduction in data file size without resulting in large visible changes in image quality.

JPEG is one of the most commonly used compression standards for storing and transmitting images on the internet. It stands for the Joint Photographic Experts Group, which created it. The JPEG compression method is *lossy*, meaning it discards data information (generally those at high-frequencies, which contribute less to image quality) to achieve its high compression rate. The compressed image is generally visually indistinguishable from the original uncompressed image. JPEG compression works by expanding the function underlying the pixel intensities using continuous and differentiable basis functions (cosine functions with varying frequencies, in particular), which is why it is suitable for images of natural scenes depicting subjects with surfaces of smoothly varying color, tone, and intensity. In contrast, texts and graphics, whose pixel intensities can be viewed as piecewise constant functions, are less suited for JPEG compression.

The purpose of this expository article[1] is to describe the basic components of JPEG compression, including the underlying linear algebra that allows for this technique to be particularly effective (see, e.g., [9, 12] for further details). The four main steps in the compression process are discussed below.

JPEG compression: A digital image is composed of pixels (a portmanteau of the words "picture" and "elements"), which represent a discrete sample of the original image. An image that is N_x pixels tall and N_y pixels wide can be represented by an $N_x \times N_y$ matrix, F, whose elements, $F_{i,j}$ with $1 \le i \le N_x$ and $1 \le j \le N_y$, correspond to the pixel intensities (brightness) at the $(i, j)^{\text{th}}$ location of the image. These intensity values are stored as B-bit strings (a string of zeros and ones of length B) corresponding to 2^B different intensity levels. Typically, a grayscale image uses $B = 8$ bits per pixels, while for color images, $B = 24$ is used (8 bits for the red, green, and blue in an RGB color pixel, for example). For most of the images displayed over the internet, these bit sample lengths are sufficient, but for some applications like medical imaging, a longer bit string might be needed. Here, we limit our discussion to grayscale images to highlight the main linear algebraic aspects of JPEG compression. (For further discussion of compressing color images, see [3].) Thus we assume that the pixel intensities take on integer values between $2^0 - 1$ and $2^8 - 1$, i.e.,

$$F_{i,j} \in \{0, 1, 2, \ldots, 255\}, \quad \text{where} \;\; 1 \le i \le N_x, \; 1 \le j \le N_y.$$

For the 256×256 pixel "cameraman" input image in Fig. 1, using $256 \times 256 \times 8 = 524,288$ bits will guarantee that this image is represented exactly by storing the image brightness pixel by pixel. This can be viewed as expressing the matrix F as a linear combination of $N_x N_y$-many matrices of the same size as F that is 1 only at one place and zero everywhere else. This representation is often referred to as the pixel basis representation. The key step in JPEG compression is to exploit the fact that adjacent pixels in natural images have correlated intensity values, which is not accounted for when a pixel basis is used. Specifically, in JPEG compression, the image is mapped onto a basis in which images with smoothly varying color, tone, and intensity, are sparse, meaning only a few of the coefficients are nonzero or significant.

The JPEG encoding process consists of the following steps: (1) subsampling the whole image into smaller nonoverlapping subimages (data units), (2) mapping the data unit values into coefficients in the cosine basis using the discrete cosine transform (DCT), (3) thresholding nonessential coefficients using quantization, and (4) encoding the quantized DCT coefficients using variable-length codes. This process is illustrated in Fig. 1 and is described more fully below.

Subdivision into subimages: The first step in the JPEG compression process is to subdivide the original image into 8×8 blocks, which are compressed separately. When the dimensions of the image are not divisible by 8, a band of zeroes is appended to the image (zero-padding) so that both the length and the width are multiples of 8. Experiments have demonstrated that segmenting the image into 8×8 subblocks leads to significant reduction in memory storage without creating an overall image with visible blocks (blocking artifacts). Using 4×4 blocks results in less obvious blocks, but this limits the ability for further compression.

[1] Editorial note: A follow-up article, covering recent advances on image reconstruction that exploits compressibility, is due to appear in the next issue of *IMAGE*.

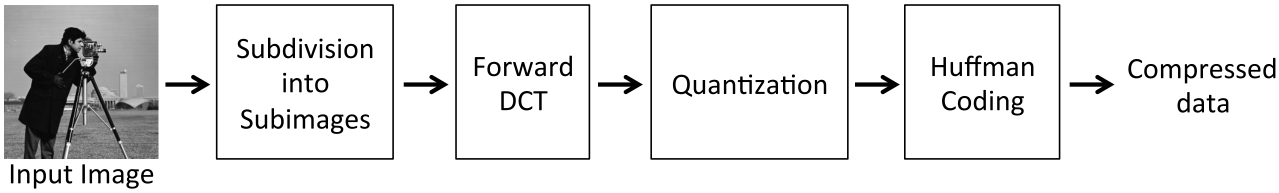


Figure 1: *The JPEG encoding system. First, the image is subsampled into nonoverlapping* 8×8 *subimages. Then each subimage is mapped onto the cosine basis function using the discrete cosine transform (DCT). The coefficients are quantized and ordered before finally being represented by variable-length codes (using Huffman coding) to obtain the final compressed data.*

Discrete cosine transform: The discrete cosine transform (DCT) is often used in signal processing especially for lossy data compression because most of the signal's energy is concentrated only on a few (low-frequency) components of the DCT. In other words, we can well approximate the signal using very few terms of the discrete cosine expansion. Here, we describe explicitly how the DCT coefficients can be computed. Specifically, we demonstrate how they can be computed efficiently since the DCT must be applied to every data subunit of the image. For simplicity, we will first consider the one-dimensional DCT.

1D DCT: Let f be an analog signal over $[0,1]$ (see Fig. 2(a)), and let f_j for $j = 1, \ldots, N$ be a uniform sampling of f at N evenly spaced intervals:

$$f_j = f(x_j), \quad \text{where} \ \ x_j = \frac{j - \frac{1}{2}}{N} = \frac{2j-1}{2N}, \quad \text{and} \ \ j = 1, 2, \ldots, N$$

(see Fig. 2(b)). The DCT can be viewed as the truncated cosine expansion of f:

$$f(x) \approx \sum_{n=1}^{N} \tilde{\theta}_n \cos\big((n-1)\pi x\big),$$

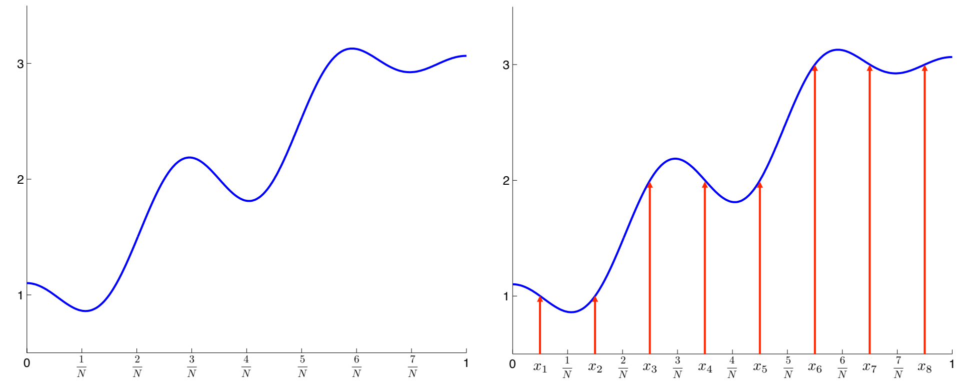
where $\tilde{\theta}_n \in \mathbb{R}$ for $n = 1, \ldots, N$ (see [1]). Specifically, the DCT determines the coefficients $\tilde{\theta}_n$ for the basis functions $\cos\big((n-1)\pi x\big)$ at each particular x_j:

$$f_j = \sum_{n=1}^{N} \tilde{\theta}_n \cos\big((n-1)\pi x_j\big) = \sum_{n=1}^{N} \tilde{\theta}_n \cos\left((n-1)\pi \frac{2j-1}{2N}\right), \quad \text{for} \ \ j = 1, 2, \ldots, N.$$

For example, for a signal of length $N = 8$, the corresponding linear system can be written as

$$\underbrace{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}}_{f} = \underbrace{\begin{bmatrix}
1 & \cos\frac{\pi}{16} & \cos\frac{2\pi}{16} & \cos\frac{3\pi}{16} & \cos\frac{4\pi}{16} & \cos\frac{5\pi}{16} & \cos\frac{6\pi}{16} & \cos\frac{7\pi}{16} \\
1 & \cos\frac{3\pi}{16} & \cos\frac{6\pi}{16} & \cos\frac{9\pi}{16} & \cos\frac{12\pi}{16} & \cos\frac{15\pi}{16} & \cos\frac{18\pi}{16} & \cos\frac{21\pi}{16} \\
1 & \cos\frac{5\pi}{16} & \cos\frac{10\pi}{16} & \cos\frac{15\pi}{16} & \cos\frac{20\pi}{16} & \cos\frac{25\pi}{16} & \cos\frac{30\pi}{16} & \cos\frac{35\pi}{16} \\
1 & \cos\frac{7\pi}{16} & \cos\frac{14\pi}{16} & \cos\frac{21\pi}{16} & \cos\frac{28\pi}{16} & \cos\frac{35\pi}{16} & \cos\frac{42\pi}{16} & \cos\frac{49\pi}{16} \\
1 & \cos\frac{9\pi}{16} & \cos\frac{18\pi}{16} & \cos\frac{27\pi}{16} & \cos\frac{36\pi}{16} & \cos\frac{45\pi}{16} & \cos\frac{54\pi}{16} & \cos\frac{63\pi}{16} \\
1 & \cos\frac{11\pi}{16} & \cos\frac{22\pi}{16} & \cos\frac{33\pi}{16} & \cos\frac{44\pi}{16} & \cos\frac{55\pi}{16} & \cos\frac{66\pi}{16} & \cos\frac{77\pi}{16} \\
1 & \cos\frac{13\pi}{16} & \cos\frac{26\pi}{16} & \cos\frac{39\pi}{16} & \cos\frac{52\pi}{16} & \cos\frac{65\pi}{16} & \cos\frac{78\pi}{16} & \cos\frac{91\pi}{16} \\
1 & \cos\frac{15\pi}{16} & \cos\frac{30\pi}{16} & \cos\frac{45\pi}{16} & \cos\frac{60\pi}{16} & \cos\frac{75\pi}{16} & \cos\frac{90\pi}{16} & \cos\frac{105\pi}{16}
\end{bmatrix}}_{\tilde{M}} \underbrace{\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \\ \tilde{\theta}_4 \\ \tilde{\theta}_5 \\ \tilde{\theta}_6 \\ \tilde{\theta}_7 \\ \tilde{\theta}_8 \end{bmatrix}}_{\tilde{\theta}}.$$

We note that the columns of $\tilde{M}$ are orthogonal to each other, and by properly scaling the columns, we can make $\tilde{M}$ an orthogonal matrix. Thus to convert a signal f of length 8, we apply the orthogonal *forward DCT* matrix to get the



(a) (b)

Figure 2: *(a) An example of an analog signal f. (b) Discrete sampling of f at uniformly spaced points $x_1, \dots, x_N$. In this case, $N = 8$.*

coefficient vector θ:

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ \theta_7\\ \theta_8\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}}\\
\frac12\cos\frac{\pi}{16} & \frac12\cos\frac{3\pi}{16} & \frac12\cos\frac{5\pi}{16} & \frac12\cos\frac{7\pi}{16} & \frac12\cos\frac{9\pi}{16} & \frac12\cos\frac{11\pi}{16} & \frac12\cos\frac{13\pi}{16} & \frac12\cos\frac{15\pi}{16}\\
\frac12\cos\frac{2\pi}{16} & \frac12\cos\frac{6\pi}{16} & \frac12\cos\frac{10\pi}{16} & \frac12\cos\frac{14\pi}{16} & \frac12\cos\frac{18\pi}{16} & \frac12\cos\frac{22\pi}{16} & \frac12\cos\frac{26\pi}{16} & \frac12\cos\frac{30\pi}{16}\\
\frac12\cos\frac{3\pi}{16} & \frac12\cos\frac{9\pi}{16} & \frac12\cos\frac{15\pi}{16} & \frac12\cos\frac{21\pi}{16} & \frac12\cos\frac{27\pi}{16} & \frac12\cos\frac{33\pi}{16} & \frac12\cos\frac{39\pi}{16} & \frac12\cos\frac{45\pi}{16}\\
\frac12\cos\frac{4\pi}{16} & \frac12\cos\frac{12\pi}{16} & \frac12\cos\frac{20\pi}{16} & \frac12\cos\frac{28\pi}{16} & \frac12\cos\frac{36\pi}{16} & \frac12\cos\frac{44\pi}{16} & \frac12\cos\frac{52\pi}{16} & \frac12\cos\frac{60\pi}{16}\\
\frac12\cos\frac{5\pi}{16} & \frac12\cos\frac{15\pi}{16} & \frac12\cos\frac{25\pi}{16} & \frac12\cos\frac{35\pi}{16} & \frac12\cos\frac{45\pi}{16} & \frac12\cos\frac{55\pi}{16} & \frac12\cos\frac{65\pi}{16} & \frac12\cos\frac{75\pi}{16}\\
\frac12\cos\frac{6\pi}{16} & \frac12\cos\frac{18\pi}{16} & \frac12\cos\frac{30\pi}{16} & \frac12\cos\frac{42\pi}{16} & \frac12\cos\frac{54\pi}{16} & \frac12\cos\frac{66\pi}{16} & \frac12\cos\frac{78\pi}{16} & \frac12\cos\frac{90\pi}{16}\\
\frac12\cos\frac{7\pi}{16} & \frac12\cos\frac{21\pi}{16} & \frac12\cos\frac{35\pi}{16} & \frac12\cos\frac{49\pi}{16} & \frac12\cos\frac{63\pi}{16} & \frac12\cos\frac{77\pi}{16} & \frac12\cos\frac{91\pi}{16} & \frac12\cos\frac{105\pi}{16}
\end{bmatrix}\begin{bmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\\ f_7\\ f_8\end{bmatrix}. \tag{1}$$

Denote the matrix in (1) by M, i.e., $\theta = Mf$, where $f = [\ f_1 \quad f_2 \quad \dots \quad f_N\ ]^T$. More generally, the signal f and DCT coefficient vector θ are related as follows:

$$\begin{aligned} \theta_n &= c_n \sum_{j=1}^{N} f_j \cos\left(\frac{2j-1}{2N}(n-1)\pi\right) \\ f_j &= \sum_{n=1}^{N} c_n \theta_n \cos\left(\frac{2j-1}{2N}(n-1)\pi\right) \end{aligned} \qquad \text{where} \quad c_n = \begin{cases} \sqrt{\dfrac{1}{N}} & \text{if } n = 1 \\ \sqrt{\dfrac{2}{N}} & \text{otherwise} \end{cases}. \tag{2}$$

We now show how multiplication by M in (1) can be done in a very efficient manner.

Fast multiplication by the DCT: Here we present a method for multiplying by M using a sequence of simple matrices. The following derivation is based on [2] and explicitly done in [8]. For other implementations and further information, see [6, 7, 10, 11]. First, we note that the entries of the fifth row of M alternate between $1/\sqrt{8}$ and $-1/\sqrt{8}$. Then if we let D be a diagonal matrix with diagonal entries $d_{1,1} = d_{5,5} = \sqrt{8}$ and $d_{j,j} = 2$ for $j \neq 1, 5$ and use the periodicity of the cosine function, we get

$$DM = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
c_1 & c_3 & c_5 & c_7 & -c_7 & -c_5 & -c_3 & -c_1\\
c_2 & c_6 & -c_6 & -c_2 & -c_2 & -c_6 & c_6 & c_2\\
c_3 & -c_7 & -c_1 & -c_5 & c_5 & c_1 & c_7 & -c_3\\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\
c_5 & -c_1 & c_7 & c_3 & -c_3 & -c_7 & c_1 & -c_5\\
c_6 & -c_2 & c_2 & -c_6 & -c_6 & c_2 & -c_2 & c_6\\
c_7 & -c_5 & c_3 & -c_1 & c_1 & -c_3 & c_5 & -c_7
\end{bmatrix}, \tag{3}$$

where $c_k = \cos(k\pi/16)$ for $k = 1, \dots, 7$. Now add the last column to the first, the seventh to the second, the sixth to the third, and the fifth to the fourth. This zeroes out half the entries in the first four columns. Similarly, add the first column to the negative of the last column, the second to the negative of the seventh, etc. Additionally, multiply by a factor of 1/2. Denote these actions by right multiplication by the matrix $\frac{1}{2}E$, where E is given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Permuting the rows by a permutation matrix P yields the matrix S given by

$$S = \frac{1}{2}PDME = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ c_2 & c_6 & -c_6 & -c_2 & 0 & 0 & 0 & 0 \\ c_6 & -c_2 & c_2 & -c_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_7 & c_5 & c_3 & c_1 \\ 0 & 0 & 0 & 0 & c_3 & c_7 & -c_1 & c_5 \\ 0 & 0 & 0 & 0 & -c_5 & -c_1 & -c_7 & c_3 \\ 0 & 0 & 0 & 0 & -c_1 & c_3 & -c_5 & c_7 \end{bmatrix}. \tag{4}$$

Thus $M = 2D^{-1}P^{-1}SE^{-1}$. Note that $E^{-1} = \frac{1}{2}E$, and since P is a permutation matrix, $P^{-1} = P^T$, $M = D^{-1}P^TSE$. Furthermore, we can write $D^{-1}P^T$ as $P^T\tilde{D}^{-1}$ where $\tilde{D}$ is a diagonal matrix with $\tilde{d}_{1,1} = \tilde{d}_{2,2} = \sqrt{8}$ and $\tilde{d}_{j,j} = 2$ for $j = 3, \ldots, 8$. Now, let $S_1 \in \mathbb{R}^{4\times 4}$ be the leading $(1,1)$ block of the matrix in (4). Note that we can write S_1 as

$$S_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ c_2 & c_6 & -c_6 & -c_2 \\ c_6 & -c_2 & c_2 & -c_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & c_6 & c_2 \\ 0 & 0 & -c_2 & c_6 \end{bmatrix}}_{\tilde{S}_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}}_{\tilde{E}_1}.$$

Let $S_2 \in \mathbb{R}^{4\times 4}$ be the 4×4 $(2,2)$ block in (4). Using the trigonometric identities

$$\begin{aligned} \cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\ \cos\alpha - \cos\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\pi}{2} - \frac{\alpha-\beta}{2}\right), \end{aligned}$$

we see that

$$\begin{aligned} \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{5\pi}{16} + \cos\tfrac{3\pi}{16}\right) &= \sqrt{2}\cos\tfrac{4\pi}{16}\cos\tfrac{\pi}{16} &= \cos\tfrac{\pi}{16}, \\ \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{7\pi}{16} + \cos\tfrac{\pi}{16}\right) &= \sqrt{2}\cos\tfrac{4\pi}{16}\cos\tfrac{3\pi}{16} &= \cos\tfrac{3\pi}{16}, \\ \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{\pi}{16} - \cos\tfrac{7\pi}{16}\right) &= \sqrt{2}\sin\tfrac{4\pi}{16}\cos\tfrac{11\pi}{16} &= \cos\tfrac{5\pi}{16}, \\ \tfrac{1}{\sqrt{2}}\left(\cos\tfrac{3\pi}{16} - \cos\tfrac{5\pi}{16}\right) &= \sqrt{2}\sin\tfrac{4\pi}{16}\cos\tfrac{9\pi}{16} &= \cos\tfrac{7\pi}{16}. \end{aligned}$$

Then S_2 can be written as

$$\begin{aligned} \begin{bmatrix} c_7 & c_5 & c_3 & c_1 \\ c_3 & c_7 & -c_1 & c_5 \\ -c_5 & -c_1 & -c_7 & c_3 \\ -c_1 & c_3 & -c_5 & c_7 \end{bmatrix} &= \begin{bmatrix} c_7 & (c_1 - c_7) & (c_1 + c_7) & c_1 \\ c_3 & (c_3 - c_5) & -(c_3 + c_5) & c_5 \\ -c_5 & -(c_3 + c_5) & -(c_3 - c_5) & c_3 \\ -c_1 & (c_1 + c_7) & -(c_1 - c_7) & c_7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_7 & -c_7 & c_1 & c_1 \\ c_3 & c_3 & -c_5 & c_5 \\ -c_5 & -c_5 & -c_3 & c_3 \\ -c_1 & c_1 & c_7 & c_7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} c_7 & 0 & 0 & c_1 \\ 0 & c_3 & c_5 & 0 \\ 0 & -c_5 & c_3 & 0 \\ -c_1 & 0 & 0 & c_7 \end{bmatrix}}_{\tilde{S}_2} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\tilde{E}_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\tilde{G}_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\tilde{H}_2}. \end{aligned}$$

If we let I_4 be the 4×4 identity matrix, we can define the block-diagonal matrices

$$\tilde{S} = \operatorname{diag}(\tilde{S}_1, \tilde{S}_2), \quad \tilde{E} = \operatorname{diag}(\tilde{E}_1, \tilde{E}_2), \quad \tilde{G} = \operatorname{diag}(I_4, \tilde{G}_2), \quad \text{and} \quad \tilde{H} = \operatorname{diag}(I_4, \tilde{H}_2),$$

so that $S = \tilde{S}\tilde{E}\tilde{G}\tilde{H}$, and therefore

$$M = P^T \tilde{D}^{-1} \tilde{S}\tilde{E}\tilde{G}\tilde{H}E. \tag{5}$$

When M is directly applied to the vector f in (1), the total number of multiplications is 64 and the number of additions is 56. However, using the decomposition in (5), by multiplying out and storing $\tilde{D}^{-1}\tilde{S}$, the total number of multiplications is 18 and the number of additions is 26.

2D DCT: Now consider an $N \times N$ image f, and suppose we want to compute its DCT coefficients. Similar to (2), the coefficients are computed as follows:

$$\begin{aligned} \theta_{m,n} &= c_m c_n \sum_{i=1}^{N}\sum_{j=1}^{N} f_{i,j} \cos\left(\frac{2i-1}{2N}(m-1)\pi\right)\cos\left(\frac{2j-1}{2N}(n-1)\pi\right) \\ f_{i,j} &= \sum_{m=1}^{N}\sum_{n=1}^{N} c_m c_n \theta_{m,n} \cos\left(\frac{2i-1}{2N}(m-1)\pi\right)\cos\left(\frac{2j-1}{2N}(n-1)\pi\right) \end{aligned} \qquad \text{where} \quad c_k = \begin{cases} \sqrt{\dfrac{1}{N}} & \text{if } k = 1, \\ \sqrt{\dfrac{2}{N}} & \text{if } k = 2, \ldots, N. \end{cases} \tag{6}$$

For an 8×8 data unit f, the 8×8 matrix of coefficients, θ, can be computed by using the M in (1):

$$\theta = MfM^T,$$

and f can easily be reconstructed from θ since M is orthogonal: $f = M^T\theta M$. For an 8×8 subimage of the "cameraman" image (see Figs. 3(a) and (b)), the matrix of pixel intensities f and the DCT coefficients θ are given as follows:

$$f = \begin{bmatrix} 184 & 187 & 193 & 180 & 170 & 171 & 166 & 65 \\ 185 & 189 & 181 & 158 & 115 & 135 & 154 & 123 \\ 155 & 118 & 90 & 77 & 44 & 28 & 77 & 138 \\ 102 & 78 & 56 & 35 & 19 & 14 & 43 & 102 \\ 139 & 104 & 47 & 25 & 36 & 90 & 140 & 141 \\ 188 & 158 & 95 & 68 & 172 & 198 & 186 & 188 \\ 198 & 193 & 164 & 154 & 201 & 209 & 204 & 210 \\ 174 & 172 & 166 & 178 & 190 & 202 & 208 & 209 \end{bmatrix}, \text{ and } \theta = \begin{bmatrix} 1092 & 21 & 134 & 38 & -12 & -1 & -9 & 7 \\ -136 & 127 & -30 & -45 & -16 & 44 & -8 & -3 \\ 322 & 22 & -133 & 16 & -34 & 19 & -8 & 2 \\ 101 & -23 & -22 & 88 & -47 & -9 & -5 & 14 \\ -96 & 0 & -20 & -8 & -10 & 23 & -4 & -4 \\ -25 & 22 & -28 & -7 & 20 & -15 & 4 & 9 \\ -18 & -12 & 8 & 21 & 17 & -11 & 0 & 9 \\ -8 & -1 & 26 & -8 & 2 & 16 & -8 & -3 \end{bmatrix}.$$

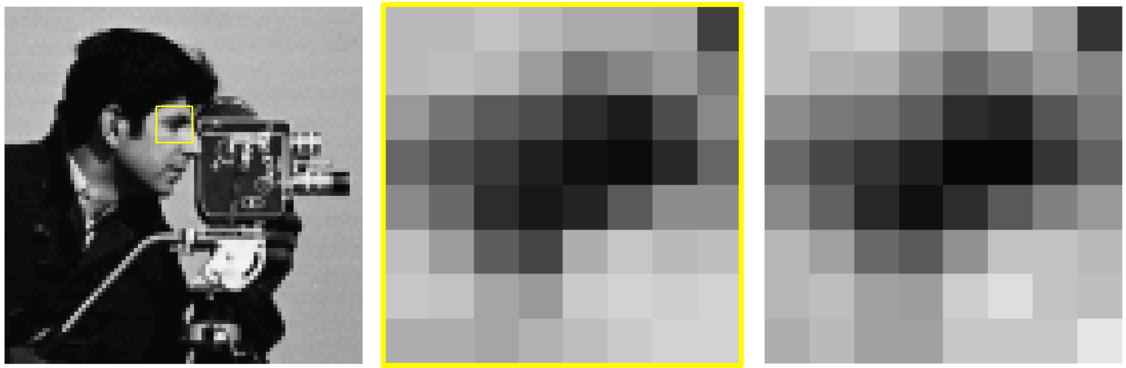
Note that the large coefficients (in magnitude) in θ are located on the upper left section of the matrix. That is because natural images generally consist of surfaces of objects, which have slowly varying values from point sample to the next. Hence, most of the signal is concentrated on low spatial frequencies, i.e., the large coefficients have low index values m and n in (6). High spatial frequencies tend to have small amplitudes. Thus they can be discarded in the encoding process, and the compressed image will still be an accurate representation of the original image.

DC coefficient: Note that the leading coefficient $\theta_{1,1}$ defined in (6) corresponds to the constant basis function. In fact, $\theta_{1,1}$ is simply the sum of the signal intensities divided by N. When normalized by an additional factor of $1/N$, this leading coefficient represents the *mean* of the signal f. In the example above, the mean of $f \approx 136.5 = \theta_{1,1}/8$. The leading coefficient of θ is often referred to as the DC coefficient while the other coefficients are called AC coefficients. (The names originate from electrical engineering, where direct current (DC) refers to constant voltage while an alternating current (AC) is sinusoidal.) Because the average value of an 8×8 data unit is highly correlated to adjacent block units, their DC coefficients are very similar. Thus, the DC coefficients are encoded as *differences* of the DC coefficients from the previous blocks.

Level shifting: Before the DCT is applied to an 8×8 subimage, the quantity $2^7 = 128$ is subtracted from each pixel value so that the elements in the resulting shifted matrix, $\bar{f}$, lie in the interval $[-128, 127]$. (By shifting the values of the subimage to this interval, the dynamic range of values needed for computing the DCT coefficients is reduced.) If $\mathbf{1}_{8\times 8}$ is the 8×8 matrix of ones and $\mathbf{1}_8$ is a vector of ones of length 8, then the DCT coefficients corresponding to this shift are given by

$$\bar{\theta} = M\Big(f - 128 \cdot \mathbf{1}_{8\times 8}\Big)M^T = MfM^T - 128 \cdot M\mathbf{1}_{8\times 8}M^T = MfM^T - 128 \cdot (M\mathbf{1}_8)(M\mathbf{1}_8)^T = MfM^T - 1024 e_1 e_1^T,$$

where e_1 is the first column of the 8×8 identity matrix. (Note that from (3), $M\mathbf{1}_8 = \sqrt{8}e_1$.) Thus shifting f by -128 is equivalent to decreasing the mean of the subimage by 128, which implies that its DCT coefficients are exactly the



(a) (b) (c)

Figure 3: *(a) A section of the "cameraman" image. (b) An 8×8 subimage of the eye of the cameraman zoomed in. (c) The decoded image from DCT compression and quantization.*

same except for the DC coefficient, which is reduced by $128 \times 8 = 1024$. In our "cameraman" subimage example, $\bar{\theta}_{1,1}$ is $\theta_{1,1} - 1024 = 68$.

Quantization: The goal of quantization is to further reduce the representation of the DCT coefficients without adversely affecting the quality of the compressed image. Rather than zeroing the coefficients of $\bar{\theta}$ by simple thresholding, i.e., letting $\bar{\theta}_{m,n} = 0$ if $|\bar{\theta}_{m,n}| \le \tau$ for some value $\tau > 0$, JPEG compression treats each coefficient $\bar{\theta}_{m,n}$ differently by assigning weights, $Q_{m,n} > 0$, to them individually:

$$\bar{\theta}^q_{m,n} = \text{round}\left(\frac{\bar{\theta}_{m,n}}{Q_{m,n}}\right).$$

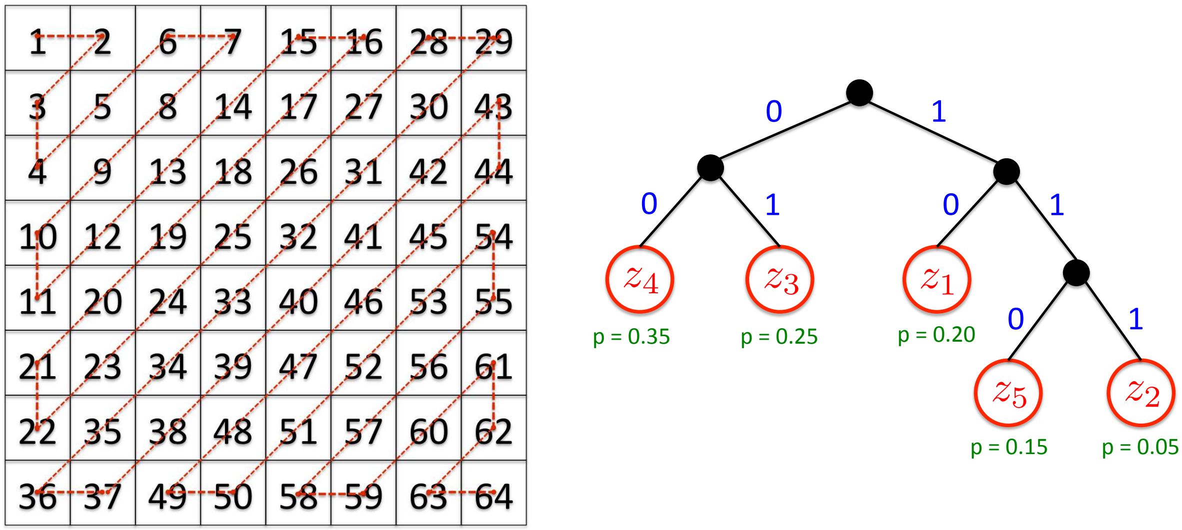
These weights are determined heuristically based on perceptual importance. Below are a typical normalization matrix, Q, (see [5]) and the (rounded) weighted DCT coefficients $\bar{\theta}^q$ after level-shifting and dividing by the weights in Q component-wise:

$$Q = \begin{bmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{bmatrix} \quad \text{and} \quad \bar{\theta}^q = \begin{bmatrix} 4 & 2 & 13 & 2 & -1 & 0 & 0 & 0 \\ -11 & 11 & -2 & -2 & -1 & 1 & 0 & 0 \\ 23 & 2 & -8 & 1 & -1 & 0 & 0 & 0 \\ 7 & -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that more than half of the coefficients now are 0. The decoded image, $\widehat{f}$, results from multiplying $\bar{\theta}^q$ by Q component-wise, applying the inverse DCT, and shifting each entry by $+128$. Fig. 3 demonstrates this concept. Fig. 3(b) is an 8×8 subimage of the eye in the "cameraman" image in Fig. 3(a). Fig. 3(c) is the decoded image from DCT compression and quantization discussed thus far. Note that the general shape of the eye is preserved, but some errors in detail are visible (the bottom-right pixel, for example). Quantitatively, the difference between the true image and the decoded image is the following:

$$f - \widehat{f} = \begin{bmatrix} -5 & -10 & -12 & 2 & 11 & -19 & 6 & 11 \\ -4 & 13 & 9 & 16 & 8 & 5 & -1 & -9 \\ 13 & -2 & -24 & -17 & -1 & -9 & -7 & 16 \\ -3 & 3 & 4 & 0 & 9 & 5 & -9 & 4 \\ 0 & 5 & 5 & 7 & -9 & -1 & 11 & -14 \\ 7 & -2 & -10 & -20 & 28 & 5 & -8 & 3 \\ 15 & 4 & 3 & -4 & -5 & -11 & 9 & 21 \\ 1 & -12 & 3 & 17 & -7 & 3 & 11 & -22 \end{bmatrix}.$$

This yields a relative error of $\|f - \hat{f}\|_F / \|f\|_F \approx 0.0716$, which is the error that is accrued for discarding about 60% of the quantized DCT coefficients. Scaling Q appropriately leads to higher compression and lower quality or lower compression but higher quality reconstruction. For example, dividing each element of Q by 2 usually results in an almost indistinguishable image from the original [9].



(a) (b)

Figure 4: (a) *Zigzag ordering for the quantized DCT coefficients starting from the upper-left* $(1, 1)$ *corner to the bottom-right* $(8, 8)$ *corner. Since the nonzero coefficients are concentrated near the upper-left corner, this ordering results in a long sequence of* 0*s at the end of the bit-string representation of the coefficients.* (b) *The Huffman coding algorithm treats each value in the string as a symbol and creates a binary tree, where the least frequently occurring symbols (e.g., those with the smallest probability of occurring) appearing farthest from the root. For each parent node, one child node is assigned* 0*, and the other,* 1*. The Huffman bit code for each symbol then is obtained by following the path from the root to the symbol and appending the value at each node to the code of the symbol.*

Zigzag ordering: The quantized DCT coefficients $\bar{\theta}^q$ are converted into a bit string where the low-frequency coefficients (which will be nonzero in general) are ordered first before the high-frequency coefficients. This is achieved using the zigzag pattern seen in Fig. 4(a). In our example, the sequence of coefficients using this ordering (not including the DC coefficient) is

$$[\,2\ \ -11\ \ 23\ \ 11\ \ 13\ \ 2\ \ -2\ \ 2\ \ 7\ \ -5\ \ -1\ \ -8\ \ -2\ \ -1\ \ 0\ \ -1\ \ 1\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 3\ \ -1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ \ -1\ \ 0\ \ -1\ \ \text{EOB}],$$

where EOB is a symbol for an end-of-block condition, meaning the rest of the coefficients are zeros.

Huffman coding: Rather than storing each element of the string above using its 8-bit representation, the JPEG encoding process uses a technique known as Huffman coding [4], which uses variable length codes and assigns shorter codes to more common values and the longer codes to less common values based on a predetermined estimated probability of occurrence. The Huffman coding algorithm treats each value in the string as a symbol and creates a binary tree, where the least frequently occurring symbols (e.g., those with the smallest probability of occurring) appearing farthest from the root. The easiest way of explaining how the binary tree is created is through an example.

Suppose we have five symbols, z_1, z_2, z_3, z_4, and z_5 with each symbol occurring with probability $p = 0.20, 0.05, 0.25, 0.35$, and 0.15, respectively. Huffman coding assigns the symbols to nodes based on decreasing likelihood of occurrence. In this case, the symbols are arranged as z_4, z_3, z_1, z_5 and z_2. Then, the last two nodes are binned onto a new node, which we denote $[z_5, z_2]$, with associated probability equal to the sum of the probabilities of z_5 and z_2 (in this case, 0.20). The new list of ordered nodes are z_4, z_3, z_1, and $[z_5, z_2]$ with probabilities $p = 0.35, 0.25, 0.20$, and 0.20, respectively. The process is repeated by binning the last two nodes z_1 and $[z_5, z_2]$ onto a new node $[z_1, z_5, z_2]$ with associated probability 0.40. Note that now this supernode $[z_1, z_5, z_2]$ has a higher probability than both z_4 and z_3. Thus the new list of ordered nodes then becomes $[z_1, z_5, z_2]$, z_4, and z_3 with probabilities $p = 0.40, 0.35$, and 0.25, respectively. The process continues by binning z_4 and z_3 together onto a new node $[z_4, z_3]$ with associated probability 0.60. The final two nodes $[z_4, z_3]$ and $[z_1, z_5, z_2]$ are then combined to get a binary tree containing all the nodes (see Fig. 4(b)).

Once all the symbols are assigned to the tree, for each parent node, one child node is assigned 0, and the other, 1. The Huffman bit code for each symbol then is obtained by following the path from the root to the symbol and appending the value at each node to the code of the symbol. Note that each symbol corresponds to a (unique) block code. For example, the code for z_4 is 00 and the code for z_5 is 110. Moreover, there is only one way of decoding a string of encoded symbols. For instance, 100111001 can only be decoded as $z_1 z_3 z_5 z_3$. And finally, each symbol can be decoded from a string without referring to succeeding symbols.

(a) (b) (c)

Figure 5: (*a*) *The uncompressed 65,536 byte "cameraman" image with the face and camera zoomed in.* (*b*) *JPEG compression using 20,014 bytes.* (*c*) *JPEG compression using 4,833 bytes. Using memory less than a third of the original image, the compressed image in* (*b*) *is visually comparable to* (*a*)*. In* (*c*)*, the compression artifacts are more noticeable, especially the ringing effects near the edges of the head and camera and the blocking artifacts on the top right.*

Example: We show the tradeoffs of JPEG compression between visual quality and memory storage on images, and in particular, on the 256×256 "cameraman" image. Fig. 5(a) is a zoomed-in section of the original uncompressed $65,536$-byte image. Note the strong (high-contrast) edges on the head and the fine details on the camera. Fig. 5(b) shows the same section but now of a JPEG compressed image that only uses $20,104$ bytes, or about 31% of the original file size. Note that the edges and details are well-preserved. Some faint ringing artifacts are visible, but it does not detract from the overall quality of the image. Fig. 5(c) shows the results for a JPEG compressed image that uses only $4,833$ bytes, or approximately 7.4% of the original file size. Here, the compression artifacts are much more noticeable. Note, in particular, the blurring of the edges and of some of the camera details, the ringing effects near the edges, and of the blocking artifacts on the top right of the image.

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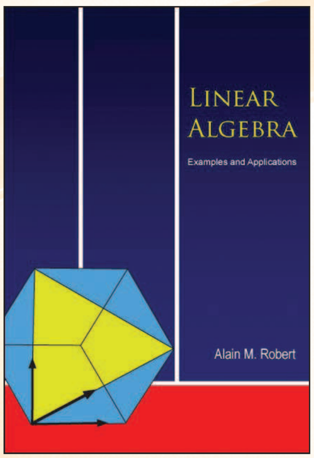

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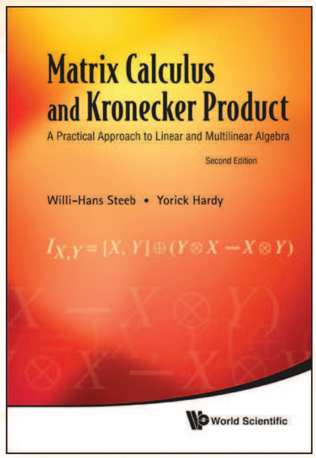

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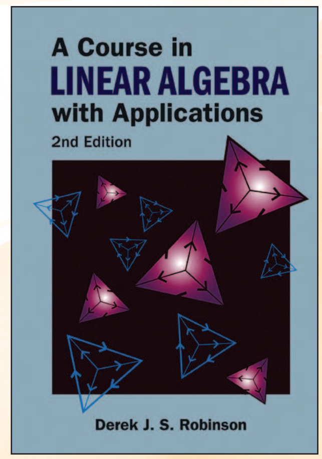

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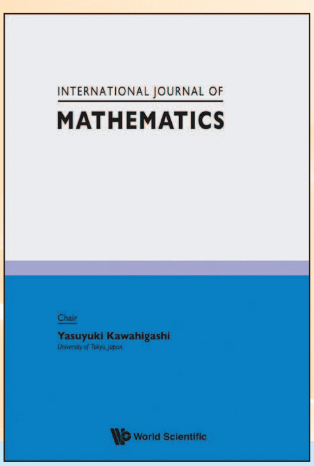

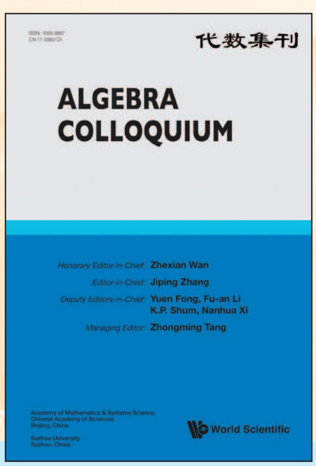

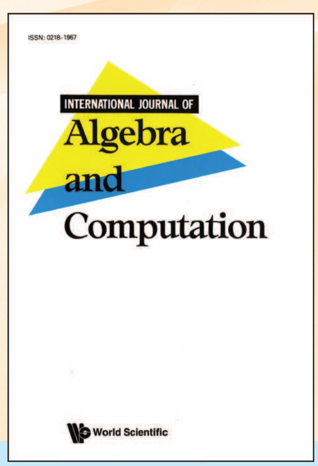

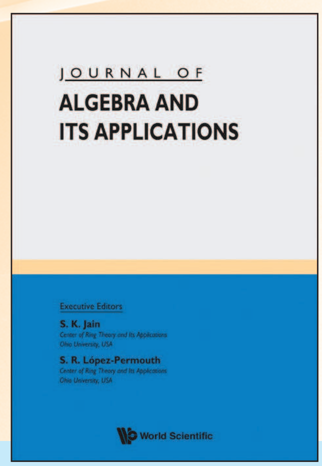

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UPCOMING CONFERENCES AND WORKSHOPS

6th de Brún Workshop, Linear Algebra and Matrix Theory: Connections, Applications and Computations NUI Galway, Ireland, December 3–7, 2012

The 6th de Brún Workshop, Linear Algebra and Matrix Theory: Connections, Applications and Computations will be held at NUI Galway, December 3–7, 2012.

The aim of the workshop is to bring together mathematicians with interests in various aspects of linear algebra and matrix theory, and their connections with other areas of pure and applied research. The workshop will include five short lecture courses given by Richard Brualdi (Wisconsin), Peter Brooksbank (Bucknell), Carlos Martins da Fonseca (Coimbra), Iain Duff (RAL and CERFACS), and Charles Johnson (William & Mary). There will also be a programme of contributed 30 minute research presentations. Graduate students are particularly encouraged to attend and participate. This workshop will be hosted by the de Brún Centre at NUI Galway, and is supported by Science Foundation Ireland.

For more details about the workshop, please see `http://www.maths.nuigalway.ie/deBrun6`. The workshop will be followed on Dec 8th by the inaugural Irish Meeting for Linear Algebra Research. For details, see `http://www.maths.nuigalway.ie/deBrun6/IrishLA.shtml`.

International Conference on Numerical Linear Algebra and its Applications (NLAA 2013) Guwahati, India, January 15–18, 2013

An International Conference on Numerical Linear Algebra and its Applications (NLAA 2013) will be held at the Indian Institute of Technology Guwahati, India, January 15–18, 2013. The conference will focus on all aspects of numerical linear algebra and its applications.

Confirmed invited speakers include: David Watkins (Washington State University, United States), Michael Overton (Courant Institute of Mathematical Sciences, United States), Volker Mehrmann (TU Berlin, Germany), Inderjit Dhillon (University of Texas, United States), Balmohan V. Limaye (IIT Bombay, India), Christian Mehl (TU Berlin, Germany), Michael Karow (TU Berlin, Germany), D. Steven Mackey (University of Western Michigan, United States), Valeria Simoncini (University of Bologna, Italy), G. W. (Pete) Stewart (University of Maryland, United States), Peter Benner (Max Planck Institute for Dynamics of Complex Technical Systems, Germany), Heike Faßbender (TU Braunschweig, Germany), Froilán M. Dopico (Universidad Carlos III de Madrid, Spain), Emre Mengi (Koc University, Turkey), G. Sajith (IIT Guwahati, India), Soumyendu Raha (Supercomputer Education and Research Centre, IISc Bangalore, India), Ivan Slapničar (University of Split, Croatia), Sk. Safique Ahmad (IIT Indore, India), Bibhas Adhikari (IIT Rajasthan, India), Biswa Nath Datta (Northern Illinois University, United States), Stephen Kirkland (Hamilton Institute, National University of Ireland, Ireland).

Early bird registration: December 15, 2012. For more details, go to `http://www.iitg.ernet.in/nlaa2013/conference/`.

New Frontiers in Numerical Analysis and Scientific Computing Kent (Ohio), United States, April 19–20, 2013

The conference New Frontiers in Numerical Analysis and Scientific Computing, will be held on April 19–20, 2013, at Kent State University, Kent, Ohio, U.S.A., on the occasion of Lothar Reichel's 60th birthday and on the 20th anniversary of the *Electronic Transactions on Numerical Analysis* (*ETNA*). Registration deadline for this conference is January 15, 2013.

The topics of this conference cover many areas of mathematics, where Lothar Reichel has made profound contributions and which are the primary areas that *ETNA* covers, including: fast iterative solutions of large systems of linear and nonlinear equations, iterative solutions of ill-posed problems, solutions of inverse problems, large-scale eigenvalue and singular value problems, structured problems in linear algebra, orthogonal polynomials, quadratures, mathematics for liquid crystals, and applications to biology.

For those who plan to organize special sessions in this conference, please send such information including the session title and tentative speakers to li@math.kent.edu.

This conference is supported by the National Science Foundation. A number of travel supports are available to encourage participation by individuals who lack other federal support, or who are students, post-doctoral scholars, junior faculty, to enhance the breadth and diversity of the participants. The support will cover all or part of the participant's transportation and local lodging expenses. Information on application of the travel support and other conference information are available on the conference website, `http://www.math.kent.edu/~li/LR60`.

18th ILAS Meeting
Providence, Rhode Island, United States, June 3–7, 2013

The 18th Conference of the International Linear Algebra Society (ILAS) will be held in Providence, Rhode Island, U.S.A. on June 3–7, 2013. Confirmed plenary speakers include:

- Alan Edelman (MIT),
- Maryam Fazel (University of Washington),
- Anne Greenbaum (University of Washington, SIAG LA Speaker),
- Ravi Kannan (Microsoft Research),
- Jean Bernard Lassere (Centre National de la Recherche Scientifique, France),
- Thomas Laffey (University College Dublin, Hans Schneider prize winner),
- Dianne O'Leary (University of Maryland),
- Ivan Oseledets (Institute of Numerical Mathematics, RAS),
- Leiba Rodman (College of William and Mary),
- Dan Spielman (Yale),
- Gilbert Strang (MIT),
- Raymond Sze (Hong Kong Polytechnic University) and
- Fuzhen Zhang (Nova Southeastern University).

The scientific organizing committee consists of Vadim Olshevsky (chair), Tom Bella (chair), Ljiljana Cvetković, Heike Faßbender, Chen Greif, J. William Helton, Olga Holtz, Steve Kirkland, Victor Pan, Panayiotis Psarrakos, Tin-Yau Tam, Paul Van Dooren and Hugo Woerdeman. The local organizers are Vadim Olshevsky (chair), Tom Bella (chair), Misha Kilmer and Steven Leon.

Titles and organizers of invited minisymposia are:

- Linear Algebra Problems in Quantum Computation (Chi-Kwong Li, Yiu Tung Poon)
- Matrices and Graph Theory (Louis Deaett, Leslie Hogben)
- Randomized Matrix Algorithms (Ravi Kannan, Michael Mahoney, Petros Drineas, Nathan Halko, Gunnar Martinsson, Joel Tropp)
- Symbolic Matrix Algorithms (Jean-Guillaume Dumas, Mark Giesbrecht)
- Krylov Subspace Methods for Linear Systems (James Baglama, Eric de Sturler)
- Matrices and Orthogonal Polynomials (Lothar Reichel)
- Matrix Methods for Polynomial Root-Finding (Dario Bini, Yuli Eidelman, Marc Van Barel, Pavel Zhlobich)
- Multilinear Algebra and Tensor Decompositions (Lieven De Lathauwer, Eugene Tyrtyshnikov)
- Nonlinear Eigenvalue Problems (Froilán M. Dopico, Volker Mehrmann, Françoise Tisseur)
- Abstract Interpolation and Linear Algebra (Joseph Ball, Vladimir Bolotnikov)
- Structured Matrix Functions and their Applications (Rien Kaashoek, Hugo Woerdeman)
- Linear Algebra Education Issues (Avi Berman, Sang-Gu Lee, Steven Leon)

In addition, the journal *Linear Algebra and its Applications* is pleased to announce a special issue devoted to papers presented at the 18th ILAS Conference.

Important deadlines:

- Minisymposia Submission: December 28, 2012; Acceptance/rejection: January 25, 2013.
- Submission of Abstracts for Contributed Talks: February 1, 2013; Acceptance/rejection date: March 1, 2013.
- All final abstracts are due March 15, 2013.
- Deadline for Early-bird Registration: March 15, 2013.

For further details, please visit the conference website at `http://www.ilas2013.com/`.

Numerical Analysis and Scientific Computation with Applications (NASCA13)
Calais, France, June 24–26, 2013

The main topics for this workshop include: large linear systems and eigenvalue problems with preconditioning; high-performance and parallel computation; linear algebra and control; multigrid and multilevel methods; numerical methods for PDEs: finite element, finite volume, and meshless methods; optimization; and applications: image processing, financial computation, and machine learning.

The plenary speakers for this workshop include: Peter Benner, Max Planck Institute, Germany; François Desbouvries, Télécom SudParis, France; Paul Van Dooren, Catholic University of Louvain, Belgium; Lothar Reichel, Kent State University, U.S.A.; Yousef Saad, University of Minnesota, U.S.A.; Roger Temam, Indiana University, U.S.A.; Ronny Ramlau, Johannes Kepler University, Austria.

The important dates are: January 15, 2013 (deadline for submission of abstracts), February 15, 2013 (authors will be notified of accepted papers), and March 1, 2013 (deadline for early registration).

For more information, please see the conference web page: `http://www-lmpa.univ-littoral.fr/NASCA13/`.

Advanced School and Workshop on Matrix Geometries and Applications
ICTP, Trieste, Italy, July 1–12, 2013

The workshop will be held at the International Centre for Theoretical Physics (ICTP). The ICTP was founded by Nobel Laureate Abdus Salam. Today it operates under a tripartite agreement among the Italian Government and two United Nations Agencies, UNESCO and IAEA. Its mission is to support the best possible science with special attention to the needs of developing countries. At the ICTP, high-level training courses, workshops and conferences are organized throughout the year. The participation of scientists from developing countries is supported by the ICTP.

This event will be organized by R. Bhatia and P. Šemrl with the help of F.R. Villegas (who works full time at the ICTP). The first week will be devoted to short introductory courses given by F. Barbaresco, R. Bhatia, D. Bini, J. Holbrook, L.-H. Lim, and P. Šemrl. Each day, four one-hour lectures will be followed by problem/tutorial sessions. The main theme is the use of ideas from geometry to solve several problems in matrix theory. Computational aspects and diverse applications (image processing, quantum mechanics, quantum information, etc.) will be presented. The workshop will take place in the second week. Talks on more advanced topics will be given by invited speakers. There will be no contributed talks in this workshop.

It is expected that besides lecturers and tutors at the Advanced School and speakers at the Workshop around 60 participants will attend this activity, approximately half from developed, and half from developing countries. The call for applications will be posted later this year at `http://cdsagenda5.ictp.trieste.it/full_display.php?ida=a12193`.

The 4$^{\text{th}}$ International Conference on Matrix Analysis and Applications
Konya, Turkey, July 2–5, 2013

The main goal of the conference is to gather experts, researchers and students together to present recent developments in this dynamic and important field. The conference also aims to stimulate research and support interactions between mathematicians and scientists by creating an environment for participants to exchange ideas and to initiate collaborations or professional partnerships. The first three meetings were held in China and the United States.

The conference is sponsored by the International Linear Algebra Society (ILAS). The conference features the keynote speaker Steve Kirkland (Stokes Professor at Hamilton Institute, National University of Ireland), and the ILAS Lecturer Alexander A. Klyachko (Bilkent University, Ankara, Turkey).

Invited speakers include Delin Chu (National University of Singapore, Singapore), Carlos Martins da Fonseca (University of Coimbra, Portugal), Fuad Kittaneh (Jordan University, Jordan), Stephan Garcia (Pomona College, California, U.S.A.), Chi-Kwong Li (College of William and Mary, U.S.A.), Mohammad Sal Moslehian (Ferdowsi University of Mashhad, Iran), Tom Pate (Auburn University, U.S.A.), Yiu-Tung Poon (Iowa State University, U.S.A.), Nung-Sing Sze (The Hong Kong Polytechnic University, China), Hugo J. Woerdeman (Drexel University, Philadelphia, U.S.A.), Pei-Yuan Wu (National Chiao Tung University, Taiwan, R.O.China), and Xiao-Dong Zhang (Shanghai Jiaotong University, China).

The scientific organizing committee (SOC) consists of Peter Šemrl, Tin-Yau Tam, Qingwen Wang, and Fuzhen Zhang. The local organizing committee (LOC) consists of Vildan Bacak, Durmus Bozkurt, Serife Burcu Bozkurt,Ahmet Sinan Çevik,Yildiray Keskin, Hasan Köse, Ayse Dilek Maden, Galip Oturanç, Vehbi Paksoy, Necati Taskara, Ramazan Turkmen, Zübeyde Ulukök, and Fatih Yilmaz.

For more information and updates, visit `http://icmaa2013.selcuk.edu.tr/`.

MatTriad 2013 – Conference on Matrix Analysis and its Applications
Herceg-Novi, Montenegro, September 16–20, 2013

The aim of this conference is to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications in other fields of mathematics and offer them a possibility to discuss current developments in these subjects. Researchers and graduate students in the area of linear algebra, statistical models and computation are particularly encouraged to attend the workshop. The format of this meeting will involve plenary talks and sessions with contributed talks. The list of invited speakers includes winners of Young Scientists Awards of MatTriad 2011 promoted by the conference. Works of young scientists continue to hold a special position at MatTriad 2013. The best poster as well as the best talk from graduate students or scientists with recently completed Ph.D.'s will be awarded. Prize-winning works will be widely publicized and promoted by the conference.

The invited speakers are Richard Brualdi (United States), Stephen Kirkland (Ireland), and the winners of Young Scientists Awards in MatTriad 2011 – Olivia Walch (United States) and Paulo Canas Rodrigues (Portugal). Lectures will be given by Siegfried M. Rump (Germany) and Adi Ben-Israel (United States).

The scientific committee consists of Tomasz Szulc (Poland) – Chair, Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany) and Simo Puntanen (Finland). The organizing committee consists of Ljiljana Cvetković (Serbia) – Chair, Francisco Carvalho (Portugal), Ksenija Doroslovački (Serbia), and Vladimir Kostić (Serbia).

Up to date information is available at `http://mattriad2013.pmf.uns.ac.rs`.

Householder Symposium XIX on Numerical Linear Algebra
Spa, Belgium, June 8–13, 2014

This symposium is the nineteenth in a series, previously called the Gatlinburg Symposia, and will be hosted by the Université Catholique de Louvain and the Katholieke Universiteit Leuven. The Symposium is very informal, with the intermingling of young and established researchers a priority. Participants are expected to attend the entire meeting. The fifteenth Householder Award will be presented for the best thesis in numerical linear algebra since January 1, 2011.

Attendance at the meeting is by invitation only. Applications will be solicited from researchers in numerical linear algebra, matrix theory, and related areas such as optimization, differential equations, signal processing, and control. Each attendee will be given the opportunity to present a talk or a poster. The application deadline will be some time in the Fall 2013. It is expected that partial support will be available for some students, early career participants, and participants from countries with limited resources.

The Householder Symposium takes place in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra. The conference website is `http://sites.uclouvain.be/HHXIX/`.

Send News for *IMAGE* Issue 50

IMAGE Issue 50 is due to appear online on June 1, 2013. Send your news for this issue to the appropriate editor by April 1, 2013. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

- Problems and Solutions to Bojan Kuzma (bojan.kuzma@upr.si)
- Feature Articles to Michael Cavers (mscavers@gmail.com)
- History of Linear Algebra to Peter Šemrl (peter.semrl@fmf.uni-lj.si)
- Book Reviews to Douglas Farenick (Doug.Farenick@uregina.ca)
- Linear Algebra Education news to Steve Leon (sleon@umassd.edu)
- Announcements and Reports of Conferences, Workshops and Journals to Minerva Catral (catralm@xavier.edu)
- Interviews of Senior Linear Algebraists to Carlos Fonseca (cmf@mat.uc.pt)
- Advertisements to Amy Wehe (awehe@fitchburgstate.edu)

All other news to Kevin N. Vander Meulen (kvanderm@redeemer.ca). This includes: honors and awards; funding opportunities; job openings; brief references to articles appearing elsewhere, of interest to the linear algebra community; announcements of new books, software, websites; transitions (new positions, obituaries); letters to the editor; small items like clever proofs, historical tidbits, cartoons, etc; and suggestions for *IMAGE*.

For past issues of *IMAGE*, please visit `http://www.ilasic.math.uregina.ca/iic/IMAGE/`.

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10/12

CONFERENCE REPORTS

Western Canada Linear Algebra Meeting (W-CLAM) 2012
Lethbridge, Canada, May 12–13, 2012

Report by Hadi Kharaghani

The 11th biennial Western Canada Linear Algebra Meeting (W-CLAM) 2012 was held at the University of Lethbridge, Lethbridge, Alberta, Canada on May 12–13, 2012. The invited speakers were: Ioana Dumitriu, University of Washington, Seattle, Washington; Chi-Kwong Li, College of William & Mary, Williamsburg, Virgina (ILAS speaker); and Heydar Radjavi, University of Waterloo, Waterloo, Ontario.

There were also 12 contributed talks and a poster presentation. The meeting was attended by 24 faculty, 10 graduate students, and 4 undergraduate students. The participants came from five Canadian provinces, the United States and elsewhere. In the words of one invited speaker, "The talks were wonderful and thought provoking. I am still working on some problems I learned at the meeting," and in words of a graduate student, "It is my first time attending [a] linear algebra conference and having the chance to meet many linear algebraists. An unforgettable experience."

The organizers (Shaun Fallat, Hadi Kharaghani, Steve Kirkland, Peter Lancaster, Michael Tsatsomeros, Pauline van den Driessche, and Wolf Holzmann) would like to thank ILAS for their generous support. Further information about the meeting may be found at `http://www.cs.uleth.ca/WCLAM/`.

Participants of the 11th W-CLAM

The 2012 SIAM Conference on Applied Linear Algebra
Valencia, Spain, June 18–22, 2012

Report by Michele Benzi, Rafael Bru Garcia, José Mas Marí

The 2012 SIAM Conference on Applied Linear Algebra, organized by the SIAM Activity Group on Linear Algebra every three years, took place on June 18–22 in Valencia, Spain, on the main campus of the Universitat Politécnica de Valencia. This was the second time the conference was held outside the United States. The main conference themes were: Iterative Methods and Preconditioning, Randomized Linear Algebra Algorithms, Inverse and Ill-posed Problems, Structured Matrices, Numerical Linear Algebra for Analysis of Complex Networks, Eigenvalue Problems, Optimization, Numerical Linear Algebra of Compressed Sensing, and Applications. These themes were discussed in twelve invited lectures, given by Hans De Sterck (University of Waterloo), Petros Drineas (Renssealer Polytechnic Institute), Lars Grasedyck (RWTH Aachen), Anshul Gupta (Thomas J. Watson Research Center, IBM), Roland Herzog (Chemnitz University of Technology), Des Higham (University of Strathclyde), Misha Kilmer (Tufts University), Juan Manuel Peña (Universidad de Zaragoza), Jared Tanner (University of Edinburgh), Françoise Tisseur (University of Manchester), Michael Tsatsomeros (Washington State University), and Chao Yang (Lawrence Berekeley National Laboratory). Tisseur and Tsatsomeros were sponsored by ILAS, while Grasedyck was sponsored by the GAMM Activity Group on Numerical Linear Algebra.

In addition to these invited lectures, there were 75 minisymposia (with typically 4 talks each) and 142 contributed talks. A novelty for this conference was the enforcement of a strict limit of one talk per person. This rule caused few complaints and was generally welcomed by the participants. The conference set an all-time record for attendance, with a total of 534 registered participants from about 50 countries covering the five continents. This number includes 113 students, a very high number which bodes well for the future of the field.

One of the highlights of the meeting was the award of the SIAG-LA Prize for the best paper in applied linear algebra over the preceding three years. The prize committee consisted of Andreas Frommer (Chair), Zlatko Drmač, Beresford Parlett, Dianne O'Leary and Steve Kirkland. Many high quality submissions were received, and the committee in the end decided to make two awards. The 2012 SIAG-LA Prize was jointly awarded to Grey Ballard, James Demmel, Olga Holtz, and

Oded Schwartz for their paper titled "Minimizing Communication in Numerical Linear Algebra" [*SIAM Journal Matrix Analysis and Applications* 32 (2011) 866–901], and to Sou-Cheng T. Choi, Christopher C. Paige, and Michael Saunders for their paper titled "MINRES-QLP: A Krylov Subspace Method for Indefinite or Singular Symmetric Systems" [*SIAM Journal on Scientific Computing*, 33 (2011) 1810–1836].

The conference dinner, held on Wednesday night (June 20) in the beautiful setting of la Vallesa de Mandor, allowed the participants to sample a variety of great Spanish foods and wines, and is likely to be fondly remembered by all who were there.

The next SIAM Conference on Applied Linear Algebra will take place in Atlanta, GA, in October 2015.

SIAM Applied Linear Algebra Conference Participants

ALAMA 2012 (3rd Biennial Meeting of the Spanish Thematic Network on Linear Algebra, Matrix Analysis and Applications) Leganés (Madrid), Spain, June 27–29, 2012

Report by Julio Moro

ALAMA 2012 *Conference Group Photograph*

The third edition of the Biennial Meeting of the Spanish Thematic Network ALAMA was held in Leganés, on the outskirts of Madrid, at the Polytechnic Campus of Universidad Carlos III de Madrid, from June 27th through June 29th, 2012. The ALAMA network was created in 2007, with the aim of gathering over one hundred spanish scientists whose research interests relate in some significant way to linear algebra, matrix analysis and/or their applications in different areas. The ALAMA Meetings have been held every other year since 2008, and are meant to be periodic get-togethers, giving members of ALAMA the chance to interact scientifically with other members of the network, as well as with participants at large (ALAMA Meetings are open to non-members).

There were over 70 participants and, although most were spaniards, the meeting was graced with the presence of colleagues from Chile, Colombia, Croatia, Germany, Israel, Santo Domingo, Tunisia and the United States. There were 38 contributed talks, and four plenary lectures, delivered by our outstanding invited speakers: Richard Brualdi, *Alternating sign matrices: origins, properties, and recent results*; Volker Mehrmann, *Hamiltonian matrices, even matrix pencils and self-adjoint operators*; Rafael Bru, *A Brauer's theorem and related results*; and Carlos Beltrán, *Some aspects of the precision problem in linear algebra.* Prof. Mehrmann acted as NICONET Plenary Speaker, thanks to generous support by the European Network on Numerics in Control.

This edition of the ALAMA Meeting was held in honor of Prof. Rafael Bru, one of the leading figures in the Spanish linear algebra community. The conference dinner, which took place inside Atocha train station, was a good opportunity to show the appreciation of ALAMA members for Prof. Bru's role in the growth and consolidation of the Spanish linear algebra network. First, Prof. José Mas gave an after-dinner speech on behalf of all colleagues from Prof. Bru's department at Universidad Politécnica de Valencia, reminiscing, both in words and with a simultaneous picture show, about more than 20 years of professional and personal contact. Then, Prof. Bru was presented with an inscribed plaque as a token of appreciation of fellow ALAMA members.

At the end of the meeting there was wide agreement among participants that this was a very fruitful meeting, with many interesting talks in a most congenial atmosphere. Further information on the meeting, including the book of abstracts, may be found at http://www.red-alama.es/encuentro2012/.

The next ALAMA Meeting is scheduled to take place in Barcelona in 2014, and will be held jointly with the annual Workshop of the GAMM Activity Group on Applied and Numerical Linear Algebra (GAMM/ANLA).

At lunchtime

Conference dinner: the new generation

The Seventh Workshop on Matrices and Operators
Harbin, Heilongjiang, China, July 13–16, 2012

Report by Chang-Jiang Bu and Chi-Kwong Li

The workshop was sponsored by the Harbin Engineering University, and was endorsed by the International Linear Algebra Society (ILAS). The workshop was organized by Chang-Jiang Bu (Harbin Engineering University, China) Chi-Kwong Li (College of William & Mary, United States), Xiaomin Tang (Hailongjiang University, China), and Qingwen Wang (Shanghai University, China). The meeting included over 40 talks on different aspects of matrix and operator theory, and related topics. The program and abstracts are available at http://people.wm.edu/~cklixx/2012maoschedule.doc and http://ilas.hrbeu.edu.cn/mao/ShowArticle.asp?ArticleID=21. There were more than 150 participants for the workshop from more 6 different countries. See http://people.wm.edu/~cklixx/mao2012.html for further information.

Matrices and Operators Group Photograph

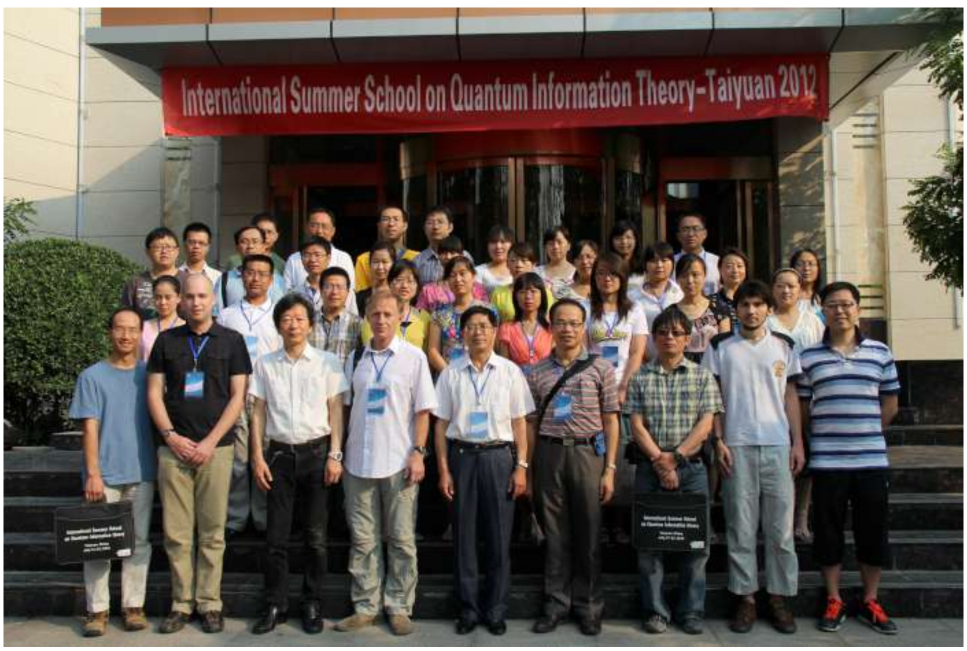
Summer Research Workshop on Quantum Information Science Taiyuan (Shanxi), China, July 17–27, 2012

Report by Jinchuan Hou and Chi-Kwong Li

The Summer Research Workshop on Quantum Information Science was held at the Taiyuan University of Technology, Taiyuan, Shanxi, China, July 17–27, 2012. The purpose of the summer workshop was to promote collaborative research on the rapidly developing interdisciplinary area of quantum information science. The focus was on the mathematical aspects of the subject.

Over 10 active researchers gave presentations at the workshop, including Yu Guo, Utkan Gungordu, Jinchuan Hou, Nathaniel Johnston, Chi-Kwong Li, Mikio Nakahara, Yiu-Tung Poon, Nung-Sing Sze, Yidun Wan, Shengjun Wu and Karol Zyczkowski. They presented recent results and open problems on different aspects of quantum information science such as entanglement, quantum operations, entropies, quantum discord, quantum error correction, NMR quantum computing, and preserver problems in quantum information science.

See the schedule at `http://cklixx.people.wm.edu/qc2012.html` and `http://www.tyut.edu.cn/mathinst/show.asp?id=90` (in Chinese) for details. Workshop reports and more were uploaded to `http://www.tyut.edu.cn/mathinst/list.asp?id=84`. There were more than 40 participants for the workshop including many graduate students; see `http://www.tyut.edu.cn/mathinst/show.asp?id=136`. Participants exchanged ideas, results and problems in a friendly atmosphere.



Summer Research Workshop on Quantum Information Science Group Photograph

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International Conference on Trends and Perspectives in Linear Statistical Inference and the 21st International Workshop on Matrices and Statistics Będlewo, Poland, July 16–20, 2012

Report by Francisco Carvalho and Katarzyna Filipiak

The 2012 edition of LinStat – the International Conference on Trends and Perspectives in Linear Statistical Inference, and the 21st IWMS – the International Workshop on Matrices and Statistics, was held at the Mathematical Research and Conference Center of the Polish Academy of Sciences in Będlewo near Poznań, Poland. Będlewo with its beautiful surroundings and privileged conditions to promote conferences made possible a get-together with highly considered specialists and researchers interested in the topics selected for these conferences.

The Scientific Committee of LinStat'2012 was chaired by Augustyn Markiewicz (Poland) and the International Organizing Committee of 21st IWMS was chaired by Simo Puntanen (Finland) holding George P.H. Styan as Honorary Chair. The Local Organizing Committee for both conferences was chaired by Katarzyna Filipiak (Poland).

The conferences were supported by the following: Stefan Banach International Mathematical Center, Institute of Mathematics of the Polish Academy of Sciences, Warsaw; Faculty of Mathematics and Computer Science, Adam Mickiewicz

University, Poznań; Institute of Socio-Economic Geography and Spatial Management, Adam Mickiewicz University, Poznań; and the Department of Mathematical and Statistical Methods, Poznań University of Life Sciences.

The purpose of the meeting was to bring together researchers sharing an interest in a variety of aspects of statistics and its applications as well as matrix analysis and its applications to statistics, and offer them a possibility to discuss current developments in these subjects.

Group Photograph of LinStat' 2012 *Conference Participants*

The plenary talks by invited speakers were delivered by Rosemary A. Bailey (U.K.), Rajendra Bhatia (India), Somnath Datta (U.S.A.), Thomas Mathew (U.S.A.), Paulo E. Oliveira (Portugal), K. Manjunatha Prasad (India) and Peter Šemrl (Slovenia) as well as by the Young Scientists awarded for the best talks and poster at LinStat'2010: Olivia Bluder (Austria), Chengcheng Hao (Sweden) and Paulo C.Rodrigues (Portugal).

Over 90 talks and 16 posters were presented, including a special session devoted to George P.H. Styan's 75$^{\text{th}}$ birthday (organized by Simo Puntanen). Other special sessions included High-Dimensional Data (organized by S. Ejaz Ahmed), Model Selection, Penalty Estimation and Applications (organized by S. Ejaz Ahmed), Robust Statistical Methods (organized by Anthony C.Atkinson), Optimum Design for Mixed Effects Regression Models (organized by Barbara Bogacka), Experimental Designs (organized by Steven Gilmour), Multivariate Analysis (organized by Dietrich von Rosen), Mixed Models (organized by Júlia Volaufová) and Matrices on Linear Models (organized by Hans Joachim Werner).

During the conference dinner, the Scientific Committee proudly announced the winners of the Young Scientists Awards for the best talks presented by Ph.D. students and young scientists: Maryna Prus (Germany), Jolanta Pielaszkiewicz (Sweden) and Fatma S. Kurnaz (Turkey). The winner for the best poster was Alena Bachratá (Slovakia). Awarded scientists will be Invited Speakers at the 2014 edition of LinStat that will be held in Linköping (Sweden).

Young Scientists Awards ceremony: (from the left) Jolanta Pielaszkiewicz, Dietrich von Rosen, Fatma S. Kurnaz, Simo Puntanen, Alena Bachratá, Maryna Prus and Augustyn Markiewicz

Special issues of *Communications in Statistics – Theory and Methods*, *Communications in Statistics – Simulation and Computation, and Discussiones Mathematicae – Probability and Statistics* will be devoted to LinStat'2012 and the 21$^{\text{st}}$ IWMS with selected papers strongly related with the presentations given at the conferences.

The participants shared the common opinion that the conference was extremely fruitful and well organized with friendly and warm atmosphere. The list of the participants of LinStat'2012 and the 21$^{\text{st}}$ IWMS, abstracts of the talks and posters, the gallery of photographs, and the other information can be found at `http://linstat2012.au.poznan.pl`.

Workshop on Structured Numerical Linear and Multilinear Algebra Problems: Analysis, Algorithms, and Applications Leuven, Belgium, September 10–14, 2012

Report by Fabio Di Benedetto, Dario Fasino, Luca Gemignani and Beatrice Meini

Belgium Workshop Participants

The workshop Structured Numerical Linear and Multilinear Algebra Problems: Analysis, Algorithms, and Application (`https://www.cs.kuleuven.be/conference/sla2012`) was held in Leuven from September 10 to 14, 2012.

The scientific committee consisted of Dario Bini (co-chair), Lieven De Lathauwer, Nicola Mastronardi, Annick Sartenaer, Marc Van Barel (co-chair), Raf Vandebril and Paul Van Dooren. There were 67 participants, and 49 talks were delivered. Participants came from 12 European countries, North America and Asia.

The subjects of the workshop were structured matrix and tensor analysis, design and analysis of algorithms for the solution of structured problems, applications in Markov chains and queuing models, image and signal processing, matrix equations, control theory, polynomial and matrix polynomial computations, polynomial eigenvalue problems.

During the meeting, many results of great interest were presented on theory, algorithms and applications concerning structured problems in numerical linear algebra. Participants had stimulating discussions on methodologies, algorithmic issues and applications which identified directions of future research and lead to improving collaborations between theoretical and applied research.

This workshop continues in the form and spirit of the series of conferences on structured matrices that were held every four years in Cortona, Italy, from 1996 until 2008. It is the plan to continue this series in the style of a "European traveling workshop" where different European countries take care of hosting and organizing the meeting. Due to the interest and the intense research activity in this area, the forthcoming meetings in this series could also be organized more frequently.

We wish to thank the organizers of the Leuven workshop, and in particular Marc Van Barel for the great organization, the high scientific quality of the conference, and for having kept alive the spirit of the Cortona conference series.

CIRM Workshop on Structured Matrix Computations in Non-Euclidean Geometries: Algorithms and Applications (SMC-NEGAA) Luminy, France, October 8–12, 2012

Report by Matthias Voigt

The CIRM Workshop on Structured Matrix Computations in Non-Euclidean Geometries: Algorithms and Applications (SMC-NEGAA) was held at the Centre International de Rencontres Mathématiques (CIRM) at the Luminy Campus of the Université d'Aix-Marseille, October 8–12, 2012. The conference was organized by Peter Benner (Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany), Lothar Reichel (Kent State University, U.S.A.), Miloud Sadkane (Université de Brest, France) and Ahmed Salam (Université Lille Nord de France, Calais, France). It brought together researchers from different fields working on applications of non-Euclidean geometries, such as geometric integration or control theory, with people focusing on the analysis and numerics of structured matrices in order to explore the usage of structured matrix computations in these applications.

The workshop had various key topics in its program. One focus was the analysis and perturbation theory of structured matrices and matrix polynomials. Talks in this field were given by Françoise Tisseur, *Structured matrix polynomials and their sign characteristics*; Christian Mehl, *Structured backward errors for eigenvalues of Hermitian pencils*; and Christian

Lubich, *Differential equations for Hamiltonian and symplectic matrix nearness problems*; among others. Another focus was on geometric integration methods with talks delivered by Marlis Hochbruck, *Exponential integrators: linear algebra aspects*; Elena Calledoni, *On geometric integrators for polynomial Hamiltonian systems*; and Peter Benner, *A step towards a symplectic exponential integrator.*

CIRM workshop participants

A particular highlight of the conference was a special session in honor of Axel Ruhe on the occasion of his 70$^{\text{th}}$ birthday. This session was devoted to recent developments in Axel Ruhe's working fields. Especially his works on rational Krylov subspaces were acknowledged by a plenary talk given by Bernhard Beckermann, *Rational Krylov revisited.*

The proceedings of the conference will be published in a special issue of the journal *BIT Numerical Mathematics* whose Editor-in-Chief is Axel Ruhe.

For more details on the program and the book of abstracts see `http://www-lmpa.univ-littoral.fr/SMC-NEGAA2012/`.

LAA Meeting
Madison, United States, October 10–14, 2012

Report by R.A. Brualdi, V. Mehrmann and P. Šemrl

Some current and former members of the LAA editorial board

After 40 years as an Editor-in-Chief of *Linear Algebra and its Applications* (LAA), Hans Schneider will step down at the end of 2012. In recognition of Hans' retirement as an Editor-in-Chief of LAA, Elsevier generously supported a three day conference in Madison for all current and former members of the LAA editorial board.

The conference was organized by Richard Brualdi. On the arrival day, October 10, a welcoming reception took place at the Lowell Center Guest House, where all the participants were staying. During the next three days, 30 half-hour talks were given by R. Varga, P. Lancaster, M. Fiedler, V. Sergeichuk, H. Bart, S. Friedland, D. Szyld, F. Dopico, L. Hogben, S. Fallat, P. Van Dooren, W. Barrett, H. J. Werner, I. Olkin, A. Boettcher, T. Laffey, B. Shader, R. Loewy, L.-H. Lim, E. Tyrtyshnikov, P. Butkovic, V. Nikiforov, F. Uhlig, A. Frommer, J. Barlow, C. Johnson, B. Lemmens, C.-K. Li, R. Horn, and V. Mehrmann. In addition to a lot of interesting mathematics, we enjoyed numerous stories illustrating Hans' contributions to LAA as well as his decisive role in the organization of the linear algebra community and the development of linear algebra as an important research field in mathematics. Most of speakers expressed their gratitude to Hans for helping them at various stages in their careers. There were also a few current and former LAA editorial board members who participated in the conference but did not give a talk: Michele Benzi, Richard Brualdi, Bryan Cain, Heike Faßbender, Joachim Rosenthal, Peter Šemrl, and of course, Hans Schneider.

On Friday afternoon a meeting of the LAA editorial board members with the LAA Editors-in-Chief and LAA Publisher Valerie Teng from Elsevier was organized. The conference ended on Saturday night with a gala LAA Editorial Board Dinner at Steenbock's at Orchard in the new Wisconsin Institute for Discovery. Richard Brualdi was the banquet speaker and he read some comments from Alan Hoffman, the first Editor-in-Chief of LAA. The dinner ended with some words of wisdom and gratitude from Hans Schneider.

Hans will now be a Distinguished Editor of LAA. We hope and expect to be able to call on him for advice and wisdom, and we wish him many more years of a happy and mathematically productive life.

HONOURS AND AWARDS

Two ILAS Board Members Receive Named Professorships

Leslie Hogben has been named the *Dio Lewis Holl Chair in Applied Mathematics* at Iowa State University. The Chair is intended for a professor who is "passionate about teaching and educating students who are going to be math teachers." Leslie Hogben is recognized for her leadership in the development of mathematics education courses at Iowa State University, both at the undergraduate level and in the Master of School of Mathematics program. She has also been actively involved in the Research Experiences for Undergraduates program in the Department of Mathematics since 2007. At the American Institute of Mathematics, where she serves as Associate Director of Diversity, she developed a program for faculty who are involved in research with undergraduates. For further information, see `http://www.math.iastate.edu/news/FP27.html`

Tin-Yau Tam has been selected as the *Lloyd and Sandra Nix Endowed Professor* at Auburn University. As noted on the department website, the named professorships support Auburn's mission to recognize "outstanding leadership in research, instruction and outreach." His extensive research record is cited, as well has his editorial work for *Linear and Multilinear Algebra* and the *Electronic Journal of Linear Algebra*. Tin-Yau Tam has received previous recognition, being named a SEC Academic Leadership Development Program Fellow for 2009-2010 and an Outstanding Graduate Mentor in 2011. He is also the new Chair of the Department of Mathematics and Statistics at Auburn. Further information can be found at `http://www.auburn.edu/academic/cosam/news/e-journey/archive/november2012/`.

MATHEMATICAL INTERLUDE

Confirmation that Column and Row Ranks of a Matrix Coincide

Oskar Maria Baksalary[1] and Götz Trenkler[2]

Let A be an $m \times n$ complex matrix and let A^* denote the conjugate transpose of A. We provide a concise confirmation of the familiar fact that $\mathrm{rk}(A) = \mathrm{rk}(A^*)$, where rk(.) stands for rank of a matrix argument.

It is known that when A and B are two conformable matrices, then $\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$; see for example [1, Lemma 2.5.2]. Another useful property is that the Moore–Penrose inverse of A satisfies both $A^\dagger = AA^\dagger A$ and $A^\dagger = A^*(AA^*)^\dagger$; see for example [1, Section 6.1]. Taking these facts into account, we have

$$\mathrm{rk}(A) = \mathrm{rk}(AA^\dagger A) \leq \mathrm{rk}(A^\dagger) = \mathrm{rk}[A^*(AA^*)^\dagger] \leq \mathrm{rk}(A^*). \tag{1}$$

Replacing A in (1) with A^* leads to the conclusion that $\mathrm{rk}(A^*) \leq \mathrm{rk}[(A^*)^*] = \mathrm{rk}(A)$, from where the equality $\mathrm{rk}(A) = \mathrm{rk}(A^*)$ follows. Recalling the fact that $AA^\dagger$ is Hermitian, instead of (1) we may also use the longer, but slightly simpler, relationship

$$\mathrm{rk}(A) = \mathrm{rk}(AA^\dagger A) \leq \mathrm{rk}(AA^\dagger) = \mathrm{rk}((AA^\dagger)^*) = \mathrm{rk}((A^\dagger)^* A^*) \leq \mathrm{rk}(A^*).$$

For alternative expositions of the result, see [1, Corollary 2.5.3], [2] or [3].

References

[1] D.S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas* (2nd ed.), Princeton University Press, Princeton, 2009.

[2] H. Liebeck, A proof of the equality of column and row rank of a matrix, in: *Linear Algebra Gems: Assets for Undergraduate Mathematics* (D. Carlson, C.R Johnson, D.C. Lay, A.D. Porter, eds.,), MAA Service Center, Washington DC, 2002, p. 171.

[3] D. Stanford, Row and column ranks are always equal, in: *Linear Algebra Gems: Assets for Undergraduate Mathematics* (D. Carlson, C.R Johnson, D.C. Lay, A.D. Porter, eds.), MAA Service Center, Washington DC, 2002, p. 169.

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Mathematics

LINEAR ALGEBRA EDUCATION

The following article is the first in a regular series of discussions on innovative and effective ways to teach linear algebra. Appropriately, we begin the series with a discussion that focuses on the first classroom lecture and illustrates how to effectively use a simple example to motivate and preview many of the major topics covered in the course. The author, David Strong, serves on the ILAS Education Committee. David is well-known for his work to promote innovations in the teaching of linear algebra. Each year he organizes the MAA (Mathematical Association of America) sessions on the teaching of linear algebra at the Annual Joint Meetings of the AMS (American Mathematical Society) and the MAA.

A Nickel and Dime Example for Introducing Several Important Linear Algebra Ideas on Day One

By David Strong, Pepperdine University, U.S.A.; David.Strong@pepperdine.edu

In any mathematics class, examples that involve multiple concepts are of great educational value. Such an example is of even more use if it is simple enough to be introduced on the first day of class. In teaching linear algebra, after the usual preliminaries of the first day of class, I give my students the following problem, which leads to a number of important questions, answers, and observations about linear algebra.

Suppose you have a large collection of nickels and dimes. Consider the following possible restrictions:

1. You have six coins.
2. You have five times as many dimes as nickels.
3. You have 75 cents.

How many nickels and dimes would you need in each of the following seven cases?

Case	Restrictions	Values of (n, d)
1	1	$(0,6),(1,5),(2,4),\ldots,(6,0)$
2	2	$(0,0),(1,5),(2,10),\ldots$
3	3	$(1,7),(3,6),(5,5),\ldots,(15,0)$
4	$1,2$	$(1,5)$
5	$1,3$	$(-3,9)$
6	$2,3$	$\left(\frac{15}{11},\frac{75}{11}\right)$
7	$1,2,3$	?

Of course I do not give the students the values of (n, d) listed in the final column.

After the students have worked for a few minutes, as a class we put solutions on the board, and come up with some of those listed in the final column above. Students can pretty easily find solutions for cases 1–4, but most students have not found solutions for cases 5–7. Regarding case 7, I simply tell them that it is a difficult problem and that we will talk about it in the next class. We end this first day of class, and I tell students to try to find solutions for all of the remaining cases. We resume our discussion the following day. By the beginning of day 2, some students have found solutions for cases 5 and/or 6. Of course none of them finds a valid solution for case 7, although many are not certain if that is because there is no solution or because they simply are not able to find it.

As much as possible, I try to allow my students to discover ideas on their own, especially when the ideas are intuitive and can be discovered about as efficiently as if I were to lecture about them. Thus as students work on this simple problem, we discuss a number of questions and we make a number of observations, most of which arise naturally, sometimes with a bit of direction from me. **Question:** What exactly is the solution to a system of linear equations? Answers from students include statements like “It’s the values of n and d which satisfy both equations” and “It’s the point where the two lines cross.” **Question:** How can you find solutions for cases 4–7 without simply guessing an answer? A few of the students have come up with the equations which correspond to the three restrictions, which as a class we now list:

$$
\begin{aligned}
n + d &= 6 \qquad (1)\\
5n - d &= 0 \qquad (2)\\
5n + 10d &= 75 \qquad (3)
\end{aligned}
$$

Observation: Each restriction can be described or represented with an equation. The restrictions describe relationships between the (unknown) values of n and d. **Observation**: It is more appropriate to write restriction (3) as $5n+10d = 75$ rather than as $d = -0.5n+7.5$ or $n = -2d+15$ as if either d or n were a function of the other. The values of n and d are *unknowns* rather than *variables*. Sometimes student mix up restriction (2) and come up with $n = 5d$ (so that $n-5d = 0$) rather than $d = 5n$.

With the equations, students can now generally find solutions for cases 5 and 6. **Observation**: Having mathematical equations (and more generally the language of mathematics) helps us to formulate problems and find solutions to these problems. Some students find a solution by solving for one variable in terms of the other in one equation and substituting that into the other equation. Others add/subtract an appropriate multiple of one equation to/from the other. **Observation**: Both methods accomplish the same thing, the elimination of a variable. I remind the class of this observation later when we discuss Gaussian elimination. **Observation**: Sometimes solutions are valid mathematically, but not in a real-world sense, as seen in both cases 5 and 6. We also see this in cases 1 through 3, in which there are mathematically an infinite number of solutions, but only a few that make sense in the real world. **Observation**: While students prefer (probably because they have been trained to find) the slope-intercept form of the equation for a line, the standard form of the equation is a more natural description of the relationship between the unknowns (rather than thinking of one unknown as a function of the other), and is much more useful in finding the solution to a system of linear equations, especially larger systems, as we see in more detail as we learn about Gaussian elimination. **Observation**: In standard form $ax + by = c$ (so that $y = -\frac{a}{b}x + \frac{c}{b}$), coefficients a and b determine the slope of the line, and for given a and b, it is c that determines the y-intercept of the line, which means that changing the value of c will shift the line up or down.

Of course a major element of linear algebra is visualization. **Observation**: Plotting the lines helps us to estimate a solution by seeing points of intersection. Of course, in the above problem, once we graph the three lines, it becomes obvious that there is no solution for case 7. **Observation**: If we have more than three unknowns, we simply can not graph the equations to find a solution. **Question**: How then do we find solutions in situations involving more than three unknowns? **Question**: Will whatever non-geometric approach we come up with also reveal to us when there is no solution or an infinite number of solutions? Both of these questions are answered with Gaussian elimination and its variations.

Observations: In the above problem with two unknowns, if needing to satisfy just one of the three restrictions, mathematically there are an infinite number of possibilities, even though only a few of them make real-world sense. With any two of the three restrictions, there is exactly one solution. With all three of the given restrictions, there is no solution. (**Observation**: In this case of no exact solution, there is still a "best" solution, that is, values of n and d which satisfy the three equations as well as possible. **Question**: How do you know exactly what the best solution is? We get to that later in the semester.) **Observation**: In general, the more restrictions there are, the more "difficult" it is for there to be a solution. More precisely, where m is the number of equations and n is the number of unknowns, typically the following occurs:

m vs. n	Number of equations	Number of solutions	Example, for $n = 2$
$m > n$	Too many	0	Three lines which do not intersect at one point.
$m = n$	Just right	1	Two lines, one point of intersection.
$m < n$	Not enough	∞	One line, and every point on line is solution

In the rightmost column, I draw the simple picture described. "Too many" means there are too many equations (restrictions) relative to the number of unknowns to determine an exact solution, that is, the system is overdetermined. "Not enough" means there are not enough equations relative to the number of unknowns to determine a unique solution, that is, the system is underdetermined.

Observation: (While they have not yet seen a rigorous proof of the fact, students already have seen in this example that) if a linear system has more than one solution it will have an infinite number of solutions. Of course it is important that at some point we give an example or two of a simple non-linear system with multiple (but still a finite number of) solutions.

Question: Is it possible to have an exact solution when there are too many equations relative to the number of unknowns?

For example, could we change the third restriction in the nickel and dime example so that there would be a number of nickels and dimes which satisfy all three restrictions? Algebraically, could we change the (right hand side of) the third equation so that there are values of n and d which satisfy all three equations? Geometrically, could we shift the third line up or down so that the three lines have a common point of intersection? The students of course discover that yes it is possible.

Question: More generally, is it possible to have 0, 1 or ∞ solutions in each of the three situations, when there are "not enough" equations, the "right number" of equations, or "too many" equations, relative to the number of unknowns? To answer this, I have my students attempt to come up with an example for each of these nine situations. **Observation**: Most, but not all, of these nine situations are possible. So we see that while the results described in the above table are common, they are not guaranteed.

All of the above seems like it could take a while to get through. However, the amount of time my students spend in discovering and observing and questioning as they solve this problem is not much more than the time spent in reading this article. And the benefit is that from day 1 the students develop the attitude of questioning and observing, and they develop some important intuition and discover (or at least become aware of) many basic principles, many of which naturally lead to other ideas and tools of linear algebra, such as augmented matrices, Gaussian elimination, pivoting, etc. My students and I make a number of observations – some quite important, some less so, but all of some worth – with this quick and simple example, and most of the observations the students are able to make on their own, with little direction, which I believe ultimately results in deeper learning.

As other concepts are subsequently introduced, we revisit this example. As they relate to the system of three equations in two unknowns corresponding to the original three restrictions, these concepts include:

- Column space: Is the right hand side $(6, 0, 75)$ in the column space of the coefficient matrix with the original three restrictions? What about in the case in which we modified the third restriction/equation, resulting in a solution which does satisfy all three restrictions/equations?
- Span: Do the two columns in the coefficient matrix span $\mathbb{R}^3$? What is the minimum number of columns from $\mathbb{R}^3$ needed to span $\mathbb{R}^3$, and how does this relate to the number of equations vs. the number of unknowns? And in general, how do pivot rows and pivot columns fit in to all of this?
- Least squares and the projection of one vector onto a collection of other vectors: If there is not an exact solution, how do you find the best solution, and what does "best" even mean?

Revisiting this example or any other at different times during the semester allows students to more easily see how the many ideas in linear algebra are connected. Of course many other examples and applications are also discussed as well.

It is likely that readers of this article will think of other observations and questions motivated by the above nickel and dime example, and I expect that some readers have effectively used their own introductory examples that involve a number of fundamental linear algebra ideas. (Indeed, I welcome feedback on additional questions and observations regarding the nickel and dime example, as well as other introductory examples, since I am currently in the process of compiling a collection of good introductory examples, both contrived – such as the nickel and dime example above – and more real-life, such as from image processing or traffic flow.) The point of this article is not this particular example, but that a solid example (or two) that can be discussed from day one and revisited during the semester is a very valuable teaching tool.

BOOK REVIEWS

Israel Gohberg and Friends, On the Occasion of his 80^{th} *Birthday*, Edited by Harm Bart, Thomas Hempfling, and Marinus A. Kaashoek

Birkhäuser, Basel-Boston-Berlin, 2008, xiv + 328 pages, ISBN 978-3-7643-8733-4.
Reviewed by Roger A. Horn, University of Utah; rhorn@math.utah.edu

A summary by the editors: "This book is dedicated to Israel Gohberg on occasion of his 80th birthday on August 23, 2008. It is an expression of esteem and friendship for a great mathematician, a remarkable person and an inspiring colleague. The book contains reflections by Gohberg himself on his own mathematical activities and those of others. It also includes contributions of colleagues and co-workers, both from his time in the Soviet Union and from when he lived and worked in the West. The contributions in question are not mathematical research papers but focus on the man Israel

Gohberg and are intended for a wide audience. To be found are letters, speeches, laudatios and reminiscences which, together, present an attractive picture of the way in which Israel Gohberg in the past four to five decades influenced the lives of many mathematicians. Biographical material such as a curriculum vitae, a list of publications, a list of Ph.D. students and information about honorary doctoral degrees are also included."

Israel Gohberg (1928-2009) was a prolific author and prodigious collaborator with more than 70 coauthors. All of his 26 books were coauthored, as were more than 90% of his 460+ published papers. In this book, Gohberg recalls a conversation with his famous mentor M.G. Krein after they wrote their first joint paper: "I suggested that his [Krein's] name appear before mine on the manuscript. I explained to M.G. that this would be to the benefit of the paper, which would be more appreciated and attract more attention. He refused, maintaining that he would not write a joint work with a person who could not be an equal partner, and that he always used the alphabetical order of names in joint papers and any change will cause speculation in the mind of the reader… In research it is not always obvious what is the more important. Sometimes it is an idea or the choice of topics or the timing, sometimes the conjecture, or the proof, sometimes the concept, sometimes a definition, a remark or finding an error, and a counterexample. M.G. would compare joint research with a team of sportsmen, especially in a long term collaboration. Each player passes the ball on to another player, and it is impossible to delineate which pass was the most important" (p. 16; see also p. 181). This anecdote is only one of many in this book which has much to offer about the art and craft of being a mathematician.

Matrix Completions, Moments, and Sums of Hermitian Squares
M. Bakonyi and H. J. Woerdeman

Princeton University Press, 2011, 536 pp., ISBN: 9780691128894
Reviewed by Harry Dym, Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel; Harry.Dym@weizmann.ac.il

As the title indicates, this book deals with matrix (and operator) completion problems, moment problems, and sums of Hermitian squares. Simple examples of these three themes illustrated below.

Completion problems: In their simplest form, completion problems are formulated in terms of partially specified $n \times n$ matrices such as

$$A = \begin{bmatrix} a_{11} & ? & ? \\ a_{21} & a_{22} & ? \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & ? \\ b_{21} & b_{22} & ? \\ ? & ? & b_{33} \end{bmatrix} = B^*,$$

and the objective is to fill in the missing entries so that the completed matrix A (respectively, B) has minimum norm or is contractive (respectively, positive semidefinite) if possible, and to parametrize all viable completions. (Other constraints, such as minimizing the rank, may also be imposed.) Typically, part of the problem is to formulate conditions on the given entries that will guarantee the existence of at least one solution. The explicit examples furnished above are for $n = 3$. Analogous problems are formulated for arbitrary positive integers n and for the case where the numbers a_{ij} and b_{ij} are replaced by matrices or bounded linear operators.

Moment problems: Given $2n+1$ complex numbers $c_{-n}, \ldots, c_n$, when does there exists a positive measure $d\mu$ on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that $c_k = \int_0^{2\pi} e^{i\theta k} d\mu(\theta)$ for $k = -n, \ldots, n$? Here too, the basic question is to formulate necessary and sufficient conditions on the data to guarantee the existence of at least one solution, and a description of all possible solutions is sought when these conditions are met. Moreover, complex coefficients may be replaced by matrices or bounded linear operators and moments may be specified by measures on $\mathbb{R}$ instead of on $\mathbb{T}$, in which case Hankel matrices come into play instead of Toeplitz matrices.

Sums of squares: The Fejér-Riesz theorem states that if $p(\zeta) = \sum_{j=-n}^{n} c_j \zeta^j$ is a trigonometric polynomial with $c_n \neq 0$, then

$$p(\zeta) \geq 0 \text{ on } \mathbb{T} \iff p(\zeta) = |g(\zeta)|^2,$$

for some stable polynomial $g(\zeta)$ of degree n. The objective is to investigate matrix and operator valued trigonometric polynomials of degree $n \geq 0$ on $\mathbb{T}$, as well as multivariable counterparts and analogous questions for polynomials on $\mathbb{R}$.

The book begins with a brief introduction to cones. The cone of positive semidefinite matrices and the cone of trigonometric polynomials, $p(\zeta) = \sum_{j=-n}^{n} c_j \zeta_j$ with $c_{-j} = \overline{c_j}$ and $p(\zeta) \geq 0$ for $\zeta \in \mathbb{T}$, serve as illustrative examples. Extreme rays are characterized and graph theoretic interpretations are discussed. Relatively quickly (by p.13) the authors enter into much more subtle issues of trigonometric polynomials $p(\zeta_1, \dots, \zeta_d)$ of many variables and explore the possibility of extending the domain of the definition of such polynomials in such a way that an associated Toeplitz matrix (with multi-indexed entries) is positive semidefinite. This is not easy stuff and it takes more than one reading to get a feeling for what is going on. The rest of the chapter is devoted to a discussion of entropy and determinant maximization as well as a short introduction to semidefinite programming.

Chapter 2 considers positive definite and positive semidefinite completions of partially specified operator matrices. Special attention is paid to the case where the given data is specified in a band (of diagonals, centered around the main diagonal) and also to the block Toeplitz case. Schur complements are used to derive an operator valued Carathéodory-Fejér theorem. The chapter explores the Hamburger moment problem based on positive semidefinite completions of Hankel matrices.

Chapter 3 begins with a brief review of the Carathéodory interpolation problem and some related moment problems for trigonometric polynomials of one variable. The main focus of the chapter, however, is on the multivariable analogs that are explored subsequently. It turns out that problems that are close in the one variable case are not so close in the multivariable case.

Chapter 4 is devoted to contractive and strictly contractive extensions of partially specified matrices and operators. A number of results rest on the corresponding extension problems in the class of positive semidefinite and positive definite matrices and operators that were treated earlier in Chapter 2 as well as the observation that

$$\|T\| \leq 1 \Longleftrightarrow \begin{bmatrix} I & T \\ T^* & I \end{bmatrix} \succeq 0.$$

Nehari problems in one and two variables are also treated. This chapter also includes an elegant solution of the Nevanlinna-Pick interpolation problem for scalar valued functions of one variable that are holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. This is followed by a discussion of the multivariable Nevanlinna-Pick interpolation problem and an application thereof to verify the two-variable von Neumann inequality.

Chapter 5 discusses an assortment of completion problems that investigate the rank, inertia, eigenvalues and singular values of the completion, among many other things.

One of the very strong features of this book is the wealth of examples, counterexamples and exercises. Roaming through, one encounters some of the major theorems in analysis in action: Hahn-Banach separation theorem, Arveson's distance formula, the Corona theorem for scalar and matrix valued functions in addition to the already mentioned von Neumann's inequality in one and two variables (i.e., Ando's theorem), the Nehari problem, optimization problems, as well as considerable information on block Toeplitz and block Hankel matrices.

The book is well written and easy to browse. It includes two lists of symbols, a reasonable index and an extensive discussion of the literature relevant to each section that cites references to related developments and a number of open problems. A very nice feature is the listing of page numbers for each cited reference in the bibliography. This is very helpful when trying to find the location of particular results when the index falls short.

The book serves as a good introduction to numerous new directions in the realm of multivariable matrix and operator valued functions. But the book is also a valuable source of examples and exercises that can serve to enrich numerous topics courses as well as courses on linear algebra, functional analysis and optimization. Among the many nuggets that I spotted and plan to use the next time I teach Linear Algebra are:

1) A very nice matrix proof of the Gauss-Lucas theorem (which states that the roots of $p'(\lambda)$ lie in the convex hull of the roots of a polynomial $p(\lambda)$) that is outlined in Exercise 49, p. 466;

2) A number of interesting inertia theorems, in Section 5.1 and elsewhere, including a nice application on p. 151 of Jordan cells of the form

$$\begin{bmatrix} \lambda & \varepsilon & 0 \\ 0 & \lambda & \varepsilon \\ 0 & 0 & \lambda \end{bmatrix}$$

with suitably small $\varepsilon > 0$ in place of the customary 1, in order to insure that $J + J^* \succ 0$ when λ is in the open right half plane;

3) Linearly constrained completion problems in Chapter 4, which include the following special case, wherein $A \in \mathbb{C}^{p\times q}$, $B \in \mathbb{C}^{p\times r}$ and $C \in \mathbb{C}^{r\times q}$,

$$\begin{bmatrix} I_p & A & B \\ A^* & C^*C & C^* \\ B^* & C & I_r \end{bmatrix} \succeq 0 \Longrightarrow \|B\| \leq 1 \text{ and } BC = A.$$

To sum up, this is a harmonious arrangement of material ranging from classical problems that were considered by some of the top ranking analysts of the early 1900s to matrix, operator and multivariable analogs that are the subject of active research today. Its value is greatly enhanced by the many examples and exercises. There appear to be very few misprints; I only found one (a missing equality sign in formula (3.2.26)), though I would have referred to Theorem 5.4.2 as a matrix version of Krein's theorem, rather than Krein's theorem. I understand that the site `http://woerdeman.wordpress.com` will be available for listing typographical errors and other points of discussion.

It really is a great pity that Mihály Bakonyi passed away prematurely and did not live to enjoy the fruits of his joint labors with Hugo Woerdeman, especially since the fruit is so tasty. Mihály and Hugo have created a very nice monograph which I highly recommend.

ILAS AND OTHER LINEAR ALGEBRA NEWS

List of Books on the ILAS-IIC Website

A list of books about linear algebra and its applications is maintained by Douglas Farenick, the Book Editor for *IMAGE*. This list has recently been updated and is accessible at `http://www.ilasic.math.uregina.ca/iic/IMAGE/`. When you find new titles that would be good to include, or informative reviews that would be appropriate to reference or if you wish to submit a book review to *IMAGE*, please contact Douglas Farenick (Doug.Farenick@uregina.ca).

Nominations for 2012 ILAS Elections

Submitted by Stephen Kirkland, ILAS President

The Nominating Committee for the 2012 ILAS elections has completed its work. Nominated for a three year term, beginning March 1, 2013, as ILAS Vice-President are:

- Shaun Fallat, University of Regina, Canada;
- Bryan Shader, University of Wyoming, United States.

Nominated for the two open three-year terms, beginning March 1, 2013, as "at-large" members of the ILAS Board of Directors are:

- Abraham Berman, Technion, Israel;
- Heike Faßbender, TU Braunschweig, Germany;
- Volker Mehrmann, TU Berlin, Germany;
- Bit-Shun Tam, Tamkang University, Taiwan.

Many thanks to the Nominating Committee for their important service to ILAS: Roger Horn, Naomi Shaked-Monderer, Helena Šmigoc (chair), Pauline van den Driessche, and Xiao-Dong Zhang. Thanks also to the nominees for agreeing to stand for election.

Voting in this election is electronic, using Votenet Solutions, with instructions sent by email.

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 48-1 through 48-7. Six new problems are on the last page; solutions are invited.

Editorial note: We obtained a notification of a partial solution to IMAGE *problem* 19-3*, part* (2) *which asks to classify pairs of linear operators* (A, B) *such that* $ABx \in \mathrm{Span}\{Ax, Bx\}$ *for every vector* x*. For the readers interested in this problem, the following information might be useful: Nir* Cohen *claims that the solution in case of matrices can be found in his recent paper* [Nir Cohen, Algebraic reflexivity and local linear dependence: generic aspects. *Oper. Theory Adv. Appl.,* 221 (2012) 219–239].

Problem 48-1: Reverse Order Law for the Core Inverse

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology*, trenkler@statistik.tu-dortmund.de

Let A be an $n \times n$ complex matrix and let $A^{\oplus}$ be the Core inverse of A defined in [O.M. Baksalary and G. Trenkler, Core inverse of matrices, *Linear Multilinear Algebra*, 58 (2010) 681–697.] to be the unique (when exists) matrix satisfying

$$AA^{\oplus} = P_A \quad \text{and} \quad \mathrm{Im}(A^{\oplus}) \subseteq \mathrm{Im}(A),$$

where P_A is the orthogonal projector onto $\mathrm{Im}(A)$. If $A^{\oplus}$, $B^{\oplus}$, and $(AB)^{\oplus}$ exist, does it follow that $(AB)^{\oplus} = B^{\oplus}A^{\oplus}$?

Solution 48-1.1 by Nir Cohen, *Dept. of Math., UFRN-Natal, Brazil*, nir@ccet.ufrn.br

Assume that $A^{\oplus}, B^{\oplus}, (AB)^{\oplus}$ exist and consider the two assertions:

$$(I) \quad (AB)^{\oplus} = B^{\oplus}A^{\oplus}, \qquad (II) \quad P_{AB} = AP_BA^{\oplus}.$$

It was required to prove or disprove (I); however (I) implies (II) since $P_{AB} = AB(AB)^{\oplus} = ABB^{\oplus}A^{\oplus} = AP_BA^{\oplus}$. So (II) is of independent interest. We show that (II) is false, hence (I) is false. To disprove (II), observe that P_{AB} is Hermitian and positive semidefinite. We provide two examples where $X := AP_BA^{\oplus}$ is either non-Hermitian, or it is Hermitian but not positive semidefinite. In all the examples, A and B are real 2×2 matrices, specifically,

$$B = P_B = B^{\oplus} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{hence} \quad X = ABA^{\oplus}.$$

First example: $X \neq X^T$. Choosing $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ non-singular, we have $A^{\oplus} = A^{-1}$ and $AB = B$ so $(AB)^{\oplus}$ exists. Hence, $X = AP_BA^{-1} = (AB)A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ which is not symmetric.

Second example: $X = X^T$ *but possesses a negative eigenvalue.* Let $A = uv^T$ ($\|u\|_2 = 1,\ c := u^Tv \neq 0$) be a rank one matrix. Then, $A^{\oplus} = \frac{1}{c}uu^T$ and with $P_B = B$ we have

$$X = ABA^{\oplus} = \frac{1}{c}(uv^T)B(uu^T) = \frac{v^TBu}{c}uu^T.$$

Thus, X is symmetric and has a negative eigenvalue whenever $\frac{v^TBu}{c}$ is negative. Our example here is

$$u = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \quad v = \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \quad A = uv^T = \begin{pmatrix} 3 & -3 \\ 4 & -4 \end{pmatrix}, \qquad \text{so} \quad c(v^TBu) = -1 \cdot 3 < 0. \ \blacksquare$$

Solution 48-1.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

No, and here is a counterexample:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A^{\oplus} = A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P_A = I_2; \qquad B^{\oplus} = P_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$AB = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \quad (AB)^{\oplus} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

However,

$$B^{\oplus}A^{\oplus} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq (AB)^{\oplus}. \ \blacksquare$$

Solution 48-1.3 by Sachindranath JAYARAMAN, *IISER - Trivandrum, CET Campus, Engineering College P.O., Trivandrum - 695016, Kerala, India*, sachindranathj@gmail.com

Let $B = \begin{pmatrix} 0&1&0\\1&1&0\\0&0&0 \end{pmatrix}, C = \begin{pmatrix} 0&1&0\\0&1&0\\0&-1&2 \end{pmatrix}$. One can verify that $X_B = \begin{pmatrix} -1&1&0\\1&0&0\\0&0&0 \end{pmatrix}$ has the same image as B and that $BX_B = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix}$ is a projection onto $\mathrm{Im}(B)$. So $B^{\oplus}$ exists and $B^{\oplus} = X_B$. Likewise, we see that $C^{\oplus}$ exists and equals $C^{\oplus} = (1/4)\begin{pmatrix} 2&2&0\\2&2&0\\1&1&2 \end{pmatrix}$. Also, $BC = \begin{pmatrix} 0&1&0\\0&2&0\\0&0&0 \end{pmatrix}$, and $(BC)^{\oplus} = (1/10)\begin{pmatrix} 1&2&0\\2&4&0\\0&0&0 \end{pmatrix} \neq C^{\oplus}B^{\oplus}$. ∎

Also solved by the proposers.

Problem 48-2: Minimality for the Nonzero Singular Values of an Idempotent Matrix

Proposed by Johanns DE ANDRADE BEZERRA, *Natal, RN, Brazil*, pav.animal@hotmail.com

Let A be an idempotent $n \times n$ complex matrix. If λ is a nonzero eigenvalue of AA^* show that $\lambda \geq 1$.

Solution 48-2.1 by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com and Götz TRENKLER, *Dortmund University of Technology*, trenkler@statistik.tu-dortmund.de

Let A be an $n \times n$ complex idempotent matrix of rank r. Then there exists a unitary matrix U such that

$$A = U \begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix} U^*, \tag{4}$$

where K is of size $r \times (n-r)$; see [1, Section 1]. From (4) it directly follows that AA^* is of the form

$$AA^* = U \begin{pmatrix} I_r + KK^* & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

Hence, the nonzero eigenvalues of AA^* are the eigenvalues of $I_r + KK^*$. Since these eigenvalues are given by $1+\lambda_j(KK^*)$, $j = 1, 2, ..., r$, where $\lambda_j(KK^*)$ denotes jth eigenvalue of KK^*, the solution is established. ∎

An additional observation is that all nonzero eigenvalues of AA^* are equal to 1 if and only if A is Hermitian, or, in other words, A is an orthogonal projector.

Reference

[1] O.M. Baksalary, G.P.H. Styan, and G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, *Linear Algebra Appl.* 430 (2009) 2798–2812.

Solution 48-2.2 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let v be an eigenvector corresponding to the eigenvalue λ. Then, since $\lambda \neq 0$, $AA^*v = \lambda v$ implies that v is in the range of A and hence orthogonal to the kernel of A^*. Since $(A^* - I)v$ is in the kernel of A^* and $A^*v = v + (A^* - I)v$, we have $\|A^*v\|^2 = \|v\|^2 + \|(A^* - I)v\|^2 \geq \|v\|^2$. Therefore

$$\lambda\|v\|^2 = \lambda\langle v, v\rangle = \langle AA^*v, v\rangle = \|A^*v\|^2 \geq \|v\|^2 \qquad \text{and so} \quad \lambda \geq 1. \ ∎$$

Solution 48-2.3 by Minghua LIN, *University of Waterloo, Ontario, Canada*, mlin87@ymail.com and Henry WOLKOWICZ, *University of Waterloo, Ontario, Canada*, hwolkowicz@uwaterloo.ca

Recall the following canonical form for idempotent matrices (see [1, Corollary 1]):
If $A^2 = A \in M_n(\mathbb{C})$ then there exists a unitary matrix U such that UAU^* has the form

$$\begin{pmatrix} 1 & \sigma_1 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & \sigma_k \\ 0 & 0 \end{pmatrix} \oplus I_m \oplus O_s,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ and $m, s \geq 0$. Moreover k, m, s and the σ_i are uniquely determined by U.

So, $UAA^*U^* = \begin{pmatrix} 1+\sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1+\sigma_k^2 & 0 \\ 0 & 0 \end{pmatrix} \oplus I_m \oplus O_s$ and the nonzero eigenvalues of AA^* are no less than 1. ∎

Reference

[1] D.Ž. Đoković, Unitary similarity of projectors, *Aequationes Math.* 42 (1991) 220–224.

Also solved by Roger A. HORN, Edward L. PEKAREV, and the proposer.

Editorial note: Edward L. Pekarev *in his solution (which is similar to the solution by Eugene A.* Herman*) observes that the claim is also valid for possibly unbounded densely defined closed operators A, acting in a Hilbert space, such that* $\mathrm{Im}(A) \subseteq \mathrm{dom}(A)$*,* $A^2 = A$*, and for which there exists* $v \in \mathrm{dom}(AA^*)$*, an eigenvector to nonzero eigenvalue of* AA^**.*

Problem 48-3: A Generalization of Geršgorin's Theorem

Proposed by Frank J. Hall, *Georgia State University, Atlanta, Georgia, U.S.A.*, fhall@gsu.edu
and Roger A. Horn, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu

Let λ be an eigenvalue of $A \in M_n(\mathbb{C})$ with geometric multiplicity at least $k \geq 1$ and let $R'_i(A) = \sum_{j \neq i} |a_{ij}|$ denote the ith deleted absolute row sum of A. Show that λ is contained in each union of $n - k + 1$ different Geršgorin discs of A, that is,

$$\lambda \in \bigcup_{j=1}^{n-k+1} \{z \in \mathbf{C} : |z - a_{i_j i_j}| \leq R'_{i_j}(A)\}$$

for *any* choice of indices $1 \leq i_1 < \cdots < i_{n-k+1} \leq n$.

Solution 48-3.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let $[v]_i$ denote the ith component of a vector $v \in \mathbb{C}^n$.

Lemma. Let V be a subspace of $\mathbb{C}^n$ with dimension $r \geq 1$. There exists a basis $\{v_1, \ldots, v_r\}$ of V and a set of distinct indices $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ such that $\|v_1\|_\infty = \cdots = \|v_r\|_\infty = 1$ and $|[v_j]_{i_j}| = 1$, $j = 1, \ldots, r$.

Proof: If $r = 1$ and $\{v\}$ is a basis of V, simply scale v so that $\|v\|_\infty = 1$. Suppose the lemma is true for some dimension $r \geq 1$, and let V be a subspace with basis $\{v_1, \ldots, v_{r+1}\}$. We may assume the basis vectors have been scaled so that for each j, $1 \leq j \leq r+1$, $\|v_j\|_\infty = 1$ and $[v_j]_{i_j} = 1$ for some index i_j. If the indices $i_1, \ldots, i_{r+1}$ are all distinct, there is nothing more to prove. If not, we may assume (by applying simultaneous permutations on the vector components) that $i_{r+1} = 1$. Let

$$w_j = v_j - [v_j]_1 \, v_{r+1}, \quad j = 1, \ldots, r.$$

Then $\{w_1, \ldots, w_r, v_{r+1}\}$ is basis of V in which the first component of the vectors $w_1, \ldots, w_r$ is 0. Applying the lemma to the subspace spanned by $\{w_1, \ldots, w_r\}$ yields a basis $\{u_1, \ldots, u_r\}$ and a corresponding set of distinct indices $\{i_1, \ldots, i_r\}$. Since none of these indices equals 1, the basis $\{u_1, \ldots, u_r, v_{r+1}\}$ and the set of indices $\{i_1, \ldots, i_r, 1\}$ constitute the conclusion of the lemma for the $(r+1)$-dimensional subspace V. ∎

Let r be the geometric multiplicity for the eigenvalue λ of A and let V be the r-dimensional eigenspace for λ. Let $\{v_1, \ldots, v_r\}$ and $\{i_1, \ldots, i_r\}$ be, respectively, a basis of V and a set of distinct indices as described in the lemma. Let $A = [a_{ij}]_{i,j=1}^n$. Then, for $j = 1, \ldots, r$, $[Av_j]_{i_j} = \lambda [v_j]_{i_j}$ and so

$$|\lambda - a_{i_j i_j}| = |\lambda - a_{i_j i_j}| \cdot |[v_j]_{i_j}| = \left| \sum_{k \neq i_j} a_{i_j k} [v_j]_k \right| \leq \sum_{k \neq i_j} |a_{i_j k}| = R'_{i_j}(A).$$

That is, λ lies in the intersection of r distinct Geršgorin disks of A. Therefore, since every union of $n - r + 1$ distinct Geršgorin disks includes at least one of these r disks, λ lies in every such union. Thus, if $k \leq r$ then every union of $n - k + 1$ distinct Geršgorin disks contains a union of $n - r + 1$ distinct Geršgorin disks and hence contains λ. ∎

Solution 48-3.2 by Denis Serre, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

By the pigeonhole principle, it is equivalent to prove that at least k disks $D_i = D(a_{ii} : R'_i(A))$ contain λ.

After a translation, we may suppose that $\lambda = 0$. Let ℓ be the number of disks D_i which do not contain the origin. Up to conjugating by a permutation matrix (this amounts to relabeling the coordinates in $\mathbb{C}^n$), we may assume that these are the disks D_i for $i \geq n - \ell + 1$. Then let us write A blockwise

$$A = \begin{pmatrix} F & G \\ H & K \end{pmatrix}, \qquad K \in \mathbf{M}_{\ell \times \ell}(\mathbb{C}).$$

By the Geršgorin's Theorem, and by assumption, K is nonsingular. Hence the $\mathrm{rk}(A) \geq \ell$.
Thus $k = \dim(\ker(A)) = n - \mathrm{rk}(A) \leq n - \ell$. ∎

Notice that the algebraic multiplicity does not satisfy the same property. For instance, $\lambda = 0$ is a double eigenvalue of the matrix $\left(\begin{smallmatrix} 1 & a^{-1} \\ -a & -1 \end{smallmatrix}\right)$, yet it is not contained in the disk D_2, if $|a| < 1$.

Also solved by Johanns de Andrade Bezerra*, Minghua* Lin *and Henry* Wolkowicz*, and the proposers.*

Problem 48-4: Characterizations of Symmetric and Skew-Symmetric Matrices
Proposed by Roger A. Horn, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu

Let A be a square complex matrix, and let $\bar{A}$ and $A^* = \bar{A}^T$ denote its complex conjugate and conjugate transpose, respectively. Show that: (1) $A = A^T$ if and only if $A\bar{A} = AA^*$, and (2) $A^T = -A$ if and only if $A\bar{A} = -AA^*$.

Solution 48-4.1 by Johanns de Andrade Bezerra, *Natal, RN, Brazil*, pav.animal@hotmail.com

Proof of (1). If $A = A^T$ then $A\overline{A} = A\overline{A}^T = AA^*$. Conversely, let $AA^* - A\overline{A} = 0$. Then $A(A^* - \overline{A}) = 0$, and so $(-A^* + \overline{A})A^T = (A(A^* - \overline{A}))^T = 0$. Hence, if $B = -A^TA^* + A^T\overline{A}$, then

$$\mathrm{Tr}(B) = \mathrm{Tr}(A^T(-A^* + \overline{A})) = \mathrm{Tr}((-A^* + \overline{A})A^T) = 0.$$

Note that

$$A(A^* - \overline{A}) + (-A^TA^* + A^T\overline{A}) = (A - A^T)(A^* - \overline{A}) = (A - A^T)(A - A^T)^* = B.$$

That is, B is Hermitian nonnegative definite with $\mathrm{Tr}(B) = 0$. Thus $B = 0$, and therefore $A - A^T = 0$.

The proof of (2) is similar. If $A = -\overline{A}$, then $AA^T = -A\overline{A}^T = AA^*$. Conversely, let $AA^* + A\overline{A} = 0$. Then $A(A^* + \overline{A}) = 0$, and so $(A(A^* + \overline{A}))^T = (A^* + \overline{A})A^T = 0$. Hence, if $C = A^TA^* + A^T\overline{A}$, then $\mathrm{Tr}(C) = \mathrm{Tr}((A^* + \overline{A})A^T) = 0$. Note that

$$A(A^* + \overline{A}) + (A^TA^* + A^T\overline{A}) = (A + A^T)(A + A^T)^* = C.$$

Thus, as before, $C = 0$ and hence $A + A^T = 0$. ∎

Solution 48-4.2 by Minghua Lin, *University of Waterloo, Ontario, Canada*, mlin87@ymail.com
and Henry Wolkowicz, *University of Waterloo, Ontario, Canada*, hwolkowicz@uwaterloo.ca

If $A = A^T$ then $A^* = \bar{A}$, so it is obvious that $AA^* = A\bar{A}$. For the converse, note that $\mathrm{Tr}(A^TA^*) = \mathrm{Tr}((A^TA^*)^T) = \mathrm{Tr}(\overline{A}A) = \mathrm{Tr}(A\overline{A})$, and $\mathrm{Tr}(A^T\overline{A}) = \mathrm{Tr}(A^*A) = \mathrm{Tr}(AA^*)$. Hence,

$$\begin{aligned}\|A - A^T\|_F^2 &= \mathrm{Tr}\big((A - A^T)(A - A^T)^*\big)\\ &= \mathrm{Tr}\big((A - A^T)(A^* - \overline{A})\big)\\ &= \mathrm{Tr}(AA^* - A\overline{A} - A^TA^* + A^T\overline{A})\\ &= 2\,\mathrm{Tr}(AA^* - A\overline{A}).\end{aligned}$$

Thus $AA^* = A\overline{A}$ implies $A = A^T$.

The other claim can be proved similarly. If $A = -A^T$ then $A^* = -\bar{A}$. Thus $AA^* = -A\bar{A}$. Conversely,

$$\|A + A^T\|_F^2 = \mathrm{Tr}\big((A + A^T)(A + A^T)^*\big) = \mathrm{Tr}(AA^* + A\overline{A} + A^TA^* + A^T\overline{A}) = 2\,\mathrm{Tr}(AA^* + A\overline{A}).$$

Thus $AA^* = -A\overline{A}$ implies $A = -A^T$. ∎

Solution 48-4.3 by Éric Pité, *Paris, France*, eric.pite@telecom-paristech.org

Let $\mathrm{ann}(B)$ be the right annihilator of a matrix B. Then, $\mathrm{ann}(AA^*) = \mathrm{ann}(A^*)$ because we have $\mathrm{ann}(A^*) \subseteq \mathrm{ann}(AA^*)$, and if $X \in \mathrm{ann}(AA^*)$, then $X^*AA^*X = 0$ which means $\|A^*X\| = 0$.

(1) Note that $A\overline{A} = AA^*$ is equivalent to $A(\overline{A - A^T}) = 0$. Hence $A\overline{A} = AA^*$ implies (i) $\overline{A}(A - A^T) = 0$ and (ii) $A\overline{A}(A - A^T) = 0$. Consequently $AA^*(A - A^T) = 0$ and (using right annihilators) $A^*(A - A^T) = 0$. Combining with (i) gives $(A - A^T)^*(A - A^T) = 0$, which implies $A = A^T$.

Conversely, if $A = A^T$ we have $A\overline{A} = AA^*$.

(2) The same method yields the desired result: $A\overline{A} = -AA^*$ is equivalent to $A(\overline{A} + \overline{A}^T) = 0$. Hence, $A\overline{A} = -AA^*$ gives (i) $\overline{A}(A + A^T) = 0$ and (ii) $A\overline{A}(A + A^T) = 0$ yielding $(\overline{A} + A^*)(A + A^T) = 0$ and thus $A + A^T = 0$.

Conversely, if $A = -A^T$ we have $A\overline{A} = -AA^*$. ∎

Also solved by Oskar Maria Baksalary *and Götz* Trenkler*, Eugene A.* Herman*, and the proposer.*

Problem 48-5: Similarity Invariant Seminorms

Proposed by Nathaniel JOHNSTON, *University of Guelph, Guelph, Canada*, njohns01@uoguelph.ca

Let $n \geq 2$. A semi-norm $\|\cdot\|$ on complex $n \times n$ matrices is similarity invariant if

$$\|X\| = \|SXS^{-1}\| \qquad \forall\, X, S \in M_n(\mathbb{C}) \text{ with } S \text{ nonsingular.}$$

Find all similarity-invariant norms and similarity-invariant seminorms on $M_n(\mathbb{C})$.

Solution 48-5.1 by Roger A. HORN, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu

Let $\|\cdot\|$ be a seminorm on $M_n(\mathbb{C})$ with $n \geq 2$. If $N \in M_n(\mathbb{C})$ is nilpotent, then N is similar to $2N$, so $\|N\| = \|2N\| = 2\,\|N\|$. It follows that $\|N\| = 0$, so $\|\cdot\|$ is not a norm. There are no similarity invariant norms on $M_n(\mathbb{C})$ if $n \geq 2$.

The matrices $E_{ij} \in M_n(\mathbb{C})$ that have a 1 in position i, j and zero entries elsewhere are nilpotent if $i \neq j$, so $\|E_{ij}\| = 0$ if $i \neq j$. If $B = [b_{ij}] \in M_n(\mathbb{C})$ has zero main diagonal entries, then $\|B\| \leq \sum_{i,j} \|b_{ij} E_{ij}\| = \sum_{i,j} |b_{ij}|\, \|E_{ij}\| = \sum_{i \neq j} |b_{ij}|\, \|E_{ij}\| = 0$, so $\|B\| = 0$.

If $A, B \in M_n(\mathbb{C})$ and $\|B\| = 0$, then $|\ ||A + B|| - ||A||\ | \leq ||B|| = 0$ (see [1, Lemma 5.1.2]), so $||A + B|| = ||A||$.

Each $A \in M_n(\mathbb{C})$ is unitarily similar to a matrix of the form $(n^{-1} \operatorname{Tr} A) I_n + B$, in which B has zero main diagonal entries (see [1, Example 2.2.3]). Consequently, $\|A\| = \left\|(n^{-1} \operatorname{Tr} A) I_n + B\right\| = \left\|(n^{-1} \operatorname{Tr} A) I_n\right\| = n^{-1} |\operatorname{Tr} A|\, \|I_n\|$, so every similarity invariant seminorm on $M_n(\mathbb{C})$ has the form $\|A\| = c|\operatorname{Tr} A|$, in which $c = n^{-1}\, \|I_n\|$. ∎

Reference

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2012.

Solution 48-5.2 by Éric PITÉ, *Paris, France*, eric.pite@telecom-paristech.org

Similarity invariant norms. Suppose that there exists a similarity invariant norm $\|\cdot\|$, hence $\|XS\| = \|SX\|$ for all $X, S \in M_n(\mathbb{C})$ with S nonsingular. As the subset of nonsingular matrices of $M_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, the previous equality holds for all $X, S \in M_n(\mathbb{C})$. Taking $X = (x_{i,j})$ with $x_{1,2} = 1$ and $x_{i,j} = 0$ otherwise, and $S = (s_{i,j})$ with $s_{1,1} = 1$ and $s_{i,j} = 0$ otherwise, we see that $XS = 0$ and $SX \neq 0$, contradicting $\|XS\| = \|SX\|$. Thus there is no such norm. ∎

Similarity invariant semi-norms. If $\alpha \geqslant 0$, then $A \mapsto \alpha|\operatorname{Tr}(A)|$ is a similarity invariant semi-norm on complex $n \times n$ matrices. We show that these are the only similarity invariant semi-norms. Let $\|\cdot\|$ be such a semi-norm.

Let A be a nilpotent matrix in $M_n(\mathbb{C})$ and write $A = PTP^{-1}$ with $t_{i,j} = 0$ if $i \geqslant j$. If k is an integer we denote by D_k the diagonal matrix with elements $(k^n, k^{n-1}, \ldots, k)$. Then $D_k^{-1} T D_k$ is upper triangular matrix $(t_{i,j} k^{i-j})$. Hence A is similar to a sequence of nilpotent matrices with zero limit. Thus $\|A\| = 0$.

Denote by $\mathcal{H}$ the hyperplane $\{M \in M_n(\mathbb{C}) \mid \operatorname{Tr}(M) = 0\}$, and by $\mathcal{N}$ the vector space spanned by nilpotent matrices. We have $\mathcal{N} \subset \mathcal{H}$. Let $E_{i,j}$ denote the canonical basis of $M_n(\mathbb{C})$. Then $\{E_{1,1} - E_{i,i}\}_{i>1} \cup \{E_{i,j}\}_{i \neq j}$ form a basis of $\mathcal{H}$. Let $F_i = E_{1,1} - E_{i,i} + E_{i,1} - E_{1,i}$ and $G_i = E_{1,1} - E_{i,i} - E_{i,1} + E_{1,i}$, for $i > 1$. These matrices have rank 1 and zero trace so they are nilpotent. The equalities $E_{1,1} - E_{i,i} = (F_i + G_i)/2$ imply that $\mathcal{N} = \mathcal{H}$. Thus $\|M\| = 0$ for all $M \in \mathcal{H}$.

Let $B \in M_n(\mathbb{C})$. Then, $B - \operatorname{Tr}(B) I_n/n \in \mathcal{H}$ and $\Big|\|B\| - |\operatorname{Tr}(B)| \cdot \|I_n\|/n\Big| \leqslant \|B - \operatorname{Tr}(B) I_n/n\| = 0.$ ∎

Also solved by Denis SERRE *and the proposer.*

Editorial note: Minghua LIN *and Henry* WOLKOWICZ *informed us that the first part of the problem also appears as Exercise IV.4.1 in* [R. Bhatia, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997].

Problem 48-6: A Determinant Inequality

Proposed by Minghua LIN, *University of Waterloo, Ontario, Canada*, mlin87@ymail.com

Let A, B be $n \times n$ positive semidefinite Hermitian matrices. Show that $\det(A^2 + AB^2A) \leq \det(A^2 + BA^2B)$.

Solution 48-6.1 by Edward L. PEKAREV, *Odessa Regional Institute of Public Administration of the National Academy of Public Administration, Office of the President of Ukraine, Ukraine*, edpekarev@yandex.ru

Let $A, B \in \mathbf{M}_n(\mathbf{C})$ and $B^* = B$. It suffices to prove the inequality $\det(A^*A + A^*B^2A) \leq \det(A^*A + BA^*AB)$ which implies the determinant inequality from Problem 48-6.

This inequality is obvious if $\det A = 0$ as the left-side is equal to 0 and $A^*A + BA^*AB \geq O$. Supposing $\det A \neq 0$, we have: $A^*A + A^*B^2A = A^*(I + B^2)A$, $A^*A + BA^*AB = A^*(I + A^{*-1}BA^*ABA^{-1})A$. It remains to show that

$\det(I+B^2) \le \det(I+A^{*-1}BA^*ABA^{-1})$ or, denoting $V = ABA^{-1}$, that $\det(I+V^2) \le \det(I+V^*V)$. But applying the Weyl Majorant Theorem (see [1, p. 544], [2, p.61] or [3, p.42]) to the matrix V with real spectrum and the function $f(x) = \ln(1+x^2)$, $x \ge 0$ ($f(e^t)$ is convex and monotone increasing in $t \ge 0$) we can find: $\Pi_{j=1}^n(1+\lambda_j^2) \le \Pi_{j=1}^n(1+s_j^2)$, where λ_j and s_j are respectively eigenvalues and singular values of V (arranged in the appropriate order). Thus equalities $\det(I+V^2) = \Pi_{j=1}^n(1+\lambda_j^2)$ and $\det(I+V^*V) = \Pi_{j=1}^n(1+s_j^2)$ end the proof. ∎

Corollary. The inequality $\det(A^*A + A^*B^kA) \le \det(A^*A + B^{\frac{k}{2}}A^*AB^{\frac{k}{2}})$ $(k = 1, 2, \ldots)$ holds for even k with arbitrary selfadjoint B and for odd k with any positive semidefinite Hermitian matrix B. ∎

References

[1] F.R. Gantmacher, *Matrix Theory*, Moskva, 1967 (in Russian).

[2] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Moskva, 1965 (in Russian).

[3] R. Bhatia, *Matrix Analysis*, Springer, 1997.

Also solved by the proposer.

Editorial note: The proposer gives a solution that relies on a majorization $\lambda^{\downarrow}(A^2+AB^2A) \succ \lambda^{\downarrow}(A^2+BA^2B)$ between eigenvalues. This was also communicated to us by Ingram Olkin *who points to* [S. Furuichi and M. Lin, A matrix trace inequality and its application, *Linear Algebra Appl.* 433 (2010) 1324–1328], *and recalls that a majorization between eigenvalues implies $\phi(\lambda^{\downarrow}(A^2+AB^2A)) \ge \phi(\lambda^{\downarrow}(A^2+BA^2B))$ for every symmetric convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, with the inequality being reversed for symmetric concave functions.*

Problem 48-7: Block-entry-wise Compressions

Proposed by Edward Poon, *Embry-Riddle Aeronautical University, Prescott, AZ, U.S.A.*, edward.poon@erau.edu

Let A be an $mn \times mn$ complex matrix with $m \times m$ block matrix entries A_{ij}, $1 \le i, j \le n$. Suppose for each unit vector $x \in \mathbb{C}^m$ the matrix U_x, defined by $(U_x)_{ij} = x^*A_{ij}x$, is unitary. Does it follow that each A_{ij} is a scalar multiple of the $m \times m$ identity matrix I_m (that is, $A_{ij} = u_{ij}I_m$ where u_{ij}, $1 \le i, j \le m$, are the entries of a unitary matrix)?

Editorial note: We thank Eugene A. Herman *for providing a correction to a misprint in the last sentence of the original problem formulation as posted in* IMAGE 48.

Solution 48-7.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

No, there are many counterexamples when $m = n = 2$. Let $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ be any 4×4 unitary matrix with 2×2 blocks A_{ij}, $1 \le i, j \le 2$. Then $AA^* = I_4$ implies

$$\begin{aligned} A_{11}A_{11}^* + A_{12}A_{12}^* &= I_2 & \qquad A_{11}A_{21}^* + A_{12}A_{22}^* &= O \\ A_{21}A_{11}^* + A_{22}A_{12}^* &= O & \qquad A_{21}A_{21}^* + A_{22}A_{22}^* &= I_2. \end{aligned}$$

Hence, if $x \in \mathbb{C}^2$ is a unit vector, then

$$U_xU_x^* = \begin{pmatrix} x^*A_{11}x & x^*A_{12}x \\ x^*A_{21}x & x^*A_{22}x \end{pmatrix} \begin{pmatrix} x^*A_{11}^*x & x^*A_{21}^*x \\ x^*A_{12}^*x & x^*A_{22}^*x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is, every U_x is unitary. However, the four 2×2 blocks A_{ij} need not be multiples of I_2, as we see in this example:

$$A = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{smallmatrix}\right). \quad ∎$$

Also solved by the proposer.

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 48 (Spring 2012), p. 40]. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Gregor Dolinar, Nung-Sing Sze, and Rajesh Pereira.

New Problems:

Problem 49-1: Generalized Eigenvalues of a Pair of Orthogonal Projectors

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology*, trenkler@statistik.tu-dortmund.de

Let $P, Q \in M_n(\mathbb{C})$ be orthogonal projectors (i.e., Hermitian idempotent matrices of size n) such that $\mathrm{Im}(P) + \mathrm{Im}(Q) = \mathbb{C}^n$. For $\lambda \in \mathbb{C}$, show that if $Q - \lambda P$ is singular then $\lambda = 0$ or $\lambda = 1$. Furthermore, prove that $\lambda = 1$ is a zero of the polynomial $\lambda \mapsto \det(Q - \lambda P)$ with multiplicity $\dim[\mathrm{Im}(P) \cap \mathrm{Im}(Q)]$.

Problem 49-2: Matrix Diagonal Entries

Proposed by Roger A. Horn, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu
and Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, Florida, U.S.A.*, zhang@nova.edu

For $A = (A_{ij}) \in M_n(\mathbb{C})$, let A^* be its conjugate transpose, and let $|A| = (A^*A)^{1/2}$ (the positive semidefinite square root).

(a) Show that $|A_{ij}|^2 \leq |A^*|_{ii}|A|_{jj}$ for all i, j. Is $|A_{ij}|^2 \leq |A|_{ii}|A^*|_{jj}$?

(b) Show that $\prod_{i=1}^n |A_{ii}|^2 = \prod_{i=1}^n |A^*|_{ii}|A|_{ii}$ if and only if (i) A has a zero row or column or (ii) $A = DP$ for some unitary diagonal matrix D and some positive semidefinite matrix P.

(c) Show that $|\det(A \circ C)|^2 \leq \prod_{i=1}^n |A^*|_{ii}|A|_{ii}$ for any $n \times n$ contraction C, where $A \circ C$ is the Hadamard (Schur, entrywise) product of A and C. (Note: A matrix is a contraction if its spectral norm is no more than 1.)

(d) Is $|A_{ii}| \leq |A|_{ii}$ for all $i = 1, 2, \ldots, n$?

Problem 49-3: Loewner Partial Order

Proposed by Minghua Lin, *University of Waterloo, Ontario, Canada*, mlin87@ymail.com

Let $A, B, C, D, X, Y \in M_n(\mathbb{C})$ such that $A > XB^{-1}X^* \geq 0$, $C > YD^{-1}Y^* \geq 0$ (in Löwner partial order). Define

$$R = B - X^*(A + CA^{-1}C)^{-1}X + Y^*(A + CA^{-1}C)^{-1}Y - X^*(C + AC^{-1}A)^{-1}Y - Y^*(C + AC^{-1}A)^{-1}X.$$

Show that $0 < R < \frac{3}{2}(B + D)$. What is the smallest constant c such that $R \leq c(B + D)$?

Problem 49-4: A Total Perron-Frobenius Theorem for Total Orders

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let A be an $n \times n$ real matrix. Show that the following are equivalent:

(a) All the eigenvalues of A are real and nonnegative.

(b) There exists a total order $\geq$ on $\mathbb{R}^n$ (partial order where any two vectors are comparable) which is preserved by A (i.e. $x \in \mathbb{R}^n$ and $x \geq 0$ implies that $Ax \geq 0$) and which makes $(\mathbb{R}^n, \geq)$ an ordered vector space.

Problem 49-5: A Property of the Electro-Magnetic Field in the Born–Infeld Model

Proposed by Denis Serre, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

In the canonical Euclidian space $\mathbb{R}^n$ $(n \geq 3)$, we give ourselves two vectors E and B, satisfying $\|E\|^2 + (E \cdot B)^2 \leq 1 + \|B\|^2$. Prove the following inequality between symmetric matrices: $EE^T + BB^T \leq (1 + \|B\|^2)I_n$.

Problem 49-6: Determinant Power Monotonicity

Proposed by Suvrit Sra, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

Let $A, B > 0$ (positive definite); let $1 \leq t \leq u$. Prove that $\det\left(\frac{A^t + B^t}{2}\right)^{1/t} \leq \det\left(\frac{A^u + B^u}{2}\right)^{1/u}$.

Solutions to Problems 48–1 *through* 48–7 *are on pp.* 46–51.