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Serving the International Linear Algebra Community

Issue Number 50, pp. 1–44, Spring 2013

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About *IMAGE*

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MATHEMATICAL PRELUDE

Another Proof of the Two-sidedness of Matrix Inverses

Oskar Maria Baksalary[1] and Götz Trenkler[2]

Let A and B be $n \times n$ complex matrices and let I_n denote the identity matrix of order n. Inspired by [1] and [2], we provide a concise proof of the known fact that when $AB = I_n$, then $BA = I_n$.

Let $AB = I_n$ and let rk(.) be the rank of a matrix argument. Then $n = \mathrm{rk}(I_n) = \mathrm{rk}(AB) \leqslant \mathrm{rk}(A) \leqslant n$. Hence $\mathrm{rk}(A) = n$. But $\mathrm{rk}(A) = \mathrm{rk}(A^\dagger A)$, where $A^\dagger$ denotes the Moore–Penrose inverse of A. Since $A^\dagger A$ is idempotent and an idempotent matrix of full rank necessarily coincides with the identity matrix, we conclude that $A^\dagger A = I_n$. On the other hand, the equation $AB = I_n$ has the solution for B as $B = A^\dagger + (I_n - A^\dagger A)Z = A^\dagger$ for some $n \times n$ complex matrix Z. In consequence, $BA = A^\dagger A = I_n$.

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[1] R.A. Beauregard, A short proof of the two-sidedness of matrix inverses, *Mathematics Magazine* 80 (2007) 135–136.
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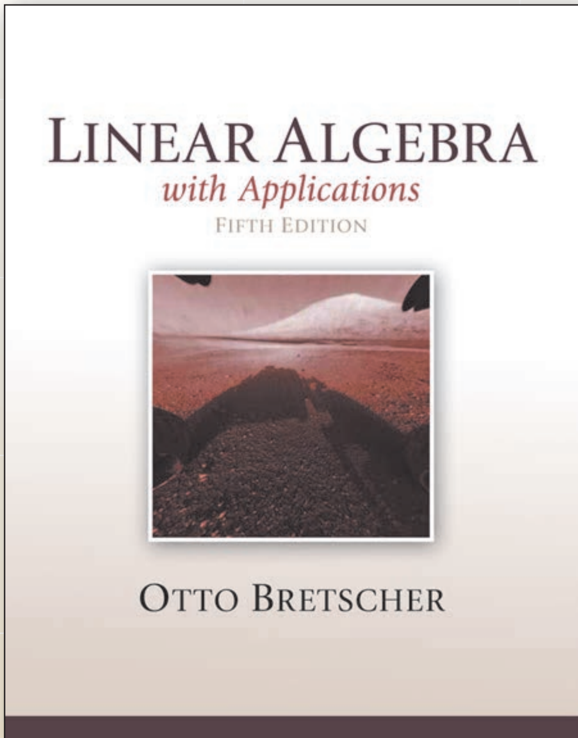

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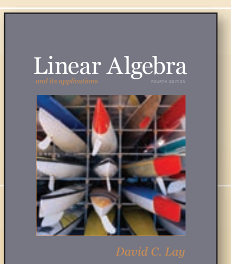

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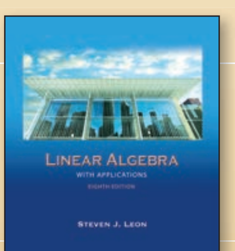

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FEATURE INTERVIEW

A Conversation with Professor Tsuyoshi Ando

Tsuyoshi Ando Interviewed by Chi-Kwong Li[1]

C-K. Li and T. Ando

Professor Ando is a well-known figure in the mathematics community. His research covers a wide range of areas including a large component in matrix and operator theory. He has won many awards including the Béla Szőkefalvi-Nagy Medal and the 2002 Hans Schneider Prize. At the age of 80, he is still very active in mathematical research, participating in conferences, as well as doing editorial and referee work.

Li - When/how did you decide to be a mathematician?

Ando - When I was in an elementary school, Japan was in war against the USA and European countries. I wanted to become a military officer or even a general as other boys did. In April 1945, having finished the first year of a middle school, I entered a Cadet school (a preliminary school for military academy). But in August of the same year Japan lost the war. Consequently, the Cadet school was closed and I returned to the middle school.

After the war, on the recommendation of the U.S.A., Japan changed its education system to more closely align with the U.S.A. In 1949, I entered Hokkaido University as one of the first students of the new system. I liked natural sciences better than humanities.

After 1.5 years of general education, students had to choose their major subjects. Among students there was excitement caused by the first Nobel prize by Japanese physicist Dr. Hideki Yukawa. Many colleagues of mine wanted to go to physics. But I decided to go to mathematics, wishing to become a researcher in university. There were only five students majoring in mathematics that year.

The Department of Mathematics had four chairs (that is, four full professors in German style); two in analysis, one in geometry and one in algebra. When I entered the university, those professors were not on good terms and three left the university, leaving only the professor of geometry. In the last year of my undergraduate study, Dr. Hidegoro Nakano was invited from Tokyo University to take a professorship in functional analysis at Hokkaido University.

Li - How did you choose your graduate school adviser and research topics?

Ando - When I got B.Sc. in mathematics, the graduate school was newly established in the university, and I was *happy* that my status as a "graduate student" was guaranteed for the next five years. This may sound a little strange. Until the new university system began, Japanese university had no established graduate programs. Generally it was considered that a doctor's degree was endowed after one had done many research works and attained a certain age, say 40.

As my family was not rich enough and the society was still in confusion, it was unthinkable for me to pursue graduate studies at other universities. Though the geometry professor was an expert of differential geometry based on Ricci calculus, I did not like it. Therefore my only choice was to study functional analysis under the supervision of Professor H. Nakano. My supervisor had made significant achievements in function space theory as well as Hilbert space operator theory during the difficult war time. His work related to unbounded normal operators is now called Wecken-Nakano theory and is the subject of P. Halmos' small book *Introduction to Hilbert spaces*. His profound contributions are seen in the construction of (abstract) function space theory from the stand point of ordered linear spaces (usually called Riesz spaces or Banach lattices). He introduced a meaningful generalization of Orlicz spaces known as "modulared semi-ordered linear spaces." His approach was always completely abstract and beautiful; he did not give any applications. I inherited my lifelong interest of order structure from Prof. H. Nakano.

In my thesis *Positive linear operators in semi-ordered linear spaces*, I gave a slight generalization of the Perron-Frobenius theorem on non-quasi-nilpotency for certain positive linear operators. Professor H. Nakano had a progressive view of the Ph.D., saying that Ph.D. should be merely a passport to the community of researchers. While not common at the time, this view is now prevalent in Japan.

I was the first Ph.D. under the new graduate program of Hokkaido University in 1958. Though it was hard to find an academic position, upon my doctorate, Prof. H. Nakano squeezed me into an assistantship at the Research Institute of Applied Electricity, which belonged to Hokkaido University. Here I was subsequently promoted to full professor, and remained until retirement! As this is a research institute, I had little teaching experience, but I did become associated with some graduate students.

[1]Ferguson Professor of Mathematics, The College of William & Mary, Williamsburg, VA 23187-8795, USA; math.ckli@gmail.com

Li - Could you talk about some important events in your career?

Ando - There are several events.

First, when I was in the course of general education at the university, Japan had not yet recovered from the damage caused by the war and the economy was not active. Modern-style books of mathematics for beginners were not published yet. By chance, in the library for citizens established by the US Army, I found a book written by G. Birkhoff and S. Mac Lane: *A Survey of Modern Algebra*. By reading this book, I came to understand the difference between a matrix which describes a linear transform (with fixed basis system) and the matrix which describes the change of basis. This is my true encounter with matrices. When I stayed for one year in CalTech (1963), I enjoyed the elementary seminar run by Olga Taussky and my interest in matrices began. Since my initial academic career focused on functional analysis, my approach to matrices was always from the stand point of functional analysis, in other words, I viewed a matrix as a specialization of a linear operator on a Hilbert space.

Second, in a seminar under the leadership of Prof. H. Nakano, an elder colleague of mine, Dr. T. Ito, introduced a problem which he had learned from an abstract of a talk at a conference in Hungary. It asks, given a pair of *commuting contractions* A, B of a Hilbert space, whether there exists a pair of *commuting unitary operators* U, V on a superspace such that

$$\langle A^n B^m x|x\rangle = \langle U^n V^m x|x\rangle \quad \text{for all } n, m = 0, 1, \ldots, \text{and for all vectors } x.$$

After a few days, I was able to give an affirmative answer as an exercise of operator theory [2]. Yet this result attracted many mathematicians and was later extended to a wide theory of the *commutant lifting theorem*, which is now a leading topic in linear systems theory. But at the time, I was not able to extend the result more broadly.

Third, in the field of Hilbert space operators, I had a joint paper [1] with a more senior colleague of mine, Professor I. Amemiya. We proved that given a finite number of contractive selfadjoint operators $T_1, \ldots, T_N$ every random product $T_{\sigma_1} T_{\sigma_2} \cdots T_{\sigma_n}$, where $\sigma_k \in \{1, 2, \ldots, N\}$ is a random choice, converges *weakly* to a limit provided every T_k $(k = 1, 2, \ldots, N)$ appears infinitely often. At first I could not imagine that random choice is possible. It was a motto of Prof. I. Amemiya that once the problem is well-formulated, then the problem itself is already almost solved. He showed me how to formulate the problem in this case. Whether *strong* convergence is possible or not in this result has stood as an open problem, attracting several mathematicians. Recently I heard some mathematician gave a counter example for strong convergence.

Fourth, I am secretly proud of a paper [6] of mine on matrix (or operator) inequalities which has few citations. Nowadays many mathematicians are interested in giving a reasonable definition of a geometric mean of an n-tuple of positive definite matrices. I read a paper of W. N. Anderson et al., in which they gave the geometric mean as the limit of successive iterations of symmetric function means. In this connection they raised several problems on matrix inequalities. Let me consider the simplest case of a triple $\mathbf{A} := (A_1, A_2, A_3)$ of positive definite matrices. Define

$$\mathfrak{S}(\mathbf{A}) := \frac{1}{2}\Big\{A_1 : (A_2 + A_3) + A_2 : (A_3 + A_1) + A_3 : (A_1 + A_2)\Big\}$$

and

$$\mathfrak{s}(\mathbf{A}) := 2\Big\{(A_1 + A_2 : A_3) : (A_2 + A_3 : A_1) : (A_3 + A_1 : A_2)\Big\}$$

where, for $C, D > 0$, the symbol $C : D$ denotes their *parallel sum*, that is, $C : D = (C^{-1} + D^{-1})^{-1}$. Therefore $2 \cdot C : D$ is their *harmonic mean*.

Anderson et al. asked whether $\mathfrak{S}(\mathbf{A}) \geq \mathfrak{s}(\mathbf{A})$ for any triple $\mathbf{A}$ of positive definite matrices. For a commuting triple, equality holds. As usual, I started to try the numerical inequalities for a given triple $\mathbf{A}$:

$$\langle \mathfrak{S}(\mathbf{A})x|x\rangle \geq \langle \mathfrak{s}(\mathbf{A})x|x\rangle \quad \text{for all vectors } x$$

One day, a new idea came to my mind. Since $\mathfrak{s}(\mathbf{A})^{-1} = \mathfrak{S}(\mathbf{A}^{-1})$, the inequality in question becomes: for fixed x, y

$$\langle \mathfrak{S}(\mathbf{A})x|x\rangle \ + \ \langle \mathfrak{S}(\mathbf{A}^{-1})y|y\rangle \ \geq \ 2|\langle x|y\rangle| \quad \text{for all } \mathbf{A}$$

which means that the *infimum* with respect to all triples $\mathbf{A}$ on the left-hand side should be not less than $2|\langle x|y\rangle|$. This approach was successful on the basis that the parallel addition is determined as an infimum:

$$\langle (C : D)x|x\rangle \ = \ \inf\Big\{\langle Cy|y\rangle + \langle Dz|z\rangle : \ x = y + z\Big\}.$$

I was happy to see that this change of approach was successful.

Li - What do you see as your main contribution to linear algebra and other topics?

Ando - First, my research in Hilbert space operators and matrices was mostly from the standpoint of order structure. Here ordering is induced by the cone of positive semi-definite operators (Löwner order). In studying the paper of M.G. Krein on extension of positive symmetric operators, I found that his result, credited to I. Gelfand, implies that the Schur complement of a positive semi-definite matrix S is characterized as the maximum among those positive semi-definite matrices dominated by S and vanishing on the suitable subspace. I wrote my first matrix paper [5]. A related observation brought me to a construction of a natural *Lebesque type decomposition* of a positive operator [3].

Second, around 1978 I had a chance to give a lecture on operator inequalities, focusing on functions which preserve the order relation via functional calculus, namely operator-monotone functions. In this connection, I studied the mathematical background of the Löwner theory of operator monotone functions. Physicists had more interest in functions of this type than mathematicians. My lecture notes were well-accepted, even by physicists. Around the same time, E. Lieb solved, affirmatively, the so-called Dyson-Yanase conjecture on joint concavity of certain bilinear functional related to fractional powers of positive operators. I came to an observation that this result can be seen as the joint concavity of the map

$$(A, B) \longmapsto A^{1-\alpha} \otimes B^{\alpha} \quad (0 \le \alpha \le 1)$$

on the space of pairs (A, B) of positive definite matrices. Starting from this observation I could prove a theorem of the following type: that the map

$$(A, B) \longmapsto f[\Phi(A)^{-1} \otimes \Psi(B)] \cdot (\Phi(A) \otimes I)$$

is *concave* if the maps $\Phi(\cdot)$ and $\Psi(\cdot)$ are *concave* and if $f(\cdot)$ is *operator monotone function.* I wrote a slightly long paper on this theme with wide applications to inequalities for Schur (Hadamard) products of positive definite matrices [4]. My interest in operator monotone functions continued. With a younger colleague, Dr. F. Kubo, I introduced the notion of a mean for a pair of positive operators, especially the *geometric mean* [12].

Third, my interest in order structure included the indefinite case. The Krein space is a Hilbert space equipped with an indefinite inner product $[x|y]$ induced by a selfajdoint involution J, that is, $J = J^* = J^{-1}$ and $[x|y] := \langle Jx|y \rangle$. Naturally J-selfadjointness and J-positivity are defined in a natural way. For instance, $A \overset{J}{\ge} B$ means that $JA \ge JB$ in the usual sense. My lecture notes on operators on Krein Spaces were warmly accepted among those who were interested in J-structure. I could prove a J-generalization of the Löwner theorem of operator monotone functions [9]. Consequently, if A, B are J-selfadjoint with all their eigenvalues in a real interval (α, β) and $f(\cdot)$ is an operator monotone function on (α, β) then $A \overset{J}{\ge} B \implies f(A) \overset{J}{\ge} f(B)$. This seems to be the first contribution in this direction.

Fourth, in the 1960s I had an interest in *majorization.* This was based on the information from a talented colleague of mine, Dr. T. Shimogaki, an active researcher in the field of functions spaces (especially the space created by G.G. Lorentz) who passed away at the age of 40. The norm in a Lorentz space is rearrangement invariant, and decreasing rearrangement comes into consideration. Meantime I was impressed by the monumental paper by Ky Fan that the Ky Fan norms control all unitarily invariant norms. I wrote several joint papers on majorization and norm inequalities with several mathematicians, including R. Horn, C.R. Johnson, F. Hiai and X. Zhan. But my most meaningful contribution to this field seems to be the comparison of eigenvalues of the usual product and those of the Schur product of two positive definite matrices [7]. The same result was independently discovered by G. Visick.

Fifth, every square matrix A admits a polar decomposition $A = V|A|$ where $|A| := (A^*A)^{\frac{1}{2}}$ and V is a partial isometry. The matrix $\Delta(A) := |A|^{\frac{1}{2}} V |A|^{\frac{1}{2}}$ is called the *Aluthge transform* of A. This gives one-step smoothing of A toward normality. In fact, any limiting element of the iterates $\Delta^n(A),\ n = 1, 2, \ldots,$ is a normal matrix with the same eigenvalues as A. I could prove that the intersection of all numerical ranges of $\Delta^n(A)$ coincides with the convex hull of the eigenvalues of A, and that the convex hull of the eigenvalues of A coincides with its numerical range if and only if the numerical range of A coincides with that of $\Delta(A)$. [8].

Li - Do you see much change in the mathematical research environment in the past few years?

Ando - The number of research papers published each year is rapidly increasing. LAA in the early years published only few hundred pages, I suppose, but now over 6,000 pages appear. This is inevitable because of the dominance of the so-called "Publish or Perish" principle: mathematicians should publish as many papers as possible each year to keep or get an academic position. Citation number is not the only indicator of the true trend of mathematical research. Though it seems hard, in guiding Ph.D. students, a professor should show his/her conviction about the direction of the future development of mathematical research.

Li - Do you want to share any interesting stories about your research, teaching, editorial work, etc.?

Ando - Story 1. Somewhere I learned that Olga Taussky told students something like the following: "If you recognize some relations hold for numbers, then try to see whether the corresponding relations hold for 2×2 matrices." When I stayed in CalTech (1963), a female graduate student was interacting with Olga regarding her thesis on a topic related to 2×2 matrices. I wondered whether a topic on (a finite number of) 2×2 matrices could deserve to be a subject of a Ph.D. thesis. Many years later I wrote a short paper with Dr. T. Yamazaki on the convergence of iterated Aluthge transforms of any 2×2 matrix (to a normal matrix) [11]. The result makes the point that the limit establishes a natural retract to the subset of normal matrices. Though it was for the case of 2×2 matrices, our proof was quite difficult and we could not solve the corresponding result for the case of $n \times n$ matrices with $n > 2$. This made me change my view on 2×2 matrices. (The general case was settled later by several mathematicians, including Prof. Tin-Yau Tam (Auburn) by completely different methods.)

Story 2. My family name begins with the character "A," the first character of the alphabet. The community of mathematics has had tradition that the names of co-authors of a joint paper are arranged according in alphabetical order. This implied that I was typically listed as the first author. In most of other natural sciences, the first author is usually the person who made the principal contributions to the growth of the paper. Therefore physicists and/or chemists are always nervous as to the number of papers for which they are first authors. Now that many mathematicians are interested in extensions of concept of geometric means of several positive definite matrices, my joint paper [10] is frequently referenced. Someone may imagine that I made the central contribution. But it is not the case. The principal part of the paper was initiated by C-K. Li and R. Mathias, and I made few comments.[2] I think that it is now time for the mathematics community to get rid of the old tradition of arranging the names of co-authors in alphabetical order.

Li - Do you have any advice for young researchers?

Ando - First suggestion. My thesis supervisor, Prof. H. Nakano, once said something like the following: "A mathematician is the one who has created or is creating his own original theory. The one who could or can only solve some problems can not be called a mathematician." Quite a severe criterion! I could not agree with him completely. Quite recently, a favorite colleague of mine, Prof. F. Hiai, wrote that the character of T. Ando has been fit to solving challenging problems rather than to developing a new theory. I agree with him. There can be mathematicians of different types. A young mathematician should find their fit, and do their best.

Another suggestion. Prof. W. Takahashi, who is an esteemed friend of mine and the chief editor of the *Journal of Nonlinear and Convex Analysis*, once told me that some percentage of the journal is reserved for young mathematicians in order to encourage them. This is a clever consideration of the editor. Even if the first paper is not of high standard, a young mathematician should always try to improve the standard at each step, not staying at the same level.

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[2] Note by C-K. Li: Readers probably know that Professor Ando is a very modest gentleman. As an author of the paper, I can testify that the insights and contributions of Ando to the paper were substantial.

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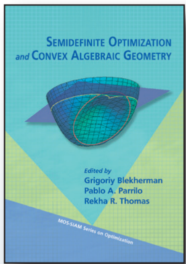

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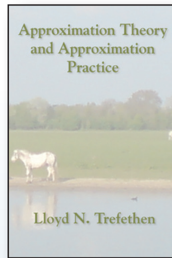

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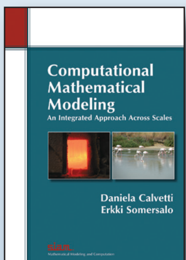

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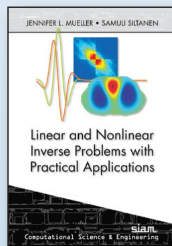

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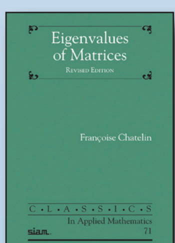

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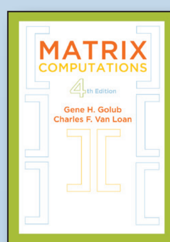

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ARTICLES

Linear Algebra, Sparsity, and Compressive Sensing

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Introduction. In numerical linear algebra, linear systems of equations, $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$, are often solved by exploiting the structure or sparsity of the coefficient matrix A. If A is upper triangular, then a simple back substitution will require $O(n^2)$, instead of $O(n^3)$, flops to solve the system. For very large systems, iterative methods such as the conjugate gradient method only uses matrix-vector products to compute an approximate solution; thus, it is highly advantageous to have a sparse and/or structured matrix A to perform these operations very quickly. In the recently developed framework called compressed sensing (CS) in signal processing, the sparsity and/or structure of the *solution*, x, and not necessarily that of A, is exploited. This framework has yielded very interesting results and has become a highly active area of research within the last few years for mathematicians, computer scientists, statisticians, and engineers because of its potential to greatly reduce the amount of data we have to collect to infer vast amounts of information. Many expository articles have been written on this subject (see e.g., [3, 9, 12, 24, 26, 33]), but they generally approach this topic from an engineering perspective. This particular article is meant to present compressed sensing to the broader linear algebra community.

Motivation. Large digital files, such as JPEG images and MP3 audio files, are often compressed to take less memory space for storage and to facilitate file transfer or download. The compression process results in a substantial reduction in file size without a perceptible loss in quality. For example, a 10 megapixel image taken from a digital camera is frequently reduced to a file size in the order of a few hundred kilobytes, with a minimal perceptual difference that is only noticeable when the image is enlarged. In JPEG compression for instance, the image is transformed using the discrete cosine transform (DCT), and only the significant nonzero coefficients are stored and transmitted (see [23] for details). Intuitively, scene objects have relatively smooth surfaces and can be represented by smooth basis functions, such as cosine functions. Since neighboring pixels in a digital image generally have similar intensity values, only very few functions are needed to represent the image. Thus, the basis coefficients are mostly zeros, i.e., the image has a sparse representation in the discrete cosine basis. Fig. 1 shows the coefficients of an image in two different bases (DCT and wavelets) and how retaining only one-fifth of the coefficients still leads to a faithful approximation to the original image.

What the *compressibility* of images in these basis functions indicates is that most of an image's energy is concentrated on only a fraction of the basis elements. In other words, a raw image contains a lot of redundant information that can be discarded in the compression stage. If that is indeed the case, then it would seem that taking direct, high-resolution point samples of a scene at specific locations is very wasteful. Rather, it would be more efficient to take fewer measurements that would identify the significant coefficients of the image in the sparsifying basis. The relatively new framework of compressed sensing (also known as compressive sampling) [11, 16] has provided mathematical insights on how to take such measurements. In particular, by taking judiciously chosen linear projections of the original scene, highly accurate representations can be obtained with very high probability.

Example: Consider the simple example of a 4×4 pixel image $F^\star$ with only 1 nonzero element with pixel intensity 255 in the (2,2) position of the image, as on line (1). (Think of a night sky with only one star represented by a single pixel. Most gray-scale image pixels are represented by 8-bit strings with values between 0 and 255, with 0 being black and 255, white.) Taking an image of $F^\star$ directly will require 16 measurements (one for each pixel to guarantee that no pixel is missed). However, by taking linear projections of $F^\star$, we can reduce the number of measurements, in particular, by using a binary search pattern (see [33]). In linear algebra notation, we can represent this as follows. First, stack the columns of $F^\star$ to obtain $f^\star \in \mathbb{R}^{16}$. For example,

$$F^\star = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 255 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \longrightarrow \quad f^\star = [\, 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 255 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,]^T. \tag{1}$$

Consider the following linear mapping $R \in \mathbb{R}^{4\times 16}$:

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

Instead of directly measuring $F^\star$ (e.g., using a camera with a 4×4 pixel photo-detector), we could compute the linear projection of $f^\star$ using R, which has the following interpretation. Let $y = Rf^\star$. Then the first element of y, y_1, is

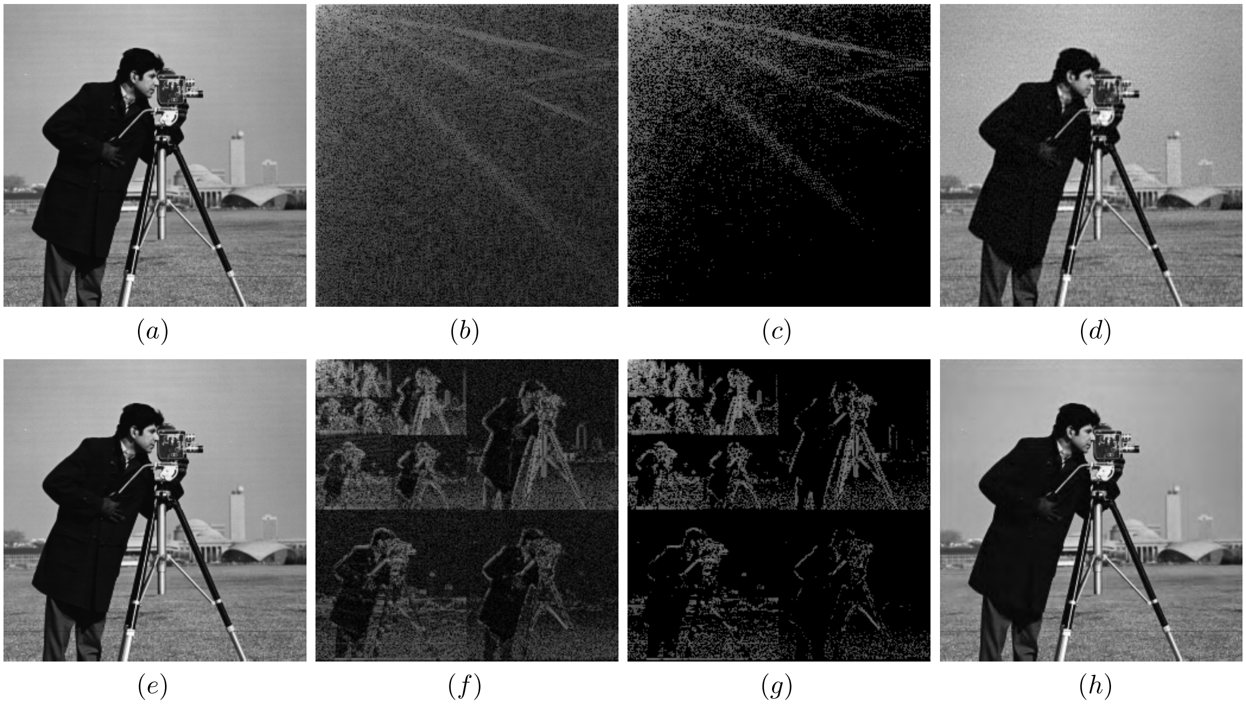


Figure 1: *A demonstration of the compressibility of the "cameraman" image in the discrete cosine transform (DCT) and wavelet bases.* (a) *Original image.* (b) *The DCT coefficients (in log scale).* (c) *The largest* 20% *DCT coefficients in magnitude.* (d) *The representation of the original image using only the top* 20% *DCT coefficients.* (e) *The same image as in* (a)*.* (f) *The wavelet coefficients of the image (in log scale).* (g) *The largest* 20% *wavelet coefficients in magnitude.* (h) *The representation of the original image using only the top* 20% *wavelet coefficients.*

the inner product of the first row of R with $f^\star$, which identifies which half of the image the nonzero pixel is located. The second measurement, y_2, identifies which quadrant; the third identifies which octant, and so forth. In this case, $y = [\ 255\ \ 0\ \ 255\ \ 0\ ]^T$, which indicates that the nonzero element has value 255 and is located on the left half, not in the first column, on the top half, and not on the first row. Using this strategy, it will take only $\log(16) = 4$ measurements to perfectly reconstruct $F^\star$. Thus, to image a 4×4 scene with only one nonzero pixel value, we can use a 2×2 pixel camera that takes linear projections of the scene rather than a 4×4 pixel camera. For real-world applications, this reduction in resolution can lead not only to substantial savings in the materials for constructing the camera and in the batteries needed to power it (e.g., for remote sensing applications), but to a significant decrease in the size and weight of the camera, features which can be crucial in some settings, e.g., its portability and maneuverability in video surveillance.

Problem formulation. For the remainder of this article, we will represent the signal or image of interest by a vector $f^\star \in \mathbb{R}^n$ and express it in some sparsifying basis $W \in \mathbb{R}^{n\times n}$, i.e., $f^\star = W\theta^\star$, where the coefficients $\theta^\star \in \mathbb{R}^n$. (In the example above, W is the identity matrix and $\theta^\star = f^\star$ since $f^\star$ is already sparse to begin with.) It is common for W to be orthogonal, such as the DCT and wavelet transform used in Fig. 1. When only $k \ll n$ coefficients in $\theta^\star$ are significant (i.e., sufficiently large in magnitude), we say that $f^\star$ is sparse or compressible in the basis W. Let $R \in \mathbb{R}^{m\times n}$ be an underdetermined linear projection matrix, and denote the low-resolution measurement by the vector $y \in \mathbb{R}^m$ obtained from projecting $f^\star$ using R, i.e., $y = Rf^\star$. Let $A = RW$ so that $y = Rf^\star = RW\theta^\star = A\theta^\star$. In practice, the measurements are corrupted by noise, which we represent by adding a zero-mean vector $w \in \mathbb{R}^m$ whose elements follow a Gaussian distribution. Thus,

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} \quad\quad A \quad\quad \end{bmatrix} \begin{bmatrix} \\ \theta^\star \\ \\ \end{bmatrix} + \begin{bmatrix} w \end{bmatrix}. \tag{3}$$

Now, in the example above, even though A is underdetermined and this inverse problem is ill-posed, we are able to solve for $\theta^\star$ because we used the *a priori* knowledge that the scene we are imaging contains only one nonzero pixel element. Compressed sensing extends this approach to vectors $\theta^\star$ with k nonzero elements (so-called k-sparse vectors) and can be summarized as follows: $\theta^\star$ can be accurately recovered provided (i) the measurement matrix A satisfies a certain statistical property (see below), (ii) the coefficients $\theta^\star$ of the high-dimensional image to be recovered is sufficiently sparse, and (iii) an appropriate optimization problem is solved. We discuss each of these items more fully.

Projection matrices. A crucial component of the CS framework is determining a measurement matrix A that leads to an accurate reconstruction of the original scene. In the example above, the measurement matrix R samples half the pixels with each measurement. For instance, y_1 is the sum of all pixel intensities on the left-half of the image. In other words, the sampling is sufficiently rich in information so that an accurate reconstruction of $F^\star$ can be obtained with only four measurements. One sufficient condition on A for recovering a k-sparse vector θ is known as the Restricted Isometry Property (RIP):

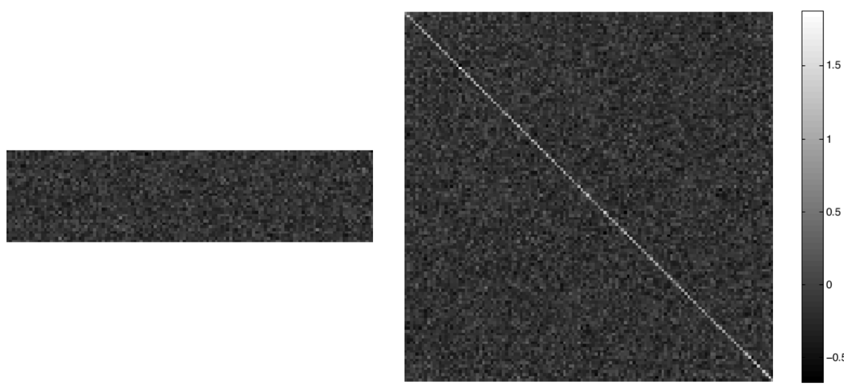
Definition: (Restricted Isometry Property [11]). The matrix A satisfies the Restricted Isometry Property of order k with parameter $\delta_k \in [0, 1)$ if

$$(1 - \delta_k)\|\theta\|_2^2 \leq \|A\theta\|_2^2 \leq (1 + \delta_k)\|\theta\|_2^2 \tag{4}$$

holds simultaneously for all sparse vectors θ having no more than k nonzero entries.

Assuming that δ_k is chosen to be the smallest value such that (4) holds, δ_k can be shown to be the largest distance from 1 of all the eigenvalues of all the matrices of the form $A_k^T A_k$, where A_k is any $m \times k$ submatrix of A (see [8]).

Matrices with this property are denoted RIP(k, δ_k). Roughly speaking, this means that A mostly preserves the Euclidean length of any vector θ that has at most k nonzero entries, i.e., when restricted to k-sparse vectors, A is almost an isometry. Alternatively, we can view RIP as a condition on A that states that the the energy of the observation vector $A\theta$ must be comparable to the energy of the original signal. For example, if A is a matrix whose entries are independent and identically drawn from a normal distribution with mean zero and variance $1/m$, then A is RIP(k, δ_k) with high probability as long as $k = O(m/\log(n))$ [4, 11]. (For instance, see Fig. 2. Note that the corresponding Gramian matrix $G = A^T A \approx I$, which intuitively explains why $\|\theta\|_2 \approx \|A\theta\|_2$ and why Eq. (4) would hold true.) Another example of a RIP-satisfying projection matrix is one whose entries are drawn from a Bernoulli distribution, taking only the values $1/\sqrt{m}$ or $-1/\sqrt{m}$ with equal probability. Finally, structured matrices like Toeplitz and circulant matrices whose entries are drawn from an appropriate distribution can be shown to be RIP-satisfying matrices as well [2].



(a) A (b) $G = A^T A$

Figure 2: *An example of a compressive sensing matrix A. (a) Here, A is 32×128 whose elements are drawn from a normal distribution with mean zero. (b) The 128×128 Gramian matrix $G = A^T A$. Note that off-diagonal entries of G are relatively small in magnitude compared to the diagonal entries.*

We note that there are other criteria on A for guaranteeing accurate signal recovery which we will not discuss here (e.g., see [17]). Given that the observation matrix A is RIP-satisfying, the following theorem guarantees that the original signal (an $\mathbb{R}^n$ vector) can be accurately approximated from an $\mathbb{R}^m$ observation vector (where $m \ll n$):

Theorem 1 [9, 10]: Let A be a matrix satisfying RIP$(2k, \delta_{2k})$ with $\delta_{2k} < \sqrt{2} - 1$, and let $y = A\theta^\star + w$ be a vector of noisy observations of any signal $\theta^\star \in \mathbb{R}^n$, where the w is a noise or error term with $\|w\|_2 \leq \varepsilon$. Let $\theta_k^\star$ be the best k-sparse approximation to $\theta^\star$, i.e., $\theta_k^\star$ is the approximation obtained by keeping the k largest entries of $\theta^\star$ and setting the others to zero. Then the estimate

$$\widehat{\theta} = \underset{\theta \in \mathbb{R}^n}{\arg\min} \;\; \|\theta\|_1 \quad \textit{subject to } \|y - A\theta\|_2 \leq \varepsilon, \tag{5}$$

satisfies

$$\|\theta^\star - \widehat{\theta}\|_2 \leq C_k \varepsilon + \widetilde{C}_k \frac{\|\theta^\star - \theta_k^\star\|_1}{\sqrt{k}},$$

where C_k and $\widetilde{C}_k$ are constants which depend on k but not on n or m.

Here ε is the variance on the noise in the data. This theorem implies that if the data are not corrupted by noise (i.e.,

$\varepsilon = 0$) and if the signal is truly k-sparse (i.e., $\theta^\star = \theta_k^\star$), then the approximation $\widehat{\theta}$ is *exact*. In other words, k-sparse signals are guaranteed to be fully recovered without error. Finally, the requirement that A is a matrix satisfying RIP$(2k, \delta_{2k})$ with $\delta_{2k} < \sqrt{2} - 1$ guarantees that θ will be reconstructed *uniquely* from y and A (see Lemma 1.3 in [11]).

Convex relaxation. To compute a sparse approximation $\widehat{\theta}$ to $\theta^\star$ from y, Theorem 1 suggests that we solve the nonlinear optimization problem (5) even though the original inverse problem (3) is linear. More specifically, it involves minimizing the ℓ_1 norm of θ to compute a sparse approximation $\widehat{\theta}$. Now, the ℓ_1 norm has long been used as an alternative norm for promoting sparsity in the solution [29] since minimizing the number of nonzero entries in θ (the so-called ℓ_0 semi-norm) is a combinatorial problem, which for large problems is computationally intractable and is therefore not a practical approach. Fig. 3 illustrates why an ℓ_1 relaxation to compute a sparse solution is intuitively a reasonable approach.

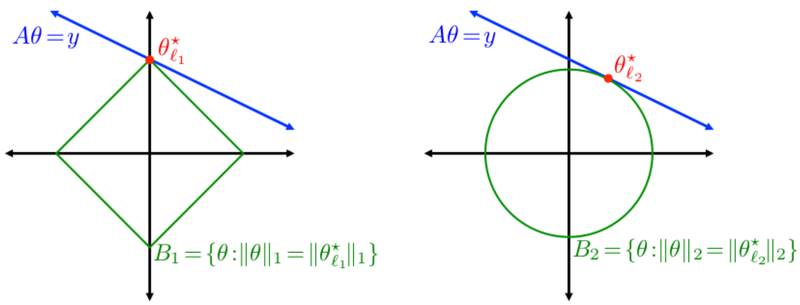


Figure 3: *A comparison between ℓ_1 and ℓ_2 minimization. Here $\theta^\star_{\ell_1}$ minimizes $\|\theta\|_1$ for all θ satisfying $A\theta = y$ and B_1 is the ℓ_1 ball with radius $\|\theta^\star_{\ell_1}\|_1$. Any θ smaller in ℓ_1 norm than $\theta^\star_{\ell_1}$ will lie inside B_1 and, therefore, will not satisfy $A\theta = y$. Note that the minimizer $\theta^\star_{\ell_1}$ will generally lie on one of the corners of an ℓ_1 ball, where some of the components will be zero and hence generate a sparse solution. In contrast, the ℓ_2 ball, B_2, is isotropic, and therefore, the minimizer will be independent of directions with any zero components. Thus the ℓ_2 minimizer, $\theta^\star_{\ell_2}$, will not generally produce a sparse solution.*

Relaxing the optimization problem using an ℓ_1 norm has several advantages. The optimization problem (5) can be written equivalently as

$$\widehat{\theta} = \underset{\theta \in \mathbb{R}^n}{\arg\min} \ \ \frac{1}{2}\|y - A\theta\|_2^2 + \tau\|\theta\|_1, \tag{6}$$

where $\tau > 0$ is a regularization parameter that balances the data-fidelity least-squares error term with the ℓ_1 sparsity-promoting regularization term. Even though this optimization problem is nonlinear, it is convex, which means that its minimizer will be the global minimizer. Also, the objective function now depends continuously on θ. Next, we will formulate the problem so that its derivatives can be computed, and hence, calculus of variations approaches can be used to solve for its minimizer. The ℓ_1 regularization term could in practice be replaced by a ℓ_2-norm Tikhonov regularization term, but as Fig. 3 shows, this approach will not yield a sparse solution.

Optimization. The ℓ_1 norm is not differentiable at 0, and consequently, gradient-based optimization methods cannot be directly applied to solve (6). However, the objective function can be transformed so that it can be differentiated by introducing a new vector $u \in \mathbb{R}^n$ with nonnegative components that serve as bounds on the magnitude of the components of θ: $-u_i \leq \theta_i \leq u_i$ for all $1 \leq i \leq n$. The ℓ_1 norm of θ can then be replaced by the sum of the u_i's, and the optimization problem becomes

$$\begin{aligned}(\widehat{\theta}, \widehat{u}) \quad = \quad & \underset{\theta, u \in \mathbb{R}^n}{\arg\min} && \frac{1}{2}\|y - A\theta\|_2^2 + \tau e^T u \\ & \text{subject to} && -u \leq \theta \leq u\end{aligned} \tag{7}$$

Here, $e \in \mathbb{R}^n$ is the vector of ones. Note that although the number of variables over which to minimize has doubled and bound constraints have been introduced, the optimization problem has become a typical quadratic programming problem. There are numerous algorithms for solving these sparse signal recovery problems, including those that use projected gradient methods [7, 18, 32], interior-point methods [19, 22], augmented Lagrangian methods [30], and iterative shrinking/thresholding methods [6, 34]. There are also non-gradient based approaches for compressed sensing. One family of such approaches is based on greedy methods called matching pursuits [14, 15, 31]. For a more complete listing of existing methods, see `http://dsp.rice.edu/cs`.

Other sparsity norms: Another widely used norm for measuring an image's sparsity is the *total variation* (TV) norm [27], which essentially measures the change in intensity values among adjacent pixels horizontally and vertically. Objects in natural scenes generally have smooth surfaces. Therefore, pixels near each other tend to have the same intensity values, and abrupt changes in pixel values generally correspond to an edge on the object. Thus, it is reasonable to store

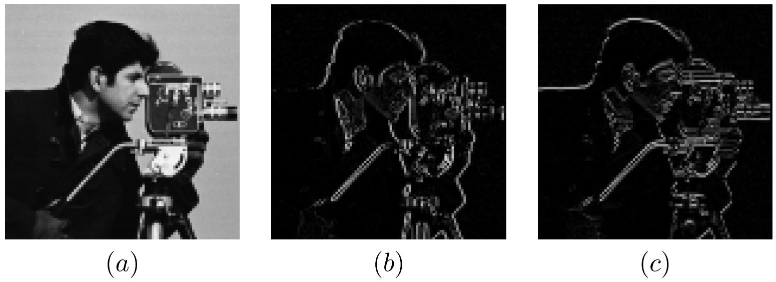


Figure 4: *An illustration of total variations as a sparsity measure.* (*a*) *A closeup of the cameraman.* (*b*) *The difference in magnitude between two adjacent pixels horizontally.* (*c*) *The difference in magnitude between two adjacent pixels vertically. Note the strong vertical edge of the inside collar; this corresponds to a large change in the horizontal direction, which is shown in* (*b*). *Similarly, the strong horizontal edge on the cameraman's right shoulder corresponds to a large change in the vertical direction, which is shown in* (*c*). *Finally, note that the edges of the diagonally-oriented pan handle part of the camera tripod appear in both* (*b*) *and* (*c*).

only where the pixel intensities *change*. The TV norm can therefore be viewed as a way of measuring the magnitude of the "gradient" of an image. Fig. 4 illustrates the changes of pixel intensities on an image both horizontally and vertically.

The TV norm can be defined in multiple ways. The anisotropic TV norm is equivalent to the ℓ_1 norm of the changes in pixel intensities in an image and is given by the following: If F is an $N \times N$ image, then

$$\|F\|_{TV} \;=\; \sum_{i=1}^{N-1}\sum_{j=1}^{N} |F_{i,j} - F_{i+1,j}| \;+\; \sum_{i=1}^{N}\sum_{j=1}^{N-1} |F_{i,j} - F_{i,j+1}|. \tag{8}$$

Many optimization methods for solving the CS problem (6) with a TV norm regularization term also exist. For example, see [5, 13, 25, 35].

Noise. In a variety of imaging applications such as medical imaging, night-sky imaging, and nocturnal wildlife observations, the measurements collected consist of counts of photons hitting a detector. Noise due to low photon count levels are modeled more accurately by Poisson processes [28] than the zero-mean additive Gaussian noise typically assumed in compressed sensing. In the Poisson model, the variance of the noisy observations is proportional to the signal intensity; and instead of using the ℓ_2 data-fitting term in (6) (which can lead to over-fitting in high-intensity regions and over-smoothing in low-intensity regions), the following penalized negative log-likelihood optimization is solved:

$$\begin{aligned} \widehat{\theta} \;&=\; \underset{\theta \in \mathbb{R}^n}{\arg\min} \quad \left\{ \sum_{j=1}^{n} (A\theta)_j - y_j \log(A\theta)_j \right\} + \tau\|\theta\|_1 \\ &\qquad \text{subject to} \quad W\theta \geq 0. \end{aligned} \tag{9}$$

Although the observations y are still linear projections of the original signal $A\theta$, much like (6), we must solve a nonlinear problem to promote sparsity in the solution and to counteract the effects of the noise. However, note that (9) is not quadratic, has potential singularities due to the log function term, and has linear constraints on the variables θ. Consequently, this is more difficult to solve than the $\ell_2 - \ell_1$ CS minimization problem in (6). Existing methods for solving (9) (or similar problems) can be found, for example, in [1, 21].

Illustration. Here, we describe one way a compressed sensing imaging system can be implemented in practice. Conventional digital cameras take direct point samples of a scene at specific locations. Thus, the resolution of the image is limited by the resolution of the focal plane (light-sensing pixel) array. One method of obtaining compressive measurements is through the use of coded apertures, which is a grid pattern of openings that can be designed so that the corresponding measurement matrix satisfies RIP (see e.g., [20]). The encoded observation does not resemble the scene; but by solving the $\ell_2 - \ell_1$ minimization problem (6) (the post-processing decoding step), a high-resolution approximation of the scene can be obtained (see Fig. 5).

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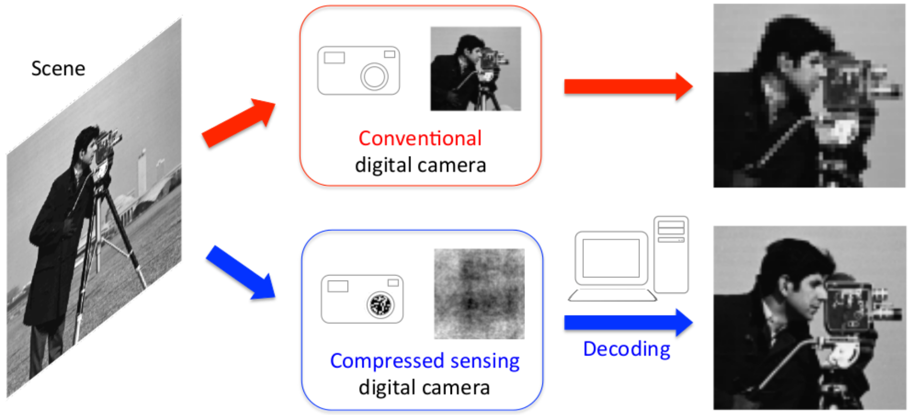

Figure 5: *An example of the compressed sensing framework. Conventional digital cameras are limited by the number of its photodetectors. Compressed sensing can overcome the pixel limitation of cameras by taking linear projections of the scene (e.g., encoded observed images) and recovering the original scene using a decoding step that involves solving a suitably formulated optimization problem (e.g.,* (6)*).*

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Comments on the Development of Linear Algebra in Poznań, Poland

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Linear algebra belongs to the parts of mathematics whose development is strongly caused by proper applications. Among the Polish mathematicians who significantly contributed to linear algebra through applications was Józef Hoene-Wroński, who used properties of matrices in the study of differential equations, introducing the Wronskian as early as 1811. Hoene-Wroński was born in Wolsztyn (close to Poznań); there is an archive located in the Kórnik castle (also close to Poznań) that includes important parts of his work. Another well-known Polish astronomer and mathematician that dealt with matrices was T. Banachiewicz, who, in 1922, introduced Cracovians (matrices with specifically defined operations) for use in astronomy; cf. [J. Grala, A. Markiewicz, and G.P.H. Styan, "Tadeusz Banachiewicz: 1882–1954," *IMAGE*, 25 (2000) 24].

Linear algebra is frequently applied in statistics, which have been developed in Poland from the beginning of 20$^{\text{th}}$ century (with large contributions by J. Neyman). During the 1970s, there was a group of statisticians, created under the direction of Professor Tadeusz Caliński at the Agricultural University in Poznań, who started intensive research into matrix methods with applications in statistics. There have been noted contributions of Jerzy K. Baksalary and Radosław Kala, along with some of their Ph.D. students, who continued the research oriented towards linear algebra with emphases on generalized inverses of matrices, projectors, partial orderings, matrix equations and inequalities, modified matrices; cf. [O.M. Baksalary, G.P.H. Styan, Some comments on the life and publications of Jerzy K. Baksalary (1944-2005), *Linear Algebra Appl.*, 410 (2005) 3–53].

Near the end of the 1980s, Tomasz Szulc began researching applications of matrix analysis in numerical analysis at the Faculty of Mathematics and Computer Science of Adam Mickiewicz University in Poznań. In the mid 1990s, Jan Hauke, Augustyn Markiewicz, and Tomasz Szulc started to cooperate by organizing seminars on linear algebra and its applications, later called the Poznań Linear Algebra Group (PLAG); cf. [*IMAGE* 34 (2005) 15]. One of the most important parts of these activities was to host visitors who would deliver speeches. This activity started in 1996, in connection with the second part of the "Linear Operators" semester held at the Banach Center in Warsaw (co-organized by Professor Jaroslav Zemánek from the Institute of Mathematics of the Polish Academy of Science). Some of the leading specialists in linear algebra and its applications visited the Faculty of Mathematics and Computer Science of Adam Mickiewicz University in Poznań within the last twenty years, one in particular being Hans Schneider. Some participants of the "Linear Operators" semester, as well as guests of the Faculty, also attended the PLAG seminar; one of them was Volker Mehrmann who encouraged us to attend the 1996 ILAS meeting in Chemnitz. The next year's visitors and collaborators from Poland included A. Smoktunowicz, Z. Woźnicki, K. Ziętak (working in the area of numerical algebra) and M. Niezgoda. Visitors from abroad included S. Puntanen, D. von Rosen, G.P.H. Styan, working on the border between linear algebra and statistics, and J. Barlow, R. Beuvens, R. Bru, L. Cvetković, L. Elsner, A. Frommer, C. Johnson, V. Mehrmann, C-K. Li, J. Peña, and S. Rump, working more directly in linear algebra.

The seminars have been regularly scheduled for every second Thursday since 1999. These group meetings were actively attended by J.K. Baksalary, his son O.M. Baksalary, K. Filipiak, and Ph.D. students. Seven of the Ph.D. students have successfully completed their theses; three from A. Markiewicz, three from T. Szulc, and one from C.R. Johnson. Our discussions have resulted in published papers, several of which are included in *Linear Algebra and Its Applications*. J.K. Baksalary has published 50 papers in this journal, some of them with O.M. Baksalary, J. Hauke, A. Markiewicz, and T. Szulc as co-authors. The PLAG members published 80 papers in this journal.

Another form of international collaboration was the organization of workshops and conferences. First came the 13$^{\text{th}}$ International Workshop On Matrices and Statistics (IWMS), held in Będlewo in celebration of Ingram Olkin's 80$^{\text{th}}$ birthday, with the participation of famous mathematicians such as T.W. Anderson, G. Golub, and Volker Mehrmann. Selected papers presented during the conference were published in special issue of *Linear Algebra and Its Applications*, Vol. 417, 2006 (with guest editors L. Elsner, A. Markiewicz and T. Szulc). Next year, encouraged by V. Mehrmann, we organized the first conference in the MatTriad series which was also held at the Będlewo Conference Center. The MatTriad conferences are devoted to applications of linear algebra, including applications in statistics. They were organized every two years: in 2007 and 2009 in Będlewo, and in 2011 in Tomar, Portugal (publishing selected papers in the *Electronic Journal of Linear Algebra* with O.M. Baksalary as one of the guests editors). The fifth conference will be held in Herceg Novi, Montenegro; `http://mattriad2013.pmf.uns.ac.rs/`. The Scientific Committee of the conference consists of Tomasz Szulc (chair), Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany), and Simo Puntanen (Finland). It became a tradition that an important part of the conference program are short courses. These were delivered by A. Cegielski and C.R. Johnson in 2009, N. Balakrishnan, Ali S. Hadi, and V. Mehrmann in 2011, and in 2013 lectures will be delivered by Adi Ben-Israel and S. Rump.

The work of young scientists has a special position in the MatTriad conferences. Awards are given to the best poster, as well as the best talk, of graduate students or scientists with a recently completed Ph.D. Prize-winning works are widely publicized and promoted by the conference and winners are invited as speakers for the next issue of the conference, Olivia Walch and Paulo Canas Rodrigues are the MatTriad 2013 invited speakers for Young Scientists Awards (YSA) in 2011.

Some participants in our PLAG seminar are also statisticians, who are involved in the organization of the LinStat conferences, in 2008 in Będlewo, in 2010 in Tomar, Portugal, and in 2012 combined with the 21$^{\text{st}}$ IWMS in Będlewo. The next LinStat conference will be held in Linköping, Sweden; `http://www.mai.liu.se/LinStat2014/`, and 22$^{\text{nd}}$ IWMS will be held in 2013 in the Fields Institute in Toronto, Canada; `http://www.fields.utoronto.ca/programs/scientific/13-14/IWMS/`.

Chebyshev, Krylov and other Legends of Linear Algebra in Russia

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There are at least two legendary Russian mathematicians who are tremendously well-cited in linear algebra papers: Pafnuty Lvovich Chebyshev (1821-1894) and Alexei Nickolaevich Krylov (1863-1945).

The marvelous Chebyshev polynomials are defined as monic polynomials on a closed interval with minimal deviation from zero. They appeared in his original memoir of 1857 with the French title "Sur les questions de minima qui se rattachent à la représentation approximative des fonctions," published in *Memoirs of Imperial Academy of Sciences*. Chebyshev's polynomials are recognized as the beginning of approximation theory. As is typical for the many other Russian mathematicians who contributed to linear algebra, it is not appropriate to say that his name belongs merely to linear algebra. In fact, Chebyshev is considered as a founding father of Russian mathematics in the whole, with thousands of scientific descendants and with quite a few of them becoming really great names, such as Dmitry Grave, Alexander Lyapunov, Andrei Markov, Georgy Voronoi, Julian Sokhotski, Egor Zolotarev and others. Besides many achievements in mathematics, mechanics and education, Pafnuty Chebyshev created the first mathematical journal in Russia which is regularly translated in English and well-known in the West as *Sbornik Mathematics*.

Alexei Krylov was primarily a naval engineer and applied mathematician. He founded the theory of magnetic and gyro-compasses and proposed a special device, called the dromoscope, to automatically calculate the deviation for a compass. By suggesting the theory of oscillating motions of the ship, he became the first foreigner in Great Britain who received the Gold Medal from the Royal Institution of Naval Architects. He put forth the theory of damping of ship rolling and pitching, and proposed the method of gyroscopic damping of the roll. As he claimed himself, his primary profession was shipbuilding, or, more accurately, application of mathematics to various questions of the maritime science. In linear algebra, the name of Alexei Krylov is given to special subspaces which are fundamental in most numerical methods pervasive in applications, such as Conjugate Gradients (CG), Generalized Minimal Residuals (GMRES), Lanczos bidiagonalization algorithm, etc.

The Krylov subspaces were born in 1931, in the paper titled "On numerical solution of the equation of technical sciences for frequencies of small oscillations in material systems". In this paper, Krylov proposed a method for the computation of the coefficients of characteristic polynomial of a matrix. Krylov's idea was to reduce the equation

$$\det \begin{bmatrix} a_{11}-\lambda & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22}-\lambda & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn}-\lambda \end{bmatrix} = 0 \tag{10}$$

to an easier calculated form

$$\det \begin{bmatrix} b_{11}-\lambda & b_{21} & \dots & b_{n1} \\ b_{12}-\lambda^2 & b_{22} & \dots & b_{n2} \\ \dots & \dots & \dots & \dots \\ b_{1n}-\lambda^n & b_{2n} & \dots & b_{nn} \end{bmatrix} = 0. \tag{11}$$

He obtained (11) from (10) by simple transformations of the equations

$$\begin{aligned} \lambda x_1 &= a_{11}x_1 + a_{21}x_2 + \ldots + a_{n1}x_n, \\ \lambda x_2 &= a_{12}x_1 + a_{22}x_2 + \ldots + a_{n2}x_n, \\ &\dots \\ \lambda x_n &= a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{nn}x_n. \end{aligned}$$

The first step is multiplication of the first equation by λ and replacing λx_1, …, λx_n with the right-hand sides of the original equations. The second step is multiplication of the new equation by λ and the substitution as previously, and so on. Denote by b_1, …, b_n the column vectors composed of the coefficients of the rows of the final equations

$$\begin{aligned} \lambda\ x_1 &= b_{11}x_1 + b_{21}x_2 + \ldots + b_{n1}x_n, \\ \lambda^2 x_1 &= b_{12}x_1 + b_{22}x_2 + \ldots + b_{n2}x_n, \\ &\dots \\ \lambda^n x_1 &= b_{1n}x_1 + b_{2n}x_2 + \ldots + b_{nn}x_n. \end{aligned}$$

Let A be a matrix with the entries a_{ij}. Then, it is not difficult to see that

$$b_1 = Ar_0, \quad b_2 = A^2 r_0, \quad \dots, \quad b_n = A^n r_0, \quad \text{where} \quad r_0 = [1, 0, \dots, 0]^\top.$$

For any r_0, the vectors r_0, Ar_0, … , $A^k r_0$ are now called the *Krylov vectors* and their linear spans for different k are known as the *Krylov subspaces* generated by r_0 and A.

The equation (10) is equivalent to (11) if the Krylov vectors r_0, Ar_0, … , $A^{n-1}r_0$ are linearly independent. If they are not, the method seems to fail. However, let

$$L_k := \operatorname{span}(r_0, Ar_0, \dots, A^{k-1}r_0)$$

and suppose that

$$\dim L_k = \dim L_{k+1} = k.$$

Then L_k is an invariant subspace with respect to A, and the method allows one to obtain the coefficients of a polynomial which is a divisor of the characteristic polynomial. In particular, one can find the minimal polynomial for A considered on L_k, and with an appropriate choice of r_0 it is always possible to get the minimal polynomial for A in the whole space.

It is interesting that Nickolai Nickolaevich Luzin (the founder of the Moscow school of theory of functions with many outstanding pupils including P. Alexandrov, A. Kolmogorov, S. Sobolev and many others) published a paper with further analysis of the Krylov method. Several subsequent papers by other authors were also devoted to a better understanding of Krylov's method.

In modern analysis of iterative methods, the names of Chebyshev and Krylov are both essential and naturally meet, because the Krylov vectors are *intrinsically linked with polynomials*. Consider a linear algebraic system $Ax = b$ with a symmetric positive definite matrix A. Then the CG method begins with an initial guess x_0, constructs the Krylov subspaces L_k with $r_0 := b - Ax_0$, and defines the kth iterate x_k so that it solves the minimization problem

$$||b - Ax_k||_{A^{-1}} = \min_{x \in x_0 + L_k} ||b - Ax||_{A^{-1}},$$

with

$$||x||_{A^{-1}} := \sqrt{(A^{-1}x, x)}.$$

Obviously, for any $x_k \in x_0 + L_k$ we have $r_k = f_k(A)r_0$ with $f_k \in F_k$, where F_k is the set of all polynomials of degree bounded by k and with the constant term equal to 1, i.e. $f_k(0) = 1$. From the above minimization property, we get

$$||b - Ax_k||_{A^{-1}} \le \min_{f_k \in F_k} ||f_k(A)r_0||_{A^{-1}} \le \min_{f_k \in F_k} \max_{m \le \lambda \le M} |f_k(\lambda)|\ ||r_0||_{A^{-1}},$$

where m and M are the minimal and maximal eigenvalues of A. The Chebyshev polynomials help to minimize the right-hand side. It is the standard way to yield the classical convergence estimates for the CG and other Krylov-space methods.

Another Russian that struck deep roots in linear algebra is Semyon Aronovich Geršhgorin (1901-1933). He had a vibrant mathematical talent, wrote several brilliant works in applied mathematics, and unfortunately died a rather young man in Leningrad. He chiefly worked on numerical and mechanical solution of PDEs. One of his works contains a careful proof of convergence for a finite-difference scheme for solving the Laplace equation. And at the same time, his localization theorems for the eigenvalues via the Geršhgorin discs belong directly to the gold reserves of linear algebra. The first Geršhgorin circle theorem states that the eigenvalues of $A = [a_{ij}]_{n\times n}$ lie in the union of the Geršhgorin discs

$$D_i := \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\}.$$

The second Geršhgorin circle theorem adds that if a union of k discs does not intersect with the other discs then it contains exactly k eigenvalues of A. These results were published in German in 1931, and Ludwig Elsner especially stresses the importance of the language for wider dissemination of good results: "It is safe to guess that this was one of the reasons that certain people outside the Soviet Union, such as Olga Taussky and Alfred Brauer, studied it and made it known in the West during the 1940s. Other important papers written in the Soviet Union in the 1930s became known in the West only much later" [L. Elsner, Review: Geršgorin and His Circles by Richard S. Varga, *Amer. Math. Monthly*, 113 (4) (2006), 379–381].

The whole picture of linear algebra, matrix theory and numerical linear algebra in the 20^{th} century cannot be complete without at least two famous encyclopedic monographs originally written in Russian, namely *Matrix Theory* by Felix Ruvimovich Gantmacher (1908-1964) and *Computational Methods of Linear Algebra* by Dmitry Konstantinovich Faddeev (1907-1989) and Vera Nickolaevna Faddeeva (1906-1983).

Felix Gantmacher was born and started his mathematical career in Odessa, a city which now belongs to Ukraine. He moved to Moscow in 1934 and worked at the Steklov Institute of Mathematics. He read lectures at the Moscow Institute of Physics and Technology since 1947, the very year of organization of this technical university, first on the base of the Lomonosov Moscow State University and then as an independent unit. This university is known for its unique system of education known as the "system of PhysTech." It is characterized by intrinsic and direct links with the research institutes of the Russian Academy of Sciences. One of founders and visionaries of PhysTech was the Nobel Laureate and physicist, Pyotr Leonidovich Kapitsa. In its rich history, PhysTech became home to eight Nobel Laureates, and due to Felix Gantmacher, PhysTech became a center for linear algebra in Russia.

Note that PhysTech was also home for Victor Borisovich Lidskii (1924-2008), who enriched linear algebra by developing his inequalities for the eigenvalues of Hermitian matrices and their sums. These are wonderful topics related to majorization theory and to the remarkably intricate theorems by Lidskii, Wieland and Mirsky. From historical remarks of G.W. Stewart and J. Sun in *Matrix Perturbation Theory*, one can learn that "Wieland proved his theorem because he was unable to succeed in completing the interesting sketch of proof given by Lidskii" in 1950. Lidskii was evidently loved by his students, and vice versa, as is witnessed in the myths and legends of PhysTech. Students liked his lectures although they did not feel safe when attending them, because he used to ask them questions concerning the topics at study and did not like it if he got no answer. In one of the PhysTech stories, he was at an exam taking place at the assembly hall of PhysTech with a grand piano on stage, and being extremely disappointed by the answer of a student, he ran to the piano and expressed himself by playing a very wrathful melody.

Felix Gantmacher is also mentioned in the myths and legends of PhysTech. When he was seriously ill in the beginning of a semester, he convinced his doctors to allow him to leave the hospital for a few hours so that he could deliver the first introductory lecture to his students at PhysTech. He wanted to demonstrate the greatness and splendor of his favourite science in at least one lecture. He arrived by ambulance and read this lecture sitting in a chair. Of course, this was an impressive act of the spirit showing devotion to the science, and welcoming the new generation to it. *Matrix Theory* by Gantmacher has had several editions on several languages and is truly recognized by experts and novices as a bible of matrix theory.

We should mention at least one more well-cited book: *Oscillation matrices and kernels and small vibrations of mechanical systems.* Written by Gantmacher and Mark Grigorievich Krein (1907-1989), the first edition appeared in 1941. Note that the so-called semiseparable structure of the inverses to tridiagonal matrices was one of many findings of Gantmacher and Krein. Far-reaching generalizations of this type of structure are one of hot topics in modern linear algebra of structured matrices.

Another bible of linear algebra in Russia was devoted to numerical linear algebra and appeared in print in 1950. *Computational Methods of Linear Algebra* was written by Vera Nickolaevna Faddeeva and then, in 1960 it was transformed into a joint book co-authored by D.K. Faddeev and V.N. Faddeeva. The two comprised an algebraic family in the direct sense. D.K. Faddeev was a prominent Russian algebraist and founder of the algebraic school of Saint-Petersburg, named Leningrad in the Soviet Union. His father, Konstantin Tikhonovich Faddeev, was noticed by A.N. Krylov, the author of the Krylov spaces, as especially talented among other maritime engineers. The son had a perfect ear and was very much inclined to pour his life into music. However, when he was a conservatory student of composition, he decided to study mathematics in parallel at Saint-Petersburg University, where his professors were I.M. Vinogradov and B.N. Delone. As a third year student, he had to make his choice between music and mathematics. As we know, mathematics won. His passion for music still accounts for the name Ludwig selected for the son of D.K. Faddeev and V.N. Faddeeva. Note that Ludwig Dmitrievich Faddeev became a distinguished mathematician and a full member of Russian Academy of Sciences, where his father was honored the status of corresponding member of the Academy.

From the historical point of view, by the claim of S.V. Vostokov and I.R. Shafarevich, the role of D.K. Faddeev in algebra and mathematics in the whole is somewhat underestimated. He worked at a time when mathematics in the West and inside Russia were separated, and mathematicians in both regions became aware of the achievements of the other with a good delay. As a result, a lot of effort was given to some fundamental things which in the end would be merely rediscovered. With his encyclopedic knowledge of mathematics and his utmost generosity and benevolence, Dmitry Konstantinovich Faddeev easily shared his ideas with others without seeking recognition for what he did. In linear algebra, he suggested his modification of the Le Verrier method for calculation of the characteristic polynomial of a matrix. The method is based on Newton sums and their recurrence relations. Later, this method was used by Csanky as a base of the best matrix inversion algorithm on a hypothetical computer with infinitely many processors and the instant communication among them. This method requires $O(log_2^2 n)$ parallel steps, and it is still not known whether it is the fastest parallel algorithm. D.K. Faddev left a nice textbook *Lectures in Algebra* and together with I.S. Sominsky he published a few popular editions of their *Collection of Problems in Higher Algebra.* Nevertheless, his most influential contribution to linear algebra was probably the book written by him together with his wife Vera.

Vera Nickolaevna Faddeeva was certainly an outstanding personality. The Russian writer V.V. Veresaev devoted a short narrative to her entitled *In Passing.* It is a story about young people who planned to make a club from a desolate building and began with long discussions how to do this. In contrast to the long talks, Vera Nickolaevna just took a mop and other necessary instruments and began to put this building to rights. It was a vivid manifestation of the maxim “The less said, the more done.”

I knew Vera Nickolaevna in person when I was a young researcher. The first memory is the couple of Vera Nickolaevna and Dmitry Konstantinovich sitting in the first row at the conference on numerical methods of linear algebra and never missing a single talk, unlike other much younger participants. Moreover, they were always eager to say a word of encouragement to younger people and colleagues, although it did not mean that their blessing was simple to obtain. Later, I remember Vera Nicholaevna doing a titanic work of collecting a bibliography on numerical linear algebra in the time before the advent of personal computers. Her other enterprise was a collection of the test matrices for numerical problems of linear algebra. She and her husband had a unique and infallible feeling of the new. When other researchers around them were only thinking about whether it is worthy or not to engage themselves in parallel computations, they simply started to study this topic and, in accordance with their encyclopedic approach to anything they do, published a short but rather comprehensive survey on parallel computations in linear algebra.

Here is one more snapshot of the personality of Vera Nickolaevna Faddeeva: “She inherited ebullient vitality, firmness of purpose and keen concern about all matters related to the Steklov Institute of Mathematics, her second home, and to her colleagues. Vera Nikolaevna also took trouble about people with whom she was not personally acquainted. She had a talent for enjoying life. Theatre, classical music, travels, and tours were only some of the many passions of this richly endowed person” [V. N. Kublanovskaya and L. Yu. Kolotilina, On the occasion of the centenary of the birthday of Vera Nikolaevna Faddeeva, *J. Math. Sci. (N.Y.)* 141 (6) (2007), 1573–1575].

In part, Faddeeva started her career at the Leningrad Institute of Constructions in 1935. For three years she belonged to the team headed by Boris Grigorievich Galerkin. Later, after the War with Hitler, she joined the Leningrad Division of Steklov Institute of Mathematics and became the head of the Laboratory of Numerical Computations. In Leningrad University, she worked with Mark Konstantinovich Gavurin and was associated with the computational mathematics unit set up by Leonid Vitalievich Kantorovich. At the Steklov Institute, together with Sergei Grigorievich Mikhlin, she headed the scientific seminar which was visited at different times by J.H. Wilkinson, G.E. Forsithe, R.S. Varga, G.H. Golub, M. Fiedler and others. The book by D.K. Faddeev and V. N. Faddeeva was kind of a bible at its time. Alston Housholder writes, “Each edition was, at the time of its appearance, by far the most comprehensive and up-to-date treatment of the subject in print.”

One of the famous colleagues of Faddeeva was Vera Nickolaevna Kublanovskaya (1920-2012). Her role is best expressed by the citation from the obituary written by Bo Kågström, Vladimir Khazanov, Frank Uhlig and Axel Ruhe: “She was the last surviving member of the generation that shaped modern numerical linear algebra, a group that included George Forsythe, Alston Householder, Heinz Rutishauser, and James Wilkinson” [*SIAM News*, July 17, 2012]. She began her career by working on a secret nuclear engineering project. In 1955, she joined the group of Leonid Vitalievich Kantorovich (1912-1986), having responsibility for the classification of matrix operations in numerical linear algebra from the viewpoint of a universal computer language that was the target set for the group. For most of her life, she worked at the Leningrad Division of the Steklov Institute of Mathematics.

Vera Kublanovskaya is most well-known as one of the inventors of the QR algorithm. Her milestone paper “On Some Algorithms for the Solution of the Complete Eigenvalue Problem” was published in 1961, in the same year as, and independently of, the two striking papers by G.F. Francis. These works together form the basis of the QR algorithm for the computation of all eigenvalues of an unsymmetric matrix. Vera Kublanovskaya suggested a quite sophisticated convergence proof using determinant theory. Later a much more transparent proof was proposed by J.H. Wilkinson.

Another celebrated paper by Vera Kublanovskaya was published in 1966. Entitled "On a Method for Solving the Complete Eigenvalue Problem for a Degenerate Matrix," it contained the basis for the staircase algorithm for the Jordan structure of a multiple eigenvalue. All forthcoming works on algorithms for the Jordan and Kronecker canonical forms of matrix pencils seem to use the findings of this paper. Vera Kublanovskaya continued working on canonical forms for polynomial and rational matrices, and published a survey titled "Methods and Algorithms of Solving Spectral Problems for Polynomial and Rational Matrices." It was first published in *Zapiski Nauchnykh Seminarov POMI* in 1997, with the English translation appearing in 2005.

Linear algebra and especially numerical linear algebra in Russia has everything to do with another great name: Valentin Vasilievich Voevodin (1934-2007). I am particularly pleased to write about him, because my first impression of linear algebra is impossible to separate from his personality. He had an amazing talent to inspire people with what he worked on and planned to do. I fell in love with linear algebra literally after the very first lecture delivered by Voevodin at the Lomonosov Moscow State University. Then he was my teacher and inspired me in every respect of science and life. His book *Computational Foundations of Linear Algebra*, published in 1977, contains a lot of his personal achievements in the analysis of round-off errors in linear algebra algorithms. In the preface, he asks whether linear algebra is a complicated science or not, and then paints the picture of the 1950s when he was among the first generation of computer scientists graduated from the Faculty of Mechanics and Mathematics at Lomonosov Moscow State University. They were encouraged to work on software for linear algebra, and from the outset that did not create much enthusiasm. The way linear algebra was taught left no doubt how everything in this branch of mathematics was supposed to be done. However, it was soon understood that linear algebra on computers differs dramatically from what the students were taught because of peculiarities of the machine arithmetic. This made it a challenging and rather young science.

All his life, Voevodin had a permanent bond with Lomonosov Moscow State University. Under the supervision of Professor Shura-Bura, he defended his 1962 candidate dissertation (analog of Ph.D.) on the topic *Solution of the algebraic eigenvalue problem by power methods*. Incidently, Wilkinson's proof for the QR algorithm reveals clearly a deep link between the QR and the power method. Seven years later, Voevodin was awarded a second doctoral degree, perhaps like habilitation in Germany, for the work *Rounding errors and stability in the direct methods of linear algebra*. The novelty of Voevodin's analysis focuses on a priori assumptions about rounding errors in two respects. First, he points out that individual arithmetic operations have small relative errors only when the result is not the machine zero. Second, in the derivation of probability estimates, he did not impose any assumption about the error distributions. He *proved* that rounding errors are asymptotically independent and uniformly distributed random quantities. For most operations, the limiting distribution turns out to be continuous; however, for addition and subtraction, it is discrete. If the numerical notation has an even base, e.g. binary, then an unremovable displacement takes place. Voevodin's rounding analysis is a uniquely subtle, laborious, and highly ingenious work.

It is important to mention at least two more of Voevodin's significant contributions to numerical linear algebra. First, he proposed various extensions of the Jacobi method with quadratic convergence, and also for unsymmetric matrices. Second, he constructed efficient block methods of linear algebra. Early on, the computers possessed a very small operative memory. However, the block methods of Voevodin allowed one to solve fairly large linear algebraic systems in a way that was at most twice longer than the time that would have been observed on an imaginary computer with the same rate of performance but with sufficiently large operative memory. Of course, this claim depended on the concrete technical characteristics of the computers in use. However, the claim is impressive, and the method can be seen behind modern approaches for the construction of algorithms using a hierarchical structure of memory.

During his work at the Lomonosov Moscow State University, Voevodin was chiefly engaged with matrices and computations. From 1956 until 1980, Voevodin worked at the Scientific Research Computer Center of the Lomonosov Moscow State University, and was the director from 1969 to 1978. From 1970 until 1981, he occupied a part-time professor position at the Faculty of Computational Mathematics and Cybernetics. His direct pupils, Khakim Dododjanovich Ikramov and Galina Dinkhovna Kim, still work at this faculty. Numerical linear algebra for grid equations was also among the interests of Alexander Andreevich Samarskii and his colleagues in the laboratory headed by Evgeny Sergeevich Nickolaev. In the 1980s, this laboratory housed a rather strong group, including Igor Evgenievich Kaporin and Andrei Borisovich Kucherov. In the 1990s, the former moved to the Computer Center of the Russian Academy of Sciences and the latter moved to the USA.

In 1981, Voevodin moved to the Institute of Numerical Mathematics of the Russian Academy of Sciences (INM), which was organized in 1980 by Guri Ivanovich Marchuk (1925-2013). Since that time, Voevodin was engaged with the analysis of parallel structures in numerical algorithms and the problem of mapping algorithms onto computational systems. And since that time, INM became one of the basic centers of numerical linear algebra in Russia. This was also because G.I. Marchuk especially appreciated the language of matrix theory in computational mathematics, which, unfortunately, is still not very typical for many others working in this area, at least in Russia.

Besides Voevodin, INM was home for Nickolai Sergeevich Bakhvalov (1934-2005). Bakhvalov had a series of outstanding results in computational mathematics, in particular the first rigorous proof for the multigrid method suggested by Radii Petrovich Fedorenko (1930-2010). Also at INM, Vyacheslav Ivanovich Lebedev (1930-2010) worked on, among other things, grid methods for PDEs and optimization of iterative algorithms developing the ideas of Chebyshev and Zolotarev. Among his pupils and younger colleagues, Andrew Knyazev now works in the USA and is known in the numerical linear algebra community by his block iterative algorithms for the minimization of the Raleigh quotient in the symmetric eigenvalue problem. Another pupil and younger colleague, Andrei Bogatyrev, works at INM and recently solved the so-called "Chebyshev program" resulting in low-parametric parametrizations of extremal polynomials and superfast algorithms for their construction. INM was home to Yu. A. Kuznetsov, who established a strong group working on fast grid solvers, partial solutions, iterations within subspaces, domain decomposition and fictitious domain technologies. In the end of 1990s, he moved to the University of Houston. His group at INM is now transformed into a group headed by his pupil Yu. V. Vassilevski.

I am personally happy to work at INM with my group which includes S.A. Goreinov, N.L. Zamarashkin, V.A. Chugunov, S.L. Stavtsev, D.A. Savostyanov and I.V. Oseledets. We accomplished a notable series of works on Toeplitz-like matrices and asymptotic distribution of eigenvalues and singular values for matrices generated by a multivariate Fourier series. These results generalize the classical theorems of Gábor Szegö and Seymour Parter. Other findings include the maximal volume concept for the low-rank approximation of matrices, cross interpolation algorithm and its application in the related mosaic-skeleton method deeply linked with the H-matrices of Hackbusch and the so-called multipole algorithms.

One of the recent achievements of this INM group is the Tensor-Train decompositions for multi-index array, and fast, reliable algorithms using tensor structures of data. A d-index array $a(i_1, \ldots, i_d)$ of size $n \times \ldots \times n$ contains n^d entries. That many entries cannot be stored on any computer, hence the main problem with d-index arrays is how to represent them. The Tensor-Train parametrization is expressed as

$$a(i_1, \ldots, i_d) = \sum_{\alpha_0, \ldots, \alpha_d} \prod_{k=1}^{d} g(\alpha_{k-1}, i_k, \alpha_k), \quad 1 \le \alpha_k \le r.$$

It is crucial that the right-hand side is defined by dnr^2 parameters instead of n^d, and equally important, that the complexity of basic algorithms using Tensor-Trains depends on d linearly, and on n and r polynomially. The leading contributions in this research belong to I.V. Oseledets, who proposed the SVD-based rounding procedure in the Tensor-Train format. It is this finding that made fast arithmetic for tensors numerically viable. The progress in this field is terrific.

Fortunately, political obstacles for international cooperation no longer exist and have been left far in the past. We established a very fruitful collaboration between INM and the group of W. Hackbusch at the Max-Planck Institute in Leipzig. During the 2000s, INM hosted several international conferences on matrix methods and operator equations. For such events, it was crucial to have the enthusiastic support from Gene Golub and Gil Strang, which I acknowledge with sincere gratitude. A recent enterprise is the Rome-Moscow school on Matrix Methods and Applied Linear Algebra, organized on the base of INM at Lomonosov Moscow State University and the University of Rome Tor Vergata. The school has already held three editions since 2010.

The picture of numerical linear algebra in Russia would be incomplete without Siberian scientists. In this regard, a very great name is Sergei Konstantinovich Godunov. Of course, he belongs to computational mathematics more broadly. In linear algebra, he revisited the rounding error analysis of its algorithms and constructed theory and algorithms evoked by the fact of the instability of the eigenvalue problem for unsymmetric matrices. He always emphasized that one should avoid using unstable quantities and reconsider the setting. As it may apply to the Jordan structure, he is sometimes rather extreme, and I personally heard in one of his lectures how he once said that "the Jordan structure should be an illegal subject in teaching our students." I think we should not take this claim too literally, yet it might be pertinent as a means to attract attention to the important problem of instability.

Of course, the list of important names should be continued. It must include A.A. Abramov, P. Chebotarev, A.E. Guterman, V.P. Il'in, I.S. Iokhvidov, L.A. Knizhnerman, Yu. I. Kuznetsov, L. Yu. Kolotilina, Yu. M. Nechepurenko, S.V. Savchenko and very many others. We should mention with pride the name of Israel Gohberg who started his outstanding career in the Soviet Union; cf. [*IMAGE*, 48 (2012) p.3]. The fundamental Gohberg-Sementsul formula for the inverses to Toeplitz matrices was obtained in that period of his life, and has inspired many researchers in Russia and all over the world.

I realize that many important names of Russian scientists who essentially contributed to linear algebra were not mentioned in this essay. A comprehensive survey was certainly beyond my purpose, and time constraints, for this work. Instead, I tried to show a few separate glimpses of brilliance according to my personal taste. Linear algebra in Russia is perhaps not a mainstream of mathematical sciences in this country, but is still remarkably versatile, profound, rich by results, and replenished by the younger generation. I rejoice in the latter, in particular as it promises a long life for this beautiful science in Russia as a part of a more and more closely connected world.

UPCOMING CONFERENCES AND WORKSHOPS

18th ILAS Meeting
Providence, Rhode Island, United States, June 3–7, 2013

The 18th Conference of the International Linear Algebra Society (ILAS) will be held in Providence, Rhode Island, U.S.A. on June 3–7, 2013. Confirmed plenary speakers include: Alan Edelman, Maryam Fazel, Anne Greenbaum, Ravi Kannan, Jean Bernard Lassere, Thomas Laffey, Dianne O'Leary, Ivan Oseledets, Leiba Rodman, Dan Spielman, Gilbert Strang, Raymond Sze, and Fuzhen Zhang,

The scientific organizing committee consists of Vadim Olshevsky (chair), Tom Bella (chair), Ljiljana Cvetković, Heike Faßbender, Chen Greif, J. William Helton, Olga Holtz, Steve Kirkland, Victor Pan, Panayiotis Psarrakos, Tin-Yau Tam, Paul Van Dooren and Hugo Woerdeman. The local organizers are Vadim Olshevsky (chair), Tom Bella (chair), Misha Kilmer and Steven Leon.

Invited minisymposia include: Linear Algebra Problems in Quantum Computation, Matrices and Graph Theory, Randomized Matrix Algorithms, Symbolic Matrix Algorithms, Krylov Subspace Methods for Linear Systems, Matrices and Orthogonal Polynomials, Matrix Methods for Polynomial Root-Finding, Multilinear Algebra and Tensor Decompositions, Nonlinear Eigenvalue Problems, Abstract Interpolation and Linear Algebra, Structured Matrix Functions and their Applications, and Linear Algebra Education Issues.

Contributed minisymposia include: Matrices over Idempotent Rings, Applications of Tropical Mathematics, Matrix Methods in Computational Systems Biology and Medicine, Advances in Combinatorial Matrix Theory and its Applications, Structure and Randomization in Matrix Computations, Generalized Inverses and Applications, Linear Least Squares Methods, Matrix Inequalities, Linear and Nonlinear Perron-Frobenius Theory, Matrices and Total Positivity, Linear Complementarity Problems and Beyond, Control and Optimization, and Sign Pattern Matrices.

The journal *Linear Algebra and its Applications* will post a special issue devoted to papers presented at the 18th ILAS Conference. For further details, please visit the conference website at `http://www.ilas2013.com/`.

Numerical Analysis and Scientific Computation with Applications (NASCA13)
Calais, France, June 24–26, 2013

The conference is organized by the "Laboratory of Pure and Applied Mathematics" (LMPA) and the Engineering School EIL Côte d'Opale. The aim of this conference is to bring together researchers working in numerical analysis, scientific computation and applications. Participants will present and discuss their latest results in this area.

The conference will take place at the Université du Littoral Côte d'Opale in Calais, France, June 24–26, 2013. The main topics include: large linear systems and preconditioning, eigenvalue problems; high-performance and parallel computation; linear algebra and control, model reduction; multigrid and multilevel methods; numerical methods for PDEs – finite element, finite volume, meshless methods; approximation, radial basis functions, scattered data approximation, splines; optimization; and applications – image processing, financial computation, and machine learning.

The plenary speakers are: Peter Benner (Max Planck Institute, Germany), François Desbouvries (Telecom SudParis, France), Paul Van Dooren (Catholic University of Louvain, Belgium), Gérard Meurant (CEA, France), Lothar Reichel (Kent State University, USA), Yousef Saad (University of Minnesota, USA), Roger Temam (Indiana University, USA), and Ronny Ramlau (Johann Radon Institute for Computational and Applied Mathematics, Austria).

For further details, please see the web page of the conference: `http://www-lmpa.univ-littoral.fr/NASCA13/` or contact the organizers at nasca13@lmpa.univ-littoral.fr.

Advanced School and Workshop on Matrix Geometries and Applications
ICTP, Trieste, Italy, July 1–12, 2013

The Advanced School and Workshop on Matrix Geometries and Applications will be held at the International Centre for Theoretical Physics (ICTP) in Trieste, Italy, July 1–12, 2013. ICTP was founded by Nobel Laureate Abdus Salam. Today, it operates under a tripartite agreement among the Italian government and two United Nations agencies, UNESCO and IAEA. Its mission is to support the best possible science with special attention to the needs of developing countries. At ICTP, high-level training courses, workshops and conferences are organized throughout the year. The participation of scientists from developing countries is supported by ICTP.

This event will be organized by R. Bhatia and P. Šemrl with the help of F.R. Villegas (who works full time at ICTP). The first week will be devoted to short introductory courses given by F. Barbaresco, R. Bhatia, D. Bini, J. Holbrook, L.-H. Lim, and P. Šemrl. Four one-hour lectures will be followed by problem/tutorial sessions each day. The main theme is the use of ideas from geometry to solve several problems in matrix theory. Computational aspects and diverse applications (image processing, quantum mechanics, quantum information, etc.) will be presented. The workshop will take place in the second week. Talks on more advanced topics will be given by invited speakers. There will be no contributed talks in this workshop.

It is expected that besides lecturers and tutors at the Advanced School, and speakers at the Workshop, around sixty participants will attend this activity, approximately half from developed, and half from developing countries. For further details, visit `http://cdsagenda5.ictp.trieste.it/full_display.php?ida=a12193`.

The 4th International Conference on Matrix Analysis and Applications Konya, Turkey, July 2–5, 2013

The 4th International Conference on Matrix Analysis and Applications will be held in Konya, Turkey, July 2–5, 2013. Online registration is ongoing: see the conference website at `http://icmaa2013.selcuk.edu.tr/`.

The main goal of the conference is to gather experts, researchers and students together to present recent developments in this dynamic and important field. The conference also aims to stimulate research and support interactions between mathematicians and scientists by creating an environment for participants to exchange ideas and to initiate collaborations or professional partnerships. The first three meetings were held in China and the USA.

The conference features keynote speaker Steve Kirkland (Stokes Professor at Hamilton Institute, National University of Ireland) and ILAS Lecturer Alexander A. Klyachko (Bilkent University, Ankara, Turkey).

Invited speakers include Delin Chu (National University of Singapore, Singapore), Carlos Martins da Fonseca (University of Coimbra, Portugal), Fuad Kittaneh (Jordan University, Jordan), Stephan Garcia (Pomona College, California, USA), Chi-Kwong Li (College of William and Mary, USA), Mohammad Sal Moslehian (Ferdowsi University of Mashhad, Iran), Tom Pate (Auburn University, USA), Yiu-Tung Poon (Iowa State University, USA), Nung-Sing Sze (The Hong Kong Polytechnic University, China), Hugo J. Woerdeman (Drexel University, Philadelphia, USA), Pei-Yuan Wu (National Chiao Tung University, Taiwan, R.O. China), and Xiao-Dong Zhang (Shanghai Jiaotong University, China).

The scientific organizing committee consists of Peter Šemrl, Tin-Yau Tam, Qingwen Wang, and Fuzhen Zhang. The local organizing committee consists of Vildan Bacak, Durmus Bozkurt, Serife Burcu Bozkurt,Ahmet Sinan Çevik,Yildiray Keskin, Hasan Köse, Ayse Dilek Maden, Galip Oturanç, Vehbi Paksoy, Necati Taskara, Ramazan Turkmen, Zübeyde Ulukök, and Fatih Yilmaz.

This conference is sponsored by ILAS. For more information and updates, visit `http://icmaa2013.selcuk.edu.tr/`.

SIAM Conference on Control and Its Applications (CT13) San Diego, California, July 8–10, 2013

This meeting is being held jointly with the 2013 SIAM Annual Meeting (`http://www.siam.org/meetings/an13/`), July 8–12, 2013, and is co-located with the SIAM Workshop on Network Science (NS13), July 7–8, 2013. The meetings will be held at the Town and Country Resort & Convention Center, San Diego, California.

The organizing committee co-chairs are: Fariba Fahro (Air Force Office of Scientific Research, USA) and Wei Kang (Naval Postgraduate School, USA). The organizing committee members are: Murat Arcak (University of California at Berkeley, USA), Francesco Bullo (University of California at Santa Barbara, USA), Tryphon Georgiou (University of Minnesota, USA), Qi Gong (University of California at Santa Cruz, USA), Daniel Hernández - Hernández (Centro de Investigación en Matemáticas, A.C. (CIMAT), Mexico), Naira Hovakimyan (University of Illinois at Urbana-Champaign, USA), Matthew James (Australian National University, Australia), William M. McEneaney (University of California, San Diego, USA (AN13 liaison)), Bozenna Pasik-Duncan (University of Kansas, USA), Maurice Robin (Digiteo, France), Shanjian Tang (Fudan University, China), Antonio Vicino (University of Siena, Italy), and Lizette Zietsman (Virginia Polytechnic Institute and State University, USA).

The invited speakers are: Nazareth Bedrossian (Halliburton, USA), John A. Burns (Virginia Tech, USA and Joint speaker with the 2013 SIAM Annual Meeting), Jean-Michel Coron (Université Pierre et Marie Curie, France), Arthur Krener (Naval Postgraduate School and University of California at Davis, USA) and Asu Ozdaglar (Massachusetts Institute of Technology, USA).

Registration and the conference program are posted at `http://www.siam.org/meetings/ct13/`.

Graduate Student Modeling Workshop (IMSM 2013) Raleigh, North Carolina, USA, July 15–23, 2013

The 19th Industrial Mathematical & Statistical Modeling (IMSM) Workshop for Graduate Students will take place at North Carolina State University, 15–23 July 2013. The workshop is sponsored by the Statistical and Applied Mathematical Science Institute (SAMSI) together with the Center for Research in Scientific Computation (CRSC) and the Department of Mathematics at North Carolina State University.

The IMSM workshop exposes graduate students in mathematics, engineering, and statistics to exciting real-world problems from industry and government. The workshop will provide students with experience in a research team environment and exposure to possible career opportunities. Local expenses and travel expenses will be covered for students at US institutions. Information is available at `http://www.samsi.info/IMSM13`, and questions can be directed to grad@samsi.info.

Workshop on Numerical Linear Algebra and Optimization
University of British Columbia, Vancouver, Canada, August 8–10, 2013

The scope of the workshop is fairly broad, involving the interplay between numerical linear algebra and optimization, with special attention to optimization problems concerning eigenvalues and singular values. The workshop will honor Michael Overton on the occasion of his 60^{th} birthday. It is organized by Ioana Dumitriu, University of Washington; Chen Greif, University of British Columbia; and Emre Mengi, Koc University.

Please see the conference website `http://www.pims.math.ca/scientific-event/130808-wnlao` for further details. The conference is hosted by PIMS (Pacific Institute for the Mathematical Sciences).

22^{nd} International Workshop on Matrices and Statistics
Fields Institute, Toronto, Canada, August 12–15, 2013

The 22^{nd} International Workshop on Matrices and Statistics, hosted at the University of Toronto by the Fields Institute, Toronto, Ontario, Canada, will be held 12–15 August 2013.

The purpose of the workshop is to bring together researchers sharing an interest in a variety of aspects of statistics and its applications as well as matrix analysis and its applications to statistics, and offer them a possibility to discuss current developments in these subjects. The workshop will bridge the gap among statisticians, computer scientists and mathematicians in understanding each other's tools. We anticipate that the workshop will stimulate research in an informal setting and foster the interaction of researchers in the interface between matrix theory and statistics. Some emphasis will be put on related numerical linear algebra issues and numerical solution methods, relevant to problems arising in statistics. The workshop will include invited talks and a special session with talks and posters by graduate students.

The international organizing committee consists of: S. Ejaz Ahmed, Chair (Canada), Augustyn Markiewicz (Poland), George P. H. Styan, Honorary Chair (Canada), Götz Trenkler (Germany), Jeffrey J. Hunter (New Zealand), Simo Puntanen, Vice Chair (Finland), Dietrich von Rosen (Sweden), Julia Volaufova (USA) and Hans Joachim Werner (Germany).

Further details can be found at `http://www.fields.utoronto.ca/programs/scientific/13-14/IWMS/`. See also the website for the IWMS series: `http://www.sis.uta.fi/tilasto/iwms/` and `http://www.sis.uta.fi/tilasto/iwms/IWMS-history.pdf`.

Note that the Joint Statistical Meetings will be held in Montreal, August 3–8, 2013: `http://www.amstat.org/meetings/jsm/2013/index.cfm`.

Frank Uhlig Retirement Colloquium
Auburn University, Alabama, USA, August 23, 2013

Frank Uhlig is retiring from Auburn University in the Fall of 2013 where he has been teaching since 1982, after teaching for ten years in Germany. Several colleagues will gather to celebrate Frank's contribution to mathematics at Auburn University on August 23, 2013. All are welcome to join. Please contact Tin-Yau Tam at tamtiny@auburn.edu if you plan to do so. The titles of the invited talks will be available in July or early August.

MatTriad 2013 – Conference on Matrix Analysis and its Applications
Herceg Novi, Montenegro, September 16–20, 2013

The Conference will be held in Herceg Novi (Montenegro), September 16–20, 2013. Up-to-date information is available at `http://mattriad2013.pmf.uns.ac.rs`. The aim is to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications in other fields of mathematics, and offer a possibility to discuss current developments in these subjects.

Researchers and graduate students in the area of linear algebra, statistical models and computation are particularly encouraged to attend the workshop. The format of this meeting will involve plenary talks and sessions with contributed talks. The list of invited speakers includes winners of Young Scientists Awards of MatTriad 2011 promoted by the conference. Works of young scientists continue to hold a special position at MatTriad 2013. The best poster, as well as the best talk, from graduate students or scientists with recently completed Ph.D.s will be awarded. Prize-winning works will be widely publicized and promoted by the conference.

The invited speakers are Richard A. Brualdi (USA), Stephen Kirkland (Ireland), and the winners of Young Scientists Awards of MatTriad 2011: Olivia Walch (USA) and Paulo Canas Rodrigues (Portugal). Lectures will also be given by Siegfried M. Rump (Germany) and Adi Ben-Israel (USA).

The scientific committee consists of Tomasz Szulc (Poland) – Chair, Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany) and Simo Puntanen (Finland). The organizing committee consists of Ljiljana Cvetković (Serbia) – Chair, Francisco Carvalho (Portugal), Ksenija Doroslovački (Serbia), and Vladimir Kostić (Serbia).

International Workshop on Operator Theory and its Applications (IWOTA 2013)
Bangalore, India, December 16–20, 2013

The XXIV$^{\text{th}}$ edition of the International Workshop on Operator Theory and Its Applications (IWOTA) will be held at the Indian Institute of Sciences, Bangalore, December 16–20, 2013. There is a rudimentary website at `http://math.iisc.ernet.in/~iwota2013`. There are plenary and semi-plenary talks. In addition, there will be special sessions at IWOTA 2013. Proposals should be submitted to iwota2013@gmail.com and should contain a brief description of the session and a preliminary list of speakers, and tentative lecture titles. There will be at least eight special sessions and each session will have between six to eight hours of time. The contact details are: Tirthankar Bhattacharyya (Main Organizer). Phone: +91 80 2293 2710. Fax: +1 509 561 1264. Email: iwota2013@gmail.com. For more details and registration, see `http://math.iisc.ernet.in/~iwota2013`.

Householder Symposium XIX on Numerical Linear Algebra
Spa, Belgium, June 8–13, 2014

The Householder Symposium XIX on Numerical Linear Algebra will be held in Spa, Belgium, June 8–13, 2014. The conference website is `http://sites.uclouvain.be/HHXIX/`. This symposium is the nineteenth in a series, previously called the Gatlinburg Symposia, and will be hosted by the Université catholique de Louvain and the Katholieke Universiteit Leuven. The symposium is very informal, giving priority to the intermingling of young and established researchers. Participants are expected to attend the entire meeting. The fifteenth Householder Award for the best thesis in numerical linear algebra since January 1, 2011 will be presented.

Attendance at the meeting is by invitation only. Applications will be solicited from researchers in numerical linear algebra, matrix theory, and related areas such as optimization, differential equations, signal processing, and control. Each attendee will be given the opportunity to present a talk or a poster. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. The application deadline will be some time in the Fall 2013. It is expected that partial support will be available for some students, early career participants, and participants from countries with limited resources.

The Householder Symposium takes place in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra.

19$^{\text{th}}$ ILAS Conference
Suwon, South Korea, August 6–9, 2014

The 19$^{\text{th}}$ ILAS conference is scheduled for August 6–9, 2014, to be held at Sungkyunkwan University, Suwon, South Korea. The ILAS conference will end less than a week before the 2014 ICM (International Congress of Mathematicians) which will be held in Seoul, Korea, August 13–21, 2014.

The ILAS organizing committee consists of: Nair Abreu, Tom Bella, Rajendra Bhatia, Richard A. Brualdi, Man-Duen Choi, Nick Higham, Leslie Hogben, Suk-Geun Hwang (chair), Steve Kirkland, Sang-Gu Lee, Helena Šmigoc, Fuzhen Zhang.

Themes selected include: Numerical Linear Algebra, Total Positivity, Approximation Algorithms, Matrix Polynomials, Quantum Computation, Matrices and Graph Theory, Random Matrix, Symbolic Matrix Algorithms, Matrices and Orthogonal Polynomials, Multilinear Algebra and Tensor Decompositions, Eigenvalue Problems, Structured Matrix, etc. There will also be the usual special lectures.

CONFERENCE REPORTS

Haifa Matrix Theory Conference
Haifa, Israel, November 12–15, 2012

Report by Avi Berman

The 2012 Haifa Matrix Theory Conference was held November 12–15, 2012 at the Technion - Israel Institute of Technology, Haifa, Israel. It was the 16$^{\text{th}}$ Haifa matrix conference in a series that started in 1984. The conference was organized by Avi Berman, Raphi Loewy and Naomi Shaked-Monderer, and included two special sessions in memory of Miki Neumann and Uri Rothblum. The ILAS speaker, Tom Laffey, presented *The characteristic polynomials of nonnegative matrices.*

Participants of the 16th Haifa Matrix Theory Conference

More than 100 mathematicians from 19 countries participated in the Haifa conference, including many of the founders of modern matrix theory, as well as quite a few young researchers. As in the previous Haifa conferences, the talks covered core linear algebra, numerical linear algebra, applications of matrix theory and teaching linear algebra. In spite of a condensed program, the participants had the opportunity to enjoy receptions hosted by the Technion Center for Mathematical Sciences and by the Mayor of Haifa, as well as a banquet dinner. The reception by the mayor, held in Madatech - The Israeli National Science Museum, was followed by a short tour of Haifa. The banquet talk, entitled *From Matrices to Space* was given by Danny Hershkowitz, who was at that time Israel's Minister of Science and Technology. It is not yet known when the next Haifa Matrix Conference will be held.

Matrices and Operators – Conference in honor of the 60th birthday of R. Bhatia
Bangalore, India, December 27–30, 2012

Report by Tirthankar Bhattacharyya

The conference was held at the Indian Institute of Science, Bangalore, India in honor of Rajendra Bhatia. It served as a meeting place for top matrix analysts and operator theorists from all over the world. The quality of talks was very high. All purposes of a good conference were fulfilled, viz., good talks, good interaction between listeners and speakers and good benefit to local participants. There was a large number of participants from Korea, Japan, Iran and Italy, but apart from these countries, there were speakers from all over the world. There were twenty-eight talks in all, each of forty minutes duration. Most speakers were collaborators of Bhatia. There was a series of three lectures by Rien Kaashoek. A large community of functional analysts from all over India came, listened to the talks, and in some cases, initiated discussion with the speakers. Some of these discussions will mature into full-scale collaborations.

The organizers (B.V.R. Bhat, T. Bhattacharyya, G. Misra and T.S.S.R.K. Rao) would like to thank ICTP and the Indian government for their support. For further information about the conference see `http://math.iisc.ernet.in/~bhatia60`.

Participants of Matrices and Operators Conference in Honor of Rajendra Bhatia

6th de Brún Workshop: Linear Algebra and Matrix Theory
NUI Galway, Ireland, December 3–7, 2012

Report by Rachel Quinlan

The 6th de Brún Workshop on Linear Algebra and Matrix Theory took place in the first week of December 2012 at the National University of Ireland, Galway, in the west of Ireland. It was followed on December 8 by the first Irish Meeting for Linear Algebra Research. The workshop featured five short lecture courses and a programme of 18 contributed talks. The short courses were given by Peter Brooksbank (Bucknell University), Richard A. Brualdi (University of Wisconsin), Iain Duff (Rutherford Appleton Laboratory and CERFACS), Carlos Fonseca (Universidade de Coimbra) and Charles Johnson (College of William and Mary). Their respective topics were Linear Methods in Computational Algebra, Combinatorial Matrix Theory, Sparse Matrices, Spectral Graph Theory, and Totally Positive Matrices. The 60 participants hailed from at least ten countries. The organisers are grateful to the main speakers and to all the participants for contributing to the success of the event. The workshop was the most recent in a series of conferences hosted by the de Brún Centre for Computational Algebra at NUI Galway. Funding was generously provided by Science Foundation Ireland (Mathematics Initiative 07/MI/007). For further details, please see `http://www.maths.nuigalway.ie/deBrun6/IrishLA.shtml`.

International Conference on Numerical Linear Algebra and its Applications (NLAA 2013) Guwahati, India, January 15–18, 2013

Report by Shreemayee Bora

An International Conference on Numerical Linear Algebra and its Applications (NLAA 2013) was organized by the Department of Mathematics, IIT Guwahati, January 15–18, 2013. This ILAS-endorsed event saw a healthy intermingling of young and experienced researchers from ten different countries. The talks delivered in the conference were mostly by invitation, with some contributed talks given by young researchers and faculty. The plenary and invited talks were delivered by some of the world's leading experts and these highlighted state of the art research in numerical linear algebra and its many applications. In addition, the young researchers got a valuable opportunity to present their work to a very eminent audience.

The conference was preceded by a workshop on numerical linear algebra at the same venue January 7–12, 2013. The workshop was fully sponsored by the National Centre for Mathematics, which is a joint initiative of IIT Bombay and TIFR Mumbai. The local organizing committee of the conference was chaired by Prof. Rafikul Alam, with Dr. Shreemayee Bora as the organizing secretary.

Participants of the International Conference on Numerical Linear Algebra and its Applications (*NLAA* 2013)

Special Sessions at the AMS Spring Central Section Meeting Ames, Iowa, April 26–28, 2013

The Spring Central Section Meeting of the American Mathematical Society was held April 26–28, 2013, at Iowa State University in Ames, Iowa. Two special sessions at the meeting focused on linear algebra. Pauline van den Driesshe and Bryan Shader gave ISU Keynote talks for these sessions.

The special session "Generalizations of Nonnegative Matrices and their Sign Patterns" was organized by Minnie Catral, Shaun Fallat and Pauline van den Driessche. There were 14 scheduled talks for the session but some of those speakers were unable to make it to the meeting. The talks were on topics related to various notions and generalizations of positivity/non-negativity, sign pattern classes of matrices, matrix factorizations and max-algebra. The session speakers were: In-Jae Kim, Pietro Paparella, Craig Erickson, Kevin Vander Meulen, Juan Manuel Peña, Olga Kushel, Chi-Kwong Li, Pablo Tarazaga and Hans Schneider. Pauline van den Driessche gave the keynote talk for this session entitled *Refined Inertia of Sign Pattern Matrices and Applications to Dynamical Systems.*

Pauline van den Driessche giving her ISU Keynote talk

The special session "Zero Forcing, Maximum Nullity/Minimum Rank, and Colin de Verdière Graph Parameters" was organized by Leslie Hogben and Bryan Shader. The session speakers were: Wayne Barrett, John Sinkovic, Keivan Monfared, Jason Grout, Tracy Hall, Michael Young, Seth Meyer, Adam Berliner, Nathaniel Dean, Nathan Warnberg, Luz DeAlba, Louis Deaett, Hein van der Holst and Shaun Fallat. Bryan Shader gave the keynote talk for this session entitled *Getting something for nothing (using transversality, linear algebra and combinatorics).*

Conference details and the full program can be found at `http://www.math.iastate.edu/Events/2013AMS/index.html`.

HONOURS AND AWARDS

ILAS member becomes SIAM Fellow and receives career award

Submitted by Stephen Kirkland

The Society for Industrial and Applied Mathematics (SIAM) has recently announced its SIAM Fellows for 2013, and long-time ILAS member Pauline van den Driessche is among those named as a class of 2013 Fellow. Professor van den Driessche was cited for her contributions to linear algebra and mathematical biology.

In addition, Professor van den Driessche is the 2013 recipient of the University of Victoria's Craigdarroch Gold Medal for Career Achievement. The Craigdarroch Gold Medal honours lifetime achievement and is awarded for a distinguished record of research which has substantially advanced the recipient's discipline.

On behalf of ILAS, I congratulate Pauline for these well-deserved acknowledgements of her mathematical accomplishments.

ANNOUNCEMENTS

LAA Special Issue on Matrix Functions: Call for Papers

Matrix functions can be broadly defined as matrices understood as changing quantities rather than given and constant. As such, the study of matrix functions encompasses a large part of linear algebra and its applications. This special issue is devoted to theoretical studies and applications of matrix functions in all their aspects. It will be open to all papers with significant new results where matrix functions play an important role and problems of linear algebraic nature are presented. Survey papers that illustrate several interconnected aspects of the theme of matrix functions and their applications are highly encouraged, as are research problems articles.

Areas and topics of interest for this special issue include, but are not limited to, the following two areas: (1) Methods and theory for: matrix polynomials, rational matrix functions, analytic and meromorphic matrix functions, matrix exponential, logarithm, square root, and others, functions of structured matrices, functions of large and sparse matrices, functions of matrices times a vector, conditioning and perturbations, interpolation. (2) Applications in: linear dynamical systems and ODE solvers, operator theory, singular systems, canonical systems of differential equations, integral equations, network analysis, control theory, model reduction, domain decomposition, mathematical physics.

The deadline for submission is July 31, 2013, and the special issue is expected to be published at the end of 2013. Papers should be submitted to the responsible Editor-in-Chief, V. Mehrmann, choosing the special issue "Matrix Functions," through the electronic submission system at `http://ees.elsevier.com/laa`. Submissions must meet the publication standards of *Linear Algebra and its Applications* and will be refereed in the usual way.

The editors of the special issue are: O. Ernst (TU Bergakademie Freiberg, Germany) ernst@math.tu-freiberg.de; C.-H. Guo (University of Regina, Canada) Chun-Hua.Guo@uregina.ca; J. Liesen (TU Berlin, Germany) liesen@math.tu-berlin.de; and L. Rodman (College of William and Mary, USA) lxrodm@math.wm.edu lxrodm@gmail.com. The responsible Editor-in-Chief for the special issue is V. Mehrmann (TU Berlin, Germany) mehrmann@math.tu-berlin.de.

Special Issues of *Annals of Functional Analysis* Dedicated to Professor Tsuyoshi Ando

A special volume consisting of two issues (2014) of *Annals of Functional Analysis* (AFA) is being dedicated to Professor Tsuyoshi Ando in celebration of his distinguished achievements in Matrix Analysis and Operator Theory. Contributors are invited by the Editor-in-Chief of AFA but a number of research articles related to the works of T. Ando by other mathematicians are also welcome. The deadline for submission to special issue is July 15, 2013. The instructions for authors can be found in the journal website `http://www.emis.de/journals/AFA/`.

LAA Special Issue in Honor of Leiba Rodman

Linear Algebra and its Applications (LAA) is pleased to announce a special issue in honor of Professor Leiba Rodman in recognition of his many important contributions in research and books to matrix theory, operator theory and applications, and on the occasion of his 65^{th} birthday in 2014. LAA solicits papers for the special issue within the entire scope of LAA or the research interests of Leiba Rodman. The deadline for submission of papers is April 1, 2014. Papers for submission will be subject to normal refereeing procedures according to the usual standards of LAA. They should be submitted via the Elsevier Editorial System (`http://ees.elsevier.com/laa/`), choosing the special issue "In Honor of Leiba Rodman" and the responsible Editor-in-Chief Richard A. Brualdi.

Authors will have the opportunity to suggest one of the following special editors for this issue to handle their submission: Joseph Ball, Virginia Polytechnic Institute and State University (USA), joball@math.vt.edu; Jussi Behrndt, Technical University Graz (Austria), behrndt@tugraz.at; Christian Mehl, Technical University Berlin (Germany), mehl@math.tu-berlin.de; Ilya Spitkovsky, College of William and Mary (USA), imspitkovsky@gmail.com.

LAA Special Issue on Statistics

Linear Algebra and Its Applications will publish a special issue on statistics devoted to both applications of linear algebra in statistics, as well as, statistical and probabilistic techniques being applied back to linear algebra. Papers should be submitted by September 30, 2013 via the Elsevier Editorial System (`http://ees.elsevier.com/laa/`), choosing the special issue called "Statistics." The responsible Editor-in-Chief is Peter Šemrl and he should be chosen as the handling Editor-in-Chief. Authors will have the opportunity to suggest one of the following guest editors to handle their submission: Mathias Drton, Lek-Heng Lim, Wei Biao Wu.

ETNA Special Volume for Lothar Reichel's 60$^{\text{th}}$ Birthday

The *Electronic Transactions on Numerical Analysis* (ETNA) announces a special volume dedicated to Lothar Reichel on the occasion of his sixtieth birthday. Papers will undergo the standard refereeing process. Submissions are being accepted starting immediately and should be sent directly to either Ronny Ramlau (ronny.ramlau@jku.at) or Daniel Szyld (szyld@temple.edu). Deadline for submissions is October 1, 2013. Inquires should be addressed to the same editors. The special volume will be published starting in January 2014. Papers will be published as soon as production of individual papers is completed.

BOOK REVIEW

Inequalities: Theory of Majorization and Its Applications, by A.W. Marshall, I. Olkin, and B.C. Arnold, Second edition

Springer Series in Statistics. Springer, New York, 2011, xxvii + 909 pages, ISBN 978-0-387-40087-7.
Reviewed by Martin Argerami, University of Regina, argerami@math.uregina.ca

Given two vectors $x, y \in \mathbb{R}^n$, we say that x *is majorized by* y (denoted $x \prec y$) if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}, \quad k = 1, \ldots, n-1, \qquad \text{and } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

Here $(x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ denotes the non-increasing rearrangement of $(x_1, \ldots, x_n)$. The book uses the notation $x_{[j]}$ instead, which is not a widespread standard. The minimal elements for the order $\prec$ are those vectors with all entries equal to each other, while the maximal are those with only one non-zero entry.

Majorization made its first appearance in the work of Lorenz on income inequality. Mathematically, I. Schur proved in 1923 that the diagonal of a matrix is majorized by the vector of its eigenvalues (together with the converse proven by A. Horn in 1954, this result makes the celebrated Schur-Horn theorem). Hardy, Littlewood, and Pólya recognized in 1929 that $x \prec y$ if and only if $g(x) \prec g(y)$ for every convex function g.

Since those early beginnings, many other characterizations of majorization are known. Here is a short list of equivalent statements for two vectors $x, y \in \mathbb{R}^n$:

1. $x \prec y$;
2. $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$, $\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow}$ for all $k = 1, \ldots, n$;
3. $\sum_j |x_j - t| \leq \sum_j |y_j - t|$ for all $t \in \mathbb{R}$;
4. $x \in \operatorname{conv}\{Sy : S \in \mathbb{S}_n\}$;
5. $x = Ay$, where A is a doubly stochastic matrix;
6. $\sum_j \phi(x_j) \leq \sum_j \phi(y_j)$ for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

All these and others, together with further facts about majorization and related notions, are considered in the first six chapters condensed in Part I. The rest of the book (14 chapters) is devoted to applications, grouped under Mathematical Applications, Stochastic Applications, and Generalizations.

The original edition of this book appeared in 1979. Since then, it has been the most complete source on majorization available, well beyond the authors' interest in statistics. Actually, only 136 of 520 pages in the first edition are explicitly dedicated to statistics (192 of 744 in the second edition). This second edition clearly looks like it has been typeset in LaTeX, although we could not find acknowledgement of such fact anywhere in the book.

The main structure and text of the original book has been preserved, with the majority of changes consisting in additional sections at the end of each chapter (the text is organized in the same 20 chapters from the original edition). In a few cases, some sections have been shuffled, and in others, titles have changed. Several of the old sections also contain some new material. The chapter "Matrix Theory" has additional sections (5), while "Stochastic Ordering" is a chapter which includes more material (43 pages, in three new sections, mostly about Lorenz Order). The "Biographies" chapter at the end of the book has been re-shuffled and slightly enlarged with the new biographies of Karamata and Jensen.

To get a glimpse of the variety of applications of majorization that appear in the book, let us mention some titles among the sections comprising the new material: Physical Interpretations of Inequalities, Top Wage Earners, Partitions, Graph Theory, Inequalities for Polygons and Simplexes, Hadamard Product, Totally Positive and M-matrices, Zeros of Polynomials, Stochastic Schur Convexity, Combining Random Variables, Concentration Inequalities for Multivariate Distributions, Optimal Experimental Design.

This is a book that should obviously appeal to matrix theorists, but it would also make an interesting addition to the libraries of statisticians and mathematicians of the most varied flavours, like those in combinatorics, inequalities, numerical analysis, probability, functional analysis, and mathematical physics. It is hard to imagine a mathematics department that could not benefit from owning this book.

ILAS AND OTHER LINEAR ALGEBRA NEWS

ILAS President/Vice President Annual Report: 2 May, 2013

Respectfully submitted by Steve Kirkland, ILAS President, stephen.kirkland@nuim.ie and Bryan Shader, ILAS Vice-President, bshader@uwyo.edu

1. The ILAS election in the fall/winter of 2012 proceeded once again via electronic voting. The following were elected in the ILAS 2012 elections to offices with terms that began on 1 March, 2013:

- Board of Directors: Avi Berman and Volker Mehrmann (term ends 29 February, 2016)
- Vice President: Bryan Shader (term ends 29 February, 2016)

The following continue in the ILAS offices to which they were previously elected:

- President: Steve Kirkland (term ends 28 February 2014)
- Secretary/Treasurer: Leslie Hogben (term ends 28 February, 2015)
- Board of Directors:
  - Dale Olesky (term ends 28 February, 2015)
  - Peter Šemrl (term ends 28 February, 2015)
  - Françoise Tisseur (term ends 28 February, 2014)
  - David Watkins (term ends 28 February, 2014)

On 28 February 2013, Chi-Kwong Li completed two consecutive terms as ILAS Vice-President, for a total of six years on the ILAS Executive Board. Chi-Kwong will now serve a one year term on the Board of Directors until 28 February, 2014. We extend sincere thanks to Chi-Kwong for his dedicated service to the society.

Christian Mehl and Tin-Yau Tam completed their three-year terms on the ILAS Board of Directors on 28 February, 2013. We thank Christian and Tin-Yau for their valuable contributions as Board members; their service to ILAS is most appreciated.

We also thank the members of the Nominating Committee - Roger Horn, Helena Šmigoc (chair), Naomi Shaked-Monderer, Pauline van den Driessche and Xiao-Dong Zhang - for their work on behalf of ILAS, and to all of the candidates that agreed to have their names stand for election.

2. The following ILAS-endorsed meetings have taken place since our last report:

- SIAM/ALA 2012 Conference, València, Spain, June 18–22, 2012
- ALAMA 2012 (3rd Biennial Meeting of the Spanish Thematic Network on Linear Algebra, Matrix Analysis and Applications), Universidad Carlos III de Madrid, Leganés (Madrid), Spain, June 27–29, 2012
- Workshop on Parallel Matrix Algorithms and Applications (PMAA 2012), Birkbeck University, London, England, June 28–30, 2012
- The Eleventh Workshop on Numerical Ranges and Numerical Radii, National Sun Yat-sen University, Kaohsiung, Taiwan, July 9–12, 2012
- 2012 Workshop on Matrices and Operators, Harbin Engineering University, July 13–16, 2012
- International Workshop on Matrices and Statistics, Będlewo, Poland, July 16–20, 2012
- Workshop on Combinatorics, University of Lisbon, Portugal, September 3–5, 2012
- Haifa Matrix Theory Conference, Technion University, Haifa, Israel, November 12–15, 2012
- International Conference on Numerical Linear Algebra and its Application, Indian Institute of Technology Guwahati, India, January 15–18, 2013

3. ILAS has endorsed the following conference of interest to ILAS members: 4th International Conference on Matrix Analysis and Applications, Selcuk University, Konya, Turkey, July 2–5, 2013.

4. The following ILAS Lectures at non-ILAS conferences have been delivered since the last report.

- Françoise Tisseur, SIAM/ALA Meeting, Universitat Politècnica de València, València, Spain, 18-22 June, 2012.
- Michael Tsatsomeros, SIAM/ALA Meeting, Universitat Politècnica de València, València, Spain, 18-22 June, 2012.

5. The following ILAS conferences are scheduled:

- 18th ILAS Conference, 3–7 June, 2013, Providence, USA. The organising committee consists of: Tom Bella (co-chair), Paul Van Dooren, Heike Faßbender, Bill Helton, Olga Holtz, Steve Kirkland, Vadim Olshevsky (co-chair), Victor Pan, Panos Psarrakos, Hugo Woerdeman, Lilliana Cvetković, Chen Greif, Tin-Yau Tam.
- 19th ILAS Conference, 6–9 August, 2014, Suwon, Korea. The organising committee consists of: Nair Abreu, Tom Bella, Rajendra Bhatia, Richard A. Brualdi, Man-Duen Choi, Nick Higham, Leslie Hogben, Suk-Geun Hwang (chair), Steve Kirkland, Sang-Gu Lee, Helena Šmigoc, Fuzhen Zhang.

6. The *Electronic Journal of Linear Algebra* (ELA) is now in its 26th volume. The Editor-in-Chief is Bryan Shader. Submissions to ELA have significantly increased, growing from 182 submissions in 210 to 244 submissions in 2012. The acceptance rate for ELA over the last 2.5 years is about 30%, which is slight lower than the historic average of about 35%.

- ELA Volume 23 (2012) is complete and contains 72 papers (1059 pages).
- ELA Volume 24 (2012) is a special volume devoted to the 2011 Directions in Matrix Theory Conference with special editors Oskar Maria Baksalary, Carlos M. Fonseca and Shmuel Friedland is complete and contains 17 paper (254 pages)
- ELA Volume 25 (2012), a special volume for the Mat Triad 2011 with special editors Oskar Maria Baksalary, Natália Bebiano, Ljiljana Cvetković and Simo Puntanen is complete and contains 10 papers (118 pages)
- ELA Volume 26 (2013) is in progress and currently has 13 papers.

ELA is indexed by *MathSciNet* and the *Science Citation Index-Expanded*, and ELA papers are reviewed in *MathReviews* and *Zentralblatt*. ELA is currently working on moving to a more friendly, robust publication platform.

7. *IMAGE* is the semi-annual bulletin for ILAS, and the Editor-in-Chief is Kevin N. Vander Meulen, supported by contributing editors Minerva Catral, Michael Cavers, Carlos Fonseca, Bojan Kuzma, Steven Leon, Peter Šemrl, and Amy Wehe, along with advisory editor Jane Day.

As of the Fall 2013, there will be a small change in the editorial staff: regarding the contributing editor for education, Steven Leon will have completed his term and we welcome David M. Strong of Pepperdine University, USA, in this role. Steven Leon has been involved with *IMAGE* since 1998 and will continue on as an Advisory Editor.

ILAS has just reached a milestone of sorts, having now published its 50th issue of *IMAGE* (as of June 1, 2013). Please note that all issues of *IMAGE* are available online at `http://www.ilasic.math.uregina.ca/iic/IMAGE/`. (The mirror sites do not have the current issues of *IMAGE* available.)

8. ILAS-NET is a moderated newsletter for mathematicians worldwide, with a focus on linear algebra; it is managed by Sarah Carnochan Naqvi. As of April 2013, there are 859 subscribers to ILAS-NET. To subscribe, unsubscribe or change your email address, please use the website: `http://leeor4.math.technion.ac.il/mailman/listinfo/ilas-net`.

Changes to ILAS-NET will be coming in 2013/2014 and information will be sent to subscribers as it becomes available.

9. The ILAS Information Centre (IIC) is at the University of Regina. All mirror sites for ILAS-IC have been discontinued and the only current site is: `http://www.ilasic.math.uregina.ca/iic/`. In 2013, we will be moving to `http://www.ilasic.org`. When the change is complete, a notice will be sent to ILAS-Net subscribers.

Respectfully submitted,

Steve Kirkland, ILAS President, stephen.kirkland@nuim.ie; and
Bryan Shader, ILAS Vice-President, bshader@uwyo.edu

Send News for *IMAGE* Issue 51

IMAGE Issue 51 is due to appear online on December 1, 2013. Send your news for this issue to the appropriate editor by October 2, 2013. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

- Problems and Solutions to Bojan Kuzma (bojan.kuzma@upr.si)
- Feature Articles to Michael Cavers (mscavers@gmail.com)
- History of Linear Algebra to Peter Šemrl (peter.semrl@fmf.uni-lj.si)
- Book Reviews to Douglas Farenick (Doug.Farenick@uregina.ca)
- Linear Algebra Education news to David Strong (David.Strong@pepperdine.edu)
- Announcements and Reports of Conferences, Workshops and Journals to Minerva Catral (catralm@xavier.edu)
- Interviews of Senior Linear Algebraists to Carlos Fonseca (cmf@mat.uc.pt)
- Advertisements to Amy Wehe (awehe@fitchburgstate.edu)

Send all other correspondence to Kevin N. Vander Meulen (kvanderm@redeemer.ca).

For past issues of *IMAGE*, please visit `http://www.ilasic.math.uregina.ca/iic/IMAGE/`.

ILAS 2012–2013 Treasurer's Report

April 1, 2012 – March 31, 2013
By Leslie Hogben

Net Account Balance on March 31, 2012

Vanguard (ST Fed Bond Fund Admiral 7548.536 Shares)		$ 81,599.66	
Checking Account - First Federal		$ 35,311.19	
Certificate of Deposit		$ 20,331.40	
Account's Payable		$ (700.00)	
			$ 136,542.25
General Fund		$ 74,854.62	
Conference Fund		$ 9,617.01	
ILAS/LAA Fund		$ 12,840.57	
Olga Taussky Todd/John Todd Fund		$ 11,212.14	
Frank Uhlig Education Fund		$ 4,949.34	
Hans Schneider Prize Fund		$ 23,068.57	
			$ 136,542.25

INCOME:

Dues	$ 9,240.00		
Corporate Sponsorship	$ 200.00		
General Fund	$ 393.42		
Conference Fund	$ 121.00		
LAMA for Speaker	$ -		
Taussky-Todd Fund	$ 306.00		
Uhlig Education Fund	$ 36.00		
Schneider Prize Fund	$ 161.00		
Interest - First Federal / Great Western	$ 64.10		
Loan at Conference (approx)			
Interest on First Federal Certificate of Deposit	$ 150.59		
Vanguard - Dividend Income	$ 670.95		
- Short Term Capital Gains	$ 740.95		
- Long Term Capital Gains	$ 302.86		
Variances in the Market	$ (630.50)		
Misc Income			
Total Income		$ 11,756.37	

EXPENSES:

Citi Bank - Credit Card Service Charges	$ 422.80		
American Express - Credit Card Service Charges	$ 52.85		
First Data - Deposit on credit card terminal	$ 175.69		
Speaker Fees	$ 2,286.45		
Treasurer's Report Preparation	$ 412.00		
Treasurer's Assistant	$ 180.00		
Conference Expenses	$ -		
Student/PostDoc Travel Support	$ 3,000.00		
Business License	$ -		
IMAGE Costs	$ 1,111.41		
Wire Transfer Fees	$ 21.20		
Misc Expenses	$ 246.37		
Ballot Costs	$ -		
Total Expenses		$ 7,908.77	

Net Account Balance on March 31, 2013

Vanguard (ST Fed Bond Fund Admiral 7648.001 Shares)		$ 82,683.93	
Checking Account - First Federal		$ 37,078.92	
Certificate of Deposit		$ 20,481.99	
Account's Payable		$ (700.00)	
			$ 139,544.84
General Fund		$ 77,233.20	
Conference Fund		$ 9,738.01	
ILAS/LAA Fund		$ 12,840.57	
Olga Taussky Todd/John Todd Fund		$ 11,518.14	
Frank Uhlig Education Fund		$ 4,985.34	
Hans Schneider Prize Fund		$ 23,229.57	
			$ 139,544.83

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 49-1 through 49-6 except Problem 49-3, for which we still seek solutions. Six new problems are on the last page; solutions are invited.

Editorial note: Dalton Couto Silva reported a typographical inaccuracy in IMAGE 48*, p.* 33*, line* 13*. Instead of "Then for all j with $k+1 \le j \le r$" it should read "Then for all j with $1 \le j \le k$."*

We were informed by Edward Poon *that solution* 48-7.1 *in* IMAGE 49 *was incorrect. A correction communicated by Eugene A.* Herman *follows.*

Corrigendum to Solution 48-7.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

For any real a and pure imaginary c with $|a|^2+|c|^2=1$, the 4–by–4 matrix consisting of 2–by–2 blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} a & -1 & -c & 1 \\ 1 & -a & 1 & c \\ c & 1 & -a & -1 \\ 1 & -c & 1 & a \end{pmatrix},$$

has the property that $A_{ij} \neq cI_2$ for any $c \in \mathbb{C}$, however a 2–by–2 matrix U_x defined by $(U_x)_{ij} = x^* A_{ij} x$ is unitary for each unit vector $x \in \mathbb{C}^2$. This is a consequence of

$$U_x = |x_1|^2 B_{11} + \overline{x_1} x_2 B_{12} + x_1 \overline{x_2} B_{21} + |x_2|^2 B_{22},$$

where $(x_1,x_2)=x$ and the (p,q) entry of B_{ij} is the (i,j) entry of A_{pq}, $1 \le i,j,p,q \le 2$. It follows that $U_x U_x^* = I_2$ for each unit vector $x \in \mathbb{C}^2$ if and only if

$$B_{11}B_{11}^* = B_{22}B_{22}^* = I, \qquad B_{11}B_{22}^* + B_{12}B_{12}^* + B_{21}B_{21}^* + B_{22}B_{11}^* = 2I,$$
$$B_{11}B_{12}^* + B_{21}B_{11}^* = B_{12}B_{22}^* + B_{22}B_{21}^* = 0, \qquad B_{21}B_{12}^* = 0. \quad \blacksquare$$

Problem 49-1: Generalized Eigenvalues of a Pair of Orthogonal Projectors
Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

Let $P, Q \in M_n(\mathbb{C})$ be orthogonal projectors (i.e., Hermitian idempotent matrices of size n) such that $\mathrm{Im}(P)+\mathrm{Im}(Q)=\mathbb{C}^n$. For $\lambda \in \mathbb{C}$, show that if $Q-\lambda P$ is singular then $\lambda=0$ or $\lambda=1$. Furthermore, prove that $\lambda=1$ is a zero of the polynomial $\lambda \mapsto \det(Q-\lambda P)$ with multiplicity $\dim[\mathrm{Im}(P)\cap\mathrm{Im}(Q)]$.

Solution 49-1.1 by Johanns de Andrade Bezerra, *Natal, RN, Brazil*, pav.animal@hotmail.com

Since P and Q are orthogonal projectors and $\mathrm{Im}(Q)+\mathrm{Im}(P)=\mathbb{C}^n$ it follows that $(\mathbb{C}^n)^\perp = [\mathrm{Im}(Q)+\mathrm{Im}(P)]^\perp = [\mathrm{Im}(Q)]^\perp \cap [\mathrm{Im}(P)]^\perp = \mathrm{Ker}(Q)\cap\mathrm{Ker}(P) = \{0\}$.

If $Q-\lambda P$ is singular with $\lambda \in \mathbb{C}$, then there exists $v \in \mathbb{C}^n \setminus \{0\}$ so that $(Q-\lambda P)v=0$, that is, $Qv=\lambda Pv$. If $v \in \mathrm{Ker}(Q)$, then $Qv=\lambda Pv=0$ and as $Pv\neq 0$ since $\mathrm{Ker}(Q)\cap\mathrm{Ker}(P)=\{0\}$ we conclude that $\lambda=0$. If $v\in\mathrm{Im}(Q)$, then $Qv=v=\lambda Pv$ and so $\lambda=1$ since P is an orthogonal projector.

Now, suppose that $v=x+y \in \mathrm{Ker}(Q-\lambda P)$ where $x\in\mathrm{Ker}(Q)$, $y\in\mathrm{Im}(Q)$, and $x,y\neq 0$. Thus, $Q(x+y)=y=\lambda Px+\lambda Py$. Hence, $Py=\lambda P^2x+\lambda P^2y=\lambda Px+\lambda Py=y$ and so $y\in\mathrm{Im}(P)$. Then $y=\lambda Px+\lambda Py=\lambda Px+\lambda y$, which implies $(1-\lambda)y=\lambda Px$. Next, $\langle y,x\rangle=0$ since $[\mathrm{Im}(Q)]^\perp=\mathrm{Ker}(Q)$, so $\langle(1-\lambda)y,x\rangle=\lambda\langle Px,x\rangle=\lambda\langle P^*Px,x\rangle=\lambda\langle Px,Px\rangle=0$, and as $\lambda\neq 0$ we deduce $Px=0$. Thus, $x\in\mathrm{Ker}(Q)\cap\mathrm{Ker}(P)=\{0\}$, a contradiction. We conclude that if $v\in\mathrm{Ker}(Q-\lambda P)$, then $v\in\mathrm{Ker}(Q)$ and $\lambda=0$, or $v\in\mathrm{Im}(Q)$ and $\lambda=1$. By symmetry, if $\lambda=1$, then $v\in\mathrm{Ker}(Q-P)=\mathrm{Ker}(P-Q)$ also gives $v\in\mathrm{Im}(P)$. Hence,

$$\mathrm{Ker}(Q-P)=\mathrm{Im}(P)\cap\mathrm{Im}(Q). \tag{$*$}$$

Furthermore, $\det(Q-\lambda P)=0$ if and only if $Q-\lambda P$ is singular and therefore $\lambda=0$ or $\lambda=1$. Hence we conclude that $\det(Q-\lambda P)=\pm\lambda^{k_0}(\lambda-1)^{k_1}$ for some $k_0,k_1\in\mathbb{N}$.

Let $\{v_1,\dots,v_k,v_{k+1},\dots,v_r,v_{r+1},\dots,v_n\}$ be an orthonormal basis in $\mathbb{C}^n$ such that $\{v_1,\dots,v_k\}$ is a basis in $\mathrm{Im}(P)\cap\mathrm{Im}(Q)$, $\{v_{k+1},\dots,v_r\}$ is a basis in $\mathrm{Im}(P)$ and $\{v_{r+1},\dots,v_n\}$ is a basis in $\mathrm{Ker}(P)$. Then, with respect to this basis,

$$(Q-\lambda P) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & A & B \\ 0 & B^* & C \end{pmatrix} - \begin{pmatrix} \lambda I_k & 0 & 0 \\ 0 & \lambda I_{r-k} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows from $(*)$ that $\dim(\mathrm{Ker}(Q-P)) = \dim(\mathrm{Im}(Q)\cap\mathrm{Im}(P)) = k$. Hence, $\left(\begin{smallmatrix} A-I_{r-k} & B \\ B^* & C\end{smallmatrix}\right)$ is nonsingular, and as $\det(Q-\lambda P) = \pm\lambda^{k_0}(\lambda-1)^{k_1}$, we conclude that $\det\left(\begin{smallmatrix} A-\lambda I_{r-k} & B \\ B^* & C\end{smallmatrix}\right) = \pm\lambda^{k_0}$, and therefore $k_1 = k$, finishing our proof. ∎

Solution 49-1.2 by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Suppose that $\lambda \notin \{0,1\}$. We will show that

$$\mathrm{Ker}(Q-\lambda P) = \mathrm{Ker}(P)\cap\mathrm{Ker}(Q). \tag{1}$$

Indeed, consider $x \in \mathrm{Ker}(Q-\lambda P)$, that is $Qx = \lambda Px$. Since $x - Qx$ and Qx are orthogonal we have $\langle x-\lambda Px, \lambda Px\rangle = 0$, or equivalently $\overline{\lambda}(1-\lambda)\|Px\|^2 = 0$. Thus $Px = 0$ and $Qx = \lambda Px = 0$. This means that x belongs to $\mathrm{Ker}(P)\cap\mathrm{Ker}(Q)$, and proves (1) since the reverse inclusion is trivial.

Now, noting that by orthogonality we have

$$\mathrm{Im}(P)+\mathrm{Im}(Q) = \mathbb{C}^n \iff \mathrm{Ker}(P)\cap\mathrm{Ker}(Q) = \{0\} \tag{2}$$

we see that if $\lambda \notin \{0,1\}$ then $\mathrm{Ker}(Q-\lambda P) = \{0\}$ and $Q-\lambda P$ is invertible. This proves the first statement.

Next, we will show that $\det(Q-\lambda P) = a(1-\lambda)^r(-\lambda)^q$, where $r = \dim(\mathrm{Im}(P)\cap\mathrm{Im}(Q))$, $q = \dim\mathrm{Ker}(Q)$, and a is some nonzero constant. Clearly, we only need to prove this for $\lambda \in \mathbb{R}$.

For $\lambda \in \mathbb{R}$, let $A_\lambda = Q-\lambda P$, and define $F = \mathrm{Im}(P)\cap\mathrm{Im}(Q)$. Then, $F^\perp = \mathrm{Ker}(P)+\mathrm{Ker}(Q)$, and the sum is direct by (2). For $x \in F$ we have $Qx = x$ and $Px = x$ so $A_\lambda x = (1-\lambda)x$. In particular, $A_\lambda(F) \subset F$. But, since λ is real, A_λ is hermitian and we have also $A_\lambda(F^\perp) \subset F^\perp$. So, let us consider a (possibly nonorthogonal) basis $(e_1,\ldots,e_n)$ of $\mathbb{C}^n$ consisting of a basis $(e_1,\ldots,e_r)$ of F, a basis $(e_{r+1},\ldots,e_{r+q})$ of $\mathrm{Ker}(Q)$ and a basis $(e_{r+q+1},\ldots,e_n)$ of $\mathrm{Ker}(P)$. Since $A_\lambda(e_j) = (1-\lambda)e_j$ for $1 \le j \le r$, $A_\lambda(e_j) = -\lambda P(e_j) \in F^\perp$ for $r < j \le r+q$, and $A_\lambda(e_j) = Q(e_j) \in F^\perp$ for $r+q < j \le n$, we conclude that

$$\det(A_\lambda) = (1-\lambda)^r(-\lambda)^q\det(e_1,\ldots,e_r,P(e_{r+1}),\ldots,P(e_{r+q}),Q(e_{r+q+1}),\ldots,Q(e_n)) = (1-\lambda)^r(-\lambda)^q\det(R)$$

where $R : F^\perp \longrightarrow F^\perp$, $Rx = Qx+Px$. Now, R is invertible since if $x \in \mathrm{Ker}(R)$ then $y := Px = Q(-x)$ belongs to $\mathrm{Im}(P)\cap\mathrm{Im}(Q) = F$, so x and y must be orthogonal, that is $\|Px\|^2 = \langle x, Px\rangle = 0$. Thus, $Px = Qx = 0$ and consequently $x \in \mathrm{Ker}(P)\cap\mathrm{Ker}(Q) = \{0\}$. It follows that $\det(R) \neq 0$. So, there is a nonzero constant $a = \det(R)$ such that $\det(Q-\lambda P) = a(1-\lambda)^r(-\lambda)^q$ where $r = \dim[\mathrm{Im}(P)\cap\mathrm{Im}(Q)]$ and $q = \dim\mathrm{Ker}(Q)$. ∎

Solution 49-1.3 by Edward L. Pekarev, *Odessa Regional Institute of Public Administration of the National Academy of Public Administration, Office of the President of Ukraine, Ukraine*, edpekarev@yandex.ru

Given orthogonal projectors $P, Q \in M_n(\mathbb{C})$ such that $\mathrm{Im}(P)+\mathrm{Im}(Q) = \mathbb{C}^n$, or equivalently $\mathrm{Ker}(P)\cap\mathrm{Ker}(Q) = \{0\}$, let $Q-\lambda P$ be singular for some $\lambda \in \mathbb{C}$, i.e., there exists a vector $x \in \mathbb{C}^n$, $x \neq 0$, such that $Qx = \lambda Px$. Then $\lambda Px, Qx \in \mathrm{Im}(P)\cap\mathrm{Im}(Q)$ and $Px \neq 0$. Denoting $\mathcal{L} = \mathrm{Im}(P)\cap\mathrm{Im}(Q)$ and $\mathcal{M} = \mathrm{Im}(P)\ominus\mathcal{L}$, the orthogonal complement of $\mathcal{L} \subseteq \mathrm{Im}(P)$, one can find w.r.t. the decomposition $\mathbb{C}^n = \mathcal{L}\oplus\mathcal{M}\oplus\mathrm{Ker}(P)$:

$$P = \begin{pmatrix} I_d & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0\end{pmatrix}, \qquad Q = \begin{pmatrix} I_d & 0 & 0 \\ 0 & A & B \\ 0 & B^* & C\end{pmatrix}, \tag{$*$}$$

where I_d, I_m are identities and A, B, C certain matrices of appropriate sizes. It is clear that $\lambda = 1$ is a zero of the polynomial $\lambda \mapsto \det(Q-\lambda P) = \det\left(\begin{smallmatrix} (1-\lambda)I_d & 0 & 0 \\ 0 & A-\lambda I_m & B \\ 0 & B^* & C\end{smallmatrix}\right)$ with multiplicity r not less then $d = \dim\mathcal{L}$. To prove the equality $r = d$ we show that $\det\left(\begin{smallmatrix} A-I_m & B \\ B^* & C\end{smallmatrix}\right) \neq 0$. In fact, if for some $f \in \mathcal{M}, g \in \mathrm{Ker}(P)$ we have $\left(\begin{smallmatrix} A-I_m & B \\ B^* & C\end{smallmatrix}\right)(f\oplus g) = 0$ then $f = Q(f\oplus g) \in \mathrm{Im}(Q)$ and in view of $f \in \mathcal{M} \subset \mathrm{Im}(P)$ we find that $f \in \mathcal{L}$ and so $f \in \mathcal{M}\cap\mathcal{L} = \{0\}$. But as $g \in \mathrm{Ker}(P)$ and $Qg = 0$ we have $g \in \mathrm{Ker}(P)\cap\mathrm{Ker}(Q) = \{0\}$, hence also $g = 0$.

To end, $(*)$ implies that for any $y \in \mathbb{C}^n$ the projections of Py and Qy on $\mathcal{L}$ are equal. So, if $\lambda \neq 0$ then $Qx = \lambda Px (\neq 0)$ implies $Qx, Px \in \mathcal{L}$, $\mathcal{L} \neq \{0\}$ and hence $Qx = Px$. This gives $\lambda = 1$. ∎

Remark. For an infinite-dimensional Hilbert space $\mathcal{H}$ and orthogonal projectors P, Q in $\mathcal{H}$ such that $\mathrm{Im}(P)+\mathrm{Im}(Q) = \mathcal{H}$ and $\mathrm{Im}(P)\cap\mathrm{Im}(Q) = \{0\}$, [T. Ando, Unbounded or bounded idempotent operators in Hilbert space, *Linear Algebra Appl.* 438 (2013) 3769–3775] proved the invertibility of operator $(P+\lambda Q)$ for any $\lambda \neq 0$ and established that $F = P(P+\lambda Q)^{-1}$ is an idempotent in $\mathcal{H}$ with $\mathrm{Im}(F) = \mathrm{Im}(P)$ and $\mathrm{Ker}(F) = \mathrm{Im}(Q)$.

Solution 49-1.4 by Hans Joachim WERNER, *University of Bonn, Bonn, Germany*, hjw.de@uni-bonn.de

Let $W = (W_1\ W_2\ W_3)$ be a unitary matrix such that $\operatorname{Im}(W_1) = \operatorname{Im}(P) \cap \operatorname{Im}(Q)$, $\operatorname{Im}(W_2) = \operatorname{Ker}(P)$, and $\operatorname{Im}(W_3) = ([\operatorname{Im}(P) \cap \operatorname{Im}(Q)] \oplus \operatorname{Ker}(P))^{\perp}$ (we omit the block W_i if the corresponding subspace is zero). Put $d := \dim(\operatorname{Im}(P) \cap \operatorname{Im}(Q))$, $s := n - \operatorname{rank}(Q)$, and $t := n - \operatorname{rank}(P)$. Then

$$W^*(Q - \lambda P)W = \begin{pmatrix} W_1^* \\ W_2^* \\ W_3^* \end{pmatrix} (Q - \lambda P) \begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix} = \begin{pmatrix} (1-\lambda)I_d & 0 & 0 \\ 0 & W_2^*QW_2 & W_2^*QW_3 \\ 0 & W_3^*QW_2 & W_3^*QW_3 - \lambda I_s \end{pmatrix}.$$

By hypothesis, $\operatorname{Im}(P) + \operatorname{Im}(Q) = \mathbb{C}^n$; so taking orthogonal complement gives $\operatorname{Ker}(P) \cap \operatorname{Ker}(Q) = \{0\}$. Therefore, $W_2^*QW_2$ is invertible. Observe that $\begin{pmatrix} W_2^*QW_2 & W_2^*QW_3 \\ W_3^*QW_2 & W_3^*QW_3 - \lambda I_s \end{pmatrix} \begin{pmatrix} I_t & -(W_2^*QW_2)^{-1}W_2^*QW_3 \\ 0 & I_s \end{pmatrix} = \begin{pmatrix} W_2^*QW_2 & 0 \\ W_3^*QW_2 & -\lambda I_s \end{pmatrix}$ where in the $(2,2)$ entry we use $-W_3^*QW_2(W_2^*QW_2)^{-1}W_2^*QW_3 + W_3^*QW_3 = W_3^*Q[I_n - QW_2(W_2^*QW_2)^{-1}W_2^*Q]QW_3$ and the fact that $QW_2(W_2^*QW_2)^{-1}W_2^*Q$ is the orthogonal projector onto $\operatorname{Im}(QW_2)$. Therefore,

$$p(\lambda) := \det(Q - \lambda P) = \det(W^*(Q - \lambda P)W) = (-1)^s \lambda^s (1-\lambda)^d \det(W_2^*QW_2),$$

thus showing that $\lambda = 0$ and $\lambda = 1$ are the zeros of polynomial $p(\lambda)$ with algebraic multiplicities $s = n - \operatorname{rank}(Q)$ and $d = \dim(\operatorname{Im}(P) \cap \operatorname{Im}(Q))$, respectively. ■

Also solved by Eugene A. HERMAN *and the proposers.*

Problem 49-2: Matrix Diagonal Entries

Proposed by Roger A. HORN, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu
and Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, U.S.A.*, zhang@nova.edu

For $A = (A_{ij}) \in M_n(\mathbb{C})$, let A^* be its conjugate transpose, and let $|A| = (A^*A)^{1/2}$ (the positive semidefinite square root).

(a) Show that $|A_{ij}|^2 \le |A^*|_{ii}|A|_{jj}$ for all i, j. Is $|A_{ij}|^2 \le |A|_{ii}|A^*|_{jj}$?

(b) Show that $\prod_{i=1}^n |A_{ii}|^2 = \prod_{i=1}^n |A^*|_{ii}|A|_{ii}$ if and only if (i) A has a zero row or column or (ii) $A = D|A|$ for some diagonal matrix D.

(c) Show that $|\det(A \circ C)|^2 \le \prod_{i=1}^n |A^*|_{ii}|A|_{ii}$ for any $n \times n$ contraction C, where $A \circ C$ is the Hadamard (Schur, entrywise) product of A and C. (Note: A matrix is a contraction if its spectral norm is no more than 1.)

(d) Is $|A_{ii}| \le |A|_{ii}$ for all $i = 1, 2, \ldots, n$?

Editorial note: The "only if" part in (b) of the original problem is false, as shown by Eugene A. HERMAN *in his solution below. The current (b) is a revision of the previous one.*

Solution 49-2.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Parts (a), (b), and (d) only. The answer to the questions in parts (a) and (d) is "no," as is shown by the example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow |A| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ |A^*| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

For the question in (a), let $(i, j) = (1, 2)$; for (d), let $i = 1$.

(a). We use the singular value decomposition $A = W\Sigma V^*$, where $W = (W_{ij})$ and $V = (V_{ij})$ are unitary and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \ge \cdots \ge \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$. Then $|A| = V\Sigma V^*$ and $|A^*| = W\Sigma W^*$, and so

$$A_{ij} = \sum_{k=1}^n W_{ik}\sigma_k \overline{V}_{jk}, \quad |A|_{jj} = \sum_{k=1}^n |V_{jk}|^2 \sigma_k, \quad |A^*|_{ii} = \sum_{k=1}^n |W_{ik}|^2 \sigma_k.$$

We use the identities

$$|\beta - \alpha|^2 = (|\beta|^2 + |\alpha|^2) - (\alpha\overline{\beta} + \beta\overline{\alpha}); \qquad (\alpha, \beta \in \mathbb{C}),$$

$$\sum_{k=1}^n \sum_{m=1}^n a_{km} = \sum_{k=1}^n a_{kk} + \sum_{k=1}^{n-1} \sum_{m=k+1}^n (a_{km} + a_{mk}); \qquad (a_{km} \in \mathbb{C})$$

to write

$$\begin{aligned}
|A^*|_{ii}|A|_{jj} - |A_{ij}|^2 &= \sum_{k=1}^{n}\sum_{m=1}^{n} |V_{jk}|^2|W_{im}|^2\sigma_k\sigma_m - \sum_{k=1}^{n}\sum_{m=1}^{n} W_{ik}\overline{V}_{jk}\,\overline{W}_{im}V_{jm}\sigma_k\sigma_m \\
&= \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} \left(|V_{jk}|^2|W_{im}|^2 + |V_{jm}|^2|W_{ik}|^2\right)\sigma_k\sigma_m \\
&\qquad - \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} \left(W_{ik}\overline{W}_{im}\,\overline{V}_{jk}V_{jm} + W_{im}\overline{W}_{ik}\,\overline{V}_{jm}V_{jk}\right)\sigma_k\sigma_m \\
&= \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} |V_{jm}W_{ik} - V_{jk}W_{im}|^2\sigma_k\sigma_m \geq 0. \qquad (1)
\end{aligned}$$

(b). The "only if" part of the original problem is false. Indeed, let $A = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$. Then, $|A| = \sqrt{\frac{5}{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $|A^*| = \sqrt{\frac{2}{5}}\left(\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right)$, so $\prod_{i=1}^{2}|A_{ii}|^2 = 4 = \prod_{i=1}^{2}|A^*|_{ii}|A|_{ii}$. However, if $A = DP$, where D is diagonal and unitary and P is positive semidefinite, then $|A|^2 = A^*A = PD^*DP = P^2$ and so $P = |A|$. It is easy to see that $A \neq D|A|$ whenever D is diagonal and unitary.

We prove the following instead:

(b'). $\prod_{i=1}^{n}|A_{ii}|^2 = \prod_{i=1}^{n}|A^*|_{ii}|A|_{ii}$ if and only if (i) A has a zero row or column or (ii) $A = D|A|$ for some diagonal matrix D. Furthermore, if A is nonsingular and $A = D|A|$ for some diagonal matrix D, then D is unitary.

By part (a), $|A_{ii}|^2 \leq |A^*|_{ii}|A|_{ii}$ for $i = 1, \ldots, n$. We may therefore use the fact that

$$0 \leq x_i \leq y_i,\ i = 1,\ldots,n \quad\Longrightarrow\quad \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i \quad \text{if and only if} \quad x_i = y_i, i = 1,\ldots,n, \text{ or } y_i = 0 \text{ for some } i. \qquad (2)$$

By formula (1),

$$|A^*|_{ii}|A|_{ii} - |A_{ii}|^2 = \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} |V_{im}W_{ik} - V_{ik}W_{im}|^2\sigma_k\sigma_m. \qquad (3)$$

Assume $\prod_{i=1}^{n}|A_{ii}|^2 = \prod_{i=1}^{n}|A^*|_{ii}|A|_{ii}$. Then, by (2), $V_{im}W_{ik} = V_{ik}W_{im}$ for all $i = 1,\ldots,n$ and all $k, m = 1,\ldots,r$ or $|A^*|_{ii}|A|_{ii} = 0$ for some i. If $|A|_{ii} = 0$ for some i, then $\sum_{k=1}^{n}|V_{ik}|^2\sigma_k = 0$ and so $V_{ik} = 0$ for $k = 1,\ldots,r$. Hence $(\Sigma V^*)_{ki} = \sigma_k\overline{V}_{ik} = 0$ for all $k = 1,\ldots,n$. That is, column i of ΣV^* is zero, and so column i of $A = W\Sigma V^*$ is zero. Similarly, if $|A^*|_{ii} = 0$ for some i, then row i of A is zero. On the other hand, if $V_{im}W_{ik} = V_{ik}W_{im}$ for all $i = 1,\ldots,n$ and all $k, m = 1,\ldots,r$, then for each $i \in \{1,\ldots,n\}$, either $V_{ik} = 0$ for $k = 1,\ldots,r$ or there exists $d_i \in \mathbb{C}$ such that $W_{im} = d_iV_{im}$ for $m = 1,\ldots,r$. As shown above, the first possibility implies that column i of A is zero. Thus, we may assume there exist $d_1,\ldots,d_n \in \mathbb{C}$ such that $W_{im} = d_iV_{im}$ for all $m = 1,\ldots,r$ and all $i = 1,\ldots,n$. That is, $\mathbf{W}_{11} = \mathbf{D}_1\mathbf{V}_{11}$ and $\mathbf{W}_{21} = \mathbf{D}_2\mathbf{V}_{21}$, where

$$W = \left(\begin{smallmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{smallmatrix}\right),\ V = \left(\begin{smallmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{smallmatrix}\right),\ D = \left(\begin{smallmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{smallmatrix}\right)$$

and where $\mathbf{D}_1 = \operatorname{diag}(d_1,\ldots,d_r), \mathbf{D}_2 = \operatorname{diag}(d_{r+1},\ldots,d_n)$ and the upper-left corner of each block matrix is $r \times r$. Since $\Sigma = \left(\begin{smallmatrix} \mathbf{\Sigma}' & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for diagonal $r \times r$ matrix $\mathbf{\Sigma}'$, we have

$$D|A| = DV\Sigma V^* = \left(\begin{smallmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{smallmatrix}\right)\Sigma V^* = \left(\begin{smallmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{smallmatrix}\right)\Sigma V^* = A.$$

For the converse implication, if column i of A is zero, then column i of $A^*A = |A|^2$ is zero and so $0 = (|A|^2)_{ii} = \sum_{k=1}^{n}|A|_{ik}|A|_{ki} = \sum_{k=1}^{n}|A|_{ik}\overline{|A|}_{ik} = \sum_{k=1}^{n}||A|_{ik}|^2$. In particular, $|A|_{ii} = 0$. Similarly, if row i of A is zero, then $|A^*|_{ii} = 0$. In either case, (2) implies that $\prod_{i=1}^{n}|A_{ii}|^2 = \prod_{i=1}^{n}|A^*|_{ii}|A|_{ii}$. On the other hand, suppose $A = D|A|$, where D is diagonal. Using the singular value decomposition $A = W\Sigma V^*$ we deduce $W\Sigma = DV\Sigma$, which we write as

$$\left(\begin{smallmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{\Sigma}' & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{\Sigma}' & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

where $\mathbf{\Sigma}'$ is nonsingular. Thus, $\mathbf{W}_{11}\mathbf{\Sigma}' = \mathbf{D}_1\mathbf{V}_{11}\mathbf{\Sigma}'$ and $\mathbf{W}_{21}\mathbf{\Sigma}' = \mathbf{D}_2\mathbf{V}_{21}\mathbf{\Sigma}'$, and so $\mathbf{W}_{11} = \mathbf{D}_1\mathbf{V}_{11}$ and $\mathbf{W}_{21} = \mathbf{D}_2\mathbf{V}_{21}$. Thus $V_{im}W_{ik} = V_{ik}W_{im}$ for all $i = 1,\ldots,n$ and all $k, m = 1,\ldots,r$, and so (3) yields $|A_{ii}|^2 = |A^*|_{ii}|A|_{ii}$ for $i = 1,\ldots,n$.

Finally, if $A = D|A|$ where D is diagonal and A is nonsingular, then $|A|^2 = A^*A = |A|D^*D|A|$. Since $|A|$ is also nonsingular, we can cancel it on the left and right to produce $I = D^*D$; that is, D is unitary. ∎

Solution 49-2.2 by the proposers Roger A. Horn, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu and Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, Florida, U.S.A.*, zhang@nova.edu

(a). Let r be the rank of A and let $\sigma_1, \ldots, \sigma_r$ be the positive singular values of A. Let $A = V\Sigma W^*$, in which $V = (v_1, \ldots, v_n)^*$ and $W = (w_1, \ldots, w_n)^*$ are $n \times r$ partial unitary (isometry) (i.e., $V^*V = I_r$ and $W^*W = I_r$), and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ is nonsingular. Thus, $|A| = W\Sigma W^*$ and $|A^*| = V\Sigma V^*$. It follows that

$$A_{ij} = v_i^* \Sigma w_j, \quad |A|_{ij} = w_i^* \Sigma w_j, \quad |A^*|_{ij} = v_i^* \Sigma v_j.$$

By the Cauchy-Schwarz inequality, we arrive at the inequalities:

$$|A_{ij}|^2 = |(\Sigma^{1/2} v_i)^* (\Sigma^{1/2} w_j)|^2 \le (v_i^* \Sigma v_i)(w_j^* \Sigma w_j) = |A^*|_{ii} |A|_{jj} \qquad (*)$$

with equality if and only if $\Sigma^{1/2} v_i$ and $\Sigma^{1/2} w_j$ (equivalently, v_i and w_i) are linearly dependent. Moreover, these inequalities ensure that $|A|$ or $|A^*|$ has a zero on its main diagonal if and only if A contains a zero column or row.

However, $|A_{ij}|^2 \le |A|_{ii} |A^*|_{jj}$ is not true in general. Take $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

(b). If A has a zero row or column, then $|A^*|$ or $|A|$ has a zero on its main diagonal, and the equality holds. If $A = D|A|$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$, then $V\Sigma W^* = DW\Sigma W^*$ which results in $V = DW$, thus, $v_i = \bar{d}_i w_i$ for $i = 1, \ldots, n$. By $(*)$, $|A_{ii}|^2 = |A^*|_{ii} |A|_{ii}$. The product equality in (b) follows immediately.

For the converse, we assume that A contains no zero row or column. Then there is no zero on the main diagonals of $|A|$ and $|A^*|$. By $(*)$, we have $|A_{ii}|^2 \le |A^*|_{ii} |A|_{ii}$. If $\prod_{i=1}^n |A_{ii}|^2 = \prod_{i=1}^n |A^*|_{ii} |A|_{ii}$, then $|A_{ii}|^2 = |A^*|_{ii} |A|_{ii} \ne 0$ for all i. Thus, v_i and w_i are linearly dependent for all i, i.e., $V = DW$ for some nonsingular diagonal matrix D. It follows that $A = V\Sigma W^* = D(W\Sigma W^*) = D|A|$, as desired. Note that if A is nonsingular, then D is unitary.

(c). Notice that $\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix} \ge 0$ and $\begin{pmatrix} I & C^* \\ C & I \end{pmatrix} \ge 0$, in which C is any $n \times n$ contraction. Taking the Hadamard product (see [1, Theorem 7.5.3]) we have

$$\begin{pmatrix} |A| \circ I & (A \circ C)^* \\ A \circ C & |A^*| \circ I \end{pmatrix} \ge 0.$$

By the Hadamard–Fischer inequality (see [1, Exercise on p. 473]) the desired inequality follows.

(d). It is not true in general. Take $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$; $|A_{11}| = 1$, $|A|_{11} = \frac{1}{\sqrt{2}}$. ∎

Reference

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2012.

Problem 49-4: A Total Perron-Frobenius Theorem for Total Orders

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let A be an n by n real matrix. Show that the following are equivalent:

(a) All the eigenvalues of A are real and nonnegative.

(b) There exists a total order $\ge$ on $\mathbb{R}^n$ (partial order where any two vectors are comparable) which is preserved by A (i.e., $x \in \mathbb{R}^n$ and $x \ge 0$ implies that $Ax \ge 0$) and which makes $(\mathbb{R}^n, \ge)$ an ordered vector space.

Solution 49-4.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

This proposition, whose proof is routine, is used in both parts of the proof:

Proposition. Let V and W be vector spaces over the reals and $\phi : V \to W$ a surjective vector space isomorphism. Let T be a linear transformation on V and $\ge$ a total order on V that is preserved by T and makes $(V, \ge)$ an ordered vector space. Define $U = \phi T \phi^{-1}$ and define a relation $\succeq$ on W by $w_2 \succeq w_1$ if and only if $\phi^{-1}(w_2) \ge \phi^{-1}(w_1)$. Then U is a linear transformation on W and $\succeq$ is a total order on W that is preserved by U and makes $(W, \succeq)$ an ordered vector space.

(a) $\Rightarrow$ (b): Let P be a nonsingular matrix such that $J = P^{-1}AP$ is a Jordan form of A. By the Proposition, it suffices to prove (b) for the matrix J. On $\mathbb{R}^n$, let $\ge$ be the reverse lexicographic order, which can be defined recursively by

$$(y_1, \ldots, y_n) \ge (x_1, \ldots x_n) \qquad \text{if and only if} \qquad y_n > x_n \text{ or } y_n = x_n \text{ and } (y_1, \ldots, y_{n-1}) \ge (x_1, \ldots, x_{n-1}).$$

This is a well-known total order on $\mathbb{R}^n$ that makes $(\mathbb{R}^n, \ge)$ an ordered vector space. It is easy to see that $x \ge 0$ implies $Jx \ge 0$ if all the diagonal entries of J are positive. If some are zero, we change basis again (proof suggested by Rajesh

Pereira). We may assume that $J = \left(\begin{smallmatrix} J_0 & 0 \\ 0 & J' \end{smallmatrix}\right)$, where J_0 consists of the nilpotent blocks of J arranged in decreasing order of size. Let d be the size of J_0 and $d-k$ be the number of ones in J_0. Claim: There exists a permutation matrix P such that $(P^{-1}J_0P)(x_1,\ldots,x_d)^T$ is a d-tuple in which each component is either a zero or one of the variables $x_{k+1},\ldots,x_d$, and these variables occur in that order, each one exactly once. Hence, a matrix

$$\left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} J_0 & 0 \\ 0 & J' \end{smallmatrix}\right)\left(\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} P^{-1}J_0P & 0 \\ 0 & J' \end{smallmatrix}\right)$$

preserves the reverse lexicographic order. To prove the claim, choose P to be the permutation matrix such that J_0P has all the zero columns of J_0 as its initial columns in their given order, then has the first nonzero column from each block of J_0 arranged in the order of the blocks until all these columns are used, then has the second nonzero column from each block arranged in the order of the blocks until all these columns are used, and so on. For example, if J_0 has blocks of size $3,2,2$ then its columns are permuted according to $(4,2)(6,3,7)$. It suffices to show that the position of 1 in nonzero rows of $P^{-1}J_0P$ increases with the index of nonzero row. Let σ be the corresponding permutation from P and write $P^{-1}J_0P = (b_{i,j})_{i,j=1,\ldots,d}$ and $J_0 = (a_{i,j})_{i,j=1,\ldots,d}$. Then $a_{i,j} = b_{\sigma(i),\sigma(j)} \in \{0,1\}$, $i,j = 1,\ldots,d$. If $a_{i,i+1}$ is the entry 1 from the first nonzero column of the jth block of J_0, then $1 = b_{\sigma(i),\sigma(i+1)}$ lies in row $\sigma(i) = j$ since P permutes column i of J_0 (which is a zero column) to the jth column. Thus, these entries of $P^{-1}J_0P$ are consecutive ones lying on a superdiagonal starting at row 1. Now consider two ones, $a_{i,i+1}$ and $a_{j,j+1}$, $i<j$ that lie in the kth nonzero column $(k>0)$ of two different Jordan blocks. Then $\sigma(i) < \sigma(j)$, since columns i and j are $(k-1)$th nonzero columns that are permuted by P in that order. Thus $b_{\sigma(i),\sigma(i+1)}$ lies in an earlier row than $b_{\sigma(j),\sigma(j+1)}$. Finally, consider two ones, $a_{i,i+1}$ and $a_{j,j+1}$ where the first lies in a kth column and the second in an mth column with $k<m$. Again, $\sigma(i) < \sigma(j)$, since either column i is zero and column j is not or both columns are nonzero with column i being a $(k-1)$th column and column j being a $(m-1)$th column.

(b) $\Rightarrow$ (a): If λ is a real eigenvalue of A with associated eigenvector x, we may assume $x \geq 0$ since otherwise we can replace x by $-x$. Hence $\lambda x = Ax \geq 0$, which implies $\lambda \geq 0$. It remains to show that A cannot have an eigenvalue $\lambda = a+bi$ with a,b real and $b \neq 0$. Suppose A has such an eigenvalue and that an associated eigenvector is $x = y+iz$ with y,z real. Then $Ax = \lambda x$ may be written as

$$Ay = ay - bz, \qquad Az = by + az. \tag{$*$}$$

We may assume $y \geq 0$ and $z \geq 0$ for the following reason: If $y<0$ and $z<0$, replace x by $-x$; if $y \geq 0$ and $z<0$, replace x by ix; and if $y<0$ and $z \geq 0$, replace x by $-ix$. Note that $\{y,z\}$ is linearly independent, since otherwise x would be a multiple of a real eigenvector and so λ would be real. Let ϕ be the isomorphism from $V = \operatorname{Span}\{y,z\}$ onto $\mathbb{R}^2$ such that $\phi(y) = e_1$ and $\phi(z) = e_2$ (the standard unit vectors). The matrix A maps V into V, and the restriction of $\geq$ to V is a total order on V preserved by the restriction of A to V. We can therefore apply the Proposition and denote the induced linear transformation and induced total order on $\mathbb{R}^2$ again by A and $\geq$. Then, equations $(*)$ imply

$$A = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{smallmatrix}\right)$$

for some $r>0$ and $\theta \in (0,2\pi)$, and so

$$A^k = r^k \left(\begin{smallmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{smallmatrix}\right), \quad k = 0,1,2,\ldots.$$

Since $y \geq 0$ and $z \geq 0$, we have $e_1 \geq 0$ and $e_2 \geq 0$ and so

$$u_k = \left(\begin{smallmatrix} \cos k\theta \\ \sin k\theta \end{smallmatrix}\right) = \tfrac{1}{r^k}A^k e_1 > 0 \quad \text{and} \quad v_k = \left(\begin{smallmatrix} -\sin k\theta \\ \cos k\theta \end{smallmatrix}\right) = \tfrac{1}{r^k}A^k e_2 > 0, \quad k = 0,1,2,\ldots.$$

We get a contradiction from these sequences of positive vectors as follows. Suppose, for some k, v_k is in the 3rd quadrant. Then u_k is in the 2nd quadrant and so $-e_1 = \alpha u_k + \beta v_k$ for some nonnegative scalars α,β. We get a similar contradiction if, for some k, v_k is in the 4th quadrant. To see that at least one of these circumstances occurs, we divide the values of θ into cases: If $0<\theta\leq\pi/2$, then v_k passes through every quadrant on each counterclockwise rotation about the origin; if $\pi/2<\theta\leq\pi$, then v_1 is in the 3rd quadrant; if $\pi<\theta\leq 3\pi/2$, then v_1 is in the 4th quadrant; and if $3\pi/2<\theta<2\pi$, we have a clockwise rotation of $2\pi-\theta \in (0,\pi/2)$, which returns us to the first case. ∎

Also solved by the proposer.

Problem 49-5: A Property of the Electro-Magnetic Field in the Born–Infeld Model

Proposed by Denis SERRE, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

In the canonical Euclidian space $\mathbb{R}^n$ $(n \geq 3)$, we give ourselves two vectors E and B, satisfying $\|E\|^2 + (E\cdot B)^2 \leq 1 + \|B\|^2$. Prove the following inequality between symmetric matrices $EE^T + BB^T \leq (1+\|B\|^2)I_n$.

Solution 49-5.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

The assumption $n \geq 3$ is not needed. Since the result is trivial for $n = 1$, we may assume $n \geq 2$. It suffices to show that $\lambda \leq 1 + \|B\|^2$ for all eigenvalues λ of $EE^T + BB^T$. There is an orthogonal matrix Q such that

$$QE = (e, 0, \dots, 0)^T, \quad QB = (b_1, b_2, 0, \dots, 0)^T.$$

Then $(QE) \cdot (QB) = E \cdot B$ and similarly for all the other relevant expressions. Hence $\|E\|^2 + (E \cdot B)^2 \leq 1 + \|B\|^2$ becomes

$$e^2 + e^2 b_1^2 \leq 1 + b_1^2 + b_2^2 \tag{$*$}$$

and

$$EE^T + BB^T = \begin{pmatrix} e^2 & 0 & \\ 0 & 0 & O \\ & O & O \end{pmatrix} + \begin{pmatrix} b_1^2 & b_1 b_2 & \\ b_1 b_2 & b_2^2 & O \\ & O & O \end{pmatrix} = \begin{pmatrix} e^2 + b_1^2 & b_1 b_2 & \\ b_1 b_2 & b_2^2 & O \\ & O & O \end{pmatrix}.$$

It therefore suffices to show that $\lambda \leq 1 + b_1^2 + b_2^2$ for all eigenvalues λ of $\left(\begin{smallmatrix} e^2+b_1^2 & b_1 b_2 \\ b_1 b_2 & b_2^2 \end{smallmatrix}\right)$. The characteristic polynomial of this 2×2 matrix is $\lambda^2 - (e^2 + b_1^2 + b_2^2)\lambda + e^2 b_2^2 = 0$ and so

$$\lambda = \frac{1}{2}\left(e^2 + b_1^2 + b_2^2 \pm \sqrt{(e^2 + b_1^2 + b_2^2)^2 - 4e^2 b_2^2}\right).$$

Since inequality $(*)$ implies $b_1^2 + b_2^2 + 2 - e^2 = 1 + e^2 b_1^2 \geq 0$ we therefore have

$$\begin{aligned} \lambda \leq 1 + b_1^2 + b_2^2 &\Leftrightarrow (e^2 + b_1^2 + b_2^2)^2 - 4e^2 b_2^2 \leq (b_1^2 + b_2^2 + 2 - e^2)^2 \\ &\Leftrightarrow e^4 + 2e^2(b_1^2 + b_2^2) - 4e^2 b_2^2 \leq 2(b_1^2 + b_2^2)(2 - e^2) + (2 - e^2)^2 \\ &\Leftrightarrow 4e^2 b_1^2 \leq 4(b_1^2 + b_2^2) + 4 - 4e^2 \\ &\Leftrightarrow e^2 + e^2 b_1^2 \leq 1 + b_1^2 + b_2^2 \end{aligned}$$

which is inequality $(*)$. Thus, the inequality $\|E\|^2 + (E \cdot B)^2 \leq 1 + \|B\|^2$ is not just sufficient but necessary. ∎

Solution 49-5.2 by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Lemma. For $m \geq 2$, let $\mathbb{R}^m$ be equipped with the canonical scalar product "$\cdot$" and norm $\|\cdot\|_2$, and let the space $M_m(\mathbb{R})$ of $m \times m$ matrices be equipped with the corresponding operator norm. Consider m vectors $e_1, e_2, \dots, e_m$ from a real scalar product space, $(H, \langle , \rangle)$. Then, for every $x \in H$, we have

$$\sum_{k=1}^{m} \langle e_k, x \rangle^2 \leq \|G\| \|x\|_H^2$$

where $G = Gram(e_1, \dots, e_m)$ is the Gram matrix of $(e_1, e_2, \dots, e_m)$.

Proof of Lemma. If $\Phi : H \longrightarrow \mathbb{R}^m$ is defined by $\Phi(x) = (\langle e_k, x \rangle)_{1 \leq k \leq m}$, then $\Phi^T : \mathbb{R}^m \longrightarrow H$ is given by $\Phi(a_1, \dots, a_m) = a_1 e_1 + \cdots + a_m e_m$ and G is the standard matrix of $\Phi\,\Phi^T$ with respect to the canonical basis of $\mathbb{R}^m$. It is well-known that $\|\Phi\| = \|\Phi^T\| = \sqrt{\|\Phi\,\Phi^T\|} = \sqrt{\|G\|}$, so the considered inequality is just expressing the fact that $\|\Phi(x)\|_2 \leq \|\Phi\| \|x\|_H$ for every $x \in H$. ∎

We will apply the Lemma to the case $H = \mathbb{R}^n$ and $m = 2$ with $e_1 = E$ and $e_2 = B$. If $x \in \mathbb{R}^n$ then

$$(E \cdot x)^2 + (B \cdot x)^2 \leq \left\| \left(\begin{smallmatrix} \|E\|^2 & E \cdot B \\ E \cdot B & \|B\|^2 \end{smallmatrix}\right) \right\| \|x\|^2.$$

But, the operator norm of a positive semi-definite real matrix is its largest eigenvalue. So

$$\left\| \left(\begin{smallmatrix} \|E\|^2 & E \cdot B \\ E \cdot B & \|B\|^2 \end{smallmatrix}\right) \right\| = \frac{1}{2}\left(\|B\|^2 + \|E\|^2 + \sqrt{(\|B\|^2 - \|E\|^2)^2 + 4(E \cdot B)^2}\right).$$

Now, using the assumption $0 \leq (E \cdot B)^2 \leq 1 + \|B\|^2 - \|E\|^2$ we see that

$$\left\| \left(\begin{smallmatrix} \|E\|^2 & E \cdot B \\ E \cdot B & \|B\|^2 \end{smallmatrix}\right) \right\| \leq \frac{1}{2}\left(\|B\|^2 + \|E\|^2 + \sqrt{(\|B\|^2 - \|E\|^2 + 2)^2}\right) = 1 + \|B\|^2.$$

So, $x^T(EE^T + BB^T)x \leq (1 + \|B\|^2)x^T x$ for every $x \in \mathbb{R}^n$, which is equivalent to the desired inequality. ∎

Solution 49-5.3 by Minghua Lin, *University of Waterloo, Ontario, Canada*, mlin87@ymail.com

Without loss of generality, we assume $B \neq 0$. We can write $E = \alpha B/\|B\| + \beta C$. Here $\alpha, \beta \in \mathbb{R}$ and $C \in \mathbb{R}^n$ is a unit vector such that $B^T C = 0$. Then $\|E\|^2 + (E^T B)^2 \leq 1 + \|B\|^2$ implies

$$\beta^2 + \alpha^2(1 + \|B\|^2) \leq 1 + \|B\|^2. \tag{$*$}$$

Our goal is to show that the eigenvalues of $EE^T + BB^T$ are no larger than $1 + \|B\|^2$. Since the nonzero eigenvalues of $EE^T + BB^T = [E \;\; B][E \;\; B]^T$ coincide with those of $[E \;\; B]^T[E \;\; B] = \left(\begin{smallmatrix} \alpha^2+\beta^2 & \alpha\|B\| \\ \alpha\|B\| & \|B\|^2 \end{smallmatrix}\right) := \mathbf{H}$ we need to show that the largest eigenvalue of $\mathbf{H}$ is no larger than $1 + \|B\|^2$.

The characteristic polynomial of $\mathbf{H}$ is $p(t) = t^2 - (\alpha^2 + \beta^2 + \|B\|^2)t + \beta^2\|B\|^2$. From $(*)$ we know $\alpha^2 \leq 1$ so $\alpha^2 + \|B\|^2 \leq 1 + \|B\|^2$. Also from $(*)$ it is obvious $\beta^2 \leq 1 + \|B\|^2$. That is, $\max\{\alpha^2 + \|B\|^2, \;\; \beta^2\} \leq 1 + \|B\|^2$. Hence, the parabola given by $p(t)$ attains its minimum at $\frac{1}{2}(\alpha^2 + \beta^2 + \|B\|^2) \leq 1 + \|B\|^2$. So it remains to show $p(1 + \|B\|^2) \geq 0$. A little calculation gives $p(1 + \|B\|^2) = 1 + \|B\|^2 - \alpha^2 - \beta^2 - \alpha^2\|B\|^2 \geq 0$. ∎

Also solved by Roger A. Horn *and the proposer.*

Problem 49-6: Determinant Power Monotonicity

Proposed by Suvrit Sra, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

Let $A, B > 0$ (positive definite); let $1 \leq t \leq u$. Prove that $\det\left(\frac{A^t+B^t}{2}\right)^{1/t} \leq \det\left(\frac{A^u+B^u}{2}\right)^{1/u}$.

Solution 49-6.1 by the proposer Suvrit Sra, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

Note that $\det\left(\frac{A^t+B^t}{2}\right)^{1/t} = \left(\det(\frac{A^t+B^t}{2})\right)^{1/t} = (\det A^t)^{1/t} \cdot (\det(I + A^{-t}B^t))^{1/t} = (\det A) \cdot (\det(I + A^{-t}B^t))^{1/t}$. Hence, using $P = A^{-1}$ we must show that

$$\prod_{j=1}^{n} \left(\frac{1 + \lambda_j(P^T B^t)}{2}\right)^{1/t} \leq \prod_{j=1}^{n} \left(\frac{1 + \lambda_j(P^u B^u)}{2}\right)^{1/u}.$$

We know that (see for example the proof of Theorem IX.2.9 in [1]):

$$\lambda^{1/t}(P^t B^t) \prec_{\log} \lambda^{1/u}(P^u B^u).$$

Then $t \mapsto \ln(1 + e^{ut})$ is a convex and monotone increasing function, therefore (see Theorem A.1 on p. 165 of [2]):

$$\prod_{j=1}^{k} \left(\frac{1 + \lambda_j^{u/t}(P^t B^t)}{2}\right)^{1/u} \leq \prod_{j=1}^{k} \left(\frac{1 + \lambda_j(P^u B^u)}{2}\right)^{1/u}, \quad 1 \leq k \leq n.$$

But, for $u \geq t$, the map $x \mapsto x^{u/t}$ is convex, whereby we have

$$\prod_{j=1}^{k} \left(\frac{1 + \lambda_j^{u/t}(P^t B^t)}{2}\right)^{1/u} \geq \prod_{j=1}^{k} \left(\left(\frac{1 + \lambda_j(P^t B^t)}{2}\right)^{u/t}\right)^{1/u} = \prod_{j=1}^{k} \left(\frac{1 + \lambda_j(P^t B^t)}{2}\right)^{1/t}. \;\blacksquare$$

References

[1] R. Bhatia, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997.

[2] A.W. Marshall, I. Olkin, and B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer, NewYork, 2011.

Editorial note: Minghua Lin *informed us that* [K.V. Bhagwat and R. Subramanian, Inequalities between means of positive operators, *Math. Proc. Camb. Phil. Soc.* 83 (1978) 393–401] *contains a more general result for which the assumptions of the problem imply* $\left(\frac{A^t+B^t}{2}\right)^{1/t} \leq \left(\frac{A^u+B^u}{2}\right)^{1/u}$ *whenever* $1 \leq t \leq u$.

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 49 (Fall 2012), p. 52] except problem 49.3, for which we still seek solutions. Submissions: Please submit proposed problems and solutions in macro-free LATEX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Gregor Dolinar, Shaun Fallat, Alexander Guterman, Nung-Sing Sze, and Rajesh Pereira.

New Problems:

Problem 50-1: An Adjugate Identity

Proposed by Khaled Aljanaideh, *University of Michigan, Ann Arbor, MI, USA*, khaledfj@umich.edu
and Dennis S. Bernstein, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $\operatorname{adj} A \in M_n(\mathbb{C})$ denote the adjugate (transposed matrix of cofactors) of $A \in M_n(\mathbb{C})$, let $A_{[i,j]} \in M_{n-1}(\mathbb{C})$ denote A with the i-th row and j-th column removed, and let $A_{[i,\cdot]} \in M_{(n-1)\times n}(\mathbb{C})$ denote A with the i-th row removed. If $e_i \in \mathbb{C}^n$ is the i-th column vector of the standard basis, show that

$$\left[(\operatorname{adj} A)_{[i,\cdot]} + (\operatorname{adj} A_{[i,i]})A_{[i,\cdot]}\right] e_i = 0_{(n-1)\times 1}; \qquad i \in \{1,\ldots,n\}.$$

Problem 50-2: Range-Hermitianness of Certain Functions of Projectors

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

(i) Let $P, Q \in M_n(\mathbb{C})$ be Hermitian idempotent matrices of order n. Show that $P + Q - PQ$ is range-Hermitian.

(ii) Let $R \in M_n(\mathbb{C})$ be an idempotent matrix of order n and let $R^\dagger$ be its Moore-Penrose inverse. Show that $I_n - R^\dagger$ is range-Hermitian, where I_n is the identity matrix of order n.

Recall that a matrix is range-Hermitian (also called EP) if its range, and the range of its conjugate transpose, coincide.

Problem 50-3: Trace Inequality for Positive Block Matrices

Proposed by Ádám Besenyei, *Department of Applied Analysis, Eötvös Loránd University, Hungary*, badam@cs.elte.hu

Let $A, B, C \in M_n(\mathbb{C})$ be such that the block matrix $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) \in M_{2n}(\mathbb{C})$ is positive-semidefinite. Show that

$$\operatorname{Tr}(AC) - \operatorname{Tr}(B^*B) \le \operatorname{Tr}(A)\operatorname{Tr}(C) - \operatorname{Tr}(B^*)\operatorname{Tr}(B).$$

Problem 50-4: Matrix Power Coefficients

Proposed by Moubinool Omarjee, *Lycée Henri IV, Paris, France*, ommou@yahoo.com

Find all real matrices A such that $A^r = (a_{ij}^r)$ for any integer r (A^r is the usual product of A r-times). What is the answer for matrices over commutative fields?

Problem 50-5: A Matrix of Divided Differences

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be an n-th degree monic polynomial with complex coefficients and let $\{x_k\}_{k=1}^n$ be a set of n pairwise distinct real numbers. Let M be the n–by–n matrix of first divided differences of p (so for $1 \le i, j \le n$ we have $m_{i,j} = (p(x_i) - p(x_j))/(x_i - x_j)$ and $m_{ii} = p'(x_i)$). Show that the determinant of M is positive if n is congruent to 0 or 1 mod 4 and negative if n is congruent to 2 or 3 mod 4.

Problem 50-6: Diagonalizable Matrices Over $\mathbb{F}_p$

Proposed by Denis Serre, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Let $p \ge 2$ be a prime number and let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$. Prove that $A \in M_n(\mathbb{F}_p)$ is diagonalizable *within* $M_n(\mathbb{F}_p)$ if and only if $A^p = A$.

Solutions to Problems 49-1 *through* 49-6*, except* 49-3*, are on pp.* 36–43.