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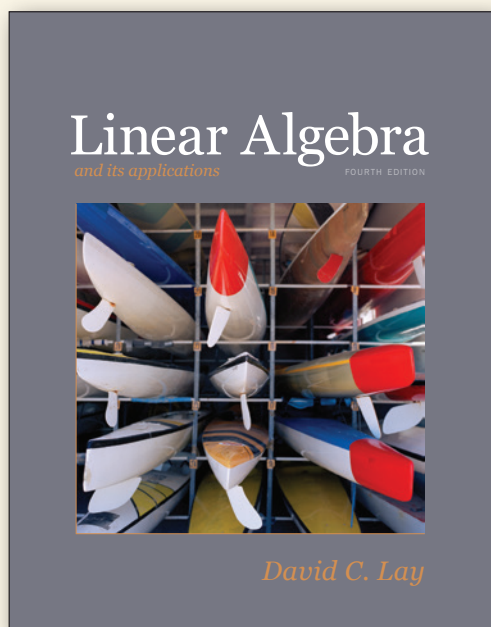
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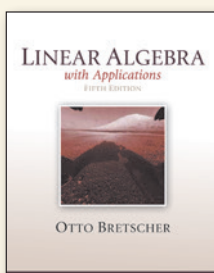
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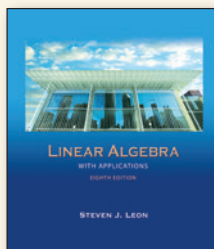
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FEATURE INTERVIEW

“Working on Classical Problems is Better than Taking the Latest Fad”

Thomas J. Laffey Interviewed by Helena Šmigoc¹



Thomas J. Laffey

Professor Thomas Laffey has made important contributions to group theory, ring theory, combinatorics, operator algebra and matrix theory. For his outstanding contribution to linear algebra he was awarded the Hans Schneider Prize at the ILAS meeting in Providence in June 2013. His influence on Irish mathematics has been profound, ranging from helping to establish Irish participation at the International Mathematical Olympiad to editorial work for the *Mathematical Proceedings of the Royal Irish Academy*.

H.S. - When did you first get interested in mathematics? Can you remember how your own interest in mathematics was aroused?

T.L. - Already in primary school, mathematics was a subject that I really liked and I was good at. At the end of primary school, I got a scholarship to go to secondary school. At the scholarship exam I got essentially full marks in mathematics. This helped me obtain the scholarship.

In the secondary school I was in, they didn't offer an honors Leaving Cert in mathematics. In Ireland you can do any subject in the Leaving Cert at either an honors or a pass level. The mathematics honors level was always considered very challenging, and a lot of schools didn't offer it because only a small fraction of the students attempted it. In the Inter Cert, halfway through secondary school, I got essentially full marks in mathematics, so the principal decided that they would try and facilitate me to do the honors level mathematics. The result was that the school arranged that a teacher from another school would come to help me on a number of Saturdays. I also got mathematics books that belonged to the school, some of them from the 19th century: various books on geometry, coordinate geometry, elementary algebra books, and so on. I taught myself the honors program; nobody in the school ever did it before. In my last years of school I was studying math that had nothing to do with the material my classmates were learning and I even ended up teaching mathematics to my classmates.

H.S. - Then you went to university in Galway?

T.L. - Yes, I went to Galway. I had to get a scholarship to be able to go to the university, and I got the state scholarship that was not means tested. Those scholarships were very competitive: there were only 20 for the country. I had to go to the university in Galway as a condition of the scholarship. NUI Galway was the smallest of the Irish universities and this was one way of ensuring that good students went there.

H.S. - Was it clear to you that you wanted to study mathematics?

T.L. - When I went to university, mathematics was not one of the subjects I intended to study. I thought I would study physics or chemistry. It transpired that in Galway there were two prestigious subjects that you could study: one was Mathematical Science, and the other was Classics (Latin and Greek). The state scholars tended to go to one of those. I suddenly realized that very few people were doing honors mathematics. I liked mathematics, I was getting high marks in it, and I enjoyed doing tricky problems. I always liked mathematics, and in university this was reinforced.

H.S. - Was there an area of mathematics that appealed to you as an undergraduate?

T.L. - In Galway there was a bias towards algebra. The number of people on the staff was quite small. Of the people that were teaching me, Sean Tobin was a group theorist and Dan McQuillan was a number theorist; they conveyed the beauty of these subjects. We didn't do as much in analysis. We did equal amounts of mathematics and mathematical physics right through for three years. Also, for my master's thesis I did half pure maths and half applied maths: Group Theory and Fluid Mechanics.

H.S. - How did you decide where to go for your Ph.D.?

T.L. - I did my masters in Galway and then I taught in Galway for one year. The tradition was that people who did well were offered a junior position for one year and during that year they could arrange to do a Ph.D. either in the UK or in the USA.

During our studies, one of the books we were using was by Ledermann and the other mathematician I heard about was Philip Hall. So I wrote to Philip Hall and I wrote to Walter Ledermann. I got a letter back from Philip Hall that I

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am sorry now that I didn't keep. He is so famous now that I would have liked to have kept the letter. And he wrote a very carefully written two-page-long letter in which he talked about what we could work on. At the end of his letter he said that he couldn't offer money directly for me to come to Cambridge, where he was, but that he was prepared to recommend me for a Gulbenkian scholarship. He said that the decision won't be taken for about six months, and he couldn't guarantee the outcome. In retrospect, I think his recommendation would probably have meant that I would have received the scholarship, but I didn't know that at the time.

I also got a letter back from Ledermann who invited me to come for an interview in Sussex. After the interview he offered me a studentship, so I took that and I didn't wait for the other one.

H.S. - What problem did you work on for your Ph.D.?

T.L. - There is a book by Curtis and Reiner entitled "Representation theory of finite groups and associative algebras" [2] that we used in our group theory course. There was a result of Schur in the book that caught my attention: if you have a finite group of matrices of size $n \times n$ over the complex numbers, then it has a subgroup H which is normal and Abelian and the index of H in G is a bounded function of n . In the Reiner and Curtis book, different bounds are presented. They are difficult to compare, so what I thought I would try to do is to get a better bound or at least make some contribution.

Ledermann had a great influence on my career. He was a very nice person, extremely helpful, and since I suggested the problem myself, he made sure that the problem was worthwhile. He was worried that the problem was too hard so he arranged for me to meet Feit when he came to a conference in the UK to give a plenary talk.

Feit thought this was a good problem and that there were aspects that I could make progress on. He also talked about variations of the problem, the index of the Abelian normal subgroups, to look at the structure of these subgroups and so on. This was a topic where Feit himself had made a contribution just about 5 or 6 years before. So that is what I worked on.

H.S. - What were your years in Sussex like?

T.L. - Sussex was a new university set up in the 1960s when the UK was very wealthy. There were very few group theorists in Sussex and I was the only student working on finite group theory. The day you arrived you got a problem to work on. There were no lectures; the graduate students were expected to attend lectures by visiting speakers and departmental seminars. Ledermann was a bit worried that I wasn't being offered enough courses, so he arranged for me to go to London every week to attend lectures by E.C. Dade who was visiting from the States and giving a series of lectures. He gave two hours of lectures each week at high speed and I tried to write my notes as fast as I could. One day Ledermann called me in his office. He wanted to check that I wasn't completely lost. He met Fröhlich, who is a big name in number theory, at a meeting in London that weekend and he told him that these lectures were difficult. There is nothing unusual about mathematicians getting lost, but if you were only learning the subject you might not realize that if you got lost, there was nothing unusual and it was part of the process.

The first year in Sussex I worked extremely hard learning the stuff and also trying to do research but at the end of the first year I had done more or less enough for a thesis, so the second year I had a more relaxed time.

H.S. - It only took you two years to finish your Ph.D.?

T.L. - Yes, it was the minimum period; I submitted it on the first allowed day. I had basically done the work the first year.

H.S. - What happened after your Ph.D.?

T.L. - I applied for a job at the University College Dublin (UCD). There was no research history at UCD when I started.

The one person that was research active at the time was Fergus Gaines. He was a student of Olga Taussky Todd and he just came to Ireland after finishing his Ph.D. at Caltech. In order to have somebody to talk to, I decided to learn linear algebra. I used to get his notes from graduate courses in Caltech. They were wonderful; I really appreciated them. They might have been delivered in a fashion that was hard to follow but you could work through the handwritten version at your leisure. I learned linear algebra over the course of two years or so. I also learned ring theory. Fergus was using essentially Wedderburn-Artin theory to do results about reducing matrices to small block size under similarity and things like that. His research had a ring-theoretic flavor.

H.S. - You spent some time in the USA?

T.L. - I went to the States for a year to Northern Illinois University where there were many group theorists; some of the biggest names of the papers that I was reading. When I was in Illinois there was a strong group in Algebra. There was Group Theory, but there was also active research in Ring Theory and in Number Theory. John Selfridge was there. He is one of the big names in number theory, one of the world experts. He used to have a lot of visitors. That is where I met Paul Erdős; he was one of the visitors. My time in Illinois was a productive period in terms of papers. The papers I wrote were mostly in Group Theory and Ring Theory.

Something else happened when I was there. The analysts in Illinois often had problems related to Linear Algebra and they would be asking the algebraists to help them out. Eventually they started asking me these questions. It turned out that, because I had learned matrix theory, I was actually able to answer their questions. That is when I realized that I had learned Linear Algebra to a point that I might be able to do some work on it, so I started to do so.

H.S. - If you had to choose one other mathematician that made a strong impression on you during your career, who would it be, and why?

T.L. - Well, several of them made impressions on me. In terms of my career probably Olga Taussky Todd. When I started to work with Fergus Gaines, one of the things we used to work on was the questions that Olga Taussky had posed when Fergus Gaines was her student. I solved a couple of problems that she suggested. When she heard about it, she used to correspond with me and suggest questions. She had a big influence because she kept corresponding, and then whenever I would go to a meeting she used to introduce me to everybody. She used to know all the big names and would bring them to her students. A lot of the questions I worked on, particularly the questions on matrices under similarity, started with questions she posed.

Shmuel Friedland was another person that I was very impressed by. I think it must have been 1977 at a meeting in Santa Barbara when I first saw him. But I had heard about him previously. He had this theorem about the fact that if you had a matrix, then you can modify the diagonal entries so it has any given eigenvalues [3]. I thought that this was a great theorem. More than any other theorem, it encouraged me to do linear algebra; it was such a nice theorem.

We had almost no conferences in Ireland with the exception of an annual Group Theory conference in Galway. In the 1980s, Fergus Gaines and I organized a couple of matrix theory conferences with almost no money. One of the people who came was Charlie Johnson and he gave wonderful lectures. Charlie Johnson was encouraging and he came with questions and problems that we used to discuss.

LeRoy Beasley spent a lot of time in Ireland. He spent a whole year one time. We wrote some papers during his visit. While he was in Ireland, he had visitors who came and gave seminars. One I remember in particular was Larry Cummings. LeRoy had a big effect in keeping my interest alive.

H.S. - For many years you were involved in the International Mathematics Olympiad.

T.L. - It started with the initiative of Finbarr Holland to set up a math competition in Ireland for bright students. He arranged for students from Ireland to take part in the American high school mathematics competition. He wrote to the organizers and they allowed us to use their exams. The three people involved in that was Finbarr Holland, Fergus Gaines and me. From that we got to the International Mathematical Olympiad (IMO).

In 1988 we got a letter from Australia, from Peter O'Halloran, who was running the IMO in Australia in 1988 and was of Irish extraction. That year Australia was celebrating its bicentennial, and O'Halloran wrote that he would like to have Ireland participate in the IMO on this occasion. He lobbied Australian ministers and he even came to Ireland, and he got help from the Australian ambassador to Ireland. The result was that even though at that time Ireland was in extreme poverty, we did get funding to send the team to Australia. In the boom years from 2000 to 2008, the department of education funded the IMO, but before, and since, we needed to get some private sponsorship as well.

H.S. - Which result are you the most proud of?

T.L. - I think the result that has had the most effect in terms of other people developing the consequences is the result that if you have two matrices whose commutator has rank one, then they are simultaneously triangularizable [4]. People like Heydar Radjavi and Peter Rosenthal, the Canadian functional analysts, have used this result. There are related questions on the variety of matrices, which B. A. Sethuraman, Robert Guralnick and others have worked on. In the book "Simultaneous Triangularization" by Radjavi and Rosenthal [5] there are several generalizations and consequences of this result.

H.S. - What problem is still challenging you?

T.L. - The nonnegative inverse eigenvalue problem (NIEP). Most of my research from the mid nineties has been on this. When Hans Schneider set up the Hans Schneider prize, I was on a committee to decide who would get the prize the first two times it was offered. The first prizes went to Gohberg, Fiedler and Friedland. The next time the prize was awarded, one of the papers that was considered for the prize was a paper of Boyle and Handelman [1]. So I had to read through it to get a feeling of what was involved and this is how I got interested in the NIEP.

H.S. - Did you ever try to prove a result that turned out not to be true?

T.L. - I suppose there are lots of those things. I tend to be pessimistic about what is true. One thing that turned out differently than I expected is the following. Suppose that you have an $n \times n$ matrix over a field which has a square root in an extension field, and you want to bound the minimum degree of an extension field for which this occurs. With Bryan Cain, we proved that the extension degree has to be very big in order for this to happen. The answer was much worse than I originally expected. I developed this further with my Ph.D. student Raja Mukherji.

H.S. - What changes in linear algebra did you see during your career?

T.L. - Linear algebra links with other subjects and there are a variety of tools coming from other areas. Linear algebra has this image that it is about solving linear systems and changing bases. Also the tools of linear algebra were dimension arguments and similarity. These methods have been replaced with methods from other areas of mathematics, such as algebraic geometry and so on.

It is the judgment part about applied mathematics that changed. I was educated with a view that pure mathematics was somewhat superior to the one that had applications. This changed in several different ways. The value judgment is gone. People now accept that applied mathematics can have very deep ideas.

H.S. - Were you involved in applied research yourself?

T.L. - One day Robert Shorten, who had been an engineering student at UCD but whom I had never seen before, turned up in my office. Together with Kumpati Narendra in Yale, he found applications of my rank one result in developing control systems theory. I had some collaboration on applied results with him after that.

H.S. - What advice would you give to young mathematicians that are starting out?

T.L. - Working on the classical problems is better, as a general policy, than following the latest fad. You will see that many mathematicians are still working on problems that are 100 years old.

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ARTICLES

The Santa Barbara School of Linear Algebra¹: Reminiscences

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The creator and master builder of the Santa Barbara school was Marvin Marcus. After serving with the navy during World War II, Marvin enrolled in UC Berkeley, graduated in 1950, earned his (differential equations) Ph.D. three years later, and accepted a position at the University of British Columbia. There he supervised the Ph.D. of Roy Westwick and the Master's thesis of Robert C. Thompson, spent a sabbatical with Morris Newman³ at the National Bureau of Standards and became acquainted with Henryk Minc. By the time he joined the UCSB faculty in 1962, the year the University of California authorized its Santa Barbara campus to grant a Ph.D. in mathematics, Marvin had published 50 articles with 17 different coauthors on topics from periodic solutions of differential equations to symmetric functions of eigenvalues, and permanents of doubly stochastic matrices. Among his first official acts on becoming department chair in 1963 was to hire Minc. A year later he hired Thompson (whose Caltech Ph.D. had been supervised by Olga Taussky). A year after that he scored another coup by persuading polymath Ky Fan to join the UCSB faculty.

I, of course, knew none of this when Bill Watkins ushered me into Marvin's office in the spring of 1965. My first impression, which endures to this day, was of a decisive figure whose courtesy seemed unlikely to extend to having his time wasted. So, coming right to the point, I asked whether I stood any chance of being admitted to the graduate program on the strength of an engineering degree from Harvey Mudd College (then in its 8th year of operation). "Transcript?" he responded. I never knew whether it was the fact that I pulled a transcript out of my shirt pocket, the "A" in complex

¹Interpreted broadly.

²With assistance from Andy Bruckner (who confirmed but was not the source of my Bruckner story), Bob Grone, Mike Mahoney, Karen Melissa Marcus, Steve Pierce, Wasin So, Ann Watkins, Bill Watkins, Gill Williamson, and Fuzhen Zhang.

³Marvin Marcus, Richard Brualdi, Stephen Pierce, Charles R. Johnson, and I were all one-time postdoctoral research associates of Morris at NBS. (It was from him that I learned group representation theory!)

variables, or Bill's tacit recommendation that did it, but Marvin agreed to admit me, subject to some vaguely stipulated remedial course work that somehow fell through the cracks.

By the time of my arrival in the fall of 1965, their many collaborations had earned Marcus and Minc the honorary student nickname, M^2 . Throw in Thompson's late night roller skating escapades, Minc's infectious conviviality, and Marcus's sporty Karmann Ghia convertible, and you have more than sufficient ingredients for all sorts of imaginative student speculation. One rumor had it that Marcus was training for Wimbledon, another that Minc and Thompson were preparing to swim the English Channel.



Steve Pierce, Bill Watkins, Russ Merris,
and Bob Grone

management of the growing pains of what would become a first rate mathematics department, Marvin taught a full complement of undergraduate courses, ran a popular research seminar (one of whose participants, Ralph Freese, made a significant discovery [1] while still an undergraduate), and took out whatever frustrations he may have had on the tennis court.

The partiality shown to linear algebra during Marvin's stewardship of the department led to hard feelings that survived his stepping down from the chairmanship in 1968. Hence the creation of a semi-autonomous *Institute for the Interdisciplinary Applications of Algebra and Combinatorics* which Marvin directed (with more than a little help from Thompson) through most of the 1970s. In parallel with its sponsorship of meetings and conferences, the Institute produced a monograph series that included an early draft of my own book *Multilinear Algebra* [6]. Then there were the journals. Over and above his commitments as one of the founding editors of *Linear Algebra and Its Applications*, Marvin (together with Thompson) launched *Linear and Multilinear Algebra*.

Despite the political convulsions and realignment of national priorities occasioned by the 1968 election, the 1970s were something of a golden age for the Santa Barbara school of linear algebra. Attracted by its growing reputation, Paul Halmos joined the department for two of those years (1976–78). In collaboration with Bill Watkins, then chair of the mathematics department at CSU Northridge, Halmos organized a Southern California Functional Analysis seminar, the success of which may have blazed a trail for the celebrated Southern California Matrix Meetings launched five years later by Steve Pierce and Bob Thompson. (Its name notwithstanding, the first SoCalM meeting was held in Toronto in 1983, the ninth in Salt Lake City in 1994, and the 17th in San Jose in 2004 where the name was changed to the Robert C. Thompson Matrix Meetings.) Meanwhile, from 1969 to 1980, Henryk Minc taught two terms a year at UCSB and one at Israel's Technion. And, although they may not have been part of the Santa Barbara School per se, other members of the faculty were occasionally drawn into its orbit; Eugene Johnsen and Adil Yaqub being examples.

The addition of Morris Newman to the mathematics faculty in 1977 marked the end of one era and the beginning of another. Driven by resentment over preferential treatment of linear algebra in the past, Newman's appointment faced serious opposition in the department, first, as to whether an offer should be made at all, and then, at what level. Once the first decision had been made, Real Analyst Andrew Bruckner moved to end a decade of bickering by insisting that the offer must be at a rank commensurate with Newman's uncontested stature. Ending Marvin's estrangement was another matter. From 1978 to 1986 he served as Vice Chancellor and Dean of Research and Academic Development. In 1987 he cut his last ties to the Mathematics Department and became a full-time member of the Computer Science Department.

Always looking for ways to promote the careers of his students, Marvin obtained a grant from the National Science Foundation in support of an International Conference on Multilinear Algebra to be held at UCSB in June of 1981. He then stepped aside and gave me a free hand as the grant's designated co-director. Expecting to drive the bus, buy the donuts, etc., I wound up choosing the invited speakers (Shmuel Friedland, then at the Hebrew University of Jerusalem; Ming-Huat Lim, Universiti Malaya but visiting Cambridge; Graciano de Oliveira, by then back at the Universidade de Coimbra; John de Pillis, UC Riverside; Neil J. A. Sloane, Bell Labs; Richard Stanley, MIT; and Gill Williamson, UC San Diego), signing the requisitions, and so on. Thanks to the support of Morris Newman and Sona MacMillan, the Institute's new director and seasoned administrative assistant, respectively, the conference was a big success. Among the other participants were Bob Grone, Massoud Malek, two of Graciano's young protégés, Eduardo Marques de Sá (the

Sa-Thompson Theorem would come later) and José A. Dias da Silva (who would make many significant contributions to multilinear algebra and was the closest thing I had to a Ph.D. student), Bob Thompson, and many others – but *not* Marvin Marcus! “It is like going to Rome,” Graciano complained, “and not seeing the pope!”

While I remember the 1981 conference especially well, for obvious reasons, it was just one of many meetings and conferences hosted over the years by the Institute and/or Department. It could well have been by way of those that I met the likes of Yik-Hoi Au-Yeung, Richard Brualdi, Miroslav Fiedler (from whom I later learned graph theory), Emilie Haynsworth, Irving (Kap) Kaplansky, and Chi-Kwong Li.

Leaving out a score of publications on topics from ancient numismatics and biblical archeology to Robert Burns and Sir Walter Scott, Henryk Minc’s Santa Barbara years saw the publication of more than 50 articles with seven different coauthors (one of whom reduced Henryk’s Erdős number to one and mine to three) and 13 books of which *Permanents* [7] and *Nonnegative Matrices* [8] are worthy of special mention. Retired by 1992, Henryk died on the Ides of July 2013.

While at UCSB, Robert C. Thompson published more than 100 articles with a dozen different coauthors and four books (two with Adil Yaqub). Bob’s research interests spanned a number of recurring themes, e.g., matrix commutators, principal submatrices, eigenvalues and singular values of matrix sums and products, and congruence and equivalence of matrices. The last two of his 10 Ph.D. students, Wasin So and Fuzhen Zhang, are especially well-known in the linear algebra community. Bob died on December 10, 1995, while waiting for a heart transplant. In a memorial eulogy to Bob, Charlie Johnson and Morris Newman opined that “strong centers of matrix research, such as those in Israel, Hong Kong, Portugal, and Spain, can trace their intellectual roots to Santa Barbara.”

Over the course of his UCSB career, Marvin Marcus published more than 200 articles and problem solutions with at least 28 different coauthors. Not counting dozens of editions, translations, Institute monographs, and microcomputer manuals (a late 1970s interest in the numerical range led him down the slippery slope from computation to computers), Marvin published more than 20 books of which the momentous two volume *Finite Dimensional Multilinear Algebra* [3] is especially significant. Omitting Paul Nikolai, whose Caltech Ph.D. was jointly supervised by H. J. Ryser, he graduated 17 Ph.D. students of whom Robert Grone (1976) is the most prominent. Marvin retired in 1991 at which time responsibility for the care and feeding of *Linear and Multilinear Algebra* was seamlessly handed off to Bill Watkins.

Morris Newman was an established mathematician (with 125 articles, four Ph.D. students, and an international reputation) before moving to Santa Barbara. His second career at UCSB saw the publication of 75 more articles with 15 different coauthors on such diverse topics as Catalan numbers, subgroup counting functions, rings of algebraic integers, diophantine equations, modular groups, Ramanujan congruences, condition numbers, doubly stochastic matrices, Kronecker products, compounds, and cospectral graphs. In addition, he co-edited the posthumous publication of an analytic number theory book by his Ph.D. supervisor, Hans Rademacher, and supervised 10 more Ph.D. students of his own. Morris, retired in 1994 but continued to advise students for another decade. He died on January 4, 2007.

Ky Fan’s first article, *Sur une représentation des fonctions abstraites continues*, appeared in 1940 in C. R. Acad. Sci. Paris. By the time of his arrival in Santa Barbara, he had produced 75 articles (the first 25 of which were in French) and 15 Ph.D. students. While at UCSB he published another 50 articles on topics ranging from locally compact Abelian groups, distributive lattices, and topological vector spaces, to normalizable operators, dissipative matrices, and sums of M-matrices. And, he supervised eight more Ph.D. students. Ky retired in 1985 and died on March 22, 2010.

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An Application of Markov Chains in Seismology

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Introduction. Seismology is the study of earthquakes and the propagation of seismic waves (i.e., waves of energy) through the earth. Linear algebra has many useful applications in seismology. One example involves using vector space ideas to obtain information about the structure of the earth by inverting seismic observations [19]. Another example occurs in techniques for determining the location of an earthquake that uses the data obtained from seismographs (devices that record the shaking of the earth's surface caused by seismic waves). One such method was introduced by Geiger in 1910, applying an iterative least-squares technique [7, 8]. Over the past century, considerable attention has been devoted by seismologists to locate earthquakes worldwide. The question of predicting or forecasting has been a topic of great interest. In this article, we first discuss some background information about earthquakes, including power law relationships between earthquake characteristics. We then turn our attention to the question of earthquake prediction and forecasting. We highlight a mathematical model of earthquake sequencing and forecasting that uses Markov chains and show how it can be applied to global earthquake data.

What causes earthquakes? The Earth's crust is always in motion and is made up of large segments called *tectonic plates*. There are seven primary tectonic plates that make up the bulk of the continents and Pacific Ocean along with various other secondary and tertiary plates. Plates shifting against each other produce a build-up of stress. When too much stress is built up, an earthquake occurs along a fault (i.e., area of weakness). Such activity is of major interest to scientists since earthquakes can cause major fractures and damage by ground-shaking. They also occasionally lead to natural disasters such as tsunamis, landslides and avalanches.

Measuring the size of an earthquake. In 1935, Charles Richter developed the familiar base-10 logarithmic scale that can be used to measure the size of an earthquake based on the amount of energy released. This magnitude scale is defined as the logarithm of the amplitude of waves recorded by seismographs. In the 1970s, the *moment magnitude scale* was developed and is currently the scale used by seismologists and the *United States Geological Survey* (USGS), one of the world's most respected and consulted agencies. The moment magnitude scale is more useful than the Richter magnitude scale for comparing the sizes of large earthquakes, but also mimics the familiar properties of the Richter scale.

Number of earthquakes per year. Based on observations since 1900, the USGS *National Earthquake Information Center* (NEIC) reports that each year there is an average of 1 great earthquake ($M \geq 8.0$), 15 major earthquakes ($7.0 \leq M \leq 7.9$), 134 large earthquakes ($6.0 \leq M \leq 6.9$), 1319 moderate earthquakes ($5.0 \leq M \leq 5.9$) and approximately 13,000 small earthquakes ($4.0 \leq M \leq 4.9$). More specifically, Table 1 shows the total number of earthquakes occurring worldwide from the years 2000 to 2012 as reported by the NEIC [21].

Magnitude	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
8.0 to 9.9	1	1	0	1	2	1	2	4	0	1	1	1	2
7.0 to 7.9	14	15	13	14	14	10	9	14	12	16	23	19	12
6.0 to 6.9	146	121	127	140	141	140	142	178	168	144	150	185	108
5.0 to 5.9	1344	1224	1201	1203	1515	1693	1712	2074	1768	1896	2209	2276	1401
4.0 to 4.9	8008	7991	8541	8462	10888	13917	12838	12078	12291	6805	10164	13315	9534

Table 1: Number of worldwide earthquakes from 2000 to 2012.

Source: US Geological Survey National Earthquake Information Center [21].

Earthquakes of magnitude 9.0 or greater are known to occur once every 10 to 50 years. Such earthquakes cause severe damage to all surrounding buildings and result in permanent changes in the ground topography. The largest recorded earthquake to date occurred in Valdivia, Chile on May 22, 1960 and measured a magnitude of 9.5.

The Gutenberg-Richter law. In the 1930s, Beno Gutenberg and Charles Francis Richter, pioneers in the field of seismology, discovered a relationship between a magnitude and the number of earthquakes of at least that magnitude in a specified region and time interval. They observed that the number of earthquakes of magnitude at least M is proportional to 10^{-bM} , for some constant b (dependent on the region) [9]. This relationship is known as the *Gutenberg-Richter law* and states:

$$\log_{10} N(M) = a - bM,$$

where $N(M)$ is the number of earthquakes having magnitude at least M , and a, b are constants. The constant b , called the b -value, has been observed to be close to $b = 1$ when considering earthquakes worldwide. Regionally, depending

on the tectonic environment of the region, the b -value typically ranges from 0.8 to 1.2. On the other hand, the a -value measures the seismicity rate of the region, that is, 10^a represents the total number of earthquakes in the region over a specified time period.

As an example, suppose $b = 1$ for a region of interest, and in this region, there is one earthquake per year of magnitude at least 6. Then using the Gutenberg-Richter law we get $a = 6$ and hence, earthquakes in the region with magnitude at least 7 occur 0.1 times per year (that is, $N(7) = 10^{-1}$). Similarly, earthquakes with magnitude at least 8 occur 0.01 times per year (that is, $N(8) = 10^{-2}$).

Aftershocks and the modified Omori's law. After a large earthquake it is common to have a cluster of smaller earthquakes occurring in the same area of the large earthquake. Such smaller earthquakes are called *aftershocks* while the larger earthquake is called a *mainshock*. In the case that a smaller earthquake occurs before a mainshock, we call the smaller earthquake a *foreshock*. By analyzing the pattern of aftershocks, the size of area that slipped during the mainshock can often be determined.

In the late 1800s, Fusakichi Omori observed that the rate of aftershocks decreases rapidly with time [17]. Today, a modified version of this result, known as the *modified Omori's law*, is commonly used. It is an empirical relationship found in the waiting time of aftershocks associated with a larger earthquake:

$$n(t) = \frac{K}{(t + c)^p},$$

where $n(t)$ is the number of aftershocks in a time interval after a mainshock, K, c are constants and p lies in the range 0.9 to 1.5 as observed from earthquake data. Typically, the values of the parameters are obtained empirically by fitting to data after the occurrence of a mainshock. Interestingly, aftershocks can follow Omori's law for hundreds of years [4]. Furthermore, their magnitude, distance and timing are independent of the mainshock magnitude [5, 18]. Note that by the Gutenberg-Richter law, the number of small aftershocks is greater than the number of large aftershocks.

Prediction versus forecasting. Forecasting earthquakes is an interesting and important problem but should be distinguished from predicting earthquakes. Earthquake prediction is a binary statement that specifies either an earthquake will or will not occur at a given location with a given magnitude range over a certain time interval. On the other hand, earthquake forecasting estimates the probabilities of an earthquake of a specified magnitude range affecting a given region within a particular time interval. Thus, in forecasting, a set of probabilities is provided similar to that as given in weather forecasts.

Public controversy over the prediction of earthquakes transpired after the occurrence of a magnitude 6.3 earthquake in L'Aquila, Italy on April 5, 2009 [15]. This is the deadliest earthquake to hit Italy since 1980 and approximately 300 people are known to have died. Before the occurrence of the earthquake, six Italian scientists and a government official downplayed the danger involved after a series of tremors prior to the main earthquake. After a year-long trial, they were each convicted of multiple manslaughter and sentenced to six years' imprisonment, though lawyers have said that they will appeal against the sentence. The verdict alarmed geoscientists worldwide who claim it sets a damaging precedent that will deter scientific experts from sharing their knowledge with the general public for fear of lawsuits.

The USGS states that earthquake prediction is currently not possible, though many theories have been tested. For earthquake *forecasting*, many mathematical models have appeared in the literature. Historically, Poisson models have been used to describe earthquake sequencing, however, recently a Markov chain model has been developed to carry out forecasting in a regional context [11, 14, 20]. Earthquake sequencing has been treated as a Markov process to study the behaviour of aftershocks [6, 10, 16, 23]. In our research, we explored such a Markov chain model in the global context which we briefly highlight in this article (see also [2, 22]).

Zone designation. The first step in developing this model is to partition a region, either local or global, into zones. In the literature, one study of Japan uses 4 zones [14] while another study of Turkey also uses 4 zones [20]. These zones are typically made up of rectangles that split up the region. In our study, we used the observation that the occurrence of earthquakes around plate boundaries falls into tectonically well-defined zones. In [1], a global model of plate boundaries on the Earth is provided, however, this model consists of 52 plates making our problem intractable. Hence, to study global seismicity, we applied the model to a simplified 5-zone plate boundary template as given by Kagan et al. [13] (see Figure 1).

State definitions and earthquake catalogue. For the five zones given, a state corresponding to a time interval, is expressed as a concatenation of binary digits $b_4b_3b_2b_1b_0$, where $b_L = 1$ (or $b_L = 0$) indicates there was (or was not) an earthquake occurrence in zone L during the specified time interval. For example, if the state 11001 is observed in a certain time interval then earthquakes were observed in zones 0, 3 and 4 during this time interval but not in zones 1 or 2. During a time interval, there are 32 possible states for the Earth depending on the zone locations of earthquake

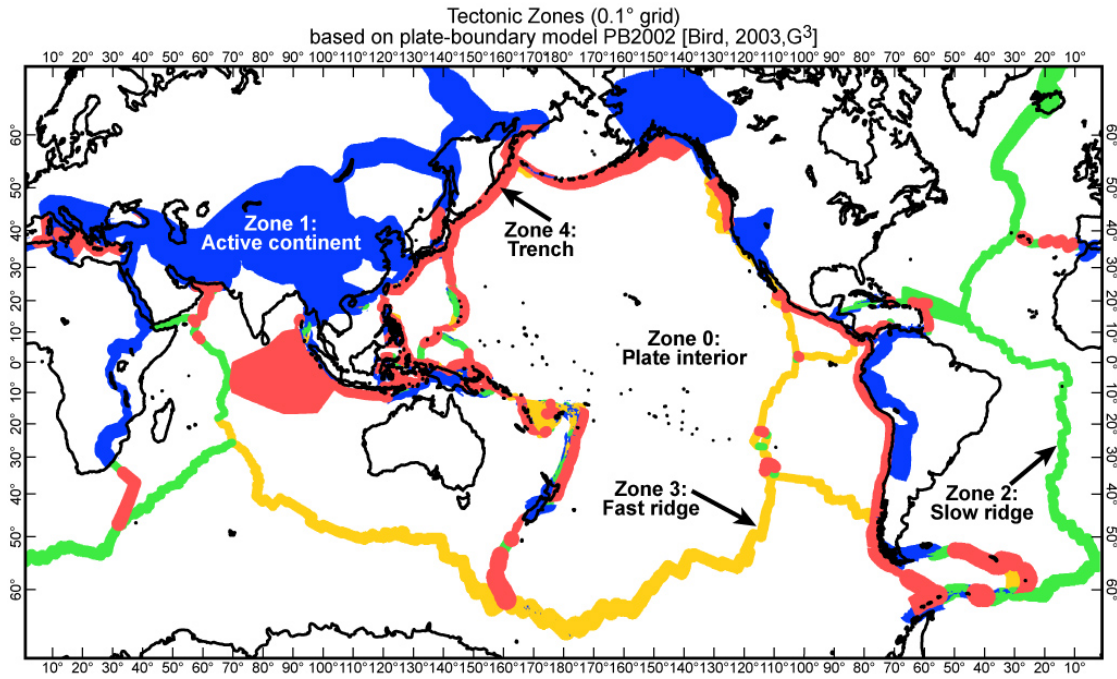


Figure 1: Five-zone plate boundary template.

Source: Kagan, Bird & Jackson (2010) [13].

events in the same time interval. These 32 states are often written in base 10 using the binary representation, for example, 5 is another representation of state 00101. We chose to use a time interval of 9 days (obtained empirically but not described here) to partition our earthquake catalogue into state-to-state transitions [2, 22]. Due to the variation of seismic hazards in the five zones, a minimum threshold magnitude of 6.0 was chosen for Zone 4 whereas a minimum threshold magnitude of 5.6 was used for the remaining zones. In other words, earthquakes with magnitude falling below these thresholds are not considered in our analysis. The catalogue used came from both the global CMT and NEIC catalogues as modified by Kagan et al. [13] consisting of earthquakes of depth at most 70 km occurring between January 1, 1982 to March 31, 2007. See Table 2 for a sample taken from this catalogue of the first 30 earthquake occurrences (column EQ) partitioned into time intervals using $\Delta t = 9$ days with a starting time of $t_0 = 723,911$ (here, time is measured as the number of days after 0 A.D.). The last column (State) represents the state of the Earth during that time interval. For example, during the first time interval the Earth is initially in state 10101 (i.e., state 21) due to the observance of earthquakes in zones 0, 2 and 4. Given the length of the catalogue used, we obtain a sequence of 1024 state-to-state transitions. From this sequence we form the transition frequency matrix defined as the 32-by-32 matrix T with T_{ij} representing the number of times the catalogue switched from state i to state j . The probability matrix P is defined similarly with P_{ij} representing the proportion of times the catalogue switched from state i to state j . The resulting transition frequency and probability matrices are displayed as colour-coded matrices in Figure 2.

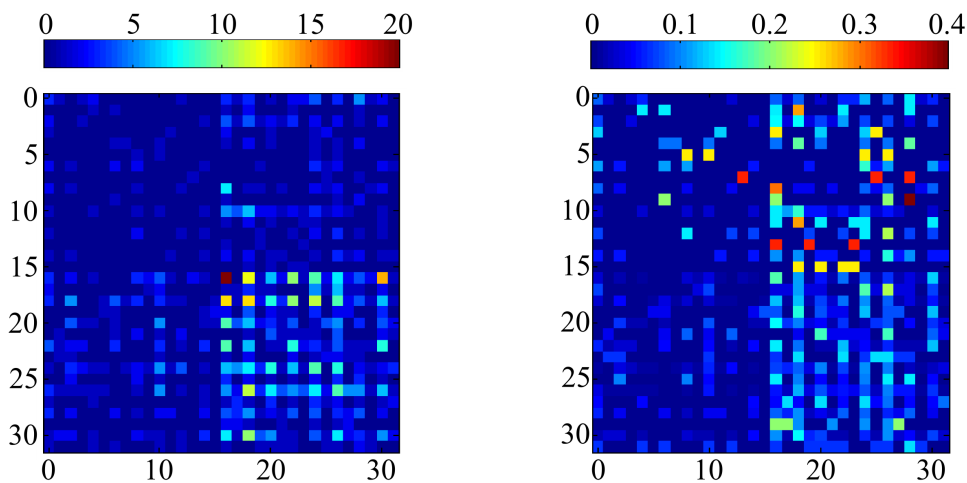


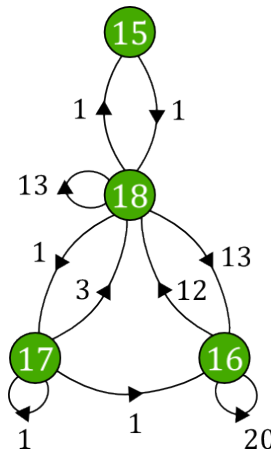
Figure 2: Transition frequency matrix (left) and probability matrix (right) for the Markov chain.

EQ	Time	Latitude	Longitude	Magnitude	Zone	State
1	723912.79	26.84	142.7	6.64	4	10101 (21)
2	723914.59	-0.96	-21.83	7.16	2	
3	723916.57	-0.49	-21.57	5.83	2	
4	723918.36	-3.34	177.5	5.93	0	
5	723922.26	13.91	124.35	7.13	4	10100 (20)
6	723923.07	-52.37	28.5	6.26	2	
7	723923.24	12.8	-87.3	6.17	4	
8	723924	-9.83	152.47	6.16	4	
9	723925.41	-9.26	151.8	6.13	4	
10	723926.58	-9.69	152.45	6.11	4	
11	723929.81	39.56	24.47	6.62	1	10011 (19)
12	723931.18	6.5	93.78	6.31	4	
13	723931.3	6.81	93.71	6.19	4	
14	723932.91	19.22	-155.6	5.72	0	
15	723932.94	19.2	-155.57	5.63	0	
16	723934.59	23.84	121.65	6.08	4	
17	723934.73	31.71	82.24	6.36	1	
18	723935.26	14.07	124.53	6.68	4	
19	723940.94	25.5	-45.29	5.94	2	00101 (5)
20	723941.11	16.71	-61.47	6.11	0	
21	723949.8	29.03	-113.03	5.64	3	01010 (10)
22	723955.83	5.81	94.68	5.78	1	
23	723956.22	-47.35	100.03	5.82	3	
24	723957.24	-3.17	177.31	6.00	0	10001 (17)
25	723962.56	-10.53	166.03	6.86	0	
26	723962.8	33.95	141.04	6.52	4	
27	723966.18	4.1	98.27	5.90	1	10010 (18)
28	723974.38	-22.54	173.66	6.15	4	
29	723981.44	-9.35	118.3	6.58	0	10001 (17)
30	723982.23	13.61	123.87	6.04	4	

Table 2: A sample of the earthquake catalogue used that shows data for the first 8 time intervals.

Source: US Geological Survey.

Statistical analysis performed on the matrices indicates that a Markov chain model for 5-zone earthquake sequencing appears to differ from the Poisson model. Our Markov chain for the catalogue is irreducible and aperiodic, hence, there is a unique stationary (limiting) distribution π where π_j is the limiting probability of state j . The n -step transition probability matrix, P^n , gives the probabilities of going from state i to state j in n transitions. We observe that P^n quickly converges to the rank one matrix $[1 \ 1 \ \cdots \ 1]\pi$ after about ten steps. Finite-state Markov chains can be represented as digraphs [12] giving a visual representation of the process. Typically the edge weights in the digraph represent probabilities (see Figure 3 for a subdigraph of our Markov chain using transition frequencies as weights).

Figure 3: Frequency counts amongst state-to-state transitions using states in $\{15, 16, 17, 18\}$.

As part of our study, properties of digraphs (and graphs) are used to draw conclusions about earthquake sequencing in a global context. One such graph property that we discuss here is that of centrality.

Centrality measures. There are various centrality measures for graphs that measure the importance of a node within the graph. Four of the most widely used in network analysis are degree centrality, closeness, betweenness and eigenvector centrality. Here we analyze the betweenness centrality [24] to obtain a ranking of states in the Markov chain. For each node v of a digraph G with node set V , the betweenness of v , denoted by $g(v)$, is defined to be

$$g(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where σ_{st} is the total number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths that pass through node v . Nodes with a high betweenness have a high probability to occur on a randomly chosen shortest path between two randomly chosen nodes. The (normalized) betweenness was computed for each node of the *underlying* digraph corresponding to the Markov chain and compared to the limiting distribution π of the Markov chain. Curiously, the betweenness of the underlying digraph provides a similar ranking of states to that of the limiting distribution (see Figure 4). This indicates that the combinatorial structure of the digraph contains important information of the earthquake sequencing, a topic we are currently exploring.

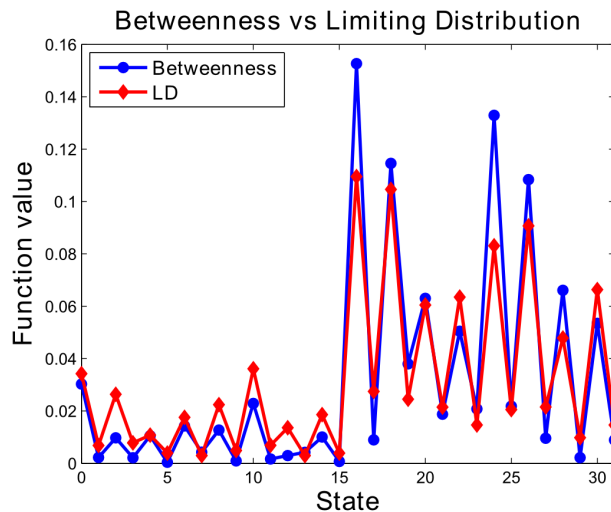


Figure 4: Betweenness for the underlying digraph and limiting distribution (LD) π of the Markov chain.

Future work. In earthquake sequencing, certain unexpected long-range behaviour enters into the forecasting problem. Long-range behaviour is an area of intense debate in seismology circles. One difficulty with the model described in this article is making causality connections between earthquakes that span large distances for forecasting purposes. To alleviate this problem, probabilities in the transition probability matrix (or corresponding weights in the Markov chain) can be updated to incorporate the spatio-temporal complexity of seismic events in relation to recurring events in the record-breaking sense (see [22] for details). This is accomplished empirically by using the recurrence network introduced in [3].

Acknowledgements. We thank the US Geological Survey National Earthquake Information Center for the data provided in Tables 1 and 2. The image in Figure 1 and a modified earthquake catalogue are provided courtesy of Kagan, Bird & Jackson [13].

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Leonid Arie Lerer: 70th Anniversary

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Leonid Lerer in his Technion office (2012)

At the ILAS conference in Providence, June 2013, a mini-symposium was dedicated to Leonid Arie Lerer on the occasion of his seventieth birthday (April 19, 2013). Now, a few months later, a special volume appeared in the book series *Operator Theory Advances and Applications*: “Advances in Structured Operator Theory and Related Areas: the Leonid Lerer Anniversary Volume” (Volume 237, Birkhäuser), with the present authors as editors.

Leonia, as he is known to his friends, is an expert in the theory of structured matrices and operators and related matrix-valued functions. He started his mathematical career in Kishinev, Ukraine, with Alek Markus and Israel Gohberg as research supervisors. His Ph.D. thesis, which he defended in 1969 in Kharkov, Ukraine, was devoted to the theory of operators acting in locally convex spaces. Asymptotic distribution of spectra and related limit theorems was one of his other research topics in these early years. In December 1973 he immigrated to Israel. Since 1981 he has been a professor at the Technion in Haifa where at present he has the status of emeritus. He has educated six Ph.D. students and five

masters students (including the third author of the present note).

Leonia’s more than 80 papers cover a wide spectrum of topics, ranging from functional analysis and operator theory, linear and multilinear algebra, ordinary differential equations, to mathematical systems and control theory. The main themes are: (1) spectral theory of matrix and operator polynomials and of rational matrix functions (common multiples/divisors, location of zeros); (2) matrix and operator equations (Lyapunov, Sylvester, Stein); (3) structured operators; (4) inverse problems, in particular, related to Szegő-Krein orthogonal polynomials and their continuous analogues; and (5) mathematical systems theory: realization problems for boundary value systems, minimality of partial realizations, including multivariable systems, completion problems. About one third of his papers have the words resultant or Bezout in the title. Leonia’s list of co-authors include: Daniel Alpay, Abraham Berman, Israel Gohberg, Iulian Haimovici, Marinus Kaashoek, Irina Karlin, Israel Koltracht, Benzion Kon, Peter Lancaster, Raphael Loewy, Igor Margulis, Vadim Olshevsky, Mark Petersen, André Ran, Leiba Rodman, Alexander Sakhnovich, Ilya Spitkovsky, Miron Tismenetsky, and Hugo Woerdeman.

As the program of the mini-symposium in Providence did, the Anniversary Volume focuses on topics that are close to Lerer’s mathematical interests. The main part of the book consists of a selection of peer-reviewed research articles presenting recent results on Toeplitz, Wiener-Hopf, and Toeplitz plus Hankel operators, Bezout equations, inertia type results, matrix polynomials, in one and severable variables, and related areas in matrix and operator theory. The first part of the book contains a picture (not the one above), Leonia’s Curriculum Vitae and List of Publications, and personal notes written by former students, mathematical friends and colleagues.

We use this occasion to wish Leonid Arie Lerer many more years of good health and happiness, and continuing mathematical work.

OBITUARY

David Allan Gregory 1942–2013



David A. Gregory

David Gregory died from cancer on July 12, 2013. He passed away peacefully in palliative care, with friends and family by his side. David was Professor Emeritus in the Department of Mathematics and Statistics at Queen’s University in Kingston, Ontario, Canada. He continued to be active in linear algebra, and participated in the joint Queen’s University and Royal Military College Discrete Mathematics Seminar. David was the author of over 50 research papers and supervised 6 Ph.D. students: Sebastian Cioabă, Randall Elzinga, Stewart Neufeld, Jian Shen, Kevin Vander Meulen, and Valerie Watts.

David enjoyed combinatorial matrix theory and eigenvalue problems in graph theory. He will be fondly remembered by his colleagues for his gentle helpfulness, his love of good combinatorial matrix theory, his well-crafted lectures, and his carefully composed writing.

Personal notes can be read and posted on a memorial page at: <http://www.uwo.edu/bshader/gregoryd>. A conference will be held to celebrate the scholarship of David Gregory: see upcoming conferences in this issue of *IMAGE*.

MATHEMATICAL OLYMPIAD SERIES

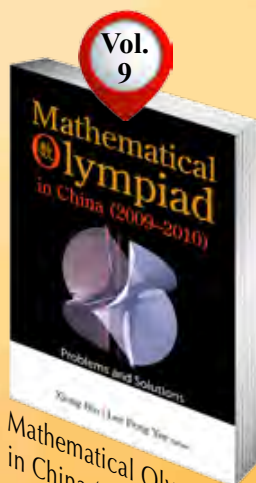
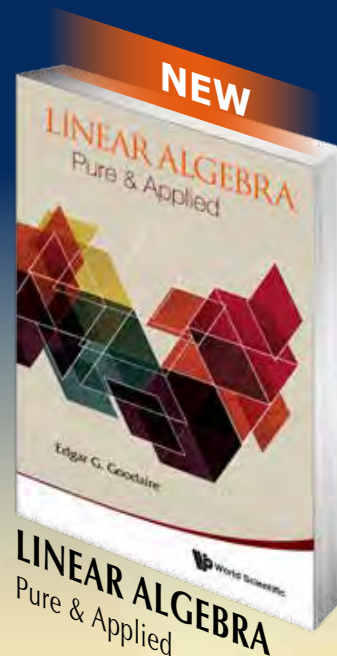
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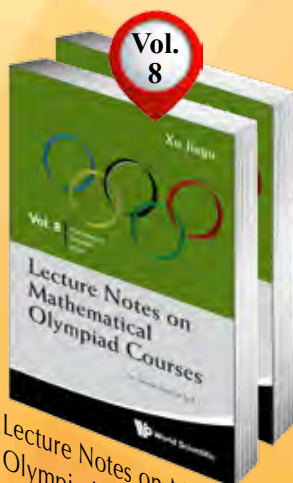
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National Institute of Education, Singapore



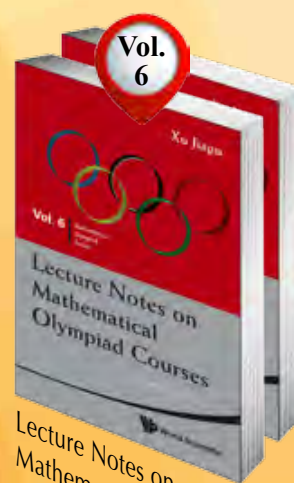
Mathematical Olympiad
in China (2009-2010)



Lecture Notes on Mathematical
Olympiad Courses



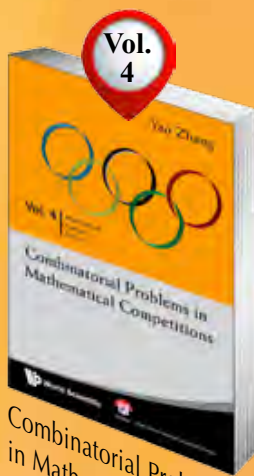
A Second Step to Mathematical
Olympiad Problems



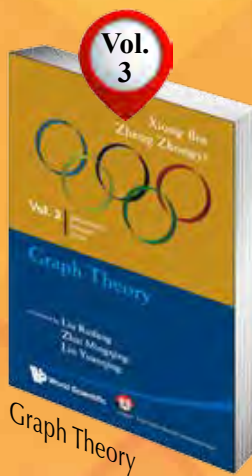
Lecture Notes on
Mathematical Olympiad
Courses



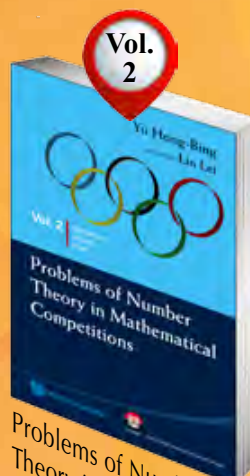
Selected Problems of the
Vietnamese Mathematical
Olympiad (1962-2009)



Combinatorial Problems
in Mathematical
Competitions



Graph Theory



Problems of Number
Theory in Mathematical
Competitions



A First Step to
Mathematical Olympiad
Problems

UPCOMING CONFERENCES AND WORKSHOPS

International Workshop on Operator Theory and its Applications (IWOTA 2013) Bangalore, India, December 16–20, 2013

The XXIVth edition of the International Workshop on Operator Theory and Its Applications (IWOTA) will be held at the Indian Institute of Sciences, Bangalore from December 16 to 20, 2013. The conference website is <http://math.iisc.ernet.in/~iwota2013>, and the contact details are: Tirthankar Bhattacharyya (Main Organizer). Phone: +91 80 2293 2710, Fax: +1 509 561 1264, email: iwota2013@gmail.com. For more details, please visit the conference website.

The 7th Seminar on Linear Algebra and its Applications Mashhad, Iran, February 26–27, 2014

The 7th Seminar on Linear Algebra and its Applications will be held at Ferdowsi University of Mashhad (Iran) during 26–27 February 2014. The seminar will provide a forum for mathematicians and scholar students to present their latest results about all aspects of linear algebra and a means to discuss their recent research with each other. Interested linear algebraists are warmly invited to propose their new ideas for enhancing the scientific level of the seminar. We look forward to meeting you in the second biggest city of Iran, Mashhad. For participating in the seminar, the interested mathematicians may contact the chair, Prof. M. S. Moslehian, via moslehian@member.ams.org or moslehian@um.ac.ir.

Western Canada Linear Algebra Meeting Regina, Canada, May 10–11, 2014

The Western Canada Linear Algebra Meeting (WCLAM) provides an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet and present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting.

Previous WCLAMs were held in Regina (1993, 2002), Lethbridge (1995, 2012), Kananaskis (1996), Victoria (1998, 2006), Winnipeg (2000, 2008), and Banff (2004, 2010).

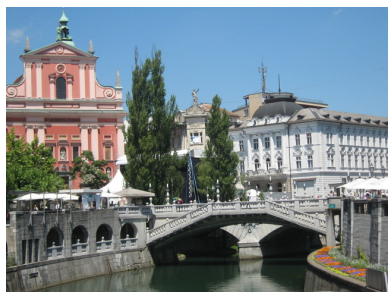
The WCLAM '14 will be held at the University of Regina on May 10–11 and the program will include the following four featured speakers: Joel Friedman (University of British Columbia), Roger Horn (University of Utah), Mitja Mastnak (Saint Mary's University) and Maya Mincheva (Northern Illinois University).

Participants who wish to present a 25-minute lecture or a poster should send a title/abstract to: western.canada.linear.algebra.meeting@uregina.ca. The deadline for abstract submission is April 15, 2014. General registration information can be found at: <http://uregina.ca/~abstsubm/index.html>. Please register by April 30, 2014.

The WCLAM Organising Committee consists of Shaun Fallat (Regina), Hadi Kharaghani (Lethbridge), Steve Kirkland (Manitoba), Peter Lancaster (Calgary), Michael Tsatsomeros (Washington State), and Pauline van den Driessche (Victoria). The local organisers are Shaun Fallat (Regina) and Douglas Farenick (Regina).

The WCLAM Organising Committee gratefully acknowledges the generous support for this meeting provided by the Pacific Institute for the Mathematical Sciences, the Department of Mathematics & Statistics, the Faculty of Science, and the President's Office at the University of Regina.

The 7th Linear Algebra Workshop (LAW'14), and the 23rd International Workshop on Matrices and Statistics (IWMS) Ljubljana, Slovenia, June 4–12, 2014 and June 8–12, 2014



Ljubljana

The main theme of the Linear Algebra Workshop will be the interplay between operator theory and algebra. The workshop will follow the usual format. A few hours of talks will be scheduled for the morning sessions, while afternoons will be reserved for work in smaller groups. LAW'14 will be organized in conjunction with IWMS that will take place in the second week. The main theme of the International Workshop on Matrices and Statistics will be the interplay between matrices and statistics. Both workshops will be held at the Faculty of Mathematics and Physics, Ljubljana, Slovenia. More information can be found at <http://www.law05.si/law14> and <http://www.law05.si/iwms/>.

Householder Symposium XIX on Numerical Linear Algebra Spa, Belgium, June 8–13, 2014

The Householder Symposium XIX on Numerical Linear Algebra will be held in Spa, Belgium, June 8–13, 2014. The conference website is <http://sites.uclouvain.be/HHXIX/>.

Attendance at the meeting is by invitation only: applications were due in October 2013. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. The symposium is very informal, with the intermingling of young and established researchers a priority. Participants are expected to attend the entire meeting. The fifteenth Householder Award for the best thesis in numerical linear algebra since January 1, 2011 will be presented. It is expected that partial support will be available for some students, early career participants, and participants from countries with limited resources.

The Householder Symposium takes place in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra.

Graph Theory, Matrix Theory and Interactions: A conference to celebrate the scholarship of David Gregory Queen's University, Canada, June 20–21, 2014

This is a preliminary announcement for a conference to celebrate the scholarship of David Gregory which will be held June 20–21, 2014, at Queen's University, Kingston, Ontario, Canada. Researchers representing any aspect of David Gregory's mathematical scholarship are invited to attend and give a presentation.

The following people have already indicated an interest to attend: Richard Brualdi, Sebastian Cioabă, Chris Godsil, Edwin van Dam, Randall Elzinga, Shaun Fallat (ILAS Lecturer), Willem Haemers, Steve Kirkland, Ram Murty, Stewart Neufeld, Bryan Shader, Claude Tardif, Kevin Vander Meulen, Valerie Watts, and David Wehlau.

The organizing committee consists of Sebastian Cioabă, Ram Murty, Bryan Shader, Claude Tardif, Kevin Vander Meulen, and David Wehlau. Please contact any committee member if you wish to attend and/or present. Information can be found at: <http://www.mast.queensu.ca/graph-theory-matrix-theory-and-interactions/>.

A special issue of the *Electronic Journal of Linear Algebra* will be dedicated to David Gregory. Details forthcoming.

19th ILAS Conference Suwon, South Korea, August 6–9, 2014

The 19th International Linear Algebra Society Conference (ILAS 2014) will be held August 6–9, 2014 at Sungkyunkwan University, Science Campus, in Suwon, Republic of Korea. The theme of the conference is *Solidarity in Linear Algebra*. ILAS 2014 is one of the satellite conferences of the 2014 International Congress of Mathematicians (ICM 2014) which will be held August 13–21, 2014 in Seoul, Korea.

ILAS 2014 plenary speakers:

- Ravindra Bapat (Indian Statistical Institute Delhi, India)
- Peter Benner (Chemnitz University of Technology, Germany)
- Dario Bini (University of Pisa, Italy), *LAA Lecturer*
- Ljiljana Cvetković (University of Novi Sad, Serbia)
- Shaun Fallat (University of Regina, Canada), *Taussky-Todd Lecturer*
- Andreas Frommer (University of Wuppertal, Germany), *SIAG/LA sponsored speaker*
- Stéphane Gaubert (École Polytechnique, France)
- Chi-Kwong Li (College of William and Mary, USA)
- Yongdo Lim (Sungkyunkwan University, Korea)
- Panayiotis Psarrakos (National Technical University of Athens, Greece)
- Vladimir Sergeichuk (Institute of Mathematics, Kiev, Ukraine)
- Bernd Sturmfels (University of California-Berkeley, USA)
- Tin-Yau Tam (Auburn University, USA)



Sungkyunkwan University

ILAS 2014 invited minisymposia:

- Combinatorial Problems in Linear Algebra (Richard Brualdi and Geir Dahl)
- Matrix Inequalities (Fuzhen Zhang and Minghua Lin)
- Spectral Theory of Graphs and Hypergraphs (Vladimir S. Nikiforov)
- Tensor Eigenvalues (Jia-Yu Shao and Liqun Qi)
- Quantum Information and Computing (Chi-Kwong Li and Yiu Tung Poon)
- Riordan Arrays and Related Topics (Gi-Sang Cheon and Lou Shapiro)
- Nonnegative Matrices and Generalizations (Judi McDonald)
- Toeplitz Matrices and Operators (Torsten Ehrhardt)

$$\begin{bmatrix} \mathcal{I} & \mathcal{L} \\ \mathcal{A} & \mathcal{S} \end{bmatrix}$$

In addition, the scientific organizing committee of ILAS 2014 welcomes proposals for contributed minisymposia.

The scientific organizing committee (SOC) consists of Suk-Geun Hwang (Chair), Nair Abreu, Tom Bella, Rajendra Bhatia, Richard Brualdi, Man-Duen Choi, Nicholas J. Higham, Leslie Hogben, Stephen Kirkland, Sang-Gu Lee, Helena Šmigoc and Fuzhen Zhang. The local organizing committee consists of Suk-Geun Hwang, Sang-Gu Lee, Gi-Sang Cheon, Yongdo Lim, Hyun-Min Kim, In-Jae Kim, and Woong Kook.

Each minisymposium speaker will be given 25 minutes (20 minutes talk plus 5 minutes discussion). Each minisymposium will have a maximum of 300 minutes (12 speakers). Proposals should be in PDF format and no more than 2 pages in total. Proposals should include the name(s) of the organizer(s), a short description of the topics of the minisymposium, and the names of proposed speakers. Proposals are due January 20, 2014, and decisions will be made by the SOC in early February, 2014.

Please email proposals directly to both of the following SOC members: Sang-Gu Lee (sglee@skku.edu) and Fuzhen Zhang (zhang@nova.edu). They will communicate with the SOC Chair and its members.

Linear Algebra and its Applications will have a special issue devoted to ILAS 2014 with the following guest editors: Ljiliana Cvetković, Andreas Frommer, Suk-Geun Hwang, Helena Šmigoc and Fuzhen Zhang.

The official homepage of ILAS 2014 is <http://www.ilas2014.org>.

The SOC wishes that many mathematicians around the world will participate and enjoy the ILAS 2014 conference.

Gene Golub SIAM Summer School 2014 Linz, Austria, August 4–15, 2014

The 2014 Gene Golub SIAM Summer School on Simulation, Optimization, and Identification in Solid Mechanics will be held on August 4–15 in Linz, Austria. The application deadline for students is February 1, 2014.

This summer school will foster advanced knowledge for the participating graduate students in several areas related to simulated materials in solid mechanics. Within this broad field the summer school will concentrate on four key issues, namely, (1) identification of material parameters from measurements, (2) material and topology optimization, (3) optimization subject to variational inequalities, and (4) adaptive discretization.

The lectures will be given by Roland Herzog (TU Chemnitz), Esther Klann (JKU Linz), Michael Sting (FAU Erlangen-Nürnberg), and Winnifried Wollner (University of Hamburg).

Please check <http://www.math.uni-hamburg.de/g2s3> for more information about the courses and how to apply.

Send News for *IMAGE* Issue 52

IMAGE Issue 52 is due to appear online on June 1, 2014. Send your news for this issue to the appropriate editor by April 2, 2014. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

- problems and solutions to Bojan Kuzma (bojan.kuzma@upr.si)
- feature articles to Michael Cavers (mcavers@ucalgary.ca)
- history of linear algebra to Naomi Shaked-Monderer (nomi@tx.technion.ac.il)
- book reviews to Douglas Farenick (Doug.Farenick@uregina.ca)
- linear algebra education news to David Strong (David.Strong@pepperdine.edu)
- announcements and reports of conferences, workshops and journals to Minerva Catral (catralm@xavier.edu)
- interviews of senior linear algebraists to Carlos Fonseca (cmf@mat.uc.pt)
- advertisements to Amy Wehe (awehe@fitchburgstate.edu).

Send all other correspondence to Kevin N. Vander Meulen (kvanderm@redeemer.ca).

For past issues of *IMAGE*, please visit <http://www.ilasic.org/IMAGE/>.



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Steve Kirkland, Chi-Kwong Li

$$\begin{aligned}\zeta^\alpha &= \sum_{\pi} \operatorname{sgn} \pi \xi^{\alpha - \operatorname{id} + \pi} \\ \zeta^\alpha \otimes \zeta^\beta &= \sum_{\pi} \operatorname{sgn} \pi \xi^{\alpha - \operatorname{id} + \pi} \otimes \zeta^\beta \\ \zeta^\alpha \otimes \zeta^\beta &= \sum_{\pi} \operatorname{sgn} \pi (\zeta^\beta \downarrow S_{\alpha - \operatorname{id} + \pi} \uparrow S_n)\end{aligned}$$



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CONFERENCE REPORTS

18th Conference of the International Linear Algebra Society Providence, Rhode Island, USA, June 3–7, 2013

Report by Tom Bella



*T. Laffey receiving prize from
L. Rodman and S. Kirkland*

The 18th Conference of the International Linear Algebra Society was held in Providence, Rhode Island, USA from June 3rd through June 7th, 2013. The conference was attended by about 340 total participants and featured a total of 308 talks. Conference sponsors included ILAS, Comsol, Taylor & Francis, Elsevier, Oxford University Press, SIAM and the SIAM Activity Group on Linear Algebra, The University of Connecticut, The University of Massachusetts Dartmouth, and The University of Rhode Island.

On the night before the first official day, a short welcome reception was held along with early registration. On Wednesday afternoon, about 150 of the participants went on the Newport Mansions excursion, and had dinner in downtown Newport.

The conference banquet was held on Thursday evening, and about 200 of the conference participants saw the presentation of the Hans Schneider Prize to Thomas Laffey. The award was presented by Leiba Rodman and Steve Kirkland, with some remarks from Hans Schneider himself following the ceremony.

There were 13 plenary talks: Anne Greenbaum, “ K -Spectral Sets and Applications”; Leiba Rodman, “Eigenvalue perturbation analysis of structured matrices”; Alan Edelman, “The Interplay of Random Matrix Theory, Numerical Linear Algebra, and Ghosts and Shadows”; Raymond Sze, “Quantum Operation and Quantum Error Correction”; Maryam Fazel, “Recovery of Structured Models with Limited Information”; Ivan Oseledets, “Numerical tensor methods and their applications”; Thomas Laffey, “Characterizing the spectra of nonnegative matrices”; Gilbert Strang, “Functions of Difference Matrices”; Fuzhen Zhang, “Generalized Matrix Functions”; Dianne O’Leary, “Eckart-Young Meets Bayes: Optimal Regularized Low Rank Inverse Approximation”; Dan Spielman, “Nearly optimal algorithms for solving linear equations in SDD matrices”; Jean Bernard Lasserre, “Tractable characterizations of nonnegativity on a closed set via Linear Matrix Inequalities”; and Ravi Kannan, “Randomized Matrix Algorithms.”



L. Livshits and P. Šemrl

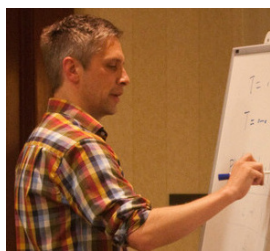
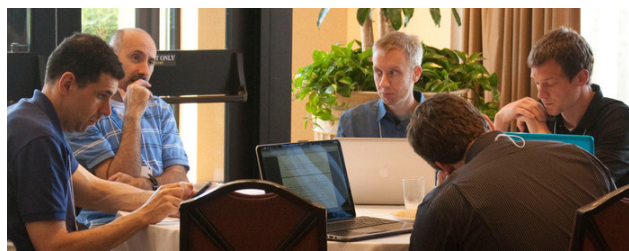


R. Sze plenary address

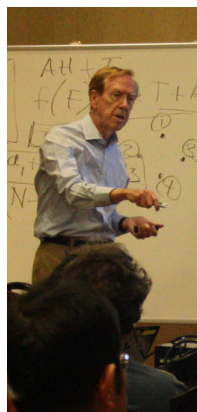
The program was packed, with up to 6 parallel sessions going at any time. The invited minisymposia with their organizers were: Linear Algebra Problems in Quantum Computation, Chi-Kwong Li, Yiu Tung Poon; Matrices and Graph Theory, Louis Deaett, Leslie Hogben; Randomized Matrix Algorithms, Ravi Kannan, Michael Mahoney, Petros Drineas, Nathan Halko, Gunnar Martinsson, Joel Tropp; Symbolic Matrix Algorithms, Jean-Guillaume Dumas, Mark Giesbrecht; Krylov Subspace Methods for Linear Systems, James Baglama, Eric de Sturler; Matrices and Orthogonal Polynomials, Jeff Geronimo, Francisco Marcellan, Lothar Reichel; Matrix Methods for Polynomial Root-Finding, Dario Bini, Yuli Eidelman, Marc Van Barel, Pavel Zhlobich; Multilinear Algebra and Tensor Decompositions, Lieven De Lathauwer, Eugene Tyrtyshnikov; Nonlinear Eigenvalue Problems, Froilan M. Dopico, Volker Mehrmann, Françoise Tisseur; Abstract Interpolation and Linear Algebra, Joseph Ball, Vladimir Bolotnikov; Structured Matrix Functions and their Applications (Dedicated to Leonia Lerer on the occasion of his 70th birthday), Rien Kaashoek, Hugo Woerdeman; and Linear Algebra Education Issues, Avi Berman, Sang-Gu Lee, Steven Leon.



ILAS 2013 group photograph

*R. Tifenbach**A. Berliner, T.S. Michael, L. Deaett
S. Meyer and G. Lisella**ILAS executive**A. Greenbaum*

The conference also included many contributed minisymposia: Matrices over Idempotent Rings, Martin Gavalec, Sergei Sergeev; Applications of Tropical Mathematics, James Hook, Marianne Johnson, Sergei Sergeev; Matrix Methods in Computational Systems Biology and Medicine, Konstantin Fackeldey, Amir Niknejad, Marcus Weber; Advances in Combinatorial Matrix Theory and its Applications, Carlos Fonseca, Geir Dahl; Structure and Randomization in Matrix Computations, Victor Pan, Jianlin Xia; Generalized Inverses and Applications, Minerva Catral, Nestor Thome, Yimin Wei; Linear Least Squares Methods: Algorithms, Analysis, and Applications, Sanzheng Qiao, Huaian Diao; Matrix Inequalities, Minghua Lin, Fuzhen Zhang; Linear and Nonlinear Perron-Frobenius Theory, Thomas J. Laffey, Bas Lemmens, Raphael Loewy; Matrices and Total Positivity, Jorge Delgado, Shaun Fallat, Juan Manuel Peña; Linear Complementarity Problems and Beyond, Sou-Cheng Choi; Linear Algebra, Control, and Optimization, Biswa Datta; and Sign Pattern Matrices, Frank Hall, Zhongshan (Jason) Li, Hein van der Holst.

*I.-J. Kim**G. Strang**T. Bella, V. Olshevsky
and I. Ipsen*

The scientific organizing committee consisted of Tom Bella and Vadim Olshevsky (chairs), Ljiljana Cvetković, Heike Faßbender, Chen Greif, J. William Helton, Olga Holtz, Steve Kirkland, Victor Pan, Panayiotis Psarrakos, Tin Yau Tam, Paul Van Dooren, and Hugo Woerdeman. More information about the conference may be found at <http://www.ilas2013.com>.

The next ILAS meeting, the 19th ILAS Conference, will be held August 6–9, 2014, at Sungkyunkwan University (Science Campus), Suwon, Korea. More details may be found on the conference website <http://www.ilas2014.org>

Advanced School and Workshop on Matrix Geometries and Applications, International Centre for Theoretical Physics (ICTP) Trieste, Italy, July 1–12, 2013

Report by Rajendra Bhatia and Peter Šemrl

The school and the workshop were organized by Rajendra Bhatia and Peter Šemrl, with the help of F. Rodriguez Villegas who works full time at ICTP. Solutions of several problems in matrix theory involve ideas from differential, algebraic, affine, and projective geometry. The results have been used in diverse applications in image processing, statistics, machine learning, elasticity, mathematical foundations of quantum mechanics, quantum information, etc. The event was devoted to the exposition and discussion of some of these ideas.

During the first week, short instructional courses were given by F. Barbaresco, R. Bhatia, D. Bini, J. Holbrook, L.-H. Lim, and P. Šemrl. Three lectures of 60 minutes in the morning were followed by one more lecture in the afternoon. Then there were two tutorial/problem sessions with B. Iannazzo, T. Jain, M. Palfia, K. Šivic, and Ke Ye acting as tutors. There were quite a few enthusiastic participants who continued to work in informal groups. Lecturers and tutors were happy to help them throughout the event to solve proposed problems.

The second week consisted of an advanced workshop where invited speakers presented current research in the mornings. The tutorial/problem sessions continued in the afternoons. The invited speakers at the workshop were J. Antezana (ILAS lecturer), S. Bonnabel, J. Holbrook, B. Iannazzo, G. Larotonda, J. Lawson, Y. Lim, B. Meini, M. Moakher, L. Molnár, F. Nielsen, M. Palfia, D. Petz, K. Šivic, and Ke Ye.

Several lecturers and speakers prepared lecture notes which can be found at http://cdsagenda5.ictp.it/full_display.php?ida=a12193.

There were 54 participants, coming from many countries: Algeria, Argentina, Azerbaijan, Burkina Faso, China, Croatia, Egypt, Ethiopia, Hungary, India, Iran, Italy, Korea, Mexico, Morocco, Nigeria, Portugal, Slovenia, Spain, Tunisia, Turkey, Ukraine, USA, Uzbekistan, and Vietnam. Lecturers and tutors agreed that some of the participants were very talented and hard working, and it is hoped that they will soon become successful mathematicians.

A similar event at ICTP was organized in 2009 and the report of that appeared in *IMAGE* 43. In that issue, one can find some more information on ICTP and its mission to support the best possible science with special attention given to the needs of developing countries.



Trieste workshop photograph

The 4th International Conference on Matrix Analysis and Applications Konya, Turkey, July 2–5, 2013

Report by Ramazan Türkmen and Fuzhen Zhang

The 4th International Conference on Matrix Analysis and Applications was successfully held July 2-5, 2013, in Konya, Turkey. In addition to the keynote speech by Steve Kirkland (Stokes Professor at Hamilton Institute, National University of Ireland) and the ILAS lecture by Alexander Klyachko (Bilkent University, Ankara, Turkey), there were ten plenary addresses and as many as eighty-eight contributed talks presented in four parallel sessions. The invited speakers were Delin Chu (National University of Singapore, Singapore), Carlos Martins da Fonseca (University of Coimbra, Portugal), Stephan Garcia (Pomona College, California, USA), Mohammad Sal Moslehian (Ferdowsi University of Mashhad, Iran), Vehbi Paksoy (Nova Southeastern University, USA), Nung-Sing Sze (The Hong Kong Polytechnic University, China), Tin-Yau Tam (Auburn University, USA), Hugo J. Woerdeman (Drexel University, Philadelphia, USA), Pei-Yuan Wu (National Chiao Tung University, Taiwan, R.O. China), and Xiao-Dong Zhang (Shanghai Jiaotong University, China).

A hundred and fourteen participants from twenty countries enjoyed the four day event in the Dedeman Hotel located in the center of Konya. The conference banquet was full of the gourmet Turkish foods and the one-day sight-seeing tour of Cappadocia was spectacular.

The main goal of the conference was to gather experts, researchers, and students together to present recent developments in the dynamic and important field of matrix analysis. The conference also aims to stimulate research and support interactions between mathematicians and scientists by creating an environment for participants to exchange ideas and to initiate collaborations or professional partnerships. The previous meetings in the sequence have taken place in China and USA.

The meeting was sponsored by ILAS and Selçuk University. The scientific organizing committee consisted of Peter Šemrl, Tin-Yau Tam, Qingwen Wang, and Fuzhen Zhang. The local organizing committee members are Vildan Bacak, Durmuş Bozkurt, Şerife Burcu Bozkurt, Ahmet Sinan Çevik, Yildiray Keskin, Hasan Köse, Ayşe Dilek Maden, Galip Oturanç, Vehbi Paksoy, Necati Taşkara, Ramazan Türkmen, Zübeyde Ulukök, and Fatih Yılmaz.

The conference website can be found at: <http://icmaa2013.selcuk.edu.tr/>



International Conference on Matrix Analysis and Applications group photograph

**The 22nd International Workshop on Matrices and Statistics (IWMS-2013),
The Fields Institute for Mathematical Research
Toronto, Canada, August 12–15, 2013**

Report by S. Ejaz Ahmed and Simo Puntanen

The purpose of the workshop was to bring together researchers sharing an interest in a variety of aspects of statistics and its applications as well as matrix analysis and its applications to statistics, and offer them a possibility to discuss current developments in these subjects. The workshop aimed to bridge the gap among statisticians, computer scientists and mathematicians in understanding each other's tools.

The International Organizing Committee was chaired by S. Ejaz Ahmed (Canada). The workshop was sponsored by the Fields Institute for Mathematical Research and the SAS Institute.

The keynote speakers were Kai-Tai Fang: "The Magic Square - Historical Review and Recent Development," and Ingram Olkin: "A Linear Algebra Biography" (his talk was read by George P. H. Styan). The invited speakers were Mohamed Amezziane, Oskar Maria Baksalary, Somnath Datta, Susmita Datta, Ali Ghodsi, Karl E. Gustafson, Abdulkadir Hussein, Tõnu Kollo, Shuangge Ma, Steven N. MacEachern, Jianxin Pan, Serge Provost, George P. H. Styan, and Fuzhen Zhang.

Invited special sessions were arranged on the following topics: Applied Probability (organized by Jeffrey J. Hunter); Statistical Inference on GLM (organized by Krishna Saha); Perspectives on High Dimensional Data Analysis (organized by Muni S. Srivastava); Open-Source Statistical Computing (organized by Reijo Sund).

Simo Puntanen organized a *Special Session to Celebrate Lynn Roy LaMotte's 70th Birthday*, with speakers: David A. Harville, Lynn Roy LaMotte, and J. N. K. Rao.



Jianxin Pan and Kai-Tai Fang



Susan Searle and Heather Selvaggio

Jeffrey J. Hunter organized a *Memorial Session and Reception to Honor Shayle R. Searle* (1928 – 2013), with speakers: David A. Harville, Jeffrey J. Hunter, J. N. K. Rao, Robert Rodriguez, Susan Searle, and Heather Selvaggio. The participants in particular appreciated the wonderful personal reminiscences of Susan Searle and Heather Selvaggio, the two daughters of Shayle R. Searle. The reception at the Fields Institute was supported by the SAS Institute, and Susan Searle and Heather Selvaggio.

The website of the workshop is <http://www.fields.utoronto.ca/programs/scientific/13-14/IWMS/> and the program is available at <http://www.fields.utoronto.ca/programs/scientific/13-14/IWMS/program.html>. A total of about 70 participants attended the conference.

Selected refereed papers presented in the workshop will be published in the *Journal of Statistical Computation and Simulation*.

For previous IWMS-workshops, see <http://www.sis.uta.fi/tilasto/iwms/> from where, for example, "A short history of the International Workshop on Matrices and Statistics" can be downloaded (<http://www.sis.uta.fi/tilasto/iwms/IWMS-history.pdf>).

The 23rd International Workshop on Matrices and Statistics (IWMS-2014) will be held in Ljubljana, Slovenia, June 8–12, 2014 (starting with a cruise on Sunday, June 8, 2014), <http://www.law05.si/iwms/>.



Participants of the IWMS-2013

International Conference on Matrix Analysis and Its Applications - MatTriad 2013 Herceg Novi, Montenegro, September 16–20, 2013

Report by Ljiljana Cvetković

The 2013 edition of MatTriad, fifth in a series of international conferences on Matrix Analysis and Its Applications, was held at the Hunguest Hotel Sun Resort Herceg Novi (Montenegro), from the 16th to the 20th of September 2013. Following the tradition of its predecessors, this meeting gathered researchers around topics in matrix theory and its profound role in theoretical and numerical linear algebra, numerical and functional analysis, and statistics. Invited lectures were presented by Richard A. Brualdi (USA) on alternating sign matrices, Stephen J. Kirkland (Ireland) on the Kemeny constant for a Markov chain, and Paulo C. Rodrigues (Portugal) on the role of matrix analysis in statistical genetics. The last of the invited talks was given by the winner of Young Scientists Award 2011 given at MatTriad 2011 held in Tomar, Portugal. As before, the work of the conference included four lectures in matrix theory, especially dedicated to young participants, given by distinguished experts in the field: Siegfried M. Rump (Germany) on normwise and componentwise structured perturbations of matrices, and Adi Ben-Israel (USA) on the Moore-Penrose inverse and its different applications.

A special issue of CEJM (*Central European Journal of Mathematics*) related to matrix problems will be devoted to the conference, with a significant part based on the best papers of participants.

Apart from the successful scientific work and professional interchange, participants especially enjoyed the beauties of the Hunguest Hotel Sun Resort on the Adriatic coast, in the town of Herceg Novi, at the entrance of Europe's southernmost fjord. At the gala lunch and the closing ceremony of the conference, two young scientists received the 2013 awards: Jaroslav Horáček (Czech Republic) for the talk "On Solvability and Unsolvability of Overdetermined Interval Linear Systems" and Maja Nedović (Serbia) for the talk "Generalizations of Diagonal Dominance and Applications to Max-norm Bounds of the Inverse." They are invited to present talks at the 6th MatTriad conference to be held in 2015 in Coimbra, Portugal.

The conference was co-organized by the Faculty of Science, Department of Mathematics and Informatics, Novi Sad and the Faculty of Mathematics and Computer Science of Adam Mickiewicz University, Poznań. The conference scientific committee consisted of Tomasz Szulc (Poland) - Chair, Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany) and Simo Puntanen (Finland), and its organising committee was Ljiljana Cvetković (Serbia) - Chair, Francisco Carvalho (Portugal), Ksenija Doroslovački (Serbia), and Vladimir Kostić (Serbia).

More details about the program, the book of abstracts, and the presentations can be found at:
<http://mattriad2013.pmf.uns.ac.rs>.



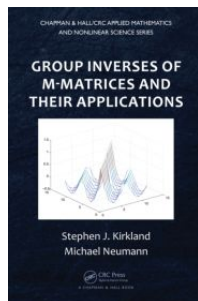
Participants of the MatTriad 2013 conference

BOOK REVIEWS

Group Inverses of M-matrices and Their Applications, by Stephen J Kirkland and Michael Neumann

Chapman & Hall/CRC, Applied Mathematics and Nonlinear Science Series,
CRC Press, Boca Raton 2013, ISBN 978-1-4398-8858-2, xv+316 pages.

Reviewed by Michael Tsatsomeros, Washington State University, tsat@wsu.edu



Have you ever thought of asking a colleague to write a book, just for the selfish reason that it would make your life that much easier? The authors of this book are friends and collaborators, and I have always looked up to them. When Miki Neumann and Steve Kirkland started working together on the topics of this book, I vowed to follow their work as it was very close to my research interests. This was probably during or just a while after the IMA year in Applied Linear Algebra (1991). Initially, I managed to stay informed, but after putting off any in-depth reading of their papers for a while, the task of catching up became quite formidable. So, I thought, why not ask them to put everything together in a book? As a matter of fact, I never did, but they must have read my mind.

To get to the point, let $Q = \rho I - A$, where A is a square entrywise nonnegative matrix and ρ is the spectral radius of A . Since by the Perron-Frobenius theorem ρ is an eigenvalue of A , Q must be a singular matrix. When the algebraic and geometric multiplicities of the eigenvalue 0 of Q coincide, there exists a unique matrix X such that

$$QXQ = Q, \quad XQX = X \quad \text{and} \quad QX = XQ.$$

The matrix Q is known as a singular M-matrix and X as the *group inverse* of Q . This book is a broad collection of properties, results and applications of group inverses of M-matrices. The applications include demographic models, Markov chains and the analysis of weighted graphs via their Laplacian matrices. One key fact in this study is the relation of the group inverse of the singular M-matrix Q to the derivatives of the spectral radius of A and of the corresponding eigenvectors as functions of the entries of A . Moreover, as it turns out, other (subdominant) eigenvalues of A can also be estimated via the group inverse of Q . As a consequence, fundamental notions in Markov chains (e.g., mean first passage times) and in graph theory (e.g., algebraic connectivity) can be studied via group inverses of M-matrices.

The essence of this book is drawn from the collaborative work of the two authors. This collaboration came as a natural outcome of their individual experiences and achievements. Indeed, the strength of the results featured in this book is a direct reflection of the strengths and interests of the two authors as research mathematicians.

Many of the concepts and conclusions in this book are fairly technical, however, the authors do a good job of presenting them in an accessible fashion. The main and supporting results flow naturally and are stated in a self-contained manner. The mathematical intuition and motivation are aided by several remarks, and ample commentary. There are also results of independent interest being quoted, several of which seem to appear in a book for the first time.

The first chapter contains two motivating examples involving Leslie matrices and stochastic matrices. In Chapter 2 there is a review of the basic theory of group inverses. Chapter 3 is about the partial derivatives of the spectral radius and the corresponding eigenvectors relative to matrix entries. Chapters 5 and 6 contain the main results on Markov chains and Chapter 7 is concerned with the applications to graphs and their Laplacian matrices. The concluding Chapter 8 regards methods to compute group inverses of M-matrices, including a fairly detailed discussion of efficiency, reliability and conditioning issues. Finally, we note that the book comes with several illustrative examples and figures, a full index and complete bibliography, as well as lists of figures and symbols.

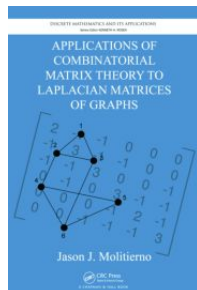
This book is worth considering for your personal collection. I certainly urge you to request your library to obtain a copy. I recommend it highly to anyone with interest in the specific applications and to the connoisseurs of nonnegative matrix theory.

Lastly, allow me to thank Steve Kirkland for completing this book after the untimely passing of Miki Neumann.

Applications of Combinatorial Matrix Theory to Laplacian Matrices of Graphs,
by Jason J. Molitierno

**Discrete Mathematics and Its Applications. CRC Press,
 Boca Raton, 2012, 405 pages, ISBN 978-1-4398-6337-4.**

Reviewed by Ryan M. Tifenbach, Memorial University of Newfoundland, Ryan.Tifenbach@mun.ca



Let G be a simple graph on the vertex set $\{1, 2, \dots, n\}$ and suppose that each edge present in G has been given a positive weight. If i and j are adjacent vertices, we represent this fact with the notation $i \sim j$, and we let $w_{ij} = w_{ji}$ be the weight of the edge joining i and j . When i and j are not adjacent, we define $w_{ij} = 0$.

The *Laplacian matrix* of the graph G is the matrix $\mathcal{L} = [l_{ij}]$ defined via

$$l_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j, \text{ and} \\ \sum_{k \sim i} w_{ik} & \text{if } i = j. \end{cases}$$

If G is unweighted, the Laplacian matrix of G is defined by assigning a weight of 1 to every edge.

The Laplacian matrix can be introduced in order to solve a graph partitioning problem. Given a graph G , the problem is to partition the vertex set V into X and $V \setminus X$ so that the sum of the weights of the edges with one endpoint in X and one endpoint not in X is as small as possible. If the graph is unweighted, we are simply attempting to find a partition where very few edges cross this partition.

A trivial solution is simply $X = V$ and $V \setminus X = \emptyset$; no edges join these sets. In order to avoid such an uninteresting solution, the problem can be recast in the following way. We wish to find a set of vertices X such that

$$\frac{|E_X|}{|X||V \setminus X|}$$

is minimal, where E_X is the number of edges joining X to $V \setminus X$. By minimising this value, we favour partitions where significant numbers of vertices are present in both members of the partition.

If G is disconnected, the problem is trivial. If G is connected, it can be shown that μ/n is a lower bound for this value, where n is the number of vertices in G and μ is the smallest positive eigenvalue of \mathcal{L} . Moreover, when equality occurs, an eigenvector associated with μ can be used to construct the desired partition.

Applications of the Laplacian matrix often have this general flavour—the spectral and algebraic properties of the Laplacian have been closely related to connectivity properties of the associated graph, such as edge density, the presence of cut vertices/edges and counting spanning trees.

The relationship between the structure of a graph and the algebraic properties of its associated Laplacian matrix is the subject of Molitierno's comprehensive monograph. The work is presented in a traditional textbook style and examines the theory of Laplacian matrices, beginning with basic graph and matrix theory. It includes large numbers of diagrams and exercise questions, and leads up to chapters discussing the most current and advanced topics in the area.

This work would make an excellent text for a graduate level reading course. The topics explored range from the most basic applications of the Laplacian to more advanced ideas, and this range is organised along a very smooth continuum. This text perfectly illustrates the versatility of the Laplacian matrix as a tool in graph analysis.

The author states that “only an undergraduate course in linear algebra and experience in proof writing are prerequisites.” However, this reviewer is inclined to disagree. Although the theory is fully developed within the book, some fundamental topics from linear algebra are given only brief treatment and the reader is required to be very familiar with a large amount of graph terminology. Graduate students with experience in these areas will find the text highly readable, but this reviewer would not recommend it as an introductory text in the area of combinatorial matrix theory.

This text is highly valuable reference concerning Laplacian matrices. It is very nice to see material that was previously somewhat scattered in the literature collected in an organised and very thorough treatment. In particular, the text includes chapters on algebraic connectivity and Fiedler vectors that this reviewer is confident will provide an invaluable resource for researchers interested in these topics.

JOURNAL ANNOUNCEMENTS

New Journal: *Journal of Complex Networks*

Submitted by Michele Benzi, Emory University

I would like to draw the attention of the ILAS community to a new journal, the *Journal of Complex Networks*, published by Oxford University Press: <http://comnet.oxfordjournals.org>. Volume 1, Issue 1 (June 2013) is now available online (all papers will be downloadable for free for the first two years of the journal).

Topics covered by the journal include (but are not limited to): mathematical and numerical analysis of networks; network theory and computer science; structural analysis of networks; networks and epidemiology; social, socio-economic and political networks; ecological networks; technological and infrastructural networks; biological and molecular networks; evolving networks; biomedicine related networks; animal social networks; climate networks; cognitive, language and informational networks.

The Editor-in-Chief of the *Journal of Complex Networks* is Professor Ernesto Estrada of the University of Strathclyde. Visit the website for more information, submission guidelines, and to sign up to receive tables of contents by email.

New Journal: *Special Matrices* (De Gruyter)

Submitted by Carlos Fonseca, University of Coimbra

I would like to inform the ILAS community of a new journal *Special Matrices*, published by Versita - De Gruyter.

The journal will be devoted to structured matrices and their applications. All papers will be freely downloadable, the first three: “Nonnegative definite hermitian matrices with increasing principal minors” by Shmuel Friedland, “Factorizable matrices” by Miroslav Fiedler and Frank J. Hall, and “A note on the determinant of a Toeplitz-Hessenberg matrix” by Mircea Merca, are already available at: <http://www.degruyter.com/view/j/spma.2013.1.issue/issue-files/spma.2013.1.issue.xml>.

The Editor of *Special Matrices* is Carlos Fonseca of the University of Coimbra. For further information and submission guidelines, please visit the journal website: <http://www.degruyter.com/view/j/spma>.

Special Issues of Journals and Research Monograph in Honor of Professor Biswa Datta

Submitted by Eric Chu, Monash University

Biswa Nath Datta has been honored by the following recent publications dedicated to him.

A special issue on “Inverse Problems in Science and Industry,” of the journal *Numerical Linear Algebra with Applications* (Vol. 20, 167–170, 2013), has been dedicated to Biswa Nath Datta. The issue contains 17 papers which address a variety of the theoretical issues and applications associated with inverse problems. The publication recognizes Biswa Datta’s many significant contributions to linear algebra and control theory. This special issue was edited by Eric Chu, W.-W. Lin and Lothar Reichel.

A research monograph entitled *Numerical Linear Algebra in Signals, Systems and Control* (Lecture Notes in Electrical Engineering 80, Springer, 2011), was dedicated to Biswa Nath Datta. The monograph was edited by Paul Van Dooren, Shankar Bhattacharyya, Raymond Chan, Vadim Olshevsky and Aurobinda Routray. It contains contributions by some of the leading researchers in these areas, including several invited talks delivered at the international workshop held at the Indian Institute of Technology, Kharagpur, India in 2007. Datta was honored at a banquet ceremony organized by the local IEEE Chapter during the conference for “his numerous contributions in the area of numerical linear algebra with control and systems theory, and signal processing.” The ceremony was presided by the President of the IEEE Chapter, IIT-Kharagpur. The speakers included: Rajendra Bhatia, the late Gene Golub, Volker Mehrmann, B. M. Mohan, and Paul Van Dooren.

A special issue of the journal *Linear Algebra and its Applications* (Vol. 434, Issue 7, 2011), was dedicated to Datta “in recognition of his contributions to linear algebra, his enthusiasm for linear algebra and its applications, and his efforts over the years to bring together researchers in linear algebra with scientists and engineers, who work on applications for which linear algebra is essential.” It was noted there that “Biswa Datta’s enthusiasm for linear algebra and its applications has been a source of inspiration for many colleagues and students.” The editors of this issue were: Richard Brualdi, Robert Plemmons, Lothar Reichel and Qiang Ye. The issue contains papers on theory, computations and applications of linear algebra, mostly based on some of the invited and contributed talks given at the IMS Conference on *Linear Algebra, Numerical Linear Algebra and Applications*, held at Northern Illinois University, August, 2011. Biswa was honored by distinguished linear algebraists in a banquet ceremony of this conference. The speakers included: Shankar Bhattacharyya, Richard Brualdi, James Nagy, Robert Plemmons, Hans Schneider, and Vadim Sokolov. The ceremony was presided by William Blair, who was the Chair of the Department of Mathematical Sciences of NIU at that time.

LINEAR ALGEBRA EDUCATION

The Sage Mathematical Software System

Jason Grout, Drake University, USA, jason.grout@drake.edu

Sage (<http://sagemath.org>) is a comprehensive free open-source mathematics software system. Anybody is welcome to freely download, modify, and redistribute copies of Sage. Alternatively, several free services exist to use Sage online, without needing to install anything. Sage can be used in teaching and researching many areas of mathematics. In this article, we will briefly discuss Sage's linear algebra capabilities, show how to use Sage through two online interfaces, and then discuss using Sage in teaching.

Sage and Linear Algebra

Rob Beezer, Robert Bradshaw, William Stein, and I recently wrote a chapter for the 2nd edition of Leslie Hogben's *Handbook of Linear Algebra*, introducing Sage's linear algebra capabilities. This chapter is also distributed under an open-source license (i.e., copying and redistribution is welcome), available from <http://artsci.drake.edu/grout/doku.php/ilas>. We highly recommend reading this more complete introduction to linear algebra in Sage.

Sage's core is a unified framework based on category theory for working with mathematical concepts. Sage can work with theoretical mathematical objects like rings, fields, vector spaces, linear transformations, and much more. Vectors and matrices in Sage know the rings in which their elements live (such as integers, rationals, finite fields, floating-point reals or complexes, or more exotic objects such as intervals, multivariate polynomials, or algebraic numbers). When a user asks Sage to do a computation, Sage automatically chooses appropriate routines for the object, often using fast standard packages included in Sage such as LAPACK, LinBox, ATLAS, IML, M4RI, etc. Users interact with Sage via Python, a very popular, easy-to-use, and powerful general-purpose programming language. Sage also includes popular open-source packages like GAP, R, Singular, and Pari, and makes it easy to use external packages from within Sage, like Octave (an open-source MATLAB clone).

In Sage, to create the vector $(1, 2, 3)$ over the rationals and assign it the name "v", type `v=vector(QQ, [1,2,3])` (the QQ tells Sage the vector is over the rationals). To create a matrix over the rationals and assign it the name "A", just specify the rows of the matrix like `A=matrix(QQ, [[1,2,3],[4,5,6],[7,8,9]])`. Vector and matrix entries are retrieved by giving indices in square brackets. Indices count from 0, as is common in many computer languages, so `A[0,0]` retrieves the upper left entry of the matrix A, and `v[2]` retrieves the third element of \vec{v} . To solve the system $A\vec{x} = \vec{v}$, do `A\v`. You can get information about an object or perform operations on an object by typing the object name, a period, and then a method name followed by parentheses. For example, the determinant and eigenvalues of A are found by doing `A.det()` and `A.eigenvalues()`, and the Euclidean norm of \vec{v} is `v.norm()`. You can get an overview of what methods are available by typing an object's name, a period, and then pressing the tab key. Sage will then display a list of possible methods you could invoke. To get help about a method, type the method name followed by a question mark, and then press tab: `A.rref?`. To see the underlying source code for a method, append two question marks and press tab: `A.rref??`.

As mentioned above, Sage understands theoretical mathematical objects like vector spaces and linear transformations. To get the (right) kernel of a matrix as a vector space and assign it the name "K", do `K=A.right_kernel()`. Among other things, you can ask for a basis for the vector space `K.basis()`, the dimension `K.dimension()`, or the coordinate vector of $(-2, 4, -2)$ relative to the basis `K.coordinate_vector((-2,4,-2))`. You can construct a vector space in many other ways, including raising a field to a power, `QQ^2`, or as the span of a list of vectors `span([v])`.

Sage also understands and can compute directly with linear transformations. You can construct a linear transformation, among other ways, by giving a matrix, like `T=linear_transformation(A,side='right')`. The `side='right'` is to tell Sage that vectors act on the right of the matrix (i.e., $A\vec{x}$). You can compute many things from linear transformations, like the image `T.image()` and the kernel `T.kernel()`. You can compose and restrict transformations to create new ones.

Sage also has many graphics capabilities: `plot(v, start=(1,2,3))+plot(A*v)` draws \vec{v} and $A\vec{v}$ (with v starting at the point $(1, 2, 3)$) in an interactive 3d graph. Doing `matrix_plot(A)` visualizes the structure of a matrix A.

Sage Cell Server: <https://sagecell.sagemath.org>

There are many ways to use Sage, including downloading an app for Apple OS X or Linux, a virtual machine image for Windows, or using the Sage apps on iPhones, iPads, or Android devices. Two of the easiest ways to use Sage are via the online Sage Cell Server and Sage Cloud. Using Sage online provides the full capabilities of Sage with no installation.



goo.gl/VaGjqt

The Sage Cell Server at <https://sagecell.sagemath.org> provides the easiest way to do short computations in Sage: just type in the code and press the Evaluate button. You can also share a snippet of code by clicking the "Share" button, which reveals permalinks and a QR code. A QR code is a square-shaped 2D barcode, such as the one above, which

encodes a permalink. A user with a QR code scanner installed on their mobile device can take a picture of the barcode and automatically go to the permalink. I also included shortened versions of the permalinks under the QR codes in this article. I have fruitfully used permalinks extensively in pdf and online notes—students click on the link to get a snippet of Sage code that they can run and modify to check their work or explore a situation.

Sage cells can also be embedded into any webpage (click the “About” link in the upper right of the Sage cell server page for instructions). Many people embed live Sage cells into webpages for their homework assignments, blogs, and online notes. For example, Rob Beezer’s open-source online linear algebra textbook uses Sage cells extensively to illustrate concepts throughout the book; see <http://linear.ups.edu/html/section-RREF.html> for an example (click on the Sage links). Since Sage provides interfaces to other programs, such as Octave (an open-source clone of MATLAB) and R (a popular statistics program), it is easy to embed examples and make permalinks using those languages as well (see the QR code to the right for an example).



goo.gl/UTrQt0

Sage Cloud: <https://cloud.sagemath.com>

William Stein, the lead developer of Sage, has been developing a new online interface to Sage, the Sage Cloud at <https://cloud.sagemath.com>. Currently in beta status, it is already a powerful computation and collaboration tool. Work is organized into projects which can be shared with others. Inside a project, you can create any number of files, folders, Sage worksheets, \LaTeX documents, code libraries, and other resources. Real-time collaborative editing allows multiple people to edit and chat about the same document simultaneously over the web. The \LaTeX editor features near real-time preview, forward and reverse search, and real-time collaboration. Also, it is easy to have Sage do computations or draw figures and have those automatically embedded into a \LaTeX document using the SageTeX package (for example, after including the `sagetex` package, typing `\sageplot{plot(sin(x))}` in a \TeX document inserts the plot of $\sin x$). A complete Linux terminal is also available from the browser to work within the project directory. Snapshots are automatically saved and backed up every few minutes to ensure work is never lost. William is rapidly adding new features, often within days of a user requesting them.

Interacts

Sage interacts make it easy to create online interfaces which include buttons, sliders, menus, etc. A visitor on the webpage (or a collaborator in the Sage Cloud) can adjust parameters and see the results immediately, without having to enter or change any Sage code. Interacts are especially effective when they are embedded into websites using the Sage cell server. The code can be hidden so the user just sees an “Evaluate” button, and clicking the button pops up the user interface. Examples of interacts can be found at <http://wiki.sagemath.org/interact> and <http://interact.sagemath.org>.

Creating interacts is straightforward. For example, suppose we want an exploration plotting \vec{v} and $A\vec{v}$ for various unit vectors $\vec{v} \in \mathbb{R}^2$. We can do that with the code:

```
angle = pi/6
A = matrix([[1,-1],[2,1]])
v = vector([cos(angle), sin(angle)])
plot(v, color='red')+plot(A*v, color='blue')
```



goo.gl/Ek3A4X

To turn this into an interact, we enclose the code in a function (put in the `def` line below and indent the code), make the variables we want the user to change into inputs of the function (give a range for numeric variables, like `angle`), and put `@interact` in front of the entire thing. We also wrap any plots in the `show()` function:

```
@interact
def myfunction(angle=(0,2*pi), A=matrix([[1,-1],[2,1]])):
    v = vector([cos(angle), sin(angle)])
    show(plot(v, color='red')+plot(A*v, color='blue'))
```



goo.gl/HQCsZ8

Sage in Teaching

Sage is being used in many ways in teaching. Sage snippets and explorations are embedded into online textbooks, notes, and homework assignments. Permalinks and QR codes to Sage snippets in pdf documents and printed material make it easy for students to explore and check their work with a single click or picture. Teachers use SageTeX to author tests and write \TeX documents that include Sage computations and graphics. The online homework system WeBwork can use Sage calculations in creating homework problems. The Sage Cloud is being used as a collaborative environment for a teacher and students to author class notes, submit and grade homework assignments, and do computational projects.

The advantages to using Sage go beyond the immediate class. The Python language that students learn when interacting with Sage is widely used in many areas outside of mathematics, and is particularly prevalent in scientific computing.

Since Sage is open-source, students can dive as deep as they want into the code to understand how algorithms work. Many undergraduate and graduate students also write and contribute code to Sage—this enhances their understanding and gives them exposure to the wider professional community.

Learn More

There are many resources and opportunities to learn about Sage and interact with the community of developers and users. Sage Days and Sage Education Days workshops happen often and new users are welcome. Funding has usually been available to help faculty and students attend. See <http://wiki.sagemath.org/Workshops> for a listing of past and future workshops, along with videos of past talks and other resources (the Education Days at the bottom of the listing have resources specifically for teaching). Additionally, the sage-support mailing list at <https://groups.google.com/forum/#!forum/sage-support> and the Ask Sage website at <http://ask.sagemath.org> provide very helpful forums for new users.

ILAS NEWS

ILAS Member Elected to Executive of SIAM

Professor Daniel Szyld (Temple University) has been elected by SIAM (*Society for Industrial and Applied Mathematics*) to a two-year term as Vice-President-at-Large, starting January 1, 2014.

Professor Szyld has been an active member of both ILAS and SIAM. A member of ILAS since its foundation, he was an ILAS board member from 2001–2004. He was instrumental in helping set up the *Electronic Journal of Linear Algebra* (ELA), and served as ELA's first managing editor from 1995–2003, as well as an associate editor from 1995–2001. He has been an advisory editor of ELA since 2001. He is currently serving on the editorial board of several other journals, including *Linear Algebra and its Applications*, *Numerical Linear Algebra with Applications*, and the *SIAM Journal on Matrix Analysis and Applications*. Professor Szyld's activities include membership on the organizing committees of a couple of ILAS conferences (1998 and 2008), and he has also been a program committee member for the SIAM Conference on Applied Linear Algebra in 2009, as well as chair of the Gene Golub Summer School (2010 – 2013) sponsored by SIAM. A member of SIAM since 1980, he chaired the SIAG-LA, the linear algebra subgroup of SIAM, from 2007–2009. Daniel notes that “the SIAG-LA has a good working relationship with ILAS” and he hopes to continue to support this positive relationship.



D. Szyld

Nominations for 2013 ILAS Elections

Submitted by Stephen Kirkland, ILAS President

The Nominating Committee for the 2013 ILAS elections has completed its work. Nominated for a three year term, beginning March 1, 2014, as ILAS President are: André Ran (Netherlands) and Peter Šemrl (Slovenia).

Nominated for the two open three-year terms, beginning March 1, 2014, as “at-large” members of the ILAS Board of Directors are: Froilán Dopico (Spain), Julio Moro (Spain), Michael Overton (USA), and Eugene Tyrtyshnikov (Russia).

Many thanks to the Nominating Committee for their important service to ILAS: Shaun Fallat, Heike Faßbender, Roger Horn (chair), Yimin Wei, and Zdenek Strakoš. Thanks also to the nominees for agreeing to stand for election.

Voting in this election is electronic, using Votenet Solutions, with instructions sent to members by email.

ILAS Website Update

Please note that ILAS has a new website address: <http://www.ilasic.org/>. Please take the time to update your bookmarks and any links you may have. If you have any suggestions for the new website, please contact Sarah Carnochan Naqvi, the ILAS-IC and ILAS-Net Manager, at ilasic@uregina.ca.

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Art credit: A. L. Traud, E. D. Kelsic, P. J. Mucha, M. A. Porter, "Comparing Community Structure to Characteristics in Online Collegiate Social Networks," SIREV Vol.53, pp.526-543.

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to all Problems 50-1 through 50-6. Six new problems are on the last page; solutions are invited.

Problem 50-1: An Adjugate Identity

Proposed by Khaled ALJANAIDEH, *University of Michigan, Ann Arbor, MI, USA*, khaledfj@umich.edu
and Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $\text{adj } A \in M_n(\mathbb{C})$ denote the adjugate (transposed matrix of cofactors) of $A \in M_n(\mathbb{C})$, let $A_{[i,j]} \in M_{n-1}(\mathbb{C})$ denote A with the i -th row and j -th column removed, and let $A_{[i,\cdot]} \in M_{(n-1) \times n}(\mathbb{C})$ denote A with the i -th row removed. If $e_i \in \mathbb{C}^n$ is the i -th column vector of the standard basis, show that

$$[(\text{adj } A)_{[i,\cdot]} + (\text{adj } A_{[i,i]})A_{[i,\cdot]}] e_i = 0_{(n-1) \times 1}; \quad i \in \{1, \dots, n\}.$$

Solution 50-1.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

We may assume $n \geq 3$, as the above identity is readily confirmed for $n = 2$. We also assume $i = 1$ to keep the notation simpler and because the proof for other values of i is similar. Let $A = (a_{jk})$. The first column of $\text{adj } (A_{[1,1]}) A_{[1,\cdot]}$ is $v := \text{adj } (A_{[1,1]}) [a_{21}, \dots, a_{n1}]^T$. Hence the k -th component of this vector is

$$v_k = \sum_{j=1}^{n-1} (\text{adj } (A_{[1,1]}))_{kj} a_{j+1,1} = \sum_{j=1}^{n-1} (-1)^{k+j} \det((A_{[1,1]})_{[j,k]}) a_{j+1,1}.$$

The first column of $(\text{adj } A)_{[1,\cdot]}$ is $u := [(\text{adj } A)_{21}, \dots, (\text{adj } A)_{n1}]^T$. Hence the k -th component of this vector can be computed by expanding the determinant along the first column of the matrix:

$$u_k = (\text{adj } A)_{k+1,1} = (-1)^{k+2} \det(A_{[1,k+1]}) = (-1)^k \sum_{j=1}^{n-1} (-1)^{j+1} \det((A_{[1,k+1]})_{[j,1]}) a_{j+1,1}.$$

Note that $(A_{[1,1]})_{[j,k]} = (A_{[1,k+1]})_{[j,1]}$ since both are obtained by deleting rows 1 and $j+1$ and columns 1 and $k+1$ from A . Hence $u_k + v_k = 0$. ■

Also solved by the proposers.

Problem 50-2: Range-Hermitianness of Certain Functions of Projectors

Proposed by Oskar Maria BAKSALARY, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz TRENKLER, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

- (i) Let $P, Q \in M_n(\mathbb{C})$ be Hermitian idempotent matrices of order n . Show that $P + Q - PQ$ is range-Hermitian.
- (ii) Let $R \in M_n(\mathbb{C})$ be an idempotent matrix of order n and let R^\dagger be its Moore-Penrose inverse. Show that $I_n - R^\dagger$ is range-Hermitian, where I_n is the identity matrix of order n .

Recall that a matrix is range-Hermitian (also called EP) when its range and the range of its conjugate transpose coincide.

Solution 50-2.1 by Arkady A. BABAJANYAN, *Institute for Informatics and Automation Problems, Armenian Academy of Sciences, Yerevan, Armenia*, arbab@list.ru

- (i) From [3, p. 302] it is known that if M and F are Hermitian idempotent matrices then $I - MF$ is an EP-matrix. Hence, (i) follows by setting $M = I - P$ and $F = I - Q$.
- (ii) By [3] a matrix $I - K$ is an EP-matrix if $\|K\| \leq 1$. But by [1, Theorem 1],

$$1 \geq \theta(\text{Im}(R), \text{Ker}(R)) = \max(\|R^\dagger\|, \|(I - R)^\dagger\|) \quad (1)$$

where $\theta(\mathcal{F}, \mathcal{G})$ denotes the gap between a pair of subspaces \mathcal{F} and \mathcal{G} in the Hilbert space. So, both $I - R^\dagger$ and $I - (I - R)^\dagger$ are EP-matrices. ■

Remark. Equality in (1) was the starting point to find necessary and sufficient conditions for an (oblique) projection

R in Hilbert space to satisfy $\|R^\dagger\| = \|(I - R)^\dagger\|$. Some sufficient conditions were given in [1,2] particularly when $\theta(\text{Im}(R), \text{Ker}(R)) < 1$. A full solution was given in [4, Theorem 13]. As an addition there are other equalities from [1,2] which connect the self-commutator of a projection, a gap, and trigonometric relations between subspaces of the Hilbert space:

$$\|RR^* - R^*R\| = \|R\|^2 \|RR^\dagger - R^\dagger R\| = \|R\|^2 \theta(\text{Im}(R), \text{Im}(R^*)) = \|R\|^2 \cos(\text{Im}(R), \text{Ker}(R)), \quad (2)$$

$$\|R - R^\dagger R\| = \|R - RR^\dagger\| = \|R\| \|RR^\dagger - R^\dagger R\| = \cot(\text{Im}(R), \text{Ker}(R)), \quad (3)$$

where $\cos(\mathcal{F}, \mathcal{G})$ and $\cot(\mathcal{F}, \mathcal{G})$ are cosine (notation as in [2]) and cotangent of the angle between subspaces \mathcal{F} and \mathcal{G} . Equalities (2), (3) show particularly that concepts of normality, EP-ness, and orthogonality are equivalent for the projections in Hilbert space.

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- [3] R.E. Hartwig and K. Spindelböck, Partial isometries, contractions and EP matrices, *Linear Multilinear Algebra* 13 (1983) 295–310.
- [4] I. Spitkovsky, Once more on algebras generated by two projections, *Lin. Alg. Appl.* 208/209 (1994) 377–395.

Solution 50-2.2 by the proposers Oskar Maria BAKSALARY and Götz TRENKLER

(i) Let $\overline{K} = I_n - K$, $K \in \{P, Q\}$. Note that $P + Q - PQ = I_n - (I_n - P)(I_n - Q) = I_n - \overline{P}\overline{Q}$. Thus, the assertion follows directly from either an observation in [2, p. 302] or from [1, Theorem 15]. The former of these results claims that $I_n - PQ$, and, consequently, $I_n - \overline{P}\overline{Q}$, are EP, whereas the latter shows that the ranges of $I_n - \overline{P}\overline{Q}$ and $P + Q$ coincide.

(ii) Since $R^\dagger = R^\dagger R R^\dagger = (R^\dagger R)(R R^\dagger)$, we arrive at $R^\dagger = PQ$, where $P = R^\dagger R$ and $Q = R R^\dagger$ are orthogonal projectors. Thus, the assertion follows on account of part (i) of the solution. ■

Remark. Note that by [3, Lemma 2.3], according to which R is idempotent if and only if there exist orthogonal projectors P and Q such that $R = (PQ)^\dagger$, the parts (i) and (ii) of the problem are actually equivalent.

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Solution 50-2.3 by Johanns DE ANDRADE BEZERRA, Natal, RN, Brazil, pav.animal@hotmail.com

Proof of (i). We will prove even more: $\text{Im}(P + Q - PQ) = \text{Im}(P + Q) = \text{Im}(P + Q - QP) = \text{Im}[(P + Q - PQ)^*]$.

Let $x \in \text{Ker}(P + Q)$. Then $0 = \langle Px + Qx, x \rangle = \langle Px, x \rangle + \langle Qx, x \rangle = \langle P^*Px, x \rangle + \langle Q^*Qx, x \rangle = \langle Px, Px \rangle + \langle Qx, Qx \rangle$, which implies $Px = Qx = 0$. Hence $\text{Ker}(P + Q) \subseteq \text{Ker}(PQ)$ and $\text{Ker}(P + Q) \subseteq \text{Ker}(QP)$, so $\text{Im}(PQ) = (\text{Ker}(QP))^\perp \subseteq (\text{Ker}(P + Q))^\perp = \text{Im}(P + Q)$. Similarly, $\text{Im}(QP) \subseteq \text{Im}(P + Q)$. Thus, $\text{Im}(P + Q - PQ) \subseteq \text{Im}(P + Q)$ and $\text{Im}(P + Q - QP) \subseteq \text{Im}(P + Q)$. Actually, the equality holds since $\text{rank}(P + Q - PQ) = \text{rank}(P + Q)$ (see [1, Theorem 3]).

Proof of (ii). According to solution of problem 47-1 in IMAGE 48,

$$\dim \text{Ker}(I - RR^*) = \dim(\text{Im}(R) \cap \text{Im}(R^*)) + \dim([\text{Im}(R) + \text{Im}(R^*)] \cap \text{Ker}(R) \cap \text{Ker}(R^*)).$$

As $[\text{Im}(R) + \text{Im}(R^*)] = (\text{Ker}(R) \cap \text{Ker}(R^*))^\perp$ we derive $\dim \text{Ker}(I - RR^*) = \dim \text{Ker}(I - R^*R) = \dim(\text{Im}(R) \cap \text{Im}(R^*))$. Note, $(\text{Im}(R) \cap \text{Im}(R^*)) \subseteq \text{Ker}(I - RR^*)$ and $(\text{Im}(R) \cap \text{Im}(R^*)) \subseteq \text{Ker}(I - R^*R)$, and so $\text{Ker}(I - RR^*) = \text{Ker}(I - R^*R) = \text{Im}(R) \cap \text{Im}(R^*)$. Let $R = U\Sigma V^*$ be a singular value decomposition of R where $\Sigma = I_r \oplus \text{diag}(s_1, \dots, s_k) \oplus 0_{n-r-k}$ for some r , $0 \leq r \leq n$, and where $1 < s_1 \leq \dots \leq s_k$ (see solution to problem 48-2 in IMAGE 49). Then, $x \in \text{Ker}(I - R^*R) \iff x = R^*Rx = V\Sigma^2V^*x \iff V^*x = \Sigma^2(V^*x)$ is an eigenvector of Σ^2 to eigenvalue 1 $\iff V^*x \in \text{Ker}(I - \Sigma^2) = \text{Lin}\{e_1, \dots, e_r\}$. So, $\text{Ker}(I - R^*R) = \text{Im}(R) \cap \text{Im}(R^*)$ is spanned by $v_1 = Ve_1, \dots, v_r = Ve_r$. Note that $v_i = Rv_i = RVe_i = U\Sigma e_i = Ue_i$ hence $R^\dagger v_i = R^\dagger Ue_i = V\Sigma^\dagger e_i = Ve_i = v_i$, so that $v_i \in \text{Ker}(I - R^\dagger)$. Thus, $\text{Ker}(I - R^*R) \subseteq \text{Ker}(I - R^\dagger)$. Conversely,

let $w \in \text{Ker}(I - R^\dagger)$. Then, $w = R^\dagger w \in \text{Im}(R^\dagger) = \text{Im}(R^*R) = \mathcal{W}' \oplus \mathcal{W}''$, with $\mathcal{W}' = \text{Ker}(I - R^*R) \subseteq \text{Ker}(I - R^\dagger)$ and $\mathcal{W}'' = \bigoplus_{i=1}^{n-r} \text{Ker}(s_i^2 I - R^*R)$. Decompose accordingly $w = w' + w''$, with $w' \in \mathcal{W}'$ and $w'' \in \mathcal{W}''$. As $R^\dagger w = w$ and $R^\dagger w' = w'$ we have $R^\dagger w'' = w''$. Then, $\|w''\|^2 = \langle (R^\dagger)^* R^\dagger w'', w'' \rangle \leq s_1^{-2} \|w''\|^2$. Since $s_1 > 1$ we get $w'' = 0$. Hence, $\text{Ker}(I - R^\dagger) = \text{Ker}(I - R^*R) = \text{Ker}(I - RR^*) = \text{Ker}(I - (R^*)^\dagger) = \text{Ker}(I - (R^\dagger)^*)$. Thus, (ii) follows from $\text{Im}(I - R^\dagger) = (\text{Ker}(I - (R^\dagger)^*))^\perp$. ■

Reference

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Solution 50-2.4 by Edward L. PEKAREV, *Odessa Regional Institute of Public Administration of the National Academy of Public Administration, Office of the President of Ukraine, Ukraine*, edpekarev@yandex.ru

(i) Given orthogonal projectors P, Q , let

$$W = P + Q - PQ = P + (I - P)Q(I - P) + (I - P)QP.$$

Then, $h \in \text{Ker}(W)$ implies $0 = PWh = Ph$ and thus $Wh = (I - P)Q(I - P)h = 0$. Hence $\|Q(I - P)h\|^2 = \langle Q(I - P)h, Q(I - P)h \rangle = \langle (I - P)Q(I - P)h, h \rangle = 0$ and so $Q(I - P)h = 0$. Combined with $Ph = 0$ we get $Qh = 0$. So $\langle h, Wx \rangle = \langle W^*h, x \rangle = \langle (P + Q - QP)h, x \rangle = 0$ for any $x \in \mathbb{C}^n$, i.e., $h \in (\text{Im}(W))^\perp$ and we have $\text{Ker}(W) \subseteq (\text{Im}(W))^\perp$.

Conversely, if $f \in (\text{Im}(W))^\perp$ then $\langle f, Wx \rangle = 0$ for any $x \in \mathbb{C}^n$. Inserting $x = (I - P)f$ gives $\langle f, (I - P)Q(I - P)f \rangle = 0$. So $Q(I - P)f = 0$. Inserting $x = Pf$ then gives $0 = \langle f, Pf + (I - P)QPf \rangle = \langle Pf, Pf \rangle + \langle Q(I - P)f, Pf \rangle = \|Pf\|^2$ so $Pf = 0$. Thus, $Wf = 0$ giving $(\text{Im}(W))^\perp \subseteq \text{Ker}(W)$. So, $(\text{Im}(W))^\perp = \text{Ker}(W)$. To end, note that $\text{Ker}(W) = (\text{Im}(W^*))^\perp$.

(ii) Let $R \in \mathcal{M}_n(\mathbb{C})$ and let R^\dagger be its Moore-Penrose inverse. Then, the operators $P = R^\dagger R$, $Q = RR^\dagger$ are orthogonal projectors and as R is an idempotent matrix, $PQ = R^\dagger$. To show that $I_n - R^\dagger$ is range-Hermitian, it is sufficient to verify the equality $\text{Ker}(I_n - R^\dagger) = \text{Ker}((I_n - R^\dagger)^*)$. Now, if $g \in \text{Ker}(I_n - R^\dagger)$ then $g = PQg$ and therefore $\|g\| = \|PQg\| \leq \|Qg\| \leq \|g\|$, i.e., $\|g\| = \|PQg\| = \|Qg\| = \|g\|$. But $\|Qg\| = \|g\|$ implies $Qg = g$ and $\|Pg\| = \|g\|$ implies $Pg = g$. So, $((I_n - R^\dagger)^*)g = (I_n - QP)g = 0$, that is $g \in \text{Ker}((I_n - R^\dagger)^*)$. This gives $\text{Ker}(I_n - R^\dagger) \subseteq \text{Ker}((I_n - R^\dagger)^*)$. Similarly, $\text{Ker}((I_n - R^\dagger)^*) \subseteq \text{Ker}(I_n - R^\dagger)$. ■

Solution 50-2.5 by Hans Joachim WERNER, *University of Bonn, Bonn, Germany*, hjw.de@uni-bonn.de

To prove claim (i) we rely on the following result which is a direct consequence of [1, Lemma 3.2]; see also [2, Eq. (2.6)].

Lemma. Let $V \in M_n(\mathbb{C})$ be a nonnegative definite Hermitian matrix, and let \mathcal{M} be a linear subspace of \mathbb{C}^n . Then $V(\mathcal{M}^\perp) \cap \mathcal{M} = \{0\}$ and $V(\mathcal{M}^\perp) + \mathcal{M} = [\mathcal{M}^\perp \cap \mathcal{N}(V)]^\perp = \mathcal{R}(V) + \mathcal{M}$.

By means of this Lemma, it is easy to prove the following.

Theorem 1. Let $P, Q \in M_n(\mathbb{C})$ be idempotent Hermitian matrices. Then $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}(P + Q - QP)$, and so, in particular, $P + Q - PQ$ is EP.

Proof. Since $P + Q - PQ = Q + P(I - Q)$, the previous Lemma, with $(P, \mathcal{R}(Q))$ in place of (V, \mathcal{M}) gives $P(\mathcal{N}(Q)) + \mathcal{R}(Q) = \mathcal{R}(P) + \mathcal{R}(Q)$. So, writing $P + Q - PQ = P(I - Q) + Q$ and decomposing $\mathbb{C}^n = \mathcal{N}(Q) \oplus \mathcal{R}(Q)$ (here, \oplus denotes the sum of orthogonal spaces) we have $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P) + \mathcal{R}(Q)$. On similar lines, $\mathcal{R}(P + Q - QP) = \mathcal{R}(P) + \mathcal{R}(Q)$. As $(P + Q - PQ)^* = P^* + Q^* - Q^*P^* = P + Q - QP$, it follows that $P + Q - PQ$ is a range-Hermitian. ■

Next, we prove claim (ii).

Theorem 2. Let $R \in M_n(\mathbb{C})$ be an idempotent matrix. Then $\mathcal{N}(I_n - R^\dagger) = \mathcal{R}(R) \cap \mathcal{R}(R^*) = \mathcal{N}(I_n - (R^\dagger)^*)$ or, equivalently, $\mathcal{R}(I_n - R^\dagger) = \mathcal{N}(R) + \mathcal{N}(R^*) = \mathcal{R}(I_n - (R^\dagger)^*)$, and so, in particular, $I_n - R^\dagger$ is EP.

Proof. If $R = 0$ then $R^\dagger = 0$ and $\mathcal{R}(I_n - R^\dagger) = \mathbb{C}^n = \mathcal{R}(I_n - (R^\dagger)^*)$. Next, let $R \neq 0$, and let $r = \text{rank}(R)$. There exist $n \times r$ matrices F and G , both of full column rank r , such that $R = FG^*$ (a full rank factorization of R). Then $\mathcal{R}(R) = \mathcal{R}(F)$, $\mathcal{N}(R) = \mathcal{N}(G^*)$, $\mathcal{R}(R^*) = \mathcal{R}(G)$, and $\mathcal{N}(R^*) = \mathcal{N}(F^*)$. Also, $R^2 = R$ implies $G^*F = I_r$. Recall that $\mathcal{R}(R^\dagger) = \mathcal{R}(R^*)$ and that $(R^\dagger)^* = (R^*)^\dagger$. Consequently, $\mathcal{R}((R^\dagger)^*) = \mathcal{R}(R)$. Further, since $R = FG^*$ is a full rank factorization of R , then $R^\dagger = (G^*)^\dagger F^\dagger$, $(G^*)^\dagger = G(G^*G)^{-1}$, and $F^\dagger = (F^*F)^{-1}F^*$. With this in mind, $x \in \mathcal{N}(I_n - R^\dagger) \implies x = R^\dagger x = G(G^*G)^{-1}(F^*F)^{-1}F^*x \implies F^*FG^*x = F^*x \implies F^*(I_n - FG^*)x = 0 \implies F^*(I_n - R)x = 0 \implies x \in \mathcal{R}(R) + [\mathcal{N}(R) \cap \mathcal{N}(F^*)] = \mathcal{R}(R) + [\mathcal{N}(R) \cap \mathcal{N}(R^*)]$. Since $x = R^\dagger x$ implies $x \in \mathcal{R}(R^*)$, we further have $x \in \mathcal{N}(I_n - R^\dagger) \implies x \in \mathcal{R}(R^*) \cap (\mathcal{R}(R) + [\mathcal{N}(R) \cap \mathcal{N}(R^*)]) = \mathcal{R}(R^*) \cap \mathcal{R}(R)$, where the last equality follows

in virtue of $\mathcal{R}(R^*) + \mathcal{R}(R) = [\mathcal{N}(R) \cap \mathcal{N}(R^*)]^\perp$. Hence, $\mathcal{N}(I_n - R^\dagger) \subseteq \mathcal{R}(R^*) \cap \mathcal{R}(R)$. To prove the converse inclusion, let $x \in \mathcal{R}(R^*) \cap \mathcal{R}(R)$. Then $x = Rx$ and as $R^\dagger R$ is a projection onto $\mathcal{R}(R^*)$, also $R^\dagger Rx = x$. So $R^\dagger x = R^\dagger Rx = x$, thus showing $(I_n - R^\dagger)x = 0$ for all $x \in \mathcal{R}(R^*) \cap \mathcal{R}(R)$. On similar lines one can prove $\mathcal{N}(I_n - (R^*)^\dagger) = \mathcal{R}(R) \cap \mathcal{R}(R^*)$. Then, $(R^\dagger)^* = (R^*)^\dagger$ implies $\mathcal{R}(I_n - R^\dagger) = [\mathcal{N}(I_n - (R^\dagger)^*)]^\perp = (\mathcal{R}(R) \cap \mathcal{R}(R^*))^\perp = [\mathcal{N}(I_n - R^\dagger)]^\perp = \mathcal{R}(I_n - (R^\dagger)^*)$. ■

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Also solved by Eugene A. HERMAN.

Problem 50-3: Trace Inequality for Positive Block Matrices

Proposed by Ádám BESENYEI, *Department of Applied Analysis, Eötvös Loránd University, Hungary*, badam@cs.elte.hu

Let $A, B, C \in M_n(\mathbb{C})$ be such that the block matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in M_{2n}(\mathbb{C})$ is positive-semidefinite. Show that

$$\text{Tr}(AC) - \text{Tr}(B^*B) \leq \text{Tr}(A) \text{Tr}(C) - \text{Tr}(B^*) \text{Tr}(B).$$

Solution 50-3.1 by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, U.S.A.*, zhang@nova.edu

The inequality is unitarily invariant, i.e., if it holds then it also holds when A, B , and C are replaced with U^*AU , U^*BU , and U^*CU for any unitary matrix U . So we may assume that A is a diagonal matrix. Then the inequality is equivalent to

$$\sum_i a_{ii} c_{ii} - \sum_{i,j} |b_{ij}|^2 \leq \sum_i a_{ii} \sum_j c_{jj} - \sum_i \bar{b}_{ii} \sum_i b_{ii},$$

which is the same as $\sum_{i \neq j} \bar{b}_{ii} b_{jj} - \sum_{i \neq j} |b_{ij}|^2 \leq \sum_{i \neq j} a_{ii} c_{jj}$. In fact the following stronger inequality holds:

$$\sum_{i \neq j} \bar{b}_{ii} b_{jj} \leq \sum_{i \neq j} a_{ii} c_{jj}. \quad (4)$$

(4) is immediate from the observation that $\begin{pmatrix} a_{ii} c_{jj} & b_{ii} \bar{b}_{jj} \\ \bar{b}_{ii} b_{jj} & c_{ii} a_{jj} \end{pmatrix} \geq 0$ since it is a principal submatrix of $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \otimes \begin{pmatrix} C & B^* \\ B & A \end{pmatrix}$. Note: $\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix} \geq 0$ implies $y + \bar{y} \leq x + z$. ■

Remark. More can be said about (4). Consider $K = \begin{pmatrix} A \otimes C & B \otimes B^* \\ B^* \otimes B & C \otimes A \end{pmatrix} \geq 0$. Then $H = \begin{pmatrix} A \circ C & B \circ B^* \\ B^* \circ B & C \circ A \end{pmatrix}$ is a principal submatrix of K (here, \circ is the Hadamard product). We get (4) by deleting all rows and columns of H in K then taking trace for each of the blocks. Similar results may be obtained by taking other principal submatrices of K .

Also solved by the proposer.

Problem 50-4: Matrix Power Coefficients

Proposed by Moubinool OMARJEE, *Lycée Henri IV, Paris, France*, ommou@yahoo.com

Find all real matrices A such that $A^r = (a_{ij}^r)$ for any positive integer r (A^r is the usual product of A r -times). What is the answer for matrices over commutative fields?

Editorial note: The original formulation of Problem 50-4 did not contain the adjective ‘positive’ integer. We thank Eugene A. HERMAN for reporting this missprint.

Solution 50-4.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices with entries in a field \mathbb{F} . For $A = (a_{ij}) \in M_n(\mathbb{F})$, denote (a_{ij}^r) by A^r . An *anchored submatrix* of A is a submatrix whose index set $I \times J$ satisfies $|I \cap J| = 1$. The unique element of $I \cap J$ is called the *anchor* of the submatrix. The *shadow* of the anchored submatrix $I \times J$ with anchor z is

$$S(I \times J) := \{(i, j) \mid j \in I \setminus \{z\} \text{ or } i \in J \setminus \{z\} \text{ or } (j = z \text{ and } i \notin I) \text{ or } (i = z \text{ and } j \notin J)\}.$$

A matrix $A \in M_n(\mathbb{F})$ is a *Hadamard power matrix* if every nonzero diagonal entry of A is the anchor of some anchored submatrix whose entries equal this diagonal entry, these submatrices are pairwise disjoint, the entries in the shadows of these submatrices equal zero, and all remaining entries are zero. Our classification will rely on the following lemma (see, e.g., [1, Lemma 2.1.15] for a proof.)

Lemma. Let $x_1, \dots, x_n \in \mathbb{F}$, a field of characteristic zero.

(a) If $\sum_{k=1}^n x_k^r = 0$ for all positive integers r , then $x_k = 0$ for all k .

(b) If $\sum_{k=1}^n x_k^r = 1$ for all positive integers r , there exists m such that $x_m = 1$ and $x_k = 0$ for all $k \neq m$.

Theorem. Let $A \in M_n(\mathbb{F})$. If A is a Hadamard power matrix, then $A^r = A^{r^*}$ for all positive integers r . If the field \mathbb{F} has characteristic zero and $A^r = A^{r^*}$ for all positive integers r , then A is a Hadamard power matrix.

The final sentence of the theorem is false if the field \mathbb{F} has prime characteristic p . For example, let A be the $(p+1) \times (p+1)$ matrix of ones. Then $A^r = A = A^{r^*}$ for all positive integers r but A is not a Hadamard power matrix.

Proof of Theorem: Suppose $A = (a_{ij}) \in M_n(\mathbb{F})$ is a Hadamard power matrix. It suffices to show $AA^{r^*} = A^{(r+1)^*}$ for all positive integers r , or equivalently: $\sum_k a_{ik}a_{kj}^r = a_{ij}^{r+1}$ whenever $1 \leq i, j \leq n$, $r = 1, 2, 3, \dots$. Consider two cases.

First case. Assume $(i, j) \in I \times J$, an anchored submatrix of A with anchor z and constant value $c \neq 0$. Then, for all $k \neq z$ we have $a_{ik} = 0$ or $a_{kj} = 0$ because $i = z$, $k \notin J$ implies $(i, k) \in S(I \times J)$ while otherwise $(k, j) \in S(I \times J)$. Hence,

$$\sum_k a_{ik}a_{kj}^r = a_{iz}a_{zj}^r = c c^r = a_{ij}^{r+1}. \quad (5)$$

Second case: Assume (i, j) belongs to no anchored submatrix. If, for some k , $a_{ik} \neq 0$ and $a_{kj} \neq 0$, then $(i, k) \in I \times J$ and $(k, j) \in I' \times J'$, anchored submatrices with anchors z and z' , respectively. Clearly, $I \times J \neq I' \times J'$, for otherwise $(i, j) \in I \times J$. So $z \neq z'$. If $k \in J \setminus \{z\}$, then $(k, j) \in S(I \times J)$ which implies $a_{kj} = 0$. If $k = z$, then $k \in I' \setminus \{z'\}$ and so $a_{ik} = 0$. Therefore, for all (i, j) and all r , equation (5) holds.

To prove the opposite direction, suppose $A = (a_{ij})$ satisfies $A^r = A^{r^*}$ for all positive integers r . From $A^r A^s = A^{r+s}$, we have $A^r A^{s^*} = A^{(r+s)^*}$ and so

$$\sum_k a_{ik}^r a_{kj}^s = a_{ij}^{r+s}; \quad 1 \leq i, j \leq n, \quad r, s = 1, 2, 3, \dots \quad (6)$$

For the most part, we use the special case when $s = r$:

$$\sum_k (a_{ik}a_{kj})^r = a_{ij}^{2r}; \quad 1 \leq i, j \leq n, \quad r = 1, 2, 3, \dots \quad (7)$$

From these identities and the lemma, we deduce the properties

$$\text{If } a_{ij} = 0 \text{ for some } i, j \text{ then for all } k, a_{ik} = 0 \text{ or } a_{kj} = 0 \quad (8)$$

$$\text{If } i \neq j \text{ then } a_{ij} = 0 \text{ or } a_{ji} = 0 \quad (9)$$

$$\begin{aligned} \text{If } a_{ij} \neq 0 \text{ for some } i \neq j \text{ then either } & a_{ij} = a_{ii}, a_{jj} = 0, a_{ik}a_{kj} = 0 \text{ for all } k \neq i \\ & \text{or } a_{ij} = a_{jj}, a_{ii} = 0, a_{ik}a_{kj} = 0 \text{ for all } k \neq j \\ & \text{or } a_{ij} = a_{mj} = a_{im} \text{ for some } m \neq i, j, a_{ii} = a_{jj} = 0, a_{ik}a_{kj} = 0 \text{ for all } k \neq m. \end{aligned} \quad (10)$$

Property (8) follows from identity (7) and part (a) of the lemma. Property (9) follows by inserting $j = i$ into (7) to obtain $\sum_{k \neq i} (a_{ik}a_{ki})^r = 0$ and then part (a) of the lemma gives $a_{ik}a_{ki} = 0$ for all $k \neq i$. If $a_{ij} \neq 0$ then (7) can be written as $\sum_k \left(\frac{a_{ik}}{a_{ij}} \cdot \frac{a_{kj}}{a_{ij}} \right)^r = 1$ for $r = 1, 2, 3, \dots$, hence, by part (b) of the lemma, there exists m such that

$$a_{im}a_{mj} = a_{ij}^2 \quad \text{and} \quad a_{ik}a_{kj} = 0 \quad \text{for all } k \neq m.$$

If $m = i$ we obtain the first of the three possible outcomes in (10); $m = j$ gives the second outcome. If $m \neq i, j$ we obtain all but the first conclusion of the third outcome. However, in addition to $a_{im}a_{mj} = a_{ij}^2$, identity (6) with $r = 1$ and $s = 2$ yields $a_{im}a_{mj}^2 = a_{ij}^3$. These two equations imply $a_{ij} = a_{mj} = a_{im}$.

Now we construct the anchored submatrices of A . Choose all indices z_k , $k = 1, \dots, q$, such that $a_{z_k z_k} \neq 0$. Define

$$I_k = \{i \mid a_{iz_k} \neq 0\}, \quad J_k = \{j \mid a_{z_k j} \neq 0\}.$$

We claim that A is the Hadamard power matrix whose anchored submatrices are $I_k \times J_k$, $k = 1, \dots, q$. By property (9), whenever $i \neq z_k$, $a_{iz_k} = 0$ or $a_{z_k i} = 0$, and so $i \notin I_k \cap J_k$; that is, $I_k \cap J_k = \{z_k\}$. Next we show that

$$a_{ij} = a_{z_k z_k} \quad \text{for all } (i, j) \in I_k \times J_k. \quad (11)$$

By construction, $a_{iz_k} \neq 0$ for $i \in I_k$ with $i \neq z_k$. Hence, by property (10), $a_{iz_k} = a_{z_k z_k}$ and $a_{ii} = 0$. Similarly, for all $j \in J_k$ with $j \neq z_k$, we have $a_{z_k j} = a_{z_k z_k}$ and $a_{jj} = 0$. Now suppose $i \neq z_k$ and $j \neq z_k$. Then $i \neq j$, since $I_k \cap J_k = \{z_k\}$. Since $a_{iz_k} = a_{z_k j} = a_{z_k z_k} \neq 0$, property (8) implies $a_{ij} \neq 0$. By property (10) and the fact that $a_{ii} = a_{jj} = 0$, there exists $m \neq i, j$ such that $a_{ij} = a_{mj} = a_{im}$ and $a_{ik}a_{kj} = 0$ for all $k \neq m$. Since $a_{iz_k}a_{z_k j} \neq 0$, we have $m = z_k$ and so $a_{ij} = a_{z_k j} = a_{z_k z_k}$, as claimed in (11).

Suppose there exists $(i, j) \in (I_k \times J_k) \cap (I_l \times J_l)$ for some $k \neq l$. Then $a_{im}a_{mj} \neq 0$ for two values of m , namely z_k and z_l contradicting property (10). So, the anchored submatrices $I_k \times J_k$ are pairwise disjoint. We claim that $a_{ij} = 0$ whenever $(i, j) \in S(I_k \times J_k)$ for some k . By construction, $a_{iz_k} = 0$ for all $i \notin I_k$ and $a_{z_k j} = 0$ for all $j \notin J_k$. Suppose $j \in I_k \setminus \{z_k\}$. If $i \in I_k$ then $a_{iz_k} \neq 0$. Since $a_{z_k z_k} \neq 0$ and $j \neq z_k$, property (10) implies $a_{ij}a_{jz_k} = 0$. Hence $a_{ij} = 0$, since $a_{jz_k} \neq 0$. If $i \notin I_k$ then $a_{iz_k} = 0$. Hence, by property (8), $a_{ij}a_{jz_k} = 0$, and so $a_{ij} = 0$ as before. Similarly, if $i \in J_k \setminus \{z_k\}$ then $a_{ij} = 0$. Finally, we show that $a_{ij} = 0$ whenever $(i, j) \notin \bigcup_k (I_k \times J_k)$. Suppose instead that $a_{ij} \neq 0$. Then $i \neq j$ since all diagonal entries of A , other than the entries $a_{z_k z_k}$, are zero. We can further assume $i \neq z_k$ and $j \neq z_k$ for $k = 1, \dots, q$, because otherwise, $(i, j) \in S(I_k \times J_k)$ and so $a_{ij} = 0$. By property (10), either $a_{ij} = a_{ii}$ or $a_{ij} = a_{jj}$ or $a_{ij} = a_{mj} = a_{im}$ for some $m \neq i, j$. The first two outcomes imply that $a_{ij} = 0$, since neither i nor j is an anchor. From the third outcome, we show that $a_{mm} \neq 0$. Suppose instead that $a_{mm} = 0$. Since $a_{ii} = 0$ and $a_{im} \neq 0$, there exists $l \neq i, m$ such that $a_{im} = a_{il} = a_{lm}$ by (10). The third outcome of property (10) for a_{ij} also says that $a_{ik}a_{kj} = 0$ for all $k \neq m$. In particular, $a_{il}a_{lj} = 0$. Since $a_{il} \neq 0$, $a_{lj} = 0$. By property (8), $a_{lk}a_{kj} = 0$ for all k . In particular, $a_{lm}a_{mj} = 0$, a contradiction. So $a_{mm} \neq 0$ and hence $m = z_k$ for some k . Thus $0 \neq a_{ij} = a_{z_k j} = a_{iz_k}$, and so $(i, j) \in I_k \times J_k$, a contradiction. ■

Reference

[1] H. Radjavi and P. Rosenthal, *Simultaneous Triangularization*, Springer-Verlag, New York 2000.

Solution 50-4.2 by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Only for real matrices. For $A = (a_{ij})_{ij}$ write $A^{(r)} = (a_{ij}^r)_{ij}$ for the Hadamard power and A^r for the standard power.

Theorem. Let $A = (a_{ij})_{ij}$ be a real n -by- n matrix. Then, $A^r = A^{(r)}$ for every positive integer r if and only if A is an adjacency matrix of a weighted digraph with the following two properties: (i) there is a path of length one from u to v if and only if there exists exactly one path of length two from u to v (loops allowed), and (ii) if $a_{ii} \neq 0$ and there is a path $u \rightarrow i \rightarrow v$ then $a_{uv} = a_{ii}$.

Proof. Sufficiency. Define $\delta(x) = \begin{cases} 0; & x = 0 \\ 1; & \text{otherwise} \end{cases}$. Condition (i) implies $X^2 = X$ where $X = (\delta(a_{ij}))_{ij}$ is a $(0, 1)$

adjacency matrix of a digraph induced by A and with weights removed. Hence, $X^r = X$ for every positive integer r and thus there is a path from u to v if and only if for every positive integer r there is a unique path of length r from u to v . Further, if $v_1 \rightarrow v_3$ and $v_1 \rightarrow v_2 \rightarrow v_3$ are paths in digraph with vertices v_1, v_2, v_3 distinct, then at least one of them forms a loop. Otherwise, by (i), there is $w \notin \{v_1, v_2\}$ with $v_1 \rightarrow w \rightarrow v_2$. If $w = v_3$ then $v_3 = w \rightarrow v_2 \rightarrow v_3$ is a path of length two from v_3 to v_3 , so by (i) there must be a path of length one, i.e., a loop $v_3 \rightarrow v_3$, contradicting the fact that $v_1 \rightarrow v_3 \rightarrow v_3$ and $v_1 \rightarrow v_2 \rightarrow v_3$ would then be two paths of length two from v_1 to v_3 . If $w \notin \{v_1, v_2, v_3\}$ then $w \rightarrow v_2 \rightarrow v_3$ is a path of length two, so by (1), $w \rightarrow v_3$ is also a path, and then $v_1 \rightarrow w \rightarrow v_3$ and $v_1 \rightarrow v_2 \rightarrow v_3$ are two paths of length two, a contradiction. So one among v_1, v_2, v_3 forms a loop and it is easy to see that $v_2 \rightarrow v_2$. By repeated use of (ii), $a_{v_1 v_3} = a_{v_2 v_2} = a_{v_1 v_2} = a_{v_2 v_3}$. Since this holds for each path $i \rightarrow k \rightarrow j$ we derive that the only path of length r connecting i to j is $i \rightarrow k \rightarrow k \rightarrow \dots \rightarrow k \rightarrow j$, so in A^r the entry at position (i, j) is $a_{kk}^r = a_{ij}^r$, giving $A^r = A^{(r)}$.

Necessity. Suppose $A^r = A^{(r)}$ for every positive integer r . Then $(\lambda A)^r = \lambda^r A^r = (\lambda A)^{(r)}$ for every scalar λ . Hence, with $|\lambda|$ small enough, $\sum_1^\infty (\lambda A)^r = \sum_1^\infty (\lambda A)^{(r)}$ which sums up into $\lambda A(I - \lambda A)^{-1} = \left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}}\right)_{ij}$. Then after rearrangements, $\left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}}\right)_{ij} \cdot \left(\frac{1}{\lambda} I - A\right) = A$, entrywise constant, for all sufficiently small $|\lambda|$.

Both factors on the left are meromorphic functions of λ , with at most finitely many poles. Hence, the equality holds for each λ outside the poles, and with $\lambda \rightarrow \infty$ we get

$$(\delta(a_{ij}))_{ij} A = A.$$

Introducing $X := (\delta(a_{ij}))_{ij}$ we can rewrite the above equation into $XA = A$. Then also $X(\lambda A)^r = (\lambda A)^r$ for every positive integer r and every λ . Thus, as above, $X \sum_{r=1}^{\infty} (\lambda A)^r = \sum_{r=1}^{\infty} (\lambda A)^r = \sum_{r=1}^{\infty} (\lambda A)^{(r)} = \left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}} \right)_{ij}$. On the other hand, $X \sum_{r=1}^{\infty} (\lambda A)^r = X \sum_{r=1}^{\infty} (\lambda A)^{(r)} = X \left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}} \right)_{ij}$.

This gives $X \left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}} \right)_{ij} = \left(\frac{\lambda a_{ij}}{1 - \lambda a_{ij}} \right)_{ij}$. Letting $\lambda \rightarrow \infty$ we obtain $X^2 = X$. Recall that X is a $(0, 1)$ matrix so it is an adjacency matrix of a digraph Γ on n points. Since $X^2 = X$, this graph has the property that between each two vertices there exists at most one path of length two (loops allowed) and it does exist if and only if the two vertices are connected with a path of length one.

Then, $A = (a_{ij})_{ij}$ is an adjacency matrix of the same graph as X but with weighted edges. Assume $x := a_{ii} \neq 0$ and there is a path of length two from u to v via i . Then, there is a path $u \rightarrow i \rightarrow i$ of length two (containing a loop $i \rightarrow i$) from u to i and so, in $A^2 = A^{(2)}$, the entry at position (u, i) equals $a_{ui}a_{ii} = a_{ui}^2$. This gives $a_{ui} = a_{ii} = x$. Likewise, $a_{iv} = x$, and so, comparing entries at (u, v) position of $A^3 = A^{(3)}$ we get $a_{ui}a_{ii}a_{iv} = a_{uv}^3$, i.e., $a_{uv}^3 = x^3$, so $a_{uv} = x$. ■

Problem 50-5: A Matrix of Divided Differences

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$ be an n -th degree monic polynomial with complex coefficients and let $\{x_k\}_{k=1}^n$ be a set of n pairwise distinct real numbers. Let M be the n -by- n matrix of first divided differences of p (so for $1 \leq i, j \leq n$ we have $m_{i,j} = (p(x_i) - p(x_j))/(x_i - x_j)$ and $m_{ii} = p'(x_i)$). Show that the determinant of M is positive if n is congruent to 0 or 1 mod 4 and negative if n is congruent to 2 or 3 mod 4.

Solution 50-5.1 by Omran KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

We will use the following lemma.

Lemma. Let $p(X) = X^n + a_1 X^{n-1} + \dots + a_{n-1} X + a_n$ be an n -th degree monic polynomial with coefficients in a commutative field \mathbb{K} , and let $Q = (q_{ij}) \in M_n(\mathbb{F})$ be the n -by- n matrix with coefficients in the field $\mathbb{F} = \mathbb{K}(X_1, \dots, X_n, Y_1, \dots, Y_n)$ of rational functions in the $2n$ variables $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ defined by

$$q_{ij} = \frac{p(X_i) - p(Y_j)}{X_i - Y_j}, \quad 1 \leq i, j \leq n.$$

Then

$$\det Q = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (X_j - X_i) \cdot \prod_{1 \leq i < j \leq n} (Y_j - Y_i).$$

Proof of the Lemma. Let V_X be the n -by- n matrix whose (i, j) -entry is X_j^{i-1} , define similarly V_Y . Also, let $\Lambda = (\lambda_{ij})$ be the n -by- n matrix defined by $\lambda_{ij} = a_{n+1-i-j}$ with $a_0 = 1$ and $a_k = 0$ if $k < 0$. Then we check easily that the (i, j) -entry of $V_X^T \Lambda V_Y$ is

$$\begin{aligned} [V_X^T \Lambda V_Y]_{ij} &= \sum_{1 \leq k, \ell \leq n} X_i^{k-1} a_{n+1-k-\ell} Y_j^{\ell-1} = \sum_{m=1}^n a_{n-m} \sum_{k+\ell=m+1} X_i^{k-1} Y_j^{\ell-1} \\ &= \sum_{m=1}^n a_{n-m} \sum_{k=1}^m X_i^{k-1} Y_j^{m-k} = \sum_{m=1}^n a_{n-m} \frac{X_i^m - Y_j^m}{X_i - Y_j} = \frac{p(X_i) - p(Y_j)}{X_i - Y_j} = q_{ij}. \end{aligned}$$

Thus $Q = V_X^T \Lambda V_Y$. In particular, $\det Q = \det \Lambda \cdot \det V_X \cdot \det V_Y$. But, clearly,

$$\det \Lambda = \det \begin{pmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & 1 & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} = (-1)^{n(n-1)/2},$$

and $\det V_X, \det V_Y$ are the well-known Vandermonde determinants. ■

Making the substitution $X_i \leftarrow x_i$ and $Y_i \leftarrow x_i$ for every i we obtain $\det M = (-1)^{n(n-1)/2} \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i)^2$ and the result follows since $\prod_{1 \leq i < j \leq n} (x_j - x_i)^2 > 0$. ■

Also solved by Eugene A. HERMAN and the proposer.

Problem 50-6: Diagonalizable Matrices Over \mathbb{F}_p

Proposed by Denis SERRE, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Let $p \geq 2$ be a prime number and let \mathbb{F}_p denote the field $\mathbb{Z}/p\mathbb{Z}$. Prove that $A \in M_n(\mathbb{F}_p)$ is diagonalizable within $M_n(\mathbb{F}_p)$ if and only if $A^p = A$.

Solution 50-6.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

We rely on Fermat's little Theorem (FLT): $k^p = k$ for all $k \in \mathbb{F}_p$. Assume A is diagonalizable. Then $A = QDQ^{-1}$ for some $Q, D \in M_n(\mathbb{F}_p)$ with Q invertible and D diagonal. By FLT, $D^p = D$, and so $A^p = QD^pQ^{-1} = QDQ^{-1} = A$.

Assume $A^p = A$. Define subspaces $V_0 = \text{Im}(I - A^{p-1})$, $V_k = \text{Im}\left(\sum_{j=1}^{p-1} k^{p-j} A^j\right)$ for $k = 1, \dots, p-1$. We claim that

$$\mathbb{F}_p^n = V_0 \oplus V_1 \oplus \dots \oplus V_{p-1} \quad (\text{direct sum})$$

and that $Av = kv$ for all $v \in V_k$, $k = 0, 1, \dots, p-1$ which implies A is diagonalizable. In fact, if $v = u - A^{p-1}u \in V_0$ then $Av = Au - A^p u = 0$. If $v = \sum_{j=1}^{p-1} k^{p-j} A^j u \in V_k$ then $Av = \sum_{j=1}^{p-1} k^{p-j} A^{j+1} u = \sum_{i=2}^{p-1} k^{p+1-i} A^i u + k A^p u = k \sum_{i=2}^{p-1} k^{p-i} A^i u + k A^{p-1} u = kv$ by FLT. Hence the second part of the claim holds.

To show that $\mathbb{F}_p^n = V_0 + V_1 + \dots + V_{p-1}$, let $v \in \mathbb{F}_p^n$. We claim there exist $c_0, c_1, \dots, c_{p-1} \in \mathbb{F}_p$ such that

$$v = c_0(I - A^{p-1})v + \sum_{k=1}^{p-1} c_k \sum_{j=1}^{p-1} k^{p-j} A^j v.$$

By choosing $c_0 = 1$, this equation becomes $A^{p-1}v = \sum_{j=1}^{p-1} \left(\sum_{k=1}^{p-1} c_k k^{p-j}\right) A^j v$. The latter equation holds if c_1, \dots, c_{p-1} satisfy $\sum_{k=1}^{p-1} c_k k = 1$ and $\sum_{k=1}^{p-1} c_k k^{p-j} = 0$, $j = 1, \dots, p-2$. This system of equations has the $(p-1) \times (p-1)$ coefficient matrix whose (j, k) entry is k^j . This is a nonsingular Vandermonde matrix, and so c_1, \dots, c_{p-1} exist.

To prove that the sum is direct, choose $v_k \in V_k$, $k = 0, 1, \dots, p-1$, such that $\sum_{k=0}^{p-1} v_k = 0$. Applying A^j , $j = 1, \dots, p-1$, to this equation yields

$$\sum_{k=1}^{p-1} k^j v_k = 0; \quad j = 1, \dots, p-1.$$

The coefficient matrix is again a nonsingular Vandermonde matrix, and so $v_1 = 0, \dots, v_{p-1} = 0$. Hence $v_0 = 0$ as well. ■

Solution 50-6.2 by Harald WIMMER, *Universität Würzburg, Germany*, wimmer@mathematik.uni-wuerzburg.de

Theorem. Let $K = \text{GF}(p^r)$ be the Galois field with $m = p^r$ elements and let $A \in K^{n \times n}$. The matrix A is diagonalizable in $K^{n \times n}$ if and only if $A^m = A$.

Proof. The following facts will be used. (i) The Galois field K is the splitting field of the polynomial $f(z) = z^m - z$. Hence all roots of $f(z)$ are simple and each element λ of K satisfies $\lambda^m = \lambda$. (ii) A matrix A over a field F is diagonalizable if and only if its characteristic polynomial splits over F and the roots of its minimal polynomial $\mu_A(z)$ are simple.

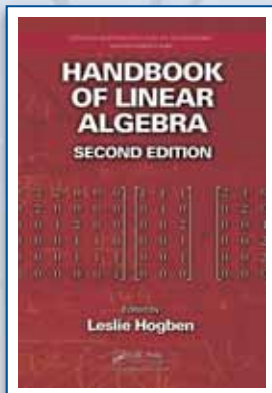
Suppose A is diagonalizable. Let S be nonsingular such that $S^{-1}AS = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then $\lambda_j \in K$, and therefore $\lambda_j^m = \lambda_j$, $j = 1, \dots, n$. Hence $A^m = A$.

Conversely, suppose $A^m = A$. Then $f(z) = z^m - z$ is an annihilating polynomial of A . Therefore $\mu_A(z) \mid f(z)$. Hence the spectrum of A is in K and the roots of $\mu_A(z)$ are simple. ■

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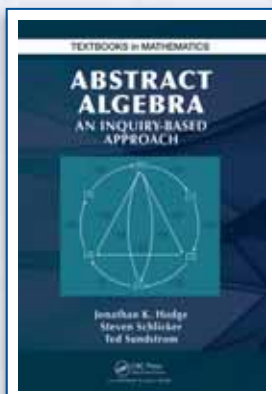
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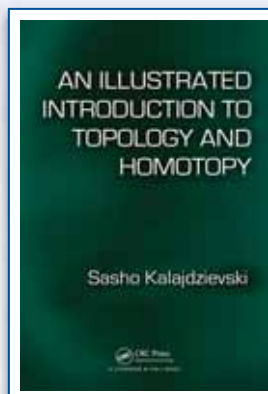
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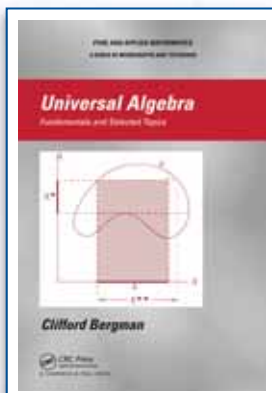
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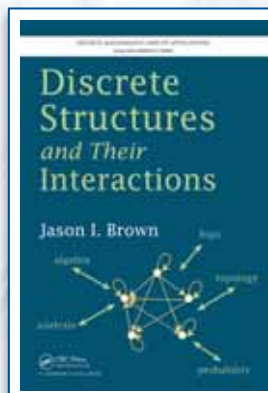
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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. **Solutions:** We present solutions to all problems in the previous issue [*IMAGE* 50 (Spring 2013), p. 44]. We still seek solutions for problem 49.3. **Submissions:** Please submit proposed problems and solutions in macro-free L^AT_EX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Steve Butler, Gregor Dolinar, Shaun Fallat, Alexander Guterman, Rajesh Pereira, and Nung-Sing Sze.

NEW PROBLEMS:

Problem 51-1: A Norm Identity

Proposed by Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu and Minghua LIN, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let a_1, \dots, a_n be nonnegative numbers, let $x_1, \dots, x_n \in \mathbb{C}^N$, and let $\|\cdot\|$ be the Euclidean norm. Show that

$$\sum_{i=1}^n a_i \left\| \sum_{j=1}^n a_j (x_j - x_i) \right\|^2 = \frac{1}{2} \left(\sum_{k=1}^n a_k \right) \sum_{i,j=1}^n a_i a_j \|x_i - x_j\|^2.$$

Problem 51-2: Commutants of Diagonal Matrices

Proposed by Gregor DOLINAR, *University of Ljubljana, Slovenia*, gregor.dolinar@fe.uni-lj.si

The commutant (also known as the centralizer) of an $m \times m$ matrix $A \in M_m(\mathbb{C})$ is $A' = \{X \in M_m(\mathbb{C}) : AX = XA\}$. Find a recursive formula for the cardinality $N_m = \#\{A' : A \text{ diagonal}\}$ of the set of all commutants of diagonal $m \times m$ matrices, as a function of m .

Problem 51-3: A Hadamard-like inequality

Proposed by Minghua LIN, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let $A = (a_{ij})$ be a real positive definite symmetric matrix of order n and let $A(i)$ be the submatrix obtained from A by deleting the i -th row and i -th column. Show that the inequality below holds for $n \geq 4$ and fails for $n = 2, 3$:

$$(n-1) \prod_{i=1}^n a_{ii} + \det A \geq \sum_{i=1}^n a_{ii} \det A(i).$$

Problem 51-4: An Adjugate Identity

Proposed by Volodymyr PROKIP, *Institute for Applied Problems of Mechanics and Mathematics, Ukraine*, v.prokip@gmail.com

Let F be a field and let $M_{m,l}(F[x])$ be the set of $m \times l$ matrices over the polynomial ring $F[x]$. Let

$$\begin{aligned} A(x) &= A_0 x^p + A_1 x^{p-1} + \dots + A_p \in M_{m,n}(F[x]); \quad p \geq 1, \\ B(x) &= B_0 x^q + B_1 x^{q-1} + \dots + B_q \in M_{m,k}(F[x]); \quad q \geq 1 \end{aligned}$$

with $\text{rank}[A_p \ B_q] = r < m$. Show that for matrices $A(x)$ and $B(x)$ there exists a common left divisor $D(x) \in M_{m,m}(F[x])$, i.e., $A(x) = D(x)\tilde{A}(x)$ and $B(x) = D(x)\tilde{B}(x)$, such that $\deg(\det D(x)) \geq m - r$.

Problem 51-5: Orthogonal Basis Subordinated to a Plane Arrangement in \mathbb{R}^3

Proposed by Denis SERRE, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Let $E_1, E_2, E_3 \subseteq \mathbb{R}^3$ be the planes of the respective equations $z_j \cdot x = 0$. Prove that there exists an orthogonal basis $\{v_1, v_2, v_3\} \subseteq \mathbb{R}^3$ such that $v_j \in E_j$ for $j = 1, 2, 3$ if and only if

$$\Delta := (\det(z_1, z_2, z_3))^2 - 4(z_1 \cdot z_2)(z_2 \cdot z_3)(z_3 \cdot z_1)$$

is non-negative. When Δ is positive, there exist two such bases, up to scaling.

Problem 51-6: The Determinant Kernel

Proposed by Suvrit SRA, *Max Planck Institute for Intelligent Systems, Tübingen, Germany*, suvrit@tuebingen.mpg.de

(i) Let X_1, X_2, \dots, X_n be $p \times p$ symmetric positive definite real matrices. Let K be the $n \times n$ matrix defined by $K = (k_{ij}) = \left(\frac{1}{\det(X_i + X_j)} \right)$. Prove that K is positive semi-definite.

(ii) Let $\beta > \frac{p-1}{2}$ be a scalar. A harder problem is to prove that the matrix $K_\beta = (k_{ij}^\beta) = \left(\frac{1}{\det((X_i + X_j)^\beta)} \right)$ is also positive semi-definite.