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IMAGE

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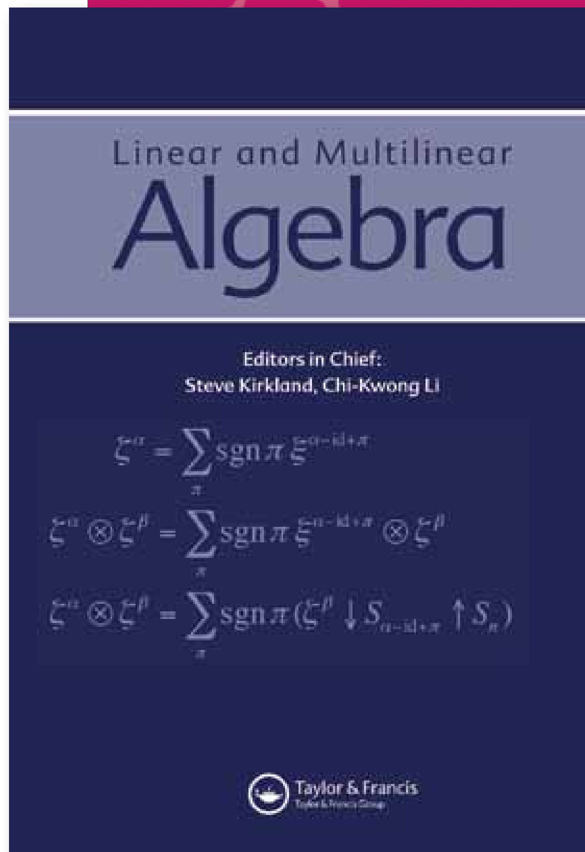

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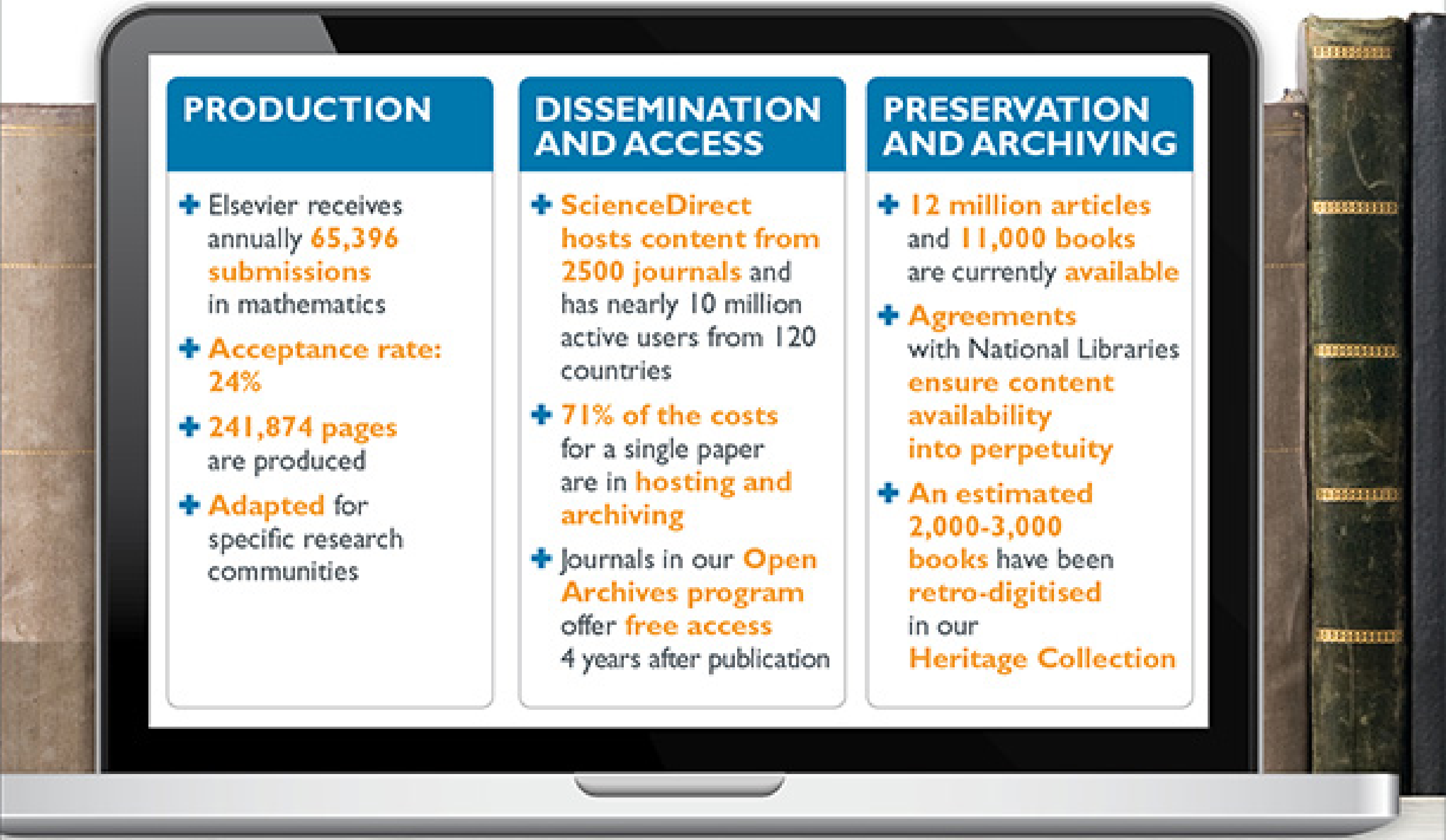

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Mathematics

FEATURE INTERVIEW

"Enjoy your work and life"

Ludwig Elsner Interviewed by Angelika Bunse-Gerstner[1]

Ludwig Elsner is well-known in the linear algebra community by his excellent work in linear and multilinear algebra, in numerical linear algebra and in particular for his inclusion theorems for eigenvalues with their concise and elegant proofs. He keeps fruitful scientific collaborations with colleagues in many different countries and has quite a number of academic children and grandchildren working in the field of linear algebra.

A.B.G. – What made you start to work in mathematical research?

L.E. – When I was studying mathematics at the University of Hamburg, my first course was taught by Lothar Collatz. At some occasion, he presented his quotient theorem, an inclusion theorem for the spectral radius of nonnegative matrices. I tested this result numerically and mentioned the results to him during an oral exam for numerical analysis. He seemed to be enthusiastic about my "check." So I had caught his attention. After my diploma and final exam in 1963, he offered me a position in his large and very active research group. The research group, among other things, ran one of the first computing centers at a German university.

A.B.G. – What was it that drew your interest to linear algebra and caused you to work in this field?

L.E. – For my diploma thesis, I worked in functional analysis considering inclusion theorems. Already as a student, I was very interested to understand how mathematical results can be implemented numerically. Later this was reinforced by my work in the computing centre and the numerous projects of our research group. Since computers can only solve linear problems, the interrelation between linear algebra, numerical analysis and functional analysis is essential for actual computation. This interplay of different fields with their different methods in the solution of a problem and its computation was challenging and fascinating for me.

A.B.G. – Are there special events which were very important for your career?

L.E. – There are three which come to my mind right now. The first and very important one is my first contact with Hans Schneider. In a paper of mine, I improved a result of Hans Schneider. I had written it in German, which was quite common at this time for a German mathematician. I sent it to Hans Schneider in this form. Based on the German sound of his name, I just assumed he would be able to read it. Well, he was able to, and he became interested in my work. The next time he visited Germany, he called me. We have kept contact since then. One year after we got to know each other, he asked me whether I would like to become an associate editor of *Linear Algebra and its Applications*. His kind interest in my work and his support was crucial for me to grow into a member of the international linear algebra community.

Another important event was certainly my participation in a linear algebra conference in Santa Barbara, California, in 1977, where I met many well-known and established researchers in linear algebra. They gave talks about results which I found very exciting, and I was thrilled to be able to get in contact with these colleagues to discuss their results, and possible extensions and improvements. A number of fruitful collaborations were started then and there.

Finally, I am thinking of the start of the 1989 DFG (Deutsche Forschungsgemeinschaft) sponsored Collaborative Research Centre "Discrete Structures in Mathematics" which ran for several years at my university in Bielefeld. Its substantial financial means provided opportunities to invite interesting researchers to Bielefeld for longer periods. I received many suggestions and stimulations by tackling problems with them during their visits, and very often this started and supported long term collaborations.

A.B.G. – What is it that you enjoy most in your research work?

L.E. – I find it most satisfying when I obtain a mathematical result which is a reasonable and useful solution to an interesting, and not-too-easy-to-solve problem in mathematics, or in an application. It is particularly satisfying if I can do this with elegant and concise proofs, reducing the mathematical arguments to their essence. This is for me the joy of mathematics. The other thing which I really enjoy is the opportunity to meet with colleagues all over the world who work on exciting and challenging problems. This has happened mostly during my sabbatical leaves, which I have often spent in Canada, with the help of Peter Lancaster and others. I loved to learn about the problems my colleagues were working on and about the various techniques they were using. This has often led to longer cooperation and sometimes to friendships for life.

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A.B.G. – Do you think that the working conditions in linear algebra research have changed over the years?

L.E. – I think that working conditions at universities in general have changed. In the past as a researcher at a university you did not feel so hemmed in. You had much more freedom in choosing your area of research and the problems you wanted to work on. Today there are quite a number of restrictions, mainly financial, which have to be considered. The teaching load at German universities is now much higher than it was before.

A.B.G. – Speaking of teaching, did you like to teach or was it more of a burden?

L.E. – Like probably most of us, I preferred teaching specialized courses; the numerical linear algebra course was one of my favorites. Every time I taught a course, it was a chance to reconsider the structure of presenting the material and the connections and dependencies between the various topics that had to be presented. When putting things together and making notes, I sometimes could find easier and more reasonable ways of deriving results; in this sense it was also a pleasure for me.

A.B.G. – Is there an advice you would give to young people in the field?

L.E. – To my experience it is essential to seek out contacts with other scientists in your field of research. Look at the problems the others are working on, and learn about the mathematical methods and techniques they use to solve them. If there are visiting researchers at your institution, go to their seminar and talk to them. Do not consider this a duty and an additional time strain, but rather as a chance to get suggestions, help and stimulations for your own work. Work on problems that you are really enthusiastic about, and enjoy your work and your life. One of the things I love about working in research is the chance to live and work abroad from time to time, to meet with inspiring people and make friends all over the world. Fortunately my wife Margrit likes this too; I am very grateful to her that in all these years she has strongly supported me in every respect. She made it possible that I could enjoy my work and at the same time have a happy family life.

LINEAR ALGEBRA EDUCATION

March MATHness in the Linear Algebra Classroom

Elizabeth L. Bouzarth[1] **and Timothy P. Chartier**[2]

Every March, a large amount of national attention in the United States turns to NCAA Division I men's basketball and the March Madness tournament. This single elimination tournament takes 68 teams from 350 schools that play men's basketball at the highest level, and crowns a national champion after three weeks of play. At the beginning of the process on Selection Sunday, the initial pairings of teams are announced and visualized as part of a tournament bracket, a portion of which is shown in Figure 1(a). The match-ups shown on the solid lines on the left side of the bracket are the

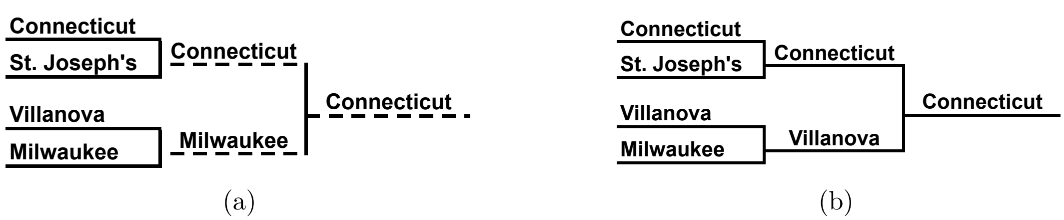

Figure 1: *A small portion of the* 2014 *March Madness bracket. Each vertical line represents a game between two teams. The match-ups on the solid lines in* (*a*) *are the known games announced at the beginning of the tournament and the teams on the dashed lines in* (*a*) *show an example of a fan's predictions for the winner of each match-up. In* (*b*), *the actual winners are indicated on the right.*

announced games and the madness of March often comes in predicting the winners of each game (dashed lines) to see who advances to eventually win the tournament. By the following Thursday, bracket predictions must be completed and entered in pools in which friends, colleagues and even strangers compete for pride, office pool winnings, or even thousands of dollars. In fact, in 2014, Warren Buffett insured a billion dollar prize for anyone who could complete a perfect bracket for the tournament. By the end of the second round (or 48 games) none of the over 8 million brackets had perfectly predicted the 2014 tournament. As the tournament progresses, game results either uphold or overturn the predictions

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of millions of fans. For instance, for the match-ups shown in Figure 1, we see in (b) that Connecticut beat St. Joseph's, which matches the prediction in (a). But Villanova beat Milwaukee, which goes against the prediction. This national interest is an opportunity to teach ideas fundamental to linear algebra and to underscore the topic's relevance in our world, which extends far beyond sports and into such areas as finding websites with Google to getting recommendations from Netflix to ranking sports teams.

In this article, we describe two rating systems, the Colley and Massey methods, and demonstrate their applicability to sports ranking and creating brackets for March Madness. The simple inherent structures of these ratings systems lend themselves well to becoming a topic of interest for linear algebra students.

First, it is important to distinguish between ratings and rankings. A *rating* is a value assigned to each team from which we can then form a *ranking*, an ordered list of teams based upon a chosen rating method. Solving the linear systems derived from the Colley and Massey methods produces the ratings and sorting the ratings in descending order creates a ranked list from first to last place.

When someone wants to rank teams, a natural rating to start with is winning percentage by computing each team's ratio of number of games won to total number of games played. However, a simple system like this only includes information about each game's end result and can lead to misleading information. For instance, if Team A only plays one basketball game and they win, then their winning percentage rating would be 1. If Team B plays 20 basketball games and goes undefeated, their winning percentage rating would also be 1. Intuitively, Team B is the more impressive team since they went undefeated for 20 games, while Team A only won once. But, they both have the same winning percentage, so this rating system would produce a tie when ranking these two teams. Yet, problems also arise even if Teams C and D both played 20 games, but C remained undefeated by playing a difficult schedule comprised of the most talented teams in the league while D played only the weakest teams. While D may be a talented team, that fact is not readily apparent from their schedule. A fundamental difference in this schedule is the strength of C's and D's opponents.

The Colley Method. To integrate strength of schedule or the quality of one's opponent into a rating system, the Colley method, as introduced in [3], takes a slightly different approach to rating teams. Here lies an important aspect of using a matrix system. Linear algebra integrates interdependence of the ratings. Specifically, it is by computing all of the ratings at one time that we can tackle the issue of strength of schedule.

The Colley method attains this through its derivation. The method starts with the assumption that every team's rating is 0.5 at the beginning of a season. It is interesting to note that even this assumption gives superior results to winning percentage. Before any games are played, the Colley method rates all teams with the value of 0.5. No games have been played so all teams are considered equal. Winning percentage, however, does not yield a number, but only the ever-troublesome 0/0.

As games are played, the Colley method adjusts its rating to incorporate game results. A team's rating is also affected by the rating of every team it plays. We now would distinguish between Team C with its undefeated season against very strong opponents and Team D with its undefeated record against very weak opponents. Colley would rate Team C higher than Team D.

An important part of the Colley method is its omission of point differentials. So, a rout by 25 points counts the same as a win in double overtime by 1 point. Some may argue that all wins are not created equal. However, it can also be that scores create random noise in a system as some games are close until the final minute or a rout is reduced with the second string playing a large portion of the game. Either way, the Massey method includes point differentials in calculating a team's rating, so that close games contribute differently from blowouts.

In both the Colley and Massey methods, the ratings of a team's opponents contribute to the team's rating. Mathematically, this results in a system of n equations and n unknowns when computing the ratings of n teams. For March Madness, we take the results of all NCAA Division I teams, which results in roughly 350 unknowns.

To derive the linear system for the Colley method, we start with the mindset that a team's desired rating is $\frac{1+w}{2+t}$, where w is the number of games the team has won and t is the total number of games the team has played. Notice that before a season starts (when $w = t = 0$), every team's rating is 0.5. This is Laplace's Rule of Succession and, loosely, considers future games as having a 1/2 probability of a win but influences this assumption by wins and total games of past games. From this equation, we can perform some creative algebra and rewrite the total number of wins as

$$\begin{aligned} w &= \frac{w}{2} + \frac{w}{2} + \frac{l}{2} - \frac{l}{2} \\ &= \frac{t}{2} + \frac{w-l}{2}, \end{aligned} \tag{1}$$

where l is the number of games a team has lost, so that $w + l = t$. To incorporate the ratings of a team's opponents, replace the $\frac{t}{2}$ term in Equation (1) with the sum of the team's opponents' ratings. Let's think about why this is a valid substitution by rewriting $\frac{t}{2}$:

$$\begin{aligned} \frac{t}{2} &= t\left(\frac{1}{2}\right) \\ &= \frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}\ (t \text{ terms}) \qquad (2) \\ &\approx r_1+r_2+\ldots+r_t \\ &= \sum_{j=1}^{t} r_j. \qquad (3) \end{aligned}$$

Since each team's rating hovers around 0.5, we are replacing each $\frac{1}{2}$ in Equation (2) with each of the opponent's ratings, r_j (even though they're likely to differ from 0.5 in actuality). Now let's incorporate Equations (1) and (3) into calculating a team's rating:

$$\begin{aligned} \frac{1+w}{1+t} &= \frac{1+\frac{t}{2}+\frac{w-l}{2}}{1+t} \\ &\approx \frac{1+\sum_{j=1}^{t} r_j+\frac{w-l}{2}}{1+t}. \qquad (4) \end{aligned}$$

Assuming equality in Equation (4), we define a team's rating, r_i, where $\mathcal{O}_i$ represents the set of team i's opponents, as:

$$r_i = \frac{1+\sum_{j\in\mathcal{O}_i} r_j+\frac{w_i-l_i}{2}}{2+t_i}.$$

We can also represent this relationship as $C\mathbf{r}=\mathbf{b}$, where C is the Colley matrix and $\mathbf{r}$ is the ratings as a vector. The diagonal entries in the Colley matrix are $c_{ii}=2+t_i$, where t_i represents the total number of games that team i has played. The off-diagonal entries convey information about the games played between two teams. That is, if g_{ij} represents the number of times teams i and j have played each other, the off-diagonal entries in the Colley matrix are $c_{ij}=-g_{ij}$. The vector $\mathbf{b}$ contains information regarding each teams' number of wins and losses: $b_i=1+\frac{w_i-l_i}{2}$. The Colley matrix C is strictly diagonally dominant, so the Colley method produces a unique rating vector r.

Note that how this topic is introduced to students depends in part on the available time and the desired goals in a classroom. One can cover simply the mechanics of going from a list of game results to the linear system. Alternatively, one can derive the system and underscore the assumption of equality in Equation (4). Some linear systems, in applied settings, require assumptions, in part because many physical phenomenon are simply nonlinear in their behavior. Nonetheless, linear approximations abound and can offer useful and important information. With this framework, linear algebra students can connect material they're learning to the field of ranking and apply it to sports of interest.

The Massey Method. While the Colley method doesn't take point differential into account, the Massey method, as introduced in [5], is inherently derived with differences in final scores in mind. Let's define the point differential for a single game between a winning team W and a losing team L as $d_{WL}=points_W-points_L$. The Massey method derives its linear system from the assumption that $r_W-r_L=d_{WL}$, where r_W and r_L are the Massey ratings for the winning and losing teams, respectively. Applying least-squares to this over-determined system results in the normal equations, $M\mathbf{r}=\mathbf{p}$, where $\mathbf{r}$ is the ratings vector. The Massey matrix is formed by placing a team's total number of games t_i on the diagonal, $m_{ii}=t_i$ and the off-diagonal elements are again $m_{ij}=-g_{ij}$. Note, $M=C-2I$, which can be a surprising connection given the differences in the methods' derivations. For the Massey method, $\mathbf{p}$ contains information regarding a team's point differential for each game: $p_i=\sum_j d_{ij}-\sum_k d_{ki}$. The sums over j and k represent the point differential from games that team i has won and lost, respectively. Since the sum over k is subtracted from the sum over j, the sign of each entry in $\mathbf{p}$ gives an indication of a team's scoring performance over the course of a season. Unfortunately, the linear system in $M\mathbf{r}=\mathbf{p}$ has infinitely many solutions for any season. To aid in this, we add the restriction that all of the ratings sum to zero by replacing the last row of $\left[\begin{array}{c|c} M & \mathbf{p}\end{array}\right]$ with $\left[\begin{array}{cccc|c} 1 & 1 & \ldots & 1 & 0\end{array}\right]$. While infinitely many solutions are still possible (for instance when the graph of the season is disconnected), many systems will now have a unique solution. An added benefit of the Massey method is the inherent meaning in the computed ratings, which connects to the method's derivation. The difference in two teams' ratings predicts the point differential of future games. That is, if $r_i-r_j>0$, the method predicts that team i will beat team j by r_i-r_j points.

Rating Madness. Let's use a fictional series of games between teams that play NCAA Division I men's basketball. To keep the example illustrative, we will restrict our discussion to four teams: College of Charleston, Furman University, Davidson College, and Appalachian State University. We will represent the records of the teams by a graph where the vertices are teams and a directed edge points from the winning team to the losing team. Although not needed for the

Colley method, the weight of an edge indicates the difference between the winning and losing scores. In Figure 2, starting in the upper left and moving clockwise, we have the College of Charleston, Furman, Davidson, and Appalachian State.

For the Colley Method, the linear system is:

$$\begin{pmatrix} 4 & -1 & 0 & -1 \\ -1 & 5 & -1 & -1 \\ 0 & -1 & 4 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} C \\ F \\ D \\ A \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \\ 1 \\ 0.5 \end{pmatrix},$$

where C, F, D, and A correspond to the ratings for the College of Charleston, Furman, Davidson, and Appalachian State, respectively. So,

$$\begin{pmatrix} C & F & D & A \end{pmatrix} = \begin{pmatrix} 0.5000 & 0.5833 & 0.5000 & 0.4167 \end{pmatrix},$$

which leads to the ranking (from best to worst) of Furman, a tie for second between Charleston and Davidson, and Appalachian State in last place.

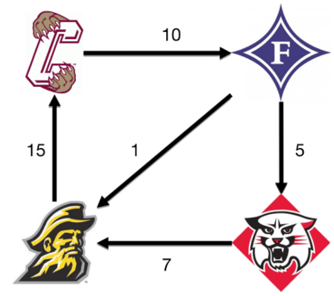

Figure 2: *A fictional season played between NCAA basketball teams. A directed edge points from the winning team to the losing team. The weight of an edge indicates the difference between the winning and losing scores.*

If one wishes to integrate scores into the method to differentiate between these teams and break the ties, the Massey method is a natural choice. For our season, depicted in Figure 2, the linear system is:

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ F \\ D \\ A \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 2 \\ 0 \end{pmatrix}.$$

Solving the linear system, we find

$$\begin{pmatrix} C & F & D & A \end{pmatrix} = \begin{pmatrix} -2.125 & -1.000 & 1.375 & 1.750 \end{pmatrix},$$

which leads to the ranking (from best to worst) of Appalachian State, Davidson, Furman, and Charleston. In comparison with the results of the Colley ranking for this collection of games, we see that the Massey ranking rewards for large wins as was the case with Appalachian State over Charleston. It also breaks the tie for second between Charleston and Davidson. This method predicts that if Charleston and Davidson were to play, Davidson would beat Charleston by $1.375 - (-2.125) = 3.5$ points.

Student Customization. There seems little madness in applying these methods to creating brackets for the NCAA basketball tournament. A bracket is formed by simply assuming a higher ranked team wins. But, this results in the entire class of students having the same bracket, which may not be desired. A simple adjustment to the system requires mathematical modeling decisions and results in personalized brackets for the students. To this end, one follows the work introduced in [2] and detailed in [4] and decides a weighting for each game. For example, one may decide the recency of a game is predictive of performance in a tournament, like March Madness. One way to measure this is to break the season into n equal parts and assign weights w_i for i from 1 to n. For example, assigning $n = 4$ breaks the season into quarters. Letting $w_1 = 0.25, w_2 = 0.6, w_3 = 1$, and $w_4 = 1.5$ assumes that a team's play increases in its predictability as the season progresses. For example, games in the first quarter of the season count as 0.25 of a game. So, it would be worth 0.25 of a win or loss for the teams involved. Similarly, in the last quarter, games count as 1.5 a win or loss. As such, games differ in their contribution to the final ratings, increasing the potential of assigning a higher rating to teams that enter the tournament with stronger records in periods of the season that one deems predictive. Further, given the underlying derivation of both the Colley and Massey methods, teams that win in such predictive parts of the season *and* do so against strong teams receive a greater reward in their increased rating. Further, the change to the linear systems is minor. Now, a game is simply counted as the weight of the game in its contribution to the linear systems. Before C and M were formed with each game counting as 1, now it is simply the associated weight. So, returning to our example of breaking a season into quarters, a game would count as 0.6 games in the second quarter of the season. As such, the total number of games becomes the total number of weighted games. The only other difference is the right-hand side of Massey. Now, the point differential in a game is a weighted point differential where d_{ij} equals the product of the weight of the game and the point differential in that game. Again returning to our example, a game in the first quarter of the season that was won by 6 points would now be recorded as a 6(0.25) or 1.5 point win.

Other aspects of a game can be weighted such as whether a team wins at home or away. Further, one might weight a team's ability to retain winning streaks. For students, the options are often exciting. Further, in the context of bracketology, one can quickly see how one's work performs against the unweighted method and also against brackets made without such methods. Both authors do this by submitting mathematically-created brackets to online tournaments, with their millions of submissions. In 2013, the first author had four students whose brackets were in the top 10% of all brackets submitted to Yahoo's online contest, the highest of which beat 96% of entries. The second author has had students in his classes produce brackets that have beaten over 97% (in 2009), 99% (in 2010), and 96% (in 2013) of the over 4 million brackets submitted each year to the ESPN online tournament. Not all brackets perform at this level and students frequently discuss the mathematical decisions that led to such results. One may also wish to compare such methods to a sample of brackets produced with coin flipping or even using the ratings produced with winning percentage, as appears in [1].

Adding Weight to the Madness. Let's return to the NCAA basketball example shown in Figure 2 to investigate the effect of incorporating recency into the Colley and Massey calculations. In Figure 3(a), the weights of the edges denote the day of the season in which a game was played. Let's take the length of the season to be seven days and weight each game such that games in first half of the season are weighted by 1/2 and in the second half games count as a full game.

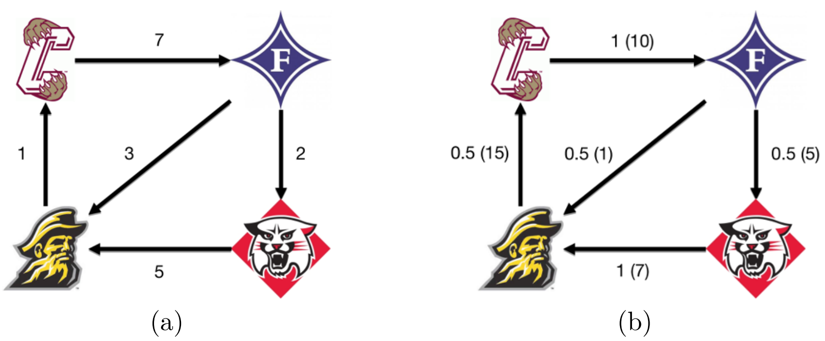

Figure 3: *A fictional season of games played between four NCAA basketball teams. A directed edge points from the winning team to the losing team. The weight of an edge in (a) indicates the day of the seven-day season that the game was played. The same season is given in (b), but now the edges are labeled with the weight for each game and the difference in the winning to losing score is given in parentheses.*

The weighted Colley method results in the linear system

$$\begin{pmatrix} 3.5 & -1 & 0 & -0.5 \\ -1 & 4 & -0.5 & -0.5 \\ 0 & -0.5 & 3.5 & -1 \\ -0.5 & -0.5 & -1 & 4 \end{pmatrix} \begin{pmatrix} C \\ F \\ D \\ A \end{pmatrix} = \begin{pmatrix} 1.25 \\ 1 \\ 1.25 \\ 0.5 \end{pmatrix}.$$

Solving the linear system, we find

$$\begin{pmatrix} C & F & D & A \end{pmatrix} = \begin{pmatrix} 0.5581 & 0.5065 & 0.5419 & 0.3935 \end{pmatrix},$$

which leads to the ranking (from best to worst) Charleston, Davidson, Furman, and Appalachian State. Incorporating recency also serves to break the tie present in the uniformly weighted Colley calculation.

Now, we turn to Massey for the same series. Figure 3(b) displays the weight for each game (derived from its day played) and the difference between the winning and losing scores is given in parentheses. Now, the linear system becomes

$$\begin{pmatrix} 1.5 & -1 & 0 & -0.5 \\ -1 & 2 & -0.5 & -0.5 \\ 0 & -0.5 & 1.5 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ F \\ D \\ A \end{pmatrix} = \begin{pmatrix} 2.5 \\ -7 \\ 4.5 \\ 0 \end{pmatrix},$$

resulting in the ratings

$$\begin{pmatrix} C & F & D & A \end{pmatrix} = \begin{pmatrix} -0.0476 & -2.8095 & 2.3810 & 0.4762 \end{pmatrix},$$

and the ranking (from best to worst) Davidson, Appalachian State, Charleston, and Furman. Notice how the effect of Appalachian State's large win over Charleston was lessened by weighting recency. Once again, if Charleston and Davidson play a game, this weighted Massey method suggests that Davidson would beat Charleston by $2.3810-(-0.0476) = 2.4286$ points.

Summary. The Colley and Massey methods provide an engaging topic for linear algebra students that underscores the applicability of mathematics in ranking methods, in particular to ranking sports teams. Depending on the prerequisites of the course and the scope of student assignments or projects, this central idea can be tailored to suit many different situations. One could assign a project where students have to program both the Colley and Massey methods to create ratings and rankings for a collection of teams or one could provide an interface for students to enter choices, such as which rating system to use and options for weightings, and then compare the methods' performances. In the end, students see an application of linear algebra to a popular topic in the United States or, depending on the sport, around the world.

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ARTICLES

History of Research in Linear Algebra in Ireland

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Currently, linear algebra is a major topic of research in Ireland. In this article, we describe how this occurred. We put the development in the context of the underlying university structure.

1. Irish universities. The University of Dublin, also called Dublin University and more often, Trinity College Dublin (TCD), was founded by Queen Elizabeth I in the sixteenth century. The most famous mathematician associated with it is Sir William Rowan Hamilton, who studied and spent his career there. There are several other well-known mathematicians associated with it, for example: James McCullough; George Salmon [4], whose book on geometry has been influential into modern times; and William Snow Burnside, who with William Panton wrote a well-known treatise on *The Theory of Equations*. The matrix theorist James Joseph Sylvester has a primary degree from TCD, but that represents a solution to recognizing his achievements as a student at Cambridge University, since, as a Jew, he was unable to sign a declaration of fidelity to Protestant principles required for graduation from Cambridge at that time. The famous group theorist, William Burnside, was awarded an honorary doctorate by TCD. It is surmised that the proposal to offer this award came with some quirkiness from his (almost) name-sake William Snow Burnside. Henry John Stephen Smith of the Smith canonical form and companion matrix, was, like Hamilton, born in Dublin, but spent his academic career at Oxford.

The next university institution established in Ireland was St. Patrick's College in Maynooth, a town just west of Dublin. This was instituted as a seminary for the training of Catholic priests at the time of the French revolution. It followed a period of religious discrimination which made it illegal to train Catholic priests in Ireland and consequently many seminarians went to France to train. Some historians, perhaps cynically, say that the decision to found St. Patrick's College represented a judgement that the ideas of the revolutionaries were the greater threat to the establishment. In the 1970s, St Patrick's College opened to general students. The seminary, which has the status of a pontifical university, retains this title, while the large secular university is now known as National University of Ireland, Maynooth (NUIM).

In the mid-nineteenth century, three new university institutions called Queen's Colleges were established; these are now known as Queen's University, Belfast (QUB), University College, Cork (UCC) and National University of Ireland, Galway (NUIG). In the late nineteenth century and early twentieth century, a scheme was in place by which some talented mathematics graduates of Oxford or Cambridge Universities (Oxbridge) spent a few years on the faculty in NUIG while waiting for a fellowship to become vacant at an Oxbridge college. One of the most well-known of these is Thomas John L'Anson Bromwich associated with the Laplace transform.

George Boole, of Boolean Algebra fame, was the first professor of Mathematics at UCC. Despite being at UCC while Hamilton was in Dublin, they had little interaction [6]. In recent times, UCC is associated with research in analysis more than in algebra.

Later, in the nineteenth century, a group of academics at Oxford University converted from Anglicanism to Roman Catholicism; the most famous of these is now known as Cardinal Newman. He had written widely on his concept of an ideal university, and he decided to try out his theories by founding a Catholic university in Ireland. While his institution failed, it led, by a very convoluted process, to the foundation of the (non-sectarian) University College Dublin (UCD), which is now the biggest university in the country with well over 20,000 students. In 1908, the UK government instituted an entity called the National University of Ireland (NUI), (somewhat similar in structure to the current University of California), with constituents UCC, NUIG and UCD. As already noted, NUIM became a constituent more recently.

Since the Anglo–Irish treaty in 1922, two further universities have been established in the Republic of Ireland: University of Limerick (UL) and Dublin City University (DCU). In Northern Ireland, the University of Ulster (UU), with campuses in Jordanstown (UUJ) and Coleraine (UUC), was established.

After independence, the Republic of Ireland was a very poor country with a predominantly rural economy and little resources to fund education. The government viewed the main purpose of universities to be the training of the teachers, doctors and engineers needed; research had no funding per se, and university academics had very large teaching loads which made little time available for research. Also, research was not a major determinant in promotion, and within Ireland, excellence in it was principally recognised through membership in the Royal Irish Academy (RIA). At an earlier time, William Rowan Hamilton famously discovered and scrawled a presentation of the quaternions on a bridge as he walked to a meeting of the RIA.

In 1940, the prime minister, Eamon de Valera, advised by Sir Edmund Whittaker and Garrett D. Birkhoff, established the research institute, Dublin Institute for Advanced Studies (DIAS). DIAS was meant to be a mini-version of the Institute for Advanced Studies at Princeton, with, on the science side, schools of Cosmic and Theoretical Physics. Erwin Schrödinger was an early member and Olga Taussky Todd occasionally visited him in Dublin during her time in Belfast. Cornelius Lanczos spent several years at DIAS. De Valera himself trained as a mathematician and taught briefly at NUIM.

During the Second World War, Olga Taussky-Todd spent some time in Belfast and worked with a student, Ernest Best, there, on group theory [2]. Best later became a theologian. Olga's husband, John Todd, was a native of Northern Ireland.

Hans Schneider, after obtaining his Ph.D. under the supervision of Alexander C. Aitken in Edinburgh, spent the period 1952–59 in QUB as a lecturer. He published several papers before moving to the US.

Some research in algebra prior to 1970 occurred in NUIG, first with Martin J. Newell, who was widely known for work on combinatorial algorithms in the representation theory of the finite symmetric groups. In order to effectively pursue research in a small department, it is often considered better to hire people with a common research interest as far as practicable; the so-called "critical mass" principle. Under the leadership of the head of the mathematics department, Seán Tobin, beginning in the 1960s, many of those hired were group theorists. NUIG established a concentration of research in group theory. This continues up to the present and has strong connections with the topic of this paper.

2. Beginnings of linear algebra research. The 1960s was a period of sustained economic growth in the Republic of Ireland. The government began to place emphasis on education as a means to build a more industrialised society and help to reduce emigration. Free second-level education for all students was instituted. This meant that far more students than before now qualified for, and aspired to enter university. Universities had to be expanded, both in capital and staff size. Several new academics were hired, and while teaching loads were still very high, there was a little more scope for doing research.

Fergus Gaines did his undergraduate training in UCD and obtained a doctorate under the supervision of Olga Taussky-Todd at Caltech and then returned to UCD as a lecturer in 1966. In TCD, there were a number of active researchers: for example, Trevor West in functional analysis, and David Simms in geometry related to quantum theory. There were good connections between the mathematicians in the two institutions. However, Fergus wished to continue his work on finite dimensional linear algebra and found that there were no colleagues whose primary interest was in this area. I was appointed to the department in 1968, having just completed a doctorate in finite group theory under the supervision of Walter Ledermann at Sussex University. When I came to Dublin and found no other finite group theorists there, I decided to learn matrix theory. I used notes of courses and seminars that Fergus had attended at Caltech, so that I could have serious discussions with him on research, though I continued to do research in group theory and ring theory. However, after spending a sabbatical in 1972–73 at Northern Illinois University and attending seminars and courses in ring theory and the algebraic theory of semi-groups in which matrix techniques were often employed, I found myself drawn to questions on matrix algebras and simultaneous triangularization, topics on which Fergus had worked with Olga. This led to the beginning of a very fruitful and influential correspondence with Olga in which she told me about unsolved questions and the related literature. Most of my research, some jointly with Fergus, was based on this.

Because of the scarcity of funds available for research, there were few conferences in mathematics in Ireland at that time. At Christmas and Easter, DIAS hosted small conferences and these were the main meeting points for researchers throughout the country. John Miller of TCD, a former student of Gil Strang at MIT, organised some meetings on numerical analysis and boundary layer problems related to semi-conductors. In NUIG, a major conference on group theory and computation took place in 1973 and led to the establishment of a series of annual international conferences, called "Groups in Galway," which continue to this day.

In 1978, an Irish Mathematical Society was founded, aiming, in particular, to encourage greater interaction between researchers in the subject in Ireland by providing a structure for organizing workshops and conferences, as well as disseminating information on new discoveries. (Those interested can find an account of this situation in greater detail in the recent issue of the *Mathematical Proceedings of the Royal Irish Academy* dedicated to the memory of Trevor West [5].)

Fergus Gaines proposed that we should hold a small matrix conference at UCD. Despite having a very tiny amount of funding, this conference was very well-attended locally, and attracted several leading figures. Charlie Johnson, Frank Uhlig and Harald Wimmer were among those at the first meeting. A bigger meeting took place in 1980; see the associated poem by Steve Barnett and Henry Power [1]. A report of the 1984 meeting appears in [3]. These meetings helped to secure the image of linear algebra research as a significant one in Ireland. In UCD, Fergus Gaines, David Lewis (whose primary work is on the abstract theory of quadratic forms), Rod Gow (who, like me, was trained as a group theorist), John Hannah (then a postdoctoral scholar with primary interest in ring theory) and I were able to discuss linear algebra problems of mutual interest. David Lewis stimulated my interest in the theory of linear preservers and this led to work with LeRoy Beasley, Charlie Johnson, Chi-Kwong Li and Raphi Loewy. LeRoy spent his sabbatical year 1985–86 at UCD and provided great inspiration and direction. During his stay in Ireland, a number of his collaborators, including Norm Pullman and Larry Cummings, visited him and added to the activity. The meeting at the Technion in Haifa in 1986 further consolidated links with researchers there. Collaboration with LeRoy Beasley, Avi Berman and Raphi Loewy still continues. Bryan Cain also visited UCD for a short period and this led to investigating questions on integer matrices, which formed the basis of the thesis of my Ph.D. student, Raja Mukherji (2009). Also, Trevor West's functional analysis group at TCD had a number of visitors, including, in particular, Bernard Aupetit, Heydar Radjavi and Jaroslav Zemánek, whose work related to simultaneous reduction and idempotents. This also influenced the direction of our research.

Because of the lack of funding for graduate students, top performing undergraduates in the Republic of Ireland aspiring to gain a Ph.D. in Mathematics generally went to the UK or the USA for graduate study. With the increased interest in research in Ireland, a number of mathematicians began to develop research programmes to enable students to do their doctorates here. In UCD, the analyst Seán Dineen blazed a trail with some Ph.D. students in the 1980s. The first Ph.D. student in linear algebra was my student Susan Lazarus (1991). Susan, who had obtained a Masters degree from University of California, Berkeley, worked on the structure of commutative matrix algebras, and, in particular, made an interesting contribution to the still unanswered question of whether a commutative algebra of $n \times n$ matrices generated by three elements can have dimension greater than n.

In the early 1990s, Hans Schneider invited me to serve on the selection committee for the awards of the first two Hans Schneider prizes. In order to assess the work of Miroslav Fiedler on the symmetric nonnegative inverse eigenvalue problem (SNIEP) and the work of Mike Boyle and David Handelman on the general nonnegative inverse eigenvalue problem (NIEP), I found it necessary to learn the literature in this area. This theory is the topic of the Ph.D. theses of my students Eleanor Meehan (1998) and Anthony Cronin (2012). It also features in the Ph.D. of Robert Reams (1994) who is now in SUNY at Plattsburg; the other topic of his thesis is integer matrix theory. Also, most of my research in recent years and my collaboration with Raphi Loewy and Helena Šmigoc is on the NIEP and SNIEP.

The first Portuguese international linear algebra conference organised by Graciano de Oliveira, took place in the early 1980s. This led to strong links between the linear algebraists in Ireland and Portugal, including several mutual visits by Graciano and Joao F. Queiró in Coimbra and José Dias Da Silva in Lisbon [7].

3. The effect of the economic boom and bust. The period 1998–2008 was one of great economic activity in Ireland. There was a huge influx of (mostly U.S. based) multinational corporations in the IT, biotechnology and financial sectors. In particular, IBM, Intel, Microsoft and Hewlett-Packard built major facilities not far from Maynooth. Apple and several other IT hardware and software companies established plants in Cork, Limerick, Galway and several other towns. Twenty-five of the top fifty financial institutions in the world established a presence in a new financial services centre in Dublin Docklands. Later, Google, Facebook and other social media companies established facilities nearby. The chemical industry, based near Cork, greatly expanded, and the primary international companies producing medical devices set up plants in Galway and many other areas. Soon, there was full employment. Many people somehow convinced themselves that the era of poverty was over forever. For the first time in history, instead of Irish people leaving to work abroad, the country became the target of a wave of immigration, especially from Eastern European countries which had recently joined the EU. This led to an increased demand for housing. Prices started to rise rapidly, leading to over-borrowing, uncontrolled financial risk-taking, followed by the well-known collapse of the economy, and the current austerity to deal with repaying debts and repairing the banking system.

However, while the boom lasted, the government had budget surpluses and the opportunity, which it took, to improve the university sector. Initially, low wage costs in Ireland was a factor in attracting labour intensive parts of the IT industry. As the economy grew, salaries quickly increased and such work moved to less expensive locations in Eastern Europe and the Far East. Thus, the level of sophistication of the Irish components of the multinationals had to increase. The government was persuaded that, in attracting such high-level industry, it was important to be able to offer multinationals a local environment in which they would find relevant high-level expertise as in the US or in other countries, such as the UK and Israel, competing for the same investment. So, as well as greatly increasing the number of places available for students in the third-level sector, the government initiated a major investment in research and in the production of more Ph.D.s. Science Foundation Ireland (SFI) was established to administer a system of research supports, to fund of projects proposed by academics, and to enable the hiring of graduate student as well as postdoctoral fellows in the area of IT and biotechnology. Some experienced members of the NSF were recruited to get the system up and running. Also, an Irish Research Council for Science, Engineering and Technology (IRCSET) was empowered to support graduate students and post-docs in these areas. Related to linear algebra, Helena Šmigoc initially came to Ireland supported as a postdoctoral scholar by SFI.

A number of research institutes were established, the most relevant to this account are the De Brún Centre in NUIG, the Shannon Institute in UCD and, most of all, the Hamilton Institute in NUIM.

The De Brún Centre for Computational Algebra was established in 2008 with support from SFI and the Marie Curie Foundation, as well as NUIG itself. As already recounted, from the 1960s onwards, NUIG was the university primarily associated with research in group theory and algebraic computation. It was a well-recognised node for the development of the GAP computer algebra system. The Centre's mission includes support for research in algebra, geometry, computation and their interdisciplinary applications. It supports graduate students in these areas and research programmes that have a teaching component delivered by international experts. The Centre has strong connections with the linear algebra researchers in Galway and, in 2012, hosted a 5-day workshop entitled *Linear Algebra and Matrix Theory: connections, applications and computations*. The principal organisers were Niall Madden and Rachel Quinlan. The main speakers, Peter Brooksbank, Richard Brualdi, Iain Duff, Carlos M. da Fonseca and Charlie Johnson, delivered short courses. Also, an Irish linear algebra society was established mainly aimed at organizing meetings of researchers, including graduate students, throughout the country. The first meeting took place immediately after the De Brún workshop.

As a result of a competition organised by SFI, the Claude Shannon Institute, under the leadership of Gary McGuire, was established in UCD in 2008. It is now part of the Computational and Adaptive Systems Laboratory (CASL). In algebra, its primary interest is in coding theory and cryptography and has a strong theoretical content. To give a linear algebra related example, research on possible dimensions of constant rank matrix subspaces, is being carried out by Rod Gow, some of it in collaboration with Rachel Quinlan in NUIG, and their students. The work of the De Brún and Shannon centres has led to a widening of the parts of linear algebra being researched in Ireland to include combinatorial matrix theory as well as linear algebra and geometry over finite fields.

The Hamilton Institute in NUIM, which came into being in 2001, was the first of the mathematically-related research institutes created during the boom. Its foundation and success owes much to the vision, as well as the managerial and persuasive skills, of Doug Leith and Robert Shorten. It describes itself as a multidisciplinary research centre that builds bridges between mathematics and its applications in ICT and biology. A distinguishing feature is its commitment to working at the interface between mathematical theory and application. Avi Berman and Volker Mehrmann serve as members of its scientific advisory board. Matrix theoretic methods pervade most of its work. For example, publications in linear algebra journals including *LAA* have resulted from its work on the design and stability of switched systems and internet congestion control protocols. Its Linear Algebra and Matrix Analysis group has researchers in combinatorial matrix theory, max algebra, nonnegative matrices, stability theory and dynamics. Designed to speed up the development of research and ensure that it lay at the cutting edge, one SFI initiative established a number of short-term distinguished research positions called Stokes Professorships. These international experts, in areas related to ICT and biotechnology, would be enticed to spend up to five years in an Irish research institute. The professorships would inspire the people there and via seminars, meetings, etc., inform workers throughout the country. The linear algebra effort was greatly assisted by the presence of Steve Kirkland as Stokes Professor in the Hamilton Institute for the period 2009–2013. In addition to frequent seminars and short courses to graduate students by experts, beginning in 2004, the Hamilton institute has held biennial workshops in linear algebra and its applications. Many of the leaders in the field have participated in these events. Volume 12 of *ELA* is devoted to the proceedings of the first meeting.

Funding for mathematical projects has become more difficult during the recession, because the government has prioritised support for projects with the potential for job creation in the short term. However, the Hamilton Institute has had its funding renewed recently.

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Reducing the Effects of Unequal Number of Games on Rankings

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Abstract. Ranking is an important mathematical process in a variety of contexts such as information retrieval, sports and business. Sports ranking methods can be applied both in and beyond the context of athletics. In both settings, once the concept of a game has been defined, teams (or individuals) accumulate wins, losses, and ties, which are then factored into the ranking computation. Many settings involve an unequal number of games between competitors. This paper demonstrates how to adapt two sports rankings methods, the Colley and Massey ranking methods, to settings where an unequal number of games are played between the teams. In such settings, the standard derivations of the methods can produce nonsensical rankings. This paper introduces the idea of including a super-user into the rankings and considers the effect of this fictitious player on the ratings. We apply such techniques to rank batters and pitchers in Major League baseball, professional tennis players, and participants in a free online social game. The ideas introduced in this paper can further the scope that such methods are applied and the depth of insight they offer.

1. Introduction. Ranking is an important mathematical process that informs decision-makers, whether they be consumers finding webpages returned from queries to search engines, the Major League Baseball (MLB) determining who will play in its playoffs, the Bowl Championship Series (BCS) selecting college football teams for the holiday bowl games, or the Association of Tennis Professionals (ATP) deciding on tournament invitations and seeding. Underneath such rankings are mathematical algorithms. Google uses variations of the classic PageRank algorithm, MLB leans on winning percentage, the BCS aggregates rankings obtained from human opinion and mathematical calculation, and the ATP awards ranking points in a way that rewards players who compete in a lot of tournaments in addition to playing well in them. This paper will demonstrate one way to adapt two specific sports ranking methods to settings in which teams or individuals play an unequal number of games.

What contexts arise where teams play unequal numbers of games? Most team sports design seasons with relative uniformity in the number of games played. For example, a season of NCAA Football Bowl Subdivision games is typically 12 to 14 games. Contrast this with the 2011 ATP Tour season which featured 304 players competing in 2566 completed matches.[5] Figure 1 displays a graph of the player by player matrix of the 2011 ATP Tour. In this matrix the players are listed in order of their ATP ranks (at the end of 2011) and entry a_{ij} equals the number of times player i played player j. In the figure, nonzero elements (nz) are colored blue. As we see, players differ widely in the number of matches played.

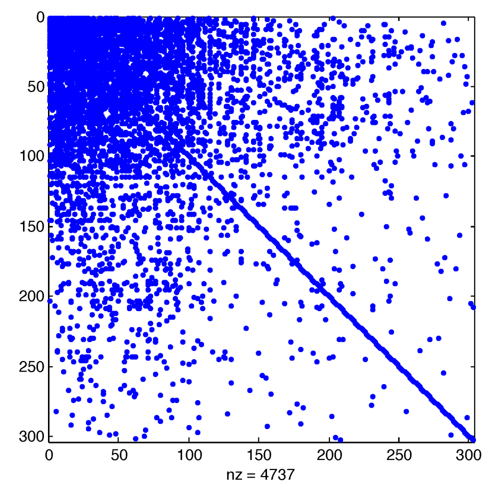

Figure 1: *A player by player matrix of the 2011 ATP Tour where a_{ij} contains a 1 if player i played player j and 0 otherwise. In the color-coded matrix above, nonzero elements are colored blue.*

Weaker players, for example, will compete less (and therefore have fewer blue pixels in their row) as they either do not qualify for many top-level tournaments without a special invitation from the tournament organizers, or when they do qualify, they are often defeated in the first round. Under these circumstances, 73 of the 304 competitors on the 2011 ATP Tour played only one match while the median number of matches per player was 6. This inherent disparity in the number of matches directly impacts the Colley and Massey ranking of professional tennis players.

To demonstrate this effect consider Jose Acasuso who in 2011 was nearing the end of a solid, if unspectacular, professional tennis career. The Argentine had an ATP ranking as high as 20$^{\text{th}}$ in 2006, but in the first few months of 2011 his rank had fallen into the 200s, and he was competing in tournaments on the second-tier Challenger Tour or trying to qualify for major tournaments he would have been invited to just a few years earlier. The one exception came in February, when the local organizers of the Buenos Aires ATP tournament awarded him a wild card entry. Acasuso, playing in his home country and on his favorite surface (clay), defeated Alexandr Dolgopolov and Pablo Cuevas in the first two rounds before losing in the quarterfinals to Nicolas Almagro. All three of his opponents were good players,

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[5]These figures include only completed matches. Matches won by walkover (i.e. forfeit) or because a player retired during a match were not counted. Also, matches at team competitions (Davis Cup and World Team Cup) were not counted.

ranked 29$^{\text{th}}$, 64$^{\text{th}}$, and 13$^{\text{th}}$ by the ATP's ranking system at the time of the tournament. Those were Acasuso's only ATP tournament matches of the year.

Both the Colley and Massey methods are designed to consider the strength of the opposition when calculating a player or team rank, and Acasuso's three matches provide a "perfect storm" for both algorithms. By winning twice as many matches as he lost (2-1) against solid opposition, Acasuso ended the year ranked 33$^{\text{rd}}$ by Colley and 14$^{\text{th}}$ by Massey.

This dynamic is not new. In fact, it is often quite common in the statistics that we use to rank athletes. Let us now consider MLB. Also in 2011, Joey Gathright made only one plate appearance for the Boston Red Sox. He made the most of it with a walk, a stolen base, and an eventual scored run. While Gathright was clearly proficient with his one plate appearance; it is difficult, if not impossible, to claim he was the best batter in the major leagues during 2011. Likewise, pitcher Jarrod Parker had a 0.00 earned run average during the 2011 season for the Arizona Diamondbacks. While this is impressive, he only faced 22 batters to achieve this ERA. There are likely dozens of major league pitchers who faced 22 batters in a row without allowing an earned run to cross the plate. Parker had an ERA below any Cy Young award winning pitcher. Yet, this ERA statistic, which is often used to compare pitchers, is not as meaningful in this case. Neither example (Parker nor Gathright) gives sufficient information in which to judge whether or not either athlete can sustain his performance. Most would agree that a batting average of 0.350 is more impressive if it can be maintained over 400 at-bats than over 40 at-bats. In fact, Rule 10.22 of the MLB Charter [1] states "The individual batting, slugging or on-base percentage champion shall be the player with the highest batting average, slugging percentage or on-base percentage, as the case may be, provided the player is credited with as many or more total appearances at the plate in league championship games as the number of games scheduled for each club in his club's league that season, multiplied by 3.1 in the case of a Major League player." If 162 games are scheduled in a given year, to win one to these batting titles a player would need $162 \cdot 3.1 = 502$ plate appearances. If no such cut-off is made for the Colley method Joey Gathright is ranked as the top batter and Jarrod Parker as the second-best pitcher in 2011.

This paper introduces adaptations to the Colley and Massey methods that aid in contexts where players play unequal numbers of games. Section 2 reviews the derivation of both the Colley and Massey methods and demonstrates how playing a small number of games can be an advantage. Section 3 introduces the idea of including a super-user into the rankings. Section 4 applies the adaptation of the Colley and Massey methods to ranking batters and pitchers in Major League Baseball. Sections 5 and 6 look at the effect of the super-user and how to choose the minimum cutoff at which point the super-user is not included for a player's ranking. Section 7 applies the super-user method to professional tennis. Finally, in Section 8 we look at the need for such methods in applications beyond sports. The concluding remarks summarize the method and its implications.

2. Ranking with the Colley and Massey Methods. This paper will focus on adapting two common sports ranking methods - the Colley method and Massey method. Both rely on linear systems to create their ratings. Both methods are also used by the Bowl Championship Series to aid in its rankings of NCAA FBS teams.

The Colley Method modifies winning percentage to create a linear system

$$C\mathbf{r} = \mathbf{b}, \tag{5}$$

which produces a rating for each team. For a full description and derivation of the Colley Method, see [12]. The linear system can be formed by the following definition of the rating r_i of team i

$$r_i = \frac{1 + (w_i - l_i)/2 + \sum_{j \in O_i} r_j}{2 + t_i}, \tag{6}$$

where w_i (l_i) represents the number of winning (losing) interactions for team i, t_i represents the total number of games involving team i, and O_i represents the set of opponents of team i. Loosely interpreted, this formula computes a team's rating as its winning percentage plus the average rating of its opponents. Equation (6) gives a row-wise description of the linear system in (5), where $\mathbf{r}$ is the vector of team ratings, $\mathbf{b} = [b_i]$ is a vector such that $b_i = 1 + \frac{1}{2}(w_i - l_i)$ and $C = T - A$, where $T = [t_{ij}]$ is a diagonal matrix in which the diagonal entries $t_{ii} = 2 + t_i$ and A is a matrix in which the (j, k)th entry is the number of times that the j and kth teams play. Solving this system of equations provides a rating for each team encoded in the ratings vector $\mathbf{r}$. This rating can be sorted to provide a ranking (or relative standing) for each team.

Another ranking approach, the Massey Method, is similar to the Colley Method in that it sets up a system of linear equations whose solution provides a rating for players. This linear system

$$M\mathbf{r} = \mathbf{p}, \tag{7}$$

is derived from the assumption that the teams' ratings will describe the point differential in their competitions. For example, if teams i and j compete with team i winning by p^{ij} points, then

$$r_i - r_j = p^{ij}.$$

However, this system is usually inconsistent and the method of least squares is employed to find a “best fit” solution. For a full derivation of the Massey Method, see [12]. Similar to the Colley Method, a team’s rating in the Massey Method can be loosely interpreted as equal to the team’s average point differential plus the average of the team’s opponents’ ratings. Again, solving this system of equations gives a rating vector that can be sorted to provide a ranking of participants.

3. Limiting the Effects of Unequal Numbers of Games. One common solution to this problem of disparate numbers of games is to simply drop athletes (or teams or items) with fewer than some minimum number of games. For example, the FIDE chess federation has required that players have at least 30 matches before being rated. While the minimum cutoffs are usually carefully chosen for the particular application, this approach is a bit arbitrary. Further, a ranking can be sensitive to the chosen cutoff. That is, a cutoff of 20, as opposed to 30, may produce quite a different ranking. Further, excluding players from rankings in this manner diminishes their contributions and reduces the visibility of the player. We seek to penalize these players without excluding them.

In this paper, we propose an alternative to the minimum cutoff approach for dealing with the disparate number of games issue for the Colley and Massey Methods. Our approach utilizes a super-user, a dummy team (athlete or item) that plays and beats every actual team that it plays. We force teams to lose to the super-user in every case, rather than a combination of wins and losses, in order to penalize teams that are perhaps artificially inflated in rank due to good performance in only a few games. Teams below the minimum cutoff face the super-user multiple times so that, with the addition of these artificial games, there is much more parity in the number of games that teams play. Because the super-user’s ranking is artificially generated and has no practical meaning, it is removed from consideration so that each actual team’s rank is incremented accordingly.

4. Ranking Baseball Players. In order to test the super-user idea in a scenario with a multitude of unequal numbers of games, we apply it to ranking pitchers and batters in the MLB. Note that the point of this exercise is not to find the best way to rank pitchers and batters but rather to explore the effects of adding a super-user to the Colley and Massey ranking methods. The Colley method can be used to rank individual players, pitchers and batters, in a similar way that it is used to rank teams. We formulate the interactions of batters and pitchers as a multi-edge bipartite (two-mode) network disregarding the interactions that National League Pitchers have with one another. This approach is similar to the mutually-antagonistic network formulation in Saavedra, et al. [14]. Bipartite networks are formulated using two disjoint sets of nodes, P (pitchers) and B (batters), and an edge connects a node x in P with a node y in B for each instance for which batter y had a plate appearance against pitcher x during the season. This network can be encoded in the Colley Matrix C where element C_{ij} equals the negative of the number of times pitcher i faced batter j during the season. Each batter-pitcher interaction results in either a hit, a walk, or an out. We ignore interactions that involve runners getting thrown out stealing to end innings, sacrifices, or that result in hit-by-pitches, and we strictly look at interactions that result in a hit, walk, or out. If the result is a hit or a walk, the hitter is awarded a “win” in that interaction and the pitcher a “loss.” If the result is an out, the pitcher is given a “win” and the batter a “loss.” From this, we can form the appropriate $\mathbf{b}$ vector, and the Colley Method, as described in the introduction, can be used to rank pitchers and batters. In using the Massey Method, the notion of a point differential p^{ij} in the interaction between pitcher i and batter j needs some explanation. There may be several ways to assign points to a pitcher-batter interaction. Here, we use the Bill James-defined RUE value [11], or runs to end of inning, to define a score for each possible plate appearance. The possible events and their corresponding score value are: generic out (0.240), strikeout (0.207), walk (0.845), single(1.025), double (1.132), triple (1.616), and home run (1.942). The batter receives the corresponding score for each walk, single, double, triple, and home run that he achieves or otherwise, he receives a score of 0. The pitcher receives a score of 0 if the batter gets a hit or a walk. Otherwise, for the pitcher, we calculate the average number of runs per inning for each year (2002-2011), call this value E and subtract the plate appearance outcome value (either 0.240 for an out or 0.207 for a strikeout) from the value E. The value of E varies from year to year. In 2011, $E = 0.478$. So, if a pitcher wins an interaction with a strikeout, he is given a value of $E - 0.027 = 0.478 - 0.207 = 0.271$. In this way the pitcher gets a score indicating how many runs were “saved” from the interaction. Using these values we define the values p^{ij} for the Massey Method.

As noted above, the number of plate appearances (or innings pitched) for baseball batters (or pitchers) can vary to a great degree, which can cause the Colley and Massey results to be skewed. For example, pinch hitters are usually called upon in later innings, if at all, to perform in certain situations. Players get injured or are called up from or sent down to the minors. Some teams assign catchers to only certain pitchers, and thus those play only when his assigned pitcher starts the game or he is called to pinch hit. Pitching rotations fluctuate throughout the year and relief pitchers have quite a bit of variability as to when they get into the game. All of these issues contribute to some players getting more plate appearances or innings pitched than others. To alleviate this, we set up two super-user nodes, the SuperBatter node and the SuperPitcher node. If a batter has $x < 300$ plate appearances during the year, he is forced to lose to the SuperPitcher $300 - x$ times. Here the cutoff of 300 was chosen to be roughly the mean/median numbers of plate appearances for batters. In 2011, the mean was 274 and the median was 291. Later, we let this vary. Likewise, if a pitcher has $y < 400$ batter interactions during the year, he is forced to lose to the SuperBatter $400 - y$ number of times. In each interaction with the SuperPitcher, the batter loses the interaction with an out, and the SuperPitcher is credited with $E - 0.240$ points. In each interaction with the SuperBatter, the SuperBatter wins the interaction with a single and is credited 1.025 points. These outcomes can change as the user sees fit though. Tables 1 and 2 show the Colley rankings of batters and pitchers both using and not using the super-user nodes for the 2011 season.

Without super-user				**With super-user**			
Rank	Batter	OBP	Interactions	Rank	Batter	OBP	Interactions
1	Joey Gathright	1.00	1	1	Miguel Cabrera	0.448	680
2	Esteban German	0.462	12	2	Jose Bautista	0.447	645
3	Gil Velazquez	0.429	6	3	Mike Napoli	0.414	427
4	Logan Schafer	0.500	4	4	Adrian Gonzalez	0.410	704
5	Antoan Richardson	0.5	4	5	Joey Votto	0.416	709
6	Russ Canzler	0.400	4	6	Lance Berkman	0.412	580
7	Miguel Cabrera	0.448	680	7	David Ortiz	0.398	603
8	Jose Bautista	0.447	645	8	Prince Fielder	0.415	676
9	Chris Parmelee	0.443	88	9	Matt Kemp	0.399	676
10	Mike Napoli	0.414	427	10	Dustin Pedroia	0.387	721
11	Adrian Gonzalez	0.410	704	11	Alex Avila	0.389	537
12	Jesus Montero	0.406	68	12	Paul Konerko	0.388	620
13	Joey Votto	0.416	709	13	Todd Helton	0.385	480
14	Cedric Hunter	0.400	5	14	Michael Young	0.380	678
15	Lance Berkman	0.412	580	15	Ryan Braun	0.397	621
16	David Ortiz	0.398	603	16	Jose Reyes	0.384	580
17	Alejandro De Aza	0.400	169	17	Victor Martinez	0.380	586
18	Leonys Martin	0.375	8	18	Carlos Beltran	0.385	591
19	Yonder Alonso	0.398	98	19	Jacoby Ellsbury	0.376	712
20	Hector Gomez	0.429	7	20	Nick Swisher	0.374	621
21	Prince Fielder	0.415	676	21	Alex Gordon	0.376	678
22	Matt Kemp	0.399	676	22	Matt Holliday	0.388	506
23	Dustin Pedroia	0.387	721	23	Chase Headley	0.374	433
24	Alex Avila	0.398	537	24	Casey Kotchman	0.378	548
25	Cole Gillespie	0.429	7	25	Yunel Escobar	0.369	574

Table 1: *Top* 25 *Colley batters with and without super-user.*

An alternative way of dealing with this issue, which we refer to as the Minimum-Game Method, is to allow every batter and pitcher to contribute to the network and to the ratings of each player, but only rank those pitchers achieving at least 400 plate appearances and those batters achieving at least 300 plate appearances. It should be noted that the results from this method are similar to the results using the super-user method. In fact, the first major discrepancy between these two Colley methods occurs at rank 40 in the Super-User Method, where Jesus Guzman appears despite having only 271 plate appearances. In 2011, Guzman appeared in 76 games, mostly as a pinch hitter. Thus we see that the Super-User Method ranks pinch hitters, players who suffered lengthy injuries, and catchers who alternate with pitchers, while the Minimum-Game Method does not. Pitchers Josh Johnson (Super-User Colley Rank 40) and Stephen Strasburg (Super-User Colly Rank 109) were having banner years until season-ending injuries. The Super-User Method allows these contributions to be noted whereas the Minimum-Game Method does not.

Without super-user				**With super-user**			
Rank	Pitcher	OBPA	Interactions	Rank	Pitcher	OBPA	Interactions
1	Justin Verlander	0.238	964	1	Justin Verlander	0.238	964
2	Jarrod Parker	0.227	20	2	Jered Weaver	0.257	913
3	Stephen Strasburg	0.193	85	3	Dan Haren	0.256	932
4	Jered Weaver	0.257	913	4	Josh Beckett	0.258	746
5	Dan Haren	0.256	932	5	Josh Tomlin	0.269	656
6	Josh Beckett	0.258	746	6	Josh Collmenter	0.266	609
7	Josh Tomlin	0.269	656	7	Guillermo Moscoso	0.266	521
8	Chris Young	0.242	96	8	Ian Kennedy	0.268	877
9	Josh Johnson	0.252	238	9	James Shields	0.267	968
10	Guillermo Moscoso	0.266	521	10	Clayton Kershaw	0.250	913
11	Josh Collmenter	0.266	609	11	Doug Fister	0.263	853
12	Ian Kennedy	0.268	877	12	Michael Pineda	0.270	685
13	Clayton Kershaw	0.250	913	13	Jeremy Hellickson	0.282	767
14	James Shields	0.267	968	14	David Price	0.278	899
15	Doug Fister	0.263	853	15	Alexi Ogando	0.277	680
16	Michael Pineda	0.270	685	16	Brandon McCarthy	0.280	677
17	Randall Delgado	0.293	146	17	Tommy Hanson	0.281	535
18	Luis Mendoza	0.267	57	18	Johnny Cueto	0.269	612
19	David Price	0.278	899	19	Tim Hudson	0.277	867
20	Alexi Ogando	0.277	680	20	Scott Baker	0.288	542
21	Jeremy Hellickson	0.282	767	21	Ricky Romero	0.279	897
22	Brandon McCarthy	0.280	677	22	Cole Hamels	0.251	848
23	Tommy Hanson	0.281	535	23	Brandon Beachy	0.289	578
24	Johnny Cueto	0.269	612	24	Daniel Hudson	0.290	911
25	Tim Hudson	0.277	867	25	Gavin Floyd	0.282	775

Table 2: *Top* 25 *Colley pitchers with and without super-user.*

These discrepancies also occur with the Massey Method. Tables 3 and 4 show the Massey rankings, the Massey Minimum Plate Appearance (dentoed minPA), and the Massey Super-User Rankings (denoted SU) for batters and pitchers in 2011.

Rank	Massey	Massey (MinPA)	Massey (SU)
1	Joey Gathright	Jose Bautista	Jose Bautista
2	Esteban German	Mike Napoli	Mike Napoli
3	Cole Gillespie	Miguel Cabrera	Miguel Cabrera
4	Russ Canzler	Matt Kemp	Matt Kemp
5	Jesus Montero	Adrian Gonzalez	Adrian Gonzalez
6	Jose Bautista	Alex Avila	Alex Avila
7	Mike Napoli	Joey Votto	Daivd Ortiz
8	Chris Parmelee	David Ortiz	Joey Votto
9	Miguel Cabrera	Curtis Granderson	Curtis Granderson
10	Jason Giambi	Giancarlo Stanton	Lance Berkman

Table 3: *Top* 10 *Massey batters with the three methods.*

Rank	Massey	Massey (MinPA)	Massey (SU)
1	Brad Peacock	Matt Cain	Matt Cain
2	Stephen Strasburg	Guillermo Moscoso	Guillermo Moscoso
3	Luis Mendoza	Roy Halladay	Roy Halladay
4	Jarrod Parker	Jordan Zimmerman	Jordan Zimmerman
5	Matt Cain	Justin Verlander	Justin Verlander
6	Josh Johnson	Doug Fister	Doug Fister
7	Chien-Ming Wang	Cole Hamels	Cole Hamels
8	Guillermo Moscoso	Ryan Vogelsong	Kyle Lohse
9	Roy Halladay	Kyle Lohse	Ryan Vogelsong
10	Chris Young	Cliff Lee	Cliff Lee

Table 4: *Top* 10 *Massey pitchers with the three methods.*

Note that the Minimum Plate Appearance approach and the super-user approach have very similar rankings with only a few transpositions of players. As with the Colley Method, we see that batters and pitchers who made the most of their limited game exposure show up high in the rankings.

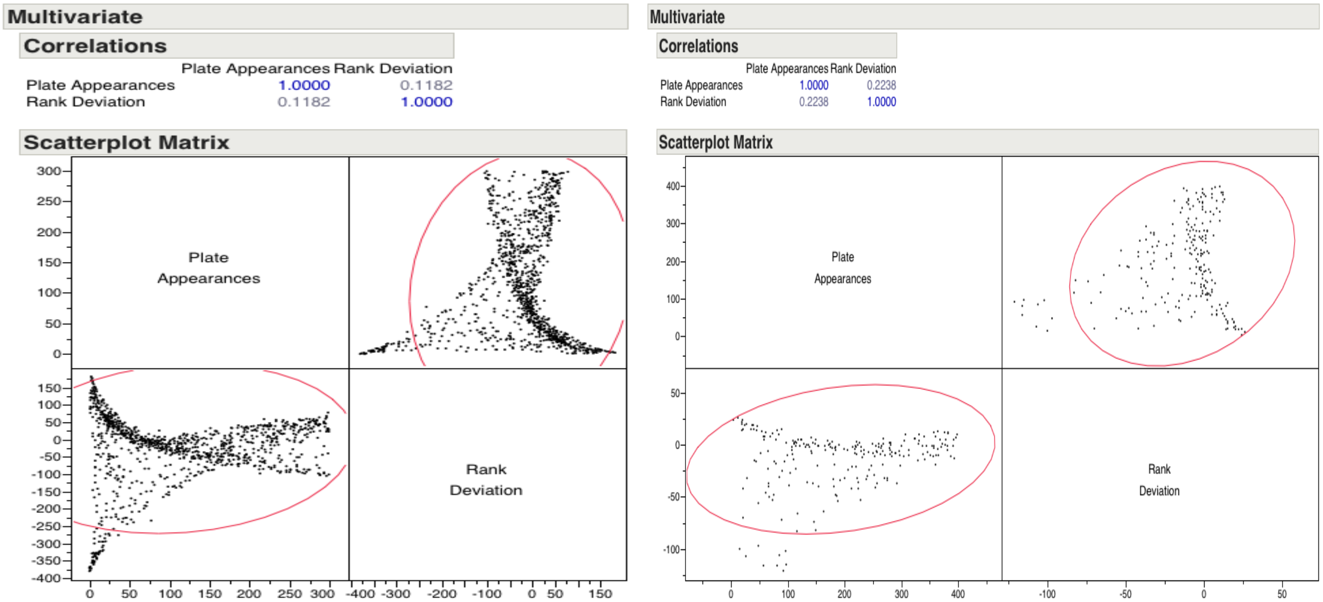
Rank	Super-User	Min Plate Appearances
20	Andruw Jones	Mark Reynolds
21	**Alejandro De Aza**	Carlos Beltran
22	Mark Reynolds	Justin Upton
23	Paul Konerko	Troy Tulowitzki
24	Carlos Gonzalez	Wilson Betemit
25	Carlos Beltran	Josh Hamilton
26	Justin Upton	Hunter Pence
27	Wilson Betemit	Matt Holliday
28	Troy Tulowitzki	Josh Willingham
29	**Allen Craig**	Carlos Pena
30	Josh Hamilton	Pablo Sandoval
31	Matt Holliday	Ryan Howard
32	Hunter Pence	Dustin Pedroia
33	Carlos Pena	Todd Helton
34	**Chris Parmelee**	Nick Swisher
35	Josh Willingham	Mike Carp
36	Pablo Sandoval	Lucas Duda
37	**Ike Davis**	Kevin Youkilis
38	Ryan Howard	Robinson Cano
39	**Jesus Montero**	Matthew Joyce
40	Dustin Pedroia	Corey Hart

Table 5: *Snapshot of Massey Super-User and MinPA Rankings.*

Table 5 gives a snapshot of the Massey Super-User and Massey Minimum Plate Appearance Methods from rank 20 to 40. The boldfaced names are those players who appear in the Massey Super-User rankings but not in the Massey Minimum Plate Appearance rankings. There are a couple of items to note. From Table 3 we see that Chris Parmelee is ranked 8^{th} in the overall Massey rankings. He had 88 plate appearances in 2011 and a batting average of 0.355 and was called up from the minors during the season. The super-user penalty that Parmelee incurred drops him in the rankings to 34, but he remains in the rankings. Alejandro De Aza is 15^{th} in the overall Massey Rankings but only drops 6 places to 21^{st} in the Massey Super-User Rankings. In 2011, De Aza appeared in only 54 games with 179 plate appearances and batted 0.329. He was an outfielder who rotated with other outfielders during the year and was used as a pinch hitter. Like Parmelee, he incurs a super-user penalty, but it is not quite as severe since he has more than twice the plate appearances as Parmelee. Finally, Allen Craig, who was instrumental in the St. Louis Cardinals winning the World Series in 2011, shows up at 29 in the Massey Super-User Rankings while not appearing at all in the Massey Minimum Plate Appearance method. In 2011, Craig had 219 plate appearances with a batting average of 0.315. He was used mostly as a utility player by the Cardinals and played 6 different positions for them throughout the season. Ike Davis was having a good year until injury struck, and Jesus Montero was a rotating designated hitter throughout the year. Contributions from these players are noted in the super-user approach but are omitted from the minimum plate appearance approach.

5. Effect of Playing the Super-User. One potential issue that could arise in the Super-User Method is whether players who are closer to the minimum cutoff appear higher in the rankings despite a poorer performance relative to other players. Could this be why players like Chris Parmelee and Alejandro De Aza moved in the rankings when a super-user node is added to produce Tables 1 and 5? To aid in answering this question, consider Figures 2 (a) and (b) of plots compiled of players from the 2009-2011 major league baseball seasons that failed to meet minimum cutoff for interactions. The number of interactions against the super-user is plotted against the overall rank change from ranking with and without the super-user nodes. The output from the statistical software package JMP is shown in the figures. In both pitchers and batters, the deviation in rank has a low correlation ($r = 0.1182$ for batters and $r = 0.2238$ for pitchers) with the number of interactions these players have against their corresponding super-user, and thus the variation in rank deviations is primarily due to other factors. This suggests that performance in the Super-User Method has more to do with where players are ranked than the number of plate appearances.

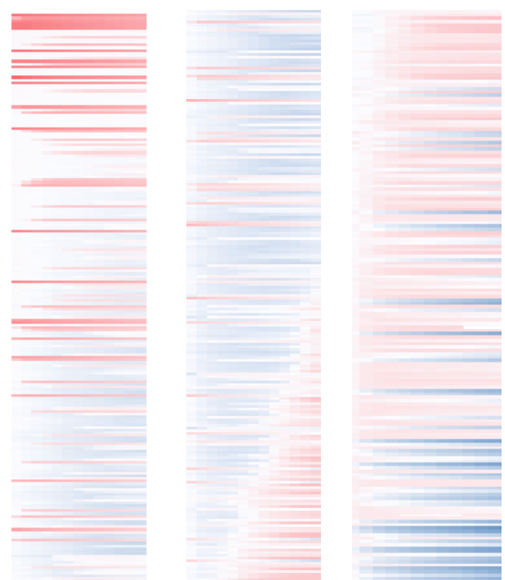
6. Choosing the Minimum Cutoff. How does one choose the minimum cutoff for games played by the super-user? Further, how sensitive is the method to such choices? Are there patterns to the perturbations seen in the rankings as evidenced by the super-user? To aid in answering these questions, let's continue our study of Major League Baseball. To simplify our discussion, we will concentrate solely on the ranking of batters.



(a) 2009-2011 Batters below cutoff (b) 2009-2011 Pitchers below cutoff

Figure 2: *Number of super-user interactions against rank change for batters (a) and pitchers (b).*

The sensitivity of rankings to changing parameters associated with these linear algebra based ranking systems has been studied in the literature. Notably, the famous PageRank method for ranking webpages results in a linear system with a scalar parameter known as the teleportation constant [8, 12]. The sensitivity of PageRank to changes in this parameter is studied in two main ways: (1) through an analysis of the derivative of the ranking with respect to this parameter [2, 7] and (2) with computational studies that vary the parameter over its domain [2, 8, 9, 10]. Because the Colley and Massey super-user ranking system also result in a linear system with one scalar parameter, namely the number of times the batter/pitcher faces their respective super-users, we consider similar sensitivity analysis. However, the first approach, a derivative analysis, will not work for the super-user application because the parameter changes depending on the number of plate appearances of each player and is not the same for each player. Thus, we apply the second type of traditional sensitivity analysis to our problem and run computational studies varying our cutoff parameter over its domain.



(a) (b) (c)

Figure 3: *Heat map showing decreases (light red indicates a slight decrease and dark red, a large decrease) and increases (blue) in player's ranks as the minimum cutoff increases where (a), (b), and (c) are the upper third, middle third, and bottom third of the rankings.*

Figure 3 shows rank variations of batters as the minimum cutoff varies from 0 to 700 incrementing by 50. We use 700 since the maximum number of plate appearances was 721 in 2011. Figure 3 is a heat map showing decreases (light red indicates a slight decrease and dark red, a large decrease) and increases (blue) in players' ranks as the minimum cutoff increases. Note that Figure 3 (a), (b), and (c) are the upper third, middle third, and bottom third of the rankings, respectively. The first column in (a), (b), and (c) shows the ranks of the players when the minimum cutoff is 0 and the last column, when the cutoff is 700. Each row in the map corresponds to an individual player and his rank changes.

Three patterns are apparent in Figure 3.

- Red dominates toward the top of the heat map (Figure 3(a)). Players with few plate appearances see an immediate penalty (severe drop in rank indicated by dark red) when the minimum is raised above their number of plate appearances. This is working as intended as these players had undeservedly high rankings due to good performances in just a few games and now with our method must face the dominant super-user multiple times.
- Conversely, blue dominates the bottom of the heat map (Figure 3(c)). This indicates that low-ranked players receive some increase in rank as the parameter increases, i.e., as they play more games against the dominant undefeated

super-user. Of course, these low-ranked players lose each time they face the super-user, so their subsequent increase in rank seems contradictory. However, this increase in rank is a result of the "strength of schedule" influence in the ranking method. Because the super-user is the #1 team, these low-ranked players receive a boost by playing the dominant team so many times.

- The middle section of the heat map (Figure 3(b)) is very interesting, showing a triangular section of decrease in red with a symmetric triangular section of increase in blue. This section justifies setting the minimum cutoff parameter somewhere between 300 and 500. In this range, players have a slight decrease in rank, followed by a stable rank period, followed by a slight increase in rank. Within this range, the penalty movement down is balanced by the strength of schedule movement up.

Finding the best minimum cutoff is difficult since the problem does not lend itself to the aforementioned derivative-based approach and must be analyzed by empirical means. Further, there may be several different objectives governing the quality of the rankings produced by the methods. Who is to say which objective ranks batters and pitchers best? For these reasons we choose to empirically find the cutoff that minimizes the squared differences in current ranking and previous ranking as the minimum cutoff is incrementally increased. For each of the deviations in the minimum cutoff we can calculate a squared deviation in ranks (or ratings) that occurs for batters. As the minimum cutoff is varied from 0 to 700 (this time by 5), the minimum squared deviation in rank occurs at a cutoff of 425. Note that this value is larger than the 300 that we prescribed but smaller than the 502 that MLB prescribes.

One can also vary how the super-user is applied. In some scenarios it might be deemed more fair for every team to play the super-user a fixed number of times (rather than ensuring everyone has a minimum number of games). If we apply this so-called uniform game method to MLB, the effects are similar to the method of minimum plate appearances. We see an immediate drop in the rankings of players with few, but successful plate appearances in both methods. In the uniform games method, these players drop into the middle third of the rankings. In the super-user method, the immediate drop is more severe, sending these players into the bottom fourth of the rankings. What is significantly different between the two applications of the super-user is that in the uniform games application those players with few, but unsuccessful plate appearances also jump into the middle third of the rankings whereas in the minimum plate appearance method these players stay near the bottom of the rankings. This difference is due to the fact that the super-user plays everyone (instead of just those with few attempts) and has thus garnered a better rating for playing a tougher strength of schedule.

7. Ranking Tennis Players. As mentioned earlier, there is a great disparity between the number of matches played by the very best professional tennis players and those players whose ability puts them on the border between the ATP Tour tournaments and lesser competitions on the Challenger Tour and Futures Series.

Unlike the ranking of baseball players, the tennis ranking problem cannot be modeled as a bipartite network. Another difference is that in tennis the super-user does not end up as the #1 player in either the Colley or Massey rankings. Evidently, in tennis the top players are so dominant, that the super-user's 2560 wins and 0 losses compiled almost exclusively against sub-par competition was only good enough for 10$^{\text{th}}$ place in the Colley rankings and 22$^{\text{nd}}$ in the Massey rankings. As the minimum cutoff is increased, however, the super-user's ranking increases as it is allowed to play competition with higher ratings and thus improve the average ratings of its opponents. For instance, as the minimum cutoff is increased to 40, the super-user's Colley ranking is 5$^{\text{th}}$. Once the minimum cutoff is increased to 80, the super-user is playing the top-ranked players and is ranked 2$^{\text{nd}}$. The super-user takes the top spot in the Colley ranking when the minimum cutoff is 95.

Player	W	L	Colley	Colley (SU)	Massey	Massey (SU)	Avg. Change
Acasuso, J.	2	1	33	96	14	110	79.5
Gonzalez, M.	3	11	224	288	193	281	76.0
De Bakker, T.	2	11	240	294	197	294	75.5
Ramirez-Hidalgo, R.	5	15	239	296	198	290	74.5
Brands, D.	4	11	202	268	195	258	64.5
Delbonis, F.	3	2	47	95	20	99	63.5
Serra, F.	3	12	257	297	217	300	61.5
Van Der Merwe, I.	3	1	51	102	53	123	60.5
Bozoljac, I.	1	1	101	137	71	148	56.5
Cervenak, P.	2	1	81	121	82	154	56.0

Table 6: *The ten players whose rankings declined the most after introduction of the super-user. Based on largest average rank decline over both the Colley and Massey methods.*

Nevertheless, the issue of players who played comparatively few matches being ranked higher than expected still arises. Though these unjustly high rankings are not as dramatic as they are in the baseball example, where players with only two or three matches appear at the very top of the rankings. In addition to Jose Acasuso, Federico Del Bonis, Izak Van Der Merwe, Ilija Bozoljac, and Pavol Cervenak, players unknown to all but the most serious tennis fans, were all able to achieve a top 100 ranking from Colley and Massey by playing in five or fewer matches and winning at least as many as they lost. Table 6 shows the decline of these players after implementation of a super-user approach with a cut-off value of 17. (The mean number of matches played per player on the 2011 ATP tour was 16.88.)

Since the Massey method requires a margin of victory for each match and tennis's scoring system does not provide an obvious one, the formula

$$\text{Margin of Victory} = 5 + 5(\text{Sets Won Differential}) + \text{Games Won Differential}$$

was used with the Massey algorithm. This formula assures that each match winner has a positive margin of victory. The winner of a five-set match by the closest score possible (0-6,0-6,7-6,7-6,7-6), gets a margin of victory equal to one. For 2011, the mean margin of victory so defined was 19.21, so the value of 19 was used as the margin of victory in each of the super-user's matches.[6]

Table 7 shows the ten players whose rankings improved the most once the super-user was introduced. These 10 players can be divided into two groups, with each group populated by a similar type of player. The three players in the first group, Andreas Vinciguerra, Alexander Sadecky, and Aljaz Bedene, lost their only completed ATP match to players whose ATP rank at the time of the match was 727^{th}, 401^{st} and 204^{th} respectively. This performance was so abysmal that the addition of sixteen losses to the super-user slightly improved their Colley and Massey ranking. Nevertheless these three still remain in the bottom third of ranked players in 2011.

Player	W	L	Colley	Colley (SU)	Massey	Massey (SU)	Avg. Change
Vinciguerra, A.	0	1	287	260	299	246	40.0
Haas, T.	7	11	141	105	132	89	39.5
Paire, B.	5	10	157	123	159	115	39.0
Falla, A.	7	14	153	116	139	100	38.0
Istomin, D.	10	20	152	117	140	103	36.0
Sadecky, A.	0	1	280	254	282	236	36.0
Gabashvili, T.	10	19	144	108	137	104	34.5
Volandri, F.	12	19	131	101	144	106	34.0
Souza, J.	7	8	126	100	147	107	33.0
Bedene, Aljaz	0	1	261	231	251	217	32.0

Table 7: *The ten players whose rankings improved the most after introduction of the super-user. Based on largest average rank improvement over both the Colley and Massey methods.*

The other group of seven players is much more interesting. Each of them played between 15 and 31 matches and had 2011 Colley/Massey rankings between 126 and 159 before the introduction of the super-user. Notice that few of these players ended up playing in any matches against the super-user, but their ranking benefited since other players originally ranked above them had their rankings fall after repeated losses to the super-user. These seven players played 70 of their combined 159 real-world matches against players in the ATP's top 50, and despite a combined record of 12-58, these "quality" losses are rewarded by Colley and Massey, especially once super-user matches were included.

Of course, it is possible that these players are merely being rewarded for having drawn tough first-round opponents, though it is interesting to note that Benoit Paire, Alejandro Falla, and Denis Istomin have all recorded their career-high ATP ranking since the 2011 season. Tommy Haas is an interesting case. The oldest player on the list, he was the ATP's #2 player in May 2002. His career has been interrupted twice by injury breaks that lasted over a year. He returned from one of these in mid-2011 and has since returned to the ATP's top 20. It appears that an increase in the Super-User Ranking may indicate future success and would be a promising area for future study.

As in the baseball example, the issue of where to place the minimum cutoff arises. If we again try to find a cutoff that minimizes the sum of the squared deviations from current rank to previous rank, one (local) minimum occurs at 17, which is where we set this cutoff. A slightly better minimum occurs when the cutoff is 30, although with this cutoff, the ranking for the top 40 players changes very little compared to when the cutoff is set at 17.

8. Games Beyond Sports. Sports ranking methods can be applied outside the context of sports. One such application is Prediculous, a free online social game where users compete with other users by predicting the future of events in sports, politics, entertainment, business, and the world. An example question regarding the prediction of a sporting event is shown in Figure 4. Players compete for points and leaderboard status. In all, approximately 18,600 unique users have played the game.

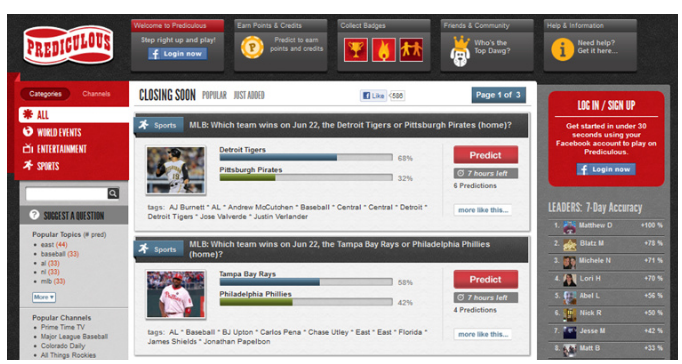


Figure 4: *A sample question contained on* `http://www.prediculous.com`.

In 2011, several authors of this paper applied the Colley method to this context with the intent of employing a more advanced ranking method in the website's leader board. The first step was defining a game. In this context, a game occurs over a prediction. Note, both players can be

[6]The Margin of Victory formula is the creation of Gettysburg College student Michael McLauglin, who studied tennis ranking for his 2012 senior capstone project.

correct, both incorrect, or one is correct in a prediction. Further, games occur between each pair of users participating in a prediction question. As such, ties must be integrated into the Colley method.

Like tennis, players (which are Prediculous users in this case) vary in the number of games played. New users begin with one prediction. Other users have over 100 predictions. The need for a super-user quickly arises. If one removes users who do not meet some minimum cut-off, new users will not receive a rank or will simply be put at the bottom until they surpass the cut-off. For Prediculous, this is undesirable since encouraging new users' play is important to growing participation on the site. Consequently, the super-user approach has business implications and is very helpful. The cut-off was chosen in consultation with Prediculous with their business goals in mind.

Earlier in the paper, other applications beyond sports were listed. If one ranks Netflix movies from user ratings, then again, movies vary in the number of ratings they receive which can elevate lesser known movies. A similar effect happens with Amazon products when a sports ranking method is applied to the user ratings of products.

9. Concluding Remarks. From sports to the Internet to businesses looking at their products, ranking is an important mathematical process. As demonstrated in this paper, the Colley and Massey methods can offer valuable insight in the ranking of batters and pitchers in Major League Baseball, of tennis players, and of participants in an online social game. These representative examples underscore how easily contexts arise in which an unequal number of games can arise. As discussed, removing data of less active players or participants removes valuable information. However, the presence of unequal numbers of games can decrease the value of the resulting rankings. To make sports ranking methods adaptable and valuable in such settings, this paper demonstrated how to adapt both the Colley and Massey ranking methods with the introduction of a super-user. This fictitious player aids in identifying strong players who have not played a significant number of games. At the same time, the effect of simply playing (and losing) a few games against strong opponents is not an immediate advantage. Further, the paper showed that such a method does not favor players approaching the cut-off and is adaptable but also not highly sensitive to the cut-off used with the super-user. The ideas introduced in this paper can further the scope that such sports ranking methods are applied and the depth of insight they offer.

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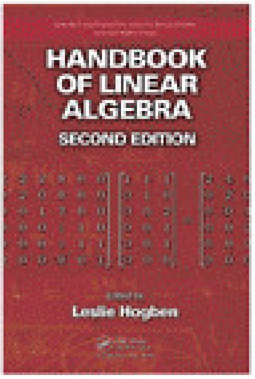

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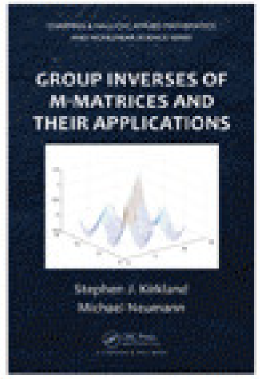

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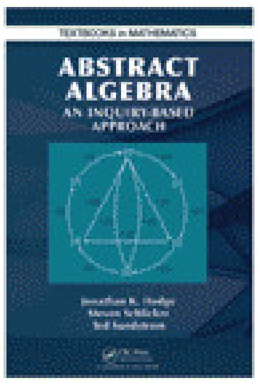

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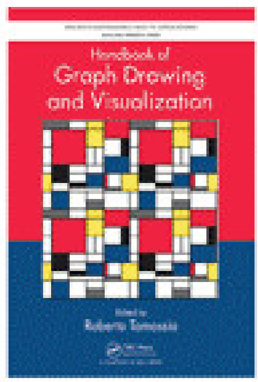

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—Tom Schulte, *MAA Reviews*, February 2014

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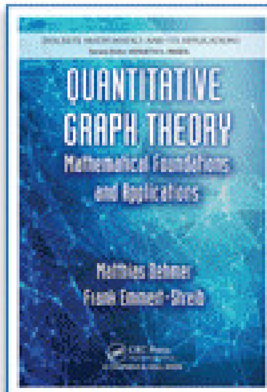

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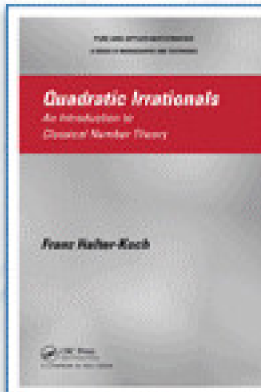

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UPCOMING CONFERENCES AND WORKSHOPS

10th International Workshop on Accurate Solution of Eigenvalue Problems Dubrovnik, Croatia, June 2–5, 2014

The original themes of this workshop arose in the intersection of two fields, matrix eigenvalue/singular value computation, and, more generally, fast and accurate matrix computation. Meanwhile, the scope has expanded, and the organizers have encouraged submissions for presentations or posters that relate to either of these two themes, including large scale and nonlinear eigenvalue problems. A prize will be given for the best poster.

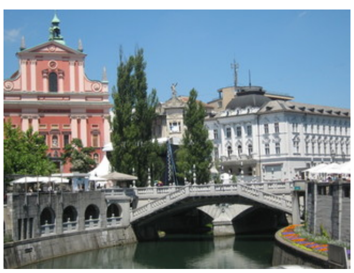
This is the tenth IWASEP workshop. Previous workshops were held in Split (1996), Penn State (1998), Hagen (2000), Split (2002), Hagen (2004), Penn State (2006), Dubrovnik (2008), Berlin (2010), and Napa Valley (2012).

The invited speakers are: Rafikul Alam (Indian Institute of Technology Guwahati), Chris Beattie (Virginia Tech), Ming Gu (UC Berkeley), Ilse Ipsen (North Carolina State University), Daniel Kressner (École Polytechnique Fédérale de Lausanne), Beresford Parlett (UC Berkeley), Danny Sorensen (Rice University), Pete Stewart (University of Maryland), Marc Van Barel (Katholieke Universiteit Leuven), and Heinrich Voss (TU Hamburg-Harburg).

The organizers of the workshop are: Jesse Barlow (The Pennsylvania State University, SIAM representative), Zlatko Drmač (University of Zagreb), Volker Mehrmann (Technical University of Berlin), Ivan Slapničar (University of Split), and Krešimir Veselić (Fernuniversität Hagen). The local organizing committee is: Ivan Slapničar, Nevena Jakovčević Stor, and Lana Periša.

For further details, check the conference website: `http://iwasep.fesb.hr/iwasep10/`.

The 7th Linear Algebra Workshop (LAW'14), and the 23rd International Workshop on Matrices and Statistics (IWMS) Ljubljana, Slovenia, June 4–12, 2014 and June 8–12, 2014



Ljubljana

The main theme of the Linear Algebra Workshop will be the interplay between operator theory and algebra. The workshop will follow the usual format. A few hours of talks will be scheduled for the morning sessions, while afternoons will be reserved for work in smaller groups. LAW'14 will be organized in conjunction with IWMS that will take place in the second week. The main theme of the International Workshop on Matrices and Statistics will be the interplay between matrices and statistics. Both workshops will be held at the Faculty of Mathematics and Physics, Ljubljana, Slovenia. More information can be found at www.law05.si/law14 and www.law05.si/iwms/.

Householder Symposium XIX on Numerical Linear Algebra Spa, Belgium, June 8–13, 2014

The Householder Symposium XIX on Numerical Linear Algebra will be held June 8–13, 2014, at the Domain Sol Cress Spa at the edge of the High Fenn Nature Park in Belgium. This meeting is the nineteenth in a series, previously called the Gatlinburg Symposia. It is hosted by Université catholique de Louvain and the Katholieke Universiteit Leuven, and is in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra. The series of conferences is named after its founder Alston S. Householder, one of the pioneers in numerical linear algebra. The meeting is very informal, with the intermingling of young and established researchers a priority. Attendance at the meeting is by invitation only. The conference website is `http://sites.uclouvain.be/HHXIX/`.

The Eleventh International Conference on Matrix Theory and Applications Shandong, China, June 13–16, 2014

The Eleventh International Conference on Matrix Theory and Applications will be held in Linyi University, Linyi City, Shandong Province, P.R. China, June 13–16, 2014. The conference aims at enhancing academic communication and promoting research among scholars, from China and abroad, who are interested in matrix theory, methods and computations, as well as applications. The conference provides a friendly platform for senior researchers and young scientists to exchange ideas on recent developments in the relevant areas. The conference is co-organized by Shanghai University, and the Academy of Mathematics and Systems Science of the Chinese Academy of Sciences (CAS). The local organizing institution is Linyi University.

The invited speakers are: Daniele Bertaccini (Università degli Studi di Roma Tor Vergata, Italy), Sou-Cheng Choi

(University of Chicago, USA), Hua Dai (Nanjing University of Aeronautics and Astronautics, China), Yong-Jian Hu (Beijing Normal University, China), Li-Ping Huang (Changsha University of Science and Technology, China), Marielba Rojas (Delft University of Technology, The Netherlands), Miroslav Rozložník (Academy of Sciences of the Czech Republic, Prague, Czech Republic), Chuan-Long Wang (Taiyuan Normal University, Shanxi, China), Mu-Sheng Wei (Shanghai Normal University, China), and Jun-Feng Yin (Tongji University, Shanghai, China).

The conference co-chairs are Er-Xiong Jiang (Shanghai University, China) and Zhong-Zhi Bai (Chinese Academy of Sciences, China). Local contact: Zi-Wu Jiang, School of Science, Linyi University, Linyi 276 005, P.R. China. Email: mta2014@126.com. Phone: +86-13665396232, 18669669676.

Graph Theory, Matrix Theory and Interactions
A conference to celebrate the scholarship of David Gregory
Queen's University, Canada, June 20–21, 2014

A conference to celebrate the scholarship of David A. Gregory will be held at Queen's University, Kingston, Ontario, Canada, June 20–21, 2014.

The program includes the following invited speakers: Richard Brualdi (University of Wisconsin), Sebastian Cioabă (University of Delaware), Randall Elzinga (Royal Military College), Chris Godsil (University of Waterloo), Willem Haemers (Tilburg University), Steve Kirkland (University of Manitoba), Ram Murty (Queen's University), Naomi Shaked-Monderer (The Max Stern Yezreel Valley College), Claude Tardif (Royal Military College), Edwin van Dam (Tilburg University), Kevin N. Vander Meulen (Redeemer University College), and David Wehlau (Queen's University). The Hans Schneider ILAS Lecturer is Shaun Fallat (University of Regina).

The Organizing Committee consists of: Sebastian Cioabă (University of Deleware), Ram Murty (Queen's University), Bryan Shader (University of Wyoming), Claude Tardif (Royal Military College), Kevin N. Vander Meulen (Redeemer University College), and David Wehlau (Queen's University).

The Organizing Committee gratefully acknowledges support from the Fields Institute for Research in Mathematical Sciences, the International Linear Algebra Society, and the Department of Mathematics and Statistics at Queen's University.

For registration and further details, please visit the conference website: www.fields.utoronto.ca/programs/scientific/13-14/interactions/index.html.

Joint ALAMA - GAMM/ANLA Meeting
Barcelona, Spain, July 14–16, 2014

This is both the fourth edition of the biennial meeting of the Spanish Linear Algebra and Matrix Analysis thematic network ALAMA, and the fourteenth edition of the annual workshop of the GAMM Activity Group on Applied and Numerical Linear Algebra, GAMM/ANLA. The goal of this meeting is to bring together scientists whose research interests are close to Matrix Theory, to either Core, Applied or Numerical Linear Algebra, or to any of their respective applications in different scientific areas. The aim is to discuss recent developments, promote cross-discipline interaction, and to foster new insights and collaborations.

On this occasion, the meeting is held in honor of Juan-Miguel Gracia, one of the pioneers of Matrix Theory in Spain, and one of the founders of the research group in Matrix Analysis and Applications of the University of the Basque Country. He was also the first Coordinator of the ALAMA Network at the moment of its foundation in September 2007, and served as its first president for two years.

The invited speakers are: María Bras (Universitat Rovira i Virgili, Spain), Geir Dahl - ILAS Speaker (University of Oslo, Norway), Melina Freitag (University of Bath, UK), Juan-Miguel Gracia (Universidad del País Vasco, Spain), and Volker Grimm (Karlsruhe Institut für Technologie, Germany).

Further details, in particular on the registration process and abstract submission, are available at www.cimne.com/alama-gamm.

Graduate Student Modeling Workshop (IMSM 2014)
Raleigh, North Carolina, July 14–22, 2014

The 19$^{\text{th}}$ Industrial Mathematical & Statistical Modeling (IMSM) Workshop for Graduate Students will take place at North Carolina State University, 14–22 July 2014. The workshop is sponsored by the Statistical and Applied Mathematical Science Institute (SAMSI) together with the Center for Research in Scientific Computation (CRSC) and the Department of Mathematics at North Carolina State University.

The IMSM workshop exposes graduate students in mathematics, engineering, and statistics to exciting real-world problems from industry and government. The workshop will provide students with experience in a research team environment and exposure to possible career opportunities. Local expenses and travel expenses will be covered for students at US institutions. Information is available at www.samsi.info/IMSM14 and questions can be directed to grad@samsi.info.

19th ILAS Conference
Seoul, South Korea, August 6–9, 2014

The 19th International Linear Algebra Society (ILAS) Conference will be held in Seoul, Korea, on the main campus of Sungkyunkwan University (note the change of the conference site), from August 6 (Wed) through August 9 (Sat), 2014. The theme of this conference is *Solidarity in Linear Algebra* and The ILAS conference is one of the satellite conferences of ICM 2014, which will be held in Seoul following the ILAS meeting.

Invited Plenary Speakers:

- Ravindra Bapat (Indian Statistical Institute, LAMA Lecturer)
- Peter Benner (Max Planck Institute)
- Dario Bini (University of Pisa, LAA Lecturer)
- Shaun Fallat (University of Regina, Taussky-Todd Lecturer)
- Andreas Frommer (University of Wuppertal, SIAG/LA Lecturer)
- Stéphane Gaubert (INRIA)
- Chi-Kwong Li (College of William and Mary)
- Yongdo Lim (Sung Kyun Kwan University)
- Panayiotis Psarrakos (National Technical University of Athens)
- Vladimir Sergeichuk (Institute of Mathematics, Kiev)
- Bernd Sturmfels (University of California-Berkeley)
- Tin-Yau Tam (Auburn University)

Invited Minisymposia:

- Combinatorial Problems in Linear Algebra (Richard Brualdi, Geir Dahl)
- Matrix Inequalities (Fuzhen Zhang, Minghua Lin)
- Spectral Theory of Graphs and Hypergraphs (Vladimir Nikiforov)
- Tensor Eigenvalues (Jia-Yu Shao, Liqun Qi)
- Quantum Information and Computing (Chi-Kwong, Yiu Tung Poon)
- Riordan Arrays and Related Topics (Gi-Sang Cheon, Donatella Merlini, Lou Shapiro, Leon Woodson)
- Nonnegative Matrices and Generalizations (Judi McDonald)
- Toeplitz Matrices and Operators (Torsten Ehrhardt)

The program also includes the following contributed minisymposia: Structured Matrix Equations; Inverse Spectral Problems; Matrix Geometry; Teaching Linear Algebra; Algebraic Combinatorics and Combinatorial Matrices; Linear Least Squares and Applications; Generalized Laplacian and Green Matrices; Eigenvalue Computations and Applications; Inequalities in Matrices, Operators, and Lie Groups; Matrix Methods in Computational Systems Biology and Medicine; Solution of Sylvester-like Equations and Canonical Forms; Generalized Matrix Inverses and Applications; Sign Pattern Matrices; and Recent Developments in Linear Preserver Problems.

The Scientific Organizing Committee consists of: Suk-Geun Hwang, Chair (Kyungpook National University), Nair Abreu (Universidade Federal do Rio de Janeiro), Tom Bella (University of Rhode Island), Rajendra Bhatia (Indian Statistical Institute), Richard Brualdi (University of Wisconsin-Madison), Man-Duen Choi (University of Toronto), Nicholas J. Higham (University of Manchester), Leslie Hogben (Iowa State University and American Institute of Mathematics), Stephen Kirkland (University of Manitoba), Sang-Gu Lee (Sungkyunkwan University), Helena Šmigoc (University College Dublin), and Fuzhen Zhang (Nova Southeastern University).

The Local Organizing Committee are: Suk-Geun Hwang (Chair), Sang-Gu Lee (Chair), Gi-Sang Cheon, Yongdo Lim, Hyun-Min Kim, and In-Jae Kim.

For more information about the ILAS conference, please visit the website www.ilas2014.org.

Gene Golub SIAM Summer School 2014
Linz, Austria, August 4–15, 2014

The 2014 Gene Golub SIAM Summer School on Simulation, Optimization, and Identification in Solid Mechanics will be held August 4–15 in Linz, Austria.

This summer school will foster advanced knowledge for the participating graduate students in several areas related to simulated materials in solid mechanics. Within this broad field the summer school will concentrate on four key issues, namely, (1) Identification of material parameters from measurements, (2) Material and topology optimization, (3) Optimization subject to variational inequalities, and (4) Adaptive discretization.

The lectures will be given by Roland Herzog (TU Chemnitz), Esther Klann (JKU Linz), Michael Sting (FAU Erlangen-Nürnberg), and Winnifried Wollner (University of Hamburg).

For further information, see: www.math.uni-hamburg.de/g2s3.

The International Conference on Trends and Perspectives in Linear Statistical Inference (LINSTAT 2014)
Linköping, Sweden, August 24–28, 2014

The *International Conference on Trends and Perspectives in Linear Statistical Inference*, LINSTAT 2014, will be held on August 24–28, 2014 in Linköping, Sweden. LINSTAT 2014 is the follow-up of the 2008, 2012 and 2010 editions held in Będlewo, Poland and in Tomar, Portugal.

The purpose of the meeting is to bring together researchers sharing an interest in a variety of aspects of statistics and its applications as well as matrix analysis and its applications to statistics, and offer them a possibility to discuss current developments in these subjects. The topics so far:

- Statistical Inference: Theory and Applications
- Multilevel models: Theory and Applications
- Model Selection and Dimension Reduction: Theory and Applications
- Design of Experiments: Theory and Applications
- Optimal Design: Theory and Applications
- Mixed Linear Models: Theory and Applications
- High-Dimensional Statistical Analysis: Theory and Applications
- Categorical Data Analysis: Theory and Applications
- Survey Methodology including Small Area Estimation
- Numerical Methods and Linear Models: Theory and Applications
- Biostatistics: Theory and Applications
- Multivariate Analysis: Theory and Applications.
- Bayesian Statistics: Theory and Applications
- Matrix Theory and Linear Models: Theory and Applications.

Researchers and graduate students in the area of linear algebra, statistical models and computation are particularly encouraged to attend the workshop. The format of this meeting will involve plenary talks and sessions with contributed talks. The Invited Speakers are Alan Agresti (USA), S. Ejaz Ahmed (Canada), Dennis Cook (USA), Sat Gupta (USA), Lynn LaMotte (USA), Emmanuel Lesaffre (Belgium), Yonghui Liu (China), Thomas Mathew (USA), Jamal Najim (France), Muni S. Srivastava (Canada), Yongge Tian (China) and Júlia Volaufová (USA), as well as the winners of the Young Scientists Awards of LinStat'2012: Maryna Prus (Germany), Jolanta Pielaszkiewicz (Sweden), Fatma Sevinç Kurnaz (Turkey) and Alena Bachratá (Slovakia).

There are several special sessions scheduled at LINSTAT 2014. One of them is to celebrate 30 years of the Polish-Nordic column spaces. The other special sessions are on the following topics: Small Area Estimation; Computation-Intensive Methods in Regression Models; The use of Kronecker Product in Statistical Modeling; Bayesian Inference for Complex Problems; and Linear spectral statistics and their applications.

The Scientific Committee will award the best presentation and best poster of young scientists. The winners will be Invited Speakers at the next edition of LINSTAT. The conference proceedings will be published in a special volume of the journal *Acta et Commentationes Universitatis Tartuensis de Mathematica*.

The Scientific Committee of LINSTAT 2014 is chaired by Dietrich von Rosen (Sweden), and the chair of the Organizing Committee is Martin Singull (Sweden).

The registration deadline for LINSTAT 2014 is July 31, 2014 and the abstracts deadline is June 30, 2014. For more information, please visit: www.mai.liu.se/LinStat2014.

4th IMA Conference on Numerical Linear Algebra and Optimisation Birmingham, UK, September 3–5, 2014

The IMA and the University of Birmingham are pleased to announce the Fourth Biennial IMA Conference on Numerical Linear Algebra and Optimisation. The meeting is co-sponsored by SIAM, whose members will receive the IMA members' registration rate.

The success of modern methods for large-scale optimisation is heavily dependent on the use of effective tools of numerical linear algebra. On the other hand, many problems in numerical linear algebra lead to linear, nonlinear or semidefinite optimisation problems. The purpose of the conference is to bring together researchers from both communities and to find and communicate points and topics of common interest.

Conference topics include any subject that could be of interest to both communities, such as: direct and iterative methods for large sparse linear systems, eigenvalue computation and optimisation, large-scale nonlinear and semidefinite programming, effect of round-off errors, stopping criteria, embedded iterative procedures, optimisation issues for matrix polynomials, fast matrix computations, compressed/sparse sensing, PDE-constrained optimisation, and applications and real time optimisation.

The invited speakers are: Michele Benzi (Emory University), Iain Duff - SIAM UKIE Speaker (Rutherford Appleton Laboratory and CERFACS Toulouse), Michael Friedlander (University of British Columbia), Andreas Grothey (University of Edinburgh), Joerg Liesen (TU Berlin), Yuri Nesterov (Université Catholique de Louvain), and Nancy Nichols (University of Reading).

The organising committee consists of: Michal Kocvara (co-chair), Daniel Loghin (co-chair), Jacek Gondzio, Nick Gould, Jennifer Scott, and Françoise Tisseur.

For further information or to register your interest, please visit the conference webpage: `http://www.ima.org.uk/conferences/conferences_calendar/4th_ima_conference_on_numerical_linear_algebra_and_optimisation.cfm`.

Structured Numerical Linear and Multilinear Algebra: Analysis, Algorithms and Applications (SLA 2014) Kalamata, Greece, September 8–12, 2014

SLA 2014, the 6th Conference on Structured Numerical Linear and Multilinear Algebra: Analysis, Algorithms and Applications, will be held in the city of Kalamata (Greece) from September 8th to September 12th 2014.

Gérard Meurant will present the 3 hour course *Matrices, Moments and Quadrature with Applications*.

SLA 2014 is a continuation of a series of meetings founded in 1996 by Dario Bini. Previous SLA meetings were held in Leuven (2012) and Cortona (2008, 2004, 2000 and 1996).

The aims of the meeting are: (1) the presentation of recent results on the theory, algorithms and applications of structured problems in numerical linear and multilinear algebra and matrix-tensor theory; (2) a review and discussion of the methodologies and the related algorithmic analysis; (3) the identification of directions of future research; (4) the improvement in the collaboration between theoretical and applied research.

The topics of the workshop are: (1) structured matrix and tensor analysis including (but not limited to) Toeplitz, Hankel, Vandermonde, banded, semiseparable, Cauchy, Hessenberg, mosaic, block, multilevel matrices and tensors, and the theoretical and applied problems in which they arise; (2) applications that involve structured matrices and tensors including (but not limited to) interpolation, integral and differential equations, least squares and regularisation, polynomial computations, matrix equations, control theory, queueing theory and Markov chains, image processing and signal processing; (3) the design and analysis of algorithms for the solution of structured problems.

The conference organisers are: Dario Bini (University of Pisa, Italy), Marilena Mitrouli (University of Athens, Greece), Marc Van Barel (Katholieke Universiteit Leuven, Belgium), and Joab Winkler (University of Sheffield, UK).

A special issue of *Linear Algebra and Its Applications* will be devoted to papers presented at the conference. The deadline for submission is December 31, 2014, and papers will be refereed according to *LAA*'s usual standards. Papers should be submitted via the Elsevier submission system ees.elsevier.com/laa/, choosing any of the special editors: Dario Bini, Marilena Mitrouli, Marc Van Barel, and Joab Winkler. The responsible Editor-in-Chief is: Volker Mehrmann.

For further details, go to `http://noether.math.uoa.gr/conferences/sla2014/?q=content/welcome-sla-2014`.

The Fifth International Conference on Numerical Algebra and Scientific Computing (NASC 2014) Shanghai, China, October 25–29, 2014

NASC is an international conference organized by the Chinese numerical algebra group starting from 2006. The conference highlights recent advances in theoretical, computational and practical aspects of linear and nonlinear numerical algebra. The aim of the conference is to gather numerical algebra and scientific computing experts to exchange ideas and discuss future developments and trends of these related fields.

The topics of NASC 2014 include, but are not limited to: solutions of linear and nonlinear equations; least-squares problems; computations of eigenvalue problems; tensor decompositions and computations; parallel computations; constructions and analyses of preconditioners; methods and theories of structured matrices; and applications of numerical algebraic techniques and algorithms.

An important event during the conference is the awarding of the Applied Numerical Algebra (ANA) Prize which is awarded to the two best papers by young Chinese scientists working on numerical algebra and scientific computing.

The registration fee for the conference is USD 350 (USD 300 for each spouse), which should be paid in cash at the conference. It includes lunches during the conference, a banquet, and the conference proceedings.

The invited keynote speakers are: Zheng-Jian Bai (Xiamen University), Martin J. Gander (University of Geneva), Chen Greif (The University of British Columbia), Ilse C.F. Ipsen (North Carolina State University), Gérard Meurant (Commissariat à l'énergie atomique at aux énergies alternatives), Alison Ramage (University of Strathclyde), Lothar Reichel (Kent State University), Kees Vuik (Delft University of Technology), Zeng-Qi Wang (Shanghai Jiaotong University), and Walter Zulehner (Johannes Kepler University).

The scientific committee are: Zhong-Ci Shi (Chair) (Chinese Academy of Sciences), Zhong-Zhi Bai (Chinese Academy of Sciences), Raymond H. Chan (Chinese University of Hong Kong), Chen Greif (The University of British Columbia), Ken Hayami (National Institute of Informatics), Ilse C.F. Ipsen (North Carolina State University), Lev A. Krukier (The Southern Federal University), Tian-Gang Lei (National Natural Science Foundation), Galina V. Muratova (The Southern Federal University), Maya G. Neytcheva (Uppsala University), Lothar Reichel (Kent State University), Miro Rozloznik (Academy of Sciences of the Czech Republic), Axel Ruhe (KTH Stockholm), Bo Yu (Dalian University of Technology), and Walter Zulehner (Johannes Kepler University).

The organizing committee are: Zhong-Zhi Bai (Chinese Academy of Sciences), Bao-Jun Bian (Tongji University), Jian-Yu Pan (The East China Normal University), Zeng-Qi Wang (Shanghai Jiaotong University), Cheng-Long Xu (Tongji University), and Jun-Feng Yin (Tongji University).

Persons who wish to submit a paper should send an extended abstract to Mr. Cun-Qiang Miao at the contact address given below before June 30th, 2014. The extended abstract should be a LaTeX file of 1-2 pages double spaced on A4 paper. Authors of all accepted papers will be notified by July 15th, 2014.

Contact Address: Mr. Cun-Qiang Miao, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, P.R. China, Email: miaocunqiang@lsec.cc.ac.cn.

The proceedings will be published as a special issue on *BIT Numerical Mathematics* (Springer, ISSN: 0006-3835). Papers in the proceedings will be refereed by the usual review procedure of the journal.

The 2014 International Conference on Tensors and Matrices and their Applications Suzhou, China, December 17–19, 2014

The 2014 conference on Tensors and Matrices and their Applications (TMA2014) will be held at Suzhou University of Science and Technology (USTS), Suzhou, China, Dec.17–19, 2014. The conference is intended for researchers in tensor and matrix communities to discuss novel methods and applications as well as theoretical advances. TMA2014 is endorsed by the International Linear Algebra Society, and will be jointly sponsored by The Hong Kong Polytechnic University, Fudan University, Jiangsu Normal University and Suzhou University of Science and Technology (USTS).

The confirmed invited speakers are from Australia, Canada, Europe, Japan, USA, Hong Kong and the Mainland of China. Open discussion sessions are also scheduled, and graduate students are encouraged to contribute and exchange their ideas for the development of a common understanding of problems and methods among the participants.

Suzhou, an ancient city with more than 2500 years of history, within 30 min driving distance from Shanghai, is attractive for its world famous garden arts and the nearby Lake Taihu.

For further conference details, please visit www.tma2014.org/.

International Conference on Linear Algebra & its Applications (ICLAA-2014)
In honor of Prof. R. B. Bapat on his 60th birthday
Manipal, India, December 18–20, 2014

ICLAA-2014 is in sequence of CMTGIM-2012 held at Manipal University, Manipal, India (www.manipal.edu). The theme of conference shall focus on (i) Matrix Methods in Statistics, (ii) Combinatorial Matrix Theory and (iii) Classical Matrix Theory covering different aspects of Linear Algebra. The conference is in sequence to the conference CMTGIM-2012 held January 10–11, 2012.

The conference will also provide a platform to many of Prof. Ravindra B. Bapat's contemporaries to come together to serve the objectives of the conference on the occasion of his 60th birthday and the conference provides an opportunity to the young scholars around, who will be the asset to the future generation, to join the team of eminent scientists.

The scientific committee consists of Rajendra Bhatia (Indian Statistical Institute, Delhi, India), Steve Kirkland (University of Manitoba, Canada), K. Manjunatha Prasad (Manipal University, Manipal India), and Simo Puntanen (University of Tampere, Finland).

For more details and registration, please visit (http://conference.manipal.edu/ICLAA2014/) or contact the organizing secretary K. Manjunatha Prasad (iclaa2014@manipal.edu or km.prasad@manipal.edu).

BOOK NEWS

Some Books on Linear Algebra and Related Topics Published in 2013 & 2014

Submitted by Douglas Farenick

1. Bapat, Ravindra B.; Kirkland, Steve J.; Prasad, K. Manjunatha; Puntanen, Simo. (Eds.) *Combinatorial matrix theory and generalized inverses of matrices.* Selected papers from the International Workshop and Conference held at Manipal University, Manipal, January 2–11, 2012. Springer, New Delhi, 2013. xviii+277 pp. ISBN: 978-81-322-1052-8; 978-81-322-1053-5.

2. Bhattacharyya, Tirthankar; Horn, Roger A.; Rao, T.S.S.R.K. (Eds.) *In the matrix mould. Expository articles by Rajendra Bhatia.* Hindustan Book Agency, New Delhi, 2013. xii+353 pp. ISBN: 978-93-80250-47-2.

3. Bleher, Pavel; Liechty, Karl. *Random matrices and the six-vertex model.* CRM Monograph Series, 32. American Mathematical Society, Providence, RI, 2014. x+224 pp. ISBN: 978-1-4704-0961-6.

4. Dym, Harry. *Linear algebra in action.* Second edition. Graduate Studies in Mathematics, 78. American Mathematical Society, Providence, RI, 2013. xx+585 pp. ISBN: 978-1-4704-0908-1.

5. Dzhamay, Anton; Muruno, Kenichi; Pierce, Virgil U. (Eds.) *Algebraic and geometric aspects of integrable systems and random matrices.* Proceedings of the AMS Special Session held in Boston, MA, January 67 2012. Contemporary Mathematics, 593. American Mathematical Society, Providence, RI, 2013. xii+345 pp. ISBN: 978-0-8218-8747-9.

6. Eidelman, Yuli; Gohberg, Israel; Haimovici, Iulian. *Separable type representations of matrices and fast algorithms.* Vol. 1. Basics. Completion problems. Multiplication and inversion algorithms. Operator Theory: Advances and Applications, 234. Birkhäuser/Springer, Basel, 2014. xvi+399 pp. ISBN: 978-3-0348-0605-3; 978-3-0348-0606-0.

7. Eidelman, Yuli; Gohberg, Israel; Haimovici, Iulian. *Separable type representations of matrices and fast algorithms.* Vol. 2. Eigenvalue method. Operator Theory: Advances and Applications, 235. Birkhäuser/Springer Basel AG, Basel, 2014. xii+359 pp. ISBN: 978-3-0348-0611-4; 978-3-0348-0612-1.

8. Golub, Gene H.; Van Loan, Charles F. *Matrix computations.* Fourth edition. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2013. xiv+756 pp. ISBN: 978-1-4214-0794-4; 1-4214-0794-9; 978-1-4214-0859-0.

9. He, Qi-Ming. *Fundamentals of matrix-analytic methods.* Springer, New York, 2014. xvi+349 pp. ISBN: 978-1-4614-7329-9; 978-1-4614-7330-5.

10. Hiai, Fumio; Petz, Dénes. *Introduction to matrix analysis and applications.* Universitext, Springer-Verlag, 2014. 332pp ISBN: 978-3-319-04150-6.

11. Hogben, Leslie. (Ed.) *Handbook of linear algebra.* Second edition. Chapman and Hall /CRC Press, 2013. ISBN:13-978-1-4665-0728-9.

12. Horn, Roger A.; Johnson, Charles R. *Matrix analysis.* Second edition. Cambridge University Press, Cambridge, 2013. xviii+643 pp. ISBN: 978-0-521-54823-6.

13. Körner, T. W. *Vectors, pure and applied. A general introduction to linear algebra.* Cambridge University Press, Cambridge, 2013. xii+444 pp. ISBN: 978-1-107-67522-3

14. Latouche, Guy; Ramaswami, Vaidyanathan; Sethuraman, Jay; Sigman, Karl; Squillante, Mark S.; Yao, David D. (Eds.) *Matrix-analytic methods in stochastic models.* Selected papers from the 7th International Conference (MAM7) held at Columbia University, New York, June 13–16, 2011. Springer Proceedings in Mathematics & Statistics, 27. Springer, New York, 2013. front matter+256 pp. ISBN: 978-1-4614-4909-6; 978-1-4614-4908-9.

15. Nielsen, Frank; Bhatia, Rajendra. (Eds.) *Matrix information geometry.* Springer, Heidelberg, 2013. xii+454 pp. ISBN: 978-3-642-30231-2.

16. Puntanen, Simo; Styan, George P. H.; Isotalo, Jarkko. *Formulas useful for linear regression analysis and related matrix theory. It's only formulas but we like them.* Springer Briefs in Statistics. Springer, Heidelberg, 2013. xii+125 pp. ISBN: 978-3-642-32930-2; 978-3-642-32931-9.

17. Shafarevich, Igor R.; Remizov, Alexey O. *Linear algebra and geometry.* Translated from the 2009 Russian original by David Kramer and Lena Nekludova. Springer, Heidelberg, 2013. xxii+526 pp. ISBN: 978-3-642-30993-9.

18. Turkington, Darrell A. *Generalized vectorization, cross-products, and matrix calculus.* Cambridge University Press, Cambridge, 2013. xii+267 pp. ISBN: 978-1-107-03200-2.

19. Uicker, John J.; Ravani, Bahram; Sheth, Pradip N. *Matrix methods in the design analysis of mechanisms and multibody systems.* Cambridge University Press, Cambridge, 2013. xx+326 pp. ISBN: 978-0-521-76109-3.

20. Zhan, Xingzhi. *Matrix theory.* Graduate Studies in Mathematics, 147. American Mathematical Society, Providence, RI, 2013. x+253 pp. ISBN: 978-0-8218-9491-0.

JOURNAL ANNOUNCEMENT

LAA Special Issue: The Legacy of Hans Schneider

The Editors-in-Chief of the journal *Linear Algebra and its Applications* (LAA) are pleased to announce a special issue, entitled "The Legacy of Hans Schneider," in recognition of the enormous contributions of Hans Schneider to the journal (forty years as an Editor-in-Chief), to research in linear algebra and related areas (nearly 200 papers over more than 60 years, and still continuing), and to the linear algebra community (organizer of the International Linear Algebra Society and its founding president).

We especially welcome contributions that have been influenced by the work of Hans Schneider.

Papers should be submitted by December 31, 2014 via the Elsevier Editorial System (EES): ees.elsevier.com/ laa, choosing the special issue indicated above and any one of the three Editors-in-Chief of *LAA*: Richard A. Brualdi, Volker Mehrmann, and Peter Šemrl.

Send News for *IMAGE* Issue 53

IMAGE Issue 53 is due to appear online on December 1, 2014. Send your news for this issue to the appropriate editor by October 1, 2014. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

- problems and solutions to Bojan Kuzma (bojan.kuzma@upr.si)
- feature articles to Michael Cavers (mcavers@ucalgary.ca)
- history of linear algebra to Naomi Shaked-Monderer (nomi@tx.technion.ac.il)
- book reviews to Douglas Farenick (Doug.Farenick@uregina.ca)
- linear algebra education news to David Strong (David.Strong@pepperdine.edu)
- announcements and reports of conferences, workshops and journals to Minerva Catral (catralm@xavier.edu)
- interviews of senior linear algebraists to Carlos Fonseca (carlos@sci.kuniv.edu.kw)
- advertisements to Amy Wehe (awehe@fitchburgstate.edu).

Send all other correspondence to Kevin N. Vander Meulen (kvanderm@redeemer.ca).

For past issues of *IMAGE*, please visit www.ilasic.org/IMAGE/.

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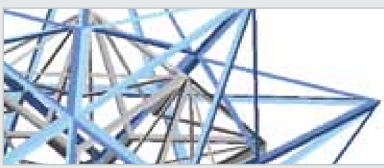

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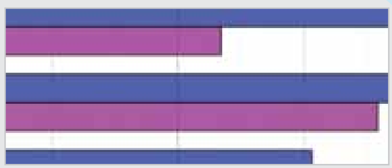

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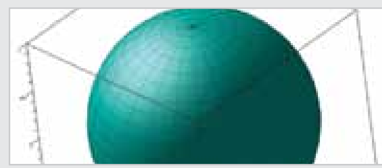

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Extensive updates to visualization abilities for 2-D and 3-D plots

Clickable Math

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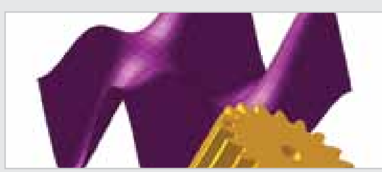

Import/Export

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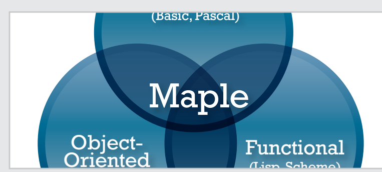

Language and Programming

Enhancements to Maple language and programming facilities include a new URL package and automatic parallelism for sorting

What else is new in Maple 18?

www.maplesoft.com/M18/IMAGE

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BESTSELLING

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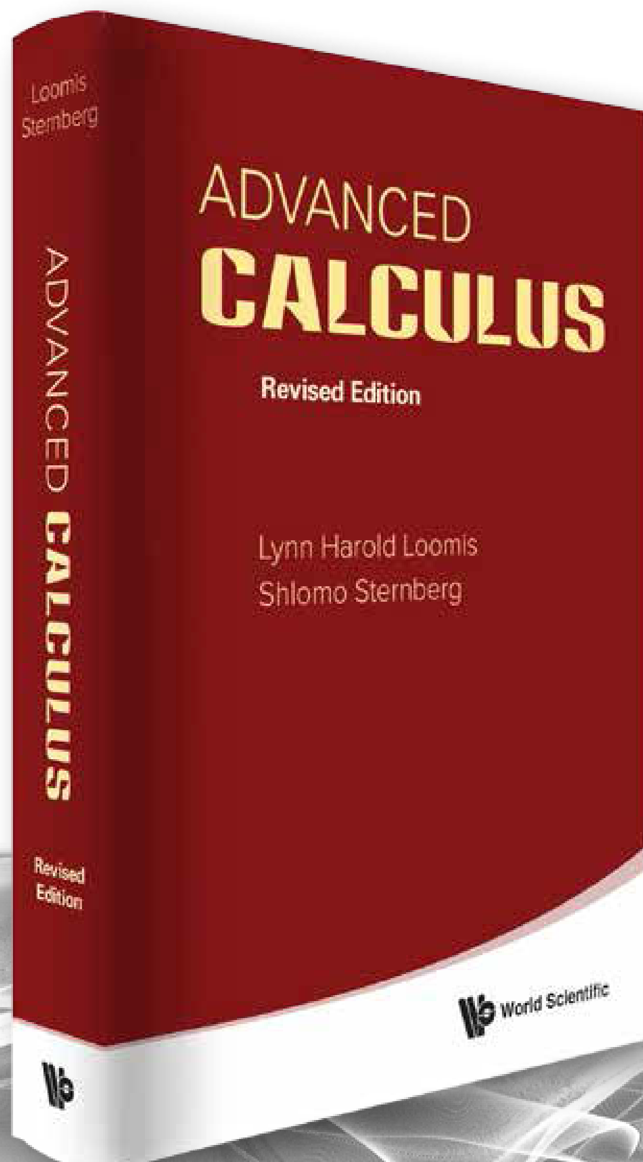

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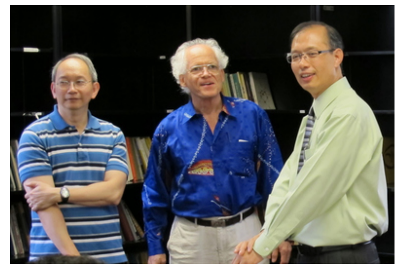
CONFERENCE REPORTS

Frank Uhlig Retirement Colloquium
Auburn University, Alabama, USA, August 23, 2013

Report by Tin-Yau Tam

The Frank Uhlig Retirement Colloquium/Workshop was held in the Department of Mathematics and Statistics at Auburn University on August 23, 2013 to honor Frank's career and achievements. Frank has served the Department for thirty-one years and he retired last fall. During the workshop, Frank received a plaque of Professor Emeritus from Nick Giordano, the Dean of the College of Sciences and Mathematics. The workshop was organized by Tin-Yau Tam as department chair. Many colleagues, friends, co-authors and students came to celebrate this special occasion with Frank and his family. The following is the schedule of the talks on August 23, 2013:



Nam-Kiu Tsing, Frank Uhlig and Tin-Yau Tam

Opening remarks by Tin-Yau Tam, followed by Frank Hall of Georgia State University with a talk on "Some graph theoretic properties of generalized complementary basic properties." Next was one of Frank's students, Chris Fuller of Cumberland University with his talk on "Frank Uhlig's influence on education." After lunch Frank entertained us with his views on life, teaching and research with an "Untitled" talk. Then Alex Fedoseyev from the CFD Research Corporation in Huntsville gave a survey on "Generalized Boltzmann equation by Alexeev with applications to unsolved problems of mechanics: turbulence, Hubble expansion, dark matter & energy," followed by Michele Benzi of Emory University who gave a talk on "Decay properties of matrix functions." Coffee was followed by Peter Nylen of Auburn University on "Eigenvalue interlacing for structured matrices" and lastly, the keynote speaker was Nam-Kiu Tsing who had flown in from the University of Hong Kong to "Revisit some generalized numerical ranges." Throughout the day there was poster viewing for the poster sent by Ivan Ganchev Ivanov of Sofia University in Bulgaria on "Numerical solvers for generalized algebraic Riccati equation."

In the aftermath, the participants enjoyed a great dinner, hosted at Frank and Dorothy's house. Frank shared a lot of great photos while we enjoyed a wonderful meal. I use this occasion to thank Frank and to wish him a happy retirement and good health.

International Workshop on Operator Theory and its Applications (IWOTA 2013)
Bangalore, India, December 16–20, 2013

Report by Tirthankar Bhattacharyya

The organizers of IWOTA look back at a very successful and enlightening conference IWOTA 2013 at the Indian Institute of Science, Bangalore. There were over 200 participants representing a diversity of countries (Canada, USA, Australia, New Zealand, South Africa, Japan, Korea, Saudi Arabia, Georgia, Netherlands, United Kingdom, Germany, France). Out of these there were 178 registrants. The plenary and semi-plenary speakers were not required to register and there were altogether 29 of them. About 70 percent of the participants were from India. Funding was successfully obtained from a combination of sources (detailed below) which helped to keep the registration fee to a moderate 100 US dollars for those who were not exempted from the registration fee. The plenary and semi-plenary speakers and a number of young researchers were exempted. The US NSF travel grant arranged through the efforts of Raul Curto and the Indo US foundation travel grants arranged by T.S.S.R.K. Rao were also very helpful.

Funding was obtained from the following sources: (1) the National Board of Higher Mathematics, India, (2) the Indian Statistical Institute, (3) the International Centre for Theoretical Physics, Trieste, Italy, (4) the National Science Foundation, USA, and (5) the Indo US Education Foundation.

T.S.S.R.K. Rao (front middle) *B.V.R. Bhat* (2nd row, 2nd from left)

T. Bhattacharyya, G. Misra, M.A. Kaashoek and J. Ball

Dinesh Singh, VC of Delhi University, inaugural lecture

The organizing team was congratulated for their excellent work by many participants and by the steering committee of IWOTA. See `http://www.math.vu.nl/~ran/iwota2014/` for the list of members of the steering committee. Birkhäuser will publish a proceedings volume for this IWOTA with Tirthankar Bhattacharyya and Michael Dritschel serving as editors. Submissions can be made to them at tirtha@member.ams.org or m.a.dritschel@ncl.ac.uk before 31 July 2014. The IWOTA 2013 photos are available at `http://math.iisc.ernet.in/~iwota2013`.

Special thanks are due to the Indian Statistical Institute for funds, excellent collaboration and joint work organization.

The 7th Seminar on Linear Algebra and its Applications Mashdad, Iran, February 26–27, 2014

Report by M. S. Moslehian

This seminar, organized by the Iranian Mathematical Society, was held at Ferdowsi University of Mashhad from February 26 to 27, 2014. The seminar provided a forum for 200 participants to present their latest results about all aspects of linear algebra. The plenary speakers were Professors Mehdi Radjabalipour, Sorina Barza and Takeaki Yamazaki. The extended abstracts can be downloaded at `http://profsite.um.ac.ir/~math/slaa7.htm`.

In the opening ceremony, the first "Radjabalipour Award" was given jointly to Dr. Kazem Ghanbari and Mr. Fardin Parvizpour for their best paper presented in the 6th Seminar.

In addition, the 3rd "Functional Analysis Award" for Ph.D. students or recent Ph.D. graduates was given jointly to Dr. Mohsen Kian and Dr. Zainab Kamali for their best papers in Functional Analysis, Harmonic Analysis and Operator Theory (including matrix analysis). The referee committee for the awards consisted of Professors Rajendra Bhatia, Raul Curto, Michael Frank, David Larson, Anthony To-Ming Lau, Volker Runde, Peter Šemrl, Brailey Sims and Dirk Werner.

7th *SLAA group photograph*

Functional Analysis Award, Mashdad, Iran

OBITUARY

Leonid Arie Lerer (April 19, 1943 – January 2, 2014)

Contributed Announcement from Hugo Woerdeman and Rien Kaashoek

Leonia, as he is known to his friends, was an expert in the theory of structured matrices and operators and related matrix-valued functions. He started his mathematical career in Kishinev, Ukraine, with Alek Markus and Israel Gohberg as research supervisors. His Ph.D. thesis, which he defended in 1969 in Kharkov, Ukraine, was devoted to the theory of operators acting in locally convex spaces. Asymptotic distribution of spectra and related limit theorems was one of his other research topics in these early years. In December 1973 he immigrated to Israel. Since 1981 he has been a professor at the Technion in Haifa where more recently he obtained the status of emeritus. He has educated six Ph.D. students and five masters students. His more than 80 papers cover a wide spectrum of topics, ranging from functional analysis and operator theory, linear and multilinear algebra, ordinary differential equations, to systems and control theory. An extended curriculum vitae and a list of publications can be found in "The Leonid Lerer Anniversary Volume" (*Operator Theory: Advances and Application*, Volume 237, Birkhäuser, 2013) which appeared last year on the occasion of his 70th birthday. Leonia is survived by his wife Berta, his daughters Hannah and Saphira, and grandson Itamar.

ILAS NEWS

ILAS members become SIAM Fellows

The Society for Industrial and Applied Mathematics (SIAM) has recently announced its SIAM Fellows for 2014. Two ILAS members are being recognized for "exemplary research" and "outstanding service to the community," namely Valeria Simoncini, and Zdeněk Strakoš.

As noted at SIAM Connect:

"Valeria Simoncini, professor of numerical analysis at Università di Bologna, is being recognized for substantial contributions to Numerical Linear Algebra including eigenvalue problems, saddle-point problems, matrix equations, matrix functions, and model order reduction. Her area of research includes matrix computations and spectral perturbation theory, with applications to PDEs, control, and multivariate statistics. She serves on the editorial boards of the *SIAM Journal on Matrix Analysis and Applications* and the *SIAM Journal on Numerical Analysis*."

"Zdeněk Strakoš, faculty of mathematics and physics at Charles University in Prague, is being honored for advances in numerical linear algebra, especially iterative methods. He previously served on the editorial board of the *SIAM Journal on Matrix Analysis and Applications* and received the SIAM Activity Group on Linear Algebra (SIAG/LA) Prize in 1994. His research interests include modeling of materials, model reduction and efficient numerical methods, applied analysis, and computational mathematics."

Tin-Yau Tam, new editor of the *Alabama Journal of Mathematics*

ILAS member, Tin-Yau Tam of the Department of Mathematics and Statistics of Auburn University has recently taken up the position of the Editor-in-Chief of the *Alabama Journal of Mathematics* `http://ajmonline.org/`.

The *Alabama Journal of Mathematics* is an online and open access journal. It is published under the auspices of the Alabama Council of Teachers of Mathematics (ACTM) and the Alabama Association of College Teachers of Mathematics (AACTM). The Alabama Journal of Mathematics is designed to meet a number of needs of the mathematics community in the State of Alabama. Specifically, the intent of the Journal is to knit together the various components of this mathematical community. As such, the journal includes research articles in mathematics and mathematics education appropriate for a general audience and activities and problems for K-16 mathematics teachers.

The journal has multiple sections in each volume including mathematics research, mathematics education research, and classroom activities. Submissions to the journal are welcome. See `http://ajmonline.org/authors.php` for submission instructions.

ILAS President/Vice President Annual Report: 8 April, 2014

Respectfully submitted by Peter Šemrl, ILAS President, peter.semrl@fmf.uni-lj.si
and Bryan Shader, ILAS Vice-President, bshader@uwyo.edu

1. The ILAS election in the fall/winter of 2013 proceeded via electronic voting. The following were elected in the ILAS 2013 elections to offices with terms that began on 1 March, 2014:

- Board of Directors: Froilán Dopico and Michael Overton (term ends 28 February, 2017) and Eugene Tyrtyshnikov (term ends 28 February, 2015)
- President: Peter Šemrl (term ends 28 February, 2017)

The following continue in the ILAS offices to which they were previously elected:

- Vice President: Bryan Shader (term ends 29 February, 2016)
- Secretary/Treasurer: Leslie Hogben (term ends 28 February, 2015)
- Board of Directors: Dale Olesky (term ends 28 February, 2015), Avi Berman (term ends 29 February, 2016), and Volker Mehrmann (term ends 29 February, 2016)

On 28 February 2014, Steve Kirkland completed two consecutive terms as ILAS President, for a total of six years on the ILAS Executive Board. Steve Kirkland will now serve a one-year term on the Board of Directors until 28 February, 2015. We extend sincere thanks to Steve for his dedicated service to the Society.

Françoise Tisseur and David Watkins completed their three-year terms on the ILAS Board of Directors on February 28, 2014. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated.

We also thank the members of the Nominating Committee - Shaun Fallat, Heike Faßbender, Roger Horn (chair), Yimin Wei, Zdeněk Strakoš - for their work on behalf of ILAS, and to all candidates that agreed to have their names stand for the elections.

2. The following ILAS-endorsed meeting has taken place since our last report: 4$^{\text{th}}$ International Conference on Matrix Analysis and Applications, Selçuk University, Konya, Turkey, July 2–5, 2013.

3. ILAS has endorsed the following conferences of interest to ILAS members:

- Conference on Graph Theory, Matrix Theory and Interactions - A Conference to celebrate the scholarship of David Gregory, Queen's University, Kingston, Canada, June 20–21, 2014
- International Conference on Linear Algebra and its Applications (in honor of Professor Ravindra B. Bapat on his 60$^{\text{th}}$ birthday), Manipal University, Manipal, India, December 18–20, 2014

4. The following ILAS Lectures at non-ILAS conferences have been delivered since the last report:

- Alexander Klyachko, International Conference on Matrix Analysis and Applications, Konya, Turkey, July 2-5, 2013
- Jorge Antezana, Workshop on Matrix Geometries and Applications, International Centre for Theoretical Physics, Trieste, Italy, July 1-12, 2013

5. The Hans Schneider ILAS Lecture Fund was established with a generous donation from ILAS's founding President, Hans Schneider. Additional contributions to the Fund are welcome. The Fund supports one speaker per year to give a lecture on linear algebra (and possibly its applications to, or connections with, a related discipline) at a non-ILAS meeting. The Hans Schneider ILAS Lecture is one of ILAS's signature programmes. Consequently, the Hans Schneider ILAS Lecture will be given by a researcher with strong expository skills whose work represents linear algebra at its highest level. Each year, the ILAS Board of Directors will evaluate proposals for ILAS Lectures at non-ILAS meetings. The top ranked proposal may be identified as the Hans Schneider ILAS Lecture. In any given year, the Board may determine that no speaker be designated as the Hans Schneider ILAS Lecturer. The first Hans Schneider ILAS Lecturer will be Shaun Fallat at the Conference on Graph Theory, Matrix Theory and Interactions - A Conference to celebrate the scholarship of David Gregory, Queen's University, Kingston, Canada, June 20–21, 2014.

6. The following ILAS conferences are scheduled:

- 20$^{\text{th}}$ ILAS Conference, July 11–15, 2016, Leuven, Belgium. The chair of the organizing committee is Raf Vanderbil.
- 21$^{\text{st}}$ ILAS Conference: Connections, July 24–28, 2017, Ames, Iowa, USA. The chair of the organizing committee is Leslie Hogben.

7. The *Electronic Journal of Linear Algebra* (ELA) is now in its 27$^{\text{th}}$ volume. The Editor-in-Chief is Bryan Shader. In 2013 *ELA* received 228 submissions, published 61 papers for a total of 961 pages. A special issue devoted to papers from the conference on Graph Theory, Matrix Theory and Interactions is planned. This is a conference to celebrate the contributions of longtime ILAS member David Gregory. Editors for the issue are: Sebastian Cioabă, Ram Murty, Claude Tardif, Kevin N. Vander Meulen and David Wehlau. *ELA* has begun the process of moving to a new platform which will provide a more modern web portal, online submission capabilities, as well as better tracking and discoverability.

8. *IMAGE* is the semi-annual bulletin for ILAS, and the Editor-in-Chief is Kevin N. Vander Meulen, supported by contributing editors Minerva Catral, Michael Cavers, Douglas Farenick, Carlos Fonseca, Bojan Kuzma, Naomi Shaked-Monderer, David Strong, and Amy Wehe.

Peter Šemrl, who spearheaded the series on the the history of linear algebra since 2008, resigned as contributing editor in the summer 2013. We are grateful to Naomi Shaked-Monderer for taking on the position of contributing editor for the history of linear algebra series.

9. ILAS-NET is a moderated newsletter for mathematicians worldwide, with a focus on linear algebra; it is managed by Sarah Carnochan Naqvi. As of April 7, 2014 there are 809 subscribers to ILAS-NET. In July, 2013, ILAS-NET moved to the GoDaddy.com Express Email Marketing system. This is an automated system which manages user accounts. We are having problems with users being removed from the system because at some point the system has received a bounced message from the user's account. There are approximately 80 ILAS-NET members who have been removed for this reason. There is a temporary archive of messages at www.ilasic.org/ilas-net/MSG.html which members can check at any time.

To send a message to ILAS-NET, please email ilasic@uregina.ca. To subscribe to ILAS-NET, please complete the form at www.ilasic.org/ilas-net/subform.html.

10. The ILAS Information Centre (IIC) is at www.ilasic.org. The website moved to GoDaddy.com at the end of 2012. The membership form is hosted at the University of Regina, but this is to be moved this year.

Respectfully submitted,

Peter Šemrl, ILAS President, peter.semrl@fmf.uni-lj.si; and
Bryan Shader, ILAS Vice-President, bshader@uwyo.edu

ILAS 2013-2014 Treasurer's Report

April 1, 2013 - March 31, 2014
By Leslie Hogben

Net Account Balance on March 31, 2013

Vanguard (ST Fed Bond Fund Admiral 7548.536 Shares)		$82,683.93	
Checking Account - First Federal/Great Western		$37,078.92	
Certificate of Deposit		$20,481.99	
Account's Payable		$(700.00)	
			$139,544.84
General Fund		$77,233.20	
Conference Fund		$9,738.01	
ILAS/LAA Fund		$12,840.57	
Olga Taussky-Todd/John Todd Fund		$11,518.14	
Frank Uhlig Education Fund		$4,985.34	
Hans Schneider Prize Fund		$23,229.57	
			$139,544.83

INCOME:

Dues	$6,740.00		
Elsevier Sponsorship (Flow Through)	$2,500.00		
General Fund	$3,026.00		
Conference Fund	$141.00		
LAMA for Speaker			
Taussky-Todd Fund	$231.00		
Uhlig Education Fund	$11.00		
Hans Schneider Prize Fund	$161.00		
Hans Schneider ILAS Lecure Fund	$15,000.00		
Interest - Great Western	$72.97		
Loan at Conference (approx)			
Interest on Great Western Certificate of Deposit	$15.32		
Vanguard - Dividend Income	$547.32		
- Short Term Capital Gains	$-		
- Long Term Capital Gains	$-		
Variances in the Market	$(536.70)		
Misc Income	$-		
Total Income		$27,908.91	

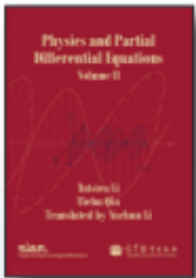
EXPENSES:

First Data - Credit Card Processing Fees	$619.62		
Speaker Fees	$973.00		
Treasurer's Report Preparation	$305.00		
Treasurer's Assistant	$177.50		
Conference Expenses	$4,450.99		
Student Travel Support - Elsevier Flowthru	$2,500.00		
Business License	$122.50		
IMAGE Costs	$1,050.54		
Wire Transfer Fees	$-		
Misc Expenses	$-		
Ballot Costs	$895.95		
Total Expenses		$11,095.10	

Net Account Balance on March 31, 2014

Vanguard (ST Fed Bond Fund Admiral 7714.044 Shares)		$82,694.55	
Checking Account - Great Western		$53,866.79	
Certificate of Deposit		$20,497.31	
Account's Payable		$(700.00)	
			$156,358.65
General Fund		$78,503.01	
Conference Fund		$9,879.01	
ILAS/LAA Fund		$12,840.57	
Olga Taussky-Todd/John Todd Fund		$11,749.14	
Frank Uhlig Education Fund		$4,996.34	
Hans Schneider Prize Fund		$23,390.57	
Hans Schneider ILAS Lecure Fund		$15,000.00	
			$156,358.64

Titles of Interest from SIAM



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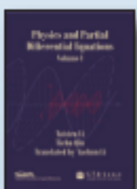

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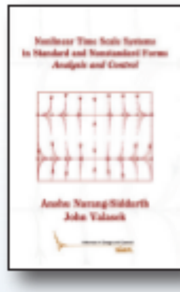

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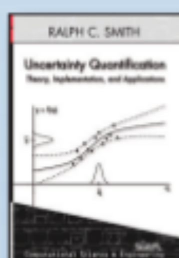

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SIAM. SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to *IMAGE* Problem 49.3 and to all Problems in 51 issue of *IMAGE* except for 51-3 and 51-6 for which we still seek a solution. Six new problems are on the last page; solutions are invited.

Problem 49-3: Loewner Partial Order

Proposed by Minghua LIN, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let $A, B, C, D, X, Y \in M_n(\mathbb{C})$ such that $A > XB^{-1}X^* \geq 0$, $C > YD^{-1}Y^* \geq 0$ (in Löwner partial order). Define

$$R = B - X^*(A + CA^{-1}C)^{-1}X + Y^*(A + CA^{-1}C)^{-1}Y - X^*(C + AC^{-1}A)^{-1}Y - Y^*(C + AC^{-1}A)^{-1}X.$$

Show that $0 < R < \frac{3}{2}(B + D)$. What is the smallest constant c such that $R \leq c(B + D)$?

Solution 49-3.1 by the proposer Minghua LIN

We need to require in addition that B and D be positive definite. Recall [1] that a complex matrix is accretive-dissipative if in its Cartesian decomposition, both real and imaginary part are positive definite. The matrix

$$T = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} + i \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix}$$

is accretive-dissipative because its real part satisfies $\left[\begin{smallmatrix} I & -XB^{-1} \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right] \left[\begin{smallmatrix} I & -XB^{-1} \\ 0 & I \end{smallmatrix}\right]^* = \left[\begin{smallmatrix} A-XB^{-1}X^* & 0 \\ 0 & B \end{smallmatrix}\right] > 0$ and likewise for its imaginary part. Also, by [1, Lemma 1], the inverse of accretive-dissipative $A + iC$ satisfies $(A + iC)^{-1} = (A + CA^{-1}C)^{-1} - i(C + AC^{-1}A)^{-1}$, and moreover, its real part, $E = (A + CA^{-1}C)^{-1}$ is positive definite, while its imaginary part $F = -(C + AC^{-1}A)^{-1}$ is negative definite. Then R given above is, by [1, Equation (39)], a real part of the Schur complement of T_{11} i.e., a real part of a matrix $T_{22} - T_{12}^* T_{11}^{-1} T_{12}$ and by discussion on [1, p. 2464], R is bounded above by $\frac{3}{2}(B+D)$. Moreover, by [2, Property 6], the Schur complement of accretive-dissipative matrix is again accretive-dissipative, which implies that R is positive definite.

The second question is open, any partial solution or comment is welcome. ∎

References

[1] Kh.D. Ikramov, Determinantal inequalities for accretive-dissipative matrices, *J. Math. Sci. (N.Y.)* 121 (2004) 2458–2464.

[2] A. George and Kh.D. Ikramov, On the properties of accretive-dissipative matrices, *Math. Notes* 77 (2005) 767–776.

Editorial note: We are still seeking an elementary proof.

Problem 51-1: A Norm Identity

Proposed by Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu
and Minghua LIN, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let $a_1, \ldots, a_n$ be nonnegative numbers, let $x_1, \ldots, x_n \in \mathbb{C}^N$, and let $\|\cdot\|$ be the Euclidean norm. Show that

$$\sum\nolimits_{i=1}^n a_i \left\| \sum\nolimits_{j=1}^n a_j(x_j - x_i) \right\|^2 = \frac{1}{2} \left(\sum\nolimits_{k=1}^n a_k \right) \sum\nolimits_{i,j=1}^n a_i a_j \|x_i - x_j\|^2.$$

Solution 51.1.1 by the proposers Dennis S. BERNSTEIN and Minghua LIN

Lagrange's second identity [1] states

$$\sum\nolimits_{i=1}^n m_i \|v - v_i\|^2 = M \left\| v - \frac{1}{M} \sum\nolimits_{i=1}^n m_i v_i \right\|^2 + \frac{1}{2M} \sum\nolimits_{i,j=1}^n m_i m_j \|v_i - v_j\|^2 \qquad (*)$$

for any points $v, v_1, \ldots, v_n \in \mathbb{C}^N$, and every family of real weights $m_i \geq 0$ with $\sum_{i=1}^n m_i = M > 0$. Take $v = \sum_{j=1}^n a_j x_j$, $m_i = a_i$, and $v_i = x_i \sum_{j=1}^n a_j$, $(i = 1, \ldots, n)$. After some calculation we get $\left\| v - \frac{1}{M} \sum_{i=1}^n m_i v_i \right\| = 0$, so the right hand side of $(*)$ equals

$$\frac{1}{2M}\sum_{i,j=1}^{n} m_i m_j\|v_i - v_j\|^2 = \frac{1}{2\sum_{i=1}^{n} a_i}\sum_{i,j=1}^{n} a_i a_j \left\|(x_i - x_j)\sum_{k=1}^{n} a_k\right\|^2 = \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)\sum_{i,j=1}^{n} a_i a_j\|x_i - x_j\|^2$$

while the left hand side of $(*)$ is

$$\sum_{i=1}^{n} m_i\|v - v_i\|^2 = \sum_{i=1}^{n} a_i\left\|\sum_{j=1}^{n} a_j x_j - x_i\sum_{j=1}^{n} a_j\right\|^2 = \sum_{i=1}^{n} a_i\left\|\sum_{j=1}^{n} a_j(x_j - x_i)\right\|^2.$$

The desired equality follows. ∎

Reference

[1] M. Gidea and C.P. Niculescu, A brief account on Lagrange's algebraic identity, *Math. Intelligencer* 34 (2012) 55–61.

Solution 51-1.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let $y_i = x_i - x_n$, $i = 1, \ldots, n$. Thus $y_n = 0$, and $x_j - x_i = y_j - y_i$ for all i and j. Also, let $s = \sum_{k=1}^{n} a_k$. The right-hand side (RHS) of the identity can then be written as

$$\begin{aligned}\text{RHS} &= s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j\|x_i - x_j\|^2 \\ &= s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j\|y_i - y_j\|^2 = s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j(\|y_i\|^2 + \|y_j\|^2 - 2\operatorname{Re}(\langle y_i, y_j\rangle)) \\ &= s\sum_{j=1}^{n} a_j\|y_j\|^2(s - a_j) - 2s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j \operatorname{Re}(\langle y_i, y_j\rangle). \qquad (8)\end{aligned}$$

The i-th term of the left-hand side (LHS_i) can be written as

$$\begin{aligned}\text{LHS}_i &= a_i\left\|\sum_{j=1,\, j\neq i}^{n} a_j(y_j - y_i)\right\|^2 = a_i\left\|\sum_{j=1,\, j\neq i}^{n-1} a_j(y_j - y_i) - a_n y_i\right\|^2 \\ &= a_i\left\|\sum_{j=1,\, j\neq i}^{n-1} a_j y_j - (s - a_i)y_i\right\|^2; \qquad (i = 1, \ldots, n). \qquad (9)\end{aligned}$$

We show that LHS = RHS by showing that the coefficients of $\|y_j\|^2$ and the coefficients of $\operatorname{Re}(\langle y_i, y_j\rangle)$ on both sides are equal. From (2), the coefficient of $\|y_j\|^2$ for $1 \le j \le n-1$ in the left-hand side is

$$a_j(s - a_j)^2 + \sum_{i=1,\, i\neq j}^{n} a_i\, a_j^2 = a_j(s - a_j)^2 + (s - a_j)a_j^2 = a_j(s - a_j)s,$$

which is also the coefficient of $\|y_j\|^2$ in (1). From (2), the terms containing $\operatorname{Re}(\langle y_j, y_k\rangle)$ for $1 \le j, k \le n-1$ with $j \neq k$ in the left-hand side are

$$\begin{aligned}&\sum_{i=1,\, i\notin\{j,k\}}^{n}\Big(a_i\, a_j a_k 2\operatorname{Re}(\langle y_j, y_k\rangle)\Big) - a_j(s - a_j)2\operatorname{Re}(\langle a_k y_k, y_j\rangle) - a_k(s - a_k)2\operatorname{Re}(\langle a_j y_j, y_k\rangle) \\ &\qquad = 2\big(a_j a_k(s - a_j - a_k) - a_j a_k(s - a_j) - a_j a_k(s - a_k)\big)\operatorname{Re}(\langle y_j, y_k\rangle) \\ &\qquad = -2s a_j a_k \operatorname{Re}(\langle y_j, y_k\rangle)\end{aligned}$$

which agrees with the corresponding term in (1). ∎

Solution 51-1.3 by Omran KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Let $s = a_1 + a_2 + \cdots + a_n$. We will just expand the two sides of the proposed identity. On the one hand we have

$$\begin{aligned}
\sum\nolimits_{i=1}^n a_i \left\|\sum\nolimits_{j=1}^n a_j(x_j - x_i)\right\|^2 &= \sum\nolimits_{i=1}^n a_i \left(\sum\nolimits_{j,k=1}^n a_j a_k (x_j - x_i)^*(x_k - x_i)\right) \\
&= \sum\nolimits_{i,j,k=1}^n a_i a_j a_k \left(x_j^* x_k - x_j^* x_i - x_i^* x_k + \|x_i\|^2\right) \\
&= s\sum\nolimits_{j,k=1}^n a_j a_k x_j^* x_k - s\sum\nolimits_{j,i=1}^n a_j a_i x_j^* x_i - s\sum\nolimits_{i,k=1}^n a_i a_k x_i^* x_k + s^2 \sum\nolimits_{i=1}^n a_i \|x_i\|^2 \\
&= s^2 \sum\nolimits_{i=1}^n a_i \|x_i\|^2 - s\left\|\sum\nolimits_{j=1}^n a_j x_j\right\|^2. \qquad (1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum\nolimits_{i,j=1}^n a_i a_j \|x_i - x_j\|^2 &= \sum\nolimits_{i,j=1}^n a_i a_j \left(\|x_i\|^2 + \|x_j\|^2 - 2\,\mathrm{Re}(x_i^* x_j)\right) \\
&= s\sum\nolimits_{i=1}^n a_i \|x_i\|^2 + s\sum\nolimits_{i=1}^n a_j \|x_j\|^2 - 2\left(\sum\nolimits_{i=1}^n a_i x_i\right)^* \left(\sum\nolimits_{j=1}^n a_j x_j\right) \\
&= 2s\sum\nolimits_{i=1}^n a_i \|x_i\|^2 - 2\left\|\sum\nolimits_{i=1}^n a_i x_i\right\|^2. \qquad (2)
\end{aligned}$$

The desired identity follows, comparing (1) and (2). ∎

Problem 51-2: Commutants of Diagonal Matrices

Proposed by Gregor DOLINAR, *University of Ljubljana, Slovenia*, gregor.dolinar@fe.uni-lj.si

Let $\mathbb{C}$ be a complex field. The commutant (also known as the centralizer) of an $m \times m$ matrix $A \in M_m(\mathbb{C})$ is the set $A' = \{X \in M_m(\mathbb{C}) :\ AX = XA\}$. Find a recursive formula for the cardinality $N_m = \#\{A' :\ A \text{ diagonal}\}$ of the set of all commutants of diagonal $m \times m$ matrices, as a function of m.

Solution 51-2.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

The scalars can belong to any field $\mathbb{F}$. Let $A = \mathrm{diag}(a_1, \ldots, a_m), X = [x_{ij}]_{i,j=1..m} \in M_m(\mathbb{F})$. Then

$$(AX - XA)_{ij} = a_i x_{ij} - a_j x_{ij} = (a_i - a_j)x_{ij}; \qquad (i, j = 1, \ldots, n).$$

Hence $X \in A'$ if and only if $x_{ij} = x_{ji} = 0$ whenever $a_i \neq a_j$. We therefore have a one-to-one correspondence between the commutants A' for A diagonal and the partitions of the set of integers $[m] = \{1, \ldots, m\}$. Specifically, if $S_1 \cup \cdots \cup S_k$ is a partition of $[m]$, then A' is the set of all matrices $X = [x_{ij}]$ such that $x_{ij} = x_{ji} = 0$ whenever i and j belong to different sets of the partition. The partition of $[m]$ is created from the matrix $A = \mathrm{diag}(a_1, \ldots, a_m)$ by putting i and j in the same set of the partition if and only if $a_i = a_j$.

Therefore $N_m = B_m$, the m-th Bell number (see [1] or [2, p. 33]). The Bell numbers can be constructed using the Bell triangle (i.e., Aitken's array) and satisfy the recurrence relation

$$B_{m+1} = \sum_{k=0}^{m} \binom{m}{k} B_k, \quad \text{for } m \geq 0$$

with initial condition $B_0 = 1$. ∎

References

[1] `http://en.wikipedia.org/wiki/Bell_number`.

[2] R.P. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, 1997.

Also solved by the proposer.

Problem 51-4: On Common Divisors of Polynomial Matrices

Proposed by Volodymyr PROKIP, *Institute for Applied Problems of Mechanics and Math., Ukraine*, v.prokip@gmail.com

Let F be a field and let $M_{m,l}(F[x])$ be the set of $m \times l$ matrices over the polynomial ring $F[x]$. Let

$$A(x) = A_0 x^p + A_1 x^{p-1} + \cdots + A_p \in M_{m,n}(F[x]); \quad p \geq 1,$$
$$B(x) = B_0 x^q + B_1 x^{q-1} + \cdots + B_q \in M_{m,k}(F[x]); \quad q \geq 1$$

with $\operatorname{rank}\begin{bmatrix} A_p & B_q \end{bmatrix} = r < m$. Show that for matrices $A(x)$ and $B(x)$ there exists a common left divisor $D(x) \in M_{m,m}(F[x])$, i.e., $A(x) = D(x)\tilde{A}(x)$ and $B(x) = D(x)\tilde{B}(x)$, such that $\deg(\det D(x)) \geq m - r$.

Editorial note: We apologize that the original problem 51-4 *was published with the incorrect title "An Adjugate Identity."*

Solution 51-4.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let E be the product of elementary matrices such that $E[A_p \ B_q]$ has $m-r$ rows of zeros at the bottom. Then the same is true for EA_p and EB_q. Hence, there exist matrices $M(x) \in M_{r,n}(F[x])$ and $N(x) \in M_{m-r,n}(F[x])$ such that

$$EA(x) = \begin{bmatrix} M(x) \\ xN(x) \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & xI_{m-r} \end{bmatrix} \begin{bmatrix} M(x) \\ N(x) \end{bmatrix}.$$

Therefore $A(x) = D(x)\widetilde{A}(x)$, where

$$D(x) = E^{-1} \begin{bmatrix} I_r & O \\ O & xI_{m-r} \end{bmatrix}, \qquad \widetilde{A}(x) = \begin{bmatrix} M(x) \\ N(x) \end{bmatrix}.$$

$B(x)$ can be factored in the same way with the same left factor $D(x)$. Since $\det D(x) = \det E^{-1} \det \left[\begin{smallmatrix} I_r & O \\ O & xI_{m-r} \end{smallmatrix}\right] = c\, x^{m-r}$ with $c \neq 0$, the degree of $D(x)$ is exactly $m-r$. ∎

Solution 51.4.2 by the proposer Volodymyr PROKIP

Denote the $n \times n$ identity matrix by I_n and the $m \times n$ zero matrix by $\mathrm{O_{m,n}}$. For the matrix $[A_p \ B_q]$ there exist linearly independent row vectors $u_1, u_2, \ldots, u_{m-r} \in M_{1,m}(F)$ such that

$$\begin{bmatrix} u_1 \\ \vdots \\ u_{m-r} \end{bmatrix} \begin{bmatrix} A_p & B_q \end{bmatrix} = [\mathrm{O_{m-r,n}} \ \mathrm{O_{m-r,k}}].$$

For the vectors $u_1, \ldots, u_{m-r}$, there exists a matrix $U_0 \in M_{r,m}(F)$ such that

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_{m-r} \\ U_0 \end{bmatrix} \in M_{m,m}(F)$$

is nonsingular. Thus, $U\begin{bmatrix} A_p & B_q \end{bmatrix} = \left[\begin{smallmatrix} \mathrm{O_{m-r,n}} & \mathrm{O_{m-r,k}} \\ \widetilde{A}_p & \widetilde{B}_q \end{smallmatrix}\right]$ where $\widetilde{A}_p \in M_{r,n}(F)$ and $\widetilde{B}_q \in M_{r,k}(F)$ and we have

$$\begin{bmatrix} A(x) & B(x) \end{bmatrix} = U^{-1}U\begin{bmatrix} A(x) & B(x) \end{bmatrix} = U^{-1}\left(\begin{bmatrix} \dot{A}(x) & \dot{B}(x) \\ \ddot{A}(x) & \ddot{B}(x) \end{bmatrix} x + \begin{bmatrix} \mathrm{O_{m-r,n}} & \mathrm{O_{m-r,k}} \\ \widetilde{A}_p & \widetilde{B}_q \end{bmatrix} \right) = U^{-1} \begin{bmatrix} xI_{m-r} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \widehat{A}(x) & \widehat{B}(x) \end{bmatrix}.$$

The matrix $D(x) = U^{-1}\mathrm{diag}\,(xI_{m-r}, I_r)$ is a common left divisor for $A(x)$ and $B(x)$ and $\deg(\det D(x)) = m - r$. ∎

Problem 51-5: Orthogonal Basis Subordinated to a Plane Arrangement in $\mathbb{R}^3$

Proposed by Denis SERRE, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Let $E_1, E_2, E_3 \subseteq \mathbb{R}^3$ be planes of respective equations $z_j \cdot x = 0$. Prove that there exists an orthogonal basis $\{v_1, v_2, v_3\} \subseteq \mathbb{R}^3$ such that $v_j \in E_j$ for $j = 1, 2, 3$ if and only if

$$\Delta := (\det(z_1, z_2, z_3))^2 - 4(z_1 \cdot z_2)(z_2 \cdot z_3)(z_3 \cdot z_1)$$

is non-negative. When Δ is positive, there exist two such bases, up to scaling.

Solution 51-5.1 by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Without loss of generality we may suppose that $\|z_1\| = \|z_2\| = \|z_3\| = 1$. First we will consider the general case: $(z_1 \cdot z_2)(z_2 \cdot z_3)(z_3 \cdot z_1) \neq 0$. There is an orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\mathbb{R}^3$ such that

$$z_1 = \mathbf{i}, \quad z_2 = \alpha\,\mathbf{i} + \beta\,\mathbf{j}, \quad z_3 = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}.$$

Let

$$v_1 = \cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k} \tag{1}$$

be an arbitrary normalized vector from E_1. A vector $v_2 \in E_2$ is orthogonal to v_1 if and only if it is proportional to $v_1 \wedge z_2$. We may take

$$v_2 = -\beta\sin\theta\,\mathbf{i} + \alpha\sin\theta\,\mathbf{j} - \alpha\cos\theta\,\mathbf{k}. \tag{2}$$

Finally, a vector v_3 is orthogonal to both v_1 and v_2, if and only if it is proportional to $v_1 \wedge v_2$, and we may take

$$v_3 = -\alpha\,\mathbf{i} - \beta\sin^2\theta\,\mathbf{j} + \beta\cos\theta\sin\theta\,\mathbf{k}. \tag{3}$$

Now, $v_3 \in E_3$ if and only if $v_3 \cdot z_3 = 0$, that is, if and only if $m := \tan\theta$ is a solution to the second degree equation:

$$-\alpha\lambda(1 + m^2) - \beta\mu m^2 + \beta\nu m = 0$$

or, in terms of z_1, z_2 and z_3:

$$(z_2 \cdot z_3)m^2 - \det(z_1, z_2, z_3)m + (z_1 \cdot z_2)(z_3 \cdot z_1) = 0.$$

This equation has a solution m if and only if its discriminant Δ is nonnegative, and when $\Delta > 0$ we find two distinct values for m each one of them yields an orthogonal basis $\{v_1, v_2, v_3\}$ which is uniquely defined by (1)—(3) up to scaling.

Next, consider the particular case $(z_1 \cdot z_2)(z_2 \cdot z_3)(z_3 \cdot z_1) = 0$. Without loss of generality we may suppose that $z_1 \cdot z_2 = 0$, and choose orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\mathbb{R}^3$ so that $z_1 = \mathbf{i}$, $z_2 = \mathbf{j}$ and $z_3 = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}$. There are several cases:

- With $v_1 = \cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k}$ for some θ such that $\sin\theta \neq 0$. As explained before, we must have $v_2 = \mathbf{i}$ and θ is determined by $\nu\cos\theta - \mu\sin\theta = 0$ (unless $\nu = \mu = 0$, that is $E_1 = E_3 \perp E_2$, in which case there are clearly infinitely many solutions to the proposed problem).

- With $v_1 = \mathbf{j}$, we can take $v_2 = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{k}$ determine φ by $\nu\cos\varphi - \lambda\sin\varphi = 0$ (unless $\nu = \lambda = 0$, that is $E_2 = E_3 \perp E_1$, and there are, in this case also infinitely many solutions to the proposed problem).

Consequently, when $\Delta > 0$ (that is $\nu \neq 0$) we also have two solutions to the proposed problem, up to scaling. ∎

Solution 51-5.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

If two of the planes are identical, there exists such an orthogonal basis if and only if the third plane is orthogonal to the others which, in this case, is equivalent to $\Delta = 0$. In fact, every pair of orthogonal vectors in the two identical planes is part of such an orthogonal basis. Thus, from now on, we may assume that each of the pairs $\{z_1, z_2\}, \{z_2, z_3\}, \{z_3, z_1\}$ is linearly independent. Next, suppose two of the planes are perpendicular; say, $z_1 \cdot z_2 = 0$, and so $\Delta \geq 0$. Then, for $v_1 \in E_1 \backslash \{0\}$ and $v_2 \in E_2 \backslash \{0\}$, the pair (v_1, v_2) is orthogonal if and only if $v_1 = z_2$ or $v_2 = z_1$ (or multiples thereof). Consider the first possibility, $v_1 = z_2$. Since $\{z_2, z_3\}$ is linearly independent, E_2 and E_3 intersect in a line L through the origin. Then (v_1, v_2, v_3) is an orthogonal basis of the desired type if and only if v_3 lies on L and v_2 lies on the line in E_2 perpendicular to L. The case $v_2 = z_1$ is similar, and so there are exactly two such orthogonal bases (up to scaling). From now on, we also assume $z_i \cdot z_j \neq 0$ for all $i \neq j$.

With a fixed nonzero vector $v_3 \in E_3$, there is, up to scaling, a unique vector $v_1 \in E_1$ perpendicular to v_3, and it equals $v_1 = v_3 \times z_1$. Likewise, there is, up to scaling, a unique vector $v_2 \in E_2$ perpendicular to v_3, and it equals $v_2 = v_3 \times z_2$. Therefore, the set $\{u \in E_3;\ (u \times z_1) \cdot (u \times z_2) = 0\}$ coincides with the set of all vectors $u = v_3$ for which there exists an orthogonal basis (v_1, v_2, v_3) of the desired type. We claim that $(u \times z_1) \cdot (u \times z_2) = 0$ is the equation of a double elliptic cone with vertex at the origin. Hence, its intersection with the plane E_3 is either the origin, a single line through the origin, or two lines through the origin. We claim that these outcomes correspond, respectively, to $\Delta < 0$, $\Delta = 0$, $\Delta > 0$.

To simplify our calculations for the proofs of these claims, we transform z_1, z_2, z_3. Since Δ is invariant under orthogonal transformations and changes only in magnitude under scale changes of z_1, z_2, z_3, we may assume

$$z_1 = (1, a, 0), \quad z_2 = (1, -a, 0), \quad z_3 = (p, q, r)$$

for some $a > 0, a \neq 1$ and some $p, q, r \in \mathbb{R}$. Hence $\Delta = 4\left(r^2a^2 - (1-a^2)(p^2 - a^2q^2)\right)$ and

$$(u \times z_1) \cdot (u \times z_2) = \big((u_1, u_2, u_3) \times (1, a, 0)\big) \cdot \big((u_1, u_2, u_3) \times (1, -a, 0)\big) = (1 - a^2)u_3^2 + u_2^2 - a^2u_1^2.$$

Therefore, if $a < 1$, the equation $a^2u_1^2 = u_2^2 + (1 - a^2)u_3^2$ defines an elliptical double cone along the u_1 axis. A cone $z^2 = a^2x^2 + b^2y^2$ and a plane $z = mx + ny$ intersect at the origin, one line through the origin, or two lines through the origin if the discriminant $a^2n^2 + b^2m^2 - a^2b^2$ is, respectively, negative, zero, or positive. Note that the equation for the plane E_3 is $pu_1 + qu_2 + ru_3 = 0$, and if $p \neq 0$, this discriminant is

$$\frac{1}{a^2}\frac{r^2}{p^2} + \left(\frac{1-a^2}{a^2}\right)\frac{q^2}{p^2} - \frac{1}{a^2}\left(\frac{1-a^2}{a^2}\right) = \frac{1}{a^4p^2}\left(r^2a^2 - (1-a^2)(p^2 - a^2q^2)\right) = \frac{\Delta}{4a^4p^2}.$$

If $p = 0$, the plane $qu_2 + ru_3 = 0$ contains the u_1 axis and thus intersects the cone in two lines. Also, $\Delta > 0$ when $p = 0$. The case $a > 1$ is similar. ∎

Solution 51.5.3 by the proposer Denis SERRE

Let Z be the matrix whose columns are the z_j's. If an orthogonal basis $\{v_1, v_2, v_3\}$ exists with $v_j \perp z_j$, we may assume that it is orthonormal. Let us define the matrix V whose columns are the v_j's; we have $V \in \mathbf{O}_3(\mathbb{R})$. By assumption, the diagonal entries of $M := V^TZ$ are zeroes. Moreover $Z^TZ = M^TM$. If m_j denotes the jth column of M, this means $m_i \cdot m_j = z_i \cdot z_j$, and this also implies that $(\det Z)^2 = (\det M)^2$. Therefore $\Delta(Z) = \Delta(M)$, where $\Delta(M) := \det(M)^2 - 4(m_1.m_2)(m_2.m_3)(m_1.m_3)$. Say that

$$M = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}. \tag{10}$$

Then $m_1 \cdot m_2 = ef$, and so on. Therefore

$$\Delta(M) = (ade + bcf)^2 - 4abcdef = (ade - bcf)^2 \geq 0.$$

This proves the necessary condition $\Delta(Z) \geq 0$.

Conversely, suppose that $\Delta(Z) \geq 0$. If $\det Z = 0$, then it follows from $\Delta(Z) \geq 0$ that at least two among vectors z_1, z_2, z_3, say z_1 and z_2, are orthogonal in which case $v_1 = z_2$, $v_2 = z_1$, and $v_3 \in \{z_1, z_2\}^\perp$ solves the problem. In the sequel we assume $\det Z \neq 0$. If all the off-diagonal entries of Z^TZ vanish, we are done, with $(v_1, v_2, v_3) = (z_2, z_3, z_1)$ for instance. If not, say $z_1 \cdot z_3 \neq 0$, then the hypothesis ensures the existence of a real solution $e, f \in \mathbb{R}$ (actually, two distinct solutions if $\Delta(Z) > 0$) of

$$ef = z_1 \cdot z_2, \qquad e(z_1 \cdot z_3) + f(z_2 \cdot z_3) = \det Z.$$

Inserting thus obtained e, f, together with $a = c = 1$, $b = z_2 \cdot z_3$, and $d = z_1 \cdot z_3$ into (10) we get a matrix M such that the off-diagonal entries of M^TM equal those of Z^TZ, and on the other hand $\det M = \det Z$. Let $p, q, r > 0$ and let $D = \operatorname{diag}(\sqrt{\frac{1}{qr}}, \sqrt{\frac{1}{pr}}, \sqrt{\frac{1}{pq}})$ and $D' = \operatorname{diag}(p, q, r)$. Then $N := DMD'$ has the same property: its diagonal is null, the off-diagonal entries of N^TN equal those of Z^TZ, and $\det N = \det Z$. We have $\|n_1\|^2 = p\left(\frac{c^2}{r} + \frac{e^2}{q}\right)$ and $\|n_2\|^2 = q\left(\frac{a^2}{r} + \frac{f^2}{p}\right)$. Hence we may choose $p = r\left(\frac{\|z_1\|^2\|z_2\|^2 - e^2f^2}{e^2 + \|z_2\|^2}\right) > 0$, $q = r\left(\frac{\|z_1\|^2\|z_2\|^2 - e^2f^2}{f^2 + \|z_1\|^2}\right) > 0$, $r > 0$, $\|n_1\|^2 = \|z_1\|^2$, and $\|n_2\|^2 = \|z_2\|^2$ (note that $p > 0$ follows from $\det Z \neq 0$ by the Cauchy-Schwarz-Bunjakowsky inequality). Then the fact that N^TN and Z^TZ have the same determinant and all their entries except perhaps the $(3, 3)$-one coincide, imply that the latter coincides too (note that its cofactor nonnegative; if it vanishes, the problem is easier to solve). The columns of the orthogonal matrix $V := ZN^{-1}$ form a solution. ∎

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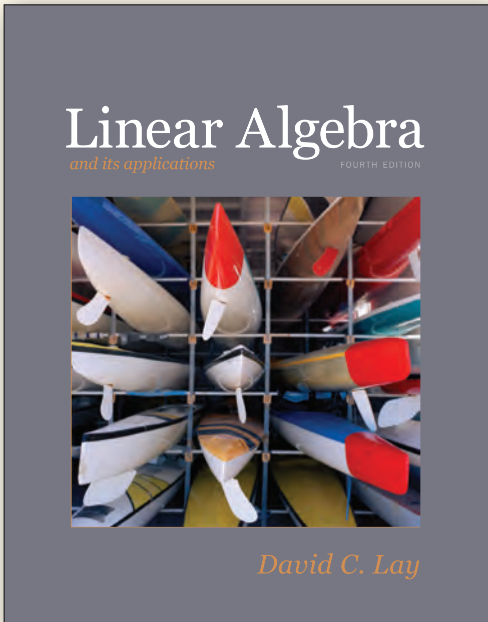

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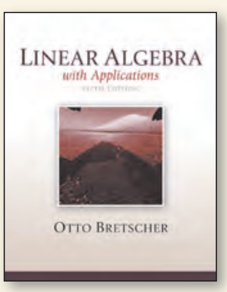

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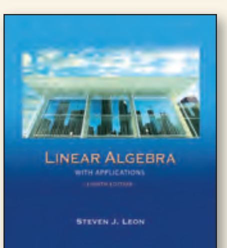

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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 51 (Fall 2013), p. 44] except problems 51.3 and 51.6 for which we still seek solutions. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Steve Butler, Gregor Dolinar, Shaun Fallat, Alexander Guterman, Rajesh Pereira, and Nung-Sing Sze.

New Problems:

Problem 52-1: Generalized Self-Inversive Operators

Proposed by Arkady A. Babajanyan, *Institute for Informatics and Automation Problems, Armenian Academy of Sciences, Yerevan, Armenia*, arbab@list.ru

Let A^+ be the Moore-Penrose inverse of a bounded operator on a complex Hilbert space. Show that $A = A^+$ if and only if $A = JQ$, where $Q = Q^* = Q^2$ is a self-adjoint projector and $J : \mathrm{Im}(Q) \to \mathrm{Im}(Q)$ is an involution or self-inversive operator (i.e., $J^{-1} = J$). When is $A = A^+ = A^*$?

Problem 52-2: Numerical Range of a Skew-Symmetric Matrix

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

Let A be a 4×4 real skew-symmetric matrix. Moreover, let $F(A)$ be the numerical range of A, which (in this case) is an ordered set. Express the exact boundary of the ordered set $F(A)$ in terms of the entries of A.

Problem 52-3: Removing Zero From Numerical Range

Proposed by Janko Bračič, *University of Ljubljana, IMFM, Slovenia*, janko.bracic@fmf.uni-lj.si
and Cristina Diogo, *Departamento de Matemática, Lisbon University Institute, Portugal*, cristina.diogo@iscte.pt

Let $A \in M_n(\mathbb{C})$. Show that 0 is not in the convex hull of the spectrum $\sigma(A)$ if and only if there exists a positive definite $P \in M_n(\mathbb{C})$ such that 0 is not in the numerical range of PA.

Problem 52-4: On a Class of Generalized Quadratic Matrices

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@Manchester.ac.uk

Let $\Omega = \{T_1, \ldots, T_m\}$ be a set of $n \times n$ real or complex matrices, none of T_i is a scalar multiple of the identity matrix I. Assume $T_i^2 \in \{-I, I\}$ and $T_iT_j + T_jT_i = 0$ for $i \neq j$. Then:

(i) Show that $I, T_1, \ldots, T_m$ are linearly independent.

(ii) Show that the minimal polynomial of each $X \in \mathrm{span}(\Omega \cup \{I\})$ has degree two.

(iii) If, in addition, for each $i \neq j$ there is a scalar c_{ij} such that $c_{ij}T_iT_j \in \Omega$, then show that $m \leq 3$.

(iv) Classify matrix triples which exhibit (iii).

Problem 52-5: Matrix Hadamard Power

Proposed by Roger A. Horn, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu
and Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, Florida, U.S.A.*, zhang@nova.edu

For a positive integer r and any $n \times n$ complex matrix $X = (x_{ij})$, let $X^{[r]} = (x_{ij}^r)$ be the Hadamard power of X and let $|X| = (|x_{ij}|)$ be the entrywise absolute value matrix of X. It is known that if A is positive semidefinite, then A^r (regular power) and $A^{[r]}$ are also positive semidefinite for any positive integer r. Moreover, $|A|^r$ and $|A|^{[r]}$ are positive semidefinite for each *even* positive integer r. Prove or disprove that $|A|^r$ and $|A|^{[r]}$ are positive semidefinite for $r = 3$.

Problem 52-6: An Additive Decomposition Involving Rank-One Matrices

Proposed by Anna A. Prach, *Middle East Technical University, Ankara, Turkey*, annaprach@me.com
and Dennis S. Bernstein, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $A \in \mathbb{R}^{n \times m}$ be nonzero, define $r := \mathrm{rank}(A)$ and let $x \in \mathbb{R}^m$ be nonzero. Show that $Ax = 0$ if and only if there exist $y_1, \ldots, y_r \in \mathbb{R}^n$ and skew-symmetric matrices $S_1, \ldots, S_r \in \mathbb{R}^{m \times m}$ such that $A = \sum_{i=1}^r y_i x^T S_i$.

Solutions to Problems 51-1 *through* 51-5, *except for* 51-3, *are on pp.* 42–27.