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# IMAGE

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## About *IMAGE*

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# FEATURE INTERVIEW

## "A Culture of Supporting, Nurturing and Sharing"

**Roger Horn Interviewed by Carlos M. da Fonseca[1] and Fuzhen Zhang[2]**

*Roger Horn*

**Q - When and how did you get interested in mathematics?**

**Horn -** In my high school, the mathematics curriculum ended with a one-semester course in solid geometry, offered in the fall of our senior year. The teacher for that course was a young man recently graduated from college who had a lot of enthusiasm, and several of us asked if he would be willing to give a one-semester calculus course in the spring. He agreed, and we had a terrific experience. When I went to college in the fall (Cornell University), I scored well on the mathematics placement exam and was put into the more rigorous of the two parallel calculus course tracks. My class met Tuesday, Thursday, and Saturday mornings at 8:00AM, and there were some snowy Saturday mornings that I wasn't so sure that accepting that placement was such a good idea. At the end of the semester, the final exam contained mostly routine problems, but one question asked for a proof that a differentiable function was continuous. Anyone who wrote a credible solution to that problem was offered placement in a new three-semester honors track. The assistant professor assigned to the course wound up staying with us for five semesters (including a semester of linear algebra) and infected many of us with his enthusiasm for mathematics. At least four students in that small class went on to earn Ph.D.'s in mathematics, and one has been elected to the National Academy of Sciences (USA).

**Q - You published your first paper in *Mathematics of Computation* on the distribution of prime numbers, in 1961, while you were still an undergraduate student. However, your Ph.D. combines analysis and matrix theory. What was the influence of your advisers, Donald C. Spencer and Charles Loewner, in this decision/move to matrix analysis? None of your supervisors worked directly on matrix theory, right?**

**Horn -** That is correct, but of course Loewner discovered monotone matrix functions in 1934 while studying analytic functions that map the upper half plane into itself and are real on some real interval. He was primarily an analyst, and made deep contributions to the theory of univalent analytic functions. He had a keen interest in matrix theory, and gave an advanced problems course in matrix theory at Stanford from time to time. That course got me thinking about linear algebra problems, and stimulated me to identify matrix analysis problems that were at the core of open problems in other areas. One of those areas had been a hot topic at Stanford for a long time: the Bieberbach Conjecture (now the De Branges Theorem) about coefficients of a normalized univalent analytic function that maps the unit disk into itself. Loewner had made a seminal contribution in 1923 and Spencer had published a book on the topic. Some classical inequalities of Grunsky characterize this class of functions. In matrix analysis language, they say that the Hadamard log of a certain positive definite matrix dominates the Hadamard log of a certain complex symmetric matrix (in the sense of a Hermitian form dominating the absolute value of a symmetric form; see Section 4.4 of *Matrix Analysis*). If one could Hadamard exponentiate those Hermitian-symmetric inequalities and tease out suitable coefficient inequalities, maybe the Bieberbach Conjecture could be resolved. I was intrigued, and the idea led to my Ph.D. thesis on infinitely divisible matrices. But I never proved the Bieberbach Conjecture.

**Q - Since then, you published several influential papers. What is (are) the result(s) that you like most? And how do you see its/their developments?**

**Horn -** I like the main theorem in a 1977 paper (with Carl FitzGerald) that identified the critical exponent for Hadamard powers of positive semidefinite matrices with nonnegative entries. This was a problem that interested Loewner, and I wish I had solved it in time to have told him about it. That was a long time ago. More recently, I like very much the canonical forms for congruence and *congruence that Vladimir Sergeichuk and I published about ten years ago. There has been some recent interest in issues related to critical exponents after 30+ years; there is an AIM workshop on the topic this fall, and a special session at the *Joint Mathematics Meetings* in San Antonio (USA) in January. It has been a real pleasure to see that several papers have used the congruence canonical forms to solve interesting (and hard!) problems in various areas.

**Q - You have a significant number of publications in top journals jointly with your wife (Dr. Susan Horn) on health care. When did you become interested in this area?**

**Horn -** That was something that began when we were both at The Johns Hopkins University. Susan is a statistician (Stanford Ph.D. also), and her appointment was in the School of Hygiene and Public Health. She and her colleagues

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[2]Division of Mathematics, Science and Technology, Nova Southeastern University, Fort Lauderdale, Florida, USA; zhang@nova.edu

encountered some interesting problems that I learned about over the dinner table and was able to make some contributions to.

**Q - The second edition of *Matrix Analysis* appeared last year. How do you evaluate the impact of the first edition in the scientific community, almost three decades after its appearance?**

**Horn -** The first edition has appeared in Romanian, Chinese, and Russian translations. (The Russian edition was priced at 3 rubles!) It has been consistently in the *MathSciNet* Top 10 list of most-frequently referenced books; in 2013 it was #7 on that list, 28 years after publication. It is a pleasure to be associated with something that many people seem to find useful.

**Q - What do you expect from this new edition?**

**Horn -** It was published one year ago; there will be a second printing soon. A China-only edition will be published in 2015. I am often asked about the cover for the second edition, as it is quite a change from the plain blue and white of the first edition. The story about the cover image is told in the Preface to the Second Edition, so I won't repeat it here.

**Q - Will there be a second edition of *Topics of Matrix Analysis*? Are you writing any other book(s)?**

**Horn -** There are no plans at this time for a second edition of *Topics*. I am coauthoring a new textbook for a second course in linear algebra. The target audience is upper division undergraduate students in science, engineering, computer science, and mathematics who have taken calculus and a first course in linear algebra. Some of these students want to acquire broader and deeper skills in linear algebra, but are not ready to dig into *Matrix Analysis*. We expect it to appear in the second half of 2015.

**Q - What is your favorite topic in linear algebra and matrix analysis?**

**Horn -** I have always been fond of canonical forms, and my work on the Grunsky inequalities led to a special interest in Hadamard products.

**Q - What is your favorite result in linear algebra? (Name one or two.)**

**Horn -** That would be the singular value decomposition (SVD). I love teaching that topic, and I am excited to be writing the SVD chapter in the new book right now. My undergraduate course in linear algebra was of the abstract coordinate-free variety, and neither our text (Hoffman & Kunze) nor our instructor ever mentioned the SVD. I learned about it in my first year of graduate study at Stanford, when I took a numerical analysis course from Gene Golub. It was Gene's first year of teaching at Stanford, and I was fortunate to be introduced to the SVD by its greatest guru. Schur triangularization and the Weyr canonical form are also close to my heart.

**Q - During the period of 1997–2001, you were the Editor of the *American Mathematical Monthly*. In this specific journal, how do you classify the importance/contribution of linear algebra?**

**Horn -** Linear algebra Articles and Notes were always popular with *Monthly* readers. It seems to be an area in which mathematicians and mathematics students from many specialties have a common interest. One of my goals as Editor was to have a linear algebra piece in every *Monthly* issue.

**Q - What would you say the relationship is between linear algebra and general mathematics?**

**Horn -** Linear algebra is a big part of the small intersection of all general mathematical areas.

**Q - While you were the chairman of the Department of Mathematical Sciences at the Johns Hopkins University in Baltimore, MD, from 1972 to 1979, you introduced the time-shared computing machines in classrooms and a new graduate course on matrix analysis. Could you describe the main ideas of your programs?**

**Horn -** At that time, "computing" primarily meant lugging a box of cards to the computing center and walking back the next day to pick up the output. Time-shared computing was just becoming available (over phone lines) from commercial vendors, so we set up some Teletype machines (yellow rolls of paper and only upper case type... noisy!) in a back hallway and started to experiment with computing laboratories in some of our courses. We had one classroom with a monitor mounted up on a wall and a phone modem under the lecture table. The idea caught on, and soon there was a DEC PDP-11 in the Electrical Engineering Department that we could use, followed by a big PDP-10 for campus-wide use and terminals everywhere. The new graduate course in matrix analysis was set up at the time the new Department of Mathematical Sciences was created at Hopkins and we had to devise a new graduate curriculum. We had a small faculty, so we looked for ways to be more efficient in our teaching. We noticed that our basic graduate courses in statistics, probability, optimization, and numerical analysis all included some matrix analysis topics that were essential for the course, but that we did not expect students to have seen before. Some topics were covered in two or more courses. We removed these topics from the course syllabi and rolled them into a new matrix analysis course for first-year graduate students. I got the assignment of putting the course together and teaching several iterations of it. There was no suitable text, so I prepared lecture notes and revised, extended, and polished them. The topical syllabus for that course became the table of contents for *Matrix Analysis*. The course became very popular with electrical engineering students, and some physics students.

**Q - How do you currently see the relevance of matrix theory in the undergraduate/graduate courses in the US?**

**Horn -** Upper division and graduate instructors tell us that they would like their students to have broader and deeper linear algebra/matrix analysis skills. But in all university programs, time is short and lists of current requirements are long, so it is hard to find time to add in another course.

**Q - In your opinion, what are the importance and the role of computers in a linear algebra classroom in our times?**

**Horn -** I still think, as I did 40 years ago at Hopkins, that carefully chosen computer laboratory exercises can enhance the learning experience in regular mathematics courses, not just in computing courses. And these days it is so much more convenient! No more trays of cards... we have MATLAB on our laptops!

**Q - What made you move from Maryland to Utah? You are a Research Professor at Utah. Do you teach any courses?**

**Horn -** After 24 years at Johns Hopkins, Susan and I were ready for new experiences. She was invited to lead a new research effort at a large health care system in Utah that wanted to implement the fruits of her academic research to improve quality and patient outcomes. It was a great opportunity for her, and we decided to do it. We have been in Salt Lake City for 22 years, and have enjoyed the lifestyle and the mathematics community. I taught both semesters last year, but am working hard on the new book now and am not teaching this fall.

**Q - You are a very generous human being always ready to help and inspire others, especially young mathematicians. What is your recommendation or advice for the young linear algebraists?**

**Horn -** Our young colleagues are fortunate to be part of a special mathematical community that has developed a culture of supporting, nurturing, and sharing with our colleagues. That culture is due in large part to the personal examples set by leaders such as Hans Schneider, Robert Thompson, Olga Taussky, Alston Householder, James Wilkinson, Gene Golub, and Richard Varga. All of us, older or younger, can do well to follow their example to grow and enrich our discipline in a spirit of cooperation and friendship.

**Q - If you had not become a mathematician, what career would you have considered?**

**Horn -** When I went to college, I wanted to be a physician, perhaps because the smartest person I knew in the little town I came from was our family doctor. But I soon found that I was a disaster in the chemistry laboratory, and decided to nurture a career direction that didn't involve glassware.

**Q - We all know your enthusiasm for matrices. Do you have any hobbies? You may say a few words about your personal life if you want to.**

**Horn -** I love classical music. We have always had symphony and opera subscriptions wherever we have lived, and I was once a pretty good trumpet player! Travel, cooking, and grandchildren are special pleasures.

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# BOOK ANNOUNCEMENT

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## Topics in Quaternion Linear Algebra
## by Leiba Rodman

**Princeton University Press, 2014, 384pp.,**
**Hardcover ISBN 978-0-6911-6185-3 | $79.50 / £55.00 | eBook ISBN 978-1-4008-5274-1**

The book features previously unpublished research results with complete proofs and many open problems at various levels, as well as more than 200 exercises to facilitate use by students and instructors. Applications presented in the book include numerical ranges, invariant semidefinite subspaces, differential equations with symmetries, and matrix equations.

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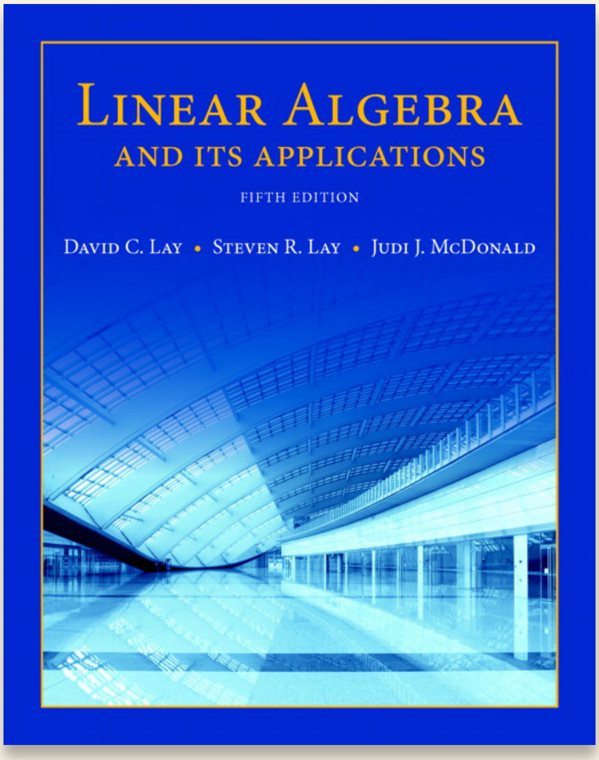


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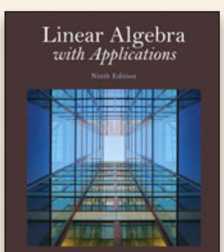


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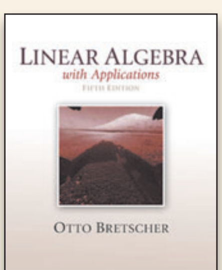


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# ARTICLES

## The Development of Linear Algebra Research in Spain

**Julio Moro, Universidad Carlos III de Madrid, Spain, jmoro@math.uc3m.es**

The purpose of this article is to describe the genesis and development of a community of researchers in the areas of linear algebra and matrix analysis in Spain over the last thirty years.

To put things in a historical context, the 1970s were a period of deep transformation for Spanish universities: research in mathematics had been a mostly marginal activity for Spanish scholars in the first half of the 20$^{\text{th}}$ century. Only for a short period in the 1920s and 1930s had there been a serious effort, championed by a group of young mathematicians around Julio Rey Pastor,[1] to bring Spanish mathematics up-to-date and into line with current developments elsewhere around the world. The Civil War destroyed any chance of success for this program and sent a whole generation of well-trained mathematicians into exile, mostly to Latin America. Research in mathematics became again the exception, instead of the rule. By the late 1960s however, Franco's dictatorial regime began to slowly open up to influence from abroad, pushed by the economic boom due, in part, to tourism at an industrial scale as one of the country's main sources of income. This relative openness and economic affluence began to change the ways of Spanish universities: for the first time in decades, funds were allocated to promote scientific research, international connections were slowly established, fellowships were granted to young undergraduates to do their Ph.D.'s abroad, and Spanish mathematics began its long journey of integrating into the scientific world at large. At the same time, the economic boom created a new middle class, who expected their offspring to go to university. Consequently, traditional universities had to grow fast in order to accommodate this new inflow of students (up to that point, university had been a prerogative of the upper crust of Spanish society).

On top of this, once Franco died in 1975, and after the consequent political turmoil, the new democratic system chose a decentralized federal-like territorial organization, shifting part of the political power to the regions. One result was the creation of several new universities in the 1980s and, especially, in the 1990s. These were promoted by the regions with the intention to attract students within each region itself, instead of from all over Spain as the handful of larger old, traditional universities had done. Both the growth of traditional universities and the creation of new ones brought the need to staff them with properly trained faculty; faculty able not only to teach, but also to do research. It is in this transitional period between the old and the new system of higher education that our story begins to unfold.

The first three places in Spain where research groups in linear algebra were formally established were (in this order) Vitoria, Valencia and Barcelona, all three in the 1980s.

The birth of the Basque group at Universidad del País Vasco (UPV-EHU) in Vitoria is inseparable from Graciano de Oliveira and his group at the University of Coimbra. The first contact took place when Juan Miguel Gracia, who was at the time at Colegio Universitario de Álava (UPV-EHU), met José Vitória, a colleague of Graciano at the Coimbra Mathematics Department, while attending the 7$^{\text{th}}$ Spanish-Portuguese Mathematical Meeting at Sant Feliú de Guíxols in May 1980. Juan Miguel had become interested in matrix analysis via his work on differential equations, and grabbed the opportunity to take advantage of the Portuguese colleagues' wide expertise in the subject. This initial contact led to a visit of Juan Miguel to Coimbra in September 1981, which resulted in Graciano committing himself to act as a sort of long-distance mentor for a linear algebra group to be created in Vitoria (see de Oliveira's own account in [5]). The initial core of this group, which amounted basically to Juan Miguel and his student, Ion Zaballa,[2] quickly enlarged, so that by the end of 1982 a weekly group seminar was taking place every Friday in Vitoria, and members of both groups in Coimbra and Vitoria began to exchange short visits on a regular basis.

From that point on, a series of meetings was held in order to bring together Spanish and Portuguese linear algebraists on a regular basis: three consecutive editions took place in 1982, 1983 and 1984 in Coimbra, Vitoria and Coimbra, respectively. Seen in retrospect, the 1983 meeting in Vitoria was one of the key moments in the creation of the Spanish linear algebra community. Besides bringing well-established invited speakers from abroad, such as Stephen Barnett, Avi Berman, Charlie Johnson and Bob Plemmons, the so-called *International Meeting on Linear Algebra and Applications* attracted numerous Spanish researchers, coming from widely differing backgrounds. This was the seed for today's linear algebra community. Among them, for instance, was Rafael Bru, from Universitat Politécnica de Valencia (UPV), who became interested in the intricacies of the Jordan canonical form by Manuel López Pellicer, a functional analyst who was his colleague at UPV's Mathematics Department. Also attending was Vicente Hernández, who would co-lead with Bru the UPV group during its first years, until he moved away from linear algebra into computer science in the 1990s. Encouraged by what they saw at the meeting, Bru and Hernández formally created a research group at UPV in a year's time, by 1984-85. This group would in time become the most numerous in Spain, with its scientific offspring spreading

[1] See [1] for a thorough investigation of the importance of Rey Pastor in bringing key linear algebraic concepts into the Spanish mathematics university curriculum.

[2] Graciano de Oliveira would ultimately become Zaballa's Ph.D. advisor.

to the newly created universities in the east of Spain.

The 1983 Vitoria meeting also helped to create a close scientific relationship between the Vitoria group and members of Universitat Politécnica de Catalunya (UPC) at Barcelona. The leading figure among the latter was Ferrán Puerta. His pathway into linear algebra came through teaching (he wrote a linear algebra textbook [8] in the 1970s, which was among the first ones written in Spain). The background of the Barcelona group, mostly algebraic and differential geometry, brought new approaches and insights into the problems in the theory of linear control systems the Basque group was working on. This was probably also responsible for the slow build-up of the Barcelona group as an established linear algebra research group, which eventually came about in the late 1980s.

The fourth edition of the Spanish-Portuguese meetings was subsumed into the *IV International Conference on Linear Algebra and Applications*, organized in Valencia by the UPV group in 1987. This was a large international conference, with over two hundred participants from all over the world, which somehow amounted to the first steps of Spain's integration of linear algebra into the international scene. Incidentally, it is worth noting that the birth of ILAS was also being conceived at the time: the 1987 Valencia conference happened to host a meeting of the international board of IMG/TILAC (the International Matrix Group/The International Linear Algebra Community), which would soon be renamed as the International Linear Algebra Society (see [2] for a very detailed report on this conference).

The 1990s was a period of consolidation for the Spanish linear algebra community. New groups were created and several meetings, both national and international, were held in Spain. Among the latter, the 1994 Meeting on Total Positivity and its Applications [3], held in Jaca in September 1994, represents a milestone in the creation of a linear algebra research group at Universidad de Zaragoza. Mariano Gasca, one of the local organizers, was one of several classical analysts who attended the previously mentioned 1983 Vitoria meeting. He became drawn to linear algebra by his research on the interpolation of functions of several real variables, which led him in turn to the study of totally positive matrices. Juan Manuel Peña, a young collaborator of Gasca at the time, continued this line of research as the leader of a group at Universidad de Zaragoza whose research exploits the connection of such matrices with applied topics.

Another very relevant international meeting was the 8$^{\text{th}}$ ILAS Conference, organized by the UPC group in Barcelona in July 1999. This was another landmark in the process of getting the Spanish linear algebra community closer to their international peers. As to more locally-focused meetings, a new series was launched, the so-called *Encuentros de Análisis Matricial - EAMA* (Matrix Analysis Meetings), which were instrumental in strengthening the ties between the emerging groups. Although in principle the EAMAs intended to further the Spanish-Portuguese connection, Portuguese attendance declined over time. Three EAMAs were organized, one in Valencia in 1989, another in Vitoria in 1994 and the last one in Sevilla in 1997, with an ILAS Conference in between, held in Lisbon in 1992. By 1994, the Spanish attendants to the EAMA in Vitoria outnumbered the Portuguese by six to one (84 to 14, see the report [4]).

Also in the 1990s, Vicente Hernández established valuable connections with researchers from all over Europe as his group at UPV joined the effort in building up the NICONET Network on Numerics in Control. This European network would officially start its activities in 1998, funded by the European Community, and helped trigger a good deal of research in High Performance Computing at UPV and, later on, also at Universitat Jaume I in Castellón (see below).

As it turned out, the 1999 ILAS meeting was the start of almost a decade in which no linear algebra meetings would be held in Spain at all. By the mid 2000s it became clear that, although some of us would still meet occasionally at linear algebra conferences abroad, or at the biennial CEDYA meetings of the Spanish Applied Mathematics Society (SEMA), the lack of some kind of regular gatherings for our community was thwarting possible collaborations, as well as any other kind of interactions between groups. Once a consensus was met that some sort of stable association would be advisable, the 2007 CEDYA meeting was chosen as the setting of a formal gathering, where members of the Spanish linear algebra research groups were summoned to agree on the terms of such an association. The outcome of this meeting was the decision to apply for official endorsement from the Spanish Ministry of Science as a Thematic Network, an administrative status bringing not only endorsement as a recognized scientific community, but also a small amount of yearly funding to subsidize the basic activities of the network. Thus the Network ALAMA (an acronym, in Spanish, for *Linear Algebra, Matrix Analysis and Applications*) was officially born when the application succeeded in 2008. So far, ALAMA has been extremely useful as an organizational umbrella under which several activities transversal to the groups are coordinated. Central among these are, of course, the ALAMA Meetings. These meetings have being held on even-numbered years since 2008, and provide an opportunity for all the groups to get together, interacting with each other and with researchers from abroad. More recently, ALAMA Spring Courses, monographically devoted to some specific topic, are also being organized on odd-numbered years at the CIEM (International Center for Mathematics Meetings) in Castro-Urdiales. So far, two of them have been held, one on Piecewise Linear Systems in 2012 and one on Matrix Polynomials in 2013. A spring course on Totally Positive Matrices is scheduled for 2015. News on these, and other activities, are reported through the network website `http://www.red-alama.es` which spreads news and notices on the activities of the network and, more generally, facilitates the exchange of information of any sort among network members. Although government funding has been discontinued since 2010 due to budgetary cuts, the network has continued supporting itself and its activities via registration fees and, especially, the generosity and hard work of network members.

So, what is the linear algebra and matrix analysis research landscape in Spain as of today? The ALAMA network has over 100 members, some of which might be called just 'sympathetic bystanders' (members of other communities who just want to stay aware of developments in ours). There is, however, a core of 60 to 70 members who regularly attend the ALAMA meetings and, occasionally, the Spring Courses. The members of this core belong to research groups based at over fifteen universities all over Spain. A brief review of the most relevant groups follows.[3]

Starting with foundational ones, the Basque group has expanded since the 1980s from Vitoria to Bilbao and San Sebastián, reaching all three campuses of UPV-EHU. Inmaculada de Hoyos, Silvia Marcaida, Alicia Roca and Francisco Velasco, among others, have joined Gracia and Zaballa in the effort of analyzing both linear control and matrix perturbation and completion problems, with recent incursions into quadratic inverse matrix polynomial problems. They have kept their close ties with the Barcelona group at UPC, which has devoted its efforts over the years to the analysis of problems in mathematical systems theory via geometric techniques. Ferrán Puerta, Josep Ferrer and M. Isabel García Planas, together with younger researchers such as Albert Compta, Dolors Magret, Marta Peña or Xavier Puerta, have both unveiled and exploited normal and canonical forms for different kinds of systems, as well as for parametrized families of systems, skillfully using Arnold's techniques on versal deformations. Quite recently, another research group with connections to linear algebra has surfaced at UPC, namely the MAPTHE group on $M$-Matrix Analysis and Potential Theory, some of whose members are Andrés Encinas, Margarida Mitjana and Ángeles Carmona. Although their scientific background is mainly potential theory, their study of discrete elliptic operators on finite networks has led them down the path to matrix analysis.

Like the Vitoria group, the one in Valencia underwent a significant expansion in the 1990s to the point of becoming the most numerous group in Spain in the 2000s. After some initial matrix-theoretic publications, the group's research soon diversified into a wider range of problems as new researchers, such as Rafael Cantó, Carmen Coll, Josep Mas, Sergio Romero, Elena Sánchez, Néstor Thomé, Juan Ramón Torregrosa and Ana Urbano joined the group. Their research interests cover many topics, such as: linear control systems, with a recent emphasis on positive ones; matrix factorizations for numerical methods; preconditioning; and mathematical modeling. Also, many core linear algebra topics, such as completion problems, H-matrices or generalized inverses, have been (and are still being) investigated at UPV. Also, a standing scientific collaboration is held with Josep Gelonch from the Universitat de Lleida.

On a different note, former UPV students of Vicente Hernández have kept working on linear algebra from a computational point of view. Such is the case, for instance, for José Román, also from UPV. He is the lead developer of SLEPC, the Scalable Library for Eigenvalue Problem Computations, a software library for the solution of large scale sparse eigenvalue problems on parallel computers. In Castellón, Enrique Quintana, also a former student of Vicente Hernández, is the group leader of the High Performance Computing and Architectures (HPCA) Group at Universitat Jaume I. Although their main goal is the optimization of (mainly parallel) numerical algorithms, several of these algorithms are meant to solve classical linear algebra problems, which requires performing detailed analyses via linear algebra techniques in order to fine-tune the algorithms.

Other young universities, created in the 1990s in the Valencia region, like Universitat Jaume I, were also populated by scientific offspring of the UPV group: Joan-Josep Climent, a former student of Bru, established a group at Universitat d'Alacant whose research, although focused mainly on cryptology and coding theory, has a strong connection with linear algebra and linear control systems. Carmen Perea, at Universidad Miguel Hernández in Orihuela, is part of that group as well.

Another group with connections to numerical analysis is the one already mentioned at Universidad de Zaragoza, led by Juan Manuel Peña. Their research exploits the connection of totally positive matrices with Neville elimination, eigenvalue localization and CAGD.[4] The Zaragoza group keeps close ties with groups at the universities of Alcalá and Oviedo. Pedro Alonso, a former student of Peña, leads a group at Universidad de Oviedo conducting research on numerical linear algebra and high performance computing for problems of large dimension, while José-Javier Martínez and Ana Marco at Universidad de Alcalá are active in exploiting classic analytic tools to devise fast and accurate algorithms for highly structured matrices.

Another group in the Madrid area, led by Nieves Castro, is based at Universidad Politécnica de Madrid, and its main research interest lies in generalized inverses. Other groups based in Madrid were somehow triggered by the EAMA meeting held at Vitoria in 1994. That was, for instance, my first personal contact with the Spanish linear algebra crowd, and there I met Alberto Borobia. He and I had obtained our undergraduate degrees together years ago from Universidad Complutense de Madrid, but had got Ph.D.'s in very different areas. During EAMA 1994, however, we were able to find some common ground in the spectral analysis of nonnegative matrices through their zero-nonzero pattern, which

[3]Although some universities host several separate research groups, I chose not to itemize them for the sake of brevity. Also, since limitations of space do not allow for the inclusion of references to specific publications for each group, an extended version of this paper, with appropriate reference to a short sample of (hopefully) representative references for each research group, will be made available at the ALAMA Network webpage (see [7]).

[4]Computer-Assisted Geometric Design.

would later lead to a long collaboration in the nonnegative inverse eigenvalue problem. Borobia and Roberto Canogar at UNED (Spain's public Open University) conduct research on several matrix completion problems as well. EAMA 1994 also gave me the chance of first meeting Froilán M. Dopico and Francisco Marcellán of Universidad Carlos III de Madrid (UC3M). Marcellán is one of the sympathetic bystanders I mentioned above. Like Gasca, he had attended the 1983 Vitoria meeting, and was well aware of the manifold connections of linear algebra with classical analysis. Although we did not talk math at the time, two years later they would both become my next-door colleagues when I was hired by the Mathematics Department in UC3M, right after a post-doc abroad in which I became interested in eigenvalue perturbation theory. Some initial papers on perturbation theory and high relative accuracy eigenvalue algorithms were followed by others, increasingly concerned with the influence of matrix structure on the properties and behavior of numerical algorithms. Right now there is an ever growing and extremely active group at UC3M, led by Dopico and with Juan Manuel Molera and Fernando de Terán as senior members, doing research in numerical linear algebra with strong matrix-theoretic foundations.

Last but not least, linear algebra also flourished in Castilla: José Ángel Hermida, who is now at Universidad de León, got his Ph.D. at Universidad de Valladolid in abstract algebra under Tomás Sánchez Giralda. One of Sánchez Giralda's research interests, which he passed on to Hermida, was the study of linear systems and, more generally, of matrices with entries belonging to a ring, an area with wide-ranging applications in control theory. Nowadays there is a young and active group at Universidad de León, which counts Miguel Carriegos, Montserrat López and Andrés Sáez-Schwedt among its members, exploring these kinds of topics. Back at Universidad de Valladolid, Carlos Marijuán and co-authors have been working for years on the nonnegative inverse eigenvalue problem (NIEP). Marijuán's previous research had been focused on algebraic graph theory. The connection between the spectral properties of nonnegative matrices and the cyclic structure of their associated digraphs is what brought him to the NIEP. One of the group's most remarkable feats was their 1997 solution of the NIEP for the case $n = 4$ using graph-theoretic techniques. Marijuán and Miriam Pisonero have lately been working on several other matrix analysis problems, such as Newton inequalities.

If we compare this rough recount with the census made in [6] in 1992, it becomes clear that we have come a long way since then: several new groups have emerged, and most of the ones which were active then remain active today. The ALAMA Network has been most useful as a stable platform to coordinate collective activities, some of them in collaboration with similar networks from abroad.[5] International connections have been strong for some time. Spain was, for instance, the country where the First SIAG/Linear Algebra Summer School was held, in July 2008 at the CIEM in Castro-Urdiales. This was just the forerunner of the present widely celebrated Gene Golub SIAM Summer Schools. Furthermore, the 11$^{\text{th}}$ SIAM Conference on Applied Linear Algebra, the second one to be held outside of the US, took place in Valencia in June 2012, organized by the group at UPV. Most Spanish researchers in linear algebra are members of some international society, such as the ILAS or SIAM (mostly through SIAG-LA), and some of them have served as members of their boards. Although our presence in the editorial boards of the journals in the area is not strong yet, I think we can safely say that our efforts to fully integrate ourselves into the international linear algebra community have largely succeeded. Unlike the pioneers in the 1980s and 90s, young Ph.D. students are being trained nowadays in constant connection with the outside world, and are enjoying all the benefits of being part of strong, established research groups. Although prospective Ph.D. students are hard to come by in our country these days, I'm hopeful for the future. I am convinced that both present members of our community, and the younger generations of researchers still to come, will manage to keep linear algebra's torch burning in Spain for at least another thirty years.

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## The Development of Linear Algebra Research in Chile

**Ricardo L. Soto, Universidad Católica del Norte, Chile, rsoto@ucn.cl**

**The establishment of mathematical research.** At the beginning of the 1960s, the academic activity in mathematics in Chilean universities consisted mainly of lecturing. The only option for people who wanted to pursue a career in mathematics was to become a math teacher or to enter an Engineering School. Towards the middle of that decade, the first Chilean mathematicians to obtain a Ph.D., Alberto Saenger and Jaime Michelow, returned to Chile from the United States, where they had obtained their degrees. Around that time, several foreign mathematicians arrived in the country to lecture at universities in Santiago and the provinces: the Germans Arno Zaddach and Kurt Legrady, the Polish Olgierd Biberstein and the Uruguayan Héctor Merklen. Then Chileans Rolando Chuaqui and Yerko Valderrama also returned, with doctoral degrees obtained overseas. All of them initially lectured at the Department of Mathematics of the Institute of Sciences (now Faculty of Sciences) at the University of Chile (Universidad de Chile) in Santiago. The arrival of these mathematicians with postgraduate education was significant to the beginning of mathematical research in Chile.

Jaime Michelow left the University of Chile a few years later to work at the State Technical University (Universidad Técnica del Estado, UTE) which is now the University of Santiago de Chile (Universidad de Santiago de Chile, USACH). There he established a postgraduate program, the Academic Degree in Mathematics (LAM for its name in Spanish), with the academic support of the American mathematician Herbert Clemens and the economic support of the Ford Foundation, under the Chile-California Plan. The LAM program offered a Masters Degree in Mathematics, which was very relevant to the advancement of the level of mathematics being taught in the tertiary centers associated with the UTE, and also in other universities in the country. The graduates of LAM contributed to the improvement of the teaching of mathematics and to the development of the discipline in Chile. Many of them continued to Ph.D. studies in developed countries overseas.

Rolando Chuaqui played a fundamental role in the foundation of the Faculty of Mathematics at the Pontifical Catholic University of Chile (Pontificia Universidad Católica de Chile, PUC). The Faculty had a rapid development, giving rise to one of the first research groups, in mathematical logic, and also resulting in the creation of a Ph.D. program in mathematics in 1972.

During the 1970s and 1980s, research groups were formed in most Chilean universities. By 1982 there were eight universities in Chile, three located in the capital and five located in the provinces. In Santiago, formal research activity developed in several areas of mathematics. In the University of Chile, research groups were formed in the Department of Mathematics of the Faculty of Sciences, and in the Mathematical Engineering Department (DIM) of the Faculty of Physical and Mathematical Sciences (FCFM). Most first-generation mathematical engineers from DIM went overseas to obtain their Ph.D.'s, mainly to France. To formalize collaboration with the industry and to extend the multidisciplinary nature of its research, FCFM founded the Mathematical Modeling Centre (Centro de Modelamiento Matemático, CMM) in 2000, with the academic support of the National Centre of Scientific Research (CNRS) in France. In the Pontifical Catholic University of Chile, research developed in the Department of Mathematics of the Faculty of Mathematics, while in the University of Santiago de Chile it developed in the Department of Mathematical and Computer Sciences of the Faculty of Science. It was in the latter, at the end of the 1970s, that Hernán Henríquez organized the first seminar on Matrix Theory for his undergraduate students.

In the provinces, professor Biberstein from the University of Concepción (Universidad de Concepción) in Southern Chile, wrote the first *Linear Algebra Notes* for his undergraduate students, while Professor Herman Alder introduced courses in numerical linear algebra and numerical calculus for engineers. These were the first seeds for the development of what is today the Numerical Analysis Research Group at the Department of Mathematical Engineering at the University of Concepción. There were also a number of active research groups at the Institute of Mathematics at the Pontifical Catholic University of Valparaíso (Pontificia Universidad Católica de Valparaíso), and at the Department of Mathematics at the Federico Santa María Technical University (Universidad Técnica Federico Santa María). At the end of the 1980s, the Northern Catholic University (Universidad Católica del Norte, UCN) in Antofagasta had its first research group organized by professors Oscar Rojo and Ricardo L. Soto in the area of Matrix Theory and Applications (MTA), which is still active today.

**Linear algebra research.** As far as I know, from the information I could collect for these notes, research on certain topics of linear algebra and matrix theory started at the beginning of the 1990s. We mention the works of Hector Miranda and Robert C. Thompson [20, 21, 22, 23] about the von Neumann trace inequality for singular values of real matrices. Miranda is a Chilean mathematician from University of Bío-Bío (Universidad del Bío-Bío) who obtained his Ph.D. under the supervision of Professor Thompson at University of California, Santa Barbara, in 1993.

People working in other fields of mathematics have also published papers related to linear algebra and matrix theory, as side topics of their interests in their own areas. We mention the works of Servet Martínez and Jaime San Martín from the

Mathematical Engineering Department (DIM) and CMM at Physical and Mathematical Sciences Faculty (FCFM) at the University of Chile, in Santiago [8]–[12], [17]–[19]. Their work concerns nonnegative strictly ultrametric matrices, new classes of inverse $M$-matrices whose graphs are trees, Hadamard functions of inverse $M$-matrices, and inverse $M$-matrices associated to random walks.

Currently there is only one research group working on Matrix Theory and Graph Theory in Chile. This is the Matrix Theory and Applications Group (MTA group) at the Department of Mathematics at the Northern Catholic University, in Antofagasta. The first works of this group, led by Oscar Rojo and Ricardo L. Soto, concern the additive inverse eigenvalue problem, eigenvalue localization for real and complex matrices, and bounds for eigenvalues and singular values. The other members of the group are colleagues Juan Egana, Raul Jiménez, María Robbiano, María Peláez, and Mario Salas. Peláez left the UCN in 2013. Since 2000, the group has published a number of papers on spectral graph theory, the nonnegative inverse eigenvalue problem, numerical reconstruction of spring-mass systems, and damage detection in rods. During this decade, some of the members of the group, led by Oscar Rojo, concentrated their research in the area of spectral graph theory, while others, led by Ricardo L. Soto, researched inverse problems for nonnegative matrices, including, in particular, the nonnegative inverse eigenvalue problem (NIEP) and the nonnegative inverse elementary divisors problem (NIEDP).

The NIEP is the problem of characterizing all possible spectra of entrywise nonnegative matrices. This problem remains unsolved. It has only been solved for lists containing no more than 4 complex numbers. The authors in [31, 3] give conditions under which a partitioned list is realizable even if some of the parts (sub-lists) are nonrealizable, provided that there are other sub-lists that are realizable and, in certain ways, compensate for the nonrealizability of the former ones. This is the idea which has been called "negativity compensation." This is done by employing an extremely useful result, due to A. Brauer (1952), which shows how to modify one single eigenvalue of a matrix via a rank-one perturbation without changing any of the remaining eigenvalues. This approach goes back to H. Perfect (1953), who first used it for the NIEP; but it was somehow abandoned for many years until Soto (2003) rediscovered it in [31], obtaining sufficient conditions for the realizability of partitioned real spectra, with the partition allowing some of its pieces to be nonrealizable. The proofs in [31] are constructive in the sense that they generate an algorithmic procedure to compute a solution matrix. The authors in [3] systematize and extend the results of [31], allowing the lists to be complex. In particular, they solve the NIEP for lists of complex numbers with negative real part and modulus that is greater than or equal to the corresponding imaginary part.

The authors in [26] solve the NIEP for circulant matrices. That is, they find a necessary and sufficient condition for a list of complex numbers to be the spectrum of some circulant nonnegative matrix. In a long-ignored paper [25] published in 1955, H. Perfect presents an extension of the Brauer result, due to R. Rado, which shows how to modify $r$ eigenvalues of a matrix of order $n$ ($n > r$) via a rank-$r$ perturbation, without changing any of the $n - r$ remaining eigenvalues. By using the Rado result, Perfect (1955) gives a realizability criterion for the real NIEP (with real eigenvalues), which is extended in [33]. The result in [33] appears to be, so far, the most general sufficient condition for the real NIEP to have a solution. It is also constructive, and always allows us, if the condition is satisfied, to compute a realizing matrix. The Rado result has been extended to the complex case in several articles by distinct authors of the MTA group [36, 37, 38]. The papers [34] and [15] introduce symmetric and normal versions of the Rado result, which allow them to obtain, in the first case, a new, more general, sufficient condition for the existence of a symmetric nonnegative matrix with prescribed spectrum, and in the second case, a new sufficient condition for the NIEP for normal matrices to have a solution. This last condition significantly improves Xu's conditions [39].

In [4], a connection is established between the problem of characterizing all possible real spectra of nonnegative matrices and a combinatorial process consisting of repeated application of three elementary transformations on a realizable list of real numbers, each of which results in a new realizable list. This defines a special kind of realizability, called $C$-realizability, closely related to the idea of negativity compensation. This criterion strictly includes, in particular, most of the previously known sufficient conditions in the literature.

Regarding the nonnegative inverse elementary divisors problem (NIEDP), the authors of [35] completely solve the NIEDP for lists of nonnegative real numbers and for lists of real numbers of Suleimanova type, that is, lists where all, except the Perron eigenvalue, are negative real numbers. Also, in [38], a complete solution is provided for the NIEDP for lists of complex numbers of Suleimanova type, that is, with negative real part and modulus that is greater than or equal to the corresponding imaginary part.

In 2003, the UCN started a Ph.D. program in mathematics. To this date, five students have obtained their Ph.D. in the area of matrix theory, with their theses addressing the nonnegative inverse eigenvalue problem and the nonnegative inverse elementary divisors problem, while three students obtained their Ph.D. with theses in spectral graph theory.

Some well-known linear algebraists have visited the Department of Mathematics at UCN. The visitors have usually given talks or seminars, and have also done collaborative work with colleagues and students of the MTA group. We mention the visit of Graham Gladwell [13, 14] in October-November of 2001; the visit of Thomas J. Laffey in 2006; the visit of Richard Brualdi, who gave a talk at the Capricorn Mathematical Congress (Congreso de Matemática Capricornio, COMCA) in August of 2003 at UCN; the visit of Ion Zaballa in July 2013; and the visit of Raphael Loewy in November 2013. The MTA group has also been visited by many researchers: regularly by Biswa N. Datta; Alberto Borobia and Julio Moro [3, 4, 32, 34]; Carlos Marijuán and Miriam Pisonero [16]; Rafael Bru, Rafael Cantó, and Ana M. Urbano [5]; Nair Abreu, Claudia Justel and Vilmar Trevisan [1, 2, 27, 28, 29]; Domingos Cardoso and Enide Martins [6, 7, 24, 30]; Ivan Gutman, Nelson Kuhl and Luis C. Santos.

In February of 2008, the MTA group organized a workshop on nonnegative matrices and related topics (WONMART), in San Pedro de Atacama, Antofagasta, with the participation of Helena Šmigoc, Alberto Borobia, Julio Moro, Carlos Marijuán, Fernando de Terán, Fernando Miranda, and the MTA group. A second version of this workshop was held in Segovia, Spain, in July of 2010. Rafael Bru, Rafael Cantó and Ana M. Urbano joined the first group of participants.

The list of people working in Chile who have done some work on linear algebra and related topics is probably far from complete and we apologize to people we failed to mention. The list of references below has been shortened for space reasons. Readers can check Google Scholar to find more articles published by the MTA Group at UCN.

I would like to thank the following colleagues who provided me with useful information for writing this note: Jorge Soto Andrade, who wrote the "History of the Department of Mathematics" for the web page of the Department of Mathematics at the Faculty of Sciences at the University of Chile; Gabriel Gatica, from University of Concepción; Hernán Henríquez, from University of Santiago of Chile; and Jaime Figueroa, from Federico Santa María Technical University.

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# LINEAR ALGEBRA EDUCATION

## Row Rank Equals Column Rank: Four Approaches

**Gilbert Strang**[1]

In teaching linear algebra, this is the first big theorem: The row rank of a matrix equals its column rank. That leads to the dimensions of all four fundamental subspaces of an $m$-by-$n$ matrix $A$ with column rank $r$: the column spaces and nullspaces of $A$ and $A^T$ have dimensions $r, n-r$ and $r$, $m-r$.

If we want the rank theorem, how should we prove it? I have collected four approaches, not all with the same appeal. My favorite is Proof 4 because it happens so suddenly. Probably Proof 2 is the cleanest, starting from basic ideas of independence. In class I depend on Proof 1, because elimination has prepared the way. (I am never formal anyway, and the 2 by 2 case is surprisingly useful—if the row $(a, b)$ is a multiple of $(c, d)$, then the column $(a, c)$ is ...)

**Proof 1.** Reduce $A$ to its row echelon form $R$. The row space stays the same. The column space changes, but its dimension stays the same. (For the columns of $A$ and $R$, exactly the same linear combinations give the zero vector.) The rank theorem is clear for $R$:

$$\begin{array}{lcl} r \text{ nonzero rows in } R & \leftrightarrow & r = \text{dimension of row space} \\ r \text{ pivot columns in } R & \leftrightarrow & r = \text{dimension of column space} \end{array}$$

**Proof 2.** [1] Suppose $\mathbf{x}_1^T, \ldots, \mathbf{x}_r^T$ is a basis for the row space of $A$. The proof will show that $A\mathbf{x}_1, \ldots, A\mathbf{x}_r$ are independent vectors in the column space. Then dim (row space) $= r \leq$ dim (column space). The same reasoning applies to $A^T$, reversing that inequality. So the two dimensions must be equal.

Suppose $c_1 A\mathbf{x}_1 + \cdots + c_r A\mathbf{x}_r = A(c_1\mathbf{x}_1 + \cdots + c_r\mathbf{x}_r) = A\mathbf{v} = \mathbf{0}$. Then $\mathbf{v}$ is in the nullspace of $A$ and $\mathbf{v}^T$ is in the row space (it is a combination of the $\mathbf{x}$'s). So $\mathbf{v}$ is orthogonal to itself. Thus $\mathbf{v} = \mathbf{0}$. All the $c$'s must be zero since the $\mathbf{x}$'s are a basis. This shows that $c_1 A\mathbf{x}_1 + \cdots + c_r A\mathbf{x}_r = 0$ requires that each $c_i = 0$. Therefore $A\mathbf{x}_1, \ldots, A\mathbf{x}_r$ are independent vectors in the column space: the dimension of the column space of $A$ is at least $r$.

**Proof 3.** (from David Gale) If $A$ has $r$ independent rows and $s$ independent columns, we can move those rows to the top of $A$ and those columns to the left. They overlap in an $r$ by $s$ submatrix $B$. Let

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}.$$

Suppose $s > r$. Since $B\mathbf{v} = \mathbf{0}$ has $r$ equations in $s$ unknowns, it has a solution $\mathbf{v} \neq \mathbf{0}$. Note that $B\mathbf{v} + C\mathbf{0} = \mathbf{0}$. But the lower rows of $A$ are combinations of the upper rows, so it follows that $D\mathbf{v} + E\mathbf{0} = \mathbf{0}$. Now a combination of the first $s$ independent columns $\begin{bmatrix} B \\ D \end{bmatrix}$ of $A$, with coefficients from $\mathbf{v}$, is producing zero. *Conclusion*: $s > r$ cannot happen. Thinking similarly for $A^T$, $r > s$ cannot happen.

**Proof 4.** (see [2]) Suppose the column space of $A$ has a basis $\mathbf{u}_1, \ldots, \mathbf{u}_r$. Then each column of $A$ is a combination of the $\mathbf{u}$'s. Column 1 of $A$ is $w_{11}\mathbf{u}_1 + \cdots + w_{r1}\mathbf{u}_r$, with some coefficients $w$. The whole matrix $A$ equals $UW$, a product of an $m$-by-$r$ matrix and an $r$-by-$n$ matrix.

$$A = \begin{bmatrix} \\ \mathbf{u}_1 & \ldots & \mathbf{u}_r \\ \\ \end{bmatrix} \begin{bmatrix} w_{11} & \ldots & w_{1n} \\ \vdots & & \vdots \\ w_{r1} & \ldots & w_{rn} \end{bmatrix} = UW.$$

*Now look differently at $A = UW$. Each row of $A$ is a combination of the $r$ rows of $W$!* Therefore the row space of $A$ has dimension at most $r$. This proves that (dimension of row space) $\leq$ (dimension of column space) for any $A$. Apply this reasoning to $A^T$, and the two dimensions must be equal. To my way of thinking, that is a really cool proof.

May I add one thought about linear algebra in the curriculum. More and more, it is taking its proper place. This subject is needed and used! Many departments, including my own, are adding linear algebra to long-established courses like differential equations. My new book (see `http://math.mit.edu/dela`) aims to help. And best of all, linear algebra courses (with applications like networks and graphs) are growing. A good thing for the students.

### References

[1] G. Mackiw, Note on the Equality of the Column and Row Rank of a Matrix, *Math. Mag.* 68(4) (1995) 285–286.

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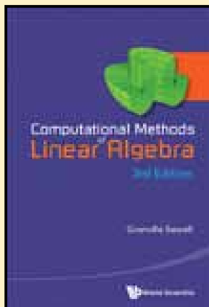


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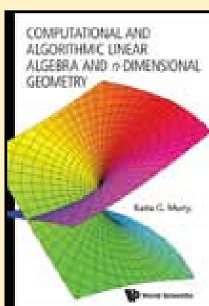


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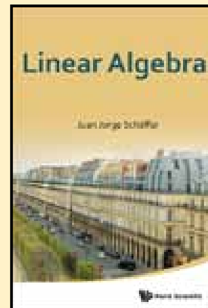


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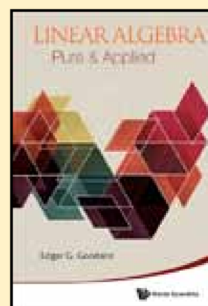


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# FEATURE ARTICLE

## A Gentle Introduction to the Normalized Laplacian

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**1. Introduction.** Graph theory at its most basic level looks at the interconnections (or edges) between various groups of objects (or nodes/vertices). This simple foundation is flexible enough to encompass a wide variety of phenomena and powerful enough to yield useful information, and so has rightly received much attention in both the pure and applied math communities. Examples of important graphs include the internet graph (each page is a vertex and each link is a (directed) edge), social networks (each individual is a vertex and a relationship is an edge), collaboration graphs (each person is a vertex and a collaboration is denoted by an edge), Cayley graphs (each element of a group is a vertex and an edge corresponds to some restricted group action), and so on.

These networks can range from dozens of nodes to billions of nodes, many of them are so large that it can be difficult to perform direct analysis with the computers currently available. This has led to a need to find an efficient way to capture information about graphs. One method that has proven successful is *spectral graph theory*. The basic idea being to associate a graph with a matrix (which can be done in a variety of ways) and then to study the spectrum, i.e., eigenvalues, of that matrix. This allows a snapshot that contains information about the matrix (and hence the graph) but at the same time knowing the eigenvalues is not sufficient to know everything about the matrix, so spectral graph theory seeks to understand what information can and cannot be revealed about a graph by use of the eigenvalues of a particular matrix.

**The four common matrices of spectral graph theory.** The most common matrix associated with a graph is the *adjacency matrix*, $A$, which for simple graphs is defined by $A_{ij} = 1$ if $i$ is connected to $j$ and 0 otherwise (see [1, 4, 17, 23, 25]). Each entry of $A^k$ corresponds to the number of walks of length $k$ between the vertices corresponding to that entry, and so the spectrum can count the number of edges (closed walks of length 2), the number of triangles (closed walks of length 3), whether a graph is bipartite (no closed walks of odd length), and other properties.

Another commonly studied matrix is the *Laplacian matrix*, $L$, which is defined by $L = D - A$, where $D$ is the diagonal matrix of degrees, $d_i$, and $A$ is the adjacency matrix (see [3, 4, 17, 29, 30]). This matrix can also be written as $Q^TQ$ where $Q$ is a signed vertex/edge incidence matrix of the graph, and thus $L$ is positive semidefinite (it will trivially have the all 1's eigenvector for the eigenvalue 0). A well-known result for the Laplacian matrix is Kirchoff's Matrix-Tree Theorem which connects the product of the nonzero-eigenvalues of the Laplacian matrix with the number of spanning trees of the graph. This matrix also has connections to measuring connectivity where the number of 0's in the spectrum indicates the number of connected components, and for connected graphs the smallest nonzero eigenvalue is known as the algebraic connectivity of the graph (see [27]).

Recently, the *signless Laplacian matrix*, $|L| = D + A$, has garnered attention (see [18, 19, 20]). This matrix shares some similarities with the Laplacian, for example it is also positive semidefinite, but also important distinctions, for example the multiplicity of 0 counts the number of *bipartite* components (see [28] for more on the smallest eigenvalue).

Finally, there is the *normalized Laplacian matrix*,[2] $\mathcal{L}$ (see [10, 14]). For simple graphs this is defined entrywise by

$$\mathcal{L}_{ij} = \begin{cases} -\dfrac{1}{\sqrt{d_i d_j}} & \text{if } i \text{ and } j \text{ are adjacent;} \\ 1 & \text{if } i = j \text{ and the vertex is not isolated;} \\ 0 & \text{otherwise.} \end{cases}$$

More commonly this is written as $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$, which is consistent when the graph has no isolated vertices.[3] This matrix is a chimera of sorts; it has connections to all three of the preceding matrices, but at the same time exhibits key differences. It can be used to measure components, bipartite-ness, and a host of other properties, and yet it seems to stumble at some of the most basic combinatorial properties about a graph that might be of interest.

To illustrate the last statement we have compiled some information about the matrices in Table 1. In this table we have the four matrices, and four properties about graphs: bipartite, number of components, number of bipartite components, and number of edges. We then indicate with a "YES" or a "no" whether the spectrum alone can determine if the graph

[1]Partially supported by an NSA Young Investigator Grant.

[2]This is also sometimes referred to (jokingly) as the "San Diego Laplacian" because of the extensive work of Fan Chung and her students using this matrix.

[3]The number of isolated vertices can be detected by using the spectrum of the normalized Laplacian, and so without loss of generality we may always assume that there are no isolated vertices.

has that property. A "YES" indicates that the spectrum of the matrix can always identify the property while a "no" indicates that there are two graphs with the same spectrum of the matrix which differ in that property. Examples of pairs of graphs with the same spectrum, also known as *cospectral graphs*, are shown in Figure 1 for each matrix.

| Matrix | bip. | # comp. | # bip. comp. | # edges |
|---|---|---|---|---|
| $A$ | YES | no | no | YES |
| $L$ | no | YES | no | YES |
| $\lvert L\rvert$ | no | no | YES | YES |
| $\mathcal{L}$ | YES | YES | YES | no |

Table 1: Simple properties that can be detected by the spectrum of a graph

One of the first surprising things to see about the normalized Laplacian matrix is that the spectrum cannot detect the number of edges. In general all complete bipartite graphs have the same spectrum so that the star $K_{1,n-1}$ (sparse) is cospectral with $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ (dense). There are other examples of this phenomenon, including examples of dense graphs which are cospectral with sparse subgraphs (see [9]).

Also, the definition of the normalized Laplacian involves square roots which can at first be off-putting. For these reasons, and others, the normalized Laplacian has not received as much attention in spectral graph theory as some of the other matrices. The goal we have here is to give an introduction to this matrix, introduce some basic tools, highlight some connections and differences of the spectrum of this matrix with other matrices, and show that this is an important matrix in spectral graph theory.

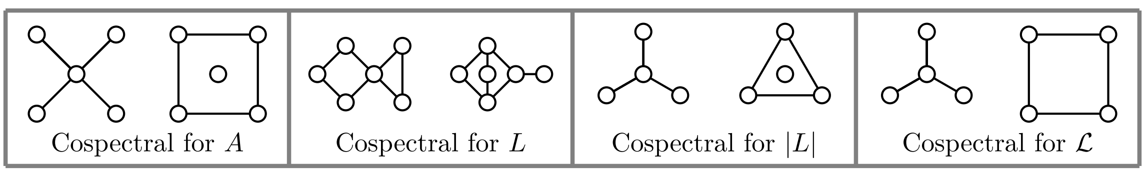


Figure 1: Pairs of cospectral graphs for each matrix

**2. Dealing with the "root" of the problem.** The "normalized" of the normalized Laplacian comes from the $D^{-1/2}$ terms which are inside its definition, i.e., $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$. This normalizing can be thought of in different ways; one useful way is that we are normalizing the vertices so that they are uniform and now placing weight on the edges (more on this later). But for now let us take a look at the $D^{-1/2}$ term and in particular some ways to think about what is happening in a way where we can remove the square roots.

**Taking a random walk.** One obvious way to get rid of the $D^{-1/2}$ term is to observe that the normalized Laplacian is similar to (and thus cospectral with) the following:

$$D^{-1/2}\mathcal{L}D^{1/2} = D^{-1}(D-A) = I - D^{-1}A.$$

This matrix is no longer symmetric so, a priori, it is not immediately clear why the spectrum should be real. However $\mathcal{L}$ is a real symmetric matrix and so we immediately know that the eigenvalues are real for both of these matrices. Written in this non-symmetric form we see that the normalized Laplacian matrix has an intimate connection with $D^{-1}A$ (i.e., $\lambda_i$ is an eigenvalue of $\mathcal{L}$ if and only if $1-\lambda_i$ is an eigenvalue of $D^{-1}A$). The matrix $D^{-1}A$ is a special matrix of the graph, namely this is the probability transition matrix of a random walk on the graph. That is, suppose we have a probability distribution $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$, where $\pi_i \geq 0$ and $\sum \pi_i = 1$. Then $\pi' = \pi D^{-1}A$ is the probability distribution after taking one step (viewed another way, if we pick vertex $j$ with probability $\pi_j$ and then randomly choose one of its neighbors with equal probability, then $\pi'_i$ is the probability that we end up at vertex $i$). One important question to ask about random walks is how quickly they diffuse, i.e., how quickly information about the initial distribution is lost as we take multiple steps.

To make this more rigorous, if a graph is connected and not bipartite (which we will show below means that all of the "nontrivial" eigenvalues of $\mathcal{L}$ lie strictly between 0 and 2), then in the long run a random walk will mix and approach the distribution $\widehat{\pi}_i = d_i/\operatorname{vol} G$ where $\operatorname{vol} G = \sum d_i$ is the *volume* of the graph. Moreover if $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are the eigenvalues of the normalized Laplacian then for any initial distribution $\pi$ we have that the distance to the limiting distribution after taking $k$ steps is bounded as follows:

$$\left\|\pi(D^{-1}A)^k - \widehat{\pi}\right\| \leq \max_{i\neq 0} |1-\lambda_i|^k \frac{\max_i \sqrt{d_i}}{\min_j \sqrt{d_j}}.$$

In particular, the eigenvalues of the normalized Laplacian can give a measure for how quickly a random walk on a graph will converge towards the distribution $\widehat{\pi}$. In this case, we want to have all of the "nontrivial eigenvalues" tightly packed around 1 to have fast convergence; we will see this phenomenon again.

**Using a Rayleigh quotient.** The Laplacian matrix is positive semidefinite and has the following quadratic form for a vector $\mathbf{x}^T = [x_1, x_2, \ldots, x_n]$:

$$\langle \mathbf{x}, L\mathbf{x}\rangle = \mathbf{x}^T L\mathbf{x} = \sum_{i\sim j}(x_i - x_j)^2,$$

where "$\sum_{i\sim j}$" is the summation over all edges $i\sim j$ in the graph. If we further have that $\mathbf{x}$ is an eigenvector of $L$ for the eigenvalue $\alpha_k$ (the $k$th eigenvalue of $L$ where $k = 0, 1, \ldots, n-1$), then

$$\alpha_k = \frac{\langle \mathbf{x}, L\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\sum_{i\sim j}(x_i - x_j)^2}{\sum x_i^2}.$$

The ratio in the middle and right hand side is known as the Rayleigh quotient. More generally, if we let $\mathcal{X}^k$ denote a $k$ dimensional subspace of $\mathbb{R}^n$, then by the Courant-Fischer Theorem (see [26])

$$\alpha_k = \min_{\mathcal{X}^{n-k-1}} \left( \max_{\substack{\mathbf{x}\perp\mathcal{X}^{n-k-1}\\ \mathbf{x}\neq\mathbf{0}}} \frac{\sum_{i\sim j}(x_i - x_j)^2}{\sum x_i^2} \right) = \max_{\mathcal{X}^k} \left( \min_{\substack{\mathbf{x}\perp\mathcal{X}^{k}\\ \mathbf{x}\neq\mathbf{0}}} \frac{\sum_{i\sim j}(x_i - x_j)^2}{\sum x_i^2} \right).$$

We can translate this idea to the normalized Laplacian by making a simple substitution $\mathbf{x} = D^{1/2}\mathbf{y}$ (which is valid as long as there are no isolated vertices). Then we have

$$\frac{\langle \mathbf{x}, \mathcal{L}\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\langle D^{1/2}\mathbf{y}, \mathcal{L}D^{1/2}\mathbf{y}\rangle}{\langle D^{1/2}\mathbf{y}, D^{1/2}\mathbf{y}\rangle} = \frac{\mathbf{y}^T D^{1/2}\big(D^{-1/2}LD^{-1/2}\big)D^{1/2}\mathbf{y}}{\mathbf{y}^T D^{1/2}D^{1/2}\mathbf{y}} = \frac{\mathbf{y}^T L\mathbf{y}}{\mathbf{y}^T D\mathbf{y}} = \frac{\sum_{i\sim j}(y_i - y_j)^2}{\sum y_i^2 d_i},$$

where $\mathbf{y}^T = [y_1, y_2, \ldots, y_n]$. Again, more generally we have the Courant-Fischer Theorem for the normalized Laplacian which states that the $k$th eigenvalue of the normalized Laplacian matrix (for $k = 0, 1, \ldots, n-1$) is given by

$$\lambda_k = \min_{\mathcal{Y}^{n-k-1}} \left( \max_{\substack{\mathbf{y}\perp\mathcal{Y}^{n-k-1}\\ \mathbf{y}\neq\mathbf{0}}} \frac{\sum_{i\sim j}(y_i - y_j)^2}{\sum y_i^2 d_i} \right) = \max_{\mathcal{Y}^k} \left( \min_{\substack{\mathbf{y}\perp\mathcal{Y}^{k}\\ \mathbf{x}\neq\mathbf{0}}} \frac{\sum_{i\sim j}(y_i - y_j)^2}{\sum y_i^2 d_i} \right).$$

Looking at the inside term, we have

$$0 \leq \frac{\sum_{i\sim j}(y_i - y_j)^2}{\sum y_i^2 d_i} \leq 2,$$

where the left inequality is by non-negativity of terms in the sum while the right inequality follows by noting that $(a-b)^2 \leq 2a^2 + 2b^2$ and simplifying. Therefore we have that the eigenvalues for the normalized Laplacian must lie between 0 and 2 inclusively. We can go even further though and note that to achieve an eigenvalue of 0 we must have all of the $y_i$ equal on a connected component which indicates that the dimension of the corresponding null space is the number of connected components (the eigenvalue of 0 is also known as the trivial eigenvalue). On the other hand, to have an eigenvalue of 2 we must have that $y_i = -y_j$ which can only happen non-trivially for a graph which is bipartite, and therefore 2 is an eigenvalue if and only if there is a bipartite component. More generally, the multiplicity of 2 is the number of bipartite components with two or more vertices.

That the spectrum lies in the closed interval between 0 and 2 helps to give a way to reasonably compare two graphs of differing number of vertices and differing number of edges. In particular, looking at the extremal eigenvalues (i.e, $\lambda_1$ and $\lambda_{n-1}$), or looking at how tightly the nontrivial eigenvalues are grouped around 0, helps to give a meaningful interpretation to say that two graphs are spectrally related when they might be of different orders.

The Courant-Fischer Theorem for the normalized Laplacian can be used to control what happens to the graph under some perturbation, e.g., adding/deleting a subgraph (see [6, 13]). In particular, we have the following result.

**Theorem.** *Let $G$ be a graph and $H$ a connected subgraph of $G$ with $|V(H)| = t$. If*

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \qquad \textit{and} \qquad \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{n-1}$$

*are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(G-H)$ respectively (where $G-H$ is the graph that results by deleting the edges of $H$*

*from G), then for $k = 0, 1, \ldots, n-1$ we have*

$$\lambda_{k-t+1} \;\le\; \theta_k \;\le\; \begin{cases} \lambda_{k+t-1} & \text{if } H \text{ is bipartite,} \\ \lambda_{k+t} & \text{otherwise,} \end{cases}$$

*where $\lambda_{-t+1} = \cdots = \lambda_{-1} = 0$ and $\lambda_n = \cdots = \lambda_{n+t} = 2$.*

This result highlights some of the difficulty in working with eigenvalues to understand the behavior of graphs. In particular, most eigenvalues will not change significantly under a small perturbation in the graph, and the ones that are most susceptible to change are those near the "ends" of the spectrum. Therefore, it is not a surprise to see that these extremal eigenvalues have been the most intensely studied for *all* matrices.

For a connected graph, an eigenvector for the trivial eigenvalue of 0 is $D^{1/2}\mathbf{1}$ (where $\mathbf{1}$ is the all 1's vector). Using this in the Courant-Fischer Theorem for the normalized Laplacian and carefully running through the orthogonality requirements, we can conclude that

$$\lambda_1 = \min_{\substack{\mathbf{y}\perp D\mathbf{1} \\ \mathbf{y}\neq\mathbf{0}}} \frac{\sum_{i\sim j}(y_i - y_j)^2}{\sum y_i^2 d_i}.$$

Now by choosing appropriate $\mathbf{y}$, various bounds on $\lambda_1$ can readily be achieved. As an example, if a graph is not the complete graph then $\lambda_1 \le 1$ which can be seen by picking two non-adjacent vertices $u$ and $v$ and letting $\mathbf{y} = d_v\mathbf{e}_u - d_u\mathbf{e}_v$, where $\mathbf{e}_w$ is the vector which is 1 on $w$ and 0 else.

Much stronger results are possible by using this as a starting point, including controlling expansion of the graph. One important method of studying expansion is through the use of the *Cheeger constant*, which is an isoperimetric measurement of the graph which captures how much a vertex set can expand. In particular, it is defined as

$$h_G = \min_{\substack{S\subseteq V \\ \operatorname{vol} S \le \frac{1}{2}\operatorname{vol} G}} \frac{e(S,\overline{S})}{\operatorname{vol} S},$$

where $e(S,\overline{S})$ indicates the number of edges that leave the subgraph on the vertices of $S$ and $\operatorname{vol}(S) = \sum_{u\in S} d_u$ is the volume of $S$. Then the following relationship between $h_G$ and $\lambda_1$ can be shown (see [14]).

**Theorem.** *If $G$ is a graph and $\mathcal{L}(\mathcal{G})$ has eigenvalues $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, then $2h_G \ge \lambda_1 \ge \dfrac{h_G^2}{2}$.*

The left inequality follows by simply taking the optimal $S$ and considering $\mathbf{y} = \frac{1}{\operatorname{vol} S}\mathbf{1}_S - \frac{1}{\operatorname{vol}\overline{S}}\mathbf{1}_{\overline{S}}$ and simplifying (here we are letting $\mathbf{1}_S$ denote the indicator vector which is 1 on $S$ and 0 otherwise, similarly for $\mathbf{1}_{\overline{S}}$). The right inequality takes more work and is found by sweeping over cuts created by an ordering of the vertices induced by an eigenvector of $\lambda_1$ for the normalized Laplacian.

**Using harmonic eigenvectors.** The previous subsection looked at the Rayleigh quotient and introduced ways to bound eigenvalues in one direction or another. There are other ways to approach eigenvalues and eigenvectors. Let us consider the adjacency matrix $A$ and suppose that $A\mathbf{x} = \beta\mathbf{x}$, i.e., that $\mathbf{x}$ is an eigenvector for the eigenvalue $\beta$. Then, by looking at the entry corresponding to the vertex $v$ on both sides, we can conclude

$$\sum_{u\sim v} x_u = \beta x_v.$$

Something similar happens when we look at the normalized Laplacian. We again start by making a substitution, $\mathbf{x} = D^{1/2}\mathbf{y}$ where $\mathbf{y}$ is known as a *harmonic eigenvector* of the graph. Then, if $\mathcal{L}\mathbf{x} = \lambda\mathbf{x}$, we get upon substitution that

$$\big(D^{-1/2}(D-A)D^{-1/2}\big)D^{1/2}\mathbf{y} = \lambda D^{1/2}\mathbf{y},$$

which simplifies to $(D-A)\mathbf{y} = \lambda D\mathbf{y}$, or, rearranged, $A\mathbf{y} = (1-\lambda)D\mathbf{y}$. Therefore we have that $\lambda$ is an eigenvalue of the normalized Laplacian matrix if and only if there exists a nonzero vector $\mathbf{y}$ which satisfies *locally* at each vertex $v$

$$\sum_{u\sim v} y_u = (1-\lambda)d_v y_v.$$

One thing that is clear when comparing the two expressions for $A$ and $\mathcal{L}$ is that if there is an eigenvector for $\beta = 0$ for the adjacency matrix, then this gives a harmonic eigenvector for $\lambda = 1$ for the normalized Laplacian, i.e., the dimension of the corresponding eigenspaces are equal. In general, it also follows from Sylvester's Law of Inertia that the number of positive eigenvalues for the adjacency matrix equals the number of eigenvalues which are *below* one for the normalized

Laplacian while the number of negative eigenvalues for the adjacency matrix equals the number of eigenvalues which are *above* one for the normalized Laplacian.

As an example of using harmonic eigenvectors, if two non-adjacent vertices $u$ and $v$ have the same neighbors, then we see that $\mathbf{y} = \mathbf{e}_u - \mathbf{e}_v$ is a (harmonic) eigenvector for the eigenvalue 1 since each vertex either is adjacent to neither $u$ and $v$ and so trivially has sum 0 or is adjacent to both $u$ and $v$ and so the summands cancel. On the other hand, if two adjacent vertices $u$ and $v$ have the same closed neighborhood and if $d$ is their common degree, then again using $\mathbf{y} = \mathbf{e}_u - \mathbf{e}_v$ we can conclude that at each vertex $w$ we have

$$\sum_{t\sim w} y_t = -y_w$$

since either $w \neq u, v$ and so both sides are 0 as before, or $w = u, v$ in which case the sum is the opposite of the initial value. So this is a harmonic eigenvector and we have that $-1 = (1-\lambda)d$, which upon rearranging gives $\lambda = 1 + 1/d$. In either of the preceding cases (both non-adjacent and adjacent) the remaining harmonic eigenvectors are perpendicular to the ones that we have given and this forces the remaining harmonic eigenvectors to always have $y_u = y_v$. This allows these vertices to be treated as a single über-vertex which can help to simplify the graph and determine the remaining eigenvalues (see [9, 31]).

Generally, these harmonic eigenvectors are useful when you want to show how the eigenvalues are maintained under some graph operation, for example "gluing" two different bipartite graphs onto a graph [11], or unfolding a bipartite graph [7], and thus forming cospectral graphs.

**3. Generalizing results to the normalized Laplacian.** Many results in spectral graph theory start by looking at what happens for regular graphs. This is for several reasons. First, such graphs are common, particularly so when looking at graphs that arise from algebraic structure. The second reason is that regular graphs have a special eigenvector, namely $\mathbf{1}$, and it is associated with an extremal eigenvalue (this is for each matrix we have talked about). Moreover it is easy to see that if the graph is regular then the spectrum of *all* of the matrices discussed in the introduction are related through simple shifting and scaling operations. As a result, you can use the tools of all the spectra simultaneously.

The question about what to do when graphs are irregular, or how to generalize the assumptions and conclusions from one type of matrix to another is the challenge. It is far from obvious what to do, but in many cases the normalization associated with the normalized Laplacian can help deal with differing situations.

The goal of this section is to look at several places where the normalized Laplacian has been able to take known results from other matrices and generalize them. We will also see an example where the normalized Laplacian fails to generalize a known result.

**Discrepancy and dealing with irregularity.** Discrepancy is a measurement for how randomly we place edges in a graph. To be more specific we will let $e(X,Y)$ denote the number of edges between $X, Y \subseteq V$ (any edge in $X \cap Y$ is counted twice). Then for *regular* graphs of degree $d$ on $n$ vertices the *discrepancy* is the minimal $\alpha$ for which the following holds for all $X$ and $Y$

$$\left| e(X,Y) - \frac{d}{n}|X||Y| \right| \leq \alpha\sqrt{|X||Y|}.$$

We have that $e(X,Y)$ by definition is the actual number of edges, the term $\frac{d}{n}|X||Y|$ is measuring the expected number of edges if the edges had been placed randomly, finally the $\sqrt{|X||Y|}$ is an error-normalizing term. Therefore, when we can take $\alpha$ small this indicates that the edges have been placed fairly randomly in the graph.

The Expander Mixing Lemma (attributed to Alon and Chung) states that if $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n = d$ are the eigenvalues of the adjacency matrix then $\alpha \leq \max\{|\nu_1|, |\nu_{n-1}|\}$. This follows by noting that for any matrix $M$ we have $|\mathbf{x}^T M\mathbf{y}| \leq \sigma(M)\|\mathbf{x}\|\|\mathbf{y}\|$ where $\sigma(M)$ is the largest singular value of $M$. The key to observe is that both $e(X,Y)$ and $\frac{d}{n}|X||Y|$ can be written in terms of inner products using the indicator functions $\mathbf{1}_X$ and $\mathbf{1}_Y$. So we then have

$$\begin{aligned}\left| e(X,Y) - \frac{d}{n}|X||Y| \right| &= \left| \mathbf{1}_X^T \left( A - d\left(\frac{\mathbf{1}}{\sqrt{n}}\right)\left(\frac{\mathbf{1}}{\sqrt{n}}\right)^T \right) \mathbf{1}_Y \right| \\ &\leq \sigma\left( A - d\left(\frac{\mathbf{1}}{\sqrt{n}}\right)\left(\frac{\mathbf{1}}{\sqrt{n}}\right)^T \right) \|\mathbf{1}_X\|\|\mathbf{1}_Y\| \\ &= \max\{|\nu_1|, |\nu_{n-1}|\}\sqrt{|X||Y|}.\end{aligned}$$

In the first line we rewrite the expression as an inner product, then we apply the singular value result noting that for the matrix

$$A - d\left(\frac{\mathbf{1}}{\sqrt{n}}\right)\left(\frac{\mathbf{1}}{\sqrt{n}}\right)^T$$

we have essentially taken the eigenvalue of $d$ for the matrix $A$ and turned it into an eigenvalue of 0 and therefore the largest singular value is the largest of the absolute values of the remaining eigenvalues.

The Expander Mixing Lemma allows us to conclude then that if $\max\{|\nu_1|, |\nu_{n-1}|\}$ is small then the discrepancy parameter will be small. Bilu and Linial [2] showed that the reverse implication holds, namely that $\max\{|\nu_1|, |\nu_{n-1}|\} \le c\alpha \log(d/\alpha)$ for some fixed constant $c$ so that when the discrepancy is small all of the eigenvalues of $A$ will be tightly packed around 0.

The problem then becomes how to generalize this to irregular graphs. For the adjacency matrix we no longer have the trivial eigenvector $\mathbf{1}$ available and so we cannot repeat the above argument in its given form. There is also a more central problem into what is the right measurement of the sets $X$ and $Y$. The above results were using the number of vertices to measure the sets, but for irregular graphs this is usually not the right measurement.

Let us take the internet graph as an example. There are many websites with differing degrees of popularity and traffic. If we compare `www.google.com` and `www.funnycatpix.com`, it is obvious that one has a much higher impact on the day-to-day traffic on the internet. This is simply a function of how connected one site is to other sites. So it is not the site but rather the connections to the site that should dictate what is important.

So let us now change our perception of the size of a set from the number of vertices ($|X|$) to the number of edges that connect to vertices in that set (the volume given by $\operatorname{vol} X = \sum_{u\in X} d_u$). So we will now define discrepancy as the minimal $\beta$ for which the following holds for all $X$ and $Y$

$$\left| e(X,Y) - \frac{\operatorname{vol} X \operatorname{vol} Y}{\operatorname{vol} G} \right| \le \beta \sqrt{\operatorname{vol} X \operatorname{vol} Y}.$$

Again, each term acts as before: $e(X,Y)$ is counting edges, $\operatorname{vol} X \operatorname{vol} Y / \operatorname{vol} G$ is now the expected number of edges joining $X$ and $Y$ if the edges were placed randomly (with edges more likely to occur between vertices of high degree than between vertices of low degree), and $\sqrt{\operatorname{vol} X \operatorname{vol} Y}$ is an error-normalizing term. (Notice that we are now making no assumptions about the graph; this statement and the result to follow apply to *all* graphs.)

When written in this format, there is a connection to the spectrum of the normalized Laplacian. We demonstrate this via the normalized adjacency matrix $D^{-1/2}AD^{-1/2}$ which has eigenvalues $1-\lambda_{n-1} \le \cdots \le 1-\lambda_1 \le 1-\lambda_0 = 1$, where the $\lambda_i$ are the eigenvalues of the normalized Laplacian and $D^{1/2}\mathbf{1}/\sqrt{\operatorname{vol} G}$ is an eigenvector for the eigenvalue of 1.

$$\begin{aligned}\left| e(X,Y) - \frac{\operatorname{vol} X \operatorname{vol} Y}{\operatorname{vol} G} \right| &= \left| (D^{1/2}\mathbf{1}_X)^T \left( D^{-1/2}AD^{-1/2} - \left(\frac{D^{1/2}\mathbf{1}}{\sqrt{\operatorname{vol} G}}\right)\left(\frac{D^{1/2}\mathbf{1}}{\sqrt{\operatorname{vol} G}}\right)^T \right) (D^{1/2}\mathbf{1}_Y) \right| \\ &\le \sigma\left( D^{-1/2}AD^{-1/2} - \left(\frac{D^{1/2}\mathbf{1}}{\sqrt{\operatorname{vol} G}}\right)\left(\frac{D^{1/2}\mathbf{1}}{\sqrt{\operatorname{vol} G}}\right)^T \right) \|D^{1/2}\mathbf{1}_X\| \|D^{1/2}\mathbf{1}_Y\| \\ &= \max\{|1-\lambda_1|, |1-\lambda_{n-1}|\}\sqrt{\operatorname{vol} X \operatorname{vol} Y}.\end{aligned}$$

As before, we can conclude that if the nontrivial eigenvalues of the normalized Laplacian are closely packed around 1, then the discrepancy parameter is small, and conversely it is known (see [5]) that if the discrepancy parameter is small then the eigenvalues are closely packed around 1, i.e., $\max\{|1-\lambda_1|, |1-\lambda_{n-1}|\} \le c\beta \log(1/\beta)$ for some fixed constant $c$.

This shows that there is some equivalence between having the eigenvalues closely packed around 1 and having a small discrepancy. This type of notion is the foundation of quasi-random graphs, which correspond to families of graph properties where if a graph satisfies one of these properties then it must satisfy them all (to within a certain error tolerance). The above two properties were examined, along with others, by Chung and Graham [16] for families of graphs with given degree sequences.

**Equitable partitions.** There are other ways that results can generalize, and we now look at what happens with equitable partitions. An equitable partition of the vertices of a simple graph $G$ is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that for each $v \in V_i$ the number of adjacent vertices in $V_j$ does not depend on the choice of $v$. One way to form an equitable partition is to take the orbits of an automorphism group, so this can be useful when dealing with graphs with high degrees of symmetry. We have the following result for adjacency matrices (see [23]).

**Theorem.** *Given an equitable partition $V = V_1 \cup \cdots \cup V_k$ of a graph $G$ with adjacency matrix $A$, form a matrix $B$ by letting $B_{ij} = e(u, V_j)$ where $u \in V_i$ (the matrix $B$ need not be symmetric and so can be interpreted as the adjacency matrix of a weighted directed graph). Then the eigenvalues of $B$ are also eigenvalues of $A$ (including multiplicity).*

In this case, when a new matrix, i.e., graph, is formed it generally leads to non-symmetry. This is easy enough to handle for the adjacency matrix, but for the normalized Laplacian there is currently no good way to handle directed graphs (though Chung [15] has made some progress in this area by focusing on flows). However, it turns out that the generalization of equitable partitions for the normalized Laplacian is even better because there is no need to lose symmetry (see [10]).

**Theorem.** *Given an equitable partition $V = V_1 \cup \cdots \cup V_k$ of a graph $G$ we can form a new graph $H$ with vertices $v_1, \ldots, v_k$ and edge weights $w(v_i, v_j) = e(V_i, V_j)$. Then the eigenvalues of $\mathcal{L}_H$ are also eigenvalues of $\mathcal{L}_G$ (including multiplicity).*

This last result implicitly introduced the idea of edge weights. This is an easy generalization to graphs wherein the entries of the adjacency matrix now become the weight of the edges between vertices and the degree is the sum of the incident weights. With this in place all of the matrices appropriately generalize. (See [10] for more information on weighted graphs). In many results for the normalized Laplacian (and even other matrices) it helps to think of graphs as weighted graphs wherein certain operations can change their weight. The discrete definitions given at the start correspond to when the edge weights of the graphs are 0 or 1.

**Cartesian products.** Not everything generalizes nicely. One example of this is the eigenvalues of a Cartesian product. Given graphs $G$ and $H$, the Cartesian product $G\square H$ is the graph having a vertex $(u, v)$ for each $u \in V(G)$ and $v \in V(H)$ and two vertices $(u_1, v_1)$, $(u_2, v_2)$ are adjacent if either $u_1 = u_2$ and $v_1{\sim}v_2$, or $u_1{\sim}u_2$ and $v_1 = v_2$. For the adjacency matrix, Laplacian matrix or signless Laplacian, the eigenvalues of $G\square H$ are found by adding the eigenvalues of $G$ with the eigenvalues of $H$ in all possible ways.

But this does *not* work for the normalized Laplacian. Which is not to say that you do not add the eigenvalues together, but rather to say there is no way to determine the eigenvalues of $G\square H$ given the eigenvalues of $G$ and $H$. To see this we only need to consider the graphs $K_2\square C_4$ (the cube) and $K_2\square K_{1,3}$ (three four cycles glued along a common edge). We saw at the beginning that $K_{2,2} = C_4$ and $K_{1,3}$ are cospectral (see Figure 1), so if there were a way to determine the spectrum of $G\square H$ from the spectrum of $G$ and $H$ the resulting graphs would have to be cospectral. But these are *not* cospectral, and have spectra $\{0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 2\}$ and $\{0, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}, 2\}$ respectively.

Some graph products do generalize. For example, consider the join of $G$ and $H$, denoted $G \vee H$, which is formed by taking the disjoint union of $G$ and $H$ and then adding an edge between each vertex of $G$ and each vertex of $H$. We then have the following results (see [7]).

- *Adjacency matrix:* If $G$ is an $r$-regular graph with eigenvalues $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n = r$ and $H$ is an $s$-regular graph with eigenvalues $\beta_1 \le \beta_2 \le \cdots \le \beta_m = s$, then the eigenvalues of $G \vee H$ are
$$\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{m-1}, \frac{r + s \pm \sqrt{(r-s)^2 + 4mn}}{2}.$$
- *Normalized Laplacian matrix:* If $G$ is an $r$-regular graph with eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ and $H$ is an $s$-regular graph with eigenvalues $0 = \theta_0 \le \theta_1 \le \cdots \le \theta_{m-1}$, then the eigenvalues of $G \vee H$ are
$$0, \frac{m + r\lambda_1}{m+r}, \ldots, \frac{m + r\lambda_{n-1}}{m+r}, \frac{n + s\theta_1}{n+s}, \ldots, \frac{n + s\theta_{m-1}}{n+s}, 2 - \frac{r}{m+r} - \frac{s}{n+s}.$$

Even though we started with regular graphs, the end result need not be regular. For example, the wheel graph on $n+1$ vertices is $W_{n+1} = K_1 \vee C_n$. The spectrum of $K_1$ is $\{0\}$, while the spectrum of $C_n$ is $1 - \cos(2\pi k/n)$ for $k = 0, 1, \ldots, n-1$. Combining, we can conclude that the spectrum of $W_{n+1}$ is
$$\Big\{0, \underbrace{1 - \frac{2}{3}\cos\frac{2\pi k}{n}}_{k=1,\ldots,n-1}, \frac{4}{3}\Big\}.$$

Another result that generalizes from the adjacency matrix to the normalized Laplacian in a special case is Godsil-McKay switching (see [22]) which is a method to locally perturb the graph in a special setting without changing the spectrum. This is the most common way to form cospectral graphs for the adjacency matrix, but for the normalized Laplacian is only guaranteed to work when the switching also preserves the degrees of the vertices (see [12]). This leaves us with the problem of how to construct cospectral graphs. Currently, the best known approach is to glue two small subgraphs which are cospectrally related onto a larger graph (see [11, 24]). Though there have been other recent advances in forming cospectral graphs for the normalized Laplacian (see [9, 31]), much work in this area remains to be done.

**4. Final remarks.** We have seen that the normalized Laplacian is a matrix with many interesting and important graph properties and is worthy of study in its own right. We refer the reader to the references given in the bibliography for more information.

Of course, for everything that we know about the normalized Laplacian, there are many more things that we don't know. One common way to understand a matrix in spectral graph theory is to understand how to construct cospectral graphs.

(These show weaknesses of the spectrum, i.e., where the two graphs differ structurally we then see that we cannot detect that structure in the graph by only using the spectrum.) Another is to understand which graphs are determined by their spectrum (see [21]).

As a simple example, for the adjacency matrix, a graph is regular if and only if the average degree (which can be determined by the number of edges, and hence the spectrum) is also the maximal eigenvalue of the matrix. Further, if the graph is regular, then the number of connected components is equal to the multiplicity of the maximal eigenvalue. From these we can conclude that the cycle graph $C_n$ is determined by its spectrum for the adjacency matrix (i.e., we can determine that it is a connected, 2-regular graph which greatly narrows down the possibilities).

For the normalized Laplacian, the cycle $C_n$ is *not* always determined by its spectrum. We already saw at the beginning that $C_4$ was cospectral with $K_{1,3}$ (see Figure 1). More generally, if we let $\gamma_{4k}$ denote the graph on $4k$ vertices formed by gluing the center vertex of $P_{2k+1}$, a path on $2k+1$ vertices, onto a vertex of $C_{2k}$, then $\gamma_{4k}$ and $C_{4k}$ are cospectral (see Figure 2).

The cospectrality of this pair of graphs is a good exercise in applying the "folding" results for the normalized Laplacian (see [8]), along with knowing the spectrum of the cycle $C_n$ (given earlier), and that the spectrum of the path $P_n$ is $1 - \cos\big(\pi k/(n-1)\big)$ for $k = 0, 1, \ldots, n-1$.

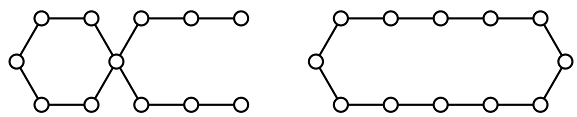

Figure 2: The cospectral graphs $\gamma_{12}$ and $C_{12}$

**Conjecture.** *For the normalized Laplacian, the only simple graphs cospectral with a cycle are the graphs $\gamma_{4k}$.*

Currently we do not have the tools to answer this, or many other questions related to the normalized Laplacian.

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# CONFERENCE REPORTS

## Western Canada Linear Algebra Meeting
## Regina, Canada, May 10–11, 2014

**Report by Douglas Farenick**

*Participants of WCLAM* 2014.

This edition of the Western Canada Linear Algebra Meeting (WCLAM) was the 12th in a series of biennial meetings that have been held at various sites throughout Western Canada, beginning with an inaugural meeting at the University of Regina in 1993. Each WCLAM provides an opportunity for researchers and mathematicians in linear algebra and related fields to meet, present accounts of their research, and to hold informal discussions. Although WCLAM has a regional base, the meetings attract a number of national and international participants. WCLAM enjoys a reputation for its welcoming atmosphere and its high quality scientific program. Young mathematicians (graduate students, postdoctoral researchers, early career faculty members) are particularly encouraged to participate.

The four featured speakers at WCLAM 2014 were Joel Friedman (University of British Columbia), Roger Horn (University of Utah), Mitja Mastnak (Saint Mary's University, Halifax), and Maya Mincheva (Northern Illinois University). Professor Mincheva was the ILAS Lecturer at the meeting. About 40 people participated in the conference, 30 of whom were from Canada and the USA.

Details of the program may be found at the WCLAM webpage `http://uregina.ca/~abstsubm/index.html`. ILAS members are welcome to join the WCLAM group page; the link to do so is `https://www.facebook.com/groups/wclam/`.

## 10th International Workshop on Accurate Solution of Eigenvalue Problems Dubrovnik, Croatia, June 2–5, 2014

**Report by Beresford Parlett**

"Eigenvalues, anyone?"

Thanks to the HBO series 'Game of Thrones,' the 'old town' part of Dubrovnik is familiar to even couch potatoes. The same is now true for some of us who work in numerical linear algebra because the IWASEP group held its biennial meeting, for the second time, in this 'pearl of the Adriatic sea.' The first occasion was in 2009. IWASEP is an acronym for *International Workshop on Accurate Solution of Eigenvalue Problems*. That odd title designated those few investigators, in 1986, who were interested in achieving high relative accuracy, when possible, in small or even tiny eigenvalues. The scene has changed steadily over the decades and the initial coterie, mainly from Croatia, Berkeley, and Penn State, has expanded along with the scope. Nevertheless, this year's meeting comprised under 50 participants who met in plenary session in one nice room for three and half intense days. Two full periods were allotted to viewing posters and a modest prize was offered to one of them, authored by Ana Susnjara of the EPF Lausanne.

We all slept in the same handsome 1901 mansion, now owned by the University of Zagreb, in which our talks took place and in whose courtyard we sipped our coffee. This impressive building is close to two swimming places and is only a five minute walk North of the 'old town.' In the evening we were free to find attractive restaurants there. Could there be a better location? On this occasion the meeting was run jointly by the University of Split (Ivan Slapničar) and the University of Zagreb (Zlatko Drmač) to whom we are all grateful for providing such a splendid environment for our interactions.

What did we talk about? Concern with high relative accuracy was still present in Chris Beattie's perturbation theory for matrix pairs and hyperbolic quadratic eigenvalue problems, as well as in the work of Heinrich Voss on the variational principles at work in specific vibration problems. Nevertheless, the variety of topics has burgeoned. Linear systems theory and transfer functions lurk in the background. Dimension reduction looms large along with special factorizations for special purposes: CUR = (a few columns)(low rank)(a few rows) for approximation and null space decomposition to reveal Jordan structure. On the theory side we heard about $\sin 2\Theta$ theorems for matrix pairs, probabilistic bounds for condition numbers, nonsymmetric preconditioners for self adjoint problems, convergence of block Jacobi methods, and some new nearness problems, namely distance to localization. We heard about Ricatti equations, Sylvester equation, contour integrals for analytic matrix functions and the virtues of arrowhead matrices (rivals to tridiagonals). There were two presentations on roots of polynomials and one on the use of GPUs. We were all ready for Wednesday's excursion to the ancient village of Ston and a fancy fish restaurant up the coast.

Exhausting, yes, but a rich experience.

The next meeting, announced for June 2016 at EPF Lausanne, will be hosted by Daniel Kressner. For more details, see `http://iwasep.fesb.hr/iwasep10`.

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## The 7th Linear Algebra Workshop (LAW'14), and the 23rd International Workshop on Matrices and Statistics (IWMS) Ljubljana, Slovenia, June 4–12, 2014 and June 8–12, 2014

**Report by Damjana Kokol Bukovšek**

*Participants of LAW-IWMS* 2014

The workshops were organized to coincide, IWMS taking place in the second week of LAW'14. The main theme of the 7th Linear Algebra Workshop workshop was the interplay between operator theory and algebra. After a few hours of talks in the morning, afternoons were used for work in smaller groups. The main theme of the 23rd International Workshop on Matrices and Statistics was the interplay between matrices and statistics. A special issue of the journal *Operators and Matrices* will be published as the proceedings of the workshops. On Tuesday, June 10, there was an invited ILAS lecture, given by Alexander Guterman from Moscow State University.

Invited speakers at LAW'14 were: A. Guterman, J. Holbrook, T. Laffey, L. Livshits, L. Marcoux, M. Mastnak, V. Müller,

H. Radjavi, L. Rodman, A. Sourour, and V. Troitsky. Invited speakers at IWMS were: F. Akdeniz, K. Conradsen, T. Dayar, A. Hassairi, S. Kirkland, I. Olkin, G. P. H. Styan, K. Šivic, P. Taylor, Y. Tian, and P. Vassiliou. There were also 14 contributed talks at LAW'14 and 17 contributed talks at IWMS. Workshops were supported by the Faculty of Mathematics and Physics, University of Ljubljana, Slovenia; the Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia; the International Linear Algebra Society; and the Jožef Stefan Institute, Slovenia. There were about 80 participants at both workshops. To see a list of those, a complete list of abstracts and some photos, please visit `http://www.law05.si/law14` and `http://www.law05.si/iwms`.

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## Graph Theory, Matrix Theory and Interactions
## A conference to celebrate the scholarship of David Gregory
## Queen's University, Canada, June 20–21, 2014

**Report by Kevin Vander Meulen and David Wehlau**

On June 20-21, the Department of Mathematics and Statistics at Queen's University (Kingston, ON, Canada) hosted a conference in honour of the late David A. Gregory. The conference, entitled "Graph Theory, Matrix Theory and Interactions," was a celebration of David's scholarship, mentoring and collaboration. Forty people attended, including three of David's former Ph.D. students.

David obtained his undergraduate mathematics degree from Queen's University, and after getting his Ph.D. from Michigan, joined the faculty at Queen's in 1967. He was a valued member of the department from 1967 until his death in the summer of 2013. David Gregory explored mathematical problems that bridged two areas of mathematics: linear algebra and discrete mathematics, especially graph theory. His insights, careful approach and writing influenced many mathematicians, especially in combinatorial matrix theory.

David's research initially focused upon vector sequence spaces but moved into topics related to ranks, exploring Boolean rank, nonnegative rank, and even introducing a new concept, Hermitian rank. He explored algebraic connections with some graph theoretic counting problems, like finding bounds on clique cover numbers, biclique partitions, and multiclique decompositions of graphs. He was fond of exploring variations on the Graham-Pollak theorem, a result which solves a purely combinatorial problem via clever algebraic means. Following along these lines, David has been keenly interested in eigenvalues of graphs and tournaments, exploring connections to inertia of graphs, spread of the eigenvalues of a graph, and most recently, relationships between eigenvalues and the size of matchings of a graph, or the independence number.

David Gregory made many contributions to these topics. The conference included presentations by both researchers who have collaborated with David (of which there are many), as well as other researchers who have been influenced by his work. A banquet was held at the University Club on June 20$^{\text{th}}$, and many fond stories of David were shared.

*Participants of the Conference in Honour of David A. Gregory*

The program included the following invited speakers: Richard Brualdi (University of Wisconsin), Sebastian Cioabă (University of Delaware), Randall Elzinga (Royal Military College), Robert Erdahl (Queen's University), Chris Godsil (University of Waterloo), Willem Haemers (Tilburg University), Steve Kirkland (University of Manitoba), Eric Moorhouse (University of Wyoming), Ram Murty (Queen's University), Naomi Shaked-Monderer (The Max Stern Yezreel Valley College), Claude Tardif (Royal Military College), Edwin van Dam (Tilburg University), Kevin N. Vander Meulen (Redeemer University College), and David Wehlau (Queen's University). Shaun Fallat (University of Regina) gave the the inaugural Hans Schneider ILAS Lecture.

The Organizing Committee consisted of: Sebastian Cioabă, Ram Murty, Bryan Shader, Claude Tardif, Kevin N. Vander Meulen, and David Wehlau. The Organizing Committee gratefully acknowledges support from the Fields Institute for Research in Mathematical Sciences, the International Linear Algebra Society, and the Department of Mathematics and Statistics at Queen's University.

A special issue of the *Electronic Journal of Linear Algebra* will be dedicated to this conference.

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## The 2014 International Workshop on Matrices and Operators Hainan, China, July 24–27, 2014

**Report by Chi-Kwong Li, Weiyan Yu, and Fuzheng Zhang**

The 2014 International Workshop on Matrices and Operators took place at the Hainan Normal University at Haikou from July 24 to 27, 2014.

*Matrices and Operators Group Photo*

A week before the workshop, Haikou was seriously hit by a typhoon, which was the strongest typhoon in 41 years. Luckily, the town recovered quickly from the damage. Despite some minor adjustments, the workshop ran well because of the hard work of the working team at Hainan Normal University.

There were about 100 participants from different regions including China, Canada, Iran, Hong Kong, Philippines, USA, Taiwan, etc. There were 35 talks, about half of them in English and half of them in Chinese. A workshop dinner took place on July 25, and a tour to the biodiversity museum was arranged on July 26. Some participants went on a tour of Haikou and Qionghai on July 27 to experience Hainan.

The 2015 workshop will take place in Shaanxi.

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## The 2014 Workshop on Numerical Ranges and Numerical Radii Sanya, China, July 28–August 1, 2014

**Report by Man-Duen Choi and Chi-Kwong Li**

The 2014 Workshop on Numerical Ranges and Numerical Radii took place at the Tsinghua Sanya International Mathematics Forum at Sanya from July 28 to August 1, 2014.

The organizing committee consisted of: Shiu-Yuen Cheng (Tsinghua University), Man-Duen Choi (University of Toronto), Jinchuan Hou (Taiyuan University of Technology), Chi-Kwong Li (College of William and Mary) and Jie Xiao (Tsinghua University).

*Numerical Ranges and Radii Group Photo*

There were 45 participants from different regions including China, Canada, Iran, Hong Kong, Philippines, USA, Taiwan, etc. There were 32 talks arranged in the five day workshop. A workshop dinner took place on July 20.

The support and hospitality of colleagues at the Tsinghua Sanya International Mathematics Forum are deeply appreciated. One may find links to the workshop photo, talk schedule, titles and abstracts at the workshop website: `http://cklixx.people.wm.edu/wonra14.html`.

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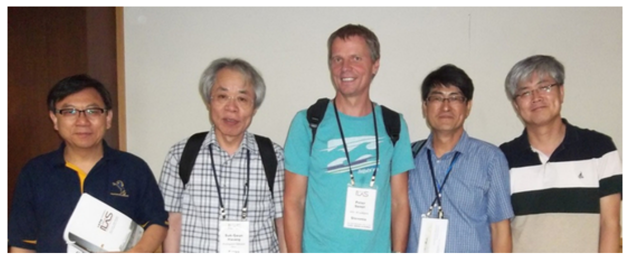
# The 19th Conference of the International Linear Algebra Society Sungkyunkwan University, Seoul, South Korea, August 6–9, 2014

**Report by Suk-Geun Hwang, In-Jae Kim and Sang-Gu Lee**

The Korean Linear Algebra Society was honored to host the 19th Conference of the International Linear Algebra Society (ILAS) at Sungkyunkwan University, Seoul, South Korea from August 6 to August 9, 2014. The conference, whose theme was *Solidarity in Linear Algebra,* was attended by 348 participants from 41 different countries, making the conference highly international. At the conference, 262 talks were presented, including 12 plenary talks. The conference was supported by many sponsors including several Korean organizations such as Sungkyunkwan University, Pusan National University, Seoul ICM 2014, the National Institute for Mathematical Science (NIMS), the National Research Foundation (NRF), the Seoul Metropolitan Government, and the Korea Tourism Organization, as well as ILAS, the SIAM Activity Group on Linear Algebra, Elsevier, and Taylor & Francis. The ILAS Conference was one of the satellite conferences of ICM 2014.

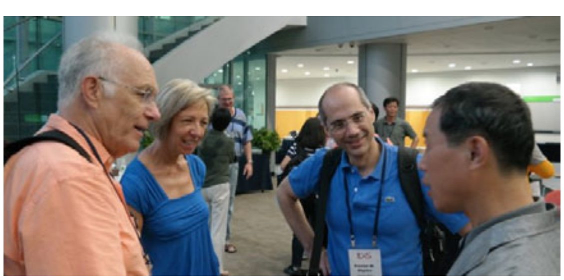


*Local Organizers with ILAS President: Gi-Sang Cheon, Suk-Geun Hwang (Co-Chair), Peter Šemrl, Sang-Gu Lee (Co-Chair), and Hyun-Min Kim.*

There were three social events prepared by the local organizing committee. The first was the welcome reception which was held the night before the conference, Tuesday, August 5, along with early registration. In the afternoon of Friday, August 8, 187 participants joined the two excursions: 74 participants went to the Seoul City excursion and 113 participants went to the Demilitarized Zone (DMZ) excursion. A lunch sponsored by Elsevier was served before the excursions. On the evening of the same day, 212 participants attended the conference banquet where a very special letter from Hans Schneider was read by Richard Brualdi.

Almost all morning and afternoon sessions started with plenary talks reflecting recent developments and new research relevant to linear algebra. There were 12 plenary addresses:

- Ravindra Bapat (LAMA Speaker), Indian Statistical Institute, Delhi, India, "Square distance matrix of a tree."
- Peter Benner, Max Planck Institute, Magdeburg, Germany, "Solving Large-Scale Matrix Equations: Recent Progress and New Applications."
- Dario Bini (LAA Lecturer), University of Pisa, Italy, "Exploiting matrix structures in polynomial eigenvalue problems: Computational Issues."
- Shaun Fallat (Taussky-Todd Lecturer), University of Regina, Canada, "On the eigenvalues of positive matrices: In the spirit of Taussky Unification."
- Andreas Frommer (SIAG/LA Lecturer), University of Wuppertal, Germany, "Arnoldi approximation for matrix functions: how to restart and when to stop."
- Stephane Gaubert, INRIA, Domaine de Voluceau, France, "From Tropical Linear Algebra to Zero-Sum Games."
- Chi-Kwong Li, College of William and Mary, USA, "Some matrix techniques, results, and problems in quantum information science."
- Yongdo Lim, Sungkyunkwan University, South Korea, "Monotonicity of the Karcher mean."
- Panayiotis Psarrakos, National Technical University of Athens, Greece, "Travelling from matrices to matrix polynomials."
- Vladmir Sergeichuk, Institute of Mathematics, Kiev, Ukraine, "Classification problems for systems of forms and linear mappings."
- Bernd Sturmfels, University of California, Berkeley, USA, "Beyond Linear Algebra."
- Tin-Yau Tam, Auburn University, USA, "Some matrix asymptotic results and their Lie counterparts."



In addition to the plenary talks there were 8 invited minisymposia, 14 contributed minisymposia, and 72 contributed talks. The 8 invited minisymposia with their organizers were:

- Combinatorial Problems in Linear Algebra, Richard A. Brualdi (USA) and Geir Dahl (Norway)
- Matrix Inequalities, Fuzhen Zhang (USA) and Minghua Lin (Canada)
- Spectral Theory of Graphs and Hypergraphs, Vladimir S. Nikiforov (USA)
- Tensor Eigenvalues, Jia-Yu Shao (China) and Liqun Qi (Hong Kong)
- Quantum Information and Computing, Chi-Kwong Li (USA) and Yiu Tung Poon (USA)
- Riordan arrays and Related Topics, Gi-Sang Cheon (South Korea) and Donatella Merlini (Italy)
- Nonnegative Matrices and Generalizations, Judi McDonald (USA)
- Toeplitz Matrices and Operators, Torsten Ehrhardt (USA)




The 14 contributed minisymposia and their organizers were:

- Structured Matrix Equations, Eric King-wah Chu (Australia)
- Inverse Spectral Problems, Carlos M. Fonseca (Kuwait)
- Matrix Geometry, Miklos Palfia (Japan) and Takeaki Yamazaki (Japan)
- Teaching Linear Algebra, Ajit Kumar (India) and Abraham Berman (Israel)
- Algebraic Combinatorics and Combinatorial Matrices, Han Hyuk Cho (South Korea), Jang Soo Kim (South Korea), and Seungjin Lee (South Korea)
- Linear Least Squares and Applications, Yimin Wei (China) and Marc Baboulin (France)
- Generalized Laplacian and Green Matrices, A. Carmona (Spain), A.M. Encinas (Spain), and M. Mitjana (Spain)
- Eigenvalue Computations and Applications, Ren-Cang Li (USA)
- Inequalities in Matrices, Operators, and Lie Groups, Natalia Bebiano (USA) and Tin-Yau Tam (USA)
- Matrix Methods in Computational Systems Biology and Medicine, Konstantin Fackeldey (Germany)
- Solution of Sylvester-like Equations and Canonical Forms, Stefan Johansson (Sweden)
- Generalized Matrix Inverses and Applications, Jeffrey Hunter (New Zealand) and Minerva Catral (USA)
- Sign Pattern Matrices, Zhongshan Li (USA) and Frank Hall (USA)
- Recent Developments in Linear Preserver Problems, LeRoy Beasley (USA) and Seok-Zun Song (South Korea)

The scientific organizing committee consisted of Suk-Geun Hwang (Chair, South Korea), Nair Abreu (Brazil), Tom Bella (USA), Rajendra Bhatia (India), Richard A. Brualdi (USA), Man-Duen Choi (Canada), Nicholas J. Higham (UK), Leslie Hogben (USA), Stephen Kirkland (Canada), Sang-Gu Lee (South Korea), Helena Šmigoc (Ireland), and Fuzhen Zhang (USA). More information on the conference may be found at the conference website, `http://www.ilas2014.org`.

Some records of the conference can be found at `http://matrix.skku.ac.kr/2014-Album/ILAS-2014/` and the official group photo can be downloaded from `http://matrix.skku.ac.kr/2014-Album/ILAS-2014/ILAS-2014-Group-Photo.JPG`. The program book of the conference can be downloaded from `http://matrix.skku.ac.kr/ILAS-Book/index.htm`. Slide shows of photos can be seen in `http://youtu.be/asJfRFYWPrk`, `http://youtu.be/bidJNagmRXQ`, `http://youtu.be/1OfDqWA-vVA`, `http://youtu.be/6IlS8U6i_8E`, and `http://youtu.be/UMwLCtSGByI`.

At the closing of the conference, ILAS President Peter Šemrl acknowledged the local organizers for their hard work, noting that the ILAS conference was very well organized, and that in the last two days of the conference he talked to many participants that shared the same sentiment.

The next ILAS meeting, the 20$^{th}$ ILAS Conference, will be held July 11–15, 2016 in Leuven, Belgium.



*Participants of the 19th ILAS Conference*

## The International Conference on Trends and Perspectives in Linear Statistical Inference (LINSTAT'2014) Linköping, Sweden, August 24–28, 2014

**Report by Francisco Carvalho and Katarzyna Filipiak**

The 2014 edition of LINSTAT – the International Conference on Trends and Perspectives in Linear Statistical Inference – was held at the Linköping University, Sweden. Linköping is a XII century city in the core of the Östergötland County, well-known for its university and its high-technology industry. With its beautiful surroundings and privileged conditions to promote conferences, Linköping made possible a get-together with highly regarded specialists and researchers interested in the topics selected for these conferences.

The Scientific Committee of LINSTAT'2014 was chaired by Dietrich von Rosen (Sweden) and the Local Organizing Committee was chaired by Martin Singull (Sweden).

*LINSTAT* 2014 *Group Photo*

The purpose of the meeting was to bring together researchers sharing an interest in a variety of aspects of statistics and its applications, as well as matrix analysis and its applications to statistics, and to offer a venue to discuss current developments in these subjects.

The plenary talks by invited speakers were delivered by Alan Agresti (USA), S. Ejaz Ahmed (Canada), Dennis Cook (USA), Sat Gupta (USA), Lynn LaMotte (USA), Emmanuel Lesaffre (Belgium), Yonghui Liu (China), Thomas Mathew (USA), Jamal Najim (France), Muni S. Srivastava (Canada), Yongge Tian (China) and Júlia Volaufová (USA), as well as the winners of the Young Scientists Awards of LINSTAT'2012: Maryna Prus (Germany), Jolanta Pielaszkiewicz (Sweden), Fatma Sevinç Kurnaz (Turkey) and Alena Bachratá (Slovakia).

Over 85 talks and 9 posters were presented, including a special session devoted to 30 years of the Polish-Nordic column spaces (organized by S. Puntanen). Other special sessions included Small Area Estimation (S. Haslett), Computation-Intensive Methods in Regression Models (M. Tez), the use of Kronecker Products in Statistical Modeling (A. Roy), Bayesian Inference for Complex Problems (M. Villani), Statistics in Life Sciences (M. Fonseca), Statistical Methods for Complex, High-Dimensional Data (T. Pavlenko), Linear Models Applied to Complex Problems (T. Holgersson) and Design and Analysis of Experiments (A. Markiewicz).

At the end of the conference, the Scientific Committee proudly announced the winners of the Young Scientists Awards for the best oral presentations and posters presented by Ph.D. students and young scientists: Matias Quiroz (Sweden), Alzbeta Kalivodova (Czech Republic), Ying Li (Sweden) and Yuli Liang (Sweden). The award winners will be Invited Speakers at the 2016 edition of LINSTAT that will be held in Istanbul (Turkey).

*The winners of Young Scientists Awards: (left to right) Yuli Liang, Alzbeta Kalivodova, Ying Li, and Matias Quiroz*

The special issue of *Acta et Commentationes Universitatis Tartuensis de Mathematica* will be devoted to LINSTAT'2014 with selected papers strongly related to the presentations given at the conference.

The participants shared the common opinion that the conference was extremely fruitful and well-organized with a friendly and warm atmosphere. The list of the participants of LINSTAT'2014, abstracts of the talks and posters, the gallery of photographs, and other information can be found at: `http://www.mai.liu.se/LinStat2014`.

# BOOK REVIEWS

## ***Totally Positive Matrices*, by Allan Pinkus**

**Cambridge University Press, 2010, ISBN 978-0-5211-9408-2, xii+182 pages.**

## ***Totally Nonnegative Matrices*, by Shaun Fallat and Charles Johnson**

**Princeton University Press, 2011, ISBN 978-0-6911-2157-4, xiii+264 pages.**

**Reviewed by Sebastian Cioabă, University of Delaware, cioaba@math.udel.edu.**

People interested in linear algebra and functional analysis have been spoiled by the appearance of the two monographs under review: *Totally Positive Matrices* by Allan Pinkus and *Totally Nonnegative Matrices* by Shaun Fallat and Charles Johnson.

Both books were inspired by the work of the pioneer researchers in this area: I.J. Schoenberg, F.R. Gantmacher, M.G. Krein and S. Karlin and build upon their works and some more modern treatment of this area of research.

There are some differences between the approaches taken in these two books.

Pinkus uses the following definition: a *totally positive matrix* or *TP matrix* is a real matrix with the property that the determinant of every square submatrix is non-negative. Fallat and Johnson call a matrix satisfying the above properties a *totally nonnegative matrix* or *TN matrix*.

The treatment of Pinkus in *Totally Positive Matrices* is classical and more reliant on totally positive functions and their associated analysis. As mentioned by its author in the introduction, this book presents the central properties of TP matrices and has only 6 chapters covering the basic properties of TP matrices, their variation-diminishing properties, various examples of TP matrices, their eigenvalue and eigenvector properties and factorization. The author confesses to omitting various applications, but provides useful references to such applications for those interested. The historical notes and comments are truly enlightening and give a clear description of the development of this subject from the early work of Schoenberg in 1930, of Gantmacher and Krein in 1950, Karlin in 1968 to the work of Ando in 1987 and the more recent work of others. The author also explains the origins of the differing terminology used: Schoenberg used the German *total positiv* in 1930 while Gantmacher and Krein, unaware of Schoenberg's work, used the French *complètement non-négative* and *complètement positive*.

The approach of Fallat and Johnson in *Totally Nonnegative Matrices* is more modern and their book relies more on matrix theory and determinantal properties and less on the analysis of functions. All the central topics are covered as in Pinkus' book, but these authors also include many other topics such as Hadamard products, completion problems associated with TN/TP matrices and a lengthy discussion on rank distribution. The underlying themes of this book are matrix factorization and the important fact that TN matrices possess a bidiagonal factorization. The authors devote an entire chapter to matrix factorization and they use the theory from this chapter throughout the book. Along with matrix factorization, the authors include planar diagrams as a conceptual combinatorial tool for analyzing TN matrices. The historical remarks are discussed at the beginning of the book and are separate from the rest of the book. The list of references is larger and more complete than Pinkus' book, which contains only the key references.

The reviewer recommends both books to those interested in learning more about TP/TN matrices. The books are well-written and the mathematics is interesting.

# *Matrix Theory*, by Xingzhi Zhan

**AMS Graduate Studies in Mathematics, Vol. 147**
**American Mathematical Society, 2013, ISBN 978-0-8218-9491-0, x+253 pages.**
**Reviewed by Douglas Farenick, University of Regina, Doug.Farenick@uregina.ca**

Xingzhi Zhan's new book on matrix theory is a relatively brief but fascinating account of a variety of topics of mainstream interest. In just 252 pages, Zhan covers a wide swath of material that ranges through matrix analysis (norms, numerical ranges, perturbation theory, majorisation, singular values), multilinear algebra (tensor products, applications to linear matrix equations), algebra (companion matrices, Newton's identities, Gröbner bases, Hilbert's Nullstellensatz), nonnegative matrices (Perron–Frobenius theorem, directed graphs), and qualitative matrix analysis (structured matrices, matrix completion problems, sign patterns).

In the Preface, the author explains his four criteria for selecting the material for inclusion in the book: a topic should be (1) important, (2) elegant, (3) ingenious, (4) interesting. To this end, Zhan succeeds wonderfully, and even the most knowledgable matrix theorist will no doubt find something appealing or new in the book. (For example, I was unaware until reading this book that the square of a real $n \times n$ matrix can have at most $n^2 - 1$ negative entries and that this upper bound is sharp.) The emphasis is on concepts and the presentation is kept lean by stressing ideas and leaving routine calculations or prerequisite material for the reader to fill in. The author is careful with citations and with giving credit to other mathematicians for certain proofs. (However, the proof of the Toeplitz–Hausdorff theorem on the convexity of the numerical range is attributed to Paul Halmos, when in fact the proof appears in Hausdorff's 1919 paper. Halmos surely knew this, but he was good at promoting and popularizing numerical ranges; eventually the community lost track of who did what early on.)

The first chapter deals with preliminary material such as determinants, norms, and the singular value decomposition. Additional preliminary material covers Gröbner bases and Carathéodory's theorem on conic combinations, which are not commonly treated in matrix theory texts.

Chapters 2 through 5 address, for the most part, topics in matrix analysis. The important work of Bhatia, Davis, and McIntosh on the existence, uniqueness, and norm of the solution $X$ to the Sylvester equation $AX - XB = C$ is treated in the chapter on tensor products, as is more standard fare such as Schur's theorem on the Hadamard product of positive semidefinite matrices. There is a very fine treatment of majorisation, matrix inequalities, unitarily invariant norms, and perturbation analysis. Although there is some overlap with Bhatia's *Matrix Analysis* regarding this suite of topics, there is also much here that appears in book form for the first time.

Chapter 6 is a look at nonnegative matrices. Most of the expected results are present, but some special topics such as totally nonnegative matrices and Hadamard exponentials are also discussed. A very elegant proof, which the author attributes to Roger Horn, is given for Hopf's bound on the moduli of the eigenvalues of positive matrices in terms of the Perron value.

Chapters 7 and 8 are devoted to qualitative aspects of matrix theory, such as sign patterns and matrix completions, and Chapter 9 treats a variety topics of interest that did not find their way into earlier chapters. For example, Chapter 9 includes a section on the geometry of numerical ranges, a discussion of the Smith canonical form, and a proof due to Audenaert of the Böttcher–Wenzel bound on the Frobenius norm of commutators.

The final chapter offers some applications of matrix theory in algebra, number theory, combinatorics, and geometry, and an appendix concludes the book with a selection of open problems and conjectures.

As a graduate text, there is a great deal here to challenge and intrigue the intelligent student, making its publication in the AMS Graduate Studies Series a particularly good outlet for this work. However, the book is not, nor does it seem intended to be, a systematic development of the theory of matrices, say in the spirit of Gantmacher, Horn & Johnson, Lancaster & Tismenetsky, or Dym. Indeed, the reader is assumed to already be fairly well versed in linear algebra, as the Jordan canonical form is invoked on page 8 and on page 24 the reader ought to know that the set of $n \times n$ diagonalisable complex matrices is dense in $M_n(\mathbb{C})$.

One of the greatest values of the book is that it draws substantially from a body of results scattered throughout the research literature and presents these results in a single volume of modest length. Stimulating and well written, I cannot really think of another book like it. Professor Zhan has written a very singular work worthy of study by students and experts alike.

# OBITUARY

## Last Words of Hans Schneider 1927–2014

Hans Schneider, a research mathematician who devoted much of his academic life to the revival of the classical field of linear algebra (aka matrix theory), died on October 18, 2014, aged 87. The cause was cancer of the esophagus.[1]

He was the author of about 160 research papers that covered many aspects of theoretical linear algebra, but particularly the theory of nonnegative matrices, an area with multiple applications. His first paper was submitted at the age of 24, his last within a year of his death. He soon discovered that research with others was productive and this led to joint publications with about 80 collaborators. In his later years, Hans would express contentment that his contributions have been taken up and developed by others, even when his original work may have been forgotten, a difficult stage to achieve.

When Hans was being considered for tenure in the mathematics department at the University of Wisconsin-Madison, a famous Russian algebraist was visiting. Hans was later told that this mathematician had informed a member of the tenure committee that in Russia, "we expect every mathematician to know linear algebra but it is not a field for research." Hans was rescued by members of his department who were outside his field. This incident left a deep mark on Hans, as he realized what he was up against, and made him committed to the revival of linear algebra as a field of research.

In 1967, he was elected Chair of the Mathematics Department, the second biggest in the University of Wisconsin, at the relatively early age of 39. He felt confident before this that he could deal with faculty problems, but soon regretted having taken this position, because he felt incompetent to deal with the student movement that was then sweeping North American campuses. He relinquished this position in 1969 with some relief.

In 1972, Hans was given the chance of being editor-in-chief of the four-year-old struggling journal "Linear Algebra and its Applications," led formerly by Alan Hoffman, a great mathematician. This is the activity for which Hans is probably best known in the profession. This commercial journal presented him with an opportunity to redress the neglect of this field by established national mathematical societies. When he retired from this position 40 years later the journal had four editors-in-chief and received about 1200 submissions annually, leading to about 5,000 pages of print.

Hans, together with some colleagues, established the International Matrix Group in 1987, which three years later was incorporated as the International Linear Algebra Society (ILAS). He was its first president from 1990 to 1996. The 19$^{\text{th}}$ meeting of the Society was held in Korea in August 2014. Hans realized that in the mathematical culture, groups tend to form around distinguished individuals. These groups flourish for a time and then typically disappear. In order to give the society permanence, a very formal structure with annual elections was established for ILAS. He wanted to make it harder to discontinue meetings in linear algebra than to continue them. Currently ILAS has about 400 members in more than 20 countries and publishes two journals. In recent years there has been much criticism of the high price of journals published by commercial publishers. Hans felt that established mathematical societies have only themselves to blame for not giving linear algebra the support it needed.

Hans Schneider was born in Vienna, Austria, on January 24, 1927 as the only child of two dentists, Hugo and (Isa)Bella (Saphir) Schneider. He led a comfortable middle class existence until the occupation of Austria by Nazi Germany in March 1938. His family had no religious affiliation but were Jewish under the Nazi racial laws. His father was one of the first to realize that further existence in Austria would be impossible. He saw that this was not a return to a limited but tolerable ghetto existence, but a new form of persecution. The family fled into an illegal, insecure, and generally miserable existence in Czechoslovakia, and then found themselves in Poland when the city in which they were living was annexed by that country following the Munich Agreement in October 1938.

In November 1938, Hans entered the Quaker school in Eerde, Netherlands, as the result of rather desperate entreaties by his mother to the Dutch authorities. Some months later, his parents gained admission to the UK thanks to special conditions arranged by the then home secretary, Lord Templewood (aka Sir Samuel Hoare), who (by his own account) had a high opinion of the Viennese school of doctors and dentists and a low opinion of the corresponding British schools. Hans was reunited with his parents in Edinburgh in August 1939, three weeks before the outbreak of the war in Europe. He ascribed his survival to his father's decisive and risky action to flee Austria followed by the good luck of leaving three countries a few months ahead of their occupation by Nazi Germany. He felt great gratitude to the British people's determination to fight Hitler even when all seemed lost.

In Edinburgh, Hans received a first-rate Scottish education at George Watson's Boys College (the equivalent of an American high school) followed by four years at Edinburgh University where he graduated M.A. with First Class Honours in 1948. He obtained a position as Scientific Officer at the Royal Observatory, Edinburgh, but was fired within two years because he had broken an expensive brand new astronomical instrument. He then returned to mathematics and had

[1] This obituary was written by Hans, who was terminally ill, on May 31, 2014. It was read by Richard Brualdi at the Banquet of the 19$^{\text{th}}$ Conference of ILAS. A memorial page has been set up as a tribute to ILAS's founding president: `http://www.ilasic.org/misc/memorial.html`

the good fortune to be the graduate student at Edinburgh University of A.C. Aitken, an idiosyncratic mathematical genius. Under pressure because of his growing family, he wrote a Ph.D. thesis within 18 months and was appointed to an assistant lectureship by S. Verblunsky at Queen's University, Belfast. In 1959 he emigrated to a position at the University of Wisconsin–Madison, where he stayed for the rest of his career. Travel was a research tool; Hans held visiting positions at several universities, including the Technion, Israel, the Technical University of Munich, and the University of Birmingham (UK). He retired from his tenure position as J.J. Sylvester Professor in the year 1993. Hans would carefully say that he retired from teaching, implying that he was not retired as a mathematician.

In 1948, Hans married Miriam Wieck, a professional violinist and member of a famous musical family. She was his constant companion and support in a marriage that lasted more than 66 years. He is survived by his wife Miriam, their three children, Barbara (Daryl), Peter (Hope), Michael (Laurie) and six grandchildren, David and Daniel Caswell, Hannah and Rebecca Schneider, Carson Rose and Kurt Schneider.

# JOURNAL ANNOUNCEMENT

## Special Issue: Legacy of Hans Scheinder

The Editors-in-Chief of the journal *Linear Algebra and its Applications (LAA)* are pleased to announce a special issue

THE LEGACY OF HANS SCHNEIDER

in recognition of the enormous contributions of Hans Schneider to the journal (forty years as an Editor-in-Chief), to research in linear algebra and related areas (nearly 200 papers over more than 60 years, and still continuing), and to the linear algebra community (organizer of the International Linear Algebra Society and its founding president). We especially welcome contributions that have been influenced by the work of Hans Schneider.

Papers should be submitted by December 31, 2014 via the Elsevier Editorial System (EES): `http://ees.elsevier.com/~laa` choosing the special issue indicated above and any one of the three Editors-in-Chief of *LAA*.

# ILAS NEWS

## 2014 ILAS Elections

Nominated for a three-year term, beginning March 1, 2015, as ILAS Secretary/Treasurer is: Leslie Hogben, USA.

Nominated for the two open three-year terms, beginning March 1, 2015, as "at-large" members of the ILAS Board of Directors are: Geir Dahl, Norway; Betrice Meini, Italy; Simo Puntanen, Finland; and Henry Wolkowicz, Canada.

Many thanks to the Nominating Committee: Dario Bini, Shaun Fallat (chair), Marko Huhtanen, Rachel Quinlan, and Xingzhi Zhan.

Electronic voting will be implemented using Votenet Solutions (the same company ILAS has used for elections since 2011). Voting concludes January 16, 2015. If you have not received your electronic ballot, please contact Leslie Hogben: hogben@aimath.org

# UPCOMING CONFERENCES AND WORKSHOPS

## Structured Matrices and Tensors: Analysis, Algorithms and Applications<br>Taipei, Taiwan, December 8–11, 2014

Structured matrices and tensors occur in many disciplines in applied mathematics and engineering. Fast algorithms for structured matrices and tensors are of particular importance in data analysis, and signal and image processing. This conference has two major objectives. (1) The intention is to improve the dialogue and collaboration between matrix and tensor theoreticians, and computational scientists. (2) The goal is to reduce the gap between researchers working on the fundamentals and those working on real-life applications, so that emerging applications will stimulate new theoretical research and the better theoretical tools will in turn be exported back to the various application fields. This meeting is the $4^{\text{th}}$ of a series of meetings on structured matrices with the first three held in Hong Kong in 2002, 2006 and 2010.

The Organizing Committee consists of: Raymond Chan (Chinese U. of Hong Kong), I-Liang Chern (National Central U.), Tsung-Ming Huang (National Taiwan Normal U.), Wen-Wei Lin (National Chiao Tung U.), Michael Ng (Hong Kong Baptist U.), Weichung Wang (National Taiwan U.). Further information can be found at: `https://sites.google.com/site/smtaaa14/`.

## The 2014 International Conference on Tensors and Matrices and their Applications (TMA2014)
## Suzhou, China, December 16–19, 2014

The 2014 conference on Tensors and Matrices and their Applications (TMA2014) will be held at Suzhou University of Science and Technology (USTS), Suzhou, China, Dec. 16–19, 2014. TMA2014 is endorsed by The International Linear Algebra Society(ILAS), and is jointly sponsored by The Hong Kong Polytechnic University, Fudan University, Jiangsu Normal University (JSNU) and Suzhou University of Science and Technology (USTS). TMA2014 also obtained sponsorship from the center of Chinese colleges (Peking University) and the Suzhou Government.

The invited speakers include: Zhaojun Bai (USA), Changjiang Bu (China), An Chang (China), Andrzej Cichocki (Japan), Joshua Cooper (USA), Yu-Hong Dai (China), Lieven De Lathauwer (Belgium), Jiu Ding (USA), Lars Elden (Sweden), Yizheng Fan (China), Yubin Gao (China), Edinah K. Gnang (USA), Chuangqing Gu (China), Zhenghai Huang (China), Zhaolin Jiang (China), Chi-Kwong Li (USA), Guoyin Li (Australia), Wen Li (China), Yaotang Li (China), Lek-Heng Lim (USA), Chen Ling (China), Michael Ng (Hong Kong), Juan Manuel Pena (Spain), Liqun Qi (Hong Kong), Sanzheng Qiao (Canada), Peter Šemrl (Slovenia), Naomi Shaked-Monderer (Israel), Jiayu Shao (China), Wenyu Sun (China), Eugene Tyrtyshnikov (Russia), Qingwen Wang (China), Yimin Wei (China), Qingzhi Yang (China), Liping Zhang (China), and Guanglu Zhou (Australia).

The Academic Committee consists of: Richard A. Brualdi, University of Wisconsin, USA; Kung-ching Chang, Peking University, China; Chi-Kwong Li, College of William & Mary, USA; Yongzhong Song, President of Nanjing Normal University, China; Zhaojun Bai, University of California at Davis, USA; A. Berman, Technion, Israel Institute of Technology, Israel; Zhongwen Chen, Soochow University, Suzhou, China; Lieven De Lathauwer, University of Leuven, Belgium; Lars Elden, Linkoping University, Sweden; M. Fiedler, Academy of Sciences of the Czech Republic; Wen Li, South China Normal University, Guangzhou, China; Lek-Heng Lim, Chicago University, USA; Michael Ng, The Hong Kong Baptist University, Hong Kong; Peter Šemrl, University of Ljubljana, Slovenia; Jiayu Shao, Tongji University, Shanghai, China; Qingwen Wang, Shanghai University, China; and Guanglu Zhou, Curtin University, Australia.

Further information can be found at: `http://www.tma2014.org/`.

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## International Conference on Linear Algebra & its Applications (ICLAA-2014)
## In honor of Prof. R. B. Bapat on his $60^{\text{th}}$ birthday
## Manipal, India, December 18–20, 2014

ICLAA-2014 will be held at Manipal University, Manipal, India (`www.manipal.edu`). The themes of the conference will focus on (i) Matrix Methods in Statistics, (ii) Combinatorial Matrix Theory and (iii) Classical Matrix Theory covering different aspects of linear algebra. The conference is a sequel to the conference CMTGIM-2012 held in January, 2012.

The conference will provide a platform to many of Prof. Ravindra B. Bapat's contemporaries to come together to serve the objectives of the conference on the occasion of his $60^{\text{th}}$ birthday. The conference also provides an opportunity to young scholars, who will be the asset to the future generation, to join the team of eminent scientists.

The scientific committee consists of Rajendra Bhatia (Indian Statistical Institute, Delhi, India), Steve Kirkland (University of Manitoba, Canada), K. Manjunatha Prasad (Manipal University, Manipal India), and Simo Puntanen (University of Tampere, Finland).

For more details and registration, please visit (`http://conference.manipal.edu/ICLAA2014/`) or contact the organizing secretary K. Manjunatha Prasad (iclaa2014@manipal.edu or km.prasad@manipal.edu).

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## Joint Mathematics Meetings
## Texas, USA, January 10–13, 2015

The Joint Mathematics Meetings is a large mathematics conference hosted by the American Mathematical Society and the Mathematical Association of America. One of the keynote addresses is on *Graphs, Vectors, and Matrices,* presented by Daniel A. Spielman of Yale University. There are several sessions devoted to topics of interest to ILAS members, including an *AMS Special Session on Graphs, Matrices, and Related Problems*, an *AMS Session on Linear and Multilinear Algebra and Matrix Theory*, an *AMS Special Session on Positivity and Matrix Inequalities*, an *MAA General Contributed Paper Session on Research in Linear Algebra*, and an *MAA Session on Innovative and Effective Ways to Teach Linear Algebra*.

For further information, visit: `http://jointmathematicsmeetings.org/jmm`.

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## Numerical Algebra, Matrix Theory, Differential-Algebraic Equations, and Control Theory
## Berlin, Germany, May 6–9, 2015

This conference aims at bringing together experts in the fields of numerical (linear) algebra, matrix theory, differential-algebraic equations and control theory. These mathematical research areas are strongly related and they often occur in the same real-world applications. Main areas where such applications emerge are computational engineering and sciences, but increasingly also social sciences and economics.

The conference is dedicated to Volker Mehrmann on the occasion of his $60^{\text{th}}$ birthday. Volker Mehrmann is a leading expert in the areas of the conference, and in a unique manner unifies expertise in the mathematical fields providing the title of this conference. For more information see: `http://www3.math.tu-berlin.de/multiphysics/VM60/`

Invited Speakers:

| | |
|---|---|
| Larry T. Biegler (Pittsburgh, PA, USA) | Matthias Bollhöfer (Braunschweig, Germany) |
| Shreemayee Bora (Guwahati, India) | Angelika Bunse-Gerstner (Bremen, Germany) |
| Stephen L. Campbell (Raleigh, NC, USA) | Etienne Emmrich (Berlin, Germany) |
| Maria J. Esteban (Paris, France) | Heike Faßbender (Braunschweig, Germany) |
| Shmuel Friedland (Chicago, IL, USA) | Martin Grötschel (Berlin, Germany), |
| Achim Ilchmann (Ilmenau, Germany) | Daniel Kressner (Lausanne, Switzerland) |
| Peter Kunkel (Leipzig, Germany) | Vu H. Linh (Hanoi, Vietnam) |
| D. Steven Mackey (Kalamazoo, MI, USA) | Agnieszka Miedlar (Lausanne, Switzerland) |
| Julio Moro (Madrid, Spain) | Nancy K. Nichols (Reading, UK) |
| Michael Overton (New York, NY, USA) | Federico Poloni (Bologna, Italy) |
| Timo Reis (Hamburg, Germany) | Valeria Simoncini (Bologna, Italy) |
| Tatjana Stykel (Augsburg, Germany) | Daniel Szyld (Philadelphia, PA, USA) |
| Caren Tischendorf (Berlin, Germany) | Françoise Tisseur (Manchester, UK) |
| David Watkins (Pullman, WA, USA) | Hongguo Xu (Lawrence, KS, USA) |

Scientific Committee: Peter Benner (Max Planck Institute for Dynamics of Complex Technical Systems), Jörg Liesen (TU Berlin), Christian Mehl (TU Berlin), Reinhard Nabben (TU Berlin), Lena Scholz (TU Berlin), and Andreas Steinbrecher (TU Berlin). Contact: vm60info@math.tu-berlin.de. Important Date: abstract submission due December 31, 2014.

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## $24^{\text{th}}$ International Workshop on Matrices and Statistics, IWMS-2015
## Haikou City, Hainan Province, China, May 25–28, 2015

The purpose of this ILAS-endorsed Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. The Workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in linear algebra and matrix theory and may exchange ideas with researchers from a wide variety of countries.

The Workshop will include Plenary Talks, Invited Minisymposia, Contributed Talks and Special Sessions honoring the $75^{\text{th}}$ birthday of Professor Kai-Tai Fang (United International College, Zhuhai, China) and the $70^{\text{th}}$ Birthday of Simo Puntanen (U. of Tampere, Finland).

Confirmed Plenary Speakers:

- Ravindra B. Bapat (Indian Statistical Institute, New Delhi, India)
- Mu-Fa Chen (Beijing Normal U., China)
- Zhi Geng (Peking U., China)
- Chris Gotwalt, SAS Lecturer (SAS Institute Inc, Cary, USA)
- Karl Gustafson, ILAS Lecturer (U. of Colorado, Boulder, USA)
- Lynn Roy LaMotte (Louisiana State U., New Orleans, USA)
- Peter Šemrl (U. of Ljubljana, Slovenia).

Invited Minisymposia topics (and organisers):

- Design and Analysis of Experiments (Augustyn Markiewicz, Poznań U. of Life Sciences, Poland)
- Generalized Inverses and Linear Models (Hans Joachim Werner, U. of Bonn, Germany)
- Linear and Mixed Models (Simo Puntanen & Julia Volaufova, Louisiana State U., New Orleans, USA):

- Magic Matrices (George P. H. Styan, McGill U., Montréal, Canada)
- Matrices in Applied Probability (Jeffrey J. Hunter, Auckland U. of Technology, New Zealand)
- Matrices Useful for Modeling Multi-level Models (Dietrich von Rosen, Swedish U. of Agriculture, Uppsala, Sweden)
- Matrices with Economic and Financial Applications (Shuangzhe Liu, U. of Canberra, Australia)
- Model Selection and Post Estimation (S. Ejaz Ahmed, Brock U., Canada)
- Statistical Modeling and Computation (Tsung-I Lin, National Chung Hsing U, Taichung, Taiwan)
- Statistical Simulation (Kai-Tai Fang, China)
- Teaching Matrices within Statistics (Kimmo Vehkalahti, U. of Helsinki, Finland)

The invited speakers in the KTF-75 Session (organised by Jianxin Pan, U. of Manchester, UK) are: Runze Li (Penn State U., USA), Min-Qian Liu (Nankai U., Tianjin, China), Dietrich von Rosen and Jianxin Pan. The invited speakers in the SP-70 Session (organised by Julia Volaufova) are: Barbora Arendacká (U. of Göttingen, Germany), Stephen J. Haslett (Massey U., Palmerston North, New Zealand) and George P. H. Styan.

The Scientific Organizing Committee consists of: Jeffrey J. Hunter (Chair), Chuanzhong Chen (Hainan Normal U., Haikou, China), S. Ejaz Ahmed, Kai-Tai Fang, Shuangzhe Liu, Yonghui Liu, Tsung-I Lin, Augustyn Markiewicz, Jianxin Pan, Simo Puntanen (Vice-chair), George P. H. Styan, Kimmo Vehkalahti, Julia Volaufova, Dietrich von Rosen and Hans Joachim Werner. The Local Organizing Committee Chair is Chuanzhong Chen: ccz0082@aliyun.com.

For more info, see: `http://iwms2015.csp.escience.cn/dct/page/1` and `http://www.sis.uta.fi/tilasto/iwms/`.

The support of the International Linear Algebra Society (ILAS) and the SAS Institute Inc is gratefully acknowledged.

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## Gene Golub SIAM Summer School 2015
## Delphi, Greece, June 15–26, 2015

The 2015 Gene Golub SIAM Summer School (G2S3), *RandNLA: Randomization in Numerical Linear Algebra*, will be held June 15–16, 2015. The summer school will be hosted by the University of Patras and held at the European Cultural Centre of Delphi. The application deadline is February 1, 2015. Further information about the 2015 G2S3 can be found at `http://www.siam.org/students/g2s3` and the archives of earlier G2S3s can be found at `http://www.siam.org/students/g2s3/archive.php`.

SIAM is calling for Letters of Intent proposing topics and organizers for the 2016 Gene Golub SIAM Summer School (G2S3). Preference will be given to a computational linear algebra theme, but Letters of Intent in all areas of applied and computational mathematics will be considered. January 31, 2015 is the deadline for submission of the Letter of Intent; full proposals will be due March 31, 2015. For more information, see `http://www.siam.org/students/g2s3/summer_call.php`.

It is expected that the program will run two weeks for 40 to 50 graduate students in 2016. SIAM will contribute up to $95,000 towards the running of the summer school. Information about the G2S3 program, including the composition of the G2S3 committee, can be found at `http://www.siam.org/about/com_golub.php`.

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## Summer School on "Exploiting Hidden Structure in Matrix Computations: Algorithms and Applications"
## Cetraro, Italy, June 22–26, 2015

During the week of June 21–26, 2015, the C.I.M.E. Foundation will organize the summer school: "Exploiting Hidden Structure in Matrix Computations: Algorithms and Applications." Lectures will be given by: Michele Benzi (Emory University, USA), Dario Bini (Università di Pisa, Italy), Daniel Kressner (EPFL, Lausanne, Switzerland), Hans Munthe-Kaas (University of Bergen, Norway) and Charles Van Loan (Cornell University, USA).

The School will be held at the Hotel San Michele in Cetraro, Italy (`http://www.sanmichele.it`). Registration will be open starting December 1, 2014 and will close on April 30, 2015. Some limited funding for lodging expenses will be available for students and postdoctoral fellows.

For further details, please visit `http://web.math.unifi.it/users/cime/`.

## MatTriad'2015 – Conference on Matrix Analysis and its Applications
## Coimbra, Portugal, September 7–11, 2015

Welcome to Coimbra, a UNESCO World Heritage site! The Conference will be held in Coimbra, Portugal, from September 7 to 11, 2015 and is endorsed by ILAS. Up-to-date information is available at: `http://www.mattriad.ipt.pt`.

The purpose of the conference is to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications, special emphasis being given to applications in other areas of science. One of the main goals is to highlight recent achievements in these mathematical domains.

The program will cover different aspects, with emphasis on: recent developments in matrix and operator theory; direct and inverse spectral problems; matrices and graphs; applications of linear algebra in statistics; matrix models in industry and sciences; linear systems and control theory; quantum computation; and combinatorial matrix theory. Researchers and graduate students interested in the scope of the conference are particularly encouraged to attend the conference. The conference will provide a friendly atmosphere for the discussion and exchange of ideas, which hopefully will lead to new scientific links among participants. The format of the meeting will involve plenary sessions and sessions with contributed talks. The list of invited speakers includes winners of Young Scientists Awards of MatTriad'2013. We are also planning two short courses delivered by leading experts. Thematic workshops are welcome.

The work of young scientists will receive special consideration in MatTriad'2015. The best poster and the best talk, of graduate students or scientists with a recently completed Ph.D., will receive awards. Prize-winning works will be widely publicized and promoted by the conference.

The lecturers are Peter Šemrl (Slovenia) and Ludwig Elsner (Germany). Confirmed invited speakers include Peter Benner (Germany), Froilán M. Dopico (Spain) and Dietrich von Rosen (Sweden). The winners of the Young Scientists Awards – MatTriad'2013 are Maja Nedović (Serbia), and Jaroslav Horáček (Czech Republic).

The Scientific Committee consists of Tomasz Szulc (Poland) – Chair, Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany) and Simo Puntanen (Finland). The Organizing Committee is: Natália Bebiano (Portugal) – Chair, Francisco Carvalho (Portugal), Susana Furtado (Portugal), Celeste Gouveia (Portugal), Rute Lemos (Portugal) and Ana Nata (Portugal).

For further information, please contact Natália Bebiano (mattriad@mat.uc.pt) or Francisco Carvalho (fpcarvalho@ipt.pt).

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## SIAM Conference on Applied Linear Algebra (LA15)
## Hyatt Regency Atlanta, Georgia, USA, October 26 – 30, 2015

The Organizing Committee Co-Chairs are Chen Greif (University of British Columbia, Canada) and James Nagy (Emory University, USA).

Organizing Committee: Zhaojun Bai, University of California, Davis, USA; Zhong-Zhi Bai, Chinese Academy of Sciences, China; Petros Drineas, Rensselaer Polytechnic Institute, USA; David Gleich, Purdue University, USA; Bruce Hendrickson, Sandia National Laboratories, USA; Beatrice Meini, Università di Pisa, Italy; Alison Ramage, University of Strathclyde, UK; Zdeněk Strakoš, Charles University of Prague, Czech Republic; Françoise Tisseur, University of Manchester, United Kingdom; Sabine Van Huffel, KU Leuven, Belgium.

Invited Plenary Speakers: Haim Avron, IBM T.J. Watson Research Center, USA; Raymond Chan, Chinese University of Hong Kong, Hong Kong; Geir Dahl,* University of Oslo, Norway; Zlatko Drmac, University of Zagreb, Croatia; Howard Elman, University of Maryland, USA; Maryam Fazel, University of Washington, USA; Melina Freitag, University of Bath, United Kingdom; Xiaoye Sherry Li, Lawrence Berkeley National Laboratory, USA; Volker Mehrmann, TU Berlin, Germany; Michael Overton,* New York University, USA; Haesun Park, Georgia Institute of Technology, USA; Eugene Tyrtyshnikov, Russian Academy of Sciences, Russia.

* These speakers are supported in cooperation with ILAS.

The banquet speaker is Iain Duff, Rutherford Appleton Laboratory, United Kingdom and CERFACS, France. For further information, check the conference website: `http://www.siam.org/meetings/la15/`.

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## The Householder Symposium XX on Numerical Linear Algebra
## Virginia Tech in Blacksburg, Virginia, USA, June 18 – 23, 2017

This symposium is the twentieth in a series, previously called the Gatlinburg Symposia, and will be hosted by the Virginia Polytechnic Institute and State University (VA Tech), in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra. Details at: `http://www.math.vt.edu/HHXX`.

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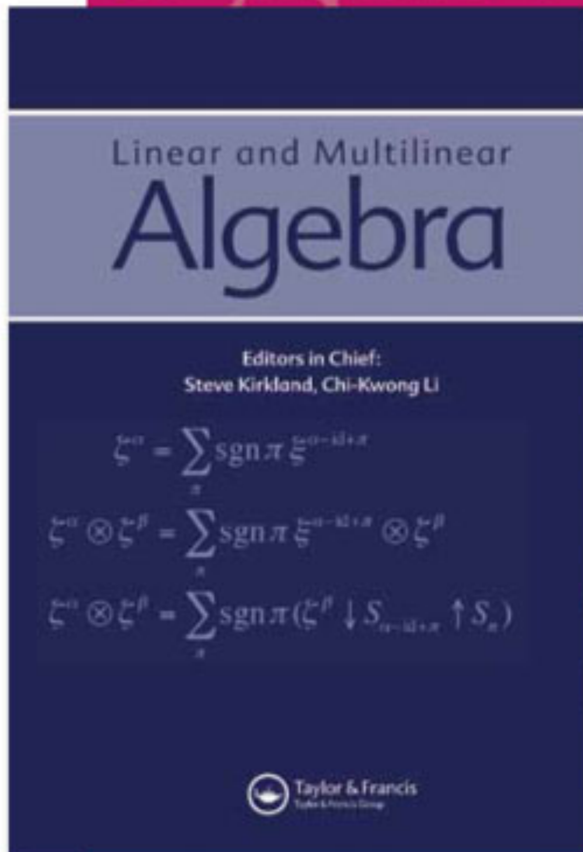


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# IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to *IMAGE* Problems 51-3, 51-6, and to all Problems in issue 52 of *IMAGE* except Problem 52-5, for which we are still seeking a solution. Six new problems are on the last page; solutions are invited.

**Problem 51-3: A Hadamard-like Inequality**
Proposed by Minghua Lin, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let $A = (a_{ij})$ be a real positive definite symmetric matrix of order $n$ and let $A(i)$ be the submatrix obtained from $A$ by deleting the $i$-th row and $i$-th column. Show that the inequality below holds for $n \geq 4$ and fails for $n = 2, 3$:

$$(n-1)\prod_{i=1}^{n} a_{ii} + \det A \geq \sum_{i=1}^{n} a_{ii} \det A(i).$$

**Solution 51-3.1** by the proposer Minghua Lin

The second immanant [1, Equation (2)] is an immanant induced by an irreducible character of the symmetric group which corresponds to the partition $[2, 1^{n-2}]$ and equals

$$d_2(A) = \sum_{i=1}^{n} a_{ii} \det A(i) - \det A.$$

Then the inequality of Problem 51-3 is equivalent to $d_2(\operatorname{diag} A) \geq d_2(A)$, which follows from a result of Merris [2]. The counterexample for $n \in \{2, 3\}$ is given by matrices $A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -1/3 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & 1 \end{pmatrix}$. ∎

*Remark.* The Hadamard inequality says $\prod_{i=1}^{n} a_{ii} \geq \det A$. If we apply the Hadamard inequality to each $A(i)$, we get $n\prod_{i=1}^{n} a_{ii} \geq \sum_{i=1}^{n} a_{ii} \det A(i)$. In this sense, the inequality of Problem 51-3 is tighter than the Hadamard inequality.

*References*

[1] R. Grone and R. Merris, A Fischer inequality for the second immanant, *Linear Algebra Appl.* 87 (1987) 77–83.

[2] R. Merris, Oppenheim's inequality for the second immanant, *Canad. Math. Bull.* 30 (1987) 367–369.

*Editorial note: We are still looking for a direct proof which does not rely on the second immanant.*

**Problem 51-6: The Determinant Kernel**
Proposed by Suvrit Sra, suvrit@gmail.com

(i) Let $X_1, X_2, \ldots, X_n$ be $p \times p$ symmetric positive definite real matrices. Let the $n \times n$ matrix be given by $K = (k_{ij}) = \left(\frac{1}{\det(X_i + X_j)}\right)$. Prove that $K$ is positive semi-definite.

(ii) Let $\beta > \frac{p-1}{2}$ be a scalar. A harder problem is to prove that the matrix $K_\beta = (k_{ij}^\beta) = \left(\frac{1}{\det((X_i+X_j)^\beta)}\right)$ is also positive semi-definite.

**Solution 51-6.1** by the proposer Suvrit Sra

To prove that $K$ given in (i) is positive definite, recall that, for positive definite $X$, the Gaussian integral states

$$\int_{\mathbb{R}^p} e^{-x^T X x}\, dx = \pi^{p/2} \det(X)^{-1/2}.$$

Now define the functions $f_i(x) := \frac{1}{\pi^{p/4}} e^{-x^T X_i x}$ for $1 \leq i \leq n$. Then $f_i \in L_2(\mathbb{R}^p)$ and

$$k_{ij} = \langle f_i, f_j \rangle := \frac{1}{\pi^{p/2}} \int_{\mathbb{R}^d} e^{-x^T (X_i + X_j) x}\, dx = \det(X_i + X_j)^{-1/2},$$

which shows that $K_{1/2}$ is positive semi-definite (P.S.D.). From the Schur product theorem we know that the elementwise product of two P.S.D. matrices is again P.S.D. So, in particular, $K_\beta$ is P.S.D. whenever $\beta$ is an integer multiple of 1/2. To extend the result to all $\beta > (p-1)/2$, we invoke another integral representation; viz. the *multivariate Gamma function*, defined as

$$\Gamma_p(\beta) := \int_{A>0} e^{-\operatorname{Tr}(A)} \det(A)^{\beta-(p+1)/2}\, dA, \qquad (\beta > (p-1)/2),$$

where the integration is over the set of $p\times p$ positive matrices. Now define the functions $f_i(A) := e^{-\operatorname{Tr}(AX_i)}$ for $1 \le i \le n$. Then, $f_i \in L_2(\mathbb{P}_p)$, where $\mathbb{P}_p$ is the set of $p \times p$ positive matrices. In this space, we see that the inner product equals

$$k_{ij} = \langle f_i, f_j\rangle := \frac{1}{\Gamma_p(\beta)} \int_{A>0} e^{-\operatorname{Tr}(A(X_i+X_j))} \det(A)^{\beta-(p+1)/2}\, dA = \det(X_i + X_j)^{-\beta},$$

which exists whenever $\beta > (p-1)/2$. This proves the positive semi-definiteness of $K_\beta$. ∎

**Problem 52-1: Generalized Self-Inversive Operators**

Proposed by Arkady A. Babajanyan, *Institute for Informatics and Automation Problems, Armenian Academy of Sciences, Yerevan, Armenia*, arbab@list.ru

Let $A^+$ be the Moore-Penrose inverse of a bounded operator on a complex Hilbert space. Show that $A = A^+$ if and only if $A = JQ$, where $Q = Q^* = Q^2$ is a self-adjoint projector and $J\colon \operatorname{Im}(Q) \to \operatorname{Im}(Q)$ is an involution or self-inversive operator (i.e., $J^{-1} = J$). When is $A = A^+ = A^*$?

**Solution 52-1.1** by the proposer Arkady A. Babajanyan

**The first part.** *Necessity.* If $A = A^+$ is a bounded linear operator on Hilbert space H, then $\operatorname{Im}(A) = \overline{\operatorname{Im}(A)}$ and $\mathrm{H} = \mathrm{Ker(A)} \oplus \mathrm{Im(A)}$. Let $\hat{A} = A|_{\operatorname{Im}(A)}\colon \operatorname{Im}(A) \to \operatorname{Im}(A)$ be the restriction of $A$ to $\operatorname{Im}(A)$, which has a bounded inverse $\hat{A}^{-1}$. Then $A^+ = \hat{A}^{-1}AA^+$ and $A = \hat{A}A^+A$. In this case, $\hat{A}A^2 = \hat{A}^{-1}A^2$ and $J = \hat{A} = \hat{A}^{-1}$ is the involution or self-inversive operator acting on $\operatorname{Im}(A)$.

*Sufficiency.* Let $A = JQ$. Since $\operatorname{Im}(J) \subseteq \operatorname{Im}(Q)$ we have $QJQ = JQ$. Hence, the following equalities hold.

$$A^3 = JQJQJQ = J^3Q = JQ = A. \qquad A^2 = JQJQ = J^2Q = Q = Q^*.$$

Thus, $A$ satisfies the conditions of the Moore-Penrose inverse for a closed linear operator and so $A = A^+$. ∎

**The second part**. We claim that $A = A^* = A^+$ if and only if $A = JQ$, where $Q = Q^* = Q^2$ and $J = J^{-1} = J^*$.

*Necessity.* Let $A = A^* = A^+$. Then, by the first part, $A = JQ$, with $J\colon \operatorname{Im}(Q) \to \operatorname{Im}(Q)$, and for each $x, u \in \mathrm{H}$ we have

$$(x, JQu) = (x, Au) = (Ax, u) = (Ax, A^+Au) = (JA^2x, A^2u) = (x, A^2J^*A^2u) = (x, J^*A^2u) = (x, J^*Qu).$$

Hence $J = J^*$.

*Sufficiency.* Let $A = JQ$ with $J^* = J\colon \operatorname{Im}(Q) \to \operatorname{Im}(Q)$. From the first part, $A = A^+$. To prove $A = A^*$ note that $(Ax, u) = (JQx, u) = (JQx, Qu) = (x, QJQu) = (x, JQu) = (x, Au)$ for each $x, u \in \mathrm{H}$. ∎

*Remark.* In [1, Theorem 3 and its Corollary] the detailed solution is given under the assumption that the operators have closed images. Both problems can be taken as the definition of the partial involution and partial orthogonal involution or partial symmetry, similar to the definition of the partial isometry [2]. In [3], the solution of the second problem in Euclidean space is defined as the extended orthogonal projector.

*References*

[1] A.A. Babajanyan and A.M. Safaryan, On pseudoinversion in Hilbert space, *Uch. Zap. Yerevan Gos. Univ. Estest. Nauki* 147 (1981) 23–30 (in Russian).

[2] M. Fildan, Sur quelques proprietes algebriques de l'inverse generalise d'un operateur, *Rev. Roum. Math. Pures et Appl.* 19 (1974) 587–603.

[3] O.M. Baksalary and G. Trenkler, On $K$-potent matrices, *Electron. J. Linear Algebra* 26 (2013) 446–470.

**Solution 52-1.2** by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let $\mathcal{H}$ denote the underlying Hilbert space. Suppose $A = JQ$ with $J$ and $Q$ as described in the formulation of the problem. Since $\operatorname{Im}(A) = \operatorname{Im}(Q) = \operatorname{Ker}(I - Q)$ is closed, $A$ has a unique Moore-Penrose inverse, $A^+$. For each $u \in \operatorname{Im}(Q)$ we have $Ju \in \operatorname{Im}(Q)$ and so $Ju = JQu = QJQu$. Then, $J^2u = J^2Qu = Qu$ since $J^2 = I$ on $\operatorname{Im}(Q)$. Therefore

$A^2 = (JQ)^2 = J(QJQ) = J^2Q = Q$, which is self-adjoint, and $A^3 = (JQ)^3 = (JQ)Q = JQ = A$. Hence, $A = JQ$ satisfies the usual four defining properties of the Moore-Penrose inverse, so $A^+ = A$.

For the converse, we are given that $A^3 = A$ and $A^2$ is self-adjoint. Thus, $A = JQ$, where $Q = A^2$ and $J = A|_{\mathrm{Im}(Q)}$. Then $Q$ is self-adjoint, and $Q^2 = A^4 = A^3A = A^2 = Q$. Also, $JQ = A^3 = QA$, and so $\mathrm{Im}(J) \subseteq \mathrm{Im}(Q)$. Then, $J^2Q = (A|_{\mathrm{Im}(Q)})^2Q = A^2Q = A^4 = Q$, so $J^2 = I_{\mathrm{Im}(Q)}$.

Finally, we show that $A = A^+ = A^*$ if and only if $A = JQ$, where $J$ and $Q$ are as in the formulation of the problem, and $J = J^*$. First, note that $J$ is self-adjoint if and only if the extension $JQ$ of $J$ to all of $\mathcal{H}$ is self-adjoint. Below, the symbol $J$ will always indicate this extended operator. Observe that $QJ = JQ$ since, for all $u \in \mathcal{H}$,

$$QJu = QJ(Qu + (u - Qu)) = QJQu = JQu.$$

Now, if $A = JQ$ and $J = J^*$, then $A^* = Q^*J^* = QJ = A$. Conversely, if $A^* = A$, then $QJ^* = JQ = QJ$, so $J = J^*$ since $\mathrm{Im}(J), \mathrm{Im}(J^*) \subseteq \mathrm{Im}(Q)$. ∎

**Problem 52-2: Numerical Range of a Skew-Symmetric Matrix**

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

Let $A$ be a $4 \times 4$ real skew-symmetric matrix. Moreover, let $F(A)$ be the numerical range of $A$, which (in this case) is an ordered set. Express the exact boundary of the ordered set $F(A)$ in terms of the entries of $A$.

**Solution 52-2.1** by the proposers Oskar Maria Baksalary and Götz Trenkler

Every $4 \times 4$ real skew-symmetric matrix $A$ can be written in the form

$$A = \begin{pmatrix} 0 & -a_3 & a_2 & b_1 \\ a_3 & 0 & -a_1 & b_2 \\ -a_2 & a_1 & 0 & b_3 \\ -b_1 & -b_2 & -b_3 & 0 \end{pmatrix},$$

with the entries $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, and $b_3$ being real numbers. Since $A = -A^t$ implies $AA^t = -A^2 = A^tA$, it is seen that every real skew-symmetric matrix is normal. Hence, $F(A)$ is the convex hull of the four eigenvalues of $A$, see [1, p. 11]. The eigenvalues are purely imaginary solutions of the equation

$$\lambda^4 + (a^ta + b^tb)\lambda^2 + (a^tb)^2 = 0,$$

where $a = (a_1, a_2, a_3)^t$ and $b = (b_1, b_2, b_3)^t$; see [2, Theorem 2]. It turns out that $\lambda_{1,2} = \pm\sqrt{x_1}$ and $\lambda_{3,4} = \pm\sqrt{x_2}$, where by Vieta's law, $x_1 + x_2 = -(a^ta + b^tb)$ and $x_1x_2 = (a^tb)^2$, and so $x_1, x_2 \leqslant 0$. As was shown in [2], we can write

$$x_1 = \alpha + \beta \quad \text{and} \quad x_2 = \alpha - \beta,$$

with

$$\alpha = -\frac{a^ta + b^tb}{2} \quad \text{and} \quad \beta = \frac{\sqrt{(a^ta + b^tb)^2 - 4(a^tb)^2}}{2}.$$

It follows that $|x_1| \leqslant |x_2|$, so the four eigenvalues of $A$ prove to be $\lambda_{1,2} = \pm|\alpha + \beta|^{1/2}i$ and $\lambda_{3,4} = \pm|\alpha - \beta|^{1/2}i$. Hence, $F(A)$ is the closed interval $\left[\, -|\alpha - \beta|^{1/2}i\,,\, |\alpha - \beta|^{1/2}i \,\right]$ on the imaginary axis. ∎

*References*

[1] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[2] G. Trenkler and D. Trenkler, The vector cross product and $4 \times 4$ skew-symmetric matrices, in: *Recent Advances in Linear Models and Related Areas – Essays in Honour of Helge Toutenburg*, (Shalabh and C. Heumann, eds.), Physica–Verlag, Heidelberg, 2008, 95–104.

**Solution 52-2.2** by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

We will show that

$$F(A) = [-i\lambda, i\lambda] = \{it : -\lambda \le t \le \lambda\} \qquad \text{where} \qquad \lambda = \frac{1}{2}\sqrt{-\operatorname{Tr}(A^2) + \sqrt{(\operatorname{Tr}(A^2))^2 - 16\det A}}.$$

Indeed, $iA$ is a Hermitian matrix, so $iF(A) = F(iA) = [\alpha, \beta]$ where $\alpha$ is the smallest eigenvalue of $iA$ and $\beta$ is the largest one. Moreover, $A$ is real skew-symmetric, so its eigenvalues are purely imaginary and appear in conjugate pairs. So, the spectrum of $A$ has the form $\{i\lambda, -i\lambda, i\mu, -i\mu\}$ with $0 \le \mu \le \lambda$. With this notation, the smallest eigenvalue of $iA$ is $-\lambda$ and the largest one is $\lambda$. Consequently,

$$F(A) = [-i\lambda, i\lambda] = \{it : -\lambda \le t \le \lambda\}.$$

Now, to determine $\lambda$ in terms of the coefficients of $A$, we note that $\det A = \lambda^2\mu^2$, and the square of the Frobenius norm of $A$ is $\operatorname{Tr}(AA^T) = -\operatorname{Tr}(A^2) = 2(\lambda^2 + \mu^2)$. Thus, $\lambda^2$ and $\mu^2$ are the solutions of the second degree equation

$$x^2 + \frac{1}{2}\operatorname{Tr}(A^2)x + \det A = 0.$$

The largest solution is $\lambda^2 = \frac{1}{2}\left(-\frac{\operatorname{Tr}(A^2)}{2} + \sqrt{\left(\frac{\operatorname{Tr}(A^2)}{2}\right)^2 - 4\det A}\right)$ and this yields the announced result. ∎

*Remark.* Explicitly, if

$$A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}$$

then $\lambda = \sqrt{\frac{1}{2}\left(u + \sqrt{u^2 - 4v^2}\right)}$ where $u = x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{23}^2 + x_{24}^2 + x_{34}^2$ and $v = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$.

*Also solved by Eugene A.* Herman *and Denis* Serre*.*

**Problem 52-3: Removing Zero From Numerical Range**

Proposed by Janko Bračič, *University of Ljubljana, IMFM, Slovenia*, janko.bracic@fmf.uni-lj.si
and Cristina Diogo, *Departamento de Matemática, Lisbon University Institute, Portugal*, cristina.diogo@iscte.pt

Let $A \in M_n(\mathbb{C})$. Show that 0 is not in the convex hull of the spectrum $\sigma(A)$ if and only if there exists a positive definite $P \in M_n(\mathbb{C})$ such that 0 is not in the numerical range of $PA$.

**Solution 52-3.1** by the proposers Janko Bračič and Cristina Diogo

Let $A \in M_n$. Suppose that $0 \in W(PA)$ for each $P > 0$. Let $S \in M_n$ be invertible. Since $0 \in W(S^*SA)$, there exists $x \in \mathbb{C}^n$, $\|x\| = 1$, such that $\langle S^*SAx, x\rangle = 0$. Let $y = \|Sx\|^{-1}Sx$. Then $\langle SAS^{-1}y, y\rangle = \|Sx\|^{-2}\langle S^*SAx, x\rangle = 0$. Hence, $0 \in W(SAS^{-1})$ for arbitrary invertible $S \in M_n$. It follows, by Hildebrant's Theorem (see [2]), that 0 is in the convex hull of $\sigma(A)$. Conversely, assume that 0 is in the convex hull of $\sigma(A)$. We have to see that $0 \in W(PA)$ for every $P > 0$. This is obvious if $A$ is not invertible. Suppose therefore that it is invertible. Since 0 is in the convex hull of $\sigma(A)$ there exist eigenvalues $\lambda_1, \dots, \lambda_d$ $(d \ge 2)$ of $A$ and numbers $0 < t_i < 1$ such that $t_1 + \cdots + t_d = 1$ and $t_1\lambda_1 + \cdots + t_d\lambda_d = 0$. Let $e_i \in \mathbb{C}^n$, $\|e_i\| = 1$, be the corresponding eigenvectors, i.e., $Ae_i = \lambda_i e_i$. Denote $\omega_i = \langle Pe_i, e_i\rangle$. Note that this is a positive number as $P$ is positive definite. Let

$$s_i = \frac{t_i}{\omega_i} \cdot \frac{1}{t_1/\omega_1 + \cdots + t_d/\omega_d} > 0 \qquad (i = 1, \dots, d).$$

It is easily seen that $s_1 + \cdots + s_d = 1$ and $s_1(\lambda_1\omega_1) + \cdots + s_d(\lambda_d\omega_d) = 0$. Hence, 0 is a convex combination of numbers $\lambda_i\omega_i = \lambda_i\langle Pe_i, e_i\rangle = \langle PAe_i, e_i\rangle \in W(PA)$, which is a convex set. We conclude that $0 \in W(PA)$. ∎

*Remark.* A more general result related to this problem can be found in [1].

*References*

[1] J. Bračič and C. Diogo, Operators with a given part of the numerical range, *Math. Slovaca*, to appear.

[2] S. Hildebrandt, Über den numerischen wertebereich eines operators, *Math. Ann.* 163 (1966) 230–247.

**Solution 52-3.2** by Denis Serre, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Because we may rotate simultaneously the spectrum of $A$ and the numerical range of $PA$ by $A \mapsto e^{i\theta}A$, it is enough to prove the equivalence of the following statements:

(i) The eigenvalues of $A$ have positive real part ($\operatorname{Re}\lambda > 0$).

(ii) There exists a positive definite Hermitian matrix $P$ such that the numerical range of $PA$ has positive real part.

That (ii) implies (i) is obvious: if $Ax = \lambda x$ with normalized vector $x$, then the assumption (ii) tells us $0 < \operatorname{Re}(x^*PAx) = (x^*Px)\operatorname{Re}\lambda$, where we used the fact that $x^*Px \in (0, +\infty)$.

The converse is a variant of Hildebrandt's Theorem; we first show that there exists a matrix $Q$ such that the numerical range of $QAQ^{-1}$ has positive real part. Then $P = Q^*Q$ works. To construct $Q$, let $3\varepsilon > 0$ be the minimum of real parts of eigenvalues of $A$, then take a positive real number $a$ large enough that the disc $D(a; a - 2\varepsilon)$ contains all the eigenvalues of $A$. Then $\rho(aI_n - A) \le a - 2\varepsilon$. Because of the Householder's Theorem, there is an operator norm $N$ such that $N(aI_n - A) < a - \varepsilon$. Actually, if $A = S + F$ is a Jordan decomposition to semisimple and nilpotent part, then $S + \eta F$ is similar to $A$ for every nonzero $\eta$, and by taking $Q$ such that $QAQ^{-1} = S + \delta F$ where $\delta > 0$ is small enough, one may assume the norm $N$ is of the form $N(X) = \|QXQ^{-1}\|_2$ where $\|\cdot\|_2$ is the standard operator norm (see also the book [D. Serre, *Matrices*, Springer-Verlag GTM **216**, 2nd edition], page 133). For every nonzero vector $x$, we have

$$\|ax - QAQ^{-1}x\|_2^2 \le \big(N(aI_n - A)\big)^2 \|x\|_2^2 \le (a-\varepsilon)^2\|x\|_2^2,$$

from which we infer $0 < \frac{2a\varepsilon - \varepsilon^2}{2a}\|x\|_2^2 \le \operatorname{Re}(x^*QAQ^{-1}x)$. ∎

**Problem 52-4: On a Class of Generalized Quadratic Matrices**

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@Manchester.ac.uk

Let $\Omega = \{T_1, \dots, T_m\}$ be a set of $n \times n$ real or complex matrices, none of $T_i$ is a scalar multiple of the identity matrix $I$. Assume $T_i^2 \in \{-I, I\}$ and $T_iT_j + T_jT_i = 0$ for $i \ne j$. Then:

(i) Show that $I, T_1, \dots, T_m$ are linearly independent.

(ii) Show that the minimal polynomial of each nonscalar $X \in \operatorname{span}(\Omega \cup \{I\})$ has degree two.

(iii) If, in addition, for each $i \ne j$ there is a scalar $c_{ij}$ such that $c_{ij}T_iT_j \in \Omega$, then show that $m \le 3$.

(iv) Classify matrix triples which exhibit (iii).

*Editorial note: In part (ii) of the original formulation, the adjective 'nonscalar' was mistakenly omitted. We thank Eugene A.* HERMAN *for pointing this out.*

**Solution 52-4.1** by the proposer Richard William FAREBROTHER

(i) By the assumptions, $T_j^2 = s_jI$ for some $s_j \in \{-1, 1\}$. Suppose $W := d_0I + d_1T_1 + \cdots + d_mT_m = 0$ where $d_0, d_1, \dots, d_m$ are scalars and not all are zero. Now,

$$T_jW + WT_j = 2d_0T_j + 2d_js_jI$$

must take a zero value for all $j = 1, 2, \dots, m$. But the matrix $T_j$ is not a multiple of $I$ so that $d_j = d_0 = 0$ for all $j = 1, 2, \dots, m$. We deduce that the $m+1$ matrices in the set $\{I, T_1, \dots, T_m\}$ are linearly independent.

(ii) Each matrix $X = c_0I + c_1T_1 + \cdots + c_mT_m \in \operatorname{span}(\Omega \cup \{I\})$ satisfies

$$(X - c_0I)^2 = (c_1T_1 + \cdots + c_mT_m)^2 = (s_1c_1^2 + \cdots + s_mc_m^2)I$$

and hence $X^2 = \alpha X + \beta I$, where $\alpha = 2c_0$ and $\beta = -c_0^2 + s_1c_1^2 + \cdots + s_mc_m^2$.

(iii) Suppose that $m \ge 4$ and that $T_1, T_2, \dots, T_m$ satisfy the conditions. Then $T_j^2 = s_jI$ where $s_j = \pm 1$. Further, suppose that $T_i = c_{12}T_1T_2$. Then

$$T_4T_i = c_{12}T_4T_1T_2 = -c12T_1T_4T_3 = c12T_1T_2T_4 = T_iT_4,$$

which, when taken in conjunction with the requirement that $T_4T_i = -T_iT_4$ implies that $T_iT_4 = 0$, and hence that $T_4 = s_iT_iT_iT_4 = 0$, contrary to hypothesis.

(iv) As a familiar example of this result, we note Farebrother, Gross and Troschke's scheme for the matrix representation of quaternions sets (see [2])

$$X = c_0I + c_1T_1 + c_2T_2 + c_3T_3$$

where $T_1, T_2$ and $T_3 = T_1T_2$ are $n \times n$ skew-involutory matrices with real or complex elements satisfying

$$T_1^2 = T_2^2 = T_3^2 = T_1T_2T_3 = -I. \qquad ∎$$

*References*

[1] R.W. Farebrother and G. Trenkler, On generalized quadratic matrices, *Linear Algebra Appl.* 410 (2005) 244–253.

[2] R.W. Farebrother, J. Groß, and S.-O. Troschke, Matrix representation of quaternions, *Linear Algebra Appl.* 362 (2002) 251–255.

**Solution 52-4.2** by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

(i) Suppose there are scalars $c, c_1, \dots, c_m$ such that $cI + c_1T_1 + \cdots + c_mT_m = 0$. Multiplying left and right by $T_i$ and adding yields

$$2cT_i + c_1(T_iT_1 + T_1T_i) + \cdots + c_m(T_iT_m + T_mT_i) = 0$$

which simplifies to $2cT_i \pm 2c_iI = 0$. Since $T_i$ and $I$ are linearly independent, we have $c = c_i = 0$ for all $i = 1, \dots, m$.

(ii) Let $A = cI + \sum_{k=1}^m c_kT_k$. If $c_k = 0$ for $k = 1, \dots, m$, the minimal polynomial of $A$ has degree 0 or 1. Otherwise,

$$A^2 = c^2I + 2c\sum_{k=1}^m c_kT_k + \sum_{k=1}^m c_k^2T_k^2 + \sum_{j,k=1,\, j\neq k}^m c_jc_k(T_jT_k + T_kT_j) = c^2I + 2c(A - cI) + dI,$$

where $d$ is a sum of terms of the form $\pm c_k^2$. Furthermore, if $A$ were a multiple of $I$, then $I, T_1, \dots, T_m$ would be linearly dependent, which contradicts (i). Thus the minimal polynomial of $A$ has degree 2 whenever $(c_1, \dots, c_m) \neq (0, \dots, 0)$.

(iii) (The argument showing that $m > 3$ is not possible was supplied by the proposer.) We will show that whenever the condition in (iii) is satisfied, then $m \in \{1, 3\}$. Since $T_i^2 = \pm I$, each $T_i$ is nonsingular. Hence, if $c_{ij}T_iT_j = T_i$, then $c_{ij} \neq 0$ and $T_j = \frac{1}{c_{ij}}I$, a contradiction. Similarly, $c_{ij}T_iT_j \neq T_j$. Thus, $c_{ij}T_iT_j = T_k$ for some $k \neq i, j$. In particular, $m = 2$ is not possible. If $m > 4$, there exist $j, k > 2$ with $k \neq j$ and $c_{12}T_1T_2 = T_j$. Multiplying on the left by $T_k$ yields $T_kT_j = c_{12}T_kT_1T_2 = -c_{12}T_1T_kT_2 = c_{12}T_1T_2T_k = T_jT_k = -T_kT_j$. Hence, $T_kT_j = 0$, which contradicts the fact that each $T_i$ is nonsingular.

(iv) For $m = 3$, we show that, up to similarity and multiplication by $\pm i$, all the possible triples are

$$T_1 = \begin{pmatrix} iI & O \\ O & -iI \end{pmatrix}, \quad T_2 = \begin{pmatrix} O & X \\ -X^{-1} & O \end{pmatrix}, \quad T_3 = \begin{pmatrix} O & iX \\ iX^{-1} & O \end{pmatrix},$$

where $n$ is even, each submatrix has size $\frac{n}{2} \times \frac{n}{2}$, and $X$ is nonsingular. Namely, let $(T_1, T_2, T_3)$ be a triple satisfying all the conditions in the problem. By replacing, if necessary, $T_j$ by $iT_j$, we may assume that $T_1^2 = T_2^2 = T_3^2 = -I$. Thus each $T_j$ has minimal polynomial $x^2 + 1 = (x - i)(x + i)$ and so it is diagonalizable with eigenvalues $\pm i$. Furthermore, $\operatorname{Tr}(T_iT_k) = \operatorname{Tr}(T_kT_i) = \operatorname{Tr}(-T_iT_k) = -\operatorname{Tr}(T_iT_k)$ implies that $n$ is even and that $i$ and $-i$ each have multiplicity $n/2$. Next, multiplying $c_{12}T_1T_2 = T_3$ on the right by $T_3$ yields $c_{12}T_1T_2T_3 = -I$. Hence, by symmetry, $c_{12} = c_{23} = c_{31}$. Also, multiplying $c_{12}T_1T_2T_3 = -I$ on the right by $c_{12}T_3T_2T_1$ yields $-c_{12}^2I = -c_{12}T_3T_2T_1 = c_{12}T_1T_2T_3 = -I$, and so $c_{12} = \pm 1$. By reordering subscripts, if necessary, we may assume $c_{12} = c_{23} = c_{31} = 1$. Thus $T_1, T_2, T_3$ are simultaneously similar to $Z = (iI_{n/2}) \oplus (-iI_{n/2})$, $P^{-1}ZP$, and $Q^{-1}ZQ$, respectively, for some nonsingular matrices $P$ and $Q$. We will denote these three new matrices again by $T_1, T_2, T_3$, respectively. Then

$$(P^{-1}ZP)^{-1}Z(P^{-1}ZP) = -P^{-1}ZPZP^{-1}ZP = -T_2T_1T_2 = -T_1 = -Z.$$

That is, $A^{-1}ZA = -Z$, where $A = P^{-1}ZP$. In partitioned form, $\left(\begin{smallmatrix} iI & O \\ O & -iI \end{smallmatrix}\right)\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) = -\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} iI & O \\ O & -iI \end{smallmatrix}\right)$. Hence, $A_{11} = A_{22} = O$, and, since $A^2 = -I$, we have $A_{12} = -A_{21}^{-1}$. Therefore, $T_2 = A$ has the desired form; also, from $T_2T_3 = T_1 = Z$, the $(1, 2)$ entry of $T_3$ equals $i$ times the $(1, 2)$ entry of $T_2$. ∎

**Problem 52-6: An Additive Decomposition Involving Rank-One Matrices**

Proposed by Anna A. Prach, *Middle East Technical University, Ankara, Turkey*, annaprach@me.com
and Dennis S. Bernstein, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $A \in \mathbb{R}^{n\times m}$ be nonzero, define $r := \operatorname{rank}(A)$ and let $x \in \mathbb{R}^m$ be nonzero. Show that $Ax = 0$ if and only if there exist $y_1, \dots, y_r \in \mathbb{R}^n$ and skew-symmetric matrices $S_1, \dots, S_r \in \mathbb{R}^{m\times m}$ such that $A = \sum_{i=1}^r y_ix^TS_i$.

**Solution 52-6.1** by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

If $A = \sum_{i=1}^r y_ix^{\mathrm{T}}S_i$ with $y_1, \dots y_r$ and $S_1, \dots, S_r$ as given, then $x^{\mathrm{T}}S_ix = (x^{\mathrm{T}}S_ix)^{\mathrm{T}} = -x^{\mathrm{T}}S_ix$, for $i = 1, \dots, r$ and so $Ax = \sum_{i=1}^r y_ix^{\mathrm{T}}S_ix = 0$.

For the converse, let $y_1, \dots, y_m$ denote the column vectors of $A$. Some set of $r$ of these vectors constitute a basis for the column space of $A$. To simplify notation, we assume that $\{y_1, \dots, y_r\}$ is such a basis. Hence, there exist scalars $c_{ji}$ with

$1 \le i \le r$ and $r+1 \le j \le m$ such that

$$y_j = \sum_{i=1}^{r} c_{ji} y_i, \quad j = r+1, \ldots, m.$$

Furthermore, with $x = (\gamma_1, \ldots, \gamma_m)^{\mathrm{T}} \in \mathbb{R}^m$, we have $0 = Ax = \sum_{i=1}^m \gamma_i y_i = \sum_{i=1}^r \gamma_i y_i + \sum_{i=r+1}^m \gamma_i \left(\sum_{j=1}^r c_{ij} y_j\right) = \sum_{i=1}^r \left(\gamma_i + \sum_{j=r+1}^m \gamma_j c_{ji}\right) y_i$ and so

$$\gamma_i + \sum_{j=r+1}^{m} \gamma_j c_{ji} = 0, \quad i = 1, \ldots, r.$$

Since $x$ is nonzero, at least one of $\gamma_{r+1}, \ldots, \gamma_m$ is nonzero. To again simplify notation, we assume $\gamma_m \neq 0$. By homogeneity, we may also assume $\gamma_m = 1$. Therefore,

$$\begin{aligned} A &= \begin{pmatrix} y_1 & \cdots & y_r & \sum_{i=1}^r c_{r+1,i} y_i & \cdots & \sum_{i=1}^r c_{mi} y_i \end{pmatrix} = y_1 (1, 0, \ldots, 0, c_{r+1,1}, \ldots, c_{m1}) + \cdots + y_r (0, \ldots, 0, 1, c_{r+1,r}, \ldots, c_{mr}) \\ &= y_1(\gamma_1, \cdots, \gamma_r, \gamma_{r+1}, \cdots, \gamma_m) S_1 + \cdots + y_r(\gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \ldots, \gamma_m) S_r = \sum_{i=1}^r y_i x^{\mathrm{T}} S_i \end{aligned}$$

where $S_i$ is the $m \times m$ skew-symmetric matrix such that the first $r$ entries of the $m$-th row are zero except for the entry at position $(m, i)$ which equals one; the remaining entries of that row are $c_{r+1,i}, \ldots, c_{m-1,i}, 0$; the $m$-th column is the negative of the transposed $m$-th row; and all other entries are zero. ∎

**Solution 52-6.2** by Rachel Quinlan, *National University of Ireland, Galway*, rachel.quinlan@nuigalway.ie

This statement holds over all fields. Let $\mathbb{F}$ be any field, and let $V$ denote the vector space of skew–symmetric $m \times m$ matrices with entries in $\mathbb{F}$. If $\operatorname{char} \mathbb{F} = 2$, then a skew–symmetric matrix over $\mathbb{F}$ is defined as a symmetric matrix whose diagonal entries are all zeros; otherwise a skew–symmetric matrix $S$ is one satisfying $S^T = -S$. We note two properties of (spaces of) skew–symmetric matrices.

*Lemma* 1. *If $S \in V$, then $x^T S x = 0$ for all $x \in \mathbb{F}^m$.*

*Proof.* If $\operatorname{char} \mathbb{F} \neq 2$, this follows from observing that $x^T S x = (x^T S x)^T = x^T S^T x = -x^T S^T x$. If $\operatorname{char} \mathbb{F} = 2$, then

$$x^T S x = \sum_{i \neq j} x_i x_j S_{ij} = \sum_{i<j} x_i x_j \underbrace{(S_{ij} + S_{ji})}_{0} = 0. \quad ∎$$

*Lemma* 2. *For a nonzero column vector $x \in \mathbb{F}^m$, let $x^\perp = \{y \in (\mathbb{F}^m)^T : yx = 0\}$ be a space of row vectors of dimension $m-1$. Then $x^\perp = x^T V$.*

*Proof.* By Lemma 1, $x^T V \subseteq x^\perp$. For the opposite inclusion, let $e_1$ be the first standard basis vector in $\mathbb{F}^m$. Since every row vector having a zero in the first position occurs as the first row of an element of $V$, the space $e_1^T V$ has dimension $m-1$ and thus is equal to $e_1^\perp$. Now let $P \in M_n(\mathbb{F})$ be a nonsingular matrix for which $e_1^T P = x^T$. Then $PVP^T = V$, so $e_1^T V = e_1^T PVP^T = x^T VP^T$. Since $P^T$ is invertible, $\dim x^T V = \dim x^T VP^T = m-1$. Thus $x^T V = x^\perp$. ∎

Now let $A \in \mathbb{F}^{n \times m}$ have rank $r$ and let $x \in \mathbb{F}^m$ be nonzero. If $A = \sum_{i=1}^r y_i x^T S_i$ for some $y_1, \ldots, y_r \in \mathbb{F}^n$ and skew–symmetric matrices $S_1, \ldots, S_r$, then $x^T S_i x = 0$ for each $i$ and hence $Ax = 0$.

On the other hand, suppose that $Ax = 0$ and let $\{y_1, \ldots, y_r\}$ be a basis for the column space of $A$. Then there exist nonzero $z_1, \ldots, z_r \in \mathbb{F}^m$ for which $A = \sum_{i=1}^r y_i z_i^T$. Since $Ax = 0$ and the $y_i$ are linearly independent, it follows for each $i$ that $z_i^T x = 0$ so $z_i^T \in x^\perp$, hence $z_i^T = x^T S_i$ for some $S_i \in V$ and $A = \sum_{i=1}^r y_i x^T S_i$. ∎

*Editorial note: The proposers' solution resembles* 52-6.2. *For real matrix $A = \sum_{i=1}^r y_i z_i^T$ they explicitly give $S_i = \frac{x z_i^T - z_i x^T}{(x^T x)}$.*

*With complex matrices one can also use skew-Hermitian $S_i$: every $A \in \mathbb{C}^{n \times m}$ of rank-$r$ can be written as $A = \sum_{i=1}^r y_i z_i^*$ for appropriate linearly independent vectors $y_i$ and vectors $z_i$. If, in addition, $Ax = 0$ for some nonzero $x \in \mathbb{C}^m$, then, as in the proof of Solution* 52-6.2 *we have $z_i^* x = x^* z_i = 0$. Thus, $A = \sum_{i=1}^r y_i x^* S_i$ for skew-Hermitian $S_i = \frac{x z_i^* - z_i x^*}{(x^* x)}$.*

*Also solved by Johanns* de Andrade Bezerra *and the proposers.*

# IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 52 (Spring 2014), p. 48] except problem 52.5 for which we still seek solutions. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Steve Butler, Gregor Dolinar, Shaun Fallat, Alexander Guterman, Rajesh Pereira, and Nung-Sing Sze.

NEW PROBLEMS:

**Problem 53-1: Matrix Proof of Dickson's Equality**

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@Manchester.ac.uk

Let $p, q, r, s$ be positive integers. Present a matrix proof of the identity

$$(p^2 - bq^2)(r^2 - bs^2) = (pr + bqs)^2 - b(qr + ps)^2 = (pr - bqs)^2 - b(qr - ps)^2.$$

*Remark.* This identity was employed by Farebrother to explain that numbers which can be expressed as the sum of two squares in two or more different ways are commonly divisible by $5, 13$ or $17$.

**Problem 53-2: AGM Inequality for Determinants**

Proposed by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, U.S.A.*, eaherman@gmail.com

Let $X_1, \ldots, X_n \in \mathbb{R}^{m\times m}$ be positive definite. Give a non-analytic proof that $\det\left(\frac{1}{n}\sum_{k=1}^n X_k\right) \geq \left(\prod_{k=1}^n \det X_k\right)^{1/n}$.

**Problem 53-3: Normal and Symmetric Matrices**

Proposed by Roger A. HORN, *University of Utah, Salt Lake City, Utah, U.S.A.*, rhorn@math.utah.edu

Let $A = VBW^* \in \mathbb{C}^{n\times n}$ be normal, in which $V$ and $W$ are unitary and $B \in \mathbb{C}^{n\times n}$. Let $\mathcal{A} = V^*AV$. First suppose that $B = B_1 \oplus \cdots \oplus B_k$ is block diagonal, and partition $\mathcal{A} = [\mathcal{A}_{ij}]_{i,j=1}^k$ conformally to $B$. Prove the following three statements:

(a) If some block $B_j$ has distinct singular values, then the corresponding block $\mathcal{A}_{jj}$ is normal.
(b) If some block $B_j$ is real and has distinct singular values, then corresponding block $\mathcal{A}_{jj}$ is normal and symmetric.
(c) If each block $B_i$ has only one singular value $\sigma_i$ and if $\sigma_i \neq \sigma_j$ $(i \neq j)$, then $\mathcal{A} = \sigma_1 Z_1 \oplus \cdots \oplus \sigma_k Z_k$ with $Z_i$ unitary.

Now suppose that $B$ is real; it need not be block diagonal. Prove the following three statements:

(d) $\mathcal{A} = Q(\sigma_1 Z_1 \oplus \cdots \oplus \sigma_d Z_d)Q^T$ for some orthogonal $Q \in \mathbb{R}^{n\times n}$ and unitary $Z_1, \ldots, Z_d$, in which $\sigma_1 > \cdots > \sigma_d \geq 0$.
(e) If $A$ has distinct singular values, then $\mathcal{A}$ is symmetric.
(f) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ have the property that $|\lambda_i| \neq |\lambda_j|$ whenever $\lambda_i \neq \lambda_j$, then $\mathcal{A}$ is symmetric.

**Problem 53-4: A Positive Semidefinite Matrix**

Proposed by Minghua LIN, *University of Victoria, BC, Canada*, mlin87@ymail.com

Let $x, y \in \mathbb{C}^n$. Show that the $(2n)$–by–$(2n)$ matrix $\begin{pmatrix} 2(x^*x)I_n - xx^* & 2(y^*x)I_n - xy^* \\ 2(x^*y)I_n - yx^* & 2(y^*y)I_n - yy^* \end{pmatrix}$ is positive semidefinite.

**Problem 53-5: Symmetric Matrices and Triangles in the Plane**

Proposed by Denis SERRE, *École Normale Supérieure de Lyon, France*, denis.serre@ens-lyon.fr

Assume $M \in \mathbf{Sym}_3(\mathbb{R})$ satisfies $m_{jj} = 1$ and $|m_{ij}| \leq 1$; $1 \leq i < j \leq 3$. Thus, if $i < j$ then $m_{ij} = \cos\theta_{ij}$.

(i) Prove that $\det M = 0$ if and only if there exist signs such that $\pm\theta_{12} \pm \theta_{13} \pm \theta_{23} \in 2\pi\mathbb{Z}$.

(ii) Let $x = (x_1, x_2, x_3)^t \in \mathbb{R}^3\backslash\{0\}$. Prove that the matrix $M$ with the above properties and with $Mx = 0$ exists if and only if $|x_i| \leq |x_k| + |x_j|$ for every pairwise distinct $i, j, k$.
**Hint**: For sufficiency reduce the problem to the case where $x_i \geq 0$ and form a triangle with edge lengths $x_1, x_2, x_3$.

**Problem 53-6: Matrix Equations Under the Ordinary Product and Kronecker Product of Matrices**

Proposed by Yongge TIAN, *Central University of Finance and Economics, Beijing, China*, yongge.tian@gmail.com

Let $A_i \in \mathbb{C}^{m_i\times n_i}$, $B_i \in \mathbb{C}^{p_i\times q_i}$, and $C_i \in \mathbb{C}^{m_i\times q_i}$ be nonzero matrices, $i = 1, 2$. Show that there exists a matrix $X$ such that $(A_1 \otimes A_2)X(B_1 \otimes B_2) = C_1 \otimes C_2$ if and only if there exist two matrices $X_1$ and $X_2$ such that $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$; here $\otimes$ denotes the Kronecker product of matrices.

*Solutions to Problems* 52-1 *through* 52-6, *except for* 52-5, *are on pp.* 45–51.