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IMAGE

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FEATURE INTERVIEW

"A Special Cooperative and Open-minded Community"

Frank Uhlig Interviewed by Tin-Yau Tam[1]

Frank Uhlig

Frank Uhlig retired in 2013 from the Department of Mathematics and Statistics at Auburn University after thirty-one years of service. A Retirement Colloquium/Workshop for Frank was organized by Tin-Yau Tam, Professor and Chair of the Department, on August 23, 2013 to honor Frank's career and achievements. Many colleagues, friends, co-authors and students came to celebrate this special occasion with Frank and his family. A report by Tin-Yau Tam appeared in *IMAGE* 52, Spring, 2014 (p. 35).

On July 15, 2014, Tin-Yau Tam conducted an interview with Frank Uhlig. It lasted for about 25 minutes and the following is a summary transcript of the interview.

Tam - *IMAGE* readers want to hear about your career in linear algebra. Would you please tell us?

Uhlig - I have thought about this a little bit. My entry into a mathematical career had quite an exceptional start, not because of the work that I have done, but by how I got interested in linear algebra. I started my studies at the Universität Köln in 1964. The number of first-year students was large, with about 320 in my first mathematics classes. These classes included mathematics, science, physics and science education majors. We had lectures in large classes with accompanying recitation sections. Towards the end of the year, I began to realize that I was not as good as some of the others appeared to be and I was not certain that I could, or should even, continue with mathematics. I was about to look at other subjects and I might have given up.

But then my teaching assistant (TA) in the advanced calculus sequence asked me if I was willing to help teach the new first-year students next year. I looked at him a little bit strangely, but I eventually took him up on this after many discussions with my friends. The main problem at Köln that year was that a sizable group of professors had retired just before my first year there and new faculty had come in to teach us, seemingly in more modern ways, and with a new strengthened curriculum. The retired faculty apparently had not taught their students well enough to become TAs for us. Two of us, young students, were hired in 1965 to fill in: to run recitation sections; to teach homework problems to students one year behind; and to grade their weekly exercises. We both, in a word, learned math swimmingly by being thrown into the pool. Our math classes in Köln were taught rigorously. Only about 80 of the beginning 320 students would pass after a year and a half. And only about 25 of my cohort eventually earned their degrees. Fifteen of these, though, became mathematics or science professors. My mathematical and intellectual education at Köln University was very challenging and it was rewarding for just a few of us. The first subject that I was asked to assist to teach was the one-year course of linear algebra for first-year students in 1965/66. Its first semester was an advanced level course as one would teach in the USA as a second linear algebra class for juniors or seniors. The second semester was suitable for graduate students in the USA. In Germany we have 13 years of schooling before university and in my last year at the gymnasium, we had studied a bit of group theory.

As a second-year student, I became a TA for a couple of old friends that I had played with on the streets years ago. (We had all moved away from the city where we grew up together). It was quite a startling and amusing situation that we met again, and I had become their teacher.

Recitation sections were held once a week. The TAs were supposed to discuss and explain, or have students explain, last week's turn-in homework problems that the TAs had handed back corrected. The problems always came in sets of five and were prepared for us by the professor in cooperation with his Ph.D.'ed main assistant. At most two of these problems were halfway routine from what had been taught in class. The rest were open questions and we generally had no initial idea how to solve them. I was only one year ahead of my recitation students in comprehension. To succeed in this system, we had to find and read books in the library, become creative and firm in the definitions and theorems, and then apply them correctly. It was harder, yet certainly rewarding, to study and to teach at the same time. In the recitation sessions, the TAs had to be swiftly intelligent and creative in leading the class to and through open discussion of their potentially faulty or correct mathematical reasoning and to learn to judge and repair their work and thoughts at the blackboard. We truly learned from each other. It was a wonderful experience. My first open-ended educational exchange between my own wits and those of the slightly 'younger' students was in linear algebra. This course was taught

[1]Department of Mathematics & Statistics, Auburn University, USA; tamtiny@auburn.edu

somewhat along the lines of Emil Artin's Hamburg Lectures I and II from the 1960s. The class presentation was modern, so modern in fact that I failed my Differential Equations course final. This happened since the later-year TAs did not understand my derivation of the Jordan normal form from the highest-grade principal vectors that I had learned from Prof. Jehne and Dr. Plewe the year before, a derivation that I was teaching to my students. This was incomprehensible for the older TA staff and got me no credit. Luckily that did not matter to me since I could substitute my Complex Variables class for Differential Equations.

My second year as TA covered advanced calculus, again for first-year mathematics and science students at Köln. Then I went to the USA on a Fulbright scholarship. In the summer of 1968, I attended a month-long NSF Summer School at Bowdoin on something algebraic and there I met Olga Taussky, who loved to swim in the pool at noon, just as I did. I had been accepted as a graduate student at Caltech for the Fall, and in early 1969 I attended the beginning graduate student seminar of Olga and Jack Todd. Eventually I asked Olga to become her student, after another seminar with her, mainly on cramped unitary and symmetric matrices. She told me that she was working on class field theory then and that I would not have quite the background to do research there. I agreed, but then mentioned our seminar, and matrices, to which she answered "but you can do matrix theory." This was my introduction to research-level linear algebra and matrix theory.

Tam - You have organized large international linear algebra conferences in Auburn (and other places as well) with themes such as "Current Trends in Matrix Theory" in 1986, "Directions in Matrix Theory" in 1990 and "Challenges in Matrix Theory" in 2002. What do you see as the main ideas and conjectures that need to be conquered or explored in linear algebra? In mathematics?

Uhlig - I come from an inspiring Ph.D. mother - Olga Taussky-Todd. Olga always gave visionary talks. She would allude to something that she would not know the answers to. With my conferences, I basically wanted to bring out more than what had been done before and what was already published and known. Instead, I tried to foster an exploration of what was still out there, intriguing and unfathomed. I wanted my colleagues to give us directions. I have not really been all that successful with that. Most people were more or less keeping their own problems closer to their chest, certainly more than Olga would. Olga was willing to share and ultimately inspire others. So I tried to bring that forward out of our group.

I really enjoy being part of the group of linear algebraists because we do not fight. The atmosphere among linear algebra researchers has been and is wonderful. I doubt that the same could be said for many other fields. We form a special cooperative and open-minded community and we all try to foster one another. The greatest overall challenges to mathematics and science are on the social level. For us, locally in matrix theory, they are to keep the linear algebra group intact and tolerant of inside and outside new ideas. Linear algebra and matrices are so useful and needed in all of our modern ways and technologies.

Tam - What are the most important applications now of linear algebra, in your view?

Uhlig - The best uses and most important applications occur in computing. All models and computational problems are best translated into matrix problems and then solved by numerical linear algebra methods. All our cell phoning, searching the Internet, communications, surveillance, and differential and discrete equations must be, and are, translated into matrix language and then solved with special matrix algorithms.

Linear algebraic methods are the atoms of our modern computer-based world.

Tam - What are your best or most cherished contributions to math and linear algebra?

Uhlig - Of course the answer to that question on any given day is always "the problem that I am currently working on." This changes with time, the most important work/result is... I just do not know. Deciding on the best in any area is not a matrix theoretic problem, is it?

But really: Once I discovered and published an elementary geometric proof for finding the derivatives of sine and cosine. Simple 3D geometry and notions of tangents, intersections, etc., made it possible. I had a most amazing and absolutely new elementary proof using no limits, no difference quotient, no continuity...

Perhaps also worthwhile: I reduced the complexity of polynomial root finding from the previously best-known $O(n^3)$ operations count for the companion matrix via the Francis method, to one of $O(n^2)$ complexity via Euclid's algorithm on the polynomial and its normalized derivative, followed by DQR on the resulting tridiagonal matrix representation of a polynomial of degree n. My algorithm moreover does not smear repeated roots as QR does – and it is forward stable, and with much higher accuracy in general. As far as I can tell, such a complexity reduction by one full order of n has never occurred or been achieved before for any problem, though there are examples of reduction from $O(n^3)$ down to $O(n^2 \log(n))$, such as the discovery of the fast Fourier transform.

Tam - You have written several books. Tell us about your passion regarding book writing.

Uhlig - I enjoyed writing books. It was a wonderful experience and I learned a lot about language and presentation. My publishers and coauthors were all wonderful.

Tam - Do you have any advice for young linear algebraists?

Uhlig - Well, from somebody trained in linear algebra and who wanted to remain in research: My transition through my mathematical life was that at first I was a somewhat narrow-minded theorist. I tried to get deeper and deeper in theory. And I was a little shy of applications. I did not understand the appliers' words, concepts and language, which appeared to me quite different from pure mathematics. Mainly I was afraid to ask, to look as dumb as I felt. Eventually, though, I was forced and ready to work with other people, going out on thin ice. They would come to me because, I guess, they saw something in me that I did not see in myself. It took me a while to ask questions back, dumb questions, or so I thought. But I found out that people in applications were really excited when I would interpret their chemistry, physics, numerics or engineering problems in my own words. Then they saw that I had penetrated some of their own understandings and that they could work with me through the mathematics, which they in turn did not quite understand and felt 'dumb' about. Our mutual openness inside the team then brought me lots of contexts and lots of happiness. I learned that I had more than one leg besides the theoretical one! Today I do not really care or wonder much about the theoretical aspects of matrices. I care more about matrix computations, about concrete mathematics and numbers just like Olga always did, but not about class field theory.

I would advise young people to think about their potential inner growth and embrace opportunities for giving up, leaving the safety of having written a thesis and knowing a topic well that helped publish three or thirteen nice papers. Extend your vision and field; have the courage to be dumb and unknowing again and again. Go into new areas and fields of discovery.

Tam - Tell us how you feel about the colloquium workshop in honor of your retirement back in 2013.

Uhlig - Oh, it was wonderful and required no work for me this time. The small workshop was very good. This was the first time our department recognized a retiree that I can recall; my thanks to you, Tam. I wish our department would start doing this more often. To bring in colleagues, friends and students so we can enjoy someone's life's work together.

Tam - Tell us your interest, hobbies, etc., beside mathematics? How do you enjoy your retirement life now?

Uhlig - Well, outside of my own field or work, we as a family do a lot with farming and research. I am heavily engaged in various prize committees for this and that prize; this is not public and there is no need to mention it. Besides, I travel, photograph, exhibit, teach, edit mathematics journals, referee, research, assess research proposals, attend music events, watch films, swim, hike, trek... There is never a dull moment.

Tam - What kind of farming are you doing? Do you mean organic farming?

Uhlig - I am not a farmer. The basic farming question today is: What is sustainable? I first noticed this quandary when I was invited by Datta some 30 or 40 years ago to give a talk at his place. I was driven through middle or western Illinois in a limousine and saw the monoculture of corn and erosion. And I wondered how many years this abuse of our fertile black soils had been going on. How many years until we would farm all of this former grassland topsoil down to nothing, just to feed ourselves today? It was clear to see outside my car window how much the midwestern topsoil was eroding and being carried down the Mississippi to poison the Caribbean Sea. Every year and for dozens of decades now, more soil in weight and volume goes down the Mississippi than the corn and grain harvested from its whole watershed. We are not living or farming sustainably. Agriculture does not have to be organic but it has to be sustainable or we shall all starve eventually.

Tam - I know you are a big traveller and you just came back from a trip to Asia – South Korea, Hong Kong, Malaysia. Looks like you enjoy your retirement life very much.

Uhlig - I have enjoyed life and living throughout my years and especially whenever I was travelling and even when attending conferences. I always set aside a few days for discovery and photography when I travel. And the mathematics will follow.

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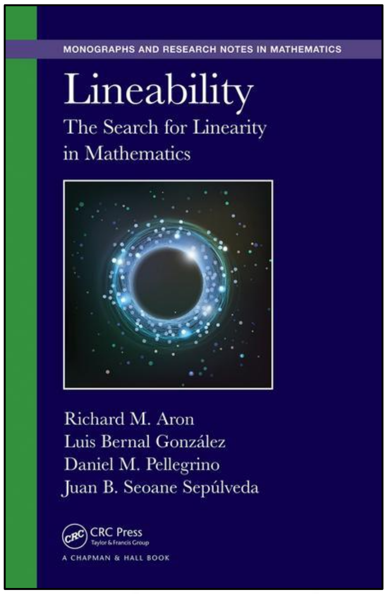

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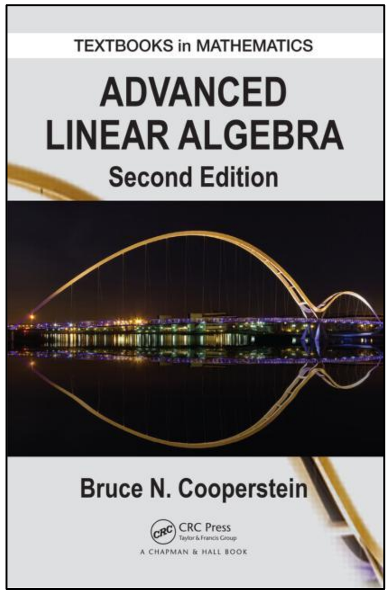

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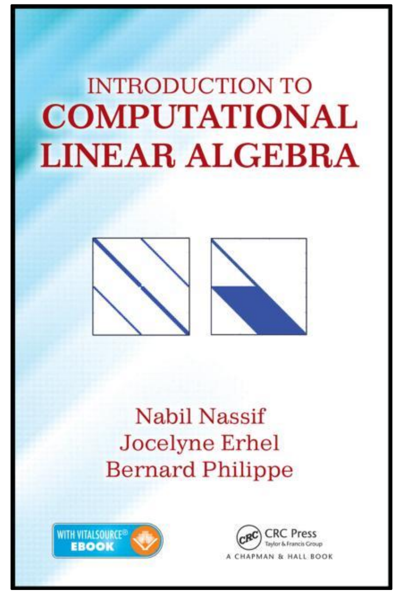

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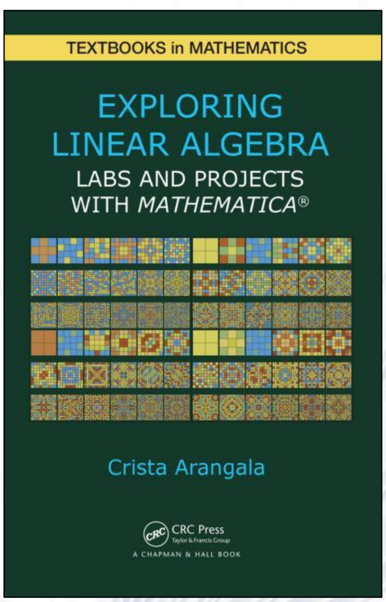

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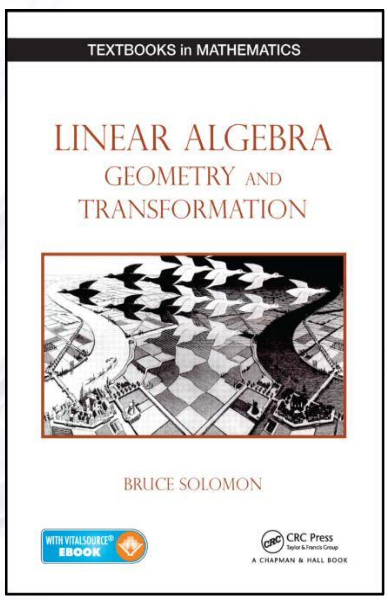

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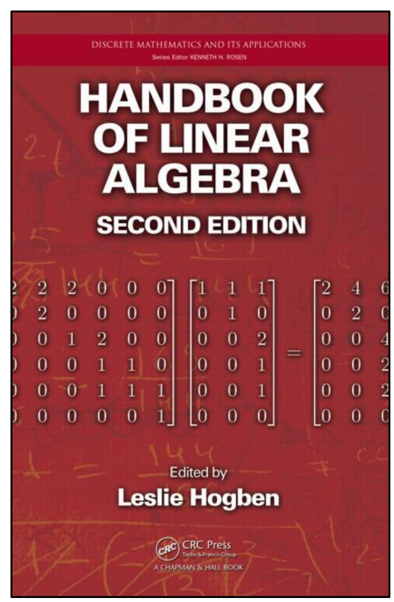

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FEATURE ARTICLE

Linear algebraic ties to quantum information theory

Sarah Plosker
Department of Mathematics and Computer Science, Brandon University,
Brandon, Manitoba, Canada
ploskers@brandonu.ca.

To me, it's impossible to think of quantum information theory (QIT) without linear algebra, and I always enjoy seeing linear algebraic techniques used in the QIT literature. I hope this light excursion into the world of QIT will whet your appetite enough that you read some of the references or attempt an open problem listed below. Unfortunately, in this short article I will not be able to touch on everything. That being said, more and more research groups focusing on quantum information are springing up all over the world, so wherever you are located geographically, you should be able to hear about the latest research and open problems in QIT at a lecture near you. If not, invite someone to come in and give a talk. There's nothing to lose, and so much to gain.

1. Some background. What is now called classical information theory was first developed by Claude Shannon in his seminal work in 1948 [22]. The quantum analogue of classical information theory is of course quantum information theory, which uses generalizations of all the mathematics developed in the classical setting. The two key properties that set QIT apart from classical information theory are superpositions and entanglement. Superpositions are essentially linear combinations of basis states (with the additional property that the coefficients $\alpha_i \in \mathbb{C}$ in the linear combination satisfy $\sum_i |\alpha_i|^2 = 1$), posing no challenge to anyone armed with basic linear algebra knowledge. Entanglement is somewhat vexing to deal with mathematically and is discussed briefly below.

In QIT, we use *kets* $|\varphi\rangle \in \mathbb{C}^n$ to represent unit (column) vectors and *bras* $\langle\varphi| := |\varphi\rangle^\dagger$ to represent the dual (row) vectors, where $(\cdot)^\dagger$ represents complex conjugate transpose. (Physicists often use $(\cdot)^*$ to represent complex conjugation.) (QIT can be discussed in the more general setting of C*-algebras, which leads to the following problem: How does one write "C†-algebra" without cringing in the same manner as when one hears fingers on a chalkboard? But I digress...) We will restrict our attention to unit vectors so that $\langle\varphi|\varphi\rangle = 1$. This notation implies the convention that inner products are conjugate linear in the first variable. These vectors $|\varphi\rangle$ represent quantum information in our quantum system $\mathbb{C}^n$ and are called quantum *states*. States can be either pure (represented by a unit vector $|\varphi\rangle \in \mathbb{C}^n$ or, equivalently, its outer product $|\varphi\rangle\langle\varphi|$, which is a rank-one projection) or mixed (represented by a density matrix $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$, where $\{p_i\}$ forms a probability distribution). Density matrices (including outer products associated to pure states) are precisely the trace-one, positive (semi-definite) matrices.

Often we are interested in a tensor product of systems: $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p \otimes \cdots$. These different systems are often labelled with subscripts $a, b, c, \ldots$ in order to identify them. At some point, physicists anthropomorphized this to Alice, Bob, Charlie,... This actually has physical significance in the sense that the different components of the tensor product could in fact be different quantum systems in different labs all over the world, controlled, respectively, by Alice, Bob, Charlie,... We focus on the bipartite case (two subsystems) herein, with some discussion of the tripartite case (three subsystems). A pure state $|\varphi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is *separable* if it can be written as an elementary tensor: $|\varphi\rangle = |\phi\rangle \otimes |\psi\rangle$ for some $|\phi\rangle \in \mathbb{C}^m$ and $|\psi\rangle \in \mathbb{C}^n$. Otherwise, $|\varphi\rangle$ is said to be *entangled*. A mixed state $\rho \in M_m \otimes M_n$ is separable if it can be written as a convex combination of separable pure states:

$$\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i| \otimes |\psi_i\rangle\langle\psi_i|,$$

where $\{p_i\}$ forms a probability distribution. Otherwise, ρ is said to be entangled. Entanglement leads to many properties and consequences that are counterintuitive, but at the same time is extremely powerful. In particular, this "spooky action at a distance" [3] is used to implement superdense coding and quantum teleportation.

Given a state $\rho \in M_m \otimes M_n$, we can look at Alice's (Bob's, respectively) reduced density matrix by "tracing out" Bob's (Alice's, respectively) subsystem. That is, we consider the *quantum marginals*

$$\rho_A = (I_m \otimes \mathrm{Tr}_n)\rho \quad \text{and} \quad \rho_B = (\mathrm{Tr}_m \otimes I_n)\rho, \tag{1}$$

where I_m is the identity map on M_m, Tr_m is the trace functional on M_m, and similarly for the subscript n. Of course, by performing the trace of a state, one loses information about the state. For a tensor product of two subsystems (belonging to Alice and Bob, respectively), given ρ_A and ρ_B, the Schmidt decomposition allows us to recover the possible "parent" states ρ for which equation (1) holds. However, for three subsystems (belonging to Alice, Bob, and Charlie, respectively), given, say, quantum marginals ρ_{AB} (where we have traced out Charlie's subsystem) and ρ_{AC} (where we have traced out Bob's system), they may be inconsistent in the sense that a parent state ρ may not exist. Or, there may exist many different parent states ρ. An interesting open problem in QIT, called the quantum marginal problem, is: given quantum marginals ρ_{AB} and ρ_{AC}, can one determine if a parent ρ exists, and if so, can one determine all possible parent states ρ? In general, it may be difficult to construct ρ, even if we know such ρ exist(s).

A *superoperator* is a linear map $\Phi : M_m \to M_n$. A superoperator Φ is positive if $\Phi(X) \geq 0$ whenever $X \geq 0$ (that is, Φ sends positive matrices to positive matrices). Of special consideration are superoperators Φ that are *completely positive*: superoperators Φ for which the induced mappings $\Phi_d : M_d \otimes M_m \to M_d \otimes M_n$, defined by $\Phi_d = \mathrm{id}_d \otimes \Phi$, are positive for all $d \in \mathbb{N}$, where id_d is the d-dimensional identity map. Equivalently, we could define a completely positive map as a linear map $\Phi : M_m \to M_n$ that can be decomposed as $\Phi(\cdot) = \sum_{k=1}^{r} A_k(\cdot)A_k^\dagger$ for some $\{A_k\}_{k=1}^r \in M_{n\times m}$ with $r \leq mn$. The A_k in this equivalence are typically called the *Choi* or *Kraus operators* of Φ. A completely positive map that is trace-preserving (i.e., $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$ for all X, or, equivalently, $\sum_{k=1}^{r} A_k^\dagger A_k = \mathrm{I}$) is called a *quantum channel*; such maps represent evolution of one's system.

A quantum channel $\Phi : M_m \to M_n$ satisfying $\Phi(I) = I$ is *unital*. Unital channels are precisely the completely positive trace preserving maps whose Kraus operators satisfy $\sum_{k=1}^{r} A_k A_k^\dagger = \mathrm{I}$. The simplest nontrivial example of a quantum channel is $\Phi : M_n \to M_n$ given by $\Phi(\rho) = U\rho U^\dagger$ (having a single Kraus operator U). Note that trace-preservation forces U to be unitary; for this reason such a quantum channel is called a unitary channel. A quantum channel of the form $\Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger$, where the U_i are unitaries and the p_i form a probability distribution, is a *mixed unitary channel* (often called a *random unitary channel*). The Kraus operators for this map can be read off from its decomposition: they are $\sqrt{p_i}U_i$. For $n = 2$ the set of all mixed unitary channels is precisely the set of all unital quantum channels, but for dimension larger than two mixed unitary channels form a proper subset of the set of all unital quantum channels [15]. The function Tr_k used in equation (1) is called the partial trace; it is in fact a linear, trace preserving, completely positive map, and as such it is a quantum channel.

Another way of thinking about a quantum channel $\Phi : M_m \to M_n$ is via the Stinespring dilation theorem. In the QIT setting, Stinespring's theorem looks slightly different: for some $k \leq mn$, there exists a partial isometry $U \in M_{nk\times mk}$, and a pure state $|\psi\rangle \in \mathbb{C}^k$ such that

$$\Phi(\rho) = \mathrm{Tr}_k(U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger) \qquad \text{for all } \rho \in M_m. \tag{2}$$

Now we define $V|\phi\rangle := U(|\phi\rangle \otimes |\psi\rangle)$ for all pure states $|\phi\rangle \in \mathbb{C}^m$ and some fixed pure state $|\psi\rangle \in \mathbb{C}^k$. Equation (2) then becomes simply

$$\Phi(\rho) = \mathrm{Tr}_k(V\rho V^\dagger).$$

The general form for U is

$$U = \left(V \mid * \right), \text{ where } V = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}.$$

That is, the Kraus operators are stacked in the left block of U; the right block of U is arbitrary since it gets traced out (and so is inconsequential).

Through the Stinespring dilation we can define a *conjugate* or *complementary* channel: given a quantum channel $\Phi : M_m \to M_n$ described by

$$\Phi(\rho) = \mathrm{Tr}_k(V\rho V^\dagger),$$

the corresponding conjugate channel is the quantum channel $\Phi^\sharp : M_m \to M_k$ described by

$$\Phi^\sharp(\rho) = \mathrm{Tr}_n(V\rho V^\dagger).$$

Armed with simply the knowledge of states, channels, and a basic understanding of entanglement, one can read and understand a surprising amount of the QIT literature.

2. Gateway Literature. Below are some gateway pieces of literature. By that I mean they are primarily math-focused and provide enough physics and QIT to get you hooked.

- In [13], the author outlines the basic objects in quantum computing, gently introducing both notation and language.

The article contains an introduction to quantum algorithms, which may be of interest to readers having a slightly more computer science background. Quantum error correction is discussed in greater detail.

- [11] is a great introduction to quantum information processing. Requiring virtually no background, the article begins with the definition of a "bit" and works its way up from there. It includes a useful glossary of commonly used terms.
- Much work has been done on higher rank numerical ranges, motivated by the study of quantum error-correcting codes. The first paper on the subject was [1], but a number of others continue to be published on the subject each year, most notably by C.K. Li, Y.T. Poon, and N.S. Sze (see, e.g., [17]).
- Nielsen [19] employed the well-known matrix theory concept of majorization to compare entanglement of different states. Much work ensued following the establishment of this link, including [14], which approaches the topic from a matrix theory point of view.
- In [4] and [10], the authors consider quantifying the ability to transform from one state to another, often called *fidelity* or *probability of state transfer*, via the use of graph theory techniques.
- For a popular online resource, see [23].
- Finally, if you want some reference texts to have handy on the shelves in your office, consider [6], [20], and, for a more mathematical approach, [21].

Many mathematicians working on the fringe of QIT seem to bring a physicist (or two!) into their circle in order to translate the papers written in physics journals into math. If you don't know any physicists offhand, I would recommend that you start by reading QIT-related articles in linear algebra journals—these are usually written by mathematicians, and may have some minor physics slant or physical applications. If the article is published in a math journal, the authors have translated the physics into math for you. This is merely to dip your toes in the water. Looking at the references in the articles and doing a forward search will take you into previously uncharted territory: *Physical Review Letters*, *Physical Review A*, *Communications in Mathematical Physics*, *Journal of Mathematical Physics*, *Journal of Physics A: Mathematical and Theoretical*, *Quantum Information & Computation*, etc. Before you know it, you'll have fully immersed yourself in physics-heavy QIT literature.

It can be frustrating reading the QIT literature. Physicists often rely on intuition rather than formal, rigorous proofs. Their publication process is more fast-paced compared to that of mathematics, which can lead to improper or missing citations. This can make it difficult to do a proper literature review, especially if you are new to the field. There are also some nuances in the physics literature that take some getting used to: typically references in the bibliography are in order of appearance rather than in alphabetical order, the authors are listed by contribution level rather than in alphabetical order, and sometimes the titles of the works in the list of references are omitted (simply authors, journal, volume, year, and page number/identifier).

3. Open problems in QIT.

There are a number of online lists of open problems in QIT, compiled around 2005–2010. Unfortunately these do not appear to have been recently updated. I will list a few open problems here; this list is heavily biased because it mainly represents open problems that I am interested in. Many seem quite innocent on the surface, but after reflecting on a particular problem for some time, you will see it is more involved than at first glance.

1. With respect to the quantum marginal problem as discussed in Section 1, even some special cases prove interesting and challenging. In the bipartite setting, it is of interest to determine all possible eigenvalues of the parent ρ such that $\rho_B = \frac{1}{n} I_n$ (the maximally mixed state). If A has dimension 2, then we are simply asking for the characterization of the eigenvalues of $2n \times 2n$ density matrices $\rho = (\rho_{ij})$, with $1 \leq i, j \leq 2$, such that $\rho_{11} + \rho_{22} = \frac{1}{n} I_n$, where ρ_{11}, ρ_{22} are $n \times n$. Not much is known for $n \geq 3$. Another open problem is: given prescribed partial states ρ_A, ρ_B, find all parents ρ with some extremal property, e.g., one with minimum von Neumann entropy, minimum rank, etc. In the bipartite setting, existence of ρ is guaranteed; for more subsystems one first needs to prove existence of ρ before considering extremal properties. A useful reference for these ideas is [18].
2. With respect to the probability of state transfer mentioned in Section 2, the ideal case is when the probability of state transfer is 1 (that is, perfect state transfer). However, it is practical in a physical setting to transfer the state quickly. In the case of perfect state transfer, it would be useful to have bounds on the readout time (after all, Bob doesn't want to have to wait 5 hours to readout the state that Alice has sent him). Furthermore, if one relaxes to allow for *pretty good state transfer* (probability of state transfer of $1 - \epsilon$), how does this affect the readout time bounds? These bounds would likely require clever use of the properties of adjacency and Laplacian matrices.

3. There are many open problems in quantum error correction and quantum cryptography. We first need some definitions in order to discuss these problems. Let $\Phi : M_n \to M_n$ be a quantum channel and let $\mathbb{C}^n = (A \otimes B) \oplus (A \otimes B)^\perp$ (we call A and B *subsystems* of $\mathbb{C}^n$; B is a *subspace* of $\mathbb{C}^n$ when A is one-dimensional). Then B is a *correctable subsystem* for Φ if there exists a quantum channel $\mathcal{R}$ such that for all σ_A, for all σ_B, there exists a τ_A such that

$$\mathcal{R} \circ \Phi(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B.$$

Here we use a subscript to denote to which subsystem a density matrix belongs. In plain language, if Alice wishes to send a message to Bob through a noisy channel, she can correct for any errors by simply selecting a message from a particular subsystem of $\mathbb{C}^n$, and then Bob is guaranteed to be able to uncover the original message. However, not all channels have a correctable subsystem so this is not always possible. On the other hand, B is a *private subsystem* for Φ if for all σ_A, for all σ_B, there exists a $\rho_0 \in M_n$ such that

$$\Phi(\sigma_A \otimes \sigma_B) = \rho_0.$$

Quantum error correction and quantum cryptography are two sides of the same coin, an intimate relationship made precise by the following theorem [12]: Given a conjugate pair of quantum channels Φ, $\Phi^\sharp$, a subsystem B is error-correctable for one if and only if it is a private subsystem for the other.

Quantum error correction has seen more development than quantum cryptography, and it would be useful to develop a systematic approach for determining private subsystem(s) given a particular quantum channel. The development of a private analogue to the celebrated stabilizer formalism in quantum error correction would be ideal, including a structure theory for the class of Pauli channels (quantum channels whose Kraus operators are tensor products of the Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, together with the identity). Furthermore, it would be of use to obtain bounds on the dimension of the ancillary and encoded Hilbert spaces (that is, the dimensions of A and B) which would serve as benchmarks for efficient private quantum communication. For example, a quantum error-correcting code of distance d protects a K-dimensional Hilbert space by encoding it into an n-qubit system. There are known bounds on d, K, and n, when the values of two of the three parameters are fixed. The goal is to maximize k and d, while minimizing n. Can we obtain analogous bounds in the quantum cryptographic setting? Recent work provides an answer to this question in a very specific setting [16].

We can also relax the quantifier from $\forall \sigma_A$ to $\exists \sigma_A$ in the above [7, 8]. Intriguingly, the complementarity theorem between error correction and privacy no longer holds in this setting. This can actually be a tremendous opportunity: creating pure states in a lab setting is difficult; it is much easier in practise to create mixed states. It would be much more practical if σ_A were a fixed mixed state rather than all (pure and mixed) states.

4. Two orthonormal bases $\{\mathbf{e}_i\}, \{\mathbf{f}_i\} \subset \mathbb{C}^d$ are said to be *unbiased* if $|\langle \mathbf{e}_i, \mathbf{f}_j \rangle|^2 = 1/d$ for all $1 \leq i, j \leq d$. A set of three or more orthonormal bases are called *mutually unbiased bases (MUBs)* if every pair of two of the bases are unbiased. A long-standing open question asks for the maximum number of MUBs in a given dimension d [2].

Some weak bounds are known for this problem. For example, it is always possible to construct 3 MUBs (in every dimension), and it is not possible to construct more than $d + 1$ MUBs. It is also known that the value of $d + 1$ is attainable when d is prime or a prime power. However, very little is known for non-prime-power dimensions. For example, when $d = 6$ it is still unknown if it is possible to construct more than 3 MUBs.

5. A similar open question is whether or not there exist symmetric, informationally complete, positive operator-valued measures (SIC-POVMs) in all dimensions. One formulation of this question asks whether or not, for all $d \geq 2$, there exists a set of d^2 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d^2}\} \subset \mathbb{C}^d$ such that all of their pairwise inner products have constant modulus. In other words, there exists some $c \in \mathbb{R}$ such that $|\langle \mathbf{v}_i, \mathbf{v}_j \rangle| = c$ for all $i \neq j$.

6. A mixed state $\rho \in M_m \otimes M_n$ is called *absolutely separable* if $U\rho U^*$ is separable for all unitary matrices $U \in M_m \otimes M_n$. In the case when $m = 2$, it is known that ρ is absolutely separable if and only if $\lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n}\lambda_{2n-2}}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n} \geq 0$ are the eigenvalues of ρ [9]. However, when $m, n \geq 3$ there is still no known complete characterization; only some partial results are known [5].

Acknowledgements. The author wishes to thank Nathaniel Johnston and Chi-Kwong Li for valuable discussions regarding the open problems.

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LINEAR ALGEBRA EDUCATION

Online resources available via the MAA's MathDL Course Communities

David Strong[1]

There is a continually-growing plethora of online resources and tools for teaching (and doing) Linear Algebra. Some of these resources are quite well conceived and created, while some are of lesser quality. While these resources and tools can be a real asset in teaching Linear Algebra, there are two significant difficulties in attempting to use them: (1) locating them and (2) finding those that are worth one's time and energy without wasting exorbitant amounts of time sifting through the mediocre and bad resources to find the really good ones.

One attempt at realizing this goal is realized in the Course Communities created by the Mathematical Association of America (MAA) as part of its Mathematical Sciences Digital Library (MathDL) project, located at `http://www.maa.org/about-maa/maa-history/mathdl`. Their Course Communities, located at `http://www.maa.org/programs/faculty-and-departments/course-communities`, currently includes resources for teaching Linear Algebra, Calculus, Differential Equations, Probability, as well as Developmental Mathematics.

While the site includes a few featured items, most resources for Linear Algebra are found via the Linear Algebra link. Upon following that link, the user will see links for the first of the 170 Linear Algebra-related resources. One can simply browse through the entire collection or else filter the collection by one of 25 different topics, which include eigenvalues/eigenvectors, matrix factorizations, online textbooks and the history of Linear Algebra. The resources include technological tools using Mathematica, Geogebra, Matlab, Maple and Java. (Quite unfortunately, due to recent changes in Java security, most of the Java-based online tools no longer run within a browser – perhaps this issue will eventually be resolved. In the meantime, there are plenty of other non-Java resources – most of those included in this collection – that are still fine.) While there are far too many resources in this Linear Algebra Course Communities collection for me to give a proper survey of them all, below I have included some of those that I think are of the most value.

There are a number of online video series for Linear Algebra, including Gil Strang's at `http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures` and the increasingly ubiquitous Kahn Academy videos at `https://www.khanacademy.org/math/linear-algebra`. There are also a growing number of online tutorials and entire textbooks included in the Course Communities, including

`http://www.numbertheory.org/book/`,

`http://joshua.smcvt.edu/linearalgebra/book.pdf`,

`https://www.math.ucdavis.edu/~linear/linear.pdf`, and

`http://www.calvin.edu/~scofield/courses/m231/S14/laNotes.pdf`.

As highlighted in the most recent issue of IMAGE, available at `http://www.ilasic.org/IMAGE/IMAGES/image54.pdf`, it is easier than ever to access past journal articles, including those related to the importance of and the teaching of Linear Algebra, and of course there are numerous article-like and textbook-like tutorials created by instructors covering every topic from adjacency matrices to vector spaces. A few currently listed at the Course Communities site include

`http://www.jstor.org/stable/i326560`,

`http://www.math.wayne.edu/~drucker/MAALinAlgResources/left%20&%20right%20inverses.pdf`,

`http://www.colorado.edu/engineering/cas/courses.d/IFEM.d/IFEM.AppP.d/IFEM.AppP.pdf`,

`http://cis-linux1.temple.edu/~latecki/Courses/RobotFall08/Papers/svdCMJ98.pdf`, and

`http://www.millersville.edu/~bikenaga/linear-algebra/matrix-exponential/matrix-exponential.html`

Further efforts at finding and categorizing by topic such articles and mini-tutorials on a wider scale would be beneficial.

Some nice tools for experimenting with and visualizing linear transformations are at

`http://mathinsight.org/applet/linear_transformation_2d`,

[1]Department of Mathematics, Pepperdine University, Malibu, CA 90263, USA, David.Strong@pepperdine.edu.

http://tube.geogebra.org/student/m2311, and

http://math.mercyhurst.edu/~lwilliams/Applets/LinearTransformations.html.

There are also other tools at their respective domains:

http://mathinsight.org/applet/,

http://tube.geogebra.org, and

http://math.mercyhurst.edu/~lwilliams/Applets/.

While I have yet to find the perfect online Gaussian Elimination (row reduction) tools, a few that do a pretty good job (with a few quirks or limitations) can be found at

http://www.idomaths.com/gauss_jordan.php,

http://www.math4all.in/public_html/linear%20algebra/RowEchelonForm/index.html,

http://matrix.reshish.com/, and

http://www.dangries.com/Flash/RowReducer/RowReducer.html.

There are dozens of nice Linear Algebra Wolfram Demonstrations, requiring the CDF (Computable Document Format) player, at http://demonstrations.wolfram.com/topic.html?topic=Linear%20Algebra. One can also simply use Wolfram Alpha to do all sorts of Linear Algebra-related computations and derivations, such as http://m.wolframalpha.com/input/?i=gram+schmidt+%7B%7B1%2C1%2C1%7D%2C%7B2%2C1%2C0%7D%2C%7B5%2C1%2C3%7D%7D. There are some on-line tools for manipulating matrices. Excel can do much of what these online tools do, but with an exception or two. Of course Matlab, Maple, Mathematica, etc,. can do all sorts of matrix manipulations. Students without Matlab or similar technology can use http://www.bluebit.gr/matrix-calculator to compute a number of types of matrices.

A listing of talks given at the Innovative and Effective Ways to Teach Linear Algebra sessions at the annual AMS/MAA Joint Meetings from 2008 to the present can be found at

http://seaver-faculty.pepperdine.edu/dstrong/LinearAlgebra/index.html.

Finally, a recent trend in university teaching is to pose questions to the entire class and solicit their immediate feedback via clickers, mobile phones, etc. A large collection of such questions for Linear Algebra was developed by Project MathQuest and is available at http://mathquest.carroll.edu/la.html.

Hopefully the above gives you a helpful sampling of some of the resources available via MAA's MathDL Course Communities for Linear Algebra. Sorting through these various resources to write this article makes me think that a more extensive survey of online resources is still warranted, one that can be sorted by topic, format or helpfulness/quality.

Send News for *IMAGE* Issue 56

IMAGE Issue 56 is due to appear online on June 1, 2016. Send your news for this issue to the appropriate editor by April 2, 2016. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

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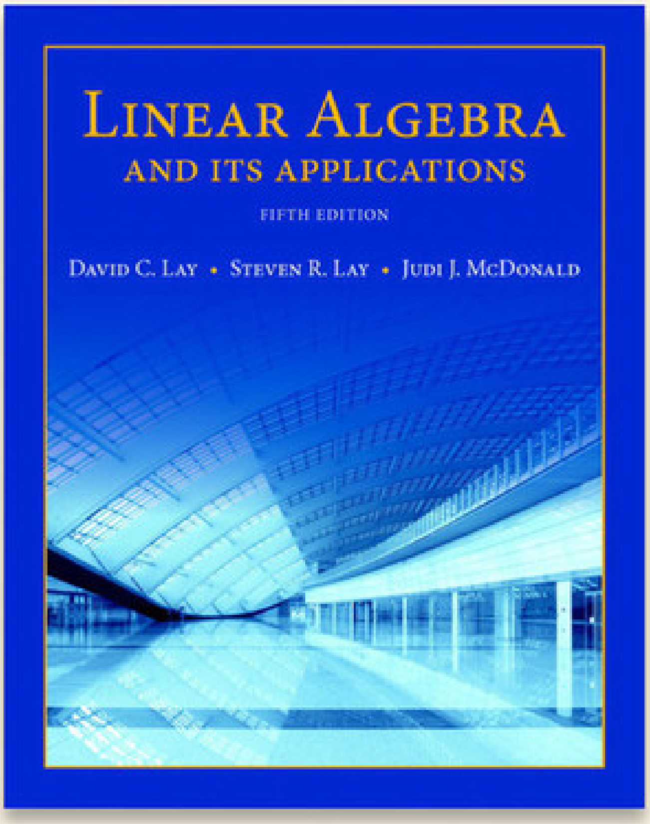

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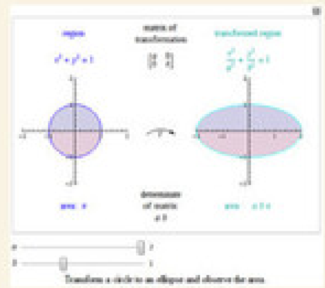

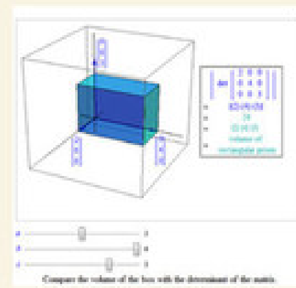

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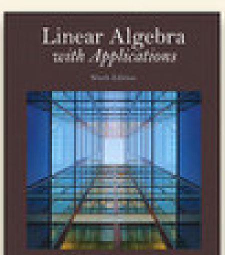

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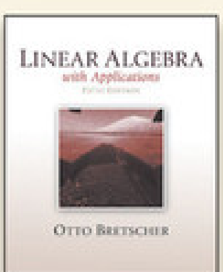

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CONFERENCE REPORTS

Numerical Algebra, Matrix Theory, Differential-Algebraic Equations, and Control Theory
Berlin, Germany, May 6–9, 2015

Report by Jörg Liesen

Berlin Conference Participants

The four-day conference was held at the Institute of Mathematics of the Technische Universität Berlin in honor of Volker Mehrmann on the occasion of his 60$^{\text{th}}$ birthday. Despite a serious strike at Deutsche Bahn, more than 140 participants made their way to Berlin and attended this special event, which featured 24 invited and 16 contributed talks. There were no parallel sessions, so that all talks were in fact plenary. This setup, rather unique for such a large conference, allowed everyone to hear and see everything, and thus get the whole picture. This was intended by the organizers in the spirit of Volker, who in a unique manner unifies expertise in the mathematical fields that provided the conference title. Let us remark that "Numerical Algebra", or rather the German "Numerische Algebra", was the title of Volker's first full professorship at the Technische Universität Chemnitz, which he was appointed to in 1993.

Eigenvalue problems and matrix polynomials, in particular structured ones, matrix factorizations, low rank approximations of matrices and tensors, as well as the theory, numerical solution and applications of DAEs played a major role in many of the talks. But there also were highly interesting other contributions. For example, Maria Esteban, ICIAM's President Elect, talked about symmetry properties of optimizers of functional inequalities. And Martin Grötschel, President Elect of the Berlin-Brandenburgische Akademie der Wissenschaften (BBAW), presented "Numerical Linear Algebra: A View from an Outsider".

A poster session with 24 posters in all areas of the conference attracted a large number of conference participants on the evening of May 7. Some of the posters have been published in a special collection of the ScienceOpen research and publishing network: `https://www.scienceopen.com/collection/2015VM60`.

Volker Mehrmann

As a part of the opening reception on May 6, the video "Top Ten Reasons it's Good to be 60" produced by Volker's friends from Kent State University was shown in the Atrium of the TU Berlin. Those who missed it can still find the video on YouTube: `https://www.youtube.com/watch?v=ucdlQGMx85A`. A special session dedicated to Volker was held on May 8. In this session, Peter Benner, Angelika Bunse-Gerstner, Christian Mehl, Nancy Nichols and Fredi Tröltzsch recalled highlights from Volker's career and shared personal memories ranging from his Ph.D. studies in Bielefeld and Kent State to his current professorship in Berlin. Afterwards the conference participants enjoyed a conference dinner held in Berlin's event location "Alte Pumpe" ("Old Pump"), followed by a dance session, where Volker proved that he is still able to rule the dance floor until early in the morning.

The organizing team consisted of Peter Benner, Max Planck Institute, Magdeburg, and Jörg Liesen, Christian Mehl, Reinhard Nabben, Lena Scholz and Andreas Steinbrecher, all TU Berlin. The organizers gratefully acknowledge support from the TU Berlin, Deutsche Forschungsgemeinschaft (DFG), Max Planck Institute, Magdeburg, European Research Council (ERC), NICONET and SynOptio.

The complete program, list of participants, Book of Abstracts and many photos taken during the event can be found at the conference website: `http://www3.math.tu-berlin.de/multiphysics/VM60/`.

24th International Workshop on Matrices and Statistics Haikou, Hainan, China, May 25–28, 2015

Report by Jeffrey J. Hunter

The 24th International Workshop on Matrices and Statistics (IWMS-2015) was held at Hainan Normal University, in Haikou, Hainan, China, 25–28 May, 2015. This Workshop was sponsored by Hainan Normal University, endorsed by the International Linear Algebra Society with an ILAS Lecturer and supported by SAS Institute, Inc. with a SAS Lecturer.

The International Organising Committee (IOC) was chaired by Jeffrey J. Hunter (New Zealand), and included Simo Puntanen (Finland) as Vice-chair, George P.H. Styan (Canada) as Honorary Chair, S. Ejaz Ahmed (Canada), Augustyn Markiewicz (Poland), Dietrich von Rosen (Sweden), Julia Volaufova (USA) and Hans Joachim Werner (Germany). The IOC was augmented by Chuanzhong Chen (China), Kai Tai Fang (China), Shuangzhe Liu (Australia), Jianxin Pan (United Kingdom), Tsung-I Lin (Taiwan), Kimmo Vehkalahti (Finland), and Yonghui Liu (China), to form the Scientific Program Committee. The Local Organising Committee was chaired by Chuanzhong Chen and assisted by Li Wang as Secretary.

Participants of the 24th International Workshop on Matrices and Statistics: Haikou, Hainan, China, 25–28 May 2015

The invited plenary speakers were:

- Ravindra B. Bapat (Indian Statistical Institute, New Delhi, India), "Moore-Penrose inverse of a Euclidean distance matrix";
- Mu-Fa Chen (Beijing Normal U, China), "Unified speed estimation of various stabilities";
- Zhi Geng (Peking U, China), "Causal effects and causal networks";
- Chris Gotwalt, SAS Lecturer (SAS Institute Inc., Cary, NC, USA), "Firth estimation in the mixed model: a new derivation of REML and improved estimates and inferences using logistic models with random effects";
- Karl Gustafson, ILAS Lecturer (U of Colorado, Boulder, CO, USA), "Antieigenvalue analysis, new applications: Continuum Mechanics, Economics, Number Theory";
- Lynn Roy LaMotte (Louisiana State U, New Orleans, LA, USA), "Multivariate inverse prediction with mixed models";
- Peter Šemrl (U of Ljubljana, Slovenia), "Adjacency and coherency preservers";
- Yoshio Takane (U of Victoria, BC, Canada), "Professor Haruo Yanai and multivariate analysis".

The Workshop Program contained two special Invited Sessions, one, organized by Jianxin Pan, Honoring Kai Tai Fang's 75th Birthday, and one, organized by Julia Volaufova, Honoring Simo Puntanen's 70th Birthday. In addition, in a departure from previous workshops, eleven minisymposia were organised:

MS1. Model Selection and Post Estimation, Organizer: S. Ejaz Ahmed;

MS2. Statistical Simulation, Organizer: Kai-Tai Fang;

MS3. Magic Matrices, Organizers: Kai-Tai Fang & George P. H. Styan;

MS4. Matrices in Applied Probability, Organizer: Jeffrey J. Hunter;

MS5. Statistical Modeling and Computation, Organizer: Tsung-I Lin;

MS6. Matrices with Economic and Financial Applications, Organizer: Shuangzhe Liu;

MS7. Design and Analysis of Experiments, Organizer: Augustyn Markiewicz;

MS8. Linear and Mixed Models, Organizers: Simo Puntanen & Julia Volaufova;

MS9. Matrices Useful for Modelling Multi-level Models, Organizer: Dietrich von Rosen;

MS10. Teaching Matrices within Statistics, Organizer: Kimmo Vehkalahti;

MS11. G-Inverses, Linear Models and Multivariate Analysis, Organizer: Hans Joachim Werner.

The plenary sessions, minisymposia and special sessions involved 76 invited speakers. As well there were five contributory paper sessions involving 24 speakers. The total number of registrants was 133 from 21 countries, making this one of the larger IWMS meetings held, exceeded only by the 2010 Workshop in Shanghai, China.

In addition to the Book of Abstracts, a Souvenir Booklet was published with the financial assistance of Professor Kai Tai Fang and technical assistance from Associate Professor Yong-Dao Zhou. This contained a history of IWMS, articles on Professor Kai Tai Fang and Simo Puntanen, and twenty-four papers that were presented at the Workshop, and was made available to all the participants at the time of registration. The Abstracts and the Souvenir Booklet are available on the website `http://iwms2015.csp.escience.cn/`.

In addition, the journal *Special Matrices* is publishing a topical issue on the Proceedings of the 24$^{\text{th}}$ International Workshop on Matrices and Statistics, with guest editors Jeffrey J. Hunter, Simo Puntanen and Dietrich Von Rosen (see `http://www.degruyter.com/view/j/spma`).

The Workshop opened with a formal opening ceremony followed by a photo session. The banquet was held on the opening evening. At the end of the workshop the hosts arranged an excursion to Sanya.

The 25$^{\text{th}}$ International Workshop on Matrices and Statistics (IWMS-2016) will be held in Funchal, Madeira, 6–9 June 2016 (see `http://www.iwms.ipt.pt`). The Scientific Program Committee is chaired by Simo Puntanen with George P. H. Styan (Honorary Chair of IWMS, Canada) and Julia Volaufova (Vice-Chair, United States of America). The Organizing Committee is chaired by Francisco Carvalho (Portugal).

2015 Shanghai International Workshop on Matrix Inequalities and Matrix Equations
Shanghai, China, June 28–30, 2015

Report by Chi-Kwong Li

Participants of the 2015 Shanghai International Workshop on Matrix Inequalities and Matrix Equations

The 2015 Shanghai International Workshop on Matrix Inequalities and Matrix Equations took place at the Shanghai University, Shanghai, China, June 28–30, 2015. Shanghai is the largest Chinese city by population, and is a global financial center.

There were about 50 participants from different regions including Canada, China, Hong Kong, India, Japan, Korea, Singapore, Taiwan, Turkey, USA, etc. There were 30 talks. See `http://math.shu.edu.cn/mime2015/` for detailed information about the workshop including the participant list, programs, photos, titles and abstracts. In particular, a workshop photo can be downloaded at `http://cklixx.people.wm.edu/2015MIME.jpg`.

Advanced Course on Combinatorial Matrix Theory
Barcelona, Spain, June 29–July 3, 2015

Report by Margarida Mitjana

The Advanced Course on *Combinatorial Matrix Theory* was held June 29 to July 3, 2015 at the Centre de Recerca Matemàtica, Bellaterra (Barcelona). It was organized by Andrés M. Encinas and Margarida Mitjana. The course consisted of five series of four lectures each, delivered by

- Richard A. Brualdi (U. of Wisconsin–Madison): *Combinatorial Matrix Theory*,
- Angeles Carmona (U. Politècnica de Catalunya): *Boundary Value Problems on Finite Networks*,
- Stephen Kirkland (U. of Manitoba): *The Group Inverse for the Laplacian Matrix of a Graph*,
- Dragan Stevanovic (Serbian Academy of Sciences and Arts): *Spectral Radius of Graphs*, and
- Pauline van den Driessche (U. of Victoria): *Sign Pattern Matrices*.

The participants hailed from many countries: Algeria, Ireland, Brazil, Japan, Canada, Serbia, USA, Spain, etc. In spite of the condensed program, the participants had two sessions to present their work informally, providing the opportunity for young and established researchers to intermingle.

A new volume of the series "Advanced Courses in Mathematics CRM Barcelona," to be published by Birkhauser in 2016, will collect an expanded version of the lectures delivered by their authors. The organizers are very grateful to the main speakers and to all the participants for contributing to the success of the event. Funding was provided by Elsevier, Societat Catalana de Matemàtiques, and Real Sociedad Española de Matemáticas.

Lecturers and participants at the Advanced Course on Combinatorial Matrix Theory

2015 Workshop on Mathematical Aspects of Quantum Information Science
Sanya, Hainan, China, July 13–17, 2015

Report by Chi-Kwong Li

The 2015 Workshop on Mathematical Aspects of Quantum Information Science took place at the Tsinghua Sanya International Mathematics Forum located at Sanya, Hainan, China from July 13–17, 2015. The Tsinghua Sanya International Mathematics Forum (TSIMF) was established to host research conferences, workshops and retreats. It will serve as an international hub for mathematicians and mathematical scientists to disseminate latest the results and to develop the newest ideas. It will be an asset for the mathematical community worldwide.

There were about 36 participants of the workshop from different regions, including Australia, Canada, China, Hong Kong, Japan, Singapore, USA, etc. There were 32 talks. See `http://msc.tsinghua.edu.cn/sanya/2015/MAQIS2015/synopsis_and_organizers.aspx` for detailed information about the workshop including the participant list, programs, photos, title and abstracts. In particular, a workshop photo can be seen at `http://cklixx.people.wm.edu/qc2015-1.jpg`.

Participants of the 2015 Workshop on Mathematical Aspects of Quantum Information Science

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2015 Workshop on Matrices and Operators
Xi'an, Shaanxi, China, July 19–21, 2015

Report by Chi-Kwong Li

The 2015 International Workshop on Matrices and Operators took place at the Shaanxi Normal University, July 19–21, 2015. This was the tenth workshop of the series. Shaanxi Normal University is located at Xi'an, the capital of the Shaanxi province of China. Xi'an is one of the oldest cities in China, and was the capital of several of the most important dynasties in Chinese history, including Zhou, Qin, Han, Sui, and Tang, and it is the starting point of the Silk Road and home to the Terracotta Army of Emperor Qin Shi Huang.

There were about 140 participants from different regions including China, Canada, Hong Kong, Japan, Korea, USA, Taiwan, etc. See `http://cklixx.people.wm.edu/mao2015list.xls` for a list of participants, and `http://cklixx.people.wm.edu/mao2015.jpg` for a workshop photo. There were 40 talks, about half of them in English and half of them in Chinese; see `http://cklixx.people.wm.edu/mao2015abs.pdf` for the titles and abstracts of the talks in English.

Participants of the 2015 Workshop on Matrices and Operators

For further information about the workshop, see `http://cklixx.people.wm.edu/mao2015.html`. The 2016 workshop will take place at Jeju National University, Korea, July 3–7, 2016.

International Conference on Matrix Analysis and Its Applications - MatTriad 2015 Coimbra, Portugal, September 7–11, 2015

Report by Natália Bebiano and Tomasz Szulc

The 2015 edition of MAT-TRIAD, sixth in a series of international conferences on Matrix Analysis and Its Applications, was held at the Department of Mathematics, University of Coimbra, Portugal, from the 7$^{\text{th}}$ to the 11$^{\text{th}}$ of September 2015. Following the tradition of its predecessors, this meeting gathered researchers around topics in matrix theory and its profound role in theoretical and numerical linear algebra, numerical and functional analysis, and statistics. Invited talks were presented by

- Peter Benner (Germany), "Numerical solution of matrix equations arising in control of bilinear and stochastic systems";
- Marija Dodig (Portugal), "Matrix pencils completions, combinatorics, and integer partitions";
- Froilán M. Dopico (Spain), "The inverse eigenstructure problems for matrix polynomials";
- Moshe Goldberg (Israel), "Radii of elements in finite-dimensional power-associative algebras";
- Christian Mehl (Germany), "Generic low rank perturbations of structured matrices";
- Dietrich von Rosen (Sweden), "The likelihood ratio test in bilinear models";
- Roman Zmyślony (Poland), "Inference in linear mixed models and Jordan algebra"; and
- Karol Życzkowski (Poland), "Joint numerical range and numerical shadow."

The winners of the Young Scientists Awards of MAT-TRIAD 2013, Jaroslav Horáček (Czech Republic) and Maja Nedović (Serbia) presented the invited talks entitled "Computational complexity and interval Linear Algebra" and "H-matrix theory and applications" respectively. As before, the work of the conference included two lectures in matrix theory, especially dedicated to young participants, given by distinguished experts in the field: Friedrich Pukelshiem (Germany) on "Matrices and the European parliament" and Peter Šemrl (Slovenia) on "Adjacency preservers."

The program included a Special Session in memory of our dearest colleague Glória Cravo about eigenvalues, multiplicities and graphs, with speaker Charles Johnson (USA).

Nine invited minisymposia were included in the program:

- Linearizations and ℓ-ifications of matrix polynomials: theory and applications. Organizers: Maribel Isabel Bueno (USA), Froilán Dopico (Spain) and Susana Furtado (Portugal);

- Spectral graph theory. Organizer: Domingos Moreira Cardoso (Portugal);
- Algebraic methods in operator theory. Organizer: M. Cristina Câmara (Portugal);
- Coding theory. Organizers: Raquel Pinto (Portugal) and Diego Napp (Portugal);
- Matrix theory, applications and engineering. Organizer: Marko Stošić (Portugal);
- Functions of matrices. Organizers: Pedro Freitas (Portugal) and Sónia Carvalho (Portugal);
- Linear preserver problems. Organizers: Henrique F. da Cruz (Portugal) and Rosário Fernandes (Portugal);
- Statistical inference, numerical and combinatorial methods. Organizers: Luís Miguel Grilo (Portugal) and Fernando Lucas Carapau (Portugal);
- Statistical models with matrix structure. Organizer: Miguel Fonseca (Portugal).

To end the five-day conference, a talk on "Linear Algebra in Portugal" was delivered by J. Vitória (Portugal). A special issue dedicated to MAT-TRIAD 2015 with selected papers of participants will be published by Springer Verlag after a refereeing procedure in a volume entitled *Applied and Computational Matrix Analysis* in the series *Proceedings of Mathematics & Statistics*.

Apart from the successful scientific work and professional interchange, participants especially enjoyed the beauties of the Forest and Royal Palace of Bussaco, and the classical music concert played by the group "Quarteto Santa Cruz de Coimbra" and "Joana Neto." At the closing ceremony of the conference, two young scientists received the 2015 awards: Ernest Šanca (Serbia) for the talk "A wider convergence area for the MSTMAOR iteration methods for LCP" and Jolanta Pielaszkiewicz (Sweden) for the talk "Test for covariance matrix with use of spectral moments." They are invited to present talks at the 7$^{\text{th}}$ MAT-TRIAD conference planned to be held in 2017 in Braunschweig, Germany.

The conference dinner included a celebration in honour of Simo Puntanen's 70$^{\text{th}}$ birthday, with speaker Alexander Kovačec. The conference ended with a visit to the "Paço das Escolas" of the University of Coimbra (UNESCO World Heritage Site).

The conference was sponsored by the International Linear Algebra Society (ILAS), Department of Mathematics, University of Coimbra (DMUC), Center for Mathematics, University of Coimbra (CMUC), Center for R&D in Mathematics and Applications (CIDMA), Center for Mathematical Analysis, Geometry and Dynamical Systems (CAMGSD), Center for Functional Analysis, Linear Structures and Applications (CEAFEL), Polytechnic Institute of Tomar, Portugal (IPT), Fundação para a Ciência e a Tecnologia (FCT), Programa Operacional Factores de Competitividade (COMPETE), Quadro de Referência Estratégica Regional (QREN), Fundo Europeu de Desenvolvimento Regional - União Europeia.

The conference scientific committee consisted of Tomasz Szulc (Poland) - Chair, Natália Bebiano (Portugal), Ljiljana Cvetković (Serbia), Heike Faßbender (Germany) and Simo Puntanen (Finland), and its organising committee was Natália Bebiano (Portugal), Francisco Carvalho (Portugal), Susana Furtado (Portugal), Celeste Gouveia (Portugal), Rute Lemos (Portugal) and Ana Nata (Portugal).

More details about the program and the book of abstracts can be found at `http://www.mattriad.ipt.pt`.

A total of 170 participants from 39 countries attended the conference.

Participants of the MAT-TRIAD 2015 conference

6th Workshop on Matrix Equations and Tensor Techniques Bologna, Italy, September 21–22, 2015

Report by Heiko Weichelt

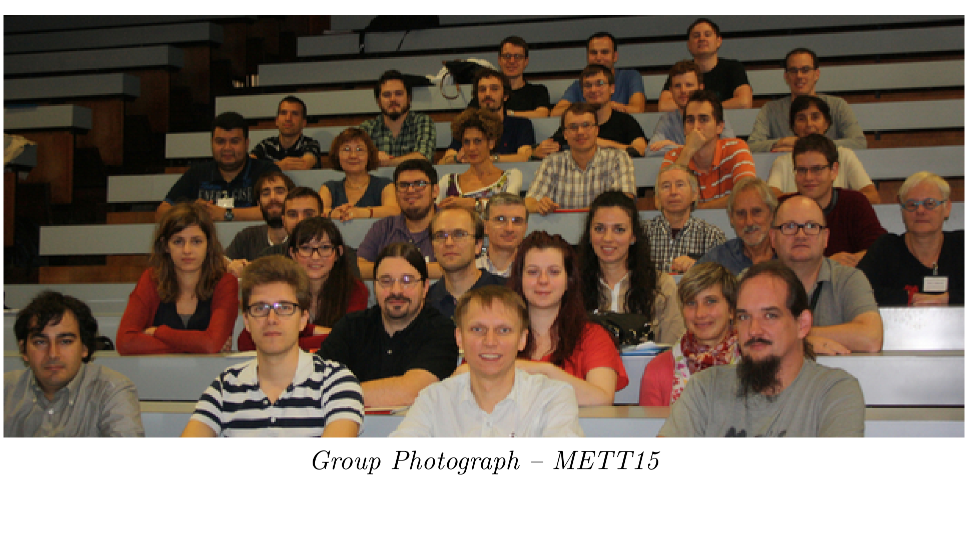
The 6th workshop on Matrix Equations and Tensor Techniques (METT) took place in Bologna, Italy, September 21–22 2015. The METT workshops evolved to an internationally established meeting for researchers in the field of tensor techniques and matrix equations during the last years. In 2005, the workshop series started in Leipzig as a small meeting for German scientists working on Riccati equations. In a two-year rhythm it was held in 2007 in Chemnitz, 2009 in Braunschweig, 2011 in Aachen, and 2013 in Lausanne, Switzerland. Since 2011, tensor techniques were included in the list of topics and two years later the workshop took place for the first time outside of Germany. The organizing committee consists of Peter Benner (MPI Magdeburg), Heike Faßbender (TU Braunschweig), Lars Grasedyck (RWTH Aachen), and Daniel Kressner (EPF Lausanne). For METT 2015, Valeria Simoncini (University of Bologna) joined the organizing committee as local organizer.

The 48 participants came from 12 different countries (two outside of Europe) to the university city Bologna in Italy, whose university was founded in 1088. Bologna is the capital of the region Emilia-Romagna and has around 350,000 inhabitants. The scientific program of the workshop was spread over two full days, where each day started with a plenary talk by a young researcher followed by three sections of 3-4 contributed talks.

The first plenary talk was given by Sergey Dolgov (MPI Magdeburg). He talked about numerical simulations of steering of complex dynamical systems, such as incompressible fluid flows, which can be very computationally demanding if the mathematical model is discretized with fine resolution. One can compress the discrete data by some techniques, such as the separation of variables, to accelerate computations. This methodology can be extended to solve systems of equations of a special form that arise from the optimal control of fluid dynamics. The first contributed session continued with the tensor techniques framework and explained, for example, how face recognition software, similar to that used by Facebook, works. The topic of matrix equations was represented in the afternoon sessions. Thereby, various methods to solve large-scale quadratic matrix equations were discussed and compared with each other. The first day was pleasantly concluded by the social dinner in the backyard of a typical Italian restaurant.

The second day started with the plenary talk by Bart Vandereycken (Université de Genève). He showed that reconstruction of large amounts of data from a few available samples or measurements is an important problem in scientific computing. To obtain a meaningful solution, some structure on the sought data needs to be imposed; in particular, the solution can be computed in the form of a low-rank matrix or its generalizations. Using the synergy of geometry of this representation and matrix analysis, an efficient algorithm for optimization and data fitting can be developed. In the subsequent contributed sessions, the topic of linear and quadratic matrix equations was addressed once again. Thereby, the solution of specially-structured Sylvester equations was one of the major topics. The workshop ended with a lively talk about image processing using polynomial computations.

In his closing remarks, Peter Benner thanked all participants for their attendance and announced the appointment of Valeria Simoncini and Beatrice Meini into the organizing committee. Beatrice Meini will act as local organizer of the 7th METT workshop that will take place in Pisa, Italy, in 2017.



Group Photograph – METT15

UPCOMING CONFERENCES AND WORKSHOPS

5th International Conference on Matrix Analysis and Applications (ICMAA) Nova Southeastern University, Fort Lauderdale, USA, December 17–20, 2015

This meeting aims to stimulate research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications, and to provide an opportunity for researchers to exchange ideas and developments on these subjects. The previous conferences were held in China (Beijing, Hangzhou), United States (Nova Southeastern University), and Turkey (Konya). Former keynote speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland and Alexander A. Klyachko (ILAS guest speaker).

The Keynote Speaker of the 5th ICMAA is Professor Shmuel Friedland, University of Illinois, Chicago, USA.

The Scientific Organizing Committee (SOC) of the 5th conference of the series consists of Shaun Fallat (University of Regina, Canada), Peter Šemrl (University of Ljubljana, Slovenia), Tin-Yau Tam (Auburn University, USA), Qingwen Wang (Shanghai University, Shanghai, China) and Fuzhen Zhang (Nova Southeastern University, USA; Chair). Local Organizing Committee members are Shahla Nasserasr (Chair), Vehbi Paksoy, as well as mathematics faculty and students of Nova Southeastern University. The conference is sponsored by Nova Southeastern University (USA) and Shanghai University (China). The De Gruyter journal *Special Matrices* will devote an issue to the conference. A special session, organized by Wasin So, will be dedicated to the memory of R. C. Thompson. Please visit `http://www.nova.edu/~zhang/15NovaMatrixMainPage.html` for details.

Tensor Decompositions and Blind Signal Separation Leuven, Belgium, January 15–16, 2016

This course is intended as a short introduction to tensor decompositions and their applications. The course will consist of 4 lectures of 2 hours each (lecturer: Lieven De Lathauwer) and 4 exercise sessions, also of 2 hours each. The course will be followed by the Workshop on Tensor Decompositions and Applications (TDA 2016). Information is available at: `http://homes.esat.kuleuven.be/~sistawww/winterschool16/index.php`.

Third Workshop on Tensor Decompositions and Applications (TDA 2016) Leuven, Belgium, January 18–22, 2016

Higher-order tensor methods are intensively studied in many disciplines nowadays. The developments gradually allow us to move from classical vector- and matrix-based methods in applied mathematics and mathematical engineering to methods that involve tensors of arbitrary order. This step from linear transformations and quadratic and bilinear forms to polynomials and multilinear forms is relevant for the most diverse applications. Furthermore, tensor methods have firm roots in multilinear algebra, algebraic geometry, numerical mathematics and optimization. This workshop will bring together researchers investigating tensor decompositions and their applications. It will feature a series of invited talks by leading experts and contributed presentations on specific problems. TDA 2016 is held in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and has been endorsed by the International Linear Algebra Society (ILAS). Part of the workshop is supported by the ERC Advanced Grant BioTensors. Information is available at: `http://www.esat.kuleuven.be/stadius/TDA2016/index.php`.

The Fourteenth Copper Mountain Conference on Iterative Methods Copper Mountain, Colorado, USA, March 20–25, 2016

The Copper Mountain Conference on Iterative Methods, held every other year, is the premiere conference on all aspects of iterative solution methods and their applications. The meeting is organized in cooperation with SIAM. The 2016 meeting features a workshop on "Iterative Linear Algebraic Data Mining Techniques" organized by Alex Breuer and Geoff Sanders and will include a tutorial by Van Emden Henson. (The tutorial is scheduled for Sunday, March 20.) The workshop focuses on numerical analysis issues associated with linear algebraic approaches to classifying, ranking, and clustering data. As in the past, there will be a Copper Mountain Special Section of the *SIAM Journal on Scientific Computing* devoted to iterative methods. It will be open to submissions from the whole community.

Important deadlines: student competition papers (January 15, 2016); author abstracts (January 22, 2016); early registration (January 29, 2016); and guaranteed lodging (March 5, 2016). Full information is at: `http://grandmaster.colorado.edu/~copper/2016/index.php`.

2016 Western Canada Linear Algebra Meeting (WCLAM)
Winnipeg, Manitoba, Canada, May 14–15, 2016

WCLAM 2016 will be held in Winnipeg at the University of Manitoba, 14–15 May, 2016. WCLAM provides an opportunity for researchers and mathematicians in linear algebra and related fields to meet, present accounts of their research, and hold informal discussions. Although the meeting has a regional base, it attracts a number of national and international participants, and enjoys a reputation for its welcoming atmosphere and high quality scientific program. Young mathematicians (graduate students, postdoctoral researchers, early career faculty) are especially encouraged to participate. WCLAM 2016 will feature three invited speakers: Rajesh Pereira (University of Guelph), Rachel Quinlan (National University of Ireland Galway) and Kevin N. Vander Meulen (Redeemer University College, ILAS Lecturer).

The meeting is organised by Shaun Fallat (Regina), Douglas Farenick (Regina), Hadi Kharaghani (Lethbridge), Steve Kirkland (Manitoba), Peter Lancaster (Calgary), Michael Tsatsomeros (Washington State) and Pauline van den Driessche (Victoria). A call for contributed talks will be distributed on ILAS-Net, as will further details regarding the meeting. The organisers gratefully acknowledge the sponsors of WCLAM 2016: ILAS, the Pacific Institute for the Mathematical Sciences, and the University of Manitoba.

IWMS 2016 – International Workshop on Matrices and Statistics
Funchal, Madeira, Portugal, June 6–9, 2016

25th IWMS'2016 Madeira Portugal

The conference will be held in Funchal, Madeira (Portugal) from June 6 to 9, 2016. Up-to-date information at `http://www.iwms.ipt.pt`.

The purpose of the workshop is to bring together researchers sharing an interest in a variety of aspects of statistics and its applications as well as matrix analysis and its applications to statistics, and offer them a possibility to discuss current developments in these subjects. The workshop will bridge the gap among statisticians, computer scientists and mathematicians in understanding each other's tools. We anticipate that the workshop will stimulate research, in an informal setting, and foster the interaction of researchers in the interface between matrix theory and statistics. Some emphasis will be put on related numerical linear algebra issues and numerical solution methods, relevant to problems arising in statistics.

The workshop will include invited talks and special sessions devoted to cutting edge research topics. A Special Session is devoted to the 75th birthday of Professor Jeffrey J. Hunter. The Invited Speakers are: Alan Agresti (USA), Ejaz S. Ahmed (Canada), Rosemary A. Bailey (UK), Radosław Kala (Poland), Alexander Kovačec (Portugal), and Jianxin Pan (UK).

The Scientific Committee consists of: Simo Puntanen (Finland) - Chair of the International Organizing Committee, George P. H. Styan (Canada) - Honorary Chair, Júlia Volaufovà (USA) - Vice-Chair, Ejaz Ahmed (Canada), Katarzyna Filipiak (Poland), Jeffrey J. Hunter (New Zealand), Augustyn Markiewicz (Poland), Dietrich von Rosen (Sweden), and Hans Joachim Werner (Germany).

The Organizing Committee consists of: Francisco Carvalho (Portugal) - Chair, Katarzyna Filipiak (Poland) - Vice-Chair, Ana Maria Abreu (Portugal), and Daniel Klein (Slovakia).

Contact: Francisco Carvalho | Email: iwms@ipt.pt. | Website: `http://www.iwms.ipt.pt`

The 20th ILAS Conference
Leuven, Belgium, July 11–15, 2016

The 20th Conference of the International Linear Algebra Society (ILAS) will take place July 11–15, 2016 in KU Leuven, Leuven, Belgium.

The Invited Plenary Speakers are Koenraad Audenaert (LAMA speaker, supported by Taylor & Francis), Pierre Comon, Paul van Dooren, Bruno Iannazzo, Monique Laurent, Elizabeth Meckes, Pablo A. Parrilo (LAA speaker, supported by Elsevier) André Ran, and Fernando de Terán (SIAG/LA speaker).

The current list of Invited Minisymposia with organizers is:

- Data-Driven Model Reduction by Athanasios C. Antoulas;
- Matrix Equations by Peter Benner and Beatrice Meini;
- Tropical Algebra in Numerical Linear Algebra by James Hook, Jennifer Pestana, and Françoise Tisseur;
- Matrix Inequalities and Operator Means by Jean-Christophe Bourin and Takeaki Yamazaki;
- Linear Algebra and Quantum Computation by Chi-Kwong Li, Raymond Sze, and Yiu Tung Poon;
- Matrix Methods for Solving Systems of Multivariate Polynomial Equations by Bernard Mourrain, Vanni Noferini, and Marc Van Barel;
- Image Restoration and Reconstruction by Marco Donatelli and Jim Nagy;
- Matrix Methods in Network Analysis by Francesco Tudisco and Dario Fasino; and
- Low-Rank Tensor Approximations by Andre Uschmajew and Bart Vandereycken.

The Scientific Committee consists of: Raf Vandebril (KU Leuven, Belgium), Wim Michiels (KU Leuven, Belgium), Peter Šemrl (University of Ljubljana, Slovenia), Froilán Dopico (Universidad Carlos III de Madrid, Spain), Heike Faßbender (Technische Universität Braunschweig, Germany), Françoise Tisseur (The University of Manchester, UK), Dario Bini (University of Pisa, Italy), Hugo Woerdeman (Drexel University, United States), Fumio Hiai (Tohoku University, Japan), Douglas Farenick (University of Regina, Canada), and Steve Kirkland (University of Manitoba, Canada). The scientific organizing committee can be reached by email: ilas2016.soc@cs.kuleuven.be.

The Local Organizing Committee consists of: Raf Vandebril (Chairman, Department of Computer Science, KU Leuven), Thomas Mach (Department of Computer Science, KU Leuven), Karl Meerbergen (Department of Computer Science, KU Leuven), Wim Michiels (Department of Computer Science, KU Leuven), Wim Vanroose (Department Mathematics and Computer Science, University of Antwerp), and Marc Van Barel (Department of Computer Science, KU Leuven). The local organizing committee can be reached by email: ilas2016@cs.kuleuven.be.

There will be a special issue of *Linear Algebra and its Applications* (LAA) for papers corresponding to talks given at the conference. The special editors of this issue are: Bas Lemmens, Douglas Farenick, Marc Van Barel, and Raf Vandebril. Papers will be refereed according to the usual high standards of LAA. The submission deadline will be 15 December 2016. More details will follow on when submission of papers will be open and how to submit papers.

The abstract submission system is now open, and proposals for contributed minisymposia and contributed talks are being accepted.

Important dates:

December 15, 2015:	Contributed minisymposium submission deadline (title and speakers).
January 31, 2016:	Notification of acceptance of the contributed minisymposia.
March 15, 2016:	Title and abstract submission deadline for all talks.
April 15, 2016:	Notification of abstract acceptance for contributed talks.
May 15, 2016:	Early bird registration deadline.

For additional information, go to `http://ilas2016.cs.kuleuven.be`.

The International Workshop on Operator Theory and Applications (IWOTA 2016) Washington University, St. Louis, USA, July 18–22, 2016

The 27$^{\text{th}}$ IWOTA will be organized by Greg Knese, Assistant Professor of Mathematics, Washington University in St. Louis; John E. McCarthy, Spencer T. Olin Professor of Mathematics, Washington University in St. Louis; and Kelly Bickel, Assistant Professor of Mathematics, Bucknell University. Further information will be posted at `http://openscholarship.wustl.edu/iwota2016/`.

The Twelfth International Conference on Matrix Theory and Applications Lanzhou University, Lanzhou City, China, July 22–26, 2016

The conference will focus on matrix theory, methods and computations, as well as their applications. Invited keynote speakers include: Richard A. Brualdi, University of Wisconsin, USA (ILAS speaker); Yu-Mei Huang, Lanzhou University, China; Wen Li, South China Normal University, China; An-Ping Liao, Hunan University, China; Lek-Heng Lim, University of Chicago, USA; Miro Rozloznik, Czech Academy of Sciences, Czech Republic; Zeng-Qi Wang, Shanghai Jiaotong University, China; Rui-Ping Wen, Taiyuan Normal University, China; and Gang Wu, China University of Mining and Technology, China. The conference is co-organized by Shanghai University and the Academy of Mathematics and Systems Science of the Chinese Academy of Sciences, P.R. China. The local organizing institution is Lanzhou University. For additional information, see: `http://icmta.lzu.edu.cn/`.

The Householder Symposium XX on Numerical Linear Algebra Virginia Tech, Blacksburg, Virginia, USA, June 18–23, 2017

The Householder Symposium XX on Numerical Linear Algebra will be held in Virginia Tech in Blacksburg, Virginia, USA, June 18–23, 2017. This symposium is the twentieth in a series, previously called the Gatlinburg Symposia, and will be hosted by the Virginia Polytechnic Institute and State University (VA Tech), in cooperation with the Society for Industrial and Applied Mathematics (SIAM) and the SIAM Activity Group on Linear Algebra. Details at: `http://www.math.vt.edu/HHXX`.

ILAS NEWS

2015 ILAS Elections

Nominated for a three-year term, beginning March 1, 2016, as ILAS Vice-President is: Bryan Shader, USA. Nominated for the two open three-year terms, beginning March 1, 2016, as "at-large" members of the ILAS Board of Directors are: Ravi Bapat, India; Naomi Shaked-Monderer, Israel; Helena Šmigoc, Ireland; and Xingzhi Zhan, China.

Many thanks to the Nominating Committee: Pauline van den Driessche, Rachel Quinlan (chair), Raf Vandebril, David Watkins, and Hugo Woerdeman. Electronic voting has been implemented using Votenet Solutions (the same company ILAS has used for elections since 2011). Voting concludes December 31, 2015. If you have not received your electronic ballot, please contact Leslie Hogben: hogben@aimath.org

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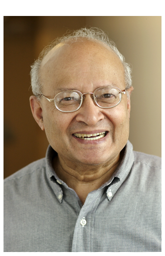
ILAS Member Receives National Medal of Science



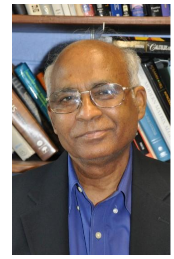
T. Kailath

Dr. Thomas Kailath was awarded the National Medal of Science, an award that is given to people who have made an outstanding contribution to science and mathematics. The award is administered by the NSF, but was presented by President Obama in late November 2014.

Dr. Kailath, who is the Hitachi America Professor of Engineering Emeritus at Stanford University, was cited "for transformative contributions to the fields of information and system science, for distinctive and sustained mentoring of young scholars, and for translation of scientific ideas into entrepreneurial ventures that have had a significant impact on industry."

ILAS Member Receives Fulbright Distinguished Chair

Reported by Richard A. Brualdi



B.N. Datta

Dr. Biswa Nath Datta, Professor of Mathematical Sciences and Distinguished Research Professor at Northern Illinois University has received the 2015 Fulbright-Nehru Distinguished Chair Award by the United States Department of State Bureau of Education and Cultural Affairs with Indian Institute of Technology-Kharagpur (IIT-KGP) as the Home Institute, for the period of four months beginning October 1, 2015.

Earlier, Dr. Datta had received two Senior Fulbright Specialists Awards in 2006 and 2008 in Mongolia and Egypt, respectively, to visit and deliver lectures on his research at National University of Mongolia and at several Universities in Egypt. The Distinguished Chair Award is of the highest level of Fulbright Awards. According to the Fulbright website: "The Fulbright Distinguished Chair Awards are viewed as among the most prestigious appointments in the Fulbright Scholar Program. Candidates should be eminent scholars and have a significant publication and teaching record." In India the award will be administered by the United States-India Educational Foundation (USIEF) which promotes "mutual understanding between the nationals of India and the U.S. through the educational exchange of outstanding scholars, professionals and students."

During the award period, in addition to delivering lectures, workshops, and courses, and conducting research at the home institute, IIT-KGP, Dr. Datta will visit several other Universities and Research Organizations in India, including IIT-Delhi, IIT-Madras, IIT-Bombay, IIT-Gujarat, IIT-Guwahati, Indian Institute of Science, Bangalore, Indian Statistical Institute in Kolkata, Bangalore, and Delhi, to deliver lectures on his current interdisciplinary research on Computational and Optimization Methods for Inverse Quadratic Eigenvalue Problems in Active Vibration Control and Finite Element Model Updating.

JOURNAL ANNOUNCEMENTS

LAA Special Issue in Honor of Rajendra Bhatia

Linear Algebra and Its Applications (*LAA*) is pleased to announce a special issue in honor of Professor Rajendra Bhatia in recognition of his many important contributions in research and books to matrix analysis and on the occasion of his 65^{th} birthday in 2017. *LAA* solicits papers for the special issue within the entire scope of *LAA* with a special emphasis on research topics related to the work of Rajendra Bhatia. The deadline for submissions of papers is March 1, 2016. All submissions will be subject to normal refereeing procedures and the usual standards of *LAA* will be applied. They should be submitted via the Elsevier Editorial System EES (`http://ees.elsevier.com/laa/`), choosing the special issue called "In Honor of Rajendra Bhatia" and the responsible Editor-in-Chief, Peter Šemrl. Authors will have the opportunity to suggest one of the following special editors to handle their submission: Ravindra B. Bapat, Shmuel Friedland, John Holbrook, Roger Horn, Fuad Kittaneh.

Special Issue of Czechoslovak Mathematical Journal in Honour of Miroslav Fiedler

In April 2016 it will be 90 years since Miroslav Fiedler was born, and a special issue of *Czechoslovak Mathematical Journal* will be devoted to this special occasion. Submissions for this special issue are being accepted. To allow for timely refereeing and processing of all submissions, the deadline for contributions is December 31, 2015, and papers are welcome to be sent just in PDF to englis@math.cas.cz.

OBITUARY NOTE

Miroslav Fiedler (April 7, 1926 – November 20, 2015)

With a deep sorrow we inform you that in the morning of November 20, 2015, Professor Miroslav Fiedler passed away peacefully at his home. You may express your sympathy to the family by sending a message to mathinst@math.cas.cz.

Jiří Rákosník, Miroslav Rozložník, Zdeněk Strakoš

BOOK REVIEWS

Linear Algebra, by Juan Jorge Schäffer

World Scientific, 2015, ISBN 978-9814623490, 131 pages.
Reviewed by Douglas Farenick, Department of Mathematics and Statistics, University of Regina, SA, S4S 0A2, Canada, Doug.Farenick@uregina.ca

When this volume arrived in the mail, I was immediately attracted by its appearance. So slim. A treatise on linear algebra, achieved in a mere 131 pages. How did the author do it, I wondered? The preface to this book informs the reader that *Linear Algebra* is intended to be a companion volume to the author's earlier book, titled *Basic Language of Mathematics*, with the goal of "providing a clear, comprehensive, and formally sound presentation of the subject matter." Having not encountered the *Basic Language of Mathematics* before, I flipped to the back of the book to look it up in the reference list to see when it appeared. My first surprise discovery: no bibliography!

Let us, therefore, begin at page 1, as I am always curious about how mathematicians start their books. What words of insight is he or she offering the reader about the subject at hand? What are the mathematical pinnacles that the author is promising to bring the reader to, enticing the reader to join the excursion and then continue forward when the going gets tough? Answer: "Throughout this work we shall assume that a field is given. In order to make plain that this field is fixed, we denote it by $\mathbb{F}$." To be fair, many good (and not so good) books start in this way. But there is something puzzling at the bottom of page 1, where I read that "scalar multiplication has priority over addition, opposition, and subtraction." Opposition? Did I miss something? What does that mean? At the top of page 2, I sense I am in trouble because opposition in a specific linear space $\mathcal{V}$ has its own notation, $\mathrm{opp}^{\mathcal{V}}$. I soon realise that my usual convention of denoting the zero vector by 0, vector addition by +, and scalar multiplication by αv is ambiguous because, in the interest of formal clarity, the author denotes these by $0^{\mathcal{V}}$, $\mathrm{add}^{\mathcal{V}}$, and $\mathrm{mult}^{\mathcal{V}}$, respectively. The very first result of the book, Proposition 11A on page 2 (11A? Really?), asserts (among other things) that $\mathrm{mult}_0^{\mathcal{V}} = (0^{\mathcal{V}})_{\mathcal{V}\to\mathcal{V}}$ and that $\mathrm{mult}_{-1}^{\mathcal{V}} = \mathrm{opp}^{\mathcal{V}}$. While this proposition is a little daunting for the novice, I am pleased, for I now understand that "opposition" means multiplication by the scalar -1. Well, perhaps not. It is likely better to say that $\mathrm{opp}^{\mathcal{V}}$ denotes the map that takes an element of $\mathcal{V}$ (commonly known as a "vector") to its additive inverse, and that Proposition 11A establishes that this map coincides with multiplication by the scalar -1. This is, I think, formally correct. Maybe. Actually, I'm not certain and I'm too scared to ask the professor.

Skipping ahead a little from page 2 and opening the book on a random page—page 10 in my case—I find that I am in the section on linear transformations. That's a relief. I am, after all, an operator theorist. Perhaps those commutative monoids on page 2 can be forgotten about. But what I find, at the bottom of the page, is Lemma 12H: $L^{<}(\mathcal{C}+\mathcal{D}) \supset L^{<}(\mathcal{C}) + L^{<}(\mathcal{D})$ for all $\mathcal{C}, \mathcal{D} \in \mathfrak{B}(\mathcal{W})$. Hmm... Better not skip ahead.

Much later, on page 80, the reader is informed that "when dealing with finite-dimensional spaces, as well as other results whose proofs relied on them, Propositions 22F.L and 22F.R and •Theorem 22G .... may be cited without the 'bullet' •." At first I thought these were major typographical errors, just like the "up-down" arrows in the margin of page 80, but maybe not. As far as I can tell, the "bullet" is first introduced on page 26, and it may signify a result that is proved using the Axiom of Choice. Looking in the index might help. Nope. The Axiom of Choice is not in the index. Nor is "bullet" in the index or in the notational index.

Alright, it is true that my little bit of fun at the author's expense (I am sorry!) does not address the qualities or content of the book. Schäffer is giving us a rigorous and formal treatment of vector spaces and their duals. Indeed, it is important, for example, that our students understand the additive inverse in vector spaces and its relation to multiplication by -1. Schäffer takes the trouble to establish notation to make such discussions possible and to provide rigorous proofs of these types of assertions. But is it worth it? Isn't it enough to denote the zero vector of V by 0 and denote $\mathrm{add}^{\mathcal{V}}$ by +, but forewarn the reader that $v_1 + v_2$ is the outcome of a certain binary operation specific to the given vector space $\mathcal{V}$? Informally, this would be intuitive and sufficient; however, this book is about formality.

And because of the emphasis on formality, this book is not easy to read, whether it be read from page 1 through to the end, or read casually by picking it up and finding your favourite theorem. But which favourite theorem? The spectral theorem? The Cayley-Hamilton theorem? These core results are not treated. (Indeed, eigenvalues are not mentioned at all.)

I do not recommend this book for the traditional linear algebra student or scholar. There is merit in the author's desire to get to the foundations of the theory of vector spaces, but the presentation and notation in this book were a challenge for me. A fuller discourse on the book's objectives would have helped considerably, and maybe even would have won over this reviewer.

***Linear Algebra Done Right, Third Edition*, by Sheldon Axler**

Springer International Publishing, 2015, ISBN 978-3-319-11079-0, xvii+34 pages
Reviewed by Mitja Mastnak, Department of Mathematics & Computing Science, Saint Mary's University, Halifax, NS, B3H 3C3, Canada, mmastnak@cs.smu.ca

I am rarely so torn about my decision on whether to recommend a book or not. For most part, I liked the presentation and the order of material very much. But unfortunately there were some issues (one in particular) that I had a hard time getting over. I will first focus on the many positives and get to the one major issue at the end.

The writing is very good – about as good as it could possibly be at places. Personally, I would prefer if the author omitted explicit statements on how much better the approach he has taken is compared to some (perhaps more standard) ways of doing things; but the slightly less than humble title of the book should already sufficiently prepare the reader for this.

I like the order of topics. The exposition starts with abstract vector spaces, then talks about linear maps between vector spaces and only then mentions matrices and their relationship with linear maps. The exposition then proceeds by discussing polynomials before moving on to the structure of linear maps, where the existence of eigenvalues, and finally the Jordan canonical form, get established without even mentioning determinants. The discussion of determinants is relegated to the very last section of the book. I like this approach very much. It is very natural and it will prepare students for follow-up courses on functional analysis and abstract algebra (where they will encounter, for example, in the study of field extensions, many similar techniques). I love the fact that important topics such as quotient spaces and invariant subspaces, that are all too often glossed over in linear algebra courses, get good coverage. The only thing missing in the exposition, in my opinion, is a "tip of the hat" to fields other than $\mathbb{R}$ and $\mathbb{C}$, and to finite fields in particular. As already mentioned above, I very much like the determinant-free approach to most things as long as it makes sense. Unfortunately, the author does not "quit while ahead" on this approach and takes it slightly too far for my personal taste. In particular, the notion of the characteristic polynomial is defined in terms of eigenvalues; I find this unnatural and would have a hard time doing it this way in my classroom.

Let me now focus on the worst feature of the book. I wish that the publisher had reprinted the second edition instead of wasting time and effort on a failed attempt to fix something that was not broken. The excellent exposition now gets drowned by the unfortunate typesetting. The typesetters went completely overboard with the use of colour and fonts. There are pages in this book (I kid you not – check page 20, for example) where I counted the use of no fewer than 6 colours (not counting the black print and white background) and 5 different fonts. The kaleidoscope of font and colour is so overwhelming that it becomes almost impossible to focus on the material. I will finish this tirade with my favourite pet peeve. Somehow, despite all the abuse of colour and fonts, a use for the mathbb font was not found: so $\mathbf{R}, \mathbf{C}$, and $\mathbf{F}$ are used instead of the standard $\mathbb{R}, \mathbb{C}$, and $\mathbb{F}$.

If you are able to get over the glossy magazine look, or if you are able to use the second edition, then I warmly recommend this textbook to you. As a textbook, the book is in some danger of falling into the gap between "too advanced for a standard second linear algebra course" and "not sufficiently advanced for an advanced linear algebra course." I hope that this does not happen. If you are among those fortunate people who get to teach a "math honours" version of second-year linear algebra, then this book is an excellent option.

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to *IMAGE* Problems 53-1 and 53-2 and to all Problems in issue 54 of *IMAGE*. Six new problems are on the last page; solutions are invited.

Problem 53-1: Matrix Proof of Dickson's Equality

Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let p, q, r, s be positive integers. Present a matrix proof of the identity

$$(p^2 - bq^2)(r^2 - bs^2) = (pr + bqs)^2 - b(qr + ps)^2 = (pr - bqs)^2 - b(qr - ps)^2.$$

Solution 53-1.1 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Taking the determinants of both sides of

$$\begin{bmatrix} p & bq \\ q & p \end{bmatrix} \begin{bmatrix} r & bs \\ s & r \end{bmatrix} = \begin{bmatrix} pr + bqs & b(qr + ps) \\ qr + ps & pr + bqs \end{bmatrix}$$

yields $(p^2 - bq^2)(r^2 - bs^2) = (pr + bqs)^2 - b(qr + ps)^2$. The second identity of the problem is obtained by taking the determinants of $\left[\begin{smallmatrix} p & bq \\ q & p \end{smallmatrix}\right] \left[\begin{smallmatrix} r & -bs \\ -s & r \end{smallmatrix}\right] = \left[\begin{smallmatrix} pr-bqs & b(qr-ps) \\ qr-ps & pr-bqs \end{smallmatrix}\right]$. □

Remark. The same technique can be used to obtain

$$(x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3)(y_1^3 + y_2^3 + y_3^3 - 3y_1y_2y_3) = t_1^3 + t_2^3 + t_3^3 - 3t_1t_2t_3$$

where $t_1 = (x_1y_1 + x_2y_3 + x_3y_2)$, $t_2 = (x_1y_2 + x_2y_1 + x_3y_3)$, and $t_3 = (x_1y_3 + x_2y_2 + x_3y_1)$, by matrix multiplication

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \\ y_3 & y_1 & y_2 \\ y_2 & y_3 & y_1 \end{bmatrix}.$$

Similarly, Euler's four-square identity

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) &= (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 \\ &\quad + (x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2)^2 + (x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1)^2\end{aligned}$$

can be derived by working on

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_2 & -x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} y_1 & -y_2 & -y_3 & -y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_2 & y_3 & -y_2 & y_1 \end{bmatrix}.$$

Also solved by the proposer.

Problem 53-2: AGM Inequality for Determinants

Proposed by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Let $X_1, \ldots, X_n \in \mathbb{R}^{m\times m}$ be positive definite. Give a non-analytic proof that $\det\left(\frac{1}{n}\sum_{k=1}^n X_k\right) \geq \left(\prod_{k=1}^n \det X_k\right)^{1/n}$.

Solution 53-2.2 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

The case for $n = 1$ is trivial. When $n = 2$, there exists a nonsingular matrix C such that $D_1 = C^TX_1C$ and $D_2 = C^TX_2C$ are both diagonal matrices with positive diagonal entries because two real symmetric matrices are simultaneously diagonalizable by congruence if at least one of them is positive definite; see, e.g., Theorem 8.7.1, page 461 in [G.H. Golub and C.F. Van Loan, *Matrix Computations*, 3rd edition, Johns Hopkins University Press, 1996]. By the AGM inequality for scalars, we have $\left(\frac{D_1(i,i)+D_2(i,i)}{2}\right)^2 \geq D_1(i,i)D_2(i,i)$ for $i = 1, \ldots, m$. Consequently, we obtain that $\left[\det\bigl((D_1 + D_2)/2\bigr)\right]^2 \geq \det D_1 \det D_2$, and hence

$$\left(\det \frac{X_1 + X_2}{2}\right)^2 = (\det C)^{-4}\left(\det \frac{D_1 + D_2}{2}\right)^2 \geq (\det C)^{-4} \det D_1 \det D_2 = \det X_1 \det X_2.$$

By induction, we conclude that the AGM inequality holds for $n = 2^q$ ($q = 0, 1, \ldots$) because

$$\left(\det \frac{1}{2}\left(\frac{1}{2^{q-1}}\sum_{k=1}^{2^{q-1}} X_k + \frac{1}{2^{q-1}}\sum_{k=2^{q-1}+1}^{2^q} X_k\right)\right)^{2^q} \geq \left(\det \frac{1}{2^{q-1}}\sum_{k=1}^{2^{q-1}} X_k \cdot \det \frac{1}{2^{q-1}}\sum_{k=2^{q-1}+1}^{2^q} X_k\right)^{2^{q-1}}$$
$$\geq \left(\prod_{k=1}^{2^{q-1}} \det X_k\right)\left(\prod_{k=2^{q-1}+1}^{2^q} \det X_k\right) = \prod_{k=1}^{2^q} \det X_k.$$

Finally, if n is not a perfect power of two, there exists a unique integer q such that $2^{q-1} < n < 2^q$. Let $A = \left(\sum_{k=1}^n X_k\right)/n$, and set $X_{n+1} = \cdots = X_{2^q} = A$. Then

$$(\det A)^{2^q} = \left(\det \frac{\sum_{k=1}^{2^q} X_k}{2^q}\right)^{2^q} \geq \prod_{k=1}^{2^q} \det X_k = (\det A)^{2^q-n} \prod_{k=1}^{n} \det X_k.$$

Therefore $(\det A)^n \geq \prod_{k=1}^n \det X_k$. □

Remark. The solution uses the technique of forward–backward induction, which was introduced by Cauchy when proving the AGM inequality on scalars.

Editorial note: A solution to IMAGE *problem 53-2, namely 53-2.1, was published in issue 54 of* IMAGE*. The present solution differs from 53-2.1 and was received too late to be included in issue 54.*

Problem 54-1: EP Product of EP Matrices

Proposed by Johanns DE ANDRADE BEZERRA, *Natal, RN, Brazil*, pav.animal@hotmail.com

A complex matrix X is EP or range-Hermitian if $\mathrm{Im}(X) = \mathrm{Im}(X^*)$. Let A, B and AB be EP matrices. Show that BA is an EP matrix only if $\mathrm{Im}(AB) = \mathrm{Im}(BA)$. Is the converse true?

Editorial note: The original formulation of the problem asked to prove that BA is an EP matrix if and only if $\mathrm{Im}(AB) = \mathrm{Im}(BA)$*. However, as shown in the solution by Meiyue* SHAO*,* $\mathrm{Im}(AB) = \mathrm{Im}(BA)$ *does not imply BA is EP.*

Solution 54-1.1 by Nebojša Č. DINČIĆ, *Faculty of Sciences and Mathematics, P.O. Box 224, University of Niš, Serbia*, ndincic@hotmail.com

Only the necessity part. However, a more general result, valid for operators with closed range on Hilbert space, will be proven. We will use the following theorem (see [1, Theorem 1] or [2, Theorem 7.2.1] for a proof).

Theorem. Let $A, B \in \mathcal{L}(H)$ be EP operators with closed ranges. Then the following statements are equivalent.
a) *AB is an EP operator.*
b) $\mathrm{Im}(AB) = \mathrm{Im}(A) \cap \mathrm{Im}(B)$ *and* $\mathrm{Ker}(AB) = \underline{\mathrm{Ker}(A) + \mathrm{Ker}(B)}$.
c) $\mathrm{Im}(AB) = \mathrm{Im}(A) \cap \mathrm{Im}(B)$ *and* $\mathrm{Ker}(AB) = \overline{\mathrm{Ker}(A) + \mathrm{Ker}(B)}$.

Remark that for finite-dimensional Hilbert spaces H the conditions (b) and (c) are the same while in general the condition (b) is stronger than (c) (see [1]).

Since A, B, and AB are EP operators, the Theorem gives $\mathrm{Im}(AB) = \mathrm{Im}(A) \cap \mathrm{Im}(B)$ and $\mathrm{Ker}(AB) = \mathrm{Ker}(A) + \mathrm{Ker}(B)$. Also, by definition, $\overline{\mathrm{Im}(AB)} = \overline{\mathrm{Im}\big((AB)^*\big)} = \mathrm{Ker}(AB)^\perp$.

Suppose that BA is EP. By the Theorem, $\mathrm{Im}(BA) = \mathrm{Im}(B) \cap \mathrm{Im}(A)$ and $\mathrm{Ker}(BA) = \mathrm{Ker}(B) + \mathrm{Ker}(A)$, so we have $\mathrm{Im}(BA) = \mathrm{Im}(AB)$ and $\mathrm{Ker}(BA) = \mathrm{Ker}(AB)$. □

References

[1] D.S. Đorđević, Products of EP operators on Hilbert spaces, *Proc. Amer. Math. Soc.* 129 (6) (2001) 1727–1731.

[2] D.S. Đorđević and V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

Solution 54-1.2 by Meiyue SHAO, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Lemma (see [1]). *A complex matrix X is EP if and only if X has a Schur decomposition of the form $X = Q\left[\begin{smallmatrix} X_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^*$ where Q is unitary and X_1 is nonsingular.*

By the lemma, there exists a unitary matrix Q_1 and a nonsingular matrix A_{11} such that $A = Q_1 \operatorname{diag}(A_{11}, 0) Q_1^*$. We partition $Q_1^*BQ_1$ conformally as $Q_1^*BQ_1 = \left[\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right]$. Then

$$AB = Q_1 \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ 0 & 0 \end{bmatrix} Q_1^* \qquad \text{and} \qquad \mathrm{B}^*\mathrm{A}^* = \mathrm{Q}_1 \begin{bmatrix} B_{11}^*A_{11}^* & 0 \\ B_{12}^*A_{11}^* & 0 \end{bmatrix} \mathrm{Q}_1^*.$$

As $\mathrm{Im}(AB) = \mathrm{Im}(B^*A^*)$, we obtain that $B_{12} = 0$.

For the necessity part, we assume that BA is EP. Applying the same analysis above to BA, we conclude that $B_{21} = 0$, as

$$BA = Q_1 \begin{bmatrix} B_{11}A_{11} & 0 \\ B_{21}A_{11} & 0 \end{bmatrix} Q_1^* \qquad \text{and} \qquad \mathrm{A}^*\mathrm{B}^* = \mathrm{Q}_1 \begin{bmatrix} A_{11}^*B_{11}^* & A_{11}^*B_{21}^* \\ 0 & 0 \end{bmatrix} \mathrm{Q}_1^*.$$

Then $B = Q_1 \operatorname{diag}(B_{11}, B_{22})Q_1^*$, indicating that both B_{11} and B_{22} are EP matrices. By the lemma, we can block diagonalize B_{11} as $B_{11} = Q_2 \operatorname{diag}(\widetilde{B}_{11}, 0)Q_2^*$, where Q_2 is unitary and $\widetilde{B}_{11}$ is nonsingular. By partitioning $Q_2^*A_{11}Q_2$ conformally as

$$Q_2^*A_{11}Q_2 = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}$$

and noticing that both $A_{11}B_{11}$ and $B_{11}A_{11}$ are EP, we conclude that $\widetilde{A}_{12} = 0$ and $\widetilde{A}_{21} = 0$ using the same technique. Let $Q = Q_1 \operatorname{diag}(Q_2, I)$. Then

$$A = Q \begin{bmatrix} \widetilde{A}_{11} & & \\ & \widetilde{A}_{22} & \\ & & 0 \end{bmatrix} Q^* \qquad \text{and} \qquad \mathrm{B} = \mathrm{Q} \begin{bmatrix} \widetilde{B}_{11} & & \\ & 0 & \\ & & B_{22} \end{bmatrix} \mathrm{Q}^*.$$

As $\widetilde{A}_{11}$ and $\widetilde{B}_{11}$ are nonsingular, we obtain that $\mathrm{Im}(AB) = \mathrm{Im}(BA)$.

The converse is false in general. Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and let $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] = AB$. Then $\mathrm{Im}(A) = \mathrm{Im}(A^*) = \mathrm{span}\{e_1, e_2\}$, $\mathrm{Im}(B) = \mathrm{Im}(B^*) = \mathrm{span}\{e_1\}$, indicating that A, B, and AB are all EP matrices. However, since $BA = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $A^*B^* = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, we have $\mathrm{Im}(BA) = \mathrm{Im}(AB) = \mathrm{span}\{e_1\} \neq \mathrm{Im}(A^*B^*)$. Therefore BA is not an EP matrix. □

Reference

[1] M. Pearl, On normal and EP_r matrices, *Michigan Math. J.* 6 (1) (1959) 1–5.

Problem 54-2: An Identity Involving the Inverse of LCL^T

Proposed by Ravindra Bapat, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in
and Aloke Dey, *Indian Statistical Institute, New Delhi, India*, aloke.dey@gmail.com

Let C be a $v \times v$ real symmetric matrix of rank $v-1$ and let L be a $(v-1) \times v$ matrix of rank $v-1$ such that $\mathcal{R}(L) \subseteq \mathcal{R}(C)$ where $\mathcal{R}$ denotes the row space. Show that (i) LCL^{T} is nonsingular, and (ii) $L = LL^{\mathrm{T}}(LCL^{\mathrm{T}})^{-1}LC$.

Solution 54-2.1 by Johanns de Andrade Bezerra, *Natal, RN, Brazil*, pav.animal@hotmail.com

Proof of (i). $\mathcal{R}(L) \subseteq \mathcal{R}(C)$ for a real symmetric matrix C implies $\mathrm{Im}(L^T) \subseteq \mathrm{Im}(C)$. Hence $(\mathrm{Im}(L^T))^\perp \supseteq (\mathrm{Im}(C))^\perp$, giving $\mathrm{Ker}(L) \supseteq \mathrm{Ker}(C)$. Since $\mathrm{rank}(C) = v-1$, it follows that $\dim \mathrm{Ker}(C) = 1$. Similarly, $\mathrm{rank}(L) = v-1$ implies $\dim \mathrm{Ker}(L) = 1$ and hence $\mathrm{Ker}(L) = \mathrm{Ker}(C)$. Thus, consider $LCL^Tw = u$, with $w \in \mathbb{R}^{v-1} \backslash \{0\}$. Note that $L^Tw \neq 0$ since L^T is of order $v \times (v-1)$ and $\mathrm{rank}(L^T) = v-1$. By the above, $L^Tw \in \mathrm{Im}(C)$ and since $\mathrm{Im}(C) \cap \mathrm{Ker}(C) = \{0\}$, it follows that $CL^Tw \neq 0$. Also, $LCL^Tw \neq 0$ since, otherwise, $CL^Tw \in \mathrm{Ker}(L) = \mathrm{Ker}(C)$ but $CL^Tw \in \mathrm{Im}(C)$, and so $CL^Tw = 0$, a contradiction. Hence, $u \neq 0$ for each $w \in \mathbb{R}^{v-1} \backslash \{0\}$, and therefore LCL^T is nonsingular.

Proof of (ii). Let $A = LL^T(LCL^T)^{-1}LC$. Then $AL^T = LL^T(LCL^T)^{-1}LCL^T = LL^T$, which implies $(A-L)L^T = 0$. Moreover, $\mathrm{rank}(C) = \mathrm{rank}(L^T)$ implies, by the proof of (i), $\mathrm{Im}(L^T) = \mathrm{Im}(C)$, and so $(A-L)x = 0$ for each $x \in \mathrm{Im}(C)$. Since $\mathrm{Ker}(L) = \mathrm{Ker}(C)$, it follows that $(A-L)y = 0$ for each $y \in \mathrm{Ker}(C)$, and as $\mathrm{Im}(C) \oplus \mathrm{Ker}(C) = \mathbb{R}^v$, we conclude that $(A-L)z = 0$ for any $z \in \mathbb{R}^v$. That is, $A = L$. □

Solution 54-2.2 by Nebojša Č. Dinčić, *Faculty of Sciences and Mathematics, P.O. Box 224, University of Niš, Serbia*, ndincic@hotmail.com

There exists orthonormal basis $e_1, \ldots, e_v$ of $\mathbb{R}^v$ such that, with respect to this basis, C has diagonal form:

$$C = \operatorname{diag}(\lambda_1, \ldots, \lambda_{v-1}, 0) = \left[\begin{smallmatrix} C_1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \qquad \lambda_k \neq 0, \quad k \in \{1, \ldots, v-1\}.$$

The relation for row spaces, $\mathcal{R}(L) \subseteq \mathcal{R}(C)$ is equivalent to the relation for column spaces $\mathcal{C}(L^T) \subseteq \mathcal{C}(C)$, which is further equivalent to $LCC^\dagger = L$, where $C^\dagger$ is the Moore-Penrose pseudoinverse of the matrix C.

Write $L = [L_1\ L_2]$, where $L_1 \in \mathbb{R}^{(v-1)\times(v-1)}$, $L_2 \in \mathbb{R}^{(v-1)\times 1}$. Since $CC^\dagger = \operatorname{diag}(I_{v-1}, 0)$, the relation $LCC^\dagger = L$ implies

$$LCC^\dagger = [L_1\ L_2]\begin{bmatrix} I_{v-1} & 0 \\ 0 & 0 \end{bmatrix} = [L_1\ 0] = L = [L_1\ L_2],$$

so $L_2 = 0$ and matrix $L = [L_1\ 0]$. Now, $\operatorname{rank} L = v-1$ implies L_1 is invertible. Hence,

(i) $LCL^T = [L_1\ 0]\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} L_1^T \\ 0 \end{bmatrix} = L_1C_1L_1^T \in \mathbb{R}^{(v-1)\times(v-1)}$, therefore LCL^T is invertible.

(ii)
$$\begin{aligned} LL^T(LCL^T)^{-1}LC &= [L_1\ 0]\begin{bmatrix} L_1^T \\ 0 \end{bmatrix}(L_1C_1L_1^T)^{-1}[L_1\ 0]\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} = L_1L_1^T(L_1C_1L_1^T)^{-1}[L_1C_1\ \ 0] \\ &= [L_1L_1^T(L_1^T)^{-1}C_1^{-1}L_1^{-1}L_1C_1\ \ 0] = [L_1\ 0] = L. \qquad \square \end{aligned}$$

Solution 54-2.3 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

More generally, we assume L has size $r \times v$ and that C and L have rank r. Since the row space of a real matrix A equals the column space of A^{T} which equals the orthogonal complement of the kernel of A, the hypothesis $\mathcal{R}(L) \subseteq \mathcal{R}(C)$ is equivalent to $\operatorname{Ker}(C) \subseteq \operatorname{Ker}(L)$. Since both kernels have dimension $v-r$, we have $\operatorname{Ker}(C) = \operatorname{Ker}(L)$. If $LCL^{\mathrm{T}}x = 0$ then $y = CL^{\mathrm{T}}x \in \operatorname{Ker}(C)$. Hence,

$$\langle y, y\rangle = \langle CL^{\mathrm{T}}x, y\rangle = \langle L^{\mathrm{T}}x, Cy\rangle = 0,$$

and so $CL^{\mathrm{T}}x = y = 0$. Thus $z = L^{\mathrm{T}}x \in \operatorname{Ker}(C) = \operatorname{Ker}(L)$, and so

$$\langle z, z\rangle = \langle L^{\mathrm{T}}x, z\rangle = \langle x, Lz\rangle = 0.$$

Therefore, $L^{\mathrm{T}}x = z = 0$, which implies $x = 0$ since L^{T} is injective. This proves (i). For (ii), we have

$$LL^{\mathrm{T}}(LCL^{\mathrm{T}})^{-1}LCL^{\mathrm{T}} = LL^{\mathrm{T}},$$

and so $LL^{\mathrm{T}}(LCL^{\mathrm{T}})^{-1}LC = L$ on the range of L^{T}. The orthogonal complement of this range is the kernel of L and hence the kernel of C as well. The identity in (ii) is clearly true on this subspace and hence everywhere on $\mathbb{R}^v$. $\square$

Solution 54-2.4 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Since $\dim \mathcal{R}(L) = \dim \mathcal{R}(C) = v-1$, we have $\mathcal{R}(L) = \mathcal{R}(C)$, with L's rows as a basis. Therefore, C can be factorized as $C = XL$ where $X \in \mathbb{R}^{v\times(v-1)}$. The symmetry of C implies that $C = C^T = L^TX^T$, which, combined with $v-1 = \operatorname{rank}(C) \le \operatorname{rank}(X) \le v-1$, gives $\mathcal{R}(X^T) = \mathcal{R}(C) = \mathcal{R}(L)$. We can hence factorize X^T as $X^T = M^TL$, where $M \in \mathbb{R}^{(v-1)\times(v-1)}$. Thus we obtain a factorization $C = L^TML$. It is obvious that M is nonsingular, as $v-1 = \operatorname{rank}(C) \le \operatorname{rank}(M) \le v-1$. Because LL^T is positive definite, $LCL^T = (LL^T)M(LL^T)$ is nonsingular. Furthermore, we have

$$LL^T(LCL^T)^{-1}LC = LL^T(LL^T)^{-1}M^{-1}(LL^T)^{-1}LL^TML = L. \qquad \square$$

Also solved by the proposers.

Problem 54-3: When Is the Adjugate a One-to-One Operator?

Proposed by Dennis S. Bernstein, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu and Khaled Aljanaideh, *University of Michigan, Ann Arbor, MI, USA*, khaledfj@umich.edu

Determine the values of n for which the adjugate operator is one-to-one on the set of $n \times n$ nonsingular real matrices.

Solution 54-3.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

The adjugate operator is one-to-one if and only if n is even. We denote the adjugate of A by A'. Suppose n is even and A, B are $n \times n$ real nonsingular matrices such that $A' = B'$. Then

$$(\det A)B = (\det A)IB = AA'B = AB'B = A(\det B)I = (\det B)A.$$

Taking the determinant of both sides yields $(\det A)^{n-1} = (\det B)^{n-1}$, and so $A = B$ since $n-1$ is odd and the determinants are real. Suppose n is odd and A, B are $n \times n$ real nonsingular matrices such that $B = -A$. Then $\det B = -\det A$ since n is odd, and $A^{-1} = -B^{-1}$. Therefore, $A' = (\det A)A^{-1} = (-\det B)(-B^{-1}) = (\det B)B^{-1} = B'$. $\square$

Solution 54-3.2 by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, rhorn@math.utah.edu

If $\operatorname{adj} A = \operatorname{adj} B$, then $(\det A)A^{-1} = (\det B)B^{-1}$ and hence $A = cB$, in which $c = (\det A)/(\det B)$. Then

$$\det A = \det(cB) = c^n \det B = \tfrac{(\det A)^n}{(\det B)^n} \det B = \tfrac{(\det A)^n}{(\det B)^{n-1}}.$$

It follows that $1 = \frac{(\det A)^{n-1}}{(\det B)^{n-1}} = c^{n-1}$. Since c is real (A and B are real), $c = 1$ if n is even and $c = \pm 1$ if n is odd and greater than 1. If $n = 1$, then $\operatorname{adj} A = 1$ for all A. If n is odd and greater than 1, then

$$\operatorname{adj}(-A) = (\det(-A))(-A)^{-1} = (-\det A)(-A^{-1}) = (\det A)A^{-1} = \operatorname{adj} A.$$

We conclude that the adjugate operator is one-to-one if and only if n is even. □

Solution 54-3.3 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

We use the convention that the adjugate of a scalar (i.e., 1×1 matrix) is 1. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. If n is odd, the adjugate operator is not one-to-one, as $\operatorname{adj}(-A) = (-1)^{n-1}\operatorname{adj}(A) = \operatorname{adj}(A)$ by definition. So we only need to consider the case when n is even. From $\operatorname{adj}(A) = \det(A)A^{-1}$, we immediately obtain that $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$ and $\operatorname{adj}(\operatorname{adj}(A)) = \det(A)^{n-2}A$. Because $\det(A)$ is real and $n-1$ is odd, we have $\det(A) = [\det(\operatorname{adj}(A))]^{1/(n-1)}$. Then $A = \operatorname{adj}(\operatorname{adj}(A))/[\det(A)]^{n-2} = \operatorname{adj}(\operatorname{adj}(A))/[\det(\operatorname{adj}(A))]^{(n-2)/(n-1)}$ is uniquely determined by $\operatorname{adj}(A)$. Therefore, the adjugate operator is one-to-one if and only if n is a positive even number. □

Also solved by Tamás F. Görbe *and the proposers.*

Problem 54-4: When k-Cycles Imply Fixed Points

Proposed by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

What conditions on the integers k, n and the field $\mathbb{F}$ ensure that each affine transformation T on an n-dimensional $\mathbb{F}$-vector space V has a fixed point in V if it induces a k-cycle (i.e., if there exist $k \geq 1$ and $v \in V$ such that $T^{(k)}(v) = v$)?

Solution 54-4.1 by the proposer Eugene A. Herman

A necessary and sufficient condition is that either $\mathbb{F}$ has characteristic zero or its characteristic does not divide k.

Suppose $\mathbb{F}$ has characteristic p and p divides k. For a nonzero vector $t \in V$, define $T(v) = v + t$ for all $v \in V$. Then $T^{(k)}(v) = v + kt = v$ (i.e., every point is part of a k-cycle), but T has no fixed point.

For the converse, assume there exists $w \in V$ such that $T^{(k)}(w) = w$. Since T is affine, there is a linear transformation A and a vector t such that $T(v) = A(v) + t$ for all $v \in V$. Hence,

$$T^{(k)}(v) = A^{(k)}(v) + \sum_{j=0}^{k-1} A^{(j)}(t) \quad \text{for all } v \in V$$

and so $T^{(k)}(w) = w$ is equivalent to $\sum_{j=0}^{k-1} A^{(j)}(t) = (I - A^{(k)})(w)$, which we write as

$$\sum_{j=0}^{k-1} A^{(j)}(t - (I - A)(w)) = 0. \tag{$*$}$$

Furthermore, T has a fixed point if and only if $t \in \operatorname{range}(I - A)$. This is guaranteed to be true if $I - A$ is invertible; so we assume it is not. Hence, 1 is an eigenvalue of A. Let m denote the dimension of the eigenspace for this eigenvalue. Since each matrix is similar to its transpose, also $\dim(I - A^{\mathrm{T}}) = m$. Therefore, there is a basis of V such that the matrix of A relative to this basis is $\left[\begin{smallmatrix} I_m & 0 \\ C & B \end{smallmatrix}\right]$. For convenience, we denote this matrix by A as well. Also, we denote the coordinate vectors that represent t and w relative to this basis by t and w, respectively. We partition all coordinate vectors $v = [v_1, v_2]^{\mathrm{T}}$ to match the partitioning of the matrix A. Thus,

$$A^j = \begin{bmatrix} I_m & 0 \\ Z_j & B^j \end{bmatrix} \quad \text{and so} \quad \sum_{j=0}^{k-1} A^j = \begin{bmatrix} kI_m & 0 \\ Z & \sum_{j=0}^{k-1} B^j \end{bmatrix}$$

for suitable matrices Z_j and Z. By $(*)$, there exists $w = [w_1, w_2]^{\mathrm{T}}$ such that

$$\begin{bmatrix} kI_m & 0 \\ Z & \sum_{j=0}^{k-1} B^j \end{bmatrix} \left(\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -C & I_{n-m} - B \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first row of this partitioned equation yields $kt_1 = 0$, and so $t_1 = 0$ by our assumption on k and the scalar field. We use this to show that $t = (I - A)v$ has a solution v. That equation can now be written as

$$\begin{bmatrix} 0 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -C & I_{n-m} - B \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -Cv_1 + (I_{n-m} - B)v_2 \end{bmatrix}.$$

If $I_{n-m} - B$ were singular, then 1 would be an eigenvalue of B, which contradicts our assumption that m is the dimension of the eigenspace for 1 with respect to the matrix A. Hence $I_{n-m} - B$ is nonsingular, and so $t_2 = -Cv_1 + (I_{n-m} - B)v_2$ has a solution $[v_1, v_2]^{\mathrm{T}}$. In fact, for every choice of v_1 there is a unique solution for v_2. □

Solution 54-4.2 by Omran KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

We will prove that the necessary and sufficient condition is that k is not a multiple of the characteristic p of the field $\mathbb{F}$.

Necessity: Let p be the characteristic of the field $\mathbb{F}$, and suppose that $p|k$ (in particular $p > 0$). In this case there is an affine transformation T having a k-cycle but no fixed points: any translation $T(v) = v + a$, where a is a fixed non-zero vector, would do. So, a necessary condition is that $p \nmid k$.

Sufficiency: Suppose that $p \nmid k$, and consider an affine transformation T having a k-cycle. There is a vector $v \in V$ such that $T^{(k)}v = v$. In this case we consider the vector u defined by

$$u = \frac{1}{k} \sum_{j=0}^{k-1} T^{(j)} v$$

(here $1/k$ is defined because $p \nmid k$). Since T is affine, there is a linear map L and a vector b with $T(x) = b + L(x)$ and so, for arbitrary vectors $a_1, \ldots, a_k$,

$$T\left(\frac{1}{k}\sum_{j=1}^{k} a_j\right) = b + L\left(\frac{1}{k}\sum_{j=1}^{k} a_j\right) = b + \frac{1}{k}\sum_{j=1}^{k} L(a_j) = b + \frac{1}{k}\sum_{j=1}^{k}(-b + T(a_j)) = \frac{1}{k}\sum_{j=1}^{k} T(a_j).$$

This gives $T(u) = u$. □

Editorial note: If $n = 0$, then $\mathbb{F}^n = 0$ and every $k \geq 1$ solves the problem. We thank Meiyue SHAO *for pointing this out.*

Problem 54-5: Inertia Formula for a Partitioned Hermitian Matrix

Proposed by Yongge TIAN, *Central University of Finance and Economics, Beijing, China*, yongge.tian@gmail.com

Let A and C be two positive semidefinite Hermitian matrices of orders m and n, respectively, and let $B \in \mathbb{C}^{m\times n}$. Show that the inertia of $T = \left(\begin{smallmatrix} A & B \\ B^* & -C \end{smallmatrix}\right)$ satisfies

$$i_+(T) = \operatorname{rank}(A\,|\,B), \quad i_-(T) = \operatorname{rank}(B^*\,|\,C) \quad \text{and} \quad \mathrm{i_0(T) = m + n - rank(A\,|\,B) - rank(B^*\,|\,C)}.$$

Solution 54-5.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

We show that $i_+(T) = \operatorname{rank}(A\,|\,B)$; the formula $i_-(T) = \operatorname{rank}(B^*\,|\,C)$ will follow by symmetry, and $i_+(T) + i_-(T) + i_0(T) = m + n$ yields $i_0(T)$. First we consider the special case when A and C are both zero matrices. Let $r = \operatorname{rank}(B)$. Then B^*B also has rank r and so has exactly r positive eigenvalues. For any nonzero value of λ, we use the Schur complement to find the characteristic polynomial of T:

$$\det(T - \lambda I) = \det\left[\begin{smallmatrix} -\lambda I & B \\ B^* & -\lambda I \end{smallmatrix}\right] = \det(-\lambda I)\det(-\lambda I - \; B^*(-1/\lambda)IB) = \tfrac{(-\lambda)^m}{\lambda^n}\det(B^*B - \lambda^2 I).$$

Thus, T has exactly r positive and r negative eigenvalues. The formulas in the problem are also readily confirmed when $m = n = 1$. If $i_+(T) = \operatorname{rank}(A\,|\,B)$ is not always true, let T be a counterexample of smallest size. So A and C are not both zero and $m + n \geq 3$. Suppose first that $A \neq 0$. Let P be a unitary matrix such that $P^*AP = \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where D is diagonal with positive diagonal entries. Then

$$\begin{bmatrix} P^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^* & -C \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^*AP & P^*B \\ B^*P & -C \end{bmatrix}.$$

We may use this new matrix in place of T, since it has the same eigenvalues as T and

$$\operatorname{rank}(P^*AP \mid P^*B) = \operatorname{rank}\left(P^* \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}\right) = \operatorname{rank}(A \mid B).$$

To simplify notation, we denote the new matrix again by $T = \begin{bmatrix} A & B \\ B^* & -C \end{bmatrix}$. Matching the partition of $A = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, we write $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and define $X = \begin{bmatrix} D^{-1}B_1 \\ 0 \end{bmatrix}$ to produce the *-congruence

$$T_1 = \begin{bmatrix} I & 0 \\ -X^* & I \end{bmatrix} \begin{bmatrix} A & B \\ B^* & -C \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B - AX \\ B^* - X^*A & X^*AX - X^*B - B^*X - C \end{bmatrix}.$$

By Sylvester's theorem, $i(T_1) = i(T)$. Since $B - AX = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$ and

$$X^*AX - X^*B - B^*X = B_1^*D^{-1}B_1 - B_1^*D^{-1}B_1 - B_1^*D^{-1}B_1 = -B_1^*D^{-1}B_1,$$

we have

$$T_1 = \begin{bmatrix} D & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & B_2^* & -B_1^*D^{-1}B_1 - C \end{bmatrix} \quad \text{or} \quad T_1 = \begin{bmatrix} D & 0 \\ 0 & -B_1^*D^{-1}B_1 - C \end{bmatrix},$$

depending on whether A is singular or nonsingular. In the latter case, T_1 is block diagonal and its positive eigenvalues are the diagonal entries of D; hence $i_+(T) = i_+(T_1) = \operatorname{rank}(D) = \operatorname{rank}(A \mid B)$, and so T is not a counterexample. In the former case, $\begin{bmatrix} 0 & B_2 \\ B_2^* & -B_1^*D^{-1}B_1 - C \end{bmatrix}$ is strictly smaller than T and so is not a counterexample. Therefore $i_+\begin{bmatrix} 0 & B_2 \\ B_2^* & -B_1^*D^{-1}B_1 - C \end{bmatrix} = \operatorname{rank}(0 \mid B_2) = \operatorname{rank}(B_2)$, and so

$$i_+(T) = i_+(T_1) = \operatorname{rank}(D) + \operatorname{rank}(B_2) = \operatorname{rank}(A \mid B - AX) = \operatorname{rank}(A \mid B)$$

and once more T is not a counterexample. If $A = 0$ and $C \neq 0$ a similar analysis yields a similar contradiction. In this case it is convenient to write $P^{-1}CP = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$, where D is again diagonal with positive diagonal entries. □

Solution 54.5-2 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Lemma 1. Let $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$ be Hermitian, with H_{11} nonsingular. Then

$$i_+(H) = i_+(H_{11}) + i_+(H_{22} - H_{12}H_{11}^{-1}H_{12}^*) \quad \text{and} \quad \mathrm{i}_-(\mathrm{H}) = \mathrm{i}_-(\mathrm{H}_{11}) + \mathrm{i}_-(\mathrm{H}_{22} - \mathrm{H}_{12}\mathrm{H}_{11}^{-1}\mathrm{H}_{12}^*).$$

Proof. The conclusion is a direct consequence of Sylvester's theorem and the congruence transformation

$$H = \begin{pmatrix} I & 0 \\ H_{12}H_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} - H_{12}H_{11}^{-1}H_{12}^* \end{pmatrix} \begin{pmatrix} I & 0 \\ H_{12}H_{11}^{-1} & I \end{pmatrix}^*. \qquad \square$$

Lemma 2. Let $H = \begin{pmatrix} 0 & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$ be Hermitian, with $H_{12} \in \mathbb{C}^{k \times k}$ nonsingular. Then H is nonsingular, and $i_+(H) = i_-(H) = k$.

Proof. Let $H_{12}^{-*}H_{22}H_{12}^{-1} = Q\Lambda Q^* = Q \operatorname{diag}(\lambda_1, \ldots, \lambda_k)Q^*$ be the spectral decomposition of $H_{12}^{-*}H_{22}H_{12}^{-1}$. Then

$$\begin{pmatrix} Q & 0 \\ 0 & H_{12}^{-1}Q \end{pmatrix}^* \begin{pmatrix} 0 & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & H_{12}^{-1}Q \end{pmatrix} = \begin{pmatrix} 0 & I_k \\ I_k & \Lambda \end{pmatrix},$$

which is permutationally equivalent to $\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & \lambda_k \end{pmatrix}\right)$. The conclusion follows from the fact that each 2×2 diagonal block of the form $\begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}$ is indefinite. □

Now we come back to the proof of the original problem. We will prove that $i_+(T) = \operatorname{rank}(A \mid B)$. Once this is shown, the rest of the proof is straightforward: As T is permutationally equivalent to $\begin{pmatrix} -C & B^* \\ B & A \end{pmatrix}$, we obtain

$$i_-(T) = i_+\begin{pmatrix} C & -B^* \\ -B & -A \end{pmatrix} = \operatorname{rank}(C \mid -B^*) = \operatorname{rank}(B^* \mid C),$$

while for $i_0(T)$ we have $i_0(T) = m + n - i_+(T) - i_-(T) = m + n - \operatorname{rank}(A \mid B) - \operatorname{rank}(B^* \mid C)$.

We use 0_k to denote a $k \times k$ zero matrix when necessary. Let $k_1 = \operatorname{rank}(A)$, $k_2 = m - k_1$, $k_3 = \operatorname{rank}(A \mid B) - k_1$, $k_4 = m - \operatorname{rank}(A \mid B) = k_2 - k_3$. First, there exists a nonsingular matrix $X \in \mathbb{C}^{m \times m}$ such that $A = X \operatorname{diag}(I_{k_1}, 0)X^*$.

By partitioning $\tilde{B} := XB$ conformally as $\tilde{B} = (B_1^* \,|\, B_2^*)^*$, where $B_i \in \mathbb{C}^{k_i \times n}$, we obtain

$$\begin{pmatrix} X & 0 \\ 0 & I_n \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ B^* & -C \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I_n \end{pmatrix}^{-*} = \begin{pmatrix} I_{k_1} & 0 & B_1 \\ 0 & 0_{k_2} & B_2 \\ B_1^* & B_2^* & -C \end{pmatrix},$$

and hence Sylvester's law of inertia applied on congruence by $\begin{pmatrix} I_{k_1} & -(0\,|\,B_1) \\ 0 & I \end{pmatrix}$ gives

$$i_+(T) = k_1 + i_+ \begin{pmatrix} 0_{k_2} & B_2 \\ B_2^* & -C - B_1^* B_1 \end{pmatrix}.$$

As $\operatorname{rank}(B_2) = k_3$, we choose nonsingular matrices $Y \in \mathbb{C}^{k_2 \times k_2}$, $Z \in \mathbb{C}^{n \times n}$ such that $B_2 = Y \operatorname{diag}(I_{k_3}, 0) Z^*$. Let $\tilde{C} = Z^{-1}(C + B_1^* B_1) Z^{-*}$ be partitioned as $\tilde{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix}$ with $C_{11} \in \mathbb{C}^{k_3 \times k_3}$. Then

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}^{-1} \begin{pmatrix} 0_{k_2} & B_2 \\ B_2^* & -C - B_1^* B_1 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix}^{-*} = \begin{pmatrix} 0_{k_3} & 0 & I_{k_3} & 0 \\ 0 & 0_{k_4} & 0 & 0 \\ I_{k_3} & 0 & -C_{11} & -C_{12} \\ 0 & 0 & -C_{12}^* & -C_{22} \end{pmatrix}.$$

Therefore,

$$i_+(T) = k_1 + i_+ \begin{pmatrix} 0_{k_3} & I_{k_3} & 0 \\ I_{k_3} & -C_{11} & -C_{12} \\ 0 & -C_{12}^* & -C_{22} \end{pmatrix}.$$

Since $\tilde{C}$ is positive semidefinite, so is C_{22}. Then the Schur complement $S := -C_{22} - \begin{pmatrix} 0 \\ -C_{12} \end{pmatrix}^* \begin{pmatrix} 0_{k_3} & I_{k_3} \\ I_{k_3} & -C_{11} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -C_{12} \end{pmatrix} = -C_{22} - (0\,|\,C_{12}^*) \begin{pmatrix} C_{11} & I_{k_3} \\ I_{k_3} & 0_{k_3} \end{pmatrix} \begin{pmatrix} 0 \\ C_{12} \end{pmatrix} = -C_{22}$ is negative semidefinite. Finally, we have

$$i_+(T) = k_1 + i_+ \begin{pmatrix} 0_{k_3} & I_{k_3} & 0 \\ I_{k_3} & -C_{11} & -C_{12} \\ 0 & -C_{12}^* & -C_{22} \end{pmatrix} = k_1 + i_+ \begin{pmatrix} 0_{k_3} & I_{k_3} \\ I_{k_3} & -C_{11} \end{pmatrix} + i_+(S) = k_1 + k_3 + 0 = \operatorname{rank}(A\,|\,B). \qquad \square$$

Solution 54-5.3 by the proposer Yongge Tian

We need some matrix inertia formulas from [1, Theorem 2.3]: If X and Z are Hermitian matrices of orders m and n, respectively, then, with $X^\dagger$ denoting the Moore-Penrose inverse,

$$i_\pm \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} = i_\pm(X) + i_\pm \begin{pmatrix} 0 & (I_m - XX^\dagger)Y \\ Y^*(I_m - XX^\dagger) & Z - Y^* X^\dagger Y \end{pmatrix}; \tag{1}$$

moreover, if X is positive semidefinite, then

$$i_+ \begin{pmatrix} X & Y \\ Y^* & 0 \end{pmatrix} = \operatorname{rank}(X\,|\,Y) \quad \text{and} \quad \mathrm{i}_- \begin{pmatrix} X & Y \\ Y^* & 0 \end{pmatrix} = \operatorname{rank}(\mathrm{Y}). \tag{2}$$

Applying (1) gives

$$i_+ \begin{pmatrix} A & B \\ B^* & -C \end{pmatrix} = i_+(A) + i_+ \begin{pmatrix} 0 & (I_m - AA^\dagger)B \\ B^*(I_m - AA^\dagger) & -(C + B^* A^\dagger B) \end{pmatrix} \quad \text{and} \tag{3}$$

$$i_- \begin{pmatrix} A & B \\ B^* & -C \end{pmatrix} = i_-(-C) + i_- \begin{pmatrix} A + BC^\dagger B^* & B(I_m - C^\dagger C) \\ (I_m - C^\dagger C)B^* & 0 \end{pmatrix}. \tag{4}$$

Note that both $C + B^* A^\dagger B$ and $A + BC^\dagger B^*$ are positive semidefinite under the given assumptions. Then by (2)

$$i_+ \begin{pmatrix} 0 & (I_m - AA^\dagger)B \\ B^*(I_m - AA^\dagger) & -(C + B^* A^\dagger B) \end{pmatrix} = i_- \begin{pmatrix} 0 & (AA^\dagger - I_m)B \\ B^*(AA^\dagger - I_m) & C + B^* A^\dagger B \end{pmatrix} = \operatorname{rank}[\,(I_m - AA^\dagger)B],$$

and

$$i_- \begin{pmatrix} A + BC^\dagger B^* & B(I_m - C^\dagger C) \\ (I_m - C^\dagger C)B^* & 0 \end{pmatrix} = \operatorname{rank}[\,(I_m - CC^\dagger)B^*].$$

Substituting them into (3) and (4) and simplifying yields

$$i_+\begin{pmatrix} A & B \\ B^* & -C \end{pmatrix} = i_+(A) + \mathrm{rank}[\,(I_m - AA^\dagger)B] = \mathrm{rank}(A) + \mathrm{rank}[\,(I_m - AA^\dagger)B] = \mathrm{rank}(A\,|\,B),$$

$$i_-\begin{pmatrix} A & B \\ B^* & -C \end{pmatrix} = i_-(-C) + \mathrm{rank}[\,(I_m - CC^\dagger)B^*] = \mathrm{rank}(C) + \mathrm{rank}[\,(I_m - CC^\dagger)B^*] = \mathrm{rank}(B^*\,|\,C),$$

establishing the formula in the problem. □

Reference

[1] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, *Linear Algebra Appl.* 433 (2010) 263–296.

Also solved by Johanns de Andrade Bezerra *and Susana* Furtado.

Problem 54-6: An Inequality for a Contraction Matrix

Proposed by Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, Florida, USA*, zhang@nova.edu

Let $n \geq 2$ and let $C = (c_{ij})$ be an $n \times n$ contraction matrix (i.e., the largest singular value of C is no more than 1). Show that

$$\sum_{i,j} |c_{ij}|^2 + 1 \leq |\det C|^2 + n.$$

When does equality occur?

Solution 54-6.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Denote the singular value decomposition of C by $C = U\Sigma V^*$ and the jth unit standard basis vector in $\mathbb{C}^n$ by e_j. Then

$$\sum_{ij} |c_{ij}|^2 = \sum_{j=1}^n \|Ce_j\|^2 = \sum_{j=1}^n \|U\Sigma V^* e_j\|^2 = \sum_{j=1}^n \|\Sigma V^* e_j\|^2.$$

With $V = (v_{ij})$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, we have $\Sigma V^* e_j = [\sigma_1 \overline{v_{j1}}\ \cdots\ \sigma_n \overline{v_{jn}}]^{\mathrm{T}}$, and so

$$\sum_{ij} |c_{ij}|^2 = \sum_{j=1}^n \sum_{i=1}^n |\sigma_i|^2 |\overline{v_{ji}}|^2 = \sum_{i=1}^n \sigma_i^2 \sum_{j=1}^n |v_{ji}|^2 = \sum_{i=1}^n \sigma_i^2.$$

Since $|\det C| = |\det \Sigma|$, the inequality in the problem becomes

$$\sum_{i=1}^n \sigma_i^2 + 1 \leq \prod_{i=1}^n \sigma_i^2 + n \qquad (*)$$

whenever $0 \leq \sigma_i \leq 1$ for $1 \leq i \leq n$. We show that equality holds if and only if $n-1$ of the singular values equal 1. Let $f_n(x) = f_n(x_1, \ldots, x_n) = \sum_{i=1}^n x_i - \prod_{i=1}^n x_i$ on the compact domain $[0,1]^n$. Using induction on n, we show that $f_n(x) \leq n-1$ on its domain with equality if and only if $n-1$ of the variables equal 1. Consider $f_2(x,y) = x + y - xy$. On the boundary of the domain, $f_2 = 1$ whenever $x = 1$ or $y = 1$ and $f_2 < 1$ otherwise. There are no interior critical points of f_2, since $\partial f_2/\partial x = 0$ and $\partial f_2/\partial y = 0$ imply $x = y = 1$. Hence $f_2 < 1$ on the interior. Now assume both assertions hold for some f_n with $n \geq 2$. On the portion of the boundary of the domain of f_{n+1} where $x_i = 1$, we have

$$f_{n+1}(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1}) = f_n(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) + 1 \leq (n-1) + 1 = n$$

and equals n exactly when $n-1$ of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}$ equal 1. Also

$$f_{n+1}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n+1}) = \sum_{j=1,\, j\neq i}^{n+1} x_j,$$

which is at most n and equals n exactly when all n of the variables $x_j, j \neq i$, equal 1. As before, f_{n+1} has no interior critical points, and so $f_{n+1} < n$ on the interior. □

Editorial note: The solution by Roger A. Horn *resembles* 54-6.1 *but uses the fact that f_n is a harmonic function because $\frac{\partial^2 f_n}{\partial x_i^2} = 0$ for $i = 1, 2, \ldots, n$ and as such, f_n cannot have interior extremal points. The proposer remarked that inequality*

$(*)$ is equivalent to the inequality $\prod_{i=1}^{n}(1-x_i)+\sum_{i=1}^{n}x_i > 1$ for $0 < x_i < 1$, $i = 1,\dots,n$, which was brought to the proposer's attention by Minghua Lin and originates from `http://www.math.purdue.edu/pow/`.

Solution 54-6.2 by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Consider the positive semidefinite matrix $A = C^*C$, and let $0 \le \lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of A arranged in a non-decreasing order. The hypothesis that C is a contraction is equivalent to $\lambda_n \le 1$.

Now, the proposed inequality is equivalent to $\mathrm{Tr}(A) + 1 \le \det(A) + n$, and in terms of the eigenvalues this means that

$$1 + \sum_{k=1}^{n} \lambda_k \le n + \prod_{k=1}^{n} \lambda_k. \tag{$*$}$$

For $1 \le k \le n$, we define

$$u_k = k + \lambda_1\lambda_2\cdots\lambda_k - 1 - (\lambda_1 + \lambda_2 + \cdots + \lambda_k).$$

It is straightforward to check that, when $k > 1$, we have $u_k - u_{k-1} = (1-\lambda_k)(1-\lambda_1\lambda_2\cdots\lambda_{k-1})$, and since $u_1 = 0$ we conclude that

$$u_n = \sum_{k=2}^{n}(1-\lambda_k)(1-\lambda_1\lambda_2\cdots\lambda_{k-1}).$$

It follows that $u_n \ge 0$ because all the summands in this formula are nonnegative, and the desired inequality $(*)$ is proved.

If $u_n = 0$, then $(1-\lambda_2)(1-\lambda_1) = 0$, but $0 \le 1-\lambda_2 \le 1-\lambda_1$, so we must have $\lambda_2 - 1 = 0$ and consequently $\lambda_k = 1$ for $k = 2, 3, \dots, n$. Conversely, it is straightforward to check that if $\lambda_k = 1$ for $k = 2, 3, \dots, n$, then $u_n = 0$. Hence, equality holds if and only if A is unitarily equivalent to $\begin{bmatrix} \lambda_1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & I_{n-1} \end{bmatrix}$ with $\lambda_1 \in [0,1]$.

So, equality holds in the proposed inequality if and only if $\mathrm{rk}(I - C^*C) \le 1$, i.e., the rank of the matrix $I - C^*C$ is less or equal to 1. □

Solution 54-6.3 by Meiyue Shao, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Let $\sigma_1 \ge \cdots \ge \sigma_n$ be the singular values of A. The desired inequality then becomes

$$\sum_{i=1}^{n} \sigma_i^2 \le n - 1 + \prod_{i=1}^{n} \sigma_i^2.$$

We claim that the equality holds if and only if $\sigma_1 = \cdots = \sigma_{n-1} = 1$. We proceed by induction. When $n = 2$, we have

$$\sigma_1^2 + \sigma_2^2 = 1 + \sigma_1^2\sigma_2^2 - (1-\sigma_1^2)(1-\sigma_2^2) \le 1 + \sigma_1^2\sigma_2^2.$$

Since $0 \le \sigma_2 \le \sigma_1$, the equality occurs if and only if $\sigma_1 = 1$. Now suppose that the inequality holds for some $n \ge 2$, and the equality occurs if and only if $\sigma_1 = \cdots = \sigma_{n-1} = 1$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} \sigma_i^2 &\le \left(n - 1 + \prod_{i=1}^{n} \sigma_i^2\right) + \sigma_{n+1}^2 = \left(n + \prod_{i=1}^{n+1} \sigma_i^2\right) - \left(1 - \sigma_{n+1}^2 - \prod_{i=1}^{n} \sigma_i^2 + \prod_{i=1}^{n+1} \sigma_i^2\right) \\ &= \left(n + \prod_{i=1}^{n+1} \sigma_i^2\right) - (1-\sigma_{n+1}^2)\left(1 - \prod_{i=1}^{n} \sigma_i^2\right) \le n + \prod_{i=1}^{n+1} \sigma_i^2.\end{aligned}$$

The equality occurs if and only if $\sigma_1 = \cdots = \sigma_{n-1} = 1$ (by induction) and $\prod_{i=1}^{n} \sigma_i^2 = 1$ (the other case, $\sigma_{n+1} = 1$, is automatically included). These conditions can be combined as $\sigma_1 = \cdots = \sigma_n = 1$. □

Also solved by Roger A. Horn *and the proposer.*

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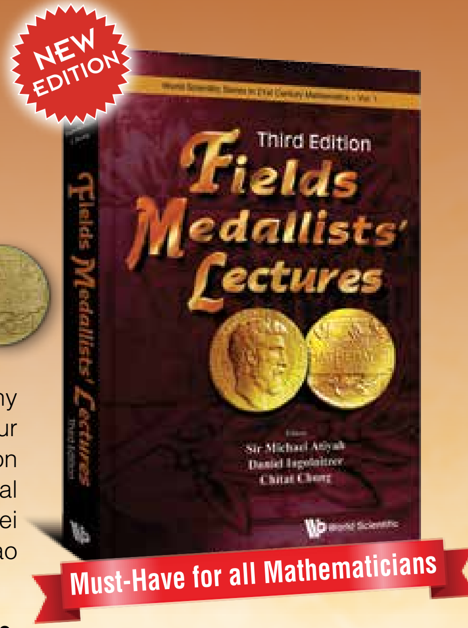

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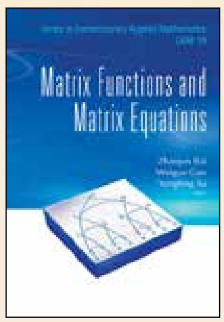

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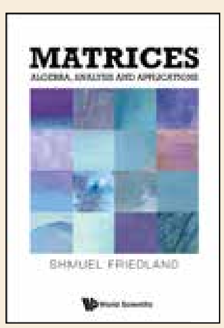

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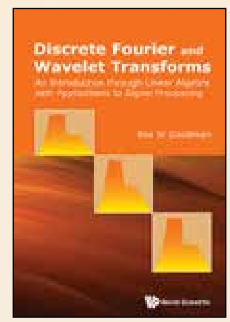

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Problems: We introduce 6 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Solutions: We present solutions to all problems in the previous issue [*IMAGE* 54 (Spring 2015), p. 44] Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Bojan Kuzma (bojan.kuzma@famnit.upr.si). The working team of the Problem Corner consists of Steve Butler, Gregor Dolinar, Shaun Fallat, Alexander Guterman, Rajesh Pereira, and Nung-Sing Sze.

New Problems:

Problem 55-1: The Perfect Aggregation of Linear Equations With M-Matrix
Proposed by Arkady A. Babajanyan, *Institute for Informatics and Automation Problems, Armenian Academy of Sciences, Yerevan, Armenia*, arbab@list.ru

Let $F \in M_n(\mathbb{R})$ be an M-matrix (i.e., $F = \alpha I - A$ where $A \geq 0$ entrywise and $\alpha \geq \rho(A)$, the spectral radius) and let C be a $(0,1)$-matrix of size $m \times n$ $(m < n)$ and full rank, where each column contains a single 1.

(A) If F is non-degenerate, construct an entrywise non-negative $m \times n$ matrix D of full rank so that $\bar{F} = DFC^+$ is again a non-degenerate M-matrix and the condition of consistency $DF = \bar{F}C$ holds (here, C^+ is a Moore-Penrose inverse).
(B) Generalize to the case of a degenerate M-matrix F.

Problem 55-2: About Equality of Dimensions of Direct Complements
Proposed by Johanns de Andrade Bezerra, *Natal, RN, Brazil*, pav.animal@hotmail.com

Let $A, B \in M_n(\mathbb{C})$ satisfy $\operatorname{rank}(A) = \operatorname{rank}(B)$. Let V and W be subspaces of $\mathbb{C}^n$ with $\operatorname{Im} A = (\operatorname{Im} A \cap \operatorname{Im} B) \oplus V$ and $\operatorname{Ker} A = (\operatorname{Ker} A \cap \operatorname{Ker} B) \oplus W$. Show that $\dim V = \dim W$ if A and B are range-Hermitian.

Problem 55-3: A Matrix Product Characterizing the Samuelson Transformation Matrix
Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let $C_n = \left(c_{ij}\right)$ be a $2^n \times (n+1)$ matrix of ± 1, with rows indexed by $i = 0, 1, \ldots, 2^n - 1$, such that $\sum_{j=0}^{n}(1 - c_{ij})2^{n-j-1} = i$ for $i = 0, \ldots, 2^n - 1$ (i.e., ith row forms a symbolic binary expansion of i).

Does there exist a $2^n \times (n+1)$ matrix E_n, with rows taken from the $(n+1) \times (n+1)$ identity matrix, so that $E_n^T C_n$ equals the $(n+1) \times (n+1)$ Samuelson transformation matrix $S_n = \left(s_{hj}\right)$ whose jth column contains the coefficients for the expansion of the polynomial

$$\sum_{h=0}^{n} s_{hj} x^{n-h} y^h = (x + y)^{n-j}(x - y)^j?$$

Problem 55-4: Orthogonality of Two Positive Semidefinite Matrices
Proposed by Minghua Lin, *Shanghai University, Shanghai, China*, mlin87@ymail.com

Let A, B be positive definite. Assume $M = \left(\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} A & Y \\ Y^* & B \end{smallmatrix}\right)$ are positive semidefinite and $MN = 0$. Show that $X = -Y$. What if A and B are only assumed to be positive semidefinite?

Problem 55-5: On the Rank of Integral Matrices
Proposed by Volodymyr Prokip, *Institute for Applied Problems of Mechanics and Mathematics, Ukraine*, v.prokip@gmail.com

Let R be an integral domain with unit element $e \neq 0$ and let $M_n(\mathrm{R})$ denote the ring of n-by-n matrices over R. Let $\operatorname{adj}(C)$ denote the classical adjoint matrix of $C \in M_n(\mathrm{R})$. Assume $A, B \in M_n(\mathrm{R})$ satisfy $\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(A + B) = n$. Show that (i) $A \operatorname{adj}(A + B) A = A \det(A + B)$ and (ii) $A \operatorname{adj}(A + B) B = 0$.

Problem 55-6: Singular Value Inequalities for 2×2 Block Matrices
Proposed by Ramazan Türkmen, *Science Faculty, Selçuk University, 42031 Konya, Turkey*, rturkmen@selcuk.edu.tr

Let $\left(\begin{smallmatrix} M & K \\ K^* & N \end{smallmatrix}\right) \geq 0$ be a positive semidefinite matrix with $M, N, K \in M_n(\mathbb{C})$. Denote by $s^{\downarrow}(K) = (s_1(K), \ldots, s_n(K))$ the vector of singular values of K and denote by $\lambda^{\downarrow}(P) = (\lambda_1(P), \ldots, \lambda_n(P))$ the vector of eigenvalues of $P \geq 0$, both arranged in decreasing order and with counted multiplicities. Prove that, with respect to weak majorization $\prec_w$, one has

$$(s^{\downarrow}(K), s^{\downarrow}(K)) \prec_w (\lambda^{\downarrow}(M), \lambda^{\downarrow}(N)).$$

Solutions to Problems 53-1, 53-2, *and* 54-1 *through* 54-6 *are on pages* 33–42.