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FEATURE INTERVIEWS

"The Field is as Exciting as Ever"

Shmuel Friedland Interviewed by Lek-Heng Lim[1]

This interview was conducted in two parts, at the Kurah Mediterranean restaurant on November 16, 2016 and at the Armand's Victory Tap restaurant on December 20, 2016.

— Scholarly work —

L.-H.L. - You have a reputation in the community for going after many daunting problems that would give most other mathematicians pause. One of these is the Jacobian Conjecture. How did you get into this problem?

S.F. - When I went to Stanford as a postdoc, I realized that linear algebra and matrix theory were not what most people were interested in (laughter). Indeed, soon after I arrived at Stanford, Paul Cohen asked me what was the most important theorem in my field and I mentioned the Perron–Frobenius Theorem. A few days later he told me that he proved it using a hint in Dunford–Schwartz [7]. So I looked around for other things to work on besides matrix theory. There was this young assistant professor in the department whose work in geometry seemed to be generating a lot of excitement and I asked him what were the important open problems in geometry. He pointed me to the Jacobian Conjecture. This assistant professor is Shing-Tung Yau and the conjecture is of course still unresolved today. Working on it was both a blessing and a curse. I invested a lot of time but I published only two papers on the topic, despite spending more effort on it than on any of my other major research interests. One good thing that came of it was my paper with Jack Milnor on plane polynomial automorphisms [27], which became my most frequently cited paper.

L.-H.L. - Yitang Zhang, who famously made a breakthrough in the twin prime conjecture, also spent a significant amount of energy and time on the Jacobian Conjecture when he was a Ph.D. student and got little out of it. In fact, even the case of two variables is open. Do you have any advice for young mathematicians who might like to cut their teeth on this problem?

S.F. - Stay away from the Jacobian Conjecture (laughter).

L.-H.L. - So what should they work on?

S.F. - They should work on one famous unsolved problem once in their career, mainly to learn something from this experience. But remember to move on to other, less ambitious problems.

L.-H.L. - Is the Jacobian Conjecture the first big open problem that you have worked on?

S.F. - No. That would be the Bieberbach Conjecture, which was the subject of my master's thesis [17]. I later wrote two joint papers [32, 33] on this conjecture with Max Schiffer in 1977. The conjecture was eventually proved by Louis de Branges [6] in 1979.

L.-H.L. - You proved the Salmon Conjecture [19] a few years ago, a problem in algebraic geometry [1] that baffled many algebraic geometers. Can you say why you succeeded when all these professionals did not?

S.F. - I would attribute this to my familiarity with linear algebra. I combined techniques in linear algebra with some very basic knowledge of algebraic geometry. In fact, for this particular problem, knowledge of advanced algebraic geometry is optional but knowledge of advanced linear algebra is not. Linear algebra contains some of the most powerful mathematical techniques ever discovered. In my view it is much more applicable than algebraic geometry.

> "Linear algebra contains some of the most powerful mathematical techniques ever discovered."

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L.-H.L. - Another notoriously difficult problem that you resolved is the simultaneous similarity of matrices [22, 21]. The Salmon Conjecture is about a special case – four 4×4 matrices – but requires that you go deeper than simultaneous similarity. So did you find your prior work in simultaneous similarity helpful?

S.F. - They required quite different techniques. I read about the simultaneous similarity problem in a paper by Gelfand and Ponomarev [36] and realized that one needs to introduce a rational invariant function, which gives the complete solution to the problem of simultaneous similarity for a pair of 2×2 matrices. I figured out that one can then generalize this to k $n \times n$ matrices using stratification theory. In fact, after my paper [22, 21] appeared, I received a letter from David Mumford saying that he never agreed with Gabriel's division of classification problems into 'tame' and 'wild'. Gelfand and Ponomarev show that the classification of a pair of $n \times n$ commuting nilpotent matrices is 'wild' but my work shows that it is nonetheless very tractable.

L.-H.L. - How did you find out about the Salmon Conjecture?

S.F. - Bernd Sturmfels posed the problem to me while we were traveling together on a bus in Berlin. I was then a visiting professor in the Berlin Mathematical School hosted by Volker Mehrmann and Bernd was also on sabbatical in Berlin.


Shmuel Friedland with his Salmon Prize

L.-H.L. - Did you enjoy your Salmon Prize?

S.F. - Very much. It was two small delicious pieces of smoked wild Alaskan salmon, caught by Elizabeth Allman and smoked by her husband John Rhodes. It tastes much better than anything you can buy from the shops. I wish there were more (laughter). I should mention that my result in [19] was later improved in a joint paper with Elizabeth Gross [24]; maybe they would send us another piece (laughter).

L.-H.L. - You also solved Peter Lax's eigenvalue crossing problem. Can you tell us about that?

S.F. - Lax gave a lecture in Wisconsin describing his problem and I was in the audience. The von Neumann–Wigner non-crossing rule states that in a generic two-dimensional subspace of $\mathsf{S}^2(\mathbb{R}^n)$, the vector space of $n \times n$ symmetric real matrices, every nonzero matrix has only simple eigenvalues. In other words, the variety of real symmetric matrices with multiple eigenvalues has codimension three. What Lax showed in his lecture is that if $n \equiv 2 \pmod 4$, then every three-dimensional subspace contains a nonzero matrix with a multiple eigenvalue, showing the converse of the von Neumann–Wigner non-crossing rule. What I showed, together with Joel Robbin and John Sylvester [31], is that the problem of finding the dimension of the minimal subspace in $\mathsf{S}^2(\mathbb{R}^n)$ such that there exists a nonzero matrix with a double eigenvalue boils down to the problem of counting the number of linearly independent vector fields on the sphere, which was of course a well-known result of the topologist Frank Adams. I have to add that my interest in the von Neumann–Wigner rule predated Lax's talk – I did related work a few years earlier with Barry Simon [34] and so I already knew the topic well.

L.-H.L. - This reminds me of your work [29] with Giorgio Ottaviani using Chern classes to determine the number of singular values of a generic tensor. How important are topological methods in linear algebra?

S.F. - I think topological techniques are very important for nonlinear problems like eigenvalue or singular value problems. Incidentally, not long ago, Doron Zeilberger wrote to tell me that he submitted a special case of our result to Neil Sloane. The number of singular values of a generic $n \times n \times n$ tensor is now in the *Online Encyclopedia of Integer Sequences*.

L.-H.L. - While we are still on the topic of eigenvalues, you are of one of the pioneers of inverse eigenvalue problems. What are your proudest contributions to this area?

S.F. - I introduced algebraic topological tools like degree theory and algebraic geometric tools like effective versions of Bezout's theorem to inverse eigenvalue problems [10]. Henry Landau's proof [38] of the inverse Toeplitz eigenvalue problem was based on my ideas of using degree theory [15] but he went far beyond what I did. I also started the systematic study of the inverse eigenvalue problem for nonnegative matrices [18]. A high point of this problem was the work of Mike Boyle and David Handelman [5], which resolves the problem modulo zeros in the spectrum. Their proof relies on symbolic dynamical systems, but Tom Laffey came up with an amazingly short proof purely in terms of matrix theory [37], although his result is slightly weaker than Boyle–Handelman's. All three of them won the Hans Schneider Prize for their contributions. On the practical side, I did some work with Jorge Nocedal and Mike Overton on algorithms for inverse eigenvalue problems [28].

L.-H.L. - Nonnegative matrices are indeed fascinating. The most famous problem in this area was probably the van der Waerden Conjecture, and you obtained some of the earliest significant results [11, 12]. What drew you to this conjecture?

S.F. - Henryk Minc introduced the van der Waerden Permanent Conjecture to me in 1969 when he was visiting the Technion, where I was obtaining my doctorate. I jumped on the conjecture right away [16]. In 1977, Minc showed me the announcement of T. Bang [4], who claimed to have proved the conjecture up to a small polynomial error, but confessed that he could not follow Bang's arguments. I thought what I did in [11] was to essentially connect the dots in Bang's arguments. But I later received a letter from Bang congratulating me on my "tour de force" (his words); so I suspect he did not completely follow his own arguments either (laughter). A year later, Egorichev [8] and Falikman [9] independently proved the van der Waerden Conjecture. Using the main tool of their proofs – Alexandrov's inequality – I proved [12] the Tverberg Conjecture, a generalization of the van der Waerden Conjecture.

L.-H.L. - Many of the biggest problems in or related to linear algebra and matrix theory have been resolved (Horn's Conjecture, Lax's Conjecture, BMV Conjecture, Kadison–Singer Conjecture) or are nearly resolved (Crouzeix's Conjecture) since the beginning of the new millennium. What else remains? And what are your recommendations for ambitious junior linear algebraists and matrix theorists?

S.F. - I would say the field is as exciting as ever, if not more so. I would mention three key words: *random*, *quantum*, *tensors*. Random matrices have already been extensively studied for many years. My own interest in the area came fairly recently. In Fall 2000, when I visited my *alma mater* Technion as a Lady Davis Professor, I gave a course on advanced topics in matrix theory; Ofer Zeitouni and his postdoc Brian Ryder were in the audience. Ofer is a world-famous probabilist and, among other things, an expert in random matrices. Thanks to him, we started collaborating on the concentration of permanent estimators [30]. I think the connection between probability and linear algebra – not just random matrix theory but also topics like randomized algorithms in numerical linear algebra – is an area with great potential and we will be seeing many more developments.

However my opinion is that by far the most exciting new area is multilinear algebra and tensors – with so many more questions than answers. Since you are the world-leading expert, this topic is best left to your interview, not mine (laughter). Instead I would say something about quantum information theory, which one may view as "finite-dimensional quantum mechanics," and therefore it naturally involves a lot of matrix theory. This has been my main interest for the past ten years or so. What I want to point out is that quantum information theory goes much further than just linear algebra; it also nicely connects with multilinear algebra and with probability theory. An example of this is my work with Gilad Gour and Vlad Gheorghiu [23], where we generalized Heisenberg's Uncertainty Principle to tensor products of probability measures using vector majorization.

— People and places —

L.-H.L. - Your first positions in the USA were at Stanford University and the Institute for Advanced Study in Princeton. Did you like your experience at these places?

S.F. - My time at Stanford was very fulfilling. I learned a lot. I learned algebraic topology by attending Hans Samelson's graduate class. I worked with Sam Karlin [26] and Max Schiffer [32, 33]. Sam introduced me to Gene Golub, and although I did not have much interaction with him then, we became good friends later in life. S. S. Chern invited me to speak in his seminar at Berkeley. His prominent student S.-T. Yau, who was an assistant professor at Stanford, told me about the Jacobian Conjecture, as I mentioned earlier. Even the students were great to work with; an example that comes to mind is Charlie Michelli.

Frankly, I got much less out of IAS than Stanford. I attended many lectures: by Griffiths, Langlands, Selberg, Weil, and many others. I saw the full breadth and depth of pure mathematics, but that was the extent of it. Nothing concrete came out of these lectures as far as my research work was concerned. I did not have much social interaction with the permanent members except this one time when Atle Selberg invited my wife and me to dinner at his home, which I think was only because he and I both came from Israel (laughter). The best thing about IAS was that it offered a lot of quiet time for my own research work without the distraction of teaching duties.

L.-H.L. - I suppose you still visit Stanford quite frequently given that your son is a professor in the medical school? How do you think the mathematics department has changed after all these years?

S.F. - The only faculty member I have been in contact with these last few years is Amir Dembo. The department has changed a lot. Looking at its faculty today, I can see that there are now many more applied mathematicians than there were 44 years ago.

L.-H.L. - After Stanford and IAS, you were appointed to the mathematics faculty at the Hebrew University of Jerusalem. Tell us more about those days.

S.F. - Joram Lindenstrauss hired me in 1975 and I worked my way up to a full professor in 1982. The mathematics department in Hebrew University is called the Institute of Mathematics. Back when I was there it was an amazing place. Everyone was excellent – and I really mean everyone. The students were exceptionally bright; the best I have seen anywhere. I wrote some papers [3, 2] with two Ph.D. students – Noga Alon and Gil Kalai. Peter Constantin was a teaching assistant for my ODE class. All of them are now world-famous mathematicians. Among the faculty, Robert Aumann later got a Nobel Prize in Economics; the dynamical systems group, helmed by Hillel Furstenberg and Benji Weiss, was especially strong. I went to many of their seminars and this paid off later when I started working in dynamical systems myself. Another big group was the one in geometry of Banach spaces, led by Joram Lindenstrauss himself, but also had other remarkable people like Lior Tzafriri. Joram was a terrific analyst and an uncompromising person. He introduced me to the Grothendieck inequality, but I was never interested in it until recently, when you explained its significance and its connection to 3-tensors to me.

L.-H.L. - Many academics originally from Israel yearn to return to the promised land, including people at places like Princeton (Elon Lindenstrauss) and Harvard (Joseph Bernstein, David Kazhdan), but you did the opposite. Why is that?

S.F. - It is the universal reason for most such relocations in academia – the two-body problem (laughter). My ex-wife went to Cornell University. So I visited Cornell for almost two years before permanently leaving Hebrew University and joining UIC. But my time at Cornell was well-spent; that was where I got into dynamical systems and ergodic theory.

L.-H.L. - Can you tell us more about that?

S.F. - I attended a series of seminars on Hénon maps by John Hubbard, which included occasional guest lectures by his former advisor Adrien Douady. I made the connection between Hénon maps and the Jacobian Conjecture and wrote a small report. On one occasion, John introduced me to Jack Milnor. I gave Jack a copy of my report and that led to my collaboration with him on the dynamics of polynomial automorphisms [27]. Jack is a fantastic mathematician in numerous ways – the one that impresses me most is his ability to make even the most complicated mathematics look simple.

Partly encouraged by this collaboration, I continued to work for a few years in dynamical systems and ergodic theory. Among other things, I generalized my results with Jack on entropy to holomorphic self-maps [14]. This in turn led to my work on multidimensional entropy [20] and for this work I should not neglect to acknowledge the help I got from Tim Gowers, who happened to be visiting IHES when I was there in 1994. That year at IHES might have been a low point in his career, as he gave me the impression of being somewhat disheartened. His magnificent talents and intellect were nonetheless obvious to me, and probably everyone else too, even at that time. The final piece of work that came out of this series of papers is [13], where I introduced a new kind of entropy that is now called "Friedland's entropy."

L.-H.L. - You said "among other things." Can you elaborate?

S.F. - Certainly. Dynamical systems occupied a substantial part of my mathematical life and I have worked on many aspects of the subject. For instance, one topic that is very far removed from my work on entropy is something I did with Sa'ar Hersonsky [25]: We combined techniques in normed algebras with techniques in dynamical systems to extend Martin's inequality for discrete nonelementary groups of Möbius transformations in higher dimensions [39] to discrete multiplicative subgroups in normed unital algebras. Aside from this, I have also worked on invariant measures, Lyapunov exponents, Hausdorff dimensions, strong hyperbolicity, monomer-dimer model, Furstenberg's 2-3 Conjecture. The last one is with Benji Weiss [35] and is my only joint work with the dynamical systems group in Hebrew University, written more than twenty years after I learned the subject from them.

L.-H.L. - Besides the people you already mentioned, were there others who played a pivotal role in your career?

S.F. - I think I owe a lot to the late Hans Schneider and, to a lesser extent, also to the late Olga Tausky-Todd. Hans recognized my talents early on in my career. When I was still an associate professor at the Hebrew University, he arranged a two-year visiting professorship for me at the University of Wisconsin–Madison. At that time, it was the world center for linear algebra and matrix theory. Those two years, 1978–80, were my best and most productive mathematical years. Olga was also very kind to me and invited me to visit Caltech several times. She inspired some of my works on the Motzkin–Tausky L-property for matrix pencils, i.e., $A + xB$ such that all eigenvalues are exactly of the form $\alpha + x\beta$ where α and β are eigenvalues of A and B. This led to my work with Nimrod Moiseyev on resonant states [40].

L.-H.L. - My last question is probably an inevitable one in such interviews: When did you realize that you wanted to become a mathematician?

S.F. - This happened when I went from my kibbutz to the Reali High School as a sophomore. I started to submit my solutions to problems in *Gilionot Matematica*, a high school mathematics periodical edited by Abraham Ginzburg. These submissions were graded and the scores published alongside the students' names in subsequent issues. It was completely voluntary but it fostered a competitive spirit among students from high schools all over Israel who partook in the activity, most of whom I had not met and knew only by name. There were only a small number of students who received higher total scores than me – one of them was the logician Saharon Shelah, who later became my colleague at the Hebrew University – and at that time I was quite encouraged by this.

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"Between Two Great Research Communities"

Henry Wolkowicz Interviewed by Abdo Alfakih[1]

H. Wolkowicz

A.A. - What was your early childhood like and how did you end up in Montreal, Canada?

H.W. - I was born in Poland after WWII to Jewish parents who had met while searching for their *first* families. We escaped from Poland when I was a 6-month-old baby, during the night. We ended up in a DP-camp in Germany and eventually made it to Paris and then to Canada. I am skipping many details that are of course interesting but I think not appropriate for this story. Suffice it to say that this history influenced my family's life every day.

A.A. - What was your undergraduate education like?

H.W. - I had a very *nonstandard* undergraduate career. I arrived at McGill University in Montreal at age 16, a year younger than all the peers that I knew. I had always liked mathematics in high school and done well in my mathematics courses. I enrolled in the general B.Sc. program at McGill University. Due to a serious sports injury I missed the first month of my first year. I remember returning to class (on crutches) and getting 100% on my first calculus midterm after learning the material on my own. But, due to my absence, I did not do so well in the other courses. I ended up dropping out of school and only returning several years later to take some courses. I remember taking a computer course using the WATFOR programming language. I did extremely well in the course and McGill allowed me to return to continue for my undergraduate degree.

A.A. - Did you major in mathematics?

H.W. - I did not major in any subject during my undergraduate work. But then in my final year I took a course on mathematical programming, by Prof. Sanjo Zlobec (I think Math 417). I liked the course very much as did almost all the students. However, when the final exam came I misunderstood the instructions. We had covered linear programming (LP) in only the last lecture and I thought Sanjo said we were not responsible for it on the final exam. But he had said the opposite. I missed several LP questions on the final exam and my final grade was just a B. I mentioned this to Sanjo and he believed me and gave me a project to do. This involved implementing the so-called *suboptimization interval linear programming code* by Robers and Ben-Israel. I ended up doing this on the diet problem (a problem actually suggested by my wife Gail). I enjoyed this nontrivial project very much and spent a lot of extra time on it. Sanjo appreciated the work and increased my grade in the course. Then as I was leaving his office he casually asked me what I was doing next year. So, this is how peoples' lives can change completely. I had no idea what I was doing the following year. So I think the only reason I ended up being a mathematician was that I accepted his invitation to do a master's thesis under his supervision. I ended up taking the honours undergraduate mathematics along with the graduate courses during this degree and continued on to doing a Ph.D. with Sanjo.

I would like to mention that George Styan's office was next to Sanjo's. While waiting to meet with Sanjo I ended up talking to and eventually working with George. George introduced me to the linear algebra community, one which I found extremely friendly and welcoming, as was George. It was fantastic to be a part of this community over the years. George and I collaborated for many years on semidefinite matrices and bounds for eigenvalues of matrices. These topics were central to the overlap between numerical linear algebra and optimization in much of my research.

A.A. - When did you know you wanted to become a mathematician?

H.W. - I always liked doing mathematics and found it easy. I did not specialize as an undergraduate. I enjoyed the computer science courses I took and if it was not for the invitation from Sanjo to do a Master's with him, I may have continued in computer science. I actually worked for a computer graphics company (Aquila BST) before returning to academic work.

A.A. - How was the academic job market when you finished your Ph.D.?

H.W. - The mathematics department at McGill actually had a poster advising graduate students to *not* go into mathematics, as the job market was so poor. This is in total contrast to the situation now. When I finished there was literally one job in all of Canada related to optimization.

Before I defended my thesis, my supervisor managed to get me a lectureship position at Dalhousie University. My hosts were Dick Sutherland and Jon Borwein. This was a tremendous opportunity as I was working with Jon who was also

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essentially just starting his career. The teaching load for both of us was high (three courses per semester) but working with Jon was an amazing experience. We now all know that Jon was an incredible mathematician and one of the top optimizers in the world. I learned an incredible amount from him and we published four papers together that year dealing with closing gaps for strong duality. This work also led to the idea of *facial reduction* for regularization of optimization problems, a topic that I am still working on and exploiting. Gail (my wife) was also working with Jon at the time. We had a truly memorable and productive year.

A.A. - How did you end up in Waterloo?

H.W. - From Dalhousie I went to University of Alberta in Edmonton and received tenure and promotion to full professor. Gail came with me and did her Ph.D. there. During this time I took three months leave and went to work with Charles Johnson at University of Maryland (also with Robert Grone and Eduardo Marques de Sa). This was a very productive time and we published five papers together including two that ended up being very influential. One involved a characterization of positive semidefinite completions using chordality of the graph. Shortly after, Gail finished her Ph.D. and obtained an NSERC postdoctoral fellowship, which she held at Emory University and Brown University. I followed her to Emory for one year and then held a position at the University of Delaware. Then, to solve our two body problem, I was very happy to follow up on an invitation from Andy Conn to come to Waterloo, and I accepted a position in the Department of Combinatorics and Optimization there; Gail accepted a position nearby at McMaster University. We have remained at these locations.

A.A. - How was the atmosphere at the time in Waterloo?

H.W. - One thing I realized at Waterloo was that it is a *very busy place*. There were an amazing number of active researchers in a large mathematics faculty and there were a steady stream of visitors coming through. I can name many, but in optimization at the beginning John Dennis visited for an extended time as an adjunct professor and we published together. Also, Andy had his co-authors Nick Gould and Philippe Toint visit regularly, as well as many others. I also met and worked with Franz Rendl, a sabbatical visitor, and we started a long-time friendship and very successful collaboration. Our work involved the overlap between continuous and discrete optimization and exploited techniques from numerical linear algebra. I also had many visitors come through, e.g., in linear algebra Charlie Johnson and George Styan. Also, Alan George was our dean for a long period, and we had other visitors such as Gene Golub. Our department at one point included Adrian Lewis, Jon Borwein, and Levent Tuncel in Continuous Optimization, just after Andy Conn had left. We had of course many influential researchers in Discrete Optimization and Graph Theory (including Jack Edmonds and Bill Tutte and many others). So our department did not lack for superstars and I considered myself extremely lucky to be involved with them.

A.A. - Which of your papers and results are you the most proud of?

H.W. - I would like to mention four. These are results that are still being applied in numerical linear algebra and in both discrete and continuous optimization. I think that a lot of my work relates to cone optimization and in particular to semidefinite programming (SDP). My early work with Jon Borwein on **(i) strong duality in cone optimization** led to facial reduction and some of our examples were on cone problems over the semidefinite cone. Then the work on **(ii) characterizing positive semidefinite completions using chordality** is also SDP in disguise and continues to find many applications in many areas. These two areas are combined in my work with Franz Rendl on **(iii) SDP relaxations of hard combinatorial problems** where surprisingly strict feasibility fails and facial reduction is required. Then with Franz and his student Christoph Helmberg and also Bob Vanderbei we developed a **(iv) search direction for solving SDP problems** with primal-dual interior-point methods. This is arguably the most successful search direction currently.

A.A. - Who would you say is the mathematician who influenced you the most?

H.W. - Definitely this was Jon Borwein. Jon has recently passed away and this is a terrible loss to many mathematical communities. I learned an amazing amount from him. Also, Jon was always ready to work with people *immediately*, as all he needed was a pad and pencil. So I learned mathematics as well as an attitude about research.

Of course, I would not be in mathematics if not for my Ph.D. supervisor Sanjo Zlobec.

A.A. - What changes have you seen in the linear algebra and optimization communities since you finished your Ph.D.?

H.W. - Both the optimization and linear algebra communities were quite small when I started. Most researchers knew each other personally and also knew each others' work. This community has grown exponentially. Also, I think the division between the theory groups and computations groups was much bigger. The theoretical side would literally not touch a computer. But we now see the two communities much more intertwined. The theory has influenced the algorithms and we see solutions of large scale hard problems that no one would have believed solvable when I started. This has happened in both linear algebra and optimization. I think that the power of optimization has grown tremendously and I think there will be a great demand for optimizers in the future as the number of applications is growing. I think that the importance of linear algebra has passed that of the calculus. This has not yet fully played out in the undergraduate curriculum but I think should in the future.

> "I think that the importance of linear algebra has passed that of the calculus."

A.A. - In your opinion what important problems in linear algebra and optimization need to be addressed?

H.W. - One problem that still needs to be addressed is accuracy and robustness in software, i.e., reliability in software. The size of problems that are solved has grown tremendously. It is still obvious that many users are happy with an output without knowing how good the solution is. I think that many algorithms are solving larger problems rather than obtaining good solutions for medium-sized problems. As with many areas of science, another problem is *reproducibility*, due to changes in software and hardware and authors not providing their code.

Yet another problem concerns the *teaching* of linear algebra and optimization. I think that using the computer while introducing students to linear algebra is essential, but a good way of doing this has, to my knowledge, still not been found. Undergraduate students have a hard time in linear algebra courses, a subject that mathematicians, I think, find extremely intuitive and easy. As for teaching optimization, one needs a background in many areas, e.g., linear algebra, calculus, analysis, computer science. This makes it difficult to present good elementary undergraduate courses.

A.A. - Do you see any major changes that are needed in the way people referee and publish, in particular since we now have the mathematics arXiv?

H.W. - Of course the number of papers being submitted/published has grown exponentially. When I started I literally stopped in the library each morning to check out *all* the new journals that arrived and looked for any interesting articles to copy; by all, I mean all areas of mathematics. Now one cannot even keep up with one specialized area such as optimization or numerical linear algebra. There are also serious problems with plagiarism. But I think the main problem with so many papers is the difficulty in getting them refereed properly. There are few journals that can guarantee a rigorous referee process as I think referees are too busy to do a proper job on the papers. And the journals are under pressure due to competition to publish *quickly.* And authors are not doing their work in doing a proper literary search and relating their work with known results. Many papers are published by students as they are also under tremendous pressure to publish papers *before* getting their degrees. And finally, we have the reproducibility crisis that many journals are addressing.

OBITUARY NOTICE

Obituary Note: Ivo Marek (Jan. 24, 1933 – Aug. 18, 2017)

Submitted by Radim Blaheta, Zdenek Strakos, and Daniel Szyld

With deep sorrow we inform the community that Ivo Marek, Professor Emeritus at Charles University and at the Czech Technical University in Prague, passed away on August 18, 2017 at the age of 84. Ivo Marek was an outstanding mathematician, who contributed over 170 papers and 3 monographs on a wide range of topics. Among those, we highlight neutron transport theory and related problems of nuclear reactor theory, numerical methods for many types of iterative methods for linear and non-linear operator equations, and especially for singular problems, including stochastic matrices.

His knowledge was remarkably broad and deep, and this can be glanced for example in his use of functional analysis tools in applied linear algebra problems. He served as editor of several journals.

Among his honors we mention that he was the recipient of the Czechoslovak National Prize in Science and the Bolzano Medal (Mathematical Sciences). Two conferences were organized in his honor, and three special issues of *Numer. Linear Algebra Appl.*, (10) 2003, (16) 2009, and (22) 2015, were dedicated to him.

His colleagues and friends will remember him for his extraordinary enthusiasm for science and education, for his friendly and generous personality, his sense of humor, his important role in integrating Czech computational mathematicians into the international community, and for his love of music, especially Mozart.

Ivo will be sorely missed. We remember him with gratitude and respect.

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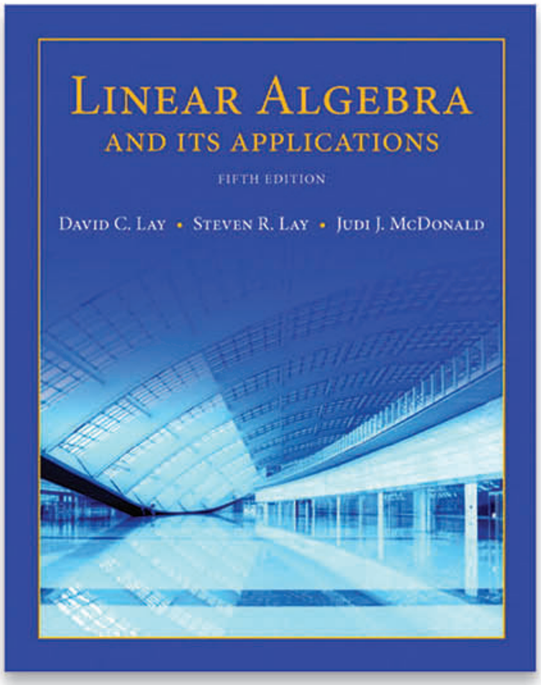

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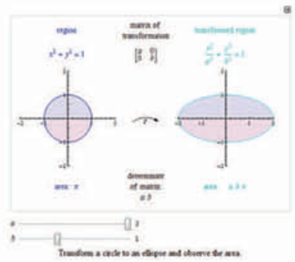

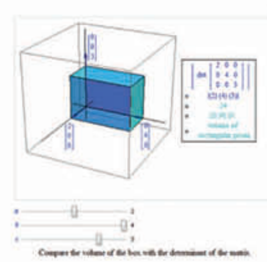

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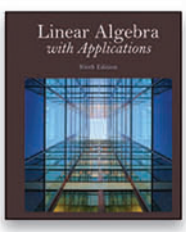

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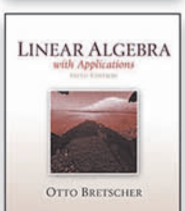

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FEATURE ARTICLE

Chebfun and Linear Algebra

Tobin A. Driscoll, University of Delaware, Newark, DE, USA, driscoll@udel.edu

1. Introduction.

The MATLAB software system deserves a great deal of credit for making vectors and matrices part of everyday thinking and practice in numerical computation. The open-source project Chebfun, written in MATLAB, extends this type of thinking to infinite-dimensional objects [10]. Chebfun extensively uses and illuminates linear algebra.

For example, one creates a *chebfun* object representing the smooth function $\sin(e^{2x})$ as follows:

```
>> f = chebfun(@(x) sin(exp(2*x)),[-1 1])
f =
chebfun column (1 smooth piece)
interval       length     endpoint values
[-1, 1]         48          0.13     0.89
```

(Here and elsewhere we make minor edits to the output in order to save space.) Henceforth we may think of this chebfun as a vector "indexed by" its argument.

```
>> f(0.5)
ans = 4.1078e-01
```

We may take the inner product of this vector with itself,

```
>> f'*f
ans = 7.9892e-01
```

which of course is the integral of the square of the function,

```
>> sum(f.^2)
ans = 7.9892e-01
```

and the square of its 2-norm.

```
>> norm(f).^2
ans = 7.9892e-01
```

A chebfun is not truly infinite-dimensional. Referring to the first output above, the chebfun `f` is a degree-47 polynomial represented in a Chebyshev basis and computed to approximate f to an accuracy close to 16 decimal digits pointwise. All chebfuns have a finite length chosen automatically to achieve (in most cases) similar accuracy. In this sense they are like floating point numbers; Chebfun aims to give numerical answers to high precision. This strategy allows Chebfun to manipulate functions quickly and very generally, unlike exact symbolic manipulation. As an illustration, consider the following:

```
>> x = chebfun('x',[0 1]);
>> g = sin(1/(x^2+1/100));
g =
chebfun column (1 smooth piece)
interval       length     endpoint values
[0, 1]          340       -0.51     0.84
>> tic, sum(g), toc
ans = 2.8294e-01
Elapsed time is 0.004731 seconds.
>> tic, r = roots(g-0.25); toc
Elapsed time is 0.017745 seconds.
>> length(r)
ans = 31
```

Chebfun took less than 0.02 seconds to find all 31 real roots of this function, which upon checking are all accurate to 16 digits. The algorithm for finding roots, by the way, rests on *colleague matrices*, which do for Chebyshev polynomials what companion matrices do for the monomials [6, 7].

2. Quasimatrices: collections of chebfuns.

You may have noticed from the output above that the chebfuns created have an explicit column shape interpretation. This allows us to horizontally concatenate them into a *quasimatrix* whose row dimension is infinite.

```
>> x = chebfun('x');  % default interval is [-1,1]
>> A = [ x^0 x x^2 x^3 x^4 ];
>> size(A)
ans = Inf  5
```

We can find the best approximation (in the least-squares sense) by a quartic polynomial of a function through the normal equations, which are just an ordinary 5×5 linear system computed through inner products.

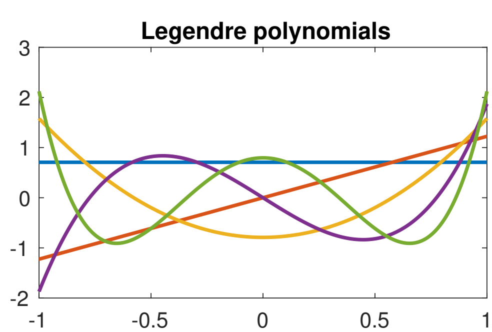
```
>> f = exp(x);   % target function
>> N = A'*A
N =
   2.0000e+00   1.7347e-17   6.6667e-01   1.3878e-17   4.0000e-01
  -6.9389e-18   6.6667e-01  -1.3878e-17   4.0000e-01            0
   6.6667e-01            0   4.0000e-01   1.3878e-17   2.8571e-01
   1.3878e-17   4.0000e-01  -2.0817e-17   2.8571e-01            0
   4.0000e-01            0   2.8571e-01            0   2.2222e-01

>> b = A'*f
b =
   2.3504e+00
   7.3576e-01
   8.7888e-01
   4.4951e-01
   5.5237e-01

>> c = N\b
c =
   1.0000e+00
   9.9795e-01
   4.9935e-01
   1.7614e-01
   4.3597e-02
>> norm(f-A*c)
ans = 4.7049e-04
```

None of that is necessary, as one can directly use the backslash to solve the quasimatrix least squares problem, just as in the discrete case:

```
>> c = A\f
c =
   1.0000e+00
   9.9795e-01
   4.9935e-01
   1.7614e-01
   4.3597e-02
```

In fact, the quasimatrix backslash does just what the ordinary backslash does for a least squares problem: QR factorization, in which the Q is a quasimatrix. Here the QR factorization leads to the Legendre polynomials:

```
>> [Q,R] = qr(A);  plot(Q)
```

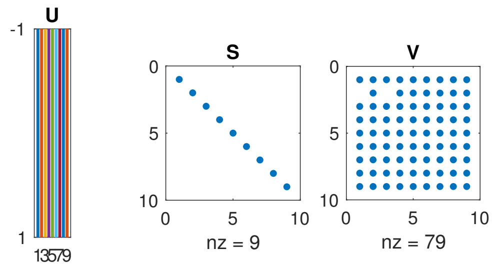


Of course, the monomials are not a particularly well-conditioned basis, as an analog-SVD computation reveals.

```
>> A = [ x^0 x x^2 x^3 x^4 x^5 x^6 x^7 x^8 ];
>> [U,S,V] = svd(A);
>> kappa = S(1,1)/S(end,end)
kappa = 5.5371e+02
```

In this SVD factorization, U is a $\infty \times 9$ quasimatrix and S and V have their usual interpretation.

```
>> subplot(131), spy(U), subplot(132), spy(S), subplot(133), spy(V)
```



Factorizations of quasimatrices are described in detail in [2, 9].

3. Chebops: operators on chebfuns.

If chebfuns are the continuous vectors, then what operates on them as matrices do on vectors? The answer is a *chebop* [4].

```
>> A = chebop(@(u) diff(u,2) + u,[-1 1])
A =
Linear operator:
u |--> diff(u,2)+u
operating on chebfun objects defined on:
[-1,1]
```

Unlike a chebfun, a chebop does not have a fixed-length underlying discrete representation. Instead, it's able to find a discrete representation of itself at any size on demand.

```
>> matrix(A,4)
ans =
   3.3035e+01  -5.1266e+01   2.7862e+01  -1.4207e+01   1.0151e+01  -4.5760e+00
  -6.0745e-01   5.7509e+00  -9.1443e+00   6.9332e+00  -3.2805e+00   1.3482e+00
   1.3482e+00  -3.2805e+00   6.9332e+00  -9.1443e+00   5.7509e+00  -6.0745e-01
  -4.5760e+00   1.0151e+01  -1.4207e+01   2.7862e+01  -5.1266e+01   3.3035e+01
```

This matrix maps interpolating polynomial values (at Chebyshev points) to interpolating polynomial values. It has shape 4×6 since the second derivative is most naturally rectangular [5]. We can "square up" the operator by adding initial or boundary conditions.

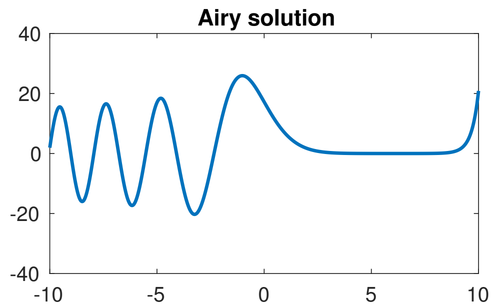
```
>> A.lbc = 0;  A.rbc = @(u) diff(u)-5;
>> matrix(A,4)
ans =
   1.0000e+00            0            0            0            0            0
  -5.0000e-01   1.1056e+00  -1.5279e+00   2.8944e+00  -1.0472e+01   8.5000e+00
   3.3035e+01  -5.1266e+01   2.7862e+01  -1.4207e+01   1.0151e+01  -4.5760e+00
  -6.0745e-01   5.7509e+00  -9.1443e+00   6.9332e+00  -3.2805e+00   1.3482e+00
   1.3482e+00  -3.2805e+00   6.9332e+00  -9.1443e+00   5.7509e+00  -6.0745e-01
  -4.5760e+00   1.0151e+01  -1.4207e+01   2.7862e+01  -5.1266e+01   3.3035e+01
```

With the boundary conditions set, we can solve a boundary-value problem using the backslash operator.

```
>> u = A\1;  norm(A*u - 1)
ans = 2.1359e-11
```

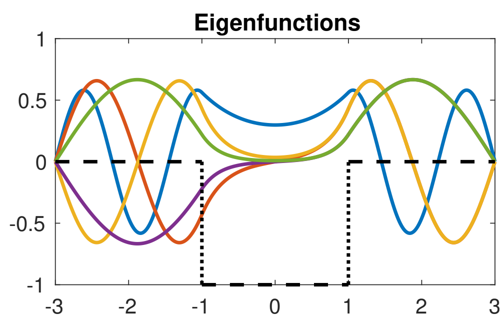
In practice, the BVP is discretized at increasing numbers of points (i.e., polynomial degree) until the coefficients of the highest degree Chebyshev polynomials are sufficiently small. One of Chebfun's strengths relative to other BVP-solving software is its facility with less-standard boundary or side conditions. For instance, here we solve the Airy equation with an interior condition $u(0) = \int xu\,dx$ and a constraint on the average value of the solution.

```
>> A = chebop(@(x,u) diff(u,2) - x.*u,[-10 10]);
>> A.bc = @(x,u) [ u(0)-sum(x.*u); mean(u)-3 ];
>> plot(A\0)
```



We can also compute eigenvalues and eigenfunctions of operators. This is as good a place as any to point out that Chebfun supports piecewise smooth functions, not just smooth ones.

```
>> x = chebfun('x',[-3 3]);   V = (abs(x) < 1);
>> A = chebop(@(x,u) diff(u,2) - 18*V*u,[-3 3]);
>> A.lbc = 0;  A.rbc = 0;
>> [X,D] = eigs(A,5);
>> plot(X), hold on, plot(-V,'k--')
```



These are the eigenfunctions of a square-well potential. The operator is self-adjoint, so they are mutually orthogonal.

```
>> X'*X
ans =
   1.0000e+00   2.3400e-11   7.1907e-11   8.2058e-12   2.7442e-11
   2.3400e-11   1.0000e+00  -4.5478e-08   3.4766e-12  -3.6299e-11
   7.1907e-11  -4.5478e-08   1.0000e+00  -1.6969e-11   4.9153e-11
   8.2058e-12   3.4766e-12  -1.6969e-11   1.0000e+00  -3.6915e-08
   2.7442e-11  -3.6299e-11   4.9153e-11  -3.6915e-08   1.0000e+00
```

Chebfun is able to find the adjoint of a linear operator, including the boundary conditions. It has the same eigenvalues as the original.

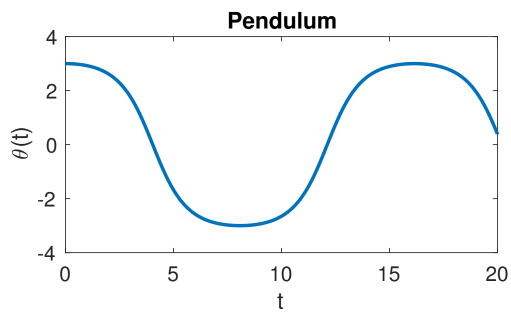
```
>> A = chebop(@(x,u) diff(u,2) + x.*diff(u) - u,[-1 1]);
>> A.lbc = @(u) diff(u) + u;  A.rbc = 0;
>> [X,D] = eigs(A,5);
>> [W,E] = eigs(A',5);
>> [diag(D) diag(E)]
ans =
  -5.0057e+01  -5.0057e+01
  -3.0323e+01  -3.0323e+01
  -1.5533e+01  -1.5533e+01
  -5.7114e+00  -5.7114e+00
   5.8819e-01   5.8819e-01
```

This particular operator does not have orthogonal eigenfunctions. But the eigenfunctions of any operator are biorthogonal with those of its adjoint.

```
>> X'*W
ans =
   9.8960e-01   8.1123e-11  -3.7174e-11  -2.6084e-11   4.3154e-11
  -4.7240e-11   9.9000e-01  -8.5784e-11  -4.6820e-11   7.0478e-11
   1.7196e-11   1.0304e-10   9.9084e-01   8.3718e-11  -9.7975e-11
   1.5576e-11   6.0083e-11  -1.0106e-10   9.9094e-01  -2.6935e-10
  -3.1304e-11  -1.0880e-10   1.3941e-10   3.2765e-10   9.8871e-01
```

The backslash in chebops generalizes MATLAB's backslash in another major way. If presented with a nonlinear operator, it is able to linearize the problem (operator and boundary conditions) with a Fréchet derivative and perform a Newton iteration, all in function space [3].

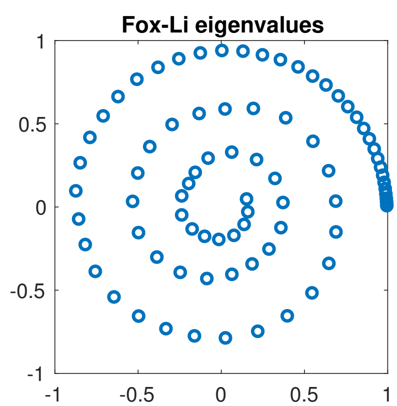
```
>> A = chebop(@(u) diff(u,2) + sin(u),[0 20]);
>> A.lbc = [3;0];
>> linearize(A,chebfun(@(x) 3*cos(x),[0 20]))
ans =
1x1 linear operator
with 2 constraints/boundary conditions.
>> plot(A\0)
```



For much more on how Chebfun's operator representations lead to insight for differential operators, see [1].

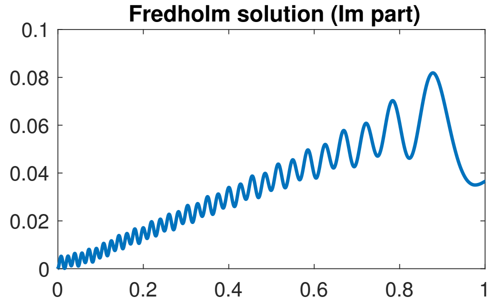
Integral operators are also supported, though only for nonsingular kernels. For example, the Fox–Li operator in optics is a Fredholm operator with an oscillatory kernel.

```
>> F = 64*pi; K = @(x,s) exp(-1i*F*(x-s).^2);
>> L = sqrt(1i*F/pi) * chebop(@(u) fred(K,u));
>> lam = eigs(L,80,'lm');  plot(lam,'o')
```



To solve an equation with this operator, we can solve discrete matrix-vector systems as in the differential case, but we have another option: GMRES, which avoids discretizing the operator altogether, instead applying it repeatedly to given chebfuns.

```
>> A = chebop(@(u) u + fred(K,u),[0 1]);
>> x = chebfun('x',[0 1]);
>> u = gmres(A,x),  plot(imag(u))
Final relative residual: 2.290e-11
u =
chebfun column (1 smooth piece)
interval       length     endpoint values
[0, 1]          199       complex values
```



4. Chebfun2: continuity squared.

Let's conclude with a yet another connection between Chebfun and linear algebra. Two-dimensional functions on rectangles can be represented as a *chebfun2* object [8].

```
>> f = chebfun2(@(x,y) tanh(2*x-y))
f =
chebfun2 object
domain                 rank       corner values
[-1, 1] x [-1, 1]       24       [-0.76   1  -1 0.76]
```

As indicated by the output, a chebfun2 uses a low-rank approximation in the form

$$f(x, y) \approx \sum_{i=1}^{r} g_i(x)h_i(y).$$

This form leverages Chebfun's 1D capabilities. If we think of g_i and h_i as the chebfuns ℓ_i and u_i, we are tempted to write

$$F \approx \sum_{i=1}^{r} \ell_i u_i^*,$$

where $\ell_i u_i^*$ is a continuous analog object called a *cmatrix*. These can be generated by an implementation of Gaussian elimination with complete pivoting [9]. The pair ℓ_1, u_1 introduces agreement with f along a pair of horizontal and vertical lines, the next pair agrees with the residual $f - \ell_1 u_1^*$ along a different pair of lines, and so on. The resulting low-rank approximation is less efficient, rank-wise, than an SVD, but much easier to obtain computationally.

Chebfun has more features and capabilities than are demonstrated here. It is a free download from `www.chebfun.org`, which also hosts a user guide and hundreds of examples.

Acknowledgement: The author thanks Nick Trefethen for his comments on this manuscript.

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LINEAR ALGEBRA EDUCATION

A Contradiction in How Introductory Textbooks Approach Matrix Multiplication?

John Paul Cook[1] and Dov Zazkis[2]

Matrix multiplication is foundational to many of the core concepts in the modern introductory linear algebra curriculum. Indeed, results concerning systems of linear equations, span, linear (in)dependence, and linear transformations can all be (and often are) formulated in terms of matrix multiplication. Though matrix multiplication is simple enough to carry out as a procedure, Harel argued in his commentary on the undergraduate linear algebra curriculum that "understanding must imply knowing *why*, not just *how*" [5, p. 111, emphasis ours]. Matrix multiplication presents some conceptual challenges for students in this regard. For example, matrix multiplication is fundamentally different than the familiar multiplication of integers: It does not 'multiply' in the colloquial sense and, moreover, it is generally not commutative. Instead, matrix multiplication is a 'multiplication' in the sense of abstract algebra: It distributes over (matrix) addition and is associative. This is potentially an unfamiliar and non-intuitive idea for students who have had no prior exposure to abstract algebra. These conceptual peculiarities — along with the importance of matrix multiplication in linear algebra — bring this question to the fore: How are instructors attempting to help students overcome the challenges associated with this unfamiliar and non-intuitive operation?

To provide some initial answers to this question, we used textbooks as a proxy for instructors' approaches and conducted a study of the presentation and exposition of matrix multiplication in 24 introductory linear algebra textbooks that are currently in use at Research-I universities in the United States. While we found substantial variation in the sequencing of the different forms of matrix multiplication and the associated explanations and uses for each, the principal finding of the study involved an apparent discrepancy across textbooks regarding the explanations that the authors used to justify matrix multiplication. Consider, for example, an explanation justifying the need for matrix multiplication in Lay, Lay, and McDonald [8, p. 36]:

> ...a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is more natural. Then you may switch from one formulation of a problem to another whenever it is most convenient. In any case, the matrix equation, the vector equation, and the system of equations are all solved in the same way.

This explanation is an example of isomorphization, which involves "[imposing] an isomorphism on two mathematical structures where one of these structures is familiar to the student" [4, p. 31]. Isomorphization introduces students to a new concept in a way that highlights how the new concept preserves the mathematical structure of a familiar concept. In this particular case, the newly introduced concept of a matrix equation is treated as isomorphic to previously-introduced (and hence more familiar) vector equations and systems of linear equations. Other examples of isomorphization in our sample included:

- "By now we are comfortable with translating back and forth between vector equations and linear systems ... $Ax = b$ is a compact form of the vector equation, which in turn is equivalent to [a] linear system." [6, p. 63]
- "We can use these new concepts to understand a system of equations $Ax = b$. If A and b are given, such a system challenges us to determine whether b is in the span of the columns of A and, if so, to find the coefficients needed to express b as a linear combination of the columns of A." [2, p. 42]
- "The equation $Ax = b$ uses the language of linear combinations right away. The vector Ax is a combination of the columns of A. The equation is asking for a combination that produces b." [9, p. vi, emphasis in original]

Our study also revealed a very different style of justification. Consider this excerpt from Kolman and Hill's textbook:

> One might ask why matrix equality and matrix addition are defined in such a natural way, while matrix multiplication appears to be much more complicated. Only a thorough understanding of the composition of functions and the relationship that exists between matrices and what are called linear transformations would show that the definition of multiplication given previously is a natural one. These topics are covered later in the book. [7, p. 24]

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This explanation is an example of postponement, a strategy that involves temporarily justifying the introduction of a new concept by commenting on its importance and magnitude because, in the estimation of the author(s), a meaningful mathematical justification is not yet accessible to the reader [4]. Other examples of postponement included:

- "Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful for most problems. Experience has led mathematicians to the following more useful definition of matrix multiplication." [1, p. 29]
- "The most natural way of multiplying two matrices might seem to be to multiply corresponding elements when the matrices are of the same size, and to say that the product does not exist if they are of different size. However, mathematicians have introduced an alternative rule that is more useful. It involves multiplying the rows of the first matrix times the columns of the second matrix in a systematic manner." [11, p. 71]
- "We now define the product of two matrices. From the way the other matrix operations have been defined, you might guess that we obtain the product of two matrices by simply multiplying corresponding entries. The definition of product given below is much more complicated than this but also considerably more useful in applications." [10, p. 90]

What makes these two types of rationale so interesting is the tension between them. On one hand, textbooks invoking isomorphization (to vector equations and linear systems) for their initial form of matrix multiplication are attempting to frame matrix multiplication in terms of familiar concepts, procedures, and ideas. On the other, textbooks invoking postponement are (oftentimes explicitly) stating that the rationale for matrix multiplication is unable to be easily understood when it is initially introduced. Informally, we might characterize these two approaches as 'this can be reasonably understood right now using familiar ideas' and 'this can only be understood later using more advanced ideas.'

This apparent discrepancy raises some interesting questions to consider and discuss. While it might be tempting to argue about which approach is better, we contend that such questions are ultimately unanswerable and should be eschewed in favor of more nuanced questions. For example: What are the affordances of each approach? How might each approach affect students' understanding of subsequent topics? Is one approach more ideally suited for an application-based linear algebra course, while the other is suited to a theory-based linear algebra course? We hope that this short report will lead to thoughtful discussion on these (and similar) points. For the full research report, the interested reader may consult our recent publication [3]; a pre-print version is available at `https://goo.gl/gPPXQ8`.

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UPCOMING CONFERENCES AND WORKSHOPS

International Conference on Linear Algebra and its Applications
Manipal, India, December 11–15, 2017

The International Conference on Linear Algebra and its Applications (ICLAA 2017) will be held December 11–15, 2017 at Manipal University, Manipal, India (`http://manipal.edu`). This ILAS endorsed conference will be held in sequel to the earlier conferences CMTGIM 2012 and ICLAA 2014 on the same theme at Manipal University in January 2012 and December 2014 respectively. The theme of the conference will focus on (i) Classical Matrix Theory covering different aspects of Linear Algebra (ii) Matrices and Graphs (iii) Combinatorial Matrix Theory (iv) Matrix & Graph Methods in Statistics, and (v) Covariance Analysis & its Applications. The scientific advisory committee consists of Ravindra B. Bapat, Steve Kirkland, K. Manjunatha Prasad, and Simo Puntanen. The present conference shall provide a platform for leading mathematicians, statisticians, and applied mathematicians working around the globe in the theme area to discuss several research issues on the topic and to introduce new innovations. The fourth DAE-BRNS Meeting on the generation and use of covariance matrices in the applications of nuclear data will start as a preconference meeting and will continue in parallel to the conference as part of Covariance Analysis & its Applications in ICLAA 2017.

Original research articles in the theme areas of (i) Linear Algebra & Matrix Theory, (ii) Combinatorial Matrix Theory, (iii) Matrices and Graphs, and (iv) Matrix and Graph Methods in Statistics may be submitted to a special issue in *Special Matrices* (`https://www.degruyter.com/view/j/spma`) dedicated to the conference. All the articles submitted to the special issue will go through the regular review process of the journal and only the articles which meet the highest standard of quality will be accepted for publication in the special issue.

For more details and registration, please visit `http://iclaa2017.com`, or contact the organizing secretary at iclaa2017@gmail.com or km.prasad@manipal.edu.

Workshop on Graph Spectra, Combinatorics and Optimization (WGSCO2018)
Aveiro, Portugal, January 25–27, 2018

The aim of WGSCO2018 is to bring together academic specialists and promote the broad exchange of information and new developments in Spectral Graph Theory, Combinatorics and Optimization, and other related areas. During the Workshop we will commemorate the 65th birthday of Domingos M. Cardoso, who is a prominent Professor at the University of Aveiro.

The scientific program of the Workshop consists of plenary talks by the Invited Speakers and 20-minute contributed talks. The topics of the Workshop are: Algebraic Combinatorics, Algebraic Graph Theory, Algorithms and Computing Techniques, Combinatorial Optimization, Communications and Control Theory, Enumerative Combinatorics, Extremal Combinatorics, Graph Theory, Optimization in Graphs, Graph Spectra and applications, Linear Optimization, Networks, and Nonlinear Optimization.

For further details, please visit the conference website at `http://wgsco2018.web.ua.pt`.

SIAM Conference on Applied Linear Algebra (SIAM-ALA18)
Hong Kong Baptist University, China, May 4–8, 2018

The SIAM Conferences on Applied Linear Algebra, organized by SIAM every three years, are the premier international conferences on applied linear algebra, which bring together diverse researchers and practitioners from academia, research laboratories, and industries all over the world to present and discuss their latest work and results on applied linear algebra.

The Scientific Committee consists of Co-Chairs – David Bindel (Cornell University, USA), Raymond Chan (The Chinese University of Hong Kong, China), and Michael Ng (Hong Kong Baptist University, China), and Members – Mario Arioli (University of Wuppertal, Germany), Haim Avron (Tel Aviv University, Israel), Zlatko Drmac (University of Zagreb, Croatia), Vladimir Druskin (Schlumberger Doll Research, USA), Melina Freitag (University of Bath, UK), Haesun Park (Georgia Tech, USA), Hans de Sterck (Monash University, Australia), Eugene Tyrtyshnikov (Russian Academy of Science, Russia), Tetsuya Sakurai (University of Tsukuba, Japan), and Chao Yang (Lawrence Berkeley National Laboratory, USA).

Further information can be found at the conference website: `http://www.math.hkbu.edu.hk/SIAM-ALA18/index.html`.

Western Canada Linear Algebra Meeting (WCLAM 2018)
Washington State University, Pullman, USA, May 26–27, 2018

The Western Canada Linear Algebra Meeting (WCLAM) provides an opportunity for mathematicians in western Canada and the USA working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting.

The participation fee is US $30, to be collected at the meeting. This fee will be waived for participating students and post-doctoral fellows. Subject to funding, support of $300 per person will be available for a limited number of participating students and post-doctoral fellows on a first-come, first-served basis.

WCLAM 2018 will have two distinguished invited speakers: Vladimir Nikiforov (University of Memphis) and Jenna Rajchgot (University of Saskatchewan). Further information is available at `http://www.math.wsu.edu/math/faculty/tsat/W-CLAM18.html`.

The organisers of WCLAM 2018 are: Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Manitoba), Sarah Plosker (Brandon University), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria). The local organisers are: Michael Tsatsomeros, Washington State University (tsat@math.wsu.edu) and Judi McDonald, Washington State University (jmcdonald@gmail.com).

Linear Algebra, Matrix Analysis and Applications (ALAMA 2018)
Sant Joan d'Alacant, Spain, May 30–June 1, 2018

The Thematic Network of Linear Algebra, Matrix Analysis and Applications (ALAMA) celebrates every two years (since 2008) the ALAMA Meetings with the objective of bringing together researchers, whether they are members of the network or not, whose work is related to linear algebra, matrix theory and matrix analysis, as well as their possible applications such as (but not restricted to) control theory, cryptography and coding theory, graph theory, numerical polynomial algebra, orthogonal polynomials, signal and image processing, optimization, statistics, computer science, data mining, structure and vibration mechanics, engineering, physics, chemistry, biology, and medicine, among others.

The ALAMA 2018 meeting, organized by the members of the ALAMA Network of the Universities of Alicante and Miguel Hernández de Elche, will be held from May 30 to June 1, 2018, at the Complejo San Juan located at number 2 of Dr. Pérez Mateos Street, in Sant Joan d'Alacant.

Further information can be found at the conference website: `https://web.ua.es/en/alama2018`.

Prairie Discrete Mathematics Workshop
Brandon University, Manitoba, Canada, June 12–15, 2018

The main objective of the PDMW is to bring together researchers in the area of discrete mathematics from across the prairie region as well as neighbouring provinces and states. The goal for the workshop is to provide opportunities for sharing research and joint research projects.

Invited Speakers: Richard Brewster (Thompson Rivers University), Rob Craigen (University of Manitoba), Shonda Gosselin (University of Winnipeg), Gary MacGillivray (University of Victoria), and Karen Meagher (University of Regina).

Contributed talks: We invite anyone working in discrete mathematics, including those working in combinatorics, graph theory, matrix analysis, quantum information theory, and theoretical computer science, to submit an abstract for a contributed talk. Early career researchers, postdocs, and graduate students are particularly encouraged to submit an abstract.

For more information, contact Shahla Nasserasr (nasserasrs@brandonu.ca) or Sarah Plosker (ploskers@brandonu.ca), or visit the conference website: `https://www.brandonu.ca/pdmw`.

International Conference on Algebra and Related Topics (ICART 2018)
Rabat, Morocco, July 2–5, 2018

The International Conference on Algebra and Related Topics (ICART 2018) will be held from the 2^{nd} to the 5^{th} of July 2018 at the Faculty of Sciences, Mohammed V University in Rabat, Morocco.

The conference will cover various research areas presented in the following three sessions: Applied and Computational Homology in Topology, Algebra and Geometry; Homological Algebra, Modules, Rings and Categories; and Linear and Multilinear Algebra and Function Spaces.

Plenary speakers in the session on Linear and Multilinear Algebra and Function Spaces include:

Abdellatif Bourhim (Syracuse University, USA)
Matej Brešar (University of Ljubljana, Slovenia)
Javad Mashreghi (Université Laval, Québec, Canada)
Mostafa Mbekhta (Université Lille, France)
Lajos Molnár (University of Debrecen, Hungary)
Rueda Pilar (Universidad de Valencia, Spain)
Peter Šemrl (University of Ljubljana, Slovenia)

Deadline for abstract submissions: April 20, 2018; Deadline for early registration: April 30, 2018; Deadline for late registration: May 31, 2018. Notification of acceptance will be communicated within one month of abstract submission. More information can be found at `http://www.fsr.ac.ma/icart2018`.

Numerical Analysis and Scientific Computation with Applications (NASCA 2018)
Kalamata, Greece, July 2–6, 2018

The aim of this conference is to bring together researchers working in numerical analysis, scientific computation and applications. Participants will present and discuss their latest results in these areas.

The main topics of the conference are: Large Linear Systems of Equations and Eigenvalue Problems with Preconditioning, Linear Algebra and Control, Model Reduction, Ill-posed Problems, Regularisation, Numerical Methods for PDEs, Approximation Theory, Radial Basis Functions, Meshless Approximation, Optimization, and Applications to Image and Signal Processing, Environment, Energy Minimization, and Internet Search Engines.

Further information can be found at the conference website: `http://nasca18.math.uoa.gr`.

ILAS 2019: Linear Algebra Without Borders
Rio de Janeiro, Brazil, July 8–12, 2019

The 22nd conference of the International Linear Algebra Society, ILAS 2019: Linear Algebra Without Borders, will be held July 8–12, 2019 in Rio de Janeiro, Brazil, at the main campus of Fundação Getúlio Vargas (FGV), a Brazilian think tank and higher education institution founded in 1944 with the aim of promoting Brazil's economic and social development.

The theme of the conference is "Linear Algebra Without Borders" and refers primarily to the fact that Linear Algebra and its myriad of applications are interwoven in a borderless unit. In the conference, the organizers plan to illustrate this by creating a program whose plenary talks and symposia represent the many scientific "countries" of Linear Algebra, and which invites participants to "visit" them. This theme also refers to the openness and inclusiveness of Linear Algebra to researchers of different backgrounds. Ongoing updates and more information about the conference will be found at `http://ilas2019.org`.

Householder Symposium XXI on Numerical Linear Algebra
Selva di Fasano, Italy, June 14–19, 2020

The next Householder Symposium will be held from June 14–19, 2020 at Hotel Sierra Silvana, Selva di Fasano (Br), Italy.

Attendance at the meeting is by invitation, and participants are expected to attend the entire meeting. Applications are solicited from researchers in numerical linear and multi-linear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control. Each attendee is given the opportunity to present a talk or a poster. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. Applications are due by October 31, 2019.

This meeting is the twenty-first in a series, previously called the Gatlinburg Symposia, but now named in honor of its founder, Alston S. Householder, a pioneer of numerical linear algebra. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1, 2017 will be presented. Nominations are due by January 31, 2020.

CONFERENCE REPORTS

The 8th Linear Algebra Workshop (LAW'17)
Ljubljana, Slovenia, June 12–16, 2017

Report by Damjana Kokol Bukovšek

The main theme of the 8th Linear Algebra Workshop was the interplay between operator theory and algebra. After plenary and contributed talks in the morning, the afternoons were devoted to work in smaller groups. A special issue of the journal *Special Matrices* will be published as the proceedings of the workshop. On Tuesday, June 13, there was the Hans Schneider ILAS lecture, given by Vladimir Müller from the Czech Academy of Sciences, Czech Republic. Invited speakers were Balint Farkas, Alexander Guterman, Dijana Ilišević, Thomas Laffey, Chi-Kwong Li, Laurent Marcoux, Mitja Mastnak, Vladimir Müller, Ahmed Sourour, Helena Šmigoc, and Vladimir Troitsky. There were also 14 contributed talks. The Workshop was supported by the Institute of Mathematics, Physics and Mechanics, Ljubljana, the International Linear Algebra Society, the Faculty of Mathematics and Physics of the University of Ljubljana, and the Faculty of Mathematics, Natural Sciences and Information Technologies of the University of Primorska. There were 53 participants attending the workshop. To see a list of those, a complete list of abstracts and some photos, please visit `http://www.law05.si`.

The 6th International Conference on Matrix Analysis and Applications (ICMAA 2017)
Duy Tan University, Da Nang City, Vietnam, June 15–18, 2017

Report by Trung Hoa Dinh

The 6th International Conference on Matrix Analysis and Its Applications (ICMAA 2017) was held in Danang, a beautiful city in Central Vietnam, June 15–18, 2017. The previous conferences were held in China (Beijing, Hangzhou), the United States (Nova Southeastern University), and Turkey (Selçuk University, Konya). Former keynote speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland, Alexander A. Klyachko (ILAS guest speaker), and Shmuel Friedland.

The scientific organizing committee of ICMAA 2017 consisted of Trung Hoa Dinh (Ho Chi Minh City University of Food Industry, Vietnam), Hiroyuki Osaka (Ritsumeikan University, Japan), Tin-Yau Tam (Auburn University, USA), Qing-Wen Wang (Shanghai University, China) and Fuzhen Zhang (Nova Southeastern University, USA).

This meeting aimed to stimulate research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications and to provide an opportunity for researchers to exchange ideas and developments on these subjects.

ICMAA 2017 was supported by Duy Tan University and ILAS. We welcomed 70 participants from 17 countries to Danang. The keynote speaker of the ICMAA 2017 was Dr. Man-Duen Choi. Three invited talks were also given by renowned mathematicians in matrix theory, Professors Tsuyoshi Ando, Abraham Berman and Richard Brualdi.

The early registration started on June 15. All talks were on June 16 and 17 at the main campus of Duy Tan University. On June 18, participants visited the famous Linh Ung Pagoda and Marble Mountains in Danang city. Dr Choi opened the conference with an interesting keynote lecture on tensor products of complex matrices and his mathematical journey. He explained the connection between tensor products and quantum entanglement and the Principle of Locality in quantum information theory. We also welcomed 46 contributed talks on different topics such as: combinatorics, numerical algebra, matrix equations, graph theory, operator algebra theory, non-commutative algebraic geometry, etc.

ICMAA 2017 was a very successful event. At the same time, it was an excellent introduction of many topics in matrix analysis and its applications to the Vietnamese math community. The journal *Acta Mathematica Vietnamica* (`https://link.springer.com/journal/40306`) will publish a special issue on Matrix Analysis and Its Applications in 2018. All are welcome to submit papers for this issue.

Preservers Everywhere
Szeged, Hungary, June 19–23, 2017

Report by György Pál Gehér

Preservers Everywhere was an international conference held in Szeged, Hungary during June 19–23, 2017, organised by Lajos Molnár (University of Szeged, Hungary) and György Pál Gehér (University of Reading, United Kingdom). The venue was the Bolyai Institute of the University of Szeged. This conference was the first international event entirely devoted to the dynamically developing topic commonly referred to as "Preserver Problems". A preserver is a transformation between mathematical structures that leaves a certain quantity/operation/relation/set/etc. invariant. The usual goal is to give a complete description of these transformations. The purpose of the meeting was to bring together mathematicians from various areas who are interested in preserver problems.

The conference attracted 54 participants from more than 20 countries from four continents, and featured 51 talks. There were 11 invited speakers, namely Maria Fernanda Botelho (The University of Memphis, Memphis, USA), Alexander Guterman (Lomonosov Moscow State University, Moscow, Russia), Jan Hamhalter (Czech Technical University of Prague, Prague, Czech Republic), Osamu Hatori (Niigata University, Niigata, Japan), Chi-Kwong Li (College of William and Mary, Williamsburg, USA), Javad Mashreghi (Université Laval, Québec, Canada), Mostafa Mbekhta (Université Lille 1, Villeneuve-d'Ascq, France), Antonio M. Peralta (Universidad de Granada, Granada, Spain), Clément de Seguins Pazzis (Université de Versailles-Saint-Quentin-en-Yvelines, Versailles, France), Peter Šemrl (University of Ljubljana, Ljubljana, Slovenia) and Ngai-Ching Wong (National Sun Yat-sen University, Kaohsiung, Taiwan).

The topics of the talks presented at the conference covered a wide range of mathematics, including linear algebra, operator theory, operator algebras, quantum physics, geometry of Banach spaces, abstract algebra, isometry theory of metric spaces, complex analysis, etc. The meeting has been very successful, it has inspired several discussions on open problems and on topics of common interests; moreover, several new collaborations have been established. The event started on Monday, June 19, at 8:40am with the opening speeches of Lajos Kemény (Vice-Rector for Research and Innovation of the University of Szeged) and Lajos Molnár, and ended on Friday, June 23, at 4:30pm with closing remarks from Lajos Molnár. The event was sponsored in part by a grant from the Hungarian Academy of Sciences; the MTA-DE "Lendület" Functional Analysis Research Group; the University of Szeged; and the Hungarian National Research, Development and Innovation Office (grant no. K115383). More information can be found on the official conference website: `http://www.math.u-szeged.hu/~gehergy/conference.html`.

Special Western Canada Linear Algebra Meeting honouring Peter Lancaster
Banff, Alberta, Canada, July 7–9, 2017

Report by Shaun Fallat

The Western Canada Linear Algebra meeting, or WCLAM, is a regular series of meetings held roughly once every two years since 1993. Professor Peter Lancaster from the University of Calgary, with others, initiated this conference series, and has been involved as a mentor, an organizer, a speaker and a participant in all of the ten or more WCLAM meetings. Two broad goals of the WCLAM series are (a) to keep the community abreast of current developments in this discipline, and (b) to bring together well-established and early-career researchers to talk about their work in a relaxed academic setting.

This particular conference brought together researchers working in a wide-range of mathematical fields, including matrix analysis, combinatorial matrix theory, applied and numerical linear algebra, combinatorics, operator theory, and operator algebras. The highlight of this meeting was the recognition of the outstanding and numerous contributions to the areas of research of Professor Peter Lancaster. Peter's work in mathematics is legendary and his footprint on mathematics in Canada is also significant.

An important special event at this meeting was a birthday celebration to honour the 88$^{\text{th}}$ birthday of Professor Lancaster. The occasion was held in the BIRS lounge on Saturday evening with all participants being involved. There was a cake brought by Peter's daughter, and the event included reading well-wishes from friends and colleagues of Peter's that could not attend the meeting at BIRS.

ILAS 2017: Connections
21$^{\text{st}}$ Conference of the International Linear Algebra Society
Iowa State University, Ames, USA, July 24–28, 2017

Report by Leslie Hogben

The 21$^{\text{st}}$ conference of the International Linear Algebra Society was held July 24-28, 2017 at Iowa State University in Ames, Iowa, USA (`https://ilas2017.math.iastate.edu`). ILAS 2017: Connections was hosted by the ISU Department of Mathematics and attracted 367 participants from 37 countries, including 95 students. The conference featured 277 talks and included a student poster session with 11 posters, for a total of 288 presentations.

Connections was an overarching theme for the conference, underscoring the fact that linear algebra is a subject of central importance in many areas of mathematics and its applications. The *Connections* theme was supported by 24 mini-symposia on a variety of topics, including Combinatorial matrix theory; Compressed sensing and matrix completion, Distance problems in linear algebra, dynamical systems and control; Distances on networks and its applications; Krylov and filtering methods for eigenvalue problems; Linear algebra aspects of association schemes; Linear algebra education; Linear algebra and geometry; Linear algebra and mathematical biology; Linear algebra and positivity with applications to data science; Linear algebra and quantum information science; Matrices, tensors and manifold optimization; Matrix analysis: inequalities, means, and majorization; Matrix polynomials; Matrix techniques in operator algebra theory; The nonnegative inverse eigenvalue problem; Numerical ranges; Random matrix theory for networks; Recent advancements in numerical methods for eigenvalue computation; Representation theory; Solving matrix equations; Spectral graph theory; Toeplitz matrices and Riemann Hilbert problems; and Zero forcing: its variations and applications.

ILAS 2017 *Conference Photograph*

The conference featured 11 plenary lectures from high-profile speakers: Rajendra Bhatia (Indian Statistical Institute, India) gave the Hans Schneider Prize Lecture, *Another metric, another mean*; Stefan Güttel (University of Manchester, UK) gave the SIAG-LA Lecture, *The nonlinear eigenvalue problem*; Miklos Palfia (Kyoto University, Japan) gave the Taussky-Todd Lecture, *On the recent advances in the multivariable theory of operator monotone functions and means*; Helena Šmigoc (University College Dublin, Ireland) gave the LAMA Lecture, *From positive matrices to negative polynomial coefficients*; and Raf Vandebril (KU Leuven, Belgium) gave the ILAS 30$^{\text{th}}$ Anniversary LAA Lecture, *The QR algorithm revisited – core chasing algorithms.* Other plenary lectures were delivered by Hal Caswell (University of Amsterdam, Netherlands), *Matrix population models: Connecting individuals and populations*; Chris Godsil (University of Waterloo, Canada), *Graph invariants from quantum walks*; Willem Haemers (Tilburg University, Netherlands), *Spectral characterizations of graphs*; Tamara Kolda (Sandia National Laboratories, USA), *Tensor decomposition: a mathematical tool for data analysis and compression*; Vern Paulsen (University of Waterloo, Canada), *Quantum chromatic numbers*; and Rachel Ward (University of Texas, USA), *Learning dynamical systems from highly corrupted measurements.*

The conference also featured talks by some of linear algebra's emerging leaders, who were designated as *LAA Early Career Speakers.* These included Gabriel Coutinho (Universidade Federal de Minas Gerais, Brazil), Andrii Dmytryshyn (Umeå University, Sweden), Mahya Ghandehari (University of Delaware, USA), Nathaniel Johnston (Mount Allison University, Canada), and Thomas Mach (Nazarbayev University, Kazakhstan).

Leslie Hogben with Jim and Judy Weaver

The reception at the end of the first day, sponsored by the ISU Department of Mathematics, showcased student posters and provided an opportunity to recognise long-time ILAS volunteers Jim and Judy Weaver and to celebrate the 30$^{\text{th}}$ anniversary of the founding of ILAS as the International Matrix Group. A mid-week excursion to Living History Farms gave participants insight into Iowa's agrarian past. The conference banquet on the penultimate day featured a buffet of Iowa specialties; the Hans Schneider Prize in Linear Algebra was presented to Professor Rajendra Bhatia at the banquet. [*Editor's note: See details on page 34.*]

The conference enjoyed generous support from Elsevier; the Fields Institute; ILAS; the Institute for Mathematics and its Applications; Iowa State University; the SIAM Activity Group in Linear Algebra; and Taylor & Francis.

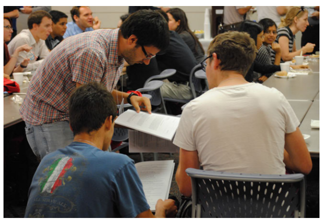
The Scientific Organizing Committee was chaired by Leslie Hogben (Iowa State University), and included Fan Chung (University of California San Diego), Mark Embree (Virginia Tech), Stephen Kirkland (University of Manitoba, Canada), Wolfgang Kliemann (Iowa State University), Bala Rajaratnam (Stanford), Joachim Rosenthal (University of Zurich, Switzerland), Peter Šemrl (University of Ljubljana, Slovenia), Bryan Shader (University of Wyoming), Tatjana Stykel (Universitat Augsburg, Germany), Jared Tanner (Oxford, UK), Raf Vandebril (KU Leuven, Belgium) and Pauline van den Driessche (University of Victoria, Canada). The members of the Local Organizing Committee (all from Iowa State University unless noted otherwise), included Leslie Hogben (chair), Cliff Bergman, Steve Butler, Stephen Kirkland (University of Manitoba), Wolfgang Kliemann, Ryan Martin, Yiu Tung Poon, Bryan Shader (University of Wyoming), Sung-Yell Song, and Michael Young. Xavier Martínez-

Rivera and Jephian C.-H. Lin were Assistant Conference Directors and Seth Selken was the Conference Assistant and Web Master.









Algebraic and Extremal Graph Theory Conference
University of Delaware, Newark, USA, August 7–10, 2017

Report by Sebastian Cioabă

Algebraic and extremal graph theory are important areas of combinatorics with a great symbiotic relationship. To celebrate this fact and to honor the research of Willem Haemers (Tilburg University, The Netherlands), Felix Lazebnik (University of Delaware, USA) and Andrew Woldar (Villanova University, USA), the *Algebraic and Extremal Graph Theory* Conference (`https://www.mathsci.udel.edu/events/conferences/aegt`) took place between August 7 and 10, 2017 at the University of Delaware (UD) in Newark, Delaware, USA.

The conference covered topics related to spectral graph theory, quantum computing, strongly regular graphs, distance-regular graphs, association schemes, Turán-type problems, Moore graphs, cages, algebraic and extremal properties of graphs/digraphs defined by systems of equations, finite groups, expanders, and designs.

The conference consisted of 14 plenary talks: Aida Abiad (Maastricht University), Camino Balbuena (Universitat Politecnica de Catalunyia), Edwin van Dam (Tilburg University), Zoltán Füredi (Rényi Institute and Univ. of Illinois Urbana-Champaign), Chris Godsil (University of Waterloo), Willem Haemers (Tilburg University), Bill Kantor (University of Oregon), Felix Lazebnik (University of Delaware), Misha Muzychuk (Netanya Academic College), Ron Solomon (Ohio State University), Andrew Thomason (Cambridge University), Andrew Woldar (Villanova University), Jason Williford (University of Wyoming) and Qing Xiang (University of Delaware), and 27 contributed talks. There were over 90 participants coming from the United States, Canada, China, India, Singapore, Thailand, Australia, Ukraine, Hungary, The

Netherlands, Israel, Germany, and Brazil, with over half of the participants being young researchers (undergraduate students, graduate students, postdocs or junior faculty).

The conference was organized by Sebastian Cioabă, Robert Coulter and Qing Xiang (all UD) and Gene Fiorini (Muhlenberg College) with the generous support of the National Science Foundation, the University of Delaware, Villanova University, DIMACS, Muhlenberg College, IMA and ILAS.

To further disseminate the knowledge presented and obtained during the conference, the slides from the presentations will be posted on the web page and there will be a special issue of *Discrete Mathematics* dedicated to the conference.

MAT-TRIAD'2017 - International Conference on Matrix Analysis and its Applications Będlewo, Poland, September 25–29, 2017

Report by Katarzyna Filipiak and Dominika Wojtera-Tyrakowska

The 2017 edition of MatTriad, seventh in a series of international conferences on Matrix Analysis and its Applications, was held September 25–29, 2017 in Będlewo (a neighborhood of Poznań, Poland) at the Mathematical Research and Conference Center of the Polish Academy of Sciences. MatTriad brought together researchers sharing an interest in a variety of aspects of matrix analysis and its applications to other areas of science. Matrix theory is used in almost all other parts of mathematics and all areas to which mathematics is applied, and, in return, other parts of mathematics can be very useful in proving things about matrices, sometimes things that are very difficult or impossible to prove using conventional matrix-theoretic methods. Many advances have been made recently, both theoretical and practical, with a lot of discussion between matrix theorists and other mathematicians.

The conference was co-organized by the International Stefan Banach Mathematical Center - Institute of Mathematics of the Polish Academy of Sciences, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland, Institute of Socio-Economic Geography and Spatial Management, Adam Mickiewicz University, Poznań, Poland and Department of Mathematical and Statistical Methods, Poznań University of Life Sciences, Poland. The Scientific Committee was chaired by Tomasz Szulc (Adam Mickiewicz University, Poznań, Poland) and comprised Natália Bebiano (University of Coimbra, Portugal), Ljiljana Cvetković (University of Novi Sad, Serbia), Heike Faßbender (Technical University Braunschweig, Germany) and Simo Puntanen (University of Tampere, Finland). The Organizing Committee was chaired by Augustyn Markiewicz (Poznań University of Life Sciences, Poland) and comprised Francisco Carvalho (Polytechnic Institute of Tomar, Portugal), Katarzyna Filipiak (Poznań University of Technology, Poland), Jan Hauke (Adam Mickiewicz University, Poznań, Poland), Aneta Sawikowska (Poznań University of Life Sciences, Poland), and Dominika Wojtera-Tyrakowska (Adam Mickiewicz University, Poznań, Poland).

The conference consisted of 2 invited lectures, 7 invited talks, 54 contributed talks, and 2 posters, in the topics chosen for this conference, especially Interval Matrices, Total Positivity, and Matrix Methods in Linear Models. There was also a session in memoriam of Prof. Jaroslav Zemánek. The invited lectures were delivered by Fuzhen Zhang (USA): *Tensors and some combinatorial properties of tensors*; and Lynn R. LaMotte (USA): *Deconstructing type III*. The invited speakers were Narayanaswamy Balakrishnan (Canada), Andreas Frommer (Germany), Volker Mehrmann (Germany), Jolanta Pielaszkiewicz (Sweden), K. Manjunatha Prasad (India), Mirloslav Rozložník (Czech Republic), and Ernest Šanca (Serbia).

During the conference dinner, the chair of the Scientific Committee proudly announced the winners of the Young Scientists Awards (YSA) for the best talks presented by Ph.D. students and young scientists: Álvaro Barreras (Spain) and Ryo Tabata (Japan). The winners will be Invited Speakers at the 2019 edition of MatTriad.

The YSA 2017 Committee and the winners: (from the left)
N. Bebiano, S. Puntanen, Á. Barreras (winner), D. von Rosen,
R. Tabata (winner), J. Volaufova, T. Szulc, and M. Rozložník

A special issue of the *Electronic Journal of Linear Algebra (ELA)* will be devoted to MAT TRIAD 2017 with selected papers strongly related with the presentations given at the conference.

The participants shared the common opinion that the conference was extremely fruitful and well organized, with a friendly and warm atmosphere. The list of the participants of MAT TRIAD 2017, abstracts of the talks and posters, the gallery of photographs, and other information can be found at `https://mattriad.wmi.amu.edu.pl`.

MAT TRIAD 2017 group photograph

JOURNAL ANNOUNCEMENT

LAA Special Issue in Honor of Vladimir Sergeichuk

Contributed announcement from Peter Šemrl

Linear Algebra and Its Applications (*LAA*) is pleased to announce a special issue in honor of Professor Vladimir Sergeichuk in recognition of his many important contributions to linear algebra and on the occasion of his 70$^{\text{th}}$ birthday in 2019. LAA solicits papers for the special issue within the entire scope of LAA with a special emphasize on research topics related to the work of Vladimir Sergeichuk. The deadline for submissions of papers is April 1, 2018. All submissions will be subject to normal refereeing procedures and the usual standards of LAA will be applied. They should be submitted via the Elsevier Editorial System EES (`http://ees.elsevier.com/laa`), choosing the special issue called "In Honor of Vladimir Sergeichuk" and the responsible Editor-in-Chief Peter Šemrl.

Authors will have the opportunity to suggest one of the following special editors to handle their submission: Natália Bebiano, Matej Brešar, Vyacheslav Futorny.

ILAS NEWS

2017 ILAS Elections: Nominations

Contributed announcement from Peter Šemrl

The Nominating Committee for the 2017 ILAS elections has completed its work. Nominated for a three year term, beginning March 1, 2018, as ILAS Secretary/Treasurer is: Leslie Hogben

Nominated for the two open three-year terms, beginning March 1, 2018, as "at-large" members of the ILAS Board of Directors are: Maria Isabel Bueno, Lajos Molnar, Abbas Salemi, and Vilmar Trevisan.

Many thanks to the Nominating Committee: Froilan Dopico (Chair), Richard Brualdi, Anne Greenbaum, Yongdo Lim, and Sarah Plosker.

Honors for ILAS Members

The American Mathematical Society (AMS) has recently announced its 2018 AMS Fellows, recognizing individuals who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among those named in 2018 are two ILAS members:

Michele Benzi from Emory University was recognized for contributions in numerical linear algebra, exposition, and service to the profession, and

Thomas Kailath from Stanford University was recognized for contributions to information theory and related areas, and for applications.

2016 Hans Schneider Prize presented to Rajendra Bhatia

P. Šemrl presenting the award to R. Bhatia

At the 21st ILAS Conference in Ames, Iowa, Rajendra Bhatia was presented with the 2016 Hans Schneider prize for lifetime contributions to linear algebra.

Laudation delivered by Richard Brualdi:

"When one thinks of Rajendra Bhatia, one thinks of an erudite and versatile mathematician, a gentle and kind person. MathSciNet list 141 papers with nearly 50 coauthors and amazing number 2831 of citations. Rajendra's versatility and breadth are evident on MathSciNet which classfies his publications into:

Differential geometry, Field theory and polynomials, Fourier analysis, Functional analysis, Geometry, Global analysis, Analysis on manifolds, History and biography, Linear and multilinear algebra and matrix theory, Numerical analysis, Operator theory, Real functions, Systems theory and control.

As evidence of his broad knowledge and accomplishments, here are a few phrases from Rajendra's many publications:

Spectral variation, Grassman powers, perturbation inequalities, singular values of operators, norm inequalities, joint spectra, matrix pencils, Schatten classes, Lyapunov equation, Banach algebras, Fréchet derivative, quantum cohomology, entropy inequalities, Riemannian geometry, noncommutative geometric means, bipolar decomposition, symplectic eigenvalues, commutators, determinant, permanent,...

His favorite coauthors, in terms of number of papers coauthored with, are Chandler Davis, Ludwig Elsner, Tanvi Jain, and Fuad Kittaneh.

However you measure it, Rajendra has had a tremendous influence in mathematics, especially in matrix analysis, positive definite matrices, matrix inequalities, and eigenvalues. His expository skills have been highly recognized, with nine books written, two lecture notes published and, in addition, six books edited. His eagerness to share ideas, his commitment to research, and his love of discussing matrix theory in all its manifestations are well-known characteristics of Rajendra by the many people who have interacted with him.

Rajendra was born in Rohtak, India in 1952 and earned a Ph.D. from the Indian Statistical Institute in Delhi in 1979. Except for the first several years after getting a Ph.D., Rajendra has been at the Indian Statistical Institute in Delhi with, of course, temporary visits elsewhere: Berkeley, Toronto, Guelph, Bielefeld, the Fields Institute, Pisa, Lisbon, ICTP-Trieste, Sungkyunkwan University in South Korea, to name only some.

In addition to his research and book-writing, Rajendra has done extensive editorial work. He has been a very important member of the editorial board of *LAA* for nearly 30 years now. He has also served on the editorial boards of the *SIAM Journal on Matrix Analysis* and *Linear and Multilinear Algebra*. He is also currently on the editorial boards of *Operators and Matrices*, the *Journal of the Ramanujan Mathematical Society*, the *Kyungpook Mathematical Journal*, and *The Mathematical Intelligencer* (as a correspondent). He was Chief Editor of the Proceedings of the ICM held in India in 2010, which was both an honor and a tremendous task.

Rajendra's many awards and honors include:

- Indian National Science Academy Medal for Young Scientists
- Shanti Swarup Prize for Science and Technology
- Fellow of the Indian National Science Academy
- J.C. Bose National Fellow.

Rajendra has so many outstanding achievements that have already been hinted at:

- Matrix Inequalities: Analogues of various classical inequalities for matrices like the Arithmetic-Geometric Mean Inequality and the Schwarz Inequality. His work here has stimulated a tremendous amount of research.
- Matrix Function Perturbations: such as square roots and other powers, absolute value, polar and QR decompositions, tensor powers.
- Sylvester Equation $AX - XB = C$: Sylvester's theorem guarantees a solution when the spectra of A and B are disjoint. Under a certain spectral condition it was shown that the solution can be represented as an integral.
- Perturbation of Eigenvalues and Eigenspaces: generalization of Weyl's inequality for the max-norm distance between ordered eigenvalue list of Hermitian matrices to normal matrices, unitary matrices and unitarily invariant norms, and Hermitian operators on infinite dimensional Hilbert spaces.
- Symplectic Eigenvalues: If A is an even order positive definite matrix, then there is a symplectic matrix S such that $S^T AS = D \oplus D$ where D is a positive diagonal matrix whose diagonal entries are the symplectic eigenvalues of A. They are of interest in quantum mechanics and recently Rajendra has proved analogues for symplectic eigenvalues of relations known for eigenvalues.

There is so much more to the work and person of Rajendra Bhatia. He is a person generous with ideas and advice, an exceptional collaborator. Rajendra, it's a great honor and pleasure for me to be part of this presentation to you of the 2016 ILAS Hans Schneider Prize. Hans would have been delighted. I am sure that everyone here thanks you for all your contributions to mathematics, your service to the ILAS community, and for your friendship."

BOOK REVIEW

***Large Truncated Toeplitz Matrices, Toeplitz Operators, and Related Topics*, The Albrecht Böttcher Anniversary Volume, edited by Dario Bini, Torsten Ehrhardt, Alexei Yu. Karlovich, and Ilya Spitkovsky**

Birkhäuser Basel, 2017, ISBN 978-3-319-49182-0, xxvi+740 pages.

Reviewed by Richard Brualdi, University of Wisconsin–Madison, brualdi@math.wisc.edu

This book is dedicated to Professor Albrecht Böttcher on the occasion of his sixtieth birthday. It is beautifully produced with some wonderful photographs. It contains a list of publications of Albrecht (222 at the time of publication) along with six delightful articles with personal reminiscences by S. Grudsky, J. Jahns, B. Silberman, F.O. Speck, I. Spitkovsky, and D. Wenzel. These articles describe each author's personal and professional relationships with Albrecht, meeting him for the first time, working with him, travelling with him and, in general, sharing with him many adventures. The titles of the articles are revealing: *Albrecht Böttcher – 20 years of Friendship and Joint Work* by Sergei Grudsky, *Salutatory with Regards from the Mathematics Students of Chemnitz* by Jonas Jahns, *Essay on Albrecht Böttcher* by Bernd Silberman, *Meeting Albrecht the Strong* by Frank-Olme Speck, *The Beginning (the way I remember it)* by Ilya Spitkovsky and *Personal Address on the Occasion of Albrecht Böttcher's 60th Birthday* by David Wenzel. Each author of these articles describes the social and working relationship that he has had with Albrecht, Albrecht's high standards for mathematics and mathematical exposition, his lecturing, his activity as a mathematician, his passion for mathematics, his strong work ethic, his tremendous mathematical knowledge and ability and, of course, his mathematical accomplishments. The article by Jahns is particularly revealing in that it is written by a mathematics student at Chemnitz (who was chosen by students at Chemnitz) and reveals the care Albrecht takes with students and his impact as a teacher.

In addition to these six laudations, there are twenty-four research articles by prominent mathematicians on topics on which Albrecht has made contributions. This volume is a fitting tribute to *LAA* senior editor Albrecht Böttcher.

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 57-2, 57-4, 58-1, 58-2a, 58-2c, 58-3.

Problem 57-2: A Matrix Proof of an Exercise in Calculus
Proposed by Meiyue SHAO, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

Present a matrix proof of the integral

$$\int_0^{\pi} \ln(2+\cos x)\,dx = \pi \ln \frac{2+\sqrt{3}}{2}.$$

Solution 57-2.1 by Meiyue SHAO, *Lawrence Berkeley National Laboratory, Berkeley, California, USA*, myshao@lbl.gov

The proof is based on the following lemma.

Lemma: Let α, β, $\gamma \in \mathbb{C}$. The eigenvalues of the $n \times n$ tridiagonal Toeplitz matrix

$$T_n(\beta,\alpha,\gamma) = \begin{bmatrix} \alpha & \gamma & & & \\ \beta & \alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \alpha & \gamma \\ & & & \beta & \alpha \end{bmatrix}$$

are

$$\lambda_k = \alpha + 2\sqrt{\beta\gamma}\cos\frac{k\pi}{n+1}, \qquad (k=1,\ldots,n).$$

A proof of this lemma can be found in, e.g., [1, Theorem 2.4].

We discretize the integral using the right Riemann sum on equally spaced nodes as

$$\int_0^{\pi} \ln(2+\cos x)\,dx = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \ln\Big(2+\cos\frac{k\pi}{n}\Big) = \lim_{n\to\infty} \frac{\pi}{n} \ln \prod_{k=1}^{n} \Big(2+\cos\frac{k\pi}{n}\Big).$$

Let $D_n = \det T_n(1/2, 2, 1/2)$, where T_n is the tridiagonal Toeplitz matrix as defined in the lemma. It then follows from the lemma that

$$\prod_{k=1}^{n} \Big(2+\cos\frac{k\pi}{n}\Big) = \prod_{k=1}^{n-1} \Big(2+\cos\frac{k\pi}{n}\Big) = D_{n-1}.$$

Expanding the determinant with respect to its first row yields the three-term recurrence

$$D_{n+1} = 2D_n - \frac{1}{4}D_{n-1}.$$

Thus, there exist constants C_1, $C_n \in \mathbb{R}$ such that

$$D_n = C_1\theta_1^n + C_2\theta_2^n \qquad (n \in \mathbb{N}_+),$$

where

$$\theta_1 = \frac{2+\sqrt{3}}{2} \qquad \text{and} \qquad \theta_2 = \frac{2-\sqrt{3}}{2}$$

are eigenvalues of the recurrence. Because D_n is rational but θ_2 is irrational, C_1 cannot be zero. In addition, the positive definiteness of $T_n(1/2, 2, 1/2)$ implies $D_n > 0$. Consequently

$$C_1 = \lim_{n\to\infty}\left[\frac{D_n}{\theta_1^n} - C_2\Big(\frac{\theta_2}{\theta_1}\Big)^n\right] = \lim_{n\to\infty} \frac{D_n}{\theta_1^n} > 0.$$

Therefore, we obtain

$$\int_0^{\pi} \ln(2+\cos x)\,dx = \lim_{n\to\infty} \frac{\pi}{n}\ln(D_{n-1}) = \lim_{n\to\infty} \frac{\pi}{n}\ln(C_1\theta_1^{n-1}) = \pi\ln\theta_1.$$

Remark: In fact, it can be determined from the initial conditions $D_1 = 2$ and $D_2 = 15/4$ that

$$C_1 = \frac{\theta_1}{\sqrt{3}} = \frac{2+\sqrt{3}}{2\sqrt{3}}, \qquad C_2 = -\frac{\theta_2}{\sqrt{3}} = \frac{-2+\sqrt{3}}{2\sqrt{3}}.$$

We show a way to avoid calculating them explicitly in this problem.

Reference

[1] A. Böttcher and S. Grudsky, *Spectral Properties of Banded Toeplitz Matrices*, SIAM, Philadelphia, PA, USA, 2005.

Problem 57-4: Factors of a Determinantal Polynomial

Proposed by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, rhorn@math.utah.edu
and Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

Let $A \in M_n(\mathbb{C})$. Show that

$$\det(xA + yA^T) = c(xy)^{\ell}(x+y)^m(x-y)^p\prod_{i=1}^{q}((x+\mu_i y)^{r_i}(y+\mu_i x)^{r_i}), \tag{1}$$

in which ℓ, m, p, q are nonnegative integers; $r_1, r_2, \ldots, r_q$ are positive integers if $q > 0$; the μ_i are distinct and each $\mu_i \in \mathbb{C}\backslash\{0, \pm 1\}$; and $c \in \mathbb{C}$ does not depend on x or y. If $\operatorname{null} A \cap \operatorname{null} A^T \neq \{0\}$, then $c = 0$. However, if

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

then $c = 0$ but $\operatorname{null} A \cap \operatorname{null} A^T = \{0\}$. Find a necessary and sufficient condition such that $c = 0$.

Solution 57-4.1 by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, rhorn@math.utah.edu
and Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

Each singular matrix $A \in \mathbb{C}^{n\times n}$ is congruent to a direct sum of nilpotent Jordan blocks and possibly a nonsingular matrix; see [2, 3] or [1, Section 4.5]. Specifically, there is a nonsingular $S \in \mathbb{C}^{n\times n}$ and a nilpotent $N \in \mathbb{C}^{n\times n}$ such that either $SAS^T = N$ or $SAS^T = M \oplus N$, in which M is nonsingular and N is a direct sum of nilpotent Jordan blocks $J_k(0)$; the sizes and numbers of nilpotent blocks are uniquely determined by A.

If A is singular and congruent to $M \oplus N$, then

$$\begin{aligned} xA + yA^T &= S\left(x\,(M\oplus N) + y\,(M\oplus N)^T\right)S^T \\ &= S\left(\left(xM + yM^T\right) \oplus \left(xN + yN^T\right)\right)S^T, \end{aligned}$$

and consequently,

$$\det\left(xA + yA^T\right) = (\det S)^2 \det\left(xM + yM^T\right)\det\left(xN + yN^T\right).$$

The decompositions in the preceding paragraphs reduce verification of (1) to two cases:

1. A is nonsingular. In this case,

$$\det(xA + yA^T) = (\det A^T)\det(xA^{-T}A + yI),$$

in which A^{-T} denotes $(A^T)^{-1}$. The inverse of $A^{-T}A$ is $A^{-1}A^T$, which is similar to its transpose, AA^{-T}, which is similar to $A^{-T}A$ since $AA^{-T} = A(A^{-T}A)A^{-1}$. Thus, $A^{-T}A$ is similar to its inverse, which means that its Jordan canonical form can contain only blocks of the form $J_k(1)$, $J_k(-1)$, or $J_k(\mu) \oplus J_k(\mu^{-1})$, in which $\mu \in \mathbb{C}\backslash\{0, \pm 1\}$. The blocks $J_k(1)$ and $J_k(-1)$ lead to factors of $\det\left(xA + yA^T\right)$ of the form $(x+y)^m$ and $(x-y)^p$, respectively, in which m and p are positive integers; each such factor is multiplied by a nonzero constant that depends only on A. A pair of blocks $J_k(\mu) \oplus J_k(\mu^{-1})$ leads to a factor of $\det\left(xA + yA^T\right)$ of the form $(x+\mu y)^k (y + \mu x)^k$ for some positive integer k, multiplied by a nonzero constant that depends only on A.

2. A is a nilpotent Jordan block. For each positive integer m, one checks that

$$\det\left(xJ_{2m}(0) + yJ_{2m}(0)^T\right) = (-1)^m (xy)^m$$

and

$$\det\left(xJ_{2m-1}(0) + yJ_{2m-1}(0)^T_{2m-1}\right) = 0.$$

If A is nonsingular, then $c \neq 0$. If A is singular and $N = J_{n_1}(0) \oplus \cdots \oplus J_{n_\tau}(0)$, then $c = 0$ if and only if at least one of the nilpotent block sizes n_i is odd. See [3, Theorem 1] for an algorithm to compute the n_i. For example, the matrix in (2) has $M = [1]$, $\tau = 1$, and $n_1 = 3$.

References

[1] R.A. Horn and C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

[2] R.A. Horn and V.V. Sergeichuk, “Congruence of a square matrix and its transpose”, *Linear Algebra Appl.* 389 (2004) 347–353.

[3] R.A. Horn and V.V. Sergeichuk, “A regularization algorithm for matrices of bilinear and sesquilinear forms”, *Linear Algebra Appl.* 412 (2006) 380-395.

Problem 58-1: Singular Value Decomposition of Involutory and Skew-Involutory Matrices
Proposed by Richard William Farebrother, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

A complex $n \times n$ matrix A is involutory if it satisfies $A^2 = I_n$ and skew-involutory if it satisfies $A^2 = -I_n$. Let A be a matrix and UDV be its singular value decomposition. Show that if there exists a complex number s with $|s| = 1$ such that $A^2 = sI_n$ then $(VU)^2 = sI_n$. In particular if A is involutory (resp. skew involutory) then VU is also involutory (resp. skew involutory).

Solution 58-1.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Let $W = VU$, which is also unitary. Then

$$sI = A^2 = (UDV)(UDV) = UDWDV \quad \text{and so} \quad DWD = sU^{-1}V^{-1} = sW^*.$$

We write this last equation in terms of the entries of the matrices $W = [w_{ij}]$ and $D = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$:

$$\sigma_i \sigma_j w_{ij} = s\overline{w_{ji}}, \quad 1 \leq i, j \leq n.$$

When $i = j$, this yields $\sigma_i^2 |w_{ii}| = |s\overline{w_{ii}}| = |w_{ii}|$, and so $\sigma_i = 1$ or $w_{ii} = 0$. Combining the last displayed equation with $\sigma_j \sigma_i w_{ji} = s\overline{w_{ij}}$ when $i \neq j$ yields

$$\sigma_j \sigma_i w_{ji} = \frac{s\bar{s}w_{ji}}{\sigma_i \sigma_j} \quad \text{and so} \quad (\sigma_i \sigma_j)^2 |w_{ji}| = |w_{ji}|.$$

Hence, when $i \neq j$, $\sigma_i \sigma_j = 1$ or $w_{ji} = 0$. By symmetry we also have, when $i \neq j$, $\sigma_i \sigma_j = 1$ or $w_{ij} = 0$.

From $DWD = sW^*$, we have $DWDW = sW^*W = sI$. We show that $DWDW = W^2$ and so $W^2 = sI$. The (i,j)-entry of $DWDW$ is $\sum_{k=1}^n \sigma_i \sigma_k w_{ik} w_{kj}$. From our above conclusions, we see that each term in this sum is either zero (when $w_{ik} = 0$ or $w_{kj} = 0$) or $w_{ik} w_{kj}$ (when $\sigma_i \sigma_k = 1$). In either case, this term equals $w_{ik} w_{kj}$.

Solution 58-1.2 by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, rhorn@math.utah.edu
and Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

Suppose that $s = e^{i\theta}$, with $\theta \in [0, 2\pi)$, and let $z = e^{i\theta/2}$. Let A be an $n \times n$ complex matrix. Then the following facts are immediate:

1. $A^2 = sI_n$ implies A is nonsingular.

2. $A^2 = sI_n$ if and only if $(z^{-1}A)^2 = I_n$.

3. $A = U\Sigma V$ is a singular value decomposition (SVD) of A if and only if $z^{-1}A = (z^{-1}U)\Sigma V$ is an SVD of $z^{-1}A$.

4. If U and V are $n\times n$ unitary matrices, then both $(VU)^2 = sI_n$ and $(UV)^2 = sI_n$ if and only if both $\big(V(z^{-1}U)\big)^2 = I_n$ and $\big((z^{-1}U)V\big)^2 = I_n$. Also, $z^{-1}U$ is unitary if and only if U is unitary.

5. A nonsingular matrix A has unique right and left polar decompositions: $A = UP$ and $A = QU$, in which P and Q are positive definite Hermitian matrices and U is unitary [1, Theorem 7.3.1]. Furthermore, $(z^{-1}U)P$ and $Q(z^{-1}U)$ are the unique right and left polar decompositions of $z^{-1}A$.

In light of $(1)-(5)$, we can restrict our attention to the case $A^2 = I_n$.

Theorem. Suppose that the $n \times n$ complex matrix A is an involution ($A^2 = I_n$). Let $A = UP$ and $A = QU$ be the unique right and left polar decompositions of A, in which case P and Q are positive definite Hermitian matrices and U is unitary. Then $Q = P^{-1}$ and U is an involution.

Proof. Since $A = A^{-1}$, we have

$$QU = A = A^{-1} = (UP)^{-1} = P^{-1}U^*.$$

The factors in the polar decompositions of A are unique, so $Q = P^{-1}$, and $U = U^*$, and $U^2 = UU^* = I_n$. □

Theorem. Suppose that A is an $n \times n$ complex matrix such that $A^2 = I_n$. Let $A = U\Sigma V$ be an SVD for A. Then VU and UV are involutions.

Proof. The right polar decomposition of A is $A = (UV)(V^*\Sigma V)$, so UV is an involution by the previous theorem. Since

$$VU = V(UV)V^*$$

is similar to UV, VU is also an involution. □

Reference

[1] R.A. Horn and C.R. Johnson, *Matrix Analysis,* 2^{nd} ed., Cambridge University Press, 2013.

Problem 58-2: A Matrix Exponential Problem

Proposed by Moubinool Omarjee, *Lycée Henri IV, Paris, France*, ommou@yahoo.com

Let A be an n by n matrix with complex entries. The matrix exponential e^A is defined using the usual power series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

We say that an n by n matrix A has the entrywise exponential property if the (i, j)-entry of e^A is $e^{a_{ij}}$ for all $1 \le i, j \le n$.

(a) Show that an n by n skew-Hermitian matrix has the entrywise exponential property only if $n = 1$.

(b) Show that an n by n Hermitian matrix has the entrywise exponential property only if $n = 1$.

(c) Show that no two by two matrix has the entrywise exponential property.

Editor's note: The conjecture that an n by n matrix has the entrywise exponential property only if $n = 1$ is an open problem.

Solution 58-2.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

(parts (a) and (c) only) If $A = [a_{ij}]$ then $e.A$ will denote the matrix $[e^{a_{ij}}]$.

(a) Since $A^* = -A$, we have $(e^A)^* = e^{-A}$, and so e^A is unitary. Thus, if $e.A = e^A$ then the diagonal entries of $e.A$, namely $e^{a_{ii}}$, have absolute value 1, since the diagonal entries of a skew-symmetric matrix are pure imaginary. Since $|e^{a_{ii}}| = 1$ and each row and column of $e.A$ has Euclidean norm one, all the off-diagonal entries of $e.A$ must be zero. But the off-diagonal entries are all in the range of the exponential function, which is never zero. Therefore, when $n \ge 2$, we have a contradiction.

(c) Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and let λ, μ denote the eigenvalues of A. Hence $a + d = \mathrm{trace}(A) = \lambda + \mu$. If $e.A = e^A$ then the eigenvalues of $e.A$ must be e^λ, e^μ. Therefore

$$e^a e^d - e^b e^c = \det(e.A) = e^\lambda e^\mu = e^{\lambda+\mu} = e^{a+d}$$

and so $e^b e^c = 0$, which is again a contradiction.

Editor's note: The method of solution to part (c) serves as a hint for the still open part (b).

Problem 58-3: Upper and Lower Bounds of Ranks of Matrices

Proposed by Yongge Tian, *Central University of Finance and Economics, Beijing, China*, yongge.tian@gmail.com

(a) Let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, where X_{11}, X_{12}, X_{21}, and X_{22} are $m_1 \times n_1$, $m_1 \times n_2$, $m_2 \times n_1$, and $m_2 \times n_2$ matrices, respectively, and assume that $\mathrm{rank}(X) = t$. Show that the rank of X_{11} satisfies the inequalities

$$\max\{0, t - m_2 - n_2\} \le \mathrm{rank}(X_{11}) \le \min\{m_1, n_1, t\}$$

and also show that these upper and lower bounds are best possible for all values of m_1, m_2, n_1, n_2, and all t with $0 \le t \le \min\{m_1 + m_2, n_1 + n_2\}$.

(b) Let X_1, X_2, and X_3 be $m_1 \times m_2$, $m_2 \times m_3$, and $m_3 \times m_4$ matrices with $\mathrm{rank}(X_1) = t_1$, $\mathrm{rank}(X_2) = t_2$, and $\mathrm{rank}(X_3) = t_3$. Show that

$$\max\{0, t_1 + t_2 + t_3 - m_2 - m_3\} \le \mathrm{rank}(X_1X_2X_3) \le \min\{t_1, t_2, t_3\},$$

and also show that the upper and lower bounds are best possible for all appropriate values of m_1, m_2, m_3, m_4, t_1, t_2, t_3.

Solution 58-3.1 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

(a) For the first inequality, we may assume $t > m_2 + n_2$. Suppose $\mathrm{rank}(X_{11}) \le t - m_2 - n_2 - 1$. Then X_{11} has at most $t - m_2 - n_2 - 1$ linearly independent columns. Also X_{12} has at most n_2 linearly independent columns, and so $[X_{11}\ X_{12}]$ has rank at most $(t - m_2 - n_2 - 1) + n_2 = t - m_2 - 1$. Since $[X_{21}\ X_{22}]$ has rank at most m_2, the rank of X is at most $(t - m_2 - 1) + m_2 = t - 1 < \mathrm{rank}(X)$, a contradiction. For the second inequality, let $E = \left[\begin{smallmatrix} I \\ O \end{smallmatrix}\right]$, where I is $n_1 \times n_1$ and the zero matrix is $n_2 \times n_1$. Then $XE = \left[\begin{smallmatrix} X_{11} \\ X_{21} \end{smallmatrix}\right]$, and so

$$\mathrm{rank}(X_{11}) \le \mathrm{rank}\left[\begin{smallmatrix} X_{11} \\ X_{21} \end{smallmatrix}\right] = \mathrm{rank}(XE) \le \mathrm{rank}(X) = t.$$

It is well known that the rank of a matrix cannot exceed its number of rows or columns, and so the second inequality holds as well.

To show that the lower bound is the best possible, we first assume that $t \le m_2 + n_2$. If X has the form $\left[\begin{smallmatrix} O & X_{12} \\ X_{21} & O \end{smallmatrix}\right]$, then $\mathrm{rank}(X) = \mathrm{rank}(X_{12}) + \mathrm{rank}(X_{21})$. Hence, if we can construct X_{12} and X_{21} such that $t \le \mathrm{rank}(X_{12}) + \mathrm{rank}(X_{21})$, we can construct X of the above form so that $\mathrm{rank}(X) = t$. For example, if $m_2 \le n_1$ and $n_2 \le m_1$, then we can construct X_{12} and X_{21} such that $\mathrm{rank}(X_{12}) = n_2$ and $\mathrm{rank}(X_{21}) = m_2$. Thus, since $t \le m_2 + n_2$, we can construct X so that $X_{11} = O$ and $\mathrm{rank}(X) = t$. Similarly, such an X can be constructed when $n_1 < m_2$ and $n_2 \le m_1$, since $t \le n_1 + n_2 < m_2 + n_2$ and since we can construct X_{12} and X_{21} such that $\mathrm{rank}(X_{12}) = n_2$ and $\mathrm{rank}(X_{21}) = n_1$. The case $m_2 \le n_1$ and $m_1 < n_2$ is analogous to the preceding case. Now consider the case when $n_1 < m_2$ and $m_1 < n_2$. If $t \le n_2$, construct $X = \left[\begin{smallmatrix} O & X_{12} \\ O & X_{22} \end{smallmatrix}\right]$, where $\mathrm{rank}\left[\begin{smallmatrix} X_{12} \\ X_{22} \end{smallmatrix}\right] = t$. This is possible since $t \le m_1 + m_2$. If $t > n_2$, let $X = \left[\begin{smallmatrix} O & I_{n_2} \\ U & O \end{smallmatrix}\right]$, where $U = \left[\begin{smallmatrix} O & O \\ I_{t-n_2} & O \end{smallmatrix}\right]$. Since $t \le m_2 + n_2$, we have $t - n_2 \le m_2$ and so X_{11} lies within the zero matrix above I_{n_2}. Also, I_{t-n_2} does lie within U since $t - n_2 \le n_1 + n_2 - n_2 = n_1$ and $t - n_2 \le m_1 + m_2 - n_2$.

Finally, assume $t > m_2 + n_2$. Then $n_2 < t - m_2 \le m_1 + m_2 - m_2 = m_1$ and, similarly, $m_2 < n_1$. Let

$$X_{11} = \begin{bmatrix} O & I_{t-m_2-n_2} \\ O & O \end{bmatrix}, \quad X_{21} = \begin{bmatrix} O & O \\ I_{m_2} & O \end{bmatrix}, \quad X_{12} = \begin{bmatrix} O & O \\ O & I_{n_2} \end{bmatrix}, \quad X_{22} = O.$$

Note that the position of I_{m_2} is to the left of the position of $I_{t-m_2-n_2}$, since $t \le n_1 + n_2$ implies $m_2 \le n_1 - (t - m_2 - n_2)$. Similarly, the position of I_{n_2} is below the position of $I_{t-m_2-n_2}$, and so $\mathrm{rank}(X) = (t - m_2 - n_2) + m_2 + n_2 = t$.

To show that the upper bound is the best possible, we consider three cases. (i) If $t \le \min(m_1, n_1)$, let $X = \left[\begin{smallmatrix} X_{11} & O \\ O & O \end{smallmatrix}\right]$, where X_{11} is any $m_1 \times n_1$ matrix of rank t. (ii) If $m_1 < t$ and $m_1 \le n_1$, let $X = \left[\begin{smallmatrix} I_{m_1} & O \\ O & U \end{smallmatrix}\right]$, where $U = \left[\begin{smallmatrix} O & O \\ O & I_{t-m_1} \end{smallmatrix}\right]$. Since $m_1 \le n_1$, $\mathrm{rank}(X_{11}) = \mathrm{rank}(I_{m_1}) = m_1$. Also, I_{t-m_1} does lie within U, since $t - m_1 \le n_1 + n_2 - m_1$ and $t - m_1 \le m_1 + m_2 - m_1 = m_2$. Case (iii), where $n_1 < t$ and $n_1 \le m_1$, is similar to case (ii).

(b) This part will follow from the proposition:

Proposition. Let X be an $m \times n$ matrix of rank t and S be a subspace of $\mathbb{C}^n$ of dimension u. Then

$$\max\{0, t+u-n\} \leq \dim(X(S)) \leq \min\{t, u\}.$$

Also, for any permissible integers m, n, t, u and any subspace $S \subseteq \mathbb{C}^n$ such that dim$(S) = u$, there exist $m \times n$ matrices X and Y of rank t such that dim$(X(S)) = \min\{t, u\}$ and dim$(Y(S)) = \max\{0, t+u-n\}$.

Proof. Since $X(S) \subseteq X(\mathbb{C}^n)$, we have $\dim(X(S)) \leq \dim(X(\mathbb{C}^n)) = t$. Also, since $X(S) = \mathrm{range}(XI)$, where I is the injection from S into $\mathbb{C}^n$, we have $\dim(X(S)) = \mathrm{rank}(XI) \leq \mathrm{rank}(I) = u$. Thus $\dim(X(S)) \leq \min\{t, u\}$. To see that this inequality is sharp, suppose first that $t \leq u$. Choose subspaces $T \subseteq S$ and $U \subseteq \mathbb{C}^m$ of dimension t. Then choose subspaces $T_1 \subseteq S$ and $S_1 \subseteq \mathbb{C}^n$ such that $S = T \oplus T_1$ and $\mathbb{C}^n = S \oplus S_1$ (direct sums). Choose X to be a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $X(T) = U$ and $X(T_1 \oplus S_1) = \{0\}$. Then $\mathrm{rank}(X) = t$ and $\dim(X(S)) = \dim(X(T)) = t$. When $u < t$, choose subspaces $T \subseteq \mathbb{C}^n$ and $U \subseteq \mathbb{C}^m$ of dimension t such that $S \subseteq T$. Also choose a subspace $T_1 \subseteq \mathbb{C}^n$ such that $\mathbb{C}^n = T \oplus T_1$. Choose X to be a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $X(T) = U$ and $X(T_1) = \{0\}$. Then $\dim(X(S)) = u$ since X is one-to-one on T.

Let K be the kernel of X and choose a subspace $L \subseteq \mathbb{C}^n$ such that $\mathbb{C}^n = K \oplus L$ (direct sum). Then $\dim(L) = t$ and $\dim(S + L) \leq n$; hence, since X is one-to-one on L,

$$\dim(X(S)) \geq \dim(S \cap L) = \dim(S) + \dim(L) - \dim(S + L) \geq u + t - n.$$

To see that this inequality is sharp, suppose first that $t + u \leq n$. Choose subspaces $T_1, T_2 \subseteq \mathbb{C}^n$ and $U \subseteq \mathbb{C}^m$ such that $\dim(T_1) = t$ and $\dim(T_2) = n - t - u$, $\mathbb{C}^n = S \oplus T_1 \oplus T_2$, and $\dim(U) = t$. Choose X to be a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $X(S) = \{0\}$, $X(T_2) = \{0\}$, and $X(T_1) = U$. When $t + u > n$, choose subspaces $S_1 \subseteq \mathbb{C}^n$, $T_1, T_2 \subseteq S$, and $U \subseteq \mathbb{C}^m$ such that $\mathbb{C}^n = S \oplus S_1$, $S = T_1 \oplus T_2$, $\dim(T_1) = n - t$, $\dim(T_2) = t + u - n$, and $\dim(U) = t$. Then choose X to be a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $X(S_1 \oplus T_2) = U$ and $X(T_1) = \{0\}$. Then $X(S) = X(T_2)$, which has dimension $t + u - n$, since X is one-to-one on T_2. $\square$

Part (b) and its generalization to any finite product of at least two matrices follow directly.

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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce 5 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (`pereirar@uoguelph.ca`).

NEW PROBLEMS:

Problem 59-1: Rationality of the Matrix Exponential
Proposed by Ovidiu FURDUI, *Technical University of Cluj-Napoca, Cluj-Napoca, Romania*, ofurdui@yahoo.com

Let $M_n(\mathbb{Q})$ denote the set of n by n matrices with rational entries. Let $n \geq 2$ be an integer and let $A \in M_n(\mathbb{Q})$. The matrix exponential e^A is defined using the usual power series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Prove that $e^A \in M_n(\mathbb{Q})$ if and only if A is a nilpotent matrix.

Problem 59-2: Root of Cauchy-Bunyakovsky-Schwarz
Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si
and Tatjana PETEK, *University of Maribor, Maribor, Slovenia*, tatjana.petek@um.si

Let $A \in M_n(\mathbb{C})$, $n \geq 3$. Let $\|\cdot\|$ denote the Euclidean norm. Show that the following conditions are equivalent:

(i) A is a scalar matrix.

(ii) $\sqrt{\|A\mathbf{t}\|^2\|\mathbf{t}\|^2 - |\mathbf{t}^*A\mathbf{t}|^2} = 0$ for every $\mathbf{t} \in \mathbb{C}^n$.

(iii) $\sqrt{\|A\mathbf{t}\|^2\|\mathbf{t}\|^2 - |\mathbf{t}^*A\mathbf{t}|^2}$ is a polynomial in $2n$ real variables of $\mathrm{Re}(\mathbf{t})$ and $\mathrm{Im}(\mathbf{t})$, $\mathbf{t} \in \mathbb{C}^n$.

(iv) The map $\mathbf{t} \mapsto \sqrt{\|A\mathbf{t}\|^2\|\mathbf{t}\|^2 - |\mathbf{t}^*A\mathbf{t}|^2}$, $\mathbf{t} \in \mathbb{C}^n$, regarded as a function of $2n$ real variables of $\mathrm{Re}(\mathbf{t})$ and $\mathrm{Im}(\mathbf{t})$, is differentiable at every eigenvector $\mathbf{t}_0$ of A and thus at every vector $\mathbf{t} \in \mathbb{C}^n$.

What happens if $n = 2$?

Problem 59-3: A Matrix Limit
Proposed by Ovidiu FURDUI, *Technical University of Cluj-Napoca, Cluj-Napoca, Romania*, ofurdui@yahoo.com

Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Evaluate the limit

$$\lim_{n\to\infty} \begin{bmatrix} \cos\left(\frac{a}{n}\right) & \frac{b}{n} \\ \frac{c}{n} & \cos\left(\frac{d}{n}\right) \end{bmatrix}^n.$$

Problem 59-4: Path Connectedness
Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Show that the set of complex n by n matrices with n distinct eigenvalues is path connected. Is the set of real n by n matrices with n distinct eigenvalues path connected?

Problem 59-5: Domination in the Derangement Graph
Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

In graph theory, a subset D of the vertex set of a graph G is called a *dominating set* if every vertex of G not in D is adjacent to a vertex in D. The derangement graph of order n, denoted Γ_n, is the graph whose vertices are the n by n permutation matrices with an edge between P and Q if and only if $\mathrm{Tr}(P^TQ) = 0$. Let C be the n by n permutation matrix with ones in the $(i, i+1)$ place for all i with addition mod n and J be the n by n permutation matrix with ones in the $(i, n-i)$ place for all i. Show that $\{C^k\}_{k=0}^{n-1} \cup \{C^kJ\}_{k=0}^{n-1}$ is a dominating set for Γ_n if and only if n is either equal to one, even or divisible by three.

Solutions to Problems 57-2, 57-4, 58-1, 58-2a, 58-2c, and 58-3 are on pages 37–42.