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FEATURE INTERVIEWS

“There were a few linear algebraists in this part of the world...”

Peter Lancaster Interviewed by Shaun Fallat¹

S.F. - What made you decide to become a mathematician?

P.L. - As a teenager my favourite subjects were art and mathematics. I took art classes through all my high school years, even though mathematics was my first love. But I must say that, when I graduated from high school, it was not entirely clear whether I would study on the scientific or the artistic side. I began my university career with a year in a school of architecture. In particular, I had to submit a sequence of design proposals in addition to coursework. Although I did well in coursework the design proposals were not accepted and it was recommended that I repeat the year. But by then, my father was fed up with funding me, so I decided to switch to mathematics – where I had a scholarship. My bachelor’s degree was in “honours mathematics” and included some applied math courses such as numerical analysis and mechanics. After this, I moved on to a job in the aircraft industry. This was my way of completing the compulsory UK national service. And it was how I came to be computing eigenvalues. I had a continuing ambition to join academia when my five years of national service were finished.

With this in mind, I obtained a master’s degree “on the side” by returning to my applied math professor at the University of Liverpool [Louis Rosenhead, FRS] and completing an external M.Sc. degree. I also wrote reports on my work which were published by the UK government.

When my national service was complete, I was appointed (in 1957) to a junior teaching/research position at the University of Malaya in Singapore (to become the University of Singapore), and was to obtain a Ph.D. from the University of Singapore.

S.F. - What brought you to the University of Calgary in 1962?

P.L. - For personal reasons, my wife Edna and I decided in 1962 that it was time to move on from the University of Singapore. At that time the universities of the British Commonwealth were still building up after the war years, and I responded to an advertisement from the University of Alberta, which was setting up a new campus in Calgary. It was near the Rockies (and we enjoyed climbing and outdoor activities) so we chose Calgary. In fact I didn’t apply for any other positions — and it worked very well for us. We built a cottage in the “wilderness,” which turned out to be a part of my mathematical life, as well as our family life. Now we could all escape to a rural environment. In that respect, at least, the move to Calgary has “paid off.” I should also say that Israel Gohberg loved to go out to the cottage, hike, and search for mushrooms. Leiba Rodman joined in too – but was a reluctant hiker.

S.F. - I want to ask you about your book, *The Theory of Matrices*, that was published in 1969. Why did you write this text and where did the influence come from? What do you think was the impact of this book?

P.L. - When I first arrived in Calgary, the department was rather small and we had some freedom in what we taught and how we taught it. The Head of Department (John Peck) agreed that I could teach a third-year linear algebra course. Not surprisingly, my course reflected my experience of the preceding ten years (five years in industry and five in Singapore). In industry I spent a lot of time computing eigenvalues (with what we now see as primitive machinery). So this certainly affected my linear algebra courses. But I could not find a text that reflected my enthusiasm and so I had to develop some course material myself. As I probably taught the course three times in the 1960s, the material evolved and became better organized, and later became the book. I was fortunate to have some freedom in designing the syllabus. I get the impression that there are stronger constraints on syllabi now.



Peter Lancaster

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The book found a wide audience. There were those in academia who sympathized with my point of view, and it was adopted as a text quite widely. Also, it was translated into Russian, and this “pirate” edition of 1978 was widely used in the USSR. I like to think that my industrial experience led to an exposition that appealed to a community outside of academia – as well as a being a stimulating textbook.

Then Miron Tismenetsky joined me as a postdoctoral fellow and he taught this third-year course. He suggested numerous changes and additions and, as a result, we produced a second, expanded edition in 1985 – which also sold quite well.

I should add that Gantmacher’s books of 1959 (two volumes on the theory of matrices) were very influential (I had them with me in Singapore), and they still sit proudly in my bookshelves.

S.F. - Can you talk about your connection and relationship with Olga Taussky-Todd and John (Jack) Todd?

P.L. - I knew of them both when I was in industry in England (from ’52 to ’57). Olga was an immigrant from central Europe and she was exercising her mathematical talents in a government research establishment. I was aware of some of her work, and I also knew of Jack Todd who was at the University of London at this time – although I had not met either of them. However, I was communicating with Ostrowski while I was in Singapore and, from that connection, I was invited to the very first Gatlinburg meeting in April of 1961. Of course, Olga and Jack Todd were there, and several other big names as well, so it was a great privilege for me to take part. Then I was invited by Olga and Jack to visit Caltech as a visiting Associate Professor in ’64–’65. While we did not collaborate closely on research, I participated in their seminars and helped in the supervision of a doctoral student. It was an important opportunity for me and critical for my career.

S.F. - Some of your long-standing and fruitful collaborations involved researchers from the Soviet Union. How did these collaborations begin and why were they so successful?

P.L. - In my Singapore days of the early 1960s, I was writing papers about the linear algebra of vibration problems, and I made a connection with M. G. Krein through a professor in London who could make direct connections with Krein. Through this intermediary, I exchanged some papers with Krein. If I recall, they would be about polynomial eigenvalue problems. It was a remote, impersonal connection, but that was where it started. Then my Ph.D. thesis was worked up and published as a monograph, “Lambda Matrices and Vibrating Systems” [Pergamon Press, 1966] and Krein was among the first to receive a copy.

After my arrival in Calgary, the connection with Krein survived via my collaboration with Israel Gohberg (who was a student and collaborator of Krein) beginning in the mid-’70s, after I.G. emigrated from Kishinev to Israel. I got to know I.G. through the good offices of Chandler Davis (of the University of Toronto). At that time there was a program allowing recent immigrants to Israel to make visits around the world. I became aware that I.G. would be visiting Canada and, with the help of Chandler Davis, he visited the University of Calgary in 1975. Then I had a sabbatical leave at the University of Dundee (finite element methods were all the rage) and we could meet again because he was spending some time in Amsterdam. Our collaboration grew quickly from that time – and soon included Leiba Rodman, who was a Ph.D. student with I.G. in Israel. Need I say, these connections had a huge influence on my career. They led to extended visits together in either Calgary or Tel Aviv, and included two years for Leiba Rodman as a postdoctoral fellow in Calgary (1978–1980).

In the late ’80s, I learned of an exchange program managed by Queen’s University and the Steklov Institute in Moscow. My request to visit 3 out of 4 locations was approved, Odessa (the home of M. G. Krein) being the only location not approved. My 1989 travels to Moscow, Leningrad (as it was), and Kishinev (former home of the Gohberg family and “school”) were approved, and were a great experience for me. In fact, during my visit to Kishinev (where I got to know Alek Markus) I did manage to travel to Odessa. However, Krein was too ill to see visitors, but his community welcomed me warmly. Sadly, Krein died shortly thereafter.

S.F. - During your time as an academic you have supervised many graduate students and postdoctoral fellows. Can you tell us some of your secrets to success in supervision?

P.L. - Certainly, graduate students and postdoctoral fellows make up integral parts of our community. For the individuals themselves, these positions can form important steps in the academic/appointment ladder. For their supervisors, they provide a stimulus to broaden and advance their research programmes – in a way that could not be done alone. Close

collaboration and guidance are generally necessary at the Ph.D. level and, ideally, the postdoctoral period allows and encourages the fellow to work independently and establish a publication record.

Although I am quite retiring by nature, I always enjoyed working with others and, in particular, welcomed opportunities to help (generally) younger people to further their careers with these positions. I don't think I actively went looking for students and postdoctoral fellows, but I was always willing to consider and (hopefully) encourage candidates.

S.F. - Leiba Rodman was a postdoctoral fellow under your guidance and a dear friend. Can you tell us some personal thoughts on Leiba?

P.L. - Leiba was a Ph.D. student with Israel Gohberg and came to Calgary as a postdoctoral fellow for the two years 1978–1980. The three of us were already well into collaboration when he arrived. At that time his mastery of English was limited and so he did no teaching in those two years. He could get absorbed in research – and he did! He really worked very hard. He had few social contacts at that time – but for my home, where he was appreciated and made welcome. All of my family knew him well. Later on, we were very glad to know his wife, Ella, to observe the development of his family life, and of his prodigious academic career.

I.G. would also visit Calgary during that two-year period and that is when our research on matrix polynomials really got going. I remember long conversations in my office while making use of the blackboard. This was to lead to our three-author *Matrix Polynomials* book published in 1982 (and again in 2009).

S.F. - How involved were you in the early years of the International Linear Algebra Society (ILAS)? For example, were you a part of the International Matrix Group (IMG)? Did you attend the 1987 meeting in Victoria, where this organization has its roots?

P.L. - I didn't play a major role in the development of ILAS, but I have been very grateful that it is there, and for the opportunities that it provides. I should add that ILAS owes a lot to the efforts of Hans Schneider.

S.F. - How did the Western Canada Linear Algebra Meeting (WCLAM) come to be and why has it been such a successful regional meeting for over 20 years?

P.L. - I believe that it began as a result of a sense of isolation for mathematicians in the western Canadian provinces, and the difficulties we had in taking advantage of conference activities. In my early days in Calgary I took advantage of programs at the Centre de Recherches Mathématiques (CRM) in Montreal and, in 1990, worked with a committee of the Ontario government in assessing proposals for hosting the Fields Institute. So I had this exposure to conference sites and felt the need for such facilities and activities in the west.

There were a few linear algebraists in this part of the world, so we decided to organize a biennial conference series. It began with a meeting in Regina in 1993. The WCLAM meetings are remarkable for their informality. Participation of younger researchers is encouraged and, over the years, I have enjoyed working with the small planning group.

Incidentally, the more broadly-based Pacific Institute for the Mathematical Sciences (PIMS) grew at about the same time and was founded in 1996. This, in turn, gave rise to the Banff International Research Station (BIRS) – founded in 2001.

S.F. - How have you enjoyed retirement?

P.L. - I have enjoyed my years of retirement. They began well in 1994 because, in that year, the University of Calgary introduced the role of “Faculty Professor” – which I have held ever since. Office facilities are provided, but there are no required teaching duties and no salary. So I have been able to maintain a research program and graduate supervision as well as some NSERC research support. I have also appreciated substantial support from agencies in the UK, Germany, Israel, Portugal, and Spain – as well as the opportunities to spend time in those countries.

S.F. - Peter, many thanks for your candid and thoughtful responses and for your time. I know you have given other interviews and written notes on your personal history, so I sure do appreciate your time today. Now, let's go for an adventure in the Rockies!



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“Well, I guess I ought to try this.”

Wayne Barrett Interviewed by Steve Butler¹

S.B. - How did you get involved with mathematics?

W.B. - I became interested in mathematics in the seventh grade. I had quite a good teacher. He used to have these big maps of yards and a house and trees and such and you were supposed to figure out how much was left for lawn or something like that. That was kind of interesting at that age, though, because you had to just use all these geometry formulas. I think in the fourth grade I liked multiplying big numbers, too. So I guess I went back kind of a ways.

S.B. - You're good with prime numbers.

W.B. - That started because I just looked at the clock in the middle of the night and wondered if something was a prime number. Finally I got tired of that and somewhere around '89 or '90 I just learned all the composites up to 600. Now we always make sure the clock is on my wife's side where I can't see it.

S.B. - You did your Ph.D. with Mark Kac. Was it statistics or probability?

W.B. - Statistical mechanics. That was on trying to explain phase transitions in a magnet.

S.B. - Your first two papers were somewhat in that area.

W.B. - Mark Kac wrote the first one without even telling me about it. It was just a two-page one for the *Proceedings of the National Academy of Sciences* [5]. That's the only reason I had an Erdős number of two for years, until you got your paper with Erdős. Now I have two reasons.

S.B. - How did you get involved with linear algebra?

W.B. - Well, it wasn't from my undergraduate class, because my algebra teacher was the worst one I had. It was linear algebra for one quarter and abstract algebra for two quarters. That was a bad experience. It was the first time I went in for help to somebody's office hours and didn't understand the answer.

I really liked functional analysis, and like Paul Halmos said, “Linear algebra is finite dimensional functional analysis.” At the Courant Institute you had to pass three written exams, to get a master's or doctorate. (For a doctorate you also has to pass two in-depth oral exams on graduate mathematics.) Unlike most places, where they test you on abstract algebra, topology, and analysis, it was linear algebra, complex analysis, and advanced calculus. So lower level, but pretty interesting problems. The first time I ever learned a matrix was positive definite if the leading principal minors were positive was when I saw it as a question on one of those linear algebra exams I took.



Wayne with a few of his students. From left to right: H. Tracy Hall, Steve Butler, Michael Barrus, Wayne Barrett, Kayla Owens, Mark Kempton, Curtis Nelson, John Sinkovic, and Chris Guzman.

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S.B. - Did you get it right?

W.B. - I don't remember how far I got on that; I don't think I got it all. It seems easy now, but I didn't know the interlacing inequalities then.² I guess you had to use some ingenuity to do it.

S.B. - You have been doing linear algebra, at least according to MathSciNet, for about forty years now.

W.B. - The first paper was around 1980. It is in *Linear Algebra and its Applications*, volume 27 [2]. I started writing it in 1978, by Hans Schneider's invitation: "Well if you ever write a pure linear algebra proof of this theorem, send it to me." Two months before a proof using probability appeared, I thought, "Well, I guess I ought to try this." It was a lot of fun so I kept doing it. It was easy to write a whole paper on it. Anybody could have done it.

S.B. - Has linear algebra changed very much in the last few decades?

W.B. - I don't think I have a big enough overview to know the answer to that question. There are trivial changes. If you go back to papers in the 1960s or '70s you never see a matrix written out. You just see "a matrix that goes with this quadratic form" and then they would write all of that out because they didn't have the typesetting abilities so they wouldn't do it. So those papers are a little unpleasant to read.

I guess combinatorial matrix theory has grown a tremendous amount in that time. I knew I liked that a lot.

S.B. - You have been involved with ILAS almost from the beginning, certainly for a very long time.

W.B. - In 1987 in Victoria, Danny Hershkowitz proposed forming it. But they called it some other name, like "International Matrix Group" (IMG) and they had a publication called *IMAGE*. So I heard this proposal at Victoria and I didn't know if it sounded important to me. But it did get formed, and then Charlie [Johnson] or Hans Schneider suggested, "Why don't we have the first meeting at Provo?"

S.B. - The first ILAS conference was at Provo. Were you heavily involved with that?

W.B. - It was a little bit painful arranging all of those things. And then we had it on campus, so we had to think about how to make it not be a bad experience for anybody, since they couldn't just get a coffee³ within five seconds of walking out of the lecture. It was pretty successful. Eighty-five people came, two from as far away as India.

S.B. - Do you have favorite problems to work on?

W.B. - I do like all this stuff that came out of minimum rank. Every time I worked in an area before that, it always came to an end. The next problem was too hard. But once I started on minimum rank it just went on and on and on; there never seemed to be an end. You can do minimum rank, then you can do inertia, then you can do more minimum rank, then you can do inverse eigenvalue problems, but then because the minimum rank problem is not solved you can always go back to that. Of course, the first time we did it, it seemed like zero forcing had nothing to do with it at the beginning, so that came four or five years later. That has exploded tremendously; you couldn't even stay up on just that one aspect. I would say the short answer is combinatorial matrix theory. That's what I always say my field is: combinatorial matrix theory.

S.B. - Do you have any favorite results?

W.B. - I do like the paper with Raphi [Loewy] and Hein van der Holst where we characterize minimum rank less than or equal to two by four forbidden subgraphs [1]. Especially finding the $K_{3,3,3}$. That was sort of tricky to find that. I liked writing the proof for that part of it.

I liked the paper called "Determinantal formulae for matrices with sparse inverses" [3], where zeroes in the inverse help you find the determinant of the original matrix. That all comes from the inverses of tridiagonal matrices, and generalizing that to inverses of banded matrices and all their properties.

I did get to publish one paper with Raphi [Loewy] and Charlie Johnson in the *Memoirs* [4]. I thought that was a pretty nice paper. That was about positive-definite completions. That had four different statements that were equivalent and it wasn't really obvious that any of them implied the other ones.

I like all of the papers I did with undergraduates, too.

²Wayne Barrett's favorite theorem is the interlacing inequalities.

³Coffee is not available on the campus of BYU.

S.B. - You have worked with a lot of young people, and are a successful mentor. How do you be a good mentor and what advice do you have for people who have to mentor?

W.B. - I'm never quite sure what I did. You should ask my students. I think you have to get committed to it just like a class. In fact, being a mentor helped me keep doing research during the semester because it was like, "Oh, my mentor class is meeting!" And when I remembered that the day before, I'd think, "Oh gee, I didn't think about that this week, I better think of some ideas for them."

S.B. - You are amazingly good at proofreading and being very careful. I've always taken advantage of that. How did you get so good at it?

W.B. - I have just always been a slow, thorough person. I used to tell students all the time, "Well, you can't stop making mistakes, all you can do is become good at finding them." I used to make lots of mistakes, up to the time I was a freshman in college, and somewhere along the way I got the idea I could recheck my work. So I actually became sort of an obsessive compulsive checker. But then, when I am reading papers I don't skip steps; I always think, "Do I believe that?" I am not easily convinced. So what some people might read in fifteen minutes might take me an hour and a half to read. That's part of it: Slow. The other part is, when I proofread it, I print it, and I have it in front of me with a red pen. Not that many people do that anymore.

S.B. - Have you had any memorable events in mathematics?

W.B. - I think going to graduate school in New York City was a memorable event, especially starting the second year. There were 180 families that lived in the building we lived in and we'd go to babysit their kids, and there would be all these books on real analysis or something on their table. I thought, "Man, growing up in Salt Lake City I would have had to go a mile and a half or two miles to find anybody else like that, and here they are just three floors below me."

Then I'd go to the classes. It'd be a Hilbert space class and there would be thirty people in it, and I would think, "Man, the graduate student classes at the U of U⁴ had three or four and they were in only a few subjects." That was a great place to focus and be with a whole bunch of people like myself. That was the best educational experience I had, going to the Courant Institute. And then I got to meet Mark Kac who was at the Rockefeller University after two years there, and that was great too. But you know all the classes there were great. I think another good thing is they only met once a week for two hours. So that gave you time to work on each class between classes. You know, when it is every Monday, Wednesday, Friday: "Oh man, there is class again, did I really do anything in between?"

S.B. - Did you go to BYU right after?

W.B. - No, I went to Madison, Wisconsin for two years. I am sure I only got an offer from Madison just because I was a student of Mark Kac so they figured I must know a lot of probability. In fact I didn't know that much, although I knew idiosyncratic facts. I told someone in the elevator at Wisconsin: "So, this is a Gaussian-Markovian process, so the inverse covariance matrix is tridiagonal," and they'd say, "Is that true? Are you sure?" Well, "Yeah." That's how I started in linear algebra, because they had written that matrix on the board at the Courant Institute. I worked it out one summer, so I guess I was working a little bit on matrices before my statistical mechanics papers appeared.

So Madison was a good experience. One year after [starting] Madison, there were five recent Ph.D.'s there that had to get new jobs. Two of them went to Israel, one of them went to Brazil, one quit mathematics and went to medical school in Miami, and one went to Texas A & M. I told my wife, "If you'd like to stay in mathematics you have to go to Texas A & M." That's what happened a year later, the chair at Texas A & M hired seventeen new lines. They were young, I guess they were all tenure-track. My wife said, "We had a lot in common with all those people; we all found out that we didn't want to be there." But it saved our academic career, it gave us a job until there was a job that opened up somewhere else. So I went to BYU six years after I got my doctorate. I was there from '81 to '14, thirty-three years.

S.B. - Do you have any advice for young people getting started in linear algebra?

W.B. - I would like to tell them to take a class from somebody who likes the subject. Try to work on some undergraduate research with someone who does linear algebra.

S.B. - A lot of good people work in linear algebra, although a lot of departments don't put a priority on hiring people in linear algebra.

W.B. - That's why I love that one problem of the matrix of Redheffer⁵, that if the determinant grows at a certain rate, the Riemann Hypothesis is true. When I was writing papers on that I used to joke and say: "Well, what you have to do is get your colleague in there and say, 'I know you don't think too much of linear algebra and you think it's pretty easy,

⁴University of Utah

⁵For a given n , the Redheffer matrix is the $n \times n$ matrix where $a_{ij} = 1$ if i divides j or if $j = 1$ and is 0 otherwise.

and I have this problem: I can't quite seem to get the right bound on the determinant. I was just wondering if you could show that it grows at this rate.' Not telling them it's the same as proving the Riemann Hypothesis."

I like that matrix. The only sad thing about that matrix is you can find out anything about it as long as it doesn't help you solve the Riemann Hypothesis.

S.B. - What are your future plans?

W.B. - I'd like to keep working. I don't feel like I am contributing quite as much anymore because I am forgetting my own results. I like coming to work with my former students, and being reminded of what I did before. I won't be paid for doing math anymore after this summer. I have to balance it out with family things and grandchildren. So probably I will not be quite as involved as before.

My uncle said after he retired he never knew how he found time to work. A lot of people say that, and I could say that too. I figure being retired is a lot easier than not. Because if you have responsibilities you can sort of default for a day or something, and say, "Oh, I didn't get that done. Oh, too bad, maybe I'll work on it tomorrow or in three days." If you have a class to teach or an exam to get ready you have an absolute deadline. You can't postpone it, so it's still easier [being retired]. So I guess I will keep doing it as long as I am contributing.

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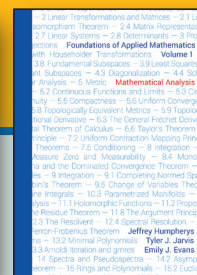
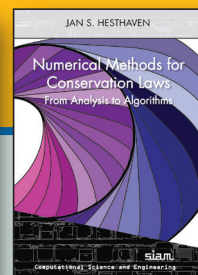
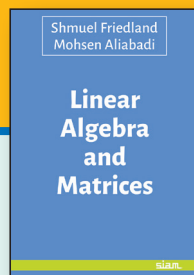
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FEATURE ARTICLE

Continuous-Time Quantum Walks in Graphs

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and
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1. A transition matrix. For us, a *qubit* is simply a 2-dimensional complex vector space, and the composite system of n qubits is $(\mathbb{C}^2)^{\otimes n}$. A *state* is a 1-dimensional subspace, which is represented by the choice of a unit vector. These qubits physically interact with one another. This interaction is mathematically defined via the choice of a $2^n \times 2^n$ real symmetric matrix H , which will be known as the Hamiltonian of the system, and which somehow encodes which pairs of qubits interact and which do not. Building blocks to obtain this matrix H will be the Pauli matrices, defined as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The understanding is that these matrices might act separately on each qubit according to the canonical basis of $\mathbb{C}^2$. More specifically, if V is the set of all qubits and $u \in V$, then X_u denotes the operator that acts as the identity at all qubits except for u , where it applies X . That is,

$$X_u = I \otimes I \otimes \cdots \otimes I \otimes \underbrace{X}_{u\text{th qubit}} \otimes I \otimes \cdots \otimes I.$$

The choice of V to denote the set of qubits is no coincidence, as in fact we will use a graph to model the network of interacting qubits. Each pair of interacting qubits will be an edge. A common and physically natural choice for the matrix H is

$$H = \frac{1}{2} \sum_{uv \in E} (X_u X_v + Y_u Y_v).$$

If the Hamiltonian is set to be constant and time-independent, and the initial state of the system is given by a unit vector ψ_0 , then Schrödinger's equation gives that the state of the system at time $t \in \mathbb{R}_+$ is

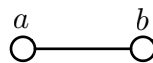
$$\psi_t = \exp(-itH)\psi_0.$$

The matrix H defined above is block diagonal. Each block is the adjacency matrix of a symmetric power of the graph. For a graph $X = (V, E)$, its k th symmetric power is the graph whose vertex set is the set of all k -subsets of V , and any two subsets are neighbours if and only if their symmetric difference is an edge of X . Note that the k th symmetric power with $k = 1$ is simply X itself. The block of H corresponding to the adjacency matrix $A(X)$ governs the evolution of states where exactly one qubit is at the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and all others are at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One-excitation subspaces are useful for many applications, and so we are motivated to study the matrix

$$U(t) = \exp(itA).$$

This is the transition matrix of a (one-excitation) continuous-time quantum walk defined on a graph with adjacency matrix A . Note that $U(t)$ is unitary, and thus its rows determine probability distributions. In fact, upon measuring the system at time t according to the canonical basis, if $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the initial state at u at time 0, then $|U(t)_{u,v}|^2$ is the probability that this state is observed at v at time t . Such an interpretation leads to many interesting questions that we will discuss in the following sections.

2. Paths on 2 and 3 vertices. We start by examining the quantum walk on P_2 , the path graph on 2 vertices, as depicted below.



First, we plot $|U(t)_{a,a}|$ and $|U(t)_{a,b}|$, for $t \in [0, 20]$. (See Figure 1.)

The wave-like and periodic behaviour is no surprise, as the adjacency matrix A of P_2 satisfies $A^2 = I$, and so

$$U(t) = \exp(itA) = \cos(t)I + i\sin(t)A.$$

This in turn leads to a demonstration that there is a time $\tau \neq 0$ such that $|U(\tau)_{a,a}| = 1$. When this occurs, a is called a periodic vertex. Perhaps more interestingly, there is also a time τ , for instance $\tau = \pi/2$, such that $|U(\tau)_{a,b}| = 1$.

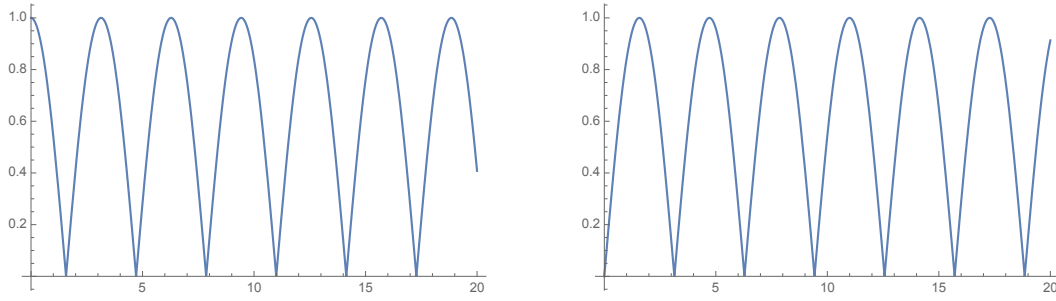
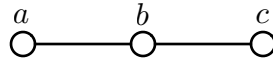


Figure 1: $|U(t)_{a,a}|$ on the left, and $|U(t)_{a,b}|$ on the right.

This phenomenon is called perfect state transfer. Also noticeable, when $\tau = \pi/4$, then all entries of $U(t)$ have the same absolute value, that is, $U(t) \circ U(t)^*$ is a multiple of the all-ones matrix. This has been defined as instantaneous uniform mixing. Periodicity, perfect state transfer, and instantaneous uniform mixing are all defined with the intent of understanding how the structure of the graph affects the dynamics of states in the quantum system. Moreover, it might be desirable for practical applications to engineer graphs that satisfy these or other properties.

Let us examine the next path, on 3 vertices.



Again, we plot two graphs. (See Figure 2.) Note here a first example where the absolute value of an entry of $U(t)$ never

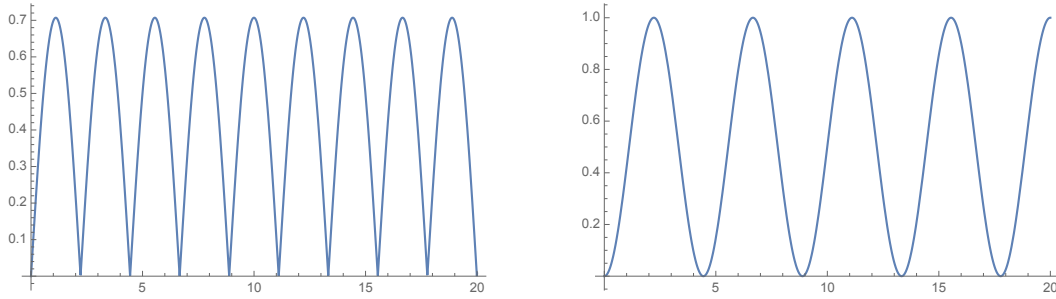


Figure 2: $|U(t)_{a,b}|$ on the left, and $|U(t)_{a,c}|$ on the right.

reaches 1. This actually tends to be the norm, as we will see later on. But first let us see that perfect state transfer happens again between the end vertices. The adjacency matrix A of P_3 admits a spectral decomposition

$$A = \sqrt{2}E_0 + 0E_1 + (-\sqrt{2})E_2,$$

where E_0 , E_1 and E_2 are the orthogonal projectors onto orthogonal eigenspaces. As such, they are orthogonal idempotents summing to the identity, which leads to

$$U(t) = e^{i\sqrt{2}t}E_0 + e^{it0}E_1 + e^{it(-\sqrt{2})}E_2.$$

As $U(t)$ is unitary and symmetric, to say that $|U(\tau)_{a,c}| = 1$ is equivalent to saying that the a th column of $U(t)$ is equal to the c th column of the canonical basis, modulo a complex phase. That is, there is a λ such that

$$U(\tau)\mathbf{e}_a = \lambda\mathbf{e}_c.$$

From the spectral decomposition, this is equivalent to having

$$e^{i\sqrt{2}t}E_0\mathbf{e}_a = \lambda E_0\mathbf{e}_c, \quad e^{it0}E_1\mathbf{e}_a = \lambda E_1\mathbf{e}_c \quad \text{and} \quad e^{it(-\sqrt{2})}E_2\mathbf{e}_a = \lambda E_2\mathbf{e}_c. \quad (1)$$

These idempotents can be computed explicitly, and it can be verified that all three equations hold true when $t = \pi/\sqrt{2}$, and $\lambda = -1$. It is also very easy to show that replacing c by b makes the second equation unsolvable.

3. Perfect state transfer. Since the seminal paper [11], the question of which graphs admit perfect state transfer has received a considerable amount of attention. Based on the two examples shown in the past section, the natural candidates were the paths on n vertices, where one would expect state transfer between extreme vertices. It turns out however that P_2 and P_3 are the only two paths that see this phenomenon happen. One way to prove this is to compute eigenvalues and eigenvectors explicitly and observe that equations such as those in (1) are incompatible for any pair of distinct vertices on paths with 4 or more vertices.

It was also very soon observed that the complete graphs other than K_2 do not admit perfect state transfer. The spectral decomposition of $A(K_n)$ is given by

$$A(K_n) = (n-1)\frac{1}{n}J + (-1)\left(I - \frac{1}{n}J\right),$$

and thus

$$U(t) = e^{it(n-1)}\frac{1}{n}J + e^{-it}\left(I - \frac{1}{n}J\right).$$

Hence there is a t such that $U(t)\mathbf{e}_a = \lambda\mathbf{e}_b$ for some λ of absolute value equal to 1 if and only if

$$e^{it(n-1)}\frac{1}{n}J\mathbf{e}_a = \lambda\frac{1}{n}J\mathbf{e}_b \quad \text{and} \quad e^{-it}\left(I - \frac{1}{n}J\right)\mathbf{e}_a = \lambda\left(I - \frac{1}{n}J\right)\mathbf{e}_b.$$

The key equation here is the second. As both e^{-it} and λ lie on the complex unit circle and each multiplies a real vector, it must be that

$$\left(I - \frac{1}{n}J\right)\mathbf{e}_a = \pm\left(I - \frac{1}{n}J\right)\mathbf{e}_b,$$

which clearly is not true unless $n = 2$.

In general, if $A(X)$ admits spectral decomposition

$$A(X) = \sum_{r=0}^d \theta_r E_r,$$

a necessary condition for perfect state transfer between vertices a and b is that, for all r ,

$$E_r\mathbf{e}_a = \pm E_r\mathbf{e}_b.$$

When this occurs, we call a and b strongly cospectral. It is surprisingly hard to find examples of pairs of strongly cospectral vertices, which of course suggests that perfect state transfer is rare. Moreover, a combinatorial characterization of this property has so far eluded us. We return to this subject in Section 7, but now we show how to find graphs with perfect state transfer.

4. Cartesian products. One of the many ways graph operations can be used to build larger systems whose quantum walk dynamics is determined by, or at least depends on, the smaller systems is by considering the cartesian product of two graphs. If X and Y are simple graphs on n and m vertices and with adjacency matrices $A(X)$ and $A(Y)$, then their cartesian product $X \square Y$ is the graph with adjacency matrix

$$A(X \square Y) = A(X) \otimes I_m + I_n \otimes A(Y).$$

Equivalently, the vertex set of $X \square Y$ is $V(X) \times V(Y)$, and (a_X, a_Y) is adjacent to (b_X, b_Y) if and only if either $a_X = b_X$ and a_Y is adjacent to b_Y in Y , or vice versa. The expression for the adjacency matrix of $X \square Y$ leads to

$$\begin{aligned} \exp(itA(X \square Y)) &= \exp(it(A(X) \otimes I)) \exp(it(I \otimes A(Y))) \\ &= (\exp(itA(X)) \otimes I) (I \otimes \exp(itA(Y))) \\ &= \exp(itA(X)) \otimes \exp(itA(Y)). \end{aligned}$$

The consequence is immediate: To know what happens in a quantum walk on $X \square Y$, one needs only to compute the walk in X and Y separately. For example, given that P_2 admits perfect state transfer between a and b at $\pi/2$, it follows that its Cartesian powers also will do so, at the same time, between vertices corresponding to $(a, a, a, \dots)$ and $(b, b, b, \dots)$. The n th Cartesian power of P_2 is the n -dimensional hypercube graph. So this is an infinite family of graphs at which perfect state transfer occurs between vertices at increasingly larger distance. The same observation, of course, works for P_3 . However it should be noted that $P_2 \square P_3$ does not admit perfect state transfer, as the times at which it occurs in each graph are different.

The hypercubes are Cayley graphs for $\mathbb{Z}_2^d$. That is to say, there is a (connection) set $C \subseteq \mathbb{Z}_2^d$ such that if you define a graph by making its vertex set the elements of $\mathbb{Z}_2^d$, and two vertices adjacent if and only if their difference lies in C , then the resulting graph is the d -dimensional hypercube. By varying C , one gets many different graphs, which we explore in the next section.

5. Cubelike graphs. We consider the $d \times d$ standard representation of $\mathbb{Z}_2^d$. To an element $a \in \mathbb{Z}_2^d$, let P_a be the corresponding permutation matrix. It follows from the definition that the Cayley graph X defined on $\mathbb{Z}_2^d$ by the connection set C has adjacency matrix

$$A(X) = \sum_{a \in C} P_a.$$

These matrices are permutation matrices of order 2 (we assume the identity does not belong to C) and they commute. Therefore

$$U(t) = \exp\left(it \sum_{a \in C} P_a\right) = \prod_{a \in C} \exp(it P_a) = \prod_{a \in C} (\cos(t)I + i \sin(t)P_a).$$

Thus, if $u \in \mathbb{Z}_2^d$ is the sum of the elements in C , it follows that

$$U(\pi/2) = i^{|C|} \prod_{a \in C} P_a = i^{|C|} P_u.$$

We just learned above that as long as you select a set of elements in $\mathbb{Z}_2^d$ that do not sum to the identity, the Cayley graph obtained by making that set the connection set admits perfect state transfer at time $\pi/2$ [7]. This result provides a new class of examples. It should be noted that it is also possible to have state transfer at smaller times, as shown first in [10] and later in [8].

6. The ratio condition. Assume that there is a pair of vertices a and b in a graph X whose adjacency matrix admits the spectral decomposition $A = \sum_{r=0}^d \theta_r E_r$ and that, for all r ,

$$E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b. \quad (2)$$

As we have seen above, this condition is necessary for perfect state transfer to occur. What else is needed? The eigenvalue support of a , denoted by Θ_a , is the set of eigenvalues θ_r such that $E_r \mathbf{e}_a \neq 0$. Clearly, the equations (2) imply that a and b have the same eigenvalue support, that is, $\Theta_a = \Theta_b$. Perfect state transfer is defined as $U(t)\mathbf{e}_a = \lambda \mathbf{e}_b$, which is equivalent that for all r we have

$$e^{it\theta_r} E_r \mathbf{e}_a = \lambda E_r \mathbf{e}_b.$$

Noting that $E_0 \mathbf{e}_a = E_0 \mathbf{e}_b \neq 0$, the equations above are equivalent to having

$$e^{it(\theta_0 - \theta_r)} = \pm 1 \quad (3)$$

for all r such that $\theta_r \in \Theta_a$. These signs will be determined by those occurring in the equations (2). For any eigenvalues θ_r and θ_s in the eigenvalue support of a and b with $\theta_s \neq \theta_0$, these conditions imply that

$$\frac{\theta_0 - \theta_r}{\theta_0 - \theta_s} \in \mathbb{Q}. \quad (4)$$

This has been called the ratio condition, and was shown in [23] to imply that these differences are either all integers or all integer multiples of the same square root. As a corollary, it was seen that for any k , there are only finitely many graphs with maximum valency k admitting perfect state transfer. Moreover, this strict restriction on the number-theoretic nature of the eigenvalues has been very effective in proving that perfect state transfer does not occur in many cases.

In addition to the ratio condition, if the integer parts of the eigenvalues satisfy certain parity conditions determined by the sign in (3), then perfect state transfer can be shown to occur.

Theorem 1. *Perfect state transfer between vertices a and b occurs at time τ if and only if the following items are true.*

(i) *Vertices a and b are strongly cospectral.*

(ii) *Let $\Theta_a = \{\theta_0, \dots, \theta_k\}$. Then the nonzero elements of Θ_a are either all integers or all quadratic integers. Moreover, there is a square-free integer Δ , an integer p , and integers $q_0, \dots, q_k$ such that*

$$\theta_r = \frac{p + q_r \sqrt{\Delta}}{2}, \quad \text{for all } r = 0, \dots, k.$$

(iii) Let $g = \gcd\{(\theta_0 - \theta_r)/\sqrt{\Delta}\}_{r=0}^k$. Then

- (a) $E_r \mathbf{e}_a = E_r \mathbf{e}_b$ if and only if $(\theta_0 - \theta_r)/(g\sqrt{\Delta})$ is even, and
- (b) $E_r \mathbf{e}_a = -E_r \mathbf{e}_b$ if and only if $(\theta_0 - \theta_r)/(g\sqrt{\Delta})$ is odd.

Moreover, if these conditions hold, the minimum time at which perfect state transfer occurs is $\pi/(g\sqrt{\Delta})$.

This characterization of perfect state transfer can be checked in polynomial time, on a classical computer [12].

7. Strongly cospectral vertices. Our definition of strongly cospectral is not always easy to use, for example, if we have to compute our spectral idempotents in floating point. Strongly cospectral vertices are cospectral, which implies, among other conditions, that for any nonnegative integer k the number of closed walks of length k at a vertex is the same for all vertices. Thus cospectral vertices must have the same degree, and lie in the same number of triangles. We have not yet been able to produce a purely combinatorial characterization of strongly cospectral vertices.

We do have some strong restrictions with a combinatorial flavour. For instance, suppose $E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b$ and let $p(x)$ be a one-variable polynomial taking values ± 1 on the spectrum of A , such that $p(A)\mathbf{e}_a = \mathbf{e}_b$. If the permutation matrix Q represents an automorphism of our graph, then $QA = AQ$ and, if this automorphism fixes a , we have

$$Q\mathbf{e}_b = Qp(A)\mathbf{e}_a = p(A)Q\mathbf{e}_a = p(A)\mathbf{e}_a = \mathbf{e}_b.$$

Thus, if a and b are strongly cospectral, then any automorphism that fixes a must also fix b . One consequence of this is, for example, that no two vertices in the Petersen graph are strongly cospectral. We also see that if a and b are strongly cospectral, then their images under an automorphism will be strongly cospectral.

If two vertices a and b of a graph are cospectral, then $(E_r)_{a,a} = (E_r)_{b,b}$. It follows that if the eigenvalues of X are simple, then two vertices are strongly cospectral if and only if they are cospectral. The following result extends this observation to a characterization. Here we are using $\phi(X, t)$ to denote the characteristic polynomial of X in the variable t [12].

Lemma 1. *If a and b are two cospectral vertices in X , they are strongly cospectral if and only if the poles of the rational function*

$$\frac{\phi(X \setminus \{a, b\}, t)}{\phi(X, t)}$$

are all simple.

Using this we can show that if a and b are twin vertices in X with a common neighbor c , and if the multiplicity of 0 as a root of $\phi(G \setminus c, t)$ is less than its multiplicity in $\phi(G, t)$, then a and b are strongly cospectral.

8. Pretty good state transfer. Imagine now that one does not require $|U(t)_{a,b}|$ to be equal to 1 at a particular time t , but only to get arbitrarily close to it. In other words, given any $\epsilon > 0$, we wish to find t such that

$$|U(t)_{a,b}| > 1 - \epsilon. \quad (5)$$

Note that this is not the same thing as expecting $|U(t)_{a,b}|$ to converge to 1, which actually never happens unless of course it reaches 1. When it is possible to find for any $\epsilon > 0$ a time t such that the inequality (5) holds we say that *pretty good state transfer* occurs between a and b .

Once it had been shown that path graphs on n vertices do not admit perfect state transfer for $n > 3$, the question of whether or not they admit pretty good state transfer became relevant. In [25, 4] the problem was completely solved for extreme vertices in paths for two different models of quantum walks. It was also studied for other graphs and also weighted paths in [33, 28]. In all of these papers, the main tool is Kronecker's Theorem on Diophantine approximation.

More specifically, vertices need be strongly cospectral for pretty good state transfer to occur [25]. Then the phenomenon is equivalent to having $e^{it\theta_r}$ arbitrarily approaching $\pm\lambda$ for some $\lambda = e^{i\delta}$, which is then equivalent to $t\theta_r$ approaching δ or $\delta + \pi$ (up to multiples of 2π). That is, for a given ϵ , we wish to find t such that, for all r ,

$$|t\theta_r - (\delta + \sigma_r\pi)| < \epsilon \pmod{2\pi},$$

where $\sigma_r = 0$ or $\sigma_r = 1$, depending on the sign of $E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b$. Let $\zeta_r = (\delta + \sigma_r\pi)$. Kronecker's Approximation Theorem (see for instance [3, Theorem 4]) establishes that such a system of linear inequalities admits a solution if and only if there do not exist integers $\ell_0, \dots, \ell_d$ such that $\sum_{r=0}^d \ell_r \theta_r = 0$ and $\sum_{r=0}^d \ell_r \zeta_r \not\equiv 0 \pmod{2\pi}$.

9. Trees. As state transfer (perfect or pretty good) has been extensively studied in paths, it is natural to ask what happens to trees. So we either wish to find examples of trees admitting interesting quantum phenomena or show that they do not happen at all. It turns out that for quantum walks defined on the Laplacian matrix, we know that no tree except for K_2 admits perfect state transfer [17].

A first observation is that in the Laplacian case, all eigenvalues in the support of vertices involved in perfect state transfer must be integers [26]. The proof is then split into three steps:

1. Show that if perfect state transfer occurs between strongly cospectral vertices a and b in a tree, then there is exactly one r such that $E_r \mathbf{e}_a = -E_r \mathbf{e}_b$, all others satisfying equality without the sign difference.
2. Show that the conclusion described above implies that the vertices must be leaves attached to the same vertex.
3. Show that leaves attached to the same vertex do not admit perfect state transfer.

Steps 2 and 3 are obtained via elementary arguments, but Step 1 brought to light a new connection between quantum walks and an old result in graph theory. More specifically, Kirchhoff's Matrix Tree theorem implies that every $(n-1) \times (n-1)$ minor of $L = L(X)$ is equal to 1 when X is a tree. Thus the rank of L over any $\mathbb{Z}_p$ with p prime is $(n-1)$. As a consequence, only the all-ones vector lies in the kernel of L over $\mathbb{Z}_p$. Thus if there is an eigenvector $\mathbf{v}$ for an integer eigenvalue λ which is not a power of 2, it cannot happen that $\mathbf{v}_a = -\mathbf{v}_b$. With a little more work to treat the powers of 2, it is easy to see how this implies Step 1 described above.

It remains an open question to decide what happens in the adjacency matrix model. We also do not know of any deep explorations in the topic of pretty good state transfer in trees.

10. Weighted paths. It is physically meaningful to consider quantum walks in weighted graphs, a fact that has been exploited since the very first paper [11] defining perfect state transfer. One very natural question is whether it is possible to engineer the weights in a path so as to achieve perfect state transfer at any desirable distance. The answer is positive, and one immediate family of examples arises from taking the quotient in a d -dimensional hypercube with respect to the distance partition of any vertex. The resulting graph is a path on $d+1$ vertices, and any pair of centrally opposed vertices will admit perfect state transfer at time $\pi/2$.

In [34] we learned that the example described above is actually not an accident. In fact, let $\theta_0 > \dots > \theta_n$ be real numbers such that $\theta_0 - \theta_r \in \mathbb{Z}$ for all r . Moreover, let g be the greatest common divisor of all differences, and suppose that, for all r even, $(\theta_0 - \theta_r)/g$ is even, and for all r odd, $(\theta_0 - \theta_r)/g$ is odd. Then there is an infinite family of directed weighted paths on $n+1$ vertices and spectrum $\{\theta_0, \dots, \theta_n\}$ admitting perfect state transfer between all centrally opposed vertices. The weights in all these examples are symmetric with regards to the centre, a feature that has been called mirror symmetry, and this in fact is a requirement. All these paths are similar, meaning that their adjacency matrices are similar, and the similarity is obtained via a diagonal matrix. Hence there is a unique example in each family which is a symmetric matrix. All this analysis becomes straightforward via the well-known correspondence between weighted paths, tridiagonal matrices, and certain sequences of orthogonal polynomials. Finally, all examples of weighted paths admitting perfect state transfer between the extreme vertices are obtained as described above.

It should be noted however that the problem of deciding whether or not a weighted path admits perfect state transfer has not been fully solved, as it could happen between inner vertices without happening between the extremes. Moreover, many other related questions have been considered recently and in most cases we have only witnessed partial progress. For example:

1. In [20] the problem of transferring from a quantum state from one extreme of a weighted path to an entangled state between the two extremes is considered. This phenomenon has been called fractional revival. Some analytical solutions are obtained via the use of orthogonal polynomials, and the connection to other quantum phenomena is established. Fractional revival in graphs which are not necessarily weighted paths is studied in [9].
2. When choosing the weights in order to achieve perfect state transfer, it has been conjectured that one could simply restrict to tweaking the weights in the loops. This is false, as see in [27], but true if one only requires pretty good state transfer.
3. Transfer of entangled states in weighted paths is analysed in many works, for instance [32]. It is possible to view this phenomenon as a generalization of perfect state transfer in weighted paths, as long as certain symmetry conditions are enforced in the weight choices.

11. Density matrices and the average mixing matrix. For matrices A and B we write $A \succcurlyeq B$ to denote that $A - B$ is positive semidefinite (i.e., that $A - B \succcurlyeq 0$).

We can describe quantum walks using density matrices, as in [21]. A density matrix is a positive semidefinite matrix with trace equal to 1. A density matrix of rank one is called a pure state; it necessarily has the form $\mathbf{z}\mathbf{z}^*$ for some unit vector $\mathbf{z}$. In terms of density matrices, the state represented by $\mathbf{e}_a$ is given by the density matrix $D_a = \mathbf{e}_a \mathbf{e}_a^T$. For any density matrix D , we define

$$D(t) = U(t)DU(-t).$$

We have perfect state transfer from a to b if there is a time t such that

$$U(t)D_aU(-t) = D_b.$$

One advantage of density matrices is that the phase factors go away. If D is a density matrix, we define

$$\Phi(D) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T D(t) dt,$$

and we call it the average state of the quantum walk starting at D . Now

$$D(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s \quad (6)$$

where the sum is over all pairs of eigenvalues, and from this it follows that

$$\Phi(D) = \sum_r E_r D E_r.$$

We see that $\Phi(D) \succcurlyeq 0$ and

$$\text{tr}(\Phi(D)) = \sum_r \text{tr}(E_r D E_r) = \sum_r \text{tr}(E_r) D = \text{tr} \left(\sum_r E_r D \right) = \text{tr}(D) = 1.$$

Hence an average state is a density matrix. We note Φ is an idempotent linear map on the space of Hermitian matrices, and is self-adjoint relative to the trace inner product. From [13], we have the following results.

Lemma 2. *If D is a density matrix, then $\Phi(D)$ is the orthogonal projection of D onto the space of matrices that commute with A .*

Lemma 3. *Vertices a and b in X are strongly cospectral if and only if $\Phi(D_a) = \Phi(D_b)$.*

The mixing matrix of a quantum walk $M(t)$ is defined by

$$M(t) = U(t) \circ \overline{U(t)} = U(t) \circ U(-t). \quad (7)$$

The results of a measurement at time t on a quantum walk are determined entirely by the entries of $M(t)$. This matrix is doubly stochastic. We have perfect state transfer from a to b at time t if and only if $M(t)\mathbf{e}_a = \mathbf{e}_b$.

We define the average mixing matrix $\widehat{M}$ by

$$\widehat{M} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T M(t) dt.$$

Since

$$M(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r \circ E_s$$

we find that

$$\widehat{M} = \sum_r E_r \circ E_r. \quad (8)$$

From this we see that $\widehat{M} \succcurlyeq 0$. The graph X is connected if and only if all entries of $\widehat{M}$ are positive.

We use $B^{\circ 2}$ to denote the Schur square $B \circ B$. As the Schur product of positive semidefinite matrices is positive semidefinite, $\widehat{M}$ is positive semidefinite. It is also doubly stochastic, since it is an average of such matrices. Another consequence of (8) is that the entries of $\widehat{M}$ are rational. All these observations were first made in [24], and the following result is from [13].

Theorem 2. *The average mixing matrix is the Gram matrix of the average states $\Phi(D_a)$ for a in $V(X)$.*

This theorem implies that two rows of $\widehat{M}$ are equal if and only if the corresponding vertices are strongly cospectral.

We have the inequalities

$$I \succcurlyeq M(t) \succcurlyeq 2\widehat{M} - I.$$

One consequence of this is that $I \succcurlyeq \widehat{M}$. Another is that all eigenvalues of $M(t)$ lie in the interval $[-1, 1]$. This is also a consequence of the fact that $M(t)$ is a principal submatrix of the unitary matrix $U(t) \otimes U(-t)$, and is therefore a contraction.

We note one more bound. A graph X is walk-regular if all nonnegative powers of its adjacency matrix have constant diagonal or, equivalently, if the diagonals of its spectral idempotents are constant. Vertex transitive graphs provide one family of examples; strongly regular graphs provide another.

Lemma 4. *If X has n vertices and the multiplicities of its eigenvalues are $m_1, \dots, m_r$, then*

$$\mathrm{tr}(\widehat{M}) \geq \frac{1}{n} \sum_i m_i^2.$$

If equality holds, then X is walk-regular.

Since $n^{-1} \sum_i m_i = 1$, we have $n^{-1} \sum_i m_i^2 \geq 1$. Hence $\mathrm{tr}(\widehat{M}) \geq 1$ in all cases. If equality holds, then the eigenvalues of X are simple, and it is known that a walk-regular graph with simple eigenvalues has at most two vertices. We also see that if $\widehat{M} = n^{-1}J$, then $|V(X)| \leq 2$; thus if $|V(X)| \geq 3$, then $\widehat{M}$ cannot be uniform.

If $\mathrm{rank}(\widehat{M}) = 1$, then $\widehat{M} = \mathbf{z}\mathbf{z}^T$ for some $\mathbf{z}$. As $\widehat{M}$ is doubly stochastic, $\widehat{M}\mathbf{1} = \mathbf{1}$ and thus $\mathbf{1} = \mathbf{z}\mathbf{z}^T\mathbf{1}$. Hence $\mathbf{z}$ is constant and $\widehat{M} = n^{-1}J$.

We view the average mixing matrix as an intriguing invariant of a graph (independent of its origin from quantum walks). Our results above show that it is interesting to investigate the relations between graph properties and parameters of $\widehat{M}$ such as rank and trace. Work on these questions is in progress; some can be found in [24] and [13].

12. Bipartite graphs and instantaneous uniform mixing. As we discussed above, the average mixing matrix is never uniform if the graph has more than 2 vertices. This raises the question of whether the mixing matrix defined in (7) can be uniform for some time t . As we announced in Section 2, this occurs for the path on 2 vertices, and it is not hard to use the conclusions of Section 4 to see that any hypercube admits instantaneous uniform mixing at time $\pi/4$. In fact, cubelike graphs are our favourite source of examples of instantaneous uniform mixing, and an extensive presentation is found in [8]. Note that if it occurs, then the adjacency algebra of the graph contains a complex Hadamard matrix. This of course presents a severe restriction, and for highly regular graphs such as those in association schemes, this is quite useful in analyzing whether instantaneous uniform mixing occurs. Nevertheless, we know much less about uniform mixing than we do about the other topics of this survey. For instance, we still cannot decide whether it occurs in the cycle on 9 vertices.

Now assume X is a bipartite graph on n vertices, and therefore we can write

$$A(X) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

which implies

$$U(t) = \begin{pmatrix} \cos(t\sqrt{BB^T}) & i\sin(t\sqrt{BB^T}B) \\ i\sin(t\sqrt{B^TB}B) & \cos(t\sqrt{B^TB}) \end{pmatrix}.$$

As a consequence, if instantaneous uniform mixing occurs at a time t , all entries in the diagonal blocks are equal to $\pm 1/\sqrt{n}$, and all entries in the off-diagonal blocks are equal to $\pm i/\sqrt{n}$. Moreover, if D is a diagonal matrix with 1s in the diagonal entries corresponding to one of the bipartite classes and i in the others, it follows that $D(\sqrt{n}U(t))D$ is a Hadamard matrix. This connection immediately implies that uniform mixing in a bipartite graph with $n > 2$ vertices implies that n must be divisible by 4 [29, Chapter 5].

Local uniform mixing occurs when the row corresponding to a vertex a in $U(t)$ is flat. In terms of the density matrices discussed in the previous section (as in (6)), this is equivalent to having a time t such that the diagonal of $D_a(t)$ is constant. If the graph is bipartite, $D_a(t)$ is algebraic, and because it is written as a linear combination of algebraic matrices which are orthogonal with respect to the trace inner product, the coefficients of this combination must also be algebraic, i.e., $e^{it(\theta_r - \theta_s)}$ is algebraic for all r and s . Applying the Gelfond-Schneider theorem, this implies that the ratio condition (4) holds for the eigenvalues in the support of vertex a (see [21]). If uniform mixing occurs in a bipartite graph, it is then easy to see that not only does the ratio condition hold for all eigenvalues of A , but these eigenvalues must all be either integers or integer multiples of a square root.

13. Constructions and examples. Here we discuss examples which illustrate some of the topics discussed above.

Perfect state transfer has been studied for some time now, and numerous examples and constructions have been found. All examples we know of fall into at least one of the following three classes.

- The distance partition relative to a vertex a involved in perfect state transfer to a vertex b is equitable. If this is the case, both a and b must be singletons in this partition [23], and the resulting partition is a weighted path that admits state transfer. As such, its parameters are fully determined by the technology described in Section 10. We should mention the extensive list of distance-regular graphs admitting perfect state transfer in [15].
- Graphs obtained from known examples and the application of some graph operations, such as lifts [2], joins and double-cones [1], and products and covers [14, 19, 30].
- Cayley graphs. Here we mentioned cubelike graphs in Section 5, but perfect state transfer has also been character-

ized for circulants [31, 6, 5].

Pretty good state transfer is less rare, but it is also harder to decide whether or not it occurs. A pair of vertices involved in pretty good state transfer must be strongly cospectral, which raises the question of finding a good characterization for such pairs of vertices. For unweighted graphs, the typical source of examples are paths. For the weighted case, some exciting work has been done recently exploiting connections to real analysis. Apart from the four references cited in Section 8, which contemplate both classes above, we mention the paths admitting pretty good state transfer between inner vertices but not between the extremes, found in [16], and the asymmetric graphs whose pretty good state transfer is induced via the addition of some weighted loops [18].

Instantaneous uniform mixing is by far the hardest phenomenon to study and by consequence the one for which we know the least about how to construct examples, even if weights are allowed. We know of some infinite families of cubelike graphs [8] admitting uniform mixing and some Cayley graphs for $\mathbb{Z}_3^d$ [22]. All examples consist of regular graphs, with the sole exception of $K_{1,3}$ and its Cartesian powers.

14. Our favourite open problems.

1. Find an example of perfect state transfer in which the phase factor is not a root of unity, or show that none exists.
2. Find an example of instantaneous uniform mixing where the eigenvalues of the transition matrix are not roots of unity, or show that none exists.
3. Provide a combinatorial characterization of strongly cospectral vertices.
4. Show that no tree on more than four vertices admits perfect state transfer according to the adjacency matrix model.
5. Provide an algorithm that checks whether a cubelike graph admits perfect state transfer that runs in polynomial time on the size of the connection set.
6. Find an example of perfect state transfer which does not fall into one of the three categories given in Section 13.
7. Find the complexity of determining whether or not pretty good state transfer occurs in a given graph.
8. Find an example of instantaneous uniform mixing in a non-regular graph which is neither $K_{1,3}$ nor one of its Cartesian powers, or prove that none exists.

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LINEAR ALGEBRA EDUCATION

An Introductory Linear Algebra Course Featuring Computation

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15. Introduction. We believe a principal strength of linear algebra is its use in computational applications to large problems. Therefore this aspect of the subject should be a key feature of an introductory course, no matter if the student is specializing in mathematics, computer science, physics, chemistry, engineering, economics, or one of the many other fields that benefit from solving problems described by linear systems or linear models. Unfortunately, the subtleties of floating-point arithmetic and numerical linear algebra can be a distraction in a first course.

In this article, we report on the design of an introductory course that integrates computations with *exact* linear algebra, using an open textbook and open software. It is based on a talk given at the Joint Mathematics Meetings in January 2018 [1].

16. Software. We use the open source software package Sage [9]. Why Sage, and not popular commercial offerings? Sage incorporates mature and powerful software packages for linear algebra such as ATLAS, BLAS, LAPACK, FFLAS-FFPACK, and LinBox, while providing a uniform and simple interface to these packages through Python syntax. Sage may be used for free, and in our course, each student has a free account on the hosting service CoCalc [5], ready to use on the first day of class. Since they then interact with Sage through their web browsers, there is never an issue with setup, licenses, or configuration. A more powerful account at CoCalc costs a student only a few dollars a month. As we will see below, the Sage Cell server provides another (free) avenue for no-hassle computations with Sage.

Sage works well for introductory linear algebra, but is powerful enough to assist with research problems, and also covers many other fields within mathematics. So it is a tool that students can use throughout their coursework and careers. The algebraic parts of Sage are centered around rings (with fields being special cases). Important here is the option to set the rational numbers (QQ in Sage syntax) as the base ring of vectors and matrices. Then all computations are performed *exactly*. “Ah, what about eigenvalues?” I hear you say. Sage implements the field of algebraic numbers, and carries an internal representation of these numbers as roots of polynomials with integer coefficients. While canonical forms and matrix decompositions may not be part of an introductory course, Sage can compute objects such as the Jordan canonical form of a matrix over the rationals *exactly*.

Computation and analysis of the reduced row-echelon form is key to the majority of the computations in an introductory course. While visualizations of lines and planes in three dimensions can be very useful in developing an intuitive understanding of linear algebra, the power comes in higher dimensions. When you row-reduce a 3×3 matrix, only a small number of cases may result, notably these four non-trivial cases:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \times \\ 0 & 1 & \times \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & \times & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & \times & \times \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We much prefer to have a student understand how to obtain a null space, row space, and column space in the more general situation of a matrix with pivot columns distributed such as in:

$$\begin{bmatrix} -1 & 1 & 5 & -1 & -5 & 0 \\ -2 & 1 & 7 & -2 & -9 & -2 \\ 1 & 2 & 4 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 3 & 3 \\ 0 & -1 & -3 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, good software is critical, since no one wants to do this by hand, and the location of the zero entries needs to be exactly right.

17. Textbook. Of course, we use our own openly-licensed introductory linear algebra textbook, *A First Course in Linear Algebra* (FCLA) [2]. The online version of the text contains 95 vignettes about Sage. These serve two purposes: to teach a student the appropriate syntax and the correct interpretation of the output, and also to illuminate the relevant concepts of the subject. In the process, we expose the student to as many of Sage’s powerful commands as we can, while emphasizing that knowledge of the commands does not replace knowledge of the underlying concepts.

These vignettes are powered by the Sage Cell Server [10]. This is an instance of Sage, up and running and available worldwide for free. Any web page can easily be enabled to send code to the server and receive back the results. Further,

the code on the page can be edited and re-evaluated. So a student reading FCLA, with no login and no configuration, can execute pre-loaded Sage examples, and tinker with them as much as they like (a page reload will recover from any messes they might make). The only downside is that there is no permanent storage, though we will see that this could in fact be a feature.

A complete list of these vignettes is available at [3]. Try out a few, or just visit the main Sage Cell page and do a simple computation. An example of computing eigenvalues would look as shown in the box below. An online version of this paper would have “live” code, so clicking on an “Evaluate” button would produce the output indicated. The method name `.fcp()` is short for “factored characteristic polynomial”. The real, irrational eigenvalue is an algebraic number. The print representation is a rational approximation, as indicated by the question mark at the end of the number. The last command merely checks that this number is $\sqrt[3]{2}$ *exactly*.

```
sage: A = matrix(QQ, [[10, -12, -11, -13],
                     [12, -13, -9, -12],
                     [-6,  9,  13,  14],
                     [ 2, -5, -11, -11]])

sage: A.fcp()

(x + 1) * (x^3 - 2)

sage: A.eigenvalues()

[-1, 1.259921049894873?, -0.6299605249474365? - 1.091123635971722?*I,
 -0.6299605249474365? + 1.091123635971722?*I]

sage: A.eigenvalues()[1]^3 == 2

True
```

While Sage can be a very useful computational assistant as a student studies, it is important that students do not confuse knowing a repertoire of commands with understanding basic concepts. For example, the `column_space()` method for a matrix will row-reduce the transpose of the matrix and report a subspace with the non-zero rows as the “echelonized” basis. It is still important for a student to know that a basis can be easily formed by row-reducing the original matrix and using the location of the pivot columns of the reduced row-echelon form to select columns of the original matrix for a basis. Similarly, for the null space, Sage computes the “natural” basis that looks like solutions to a homogeneous system when free variables are set individually to one, while the others are held at zero. But then Sage puts these vectors into a matrix as rows, row-reduces, and keeps the non-zero rows as the basis of the subspace. Why obscure the natural basis? The uniqueness of reduced row-echelon form provides a canonical choice of a basis for any (finite-dimensional) subspace, and thus allows Sage to reliably test subspace equality. This is a good occasion to show students how a finite computer can test the equality of what the student views properly as infinite vector spaces.

The online version of our textbook is authored in PreTeXt [8], which produces HTML pages with a fixed text width that leaves ample room in the right margin for pointers to additional resources. By tagging the source for the textbook with generic topic identifiers, we are able to automatically list relevant resources maintained by the CuratedCourses project [6] that are aligned closely with the textbook.

In Figure 3, the theorem about possible solutions sets to a linear system has been tagged by the author with the identifier `math.la.t.linsys.zoi` from [7]. When a student loads this section of the textbook, the CuratedCourses site is queried, and five links populate the right margin as a result. The first is the human-readable version of the topic identified by

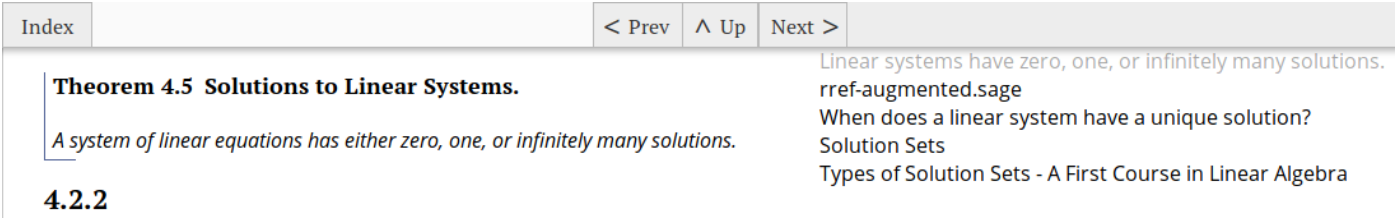


Figure 3: Linear algebra resources in the margin of an electronic textbook

`math.la.t.linsys.zoi` and linked to the CuratedCourses site. The next four links point to specific resources collected by CuratedCourses: a Sage example, a video, a handout, and a statement of the theorem in an online textbook.

18. Lectures. To expose students to the use of Sage as an exploratory tool that complements their other activities, we have developed a series of seventeen worksheets that we work through as part of a lecture. These take from five to twenty minutes when presented in class. They help students learn Sage syntax and output, and they also see how Sage can be used in the hands of an experienced user to investigate the results of complicated computations, verify theorems, and test conjectures. Student feedback suggests that students gain little from watching an instructor simply evaluate a sequence of code snippets. So the worksheets are necessarily incomplete, and meant to be used more dynamically in class. When a randomly-generated nonsingular matrix is sufficient to demonstrate some theorem (say, that it has a non-zero determinant), then such a matrix is generated, demonstrating the wide applicability of the theorem. When a carefully-engineered matrix with specific properties is required, then that is pre-built in the worksheet. These sessions often produce very good questions along the lines of “What would happen if we...?” The answer is always, “Well, let’s see...,” followed by some experimentation.

These worksheets are freely available in a Git repository [4] in three formats. The first is PDF, which is not very useful. They are also available as HTML pages, where the code is all live via the Sage Cell server. Finally, they also may be installed into a student’s CoCalc account as Jupyter notebooks with a single click in the CoCalc Library. Posting the completed instructor’s Jupyter notebook as a “transcript” of the day’s presentation is potentially useful to some students.

19. Examinations. If students are encouraged to use computational tools, and are comfortable exploring properties of 6×9 matrices, what does an examination look like? We require students to bring a laptop (or tablet with a keyboard) to examinations. We build a web page in advance that has:

- three empty instances of the Sage Cell server
- verbatim Python lists with the entries of some matrices
- a cryptic URL that cannot be guessed in advance
- a distinctive background image, that is different for each exam

Students are told to have their browser full-screen, no other tabs open, no chat windows, etc. Since the Sage Cell Server does not allow any local storage, we have found that (to our knowledge!) careful monitoring has prevented any instances of cheating. By providing the entries of matrices on the exam (along with a few red herrings, and some transposes) cut-and-paste prevents keyboard mistakes, and allows students to analyze non-trivial questions quickly.

We also very carefully limit which commands a student may use, since we want to test understanding, not Sage syntax. So instructions are very clear about requiring students to list input, commands used, and resulting output. We also limit commands to

- basic vector and matrix operations (such as matrix multiplication)
- reduced row-echelon form
- computations of the determinant
- characteristic polynomials and their factorization
- eigenvalues, eigenvectors, and eigenspaces

Late in the course, when we ask for a matrix representation (diagonal or otherwise), we might allow a student to use any of these commands. Earlier, we might have a problem where Sage could be used to factor the characteristic polynomial of a matrix, but we expect students to obtain a basis for the eigenspace by row-reducing and analyzing the correct singular matrix. Of course, it would be hard to limit a student from checking their final answers with Sage (and so we even encourage this), but we make it clear that more advanced commands cannot be used to *justify* their answers.

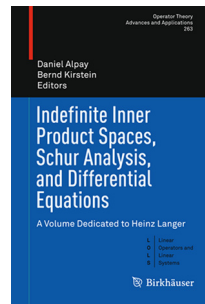
20. Conclusion. A student who uses linear algebra after an introductory course is sure to encounter computational applications. So the sooner they are exposed to computational aspects of the subject the better, and computational assistance can help them learn the basic concepts. Numerical linear algebra can wait for a second course. The broad coverage of exact linear algebra offered by Sage makes it an excellent addition to a first course. Since Sage is freely available via the Sage Cell server (even semi-securely during examinations) and is freely available (or at very low cost) conveniently at CoCalc, it is easy and effective to use for both instructors and students. Open resources, such as textbooks and worksheets, either in online interactive forms with embedded Sage cells or as Jupyter notebooks, make computation an integral part of the course.

Acknowledgment. Partial support for this work was provided by the National Science Foundation's Improving Undergraduate STEM Education (IUSE) program under Award Nos. 1626455, 1505246. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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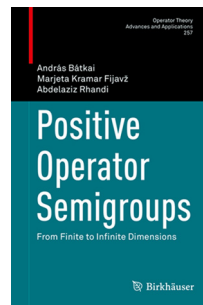
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Indefinite Inner Product Spaces, Schur Analysis, and Differential Equations

A Volume Dedicated to Heinz Langer (2018)
Daniel Alpay, Bernd Kirstein (Eds.)
ISBN 978-3-319-68848-0



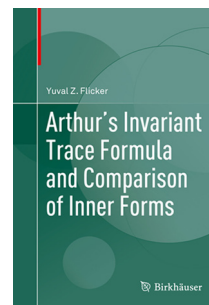
Positive Operator Semigroups

From Finite to Infinite Dimensions (2017)
András Bátkai, Marjeta Kramar Fijavž, Abdelaziz Rhandi
ISBN 978-3-319-42811-6



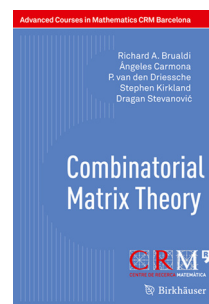
Operator Theory, Function Spaces, and Application

IWOTA, Amsterdam, July 2014 (2016)
Tanja Eisner, Birgit Jacob, André Ran, Hans Zwart (Eds.)
ISBN 978-3-319-31381-8



Arthur's Invariant Trace Formula and Comparison of Inner Forms

(2016)
Yuval Z. Flicker
ISBN 978-3-319-31591-1



Combinatorial Matrix Theory

(2018)
Richard A. Brualdi, Ángeles Carmona, P. van den Driessche, Stephen Kirkland, Dragan Stevanović
ISBN 978-3-319-70952-9

CuratedCourses: A Fine-Grained Educational Repository Demonstrated with Linear Algebra Resources

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 Jim Fowler, The Ohio State University, Columbus, OH, USA, fowler@math.osu.edu.

1. Introduction. The ability to find linear algebra resources that are available online, released under an open license, context-appropriate, and of high quality would be useful for instructors and students alike. The authors are part of a broader team designing and building a platform called CuratedCourses, available at curatedcourses.org, which organizes and curates open educational resources for linear algebra. The key difference between CuratedCourses and other repositories is that resources at CuratedCourses are not only reviewed by mathematicians, but also tagged at a level of granularity commensurate with the instructional detail of linear algebra content. This granularity enables students and faculty to find high-quality, appropriate resources more easily, thereby lowering the barrier to utilizing more ambitious techniques in the classroom.

2. The Problem. There are a variety of scenarios in which an instructor or student may want to find open educational resources. An instructor may wish to find a good video related to a theorem discussed in an upcoming lecture. Providing a short video to the class can provide students with a contrasting but helpful perspective. More ambitiously, an instructor may wish to find pre-class quiz questions or in-class activities. A student may wish to find additional resources (like extra online practice problems) for the current section of the textbook. In all these cases, a user – either an instructor or a student – is seeking resources *aligned* to the textbook.

Finding online resources for the teaching of college mathematics is not difficult. Resources abound. However, finding resources that are readily available, field-tested, and known to be of high quality is more challenging. For example, imagine the challenge of finding a short, high-quality video that addresses the uniqueness of the inverse of an invertible matrix. In performing this search, the authors found that only one of the first ten search results was on the desired theorem; the other videos were either not linear algebra videos, or were videos about other theorems like uniqueness of solutions to $A\vec{v} = \vec{b}$ when A is invertible, or about the fact that the product of invertible matrices is invertible. Even if a search provides a short video about the given topic, the video may not be suitable. It may have issues related to differing or confusing notation, require unfamiliar prerequisites, or simply present a proof while failing to include any scaffolding or any hint as to how a learner might invent such a proof.

For an ambitious instructor wanting to share appropriate resources alongside every lesson, the aforementioned pain points become distressingly repetitive. Often, heroic instructors, feeling the tension between curation versus creation, decide that a “solution” is to create their own videos and activities which they may post online. Often other instructors then may not be able to find those resources for the aforementioned reasons.

3. Existing solutions are incomplete. To curate linear algebra resources, an ideal solution would be a website where resources are collected, linked to a fine-grained hierarchy, and reviewed by experts.

Is it not the case that such a website already exists? It is true that there are a number of repositories of open educational resources. Well-known examples include MERLOT, the OER Commons, and Curriki. Some repositories even cater to math content, like Open Math Notes of the AMS. However, these platforms operate at a level of granularity, for both content and tags, much coarser than a single topic or a single learning outcome. Rather than a single video or a single activity, one record in existing repositories might encompass an entire course, and such a record might be tagged as “linear algebra.” With repositories like this, it is no wonder people often turn to search engines!

4. The CuratedCourses solution. In contrast to these existing repositories, CuratedCourses is organized around fine-grained tags attached to small and reusable resources. Linear algebra was chosen as a domain in which to demonstrate

the effectiveness of the concept. Our team led the organization of hundreds of tags for linear algebra, mapping out the subject with the appropriate level of detail needed to connect resources to underlying topics.

Tag hierarchies are not the only organizing principle of CuratedCourses. Linear algebra textbooks provide a second organizing principle. By attaching topic tags to each section of popular textbooks, a user visiting curatedcourses.org can view the table of contents of popular linear algebra texts and, under each section heading, view related curated resources.

Critically, the connection between resources and textbooks requires no additional work; once textbooks are aligned to the tag hierarchy and once resources are likewise tagged, then textbooks and resources can be automatically aligned to each other. This same technique permits the visualization of the alignment of popular linear algebra textbooks with each other, meaning that an instructor or student can see where a theorem from one textbook appears in other texts.

The result of this project is a platform that helps users find content related to particular sections from popular textbooks. Beyond simply finding content, by virtue of appearing on curatedcourses.org, the content has been reviewed by other mathematicians.

5. Sustainability and scale. The expansion of the peer review mechanism is one component of a plan for sustainability. By creating a place to “publish teaching,” CuratedCourses aims to engage the community with mechanisms similar to academic research journals, e.g., editorial boards which solicit peer review of submitted materials and which, by virtue of their selectivity, provide external evidence of excellence for selected materials.

Another component of sustainability is something akin to an advertising network. Instead of curatedcourses.org being a walled garden, the resources from curatedcourses.org can be placed in any webpage. Page authors need only include a little bit of HTML (e.g., a `<span>` with a topic tag and a `<script>` pointing to curatedcourses.org) and then CuratedCourses will add to the webpage links to high-quality relevant resources from the reviewed, topic-aligned repository at curatedcourses.org. Conversely, when page authors mark a page with a `<span>` as being about a specific topic tag, then that page can also be automatically entered into a moderation queue at curatedcourses.org for review and possible inclusion in the repository.

The CuratedCourses solution addresses the tension between curation and creation. By curating existing content, reviewing content for quality, and aligning content to a detailed tag hierarchy, CuratedCourses makes it easier to find appropriate high-quality, openly-licensed resources. The project has also revealed areas in which linear algebra resources were missing, meaning that the community can now avoid duplication of effort and can organize creation efforts towards new resources addressing content topics or resource types which were truly missing. The outcome is to make it easier for instructors to adopt evidence-based instructional practices through the use of open educational resources. After all, the promise of open resources is not simply that they are cheaper, but ultimately that they enable customized, more effective classroom experiences for each learner.

6. Future Work. We are hoping to expand the CuratedCourses content management model beyond linear algebra content. Our vision is to eventually have all of mathematics tagged, so that one could look up any fine-grained topic and find appropriate resources. Additionally, by mapping to textbooks and flagging required prerequisites, one can envision a system where a student could pick a topic they want to learn, and then identify a custom path of open resources to help them get from their current knowledge to mastery of the desired topic.

We welcome contributions of new linear algebra resources to the CuratedCourses ecosystem. We also appreciate comments and feedback on the organization, usability and utility of the curatedcourses.org site. If you are interested in CuratedCourses and would like to be involved in its growth to new areas of mathematics, please let us know via the contact button on the website.

Acknowledgments. The authors received support from the NSF under DUE-1505246. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

BOOK REVIEW

Matrices: Algebra, Analysis and Applications, by Shmuel Friedland

World Scientific, 2016, ISBN 978-981-4667-96-8/hbk; 978-981-4667-98-2/ebook, xii+582 pages.

Reviewed by Igor Klep, University of Auckland, igor.klep@auckland.ac.nz

Shmuel Friedland is one of the world's leading researchers in linear algebra and matrix theory. In his more than 200 publications he has solved many famous open problems, such as the notoriously “wild” problem of classifying tuples of matrices up to simultaneous similarity, the Salmon Conjecture, Lax's eigenvalue crossing problem, and many others. Friedland has also contributed significantly to the theory of inverse eigenvalue problems, dynamical systems and ergodic theory, and nonnegative matrices, to name just a few.

His breadth of expertise is also on clear display in this impressive, almost 600-page book, *Matrices*. In addition to classical results, the book includes a substantial amount of material which was previously only available in the form of research articles – much of it the author's own work. Friedland successfully combines tools from various branches of mathematics, such as algebra, functional analysis and operator theory, complex analysis, algebraic geometry, and numerical analysis to tackle problems in matrix theory.

The book is divided into seven chapters: Domains, Modules and Matrices; Canonical Forms for Similarity; Functions of Matrices and Analytic Similarity; Inner Product Spaces; Elements of Multilinear Algebra; Non-Negative Matrices; and Various Topics. Each chapter of the book concludes with a section on historical remarks containing bibliographic information and references to the original literature. Numerous exercises (of varying difficulty) are dispersed throughout the book to challenge the reader.

Chapter 1 contains a study of properties of matrices with entries in integral domains and canonical forms for similarity. Of particular interest are matrices $A(\omega)$ whose entries are holomorphic functions $H(\Omega)$ on an open domain $\Omega \subseteq \mathbb{C}^n$. To give the reader a feel for the depth of the material: In Theorem 1.16.6, the author uses sheaf cohomology to characterize a class of domains Ω over which local (i.e., pointwise) solutions to $A(\omega)u = b(\omega)$ lift to solutions $u(\omega)$ holomorphic on Ω .

Chapter 2 is on canonical forms of similarity, another area where Friedland is one of the leading experts. While most of the results here are standard, there are a few new tidbits, mostly from the author's own work.

Chapter 3 returns to the topic of holomorphic matrices and thus has a decidedly complex analytic flavor. Three notions of similarity are considered for matrix-valued holomorphic functions: pointwise, analytic and rational (which could perhaps be called meromorphic). The final part of the chapter also investigates matrix polynomials.

Chapter 4 on inner product spaces is the longest. It includes standard topics on hermitian and symmetric matrices, their eigenvalues, positive (semi)definite matrices, trace inequalities, the singular value decomposition, the Moore-Penrose generalized inverse, low-rank approximations, hermitian pencils, eigenvalues of sums of hermitian matrices (without the Klyachko & Knutson–Tao solution of the Horn Conjecture), and perturbation theory.

Chapter 5 on tensors is perhaps the most standard one in the book, but still contains plenty of material previously available only in research form. A highlight is the proof of the Golden-Thompson Inequality,

$$\mathrm{tr} \exp(A + B) \leq \mathrm{tr}(\exp A \exp B)$$

for hermitian matrices A and B .

Chapter 6 deals with nonnegative matrices. After a brief introduction to graph theory, it gives the Perron-Frobenius Theorem. Further topics here include stochastic matrices and Markov chains, Z - and M -matrices, as well as a (nontrivial) application to cellular communication.

Finally, the last chapter discusses numerical ranges, operator norms, tensor products of convex sets (e.g., for the convex set Ω_n of doubly stochastic matrices), permanents, and, of course, the inverse eigenvalue problem for nonnegative matrices.

The selection of topics, terse style, and lack of motivation make it difficult to recommend as a textbook for anything other than an advanced graduate course in matrix theory. It will however, because of its breadth, be a mainstay of the linear algebraist's library and an invaluable reference tool for researchers in mathematics and allied disciplines who have an interest in matrix theory.

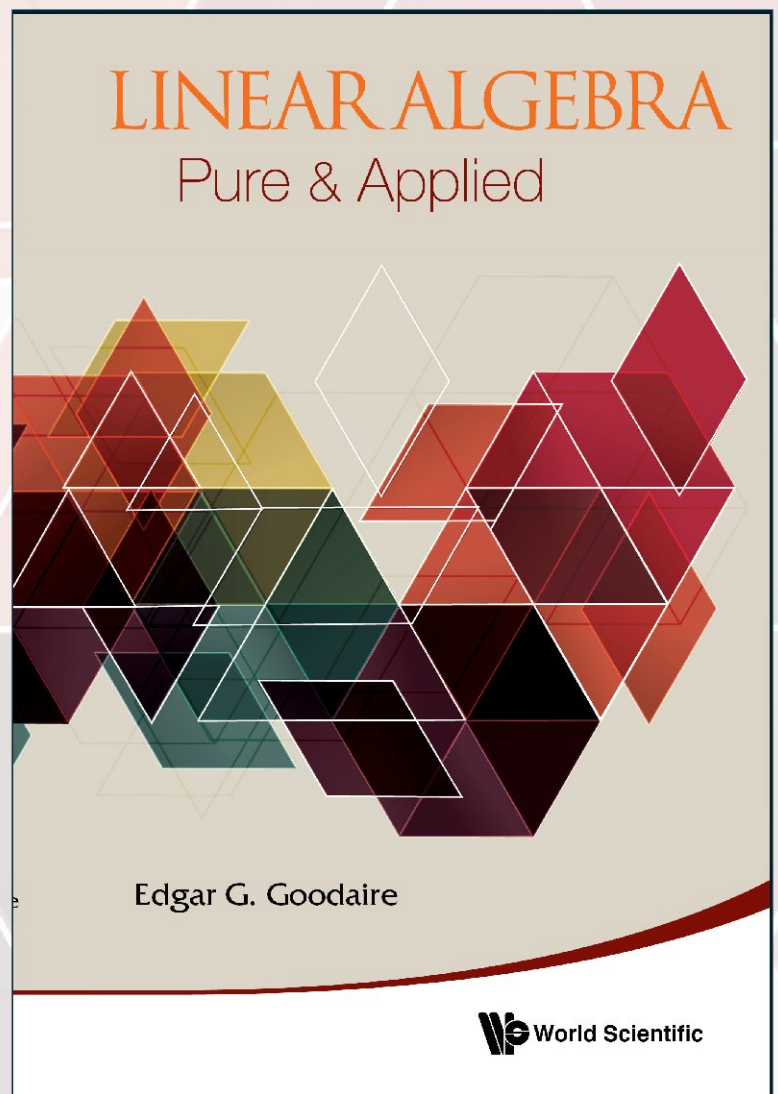
“

Goodaire's book gives a proper balance to logic and computation. This book is worth considering if you feel the need to change books for your course.

Mathematical Association of America

Key Features:

- The author is a pure mathematician who regularly asks “Why” and expects the student to answer this question regularly and precisely
- The book encourages interactive reading via “Reading Checks,” short questions most of which can be answered mentally, that appear regularly throughout the exposition
- The book provides several ways in which the reader can assess his or her progress, including true-false questions at every section and a vocabulary check at every chapter
- The absolutely crucial concept of eigenvalues and eigenvectors is introduced early enough that the topic can be covered easily in a first course in linear algebra
- There is a glossary at the back of the book every definition accompanied by an example of something that is and something that isn't



More information at

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CONFERENCE REPORTS

International Conference on Matrix Inequalities and Matrix Equations (MIME2017) Shanghai, China, June 6–8, 2017

Report by Minghua Lin

MIME2017 took place over three days at the New Lehu Hotel of Shanghai University. It was sponsored by Shanghai University, the International Research Center for Tensor and Matrix Theory (IRCTMT) and Gaoyuan Discipline of Shanghai. It was attended by over one hundred faculty members and graduate students from Belarus, Canada, Japan, Portugal, Russia, Serbia, the USA and mainland China, Hong Kong, Macau, and Taiwan.



Participants of MIME2017

Conference pictures and the conference program are available at the website: <http://math.shu.edu.cn/mime2017>

The Scientific Committee for this conference consisted of:

- Delin Chu, National University of Singapore, Singapore
- Alexander E. Guterman, Moscow State University, Russia
- Chi-Kwong Li, College of William and Mary, USA
- Tin-Yau Tam, Auburn University, USA
- Qing-Wen Wang, Shanghai University, China
- Fuzhen Zhang, Nova Southeastern University, USA
- Yang Zhang, University of Manitoba, Canada

The next MIME conference will be held at Shanghai University from June 7–10, 2018. (See the conference announcements that follow.)

UPCOMING CONFERENCES AND WORKSHOPS

The 26th International Workshop on Matrices and Statistics (IWMS-2018) Montreal, Canada, June 5–7, 2018

The 26th International Workshop on Matrices and Statistics will take place at the Multimedia Centre at Dawson College, Westmount (Montréal).

On Tuesday, June 5th there will be a mini-symposium for T. W. Anderson (1918–2016) and on Thursday, June 7th the 3rd mini-symposium on “Magic squares, prime numbers and postage stamps” will be held.

For more information on IWMS-2018, please visit the website: <https://profchu.wixsite.com/mysite>

International Conference on Matrix Inequalities and Matrix Equations (MIME2018) Shanghai University, Shanghai, China, June 8–10, 2018

The purpose of this conference is to stimulate research and foster interactions between researchers interested in matrix inequalities, matrix equations, and their applications. It is hoped that this informal conference will provide an opportunity for researchers from different areas to exchange ideas and to share information.

The conference is hosted by the International Research Center for Tensor and Matrix Theory (IRCTMT) at Shanghai University, and supported by Gaoyuan Discipline of Shanghai – Mathematics, Shanghai City, China.

The Scientific Organizing Committee consists of:

- Delin Chu (National University of Singapore, Singapore)
- Chi-Kwong Li (College of William & Mary, USA)
- Tin-Yau Tam (Auburn University, USA)
- Qing-Wen Wang (Chair, Shanghai University, China)
- Fuzhen Zhang (Nova Southeastern University, USA)
- Yang Zhang (University of Manitoba, Canada)

For more information, please visit the conference website: <http://math.shu.edu.cn/mime2018>

Prairie Discrete Mathematics Workshop Brandon University, Manitoba, Canada, June 12–15, 2018

The main objective of the PDMW is to bring together researchers in the area of discrete mathematics from across the prairie region as well as neighbouring provinces and states. The goal for the workshop is to provide opportunities for sharing research and joint research projects.

Invited Speakers: Richard Brewster (Thompson Rivers University), Rob Craigen (University of Manitoba), Shonda Goselin (University of Winnipeg), Gary MacGillivray (University of Victoria), and Karen Meagher (University of Regina).

For more information, contact Shahla Nasserassr (nasserassrs@brandonu.ca) or Sarah Plosker (ploskers@brandonu.ca), or visit the conference website: <https://www.brandonu.ca/pdmw>.

The Fourteenth Workshop on Numerical Ranges and Numerical Radii Munich, Germany, June 13–17, 2018

The year 2018 marks the 100th anniversary of the celebrated Toeplitz-Hausdorff Theorem asserting that the numerical range of an operator is always convex. There has been a lot of research activity on the topic since this fundamental result was established, due to the many connections of the subject to different pure and applied areas.

The purpose of this conference is to stimulate research and foster interactions between researchers interested in the subject. The informal workshop atmosphere will facilitate the exchange of ideas from different research areas and, hopefully, the participants will leave informed of the latest developments and newest ideas.

The conference organizers are:

- Douglas Farenick (University of Regina, Canada)
- David Kribs (University of Guelph, Canada)
- Chi-Kwong Li (College of William & Mary, USA)
- Sarah Plosker (Brandon University, Canada)
- Thomas Schulte-Herbruggen (Technical University, Munich)

For updates and registration details, one may visit <http://cklixx.people.wm.edu/wonra18.html>.

The 7th International Conference on Matrix Analysis and Applications (ICMAA2018) Shinshu University, Japan, June 22–25, 2018

This meeting aims to stimulate research and foster interactions between mathematicians in all aspects of linear and multilinear algebra, matrix analysis, operator theory, graph theory, combinatorics, and their applications, and to provide an opportunity for researchers to exchange ideas and discuss developments in these subjects.

The previous conferences were held in China (Beijing, Hangzhou), the United States (Nova Southeastern University), Turkey (Selçuk University, Konya), and Vietnam (Duy Tan University, Da Nang). Former keynote invited speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland, Alexander A. Klyachko (ILAS guest speaker), Shmuel Friedland, and Man-Duen Choi.

The invited speakers of ICMAA2018 are:

- Tsuyoshi Ando, Hokkaido University (Emeritus), Japan
- Fumio Hiai, Tohoku University, Japan

The scientific organizing committee (SOC) consists of

- Hiromichi Ohno, Shinshu University, Japan.
- Hiroyuki Osaka, Ritsumeikan University, Japan.
- Tin-Yau Tam, Auburn University, USA.
- Qing-Wen Wang, Shanghai University, China.
- Fuzhen Zhang, Nova Southeastern University, USA.

Details at:

<http://www.shinshu-u.ac.jp/faculty/engineering/appl/2017/math/ohno/icmaa.htm>

The conference is sponsored by JSPS.

The 6th IMA Conference on Numerical Linear Algebra and Optimization Birmingham, UK, June 27–29 2018

The IMA and the University of Birmingham are pleased to announce the Sixth IMA Conference on Numerical Linear Algebra and Optimization. The meeting is organised in cooperation with SIAM, whose members will receive the IMA member registration rate.

The success of modern codes for large-scale optimization is heavily dependent on the use of effective tools of numerical linear algebra. On the other hand, many problems in numerical linear algebra lead to linear, nonlinear or semidefinite optimization problems. The purpose of this conference is to bring together researchers from both communities and to find and communicate points and topics of common interest.

Conference topics include any subject that could be of interest to both communities, such as: direct and iterative methods for large sparse linear systems; eigenvalue computation and optimization; large-scale nonlinear and semidefinite programming; effect of round-off errors, stopping criteria, embedded iterative procedures; optimization issues for matrix polynomials; fast matrix computations; compressed/sparse sensing; PDE-constrained optimization; distributed computing and optimization; applications and real time optimization.

The invited speakers are:

- Amir Beck (Tel Aviv University)
- Julian Hall (University of Edinburgh)
- Misha Kilmer (Tufts University)
- Dominique Orban (Polytechnique Montréal)
- Alison Ramage (University of Strathclyde)
- Françoise Tisseur (University of Manchester)
- Luis Nunes Vicente (University of Coimbra)
- Steve Wright (University of Wisconsin)

More information is available at <https://ima.org.uk/7149/6thIMANLA0>, or contact Lizzi Lake, Conference Officer (e-mail: conferences@ima.org.uk, tel: +44 (0) 1702 354 020), Institute of Mathematics and its Applications, Catherine Richards House, 16 Nelson Street, Southend-on-Sea, Essex, SS1 1EF, UK.

Riordan Arrays and Related Topics (5RART 2018) Busan, Republic of Korea, June 25–29, 2018



The purpose of the conference is to stimulate research and foster interaction of researchers interested in Riordan arrays and related topics. The general topic of the conference is the theory and application of Riordan arrays. Particular areas of interest include (but are not limited to) the algebraic structure of the Riordan group, Riordan arrays and combinatorial sums, Sheffer sequences, umbral calculus, lattice paths, combinatorics of posets, lattices and words, graph theory, and many other aspects of combinatorics and their relationships with other areas of mathematics, computer science, physics, biology, and other fields. It is our hope that the conference will provide a convenient platform for the exchange of research experiences and ideas from different research areas related to Riordan arrays and the Riordan group.

Invited speakers:

- Richard A. Brualdi (University of Wisconsin–Madison, USA)
- Miklós Bóna (University of Florida, USA)
- Sergey Kitaev (University of Strathclyde, UK)
- Manuel Alonso Morón (Universidad Complutense de Madrid, Spain)
- Emanuele Munarini (Politecnico di Milano, Italy)
- Yi Wang (Dalian University of Technology, China)
- Ian Wanless (University of Monash, Australia)

More information is available at <https://sites.google.com/view/5rart/home>.

International Conference on Algebra and Related Topics (ICART 2018) Rabat, Morocco, July 2–5, 2018

The International Conference on Algebra and Related Topics (ICART 2018) will be held from the 2nd to the 5th of July 2018 at the Faculty of Sciences, Mohammed V University in Rabat, Morocco.

The conference will cover various research areas presented in the following three sessions: Applied and Computational Homology in Topology, Algebra and Geometry; Homological Algebra, Modules, Rings and Categories; and Linear and Multilinear Algebra and Function Spaces.

Plenary speakers in the session on Linear and Multilinear Algebra and Function Spaces include:

- Abdellatif Bourhim (Syracuse University, USA)
- Matej Brešar (University of Ljubljana, Slovenia)
- Javad Mashregi (Université Laval, Québec, Canada)
- Mostafa Mbekhta (Université Lille, France)
- Lajos Molnár (University of Debrecen, Hungary)
- Rueda Pilar (Universidad de Valencia, Spain)
- Peter Šemrl (University of Ljubljana, Slovenia)

More information can be found at <http://www.fsr.ac.ma/icart2018>.

Numerical Analysis and Scientific Computation with Applications (NASCA 2018) Kalamata, Greece, July 2–6, 2018

The aim of this conference is to bring together researchers working in numerical analysis, scientific computation and applications. Participants will present and discuss their latest results in these areas.

The main topics of the conference are: Large Linear Systems of Equations and Eigenvalue Problems with Preconditioning, Linear Algebra and Control, Model Reduction, Ill-posed Problems, Regularisation, Numerical Methods for PDEs, Approximation Theory, Radial Basis Functions, Meshless Approximation, Optimization, and Applications to Image and Signal Processing, Environment, Energy Minimization, and Internet Search Engines.

Further information can be found at the conference website: <http://nasca18.math.uoa.gr>.

The 24th Industrial Mathematical & Statistical Modeling (IMSM) Workshop for Graduate Students North Carolina State University, USA, July 15–25, 2018

The Industrial Mathematical & Statistical Modeling (IMSM) Workshop for Graduate Students exposes graduate students in mathematics, engineering, and statistics to exciting real-world problems from industry and government. The workshop provides students with experience in a research team environment and exposure to possible career opportunities. On the first day, a Software Carpentry bootcamp will bring students up-to-date on their programming skills in Python/Matlab and R, and introduce them to version control systems and software repositories.

This workshop is sponsored by the Statistical and Applied Mathematical Sciences Institute (SAMSI) and the Center for Research in Scientific Computation (CRSC) and the Department of Mathematics at North Carolina State University.

More information is available at <http://www.samsi.info/ims-18> and questions can be directed to grad@samsi.info.

The Linear Algebra and Applications Workshop (LAAW2018) Niterói, RJ, Brazil, July 30–31, 2018

The Linear Algebra and Applications Workshop (LAAW2018) will be held at Fluminense Federal University (UFF). LAAW2018 is a satellite event of ICM 2018 and a warm up for ILAS 2019, which will take place in Rio as well.

Invited speakers include:

- Carlos Tomei (PUC-RJ, Brazil)
- Chris Godsil (University of Waterloo, Canada)
- Dragan Stevanovic (Serbian Academy of Sciences and Arts, Serbia)
- José Roberto P. Rodrigues (Petrobras/Cenpes, Brazil)

The conference website is <https://laaw2018.org>.

ILAS 2019: Linear Algebra Without Borders Rio de Janeiro, Brazil, July 8–12, 2019



The 22nd conference of the International Linear Algebra Society, ILAS 2019: Linear Algebra Without Borders, will be held July 8–12, 2019 in Rio de Janeiro, Brazil, at the main campus of Fundação Getúlio Vargas (FGV), a Brazilian think tank and higher education institution founded in 1944 with the aim of promoting Brazil's economic and social development.

The theme of the conference is “Linear Algebra Without Borders” and refers primarily to the fact that Linear Algebra and its myriad of applications are interwoven in a borderless unit. In the conference, the organizers plan to illustrate this by creating a program whose plenary talks and symposia represent the many scientific “countries” of Linear Algebra, and which invites participants to “visit” them. This theme also refers to the openness and inclusiveness of Linear Algebra to researchers of different backgrounds. Ongoing updates and more information about the conference will be found at <http://ilas2019.org>.

Householder Symposium XXI on Numerical Linear Algebra Selva di Fasano, Italy, June 14–19, 2020

The next Householder Symposium will be held from June 14–19, 2020 at Hotel Sierra Silvana, Selva di Fasano (Br), Italy.

Attendance at the meeting is by invitation, and participants are expected to attend the entire meeting. Applications are solicited from researchers in numerical linear and multi-linear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control. Each attendee is given the opportunity to present a talk or a poster. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. Applications are due by October 31, 2019.

This meeting is the twenty-first in a series, previously called the Gatlinburg Symposia, but now named in honor of its founder, Alston S. Householder, a pioneer of numerical linear algebra. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1, 2017 will be presented. Nominations are due by January 31, 2020.

ILAS NEWS

Françoise Tisseur to deliver Olga Taussky-Todd Lecture at ICIAM 2019



Françoise Tisseur

The International Council for Industrial and Applied Mathematics has selected Françoise Tisseur of the School of Mathematics, University of Manchester, to deliver the Olga Taussky-Todd Lecture at the International Congress on Industrial and Applied Mathematics, ICIAM 2019, in Valencia, Spain. For more details see:

<http://www.iciam.org/olga-taussky-todd-lecture-iciam-2019>

Françoise has been a member of ILAS for many years and served on the ILAS Board of Directors from 2011 until 2014.

Nominations Sought for Hans Schneider Prize

Contributed announcement from Peter Šemrl

The Hans Schneider Prize in Linear Algebra is awarded by the International Linear Algebra Society for research contributions and achievements at the highest level of Linear Algebra. The Prize may be awarded for either outstanding scientific achievement or for a lifetime contribution. The Prize is awarded every three years, and the next winner will give the Hans Schneider Prize Lecture at the ILAS conference in Rio de Janeiro, Brazil, July 2019.

The committee charged with selection of the next Hans Schneider Prize winner has been appointed by the ILAS President upon the advice of the ILAS Executive Board. The chair is Rajendra Bhatia and the members are: Richard Brualdi, Shmuel Friedland, Tom Laffey, Paul Van Dooren, and Peter Šemrl (ex officio - ILAS president).

Nominations, of distinguished individuals judged worthy of consideration for the Prize, are now being invited from members of ILAS and the scientific community in general. In nominating an individual, the nominator should include:

- a brief biographical sketch of the nominee, and
- a statement explaining why the nominee is considered worthy of the prize, including references to publications or other contributions of the nominee which are considered most significant in making this assessment.

The prize guidelines can be found at

<http://www.ilasic.math.uregina.ca/iic/misc/hsguidelines.html>

and the list of all Hans Schneider Prize winners at

<http://www.ilasic.math.uregina.ca/iic/misc/hsall.html>

Nominations are open until December 1, 2018, and should be sent to the committee chair, Rajendra Bhatia, at rajenbhatia@gmail.com.

ILAS Election Results

Leslie Hogben was re-elected to the position of Secretary/Treasurer for a three-year term, beginning on March 1, 2018; Maria Isabel Bueno and Vilmar Trevisan were elected to three-year terms as members of the ILAS Board, beginning on March 1, 2018.

ILAS President/Vice President Annual Report: March 31, 2018

Respectfully submitted by Peter Šemrl, ILAS President, peter.semrl@fmf.uni-lj.si
and Hugo Woerdeman, ILAS Vice President, hugo@math.drexel.edu

1. Board-approved actions since the last report include:

- The board approved the appointment of Louis Deaett as Editor-in-Chief of *IMAGE* starting August 1, 2017. To ensure a smooth transition, Kevin Vander Meulen served as co-Editor-in-Chief for the Fall 2017 issue, and will continue as Advisory Editor.
- The membership was informed that *IMAGE* will no longer provide hard copies. Issue 59 will be the last issue of *IMAGE* with a print edition. Starting with issue 60, *IMAGE* will only be produced in an electronic format. Accommodations will be made for the two ILAS members who prepaid for *IMAGE* hardcopies.
- The membership was informed that although dues are due January 1 of each year, from now on the membership year will have a grace period until February 28/29; in other words, a membership in year n is good until Feb 28/29 year $n + 1$. This conforms to previous practice (except in 2017) and avoids confusion at election time, which ends in February. Although there was never an official grace period, the membership list was typically updated in early March. It was observed that the by-laws are silent on this issue, and that no other ILAS business is affected by this change.
- The membership was informed that the ILAS 2020 meeting will be held at National University of Ireland, Galway, June 22–26.
- An agreement between Springer, IWOTA, and ILAS was signed. Springer received a commitment from IWOTA to continue to publish their proceedings with Springer. In return Springer will make donations to the Israel Gohberg ILAS-IWOTA Fund.
- The board approved support of ILAS conferences at a higher financial level. This change was made possible by ILAS's healthy financial shape.
- The board approved an increase of support for ILAS speakers at SIAM ALA conferences, equalizing contributions to the SIAM ALA - ILAS Speaker Exchange program.

2. ILAS elections ran from December 18, 2017 to February 1, 2018 and proceeded via electronic voting. The following were (re-)elected to offices with three-year terms that began on March 1, 2018:

- Secretary/Treasurer: Leslie Hogben
- Board of Directors: Maria Isabel Bueno and Vilmar Trevisan.

The following continue in the ILAS offices which they currently hold:

- President: Peter Šemrl (term ends 29 February 2020)
- Vice President: Hugo Woerdeman (term ends 28 February 2019)
- Second Vice President (for ILAS conferences): Steve Kirkland (reappointed by ILAS President for a second term, ending 29 February 2020)
- Board of Directors: Ravindra Bapat (term ends 28 February 2019), James Nagy (term ends 29 February 2020), Rachel Quinlan (term ends 29 February 2020), and Helena Šmigoc (term ends 28 February 2019).

Beatrice Meini and Henry Wolkowicz completed their terms on the ILAS Board of Directors on 28 February 2018. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated. We also thank the members of the Nominating Committee—Froilán Dopico (chair), Richard Brualdi, Anne Greenbaum, Yongdo Lim, Sarah Plosker—for their work on behalf of ILAS, and also extend gratitude to all candidates that agreed to have their names stand for the elections.

3. The following ILAS-endorsed meetings have taken place since our last report:

- 8th Linear Algebra Workshop, Ljubljana, Slovenia, June 12–16, 2017. Vladimir Muller was a Hans Schneider ILAS lecturer. <http://www.law05.si/law17>
- 6th International Conference on Matrix Analysis and Applications, Duy Tan University, Da Nang City, Vietnam, June 15–18, 2017. <http://icmaa-2017.duytan.edu.vn/>
- Householder Symposium XX, Virginia Tech, Blacksburg, VA, USA, June 18–23, 2017. Paul van Dooren was an ILAS lecturer. <http://www.math.vt.edu/HHXX>
- The Second Malta Conference in Graph Theory and Combinatorics, Qawra, Malta, June 26–30, 2017. <http://www.um.edu.mt/events/2mcgtc2017/>

- Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics, Denver, CO, USA, July 9–22, 2017. <https://sites.google.com/site/rmgpgrwc/>
- Algebraic and Extremal Graph Theory Conference, University of Delaware, Newark, DE, USA, August 7–10, 2017. <https://www.mathsci.udel.edu/events/conferences/aegt>
- MatTriad-2017, Bedlewo, Poland, September 25–29, 2017. <https://mattriad.wmi.amu.edu.pl/>
- International Conference on Linear Algebra and its Applications, Manipal University, Manipal, India, December 11–15, 2017. <http://iclaa2017.com/>
- Workshop on Graph Spectra, Combinatorics and Optimization, University of Aveiro, Aveiro, Portugal, January 25–27, 2018. <http://wgsc2018.web.ua.pt/>

4. ILAS has endorsed the following conferences of interest to ILAS members:

- SIAM Conference on Applied Linear Algebra (SIAM-ALA18), Hong Kong Baptist University, Hong Kong, May 4–8 2018. Mark Embree will be an ILAS lecturer and Valeria Simoncini a Hans Schneider ILAS lecturer. <http://www.math.hkbu.edu.hk/siam-ala18/>
- Western Canadian Linear Algebra Meeting, WCLAM 18, Washington State University, Pullman, Washington, USA, May 26–27, 2018. <http://www.math.wsu.edu/math/faculty/tsat/W-CLAM18.html>
- Linear Algebra, Matrix Analysis and Applications, ALAMA2018, Complejo San Juan, Sant Joan d'Alacant, Spain, May 30 – June 1, 2018. <https://web.ua.es/en/alama2018/home.html>
- Riordan Arrays and Related Topics (5RART 2018), Busan, Republic of Korea, June 25–29, 2018. <https://sites.google.com/view/5rart/home>
- International Conference on Algebra and Related Topics (ICART 2018), Rabat, Morocco, July 2–5, 2018. <http://nasca18.math.uoa.gr/>
- Numerical Analysis and Scientific Computation with Applications (NASCA 2018), Kalamata, Greece, July 2–6, 2018. Michael Tsatsomeros will be an ILAS lecturer. <http://nasca18.math.uoa.gr/>

5. The following ILAS conferences are scheduled:

- 22nd ILAS Conference: Linear Algebra Without Borders, 8–12 July, 2019, Rio de Janeiro, Brazil. The co-chairs of the organizing committee are Nair Abreu and Vilmar Trevisan. <http://ilas2019.org/>
- 23rd ILAS Conference: ILAS 2020 will be held at National University of Ireland, Galway, June 22–26, 2020. The chair of the organizing committee is Rachel Quinlan. <http://www.maths.nuigalway.ie/ILAS2020/>

6. The Electronic Journal of Linear Algebra (*ELA*) is now in its 34th volume. *ELA*'s url is <http://repository.uwo.edu/ela/>.

Volume 32 was published in 2017 and contains 40 papers. Volume 33 is a special issue of *ELA* devoted to the International Conference on Matrix Analysis and Its Applications – MatTriad-2017, with editors Oskar Maria Baksalary, Natália Bebiano, Heike Faßbender, and Simo Puntanen.

Bryan Shader (University of Wyoming) and Michael Tsatsomeros (Washington State University) continue as the Editors-in-Chief.

7. *IMAGE* is the semi-annual bulletin for ILAS available online at <http://ilasic.org/IMAGE/>. As of January 2018, the Editor-in-Chief is Louis Deaett (Quinnipiac University).

8. ILAS-NET is a moderated newsletter for mathematicians worldwide, with a focus on linear algebra; it is managed by Sarah Carnochan Naqvi.

An archive of ILAS-NET messages is available at <http://www.ilasic.org/ilas-net/>. To send a message to ILAS-NET, please send the message (preferably in text format) in an email to ilasic@uregina.ca indicating that you would like it to be posted on ILAS-NET. If the message is approved, it will be posted soon afterwards. To subscribe to ILAS-NET, please complete the form at <http://ilasic.us10.list-manage.com/subscribe?u=6f8674f5d780d2dc591d397c9&id=dbda1af1a5>.

9. ILAS' website, known as the ILAS Information Centre (IIC), is located at <http://www.ilasic.org> and provides general information about ILAS (e.g., ILAS officers, By-laws, Special Lecturers), as well as links to pages of interest to the ILAS community.

Respectfully submitted,

Peter Šemrl, ILAS President (peter.semrl@fmf.uni-lj.si); and
Hugo J. Woerdeman, ILAS Vice-President (hugo@math.drexel.edu).

ILAS 2017–2018 Treasurer's Report

April 1, 2017 – March 31, 2018

By Leslie Hogben

Net Account Balance on March 31, 2017

Vanguard (ST Fed Bond Fund Admiral 7876.686 Shares)	\$ 85,505.08	
Checking Account - Great Western	\$ 56,420.16	
Certificate of Deposit	\$ 45,905.27	
Accounts Payable	\$ 154.73	
		<u>\$ 187,985.24</u>
General Fund	\$ 98,101.40	
Israel Gohberg ILAS-IWOTA Lecture	\$ 3,500	
Conference Fund	\$ 9,874.55	
Olga Taussky Todd/John Todd Fund	\$ 12,331.68	
Hans Schneider Lecture Fund	\$ 13,227.54	
Frank Uhlig Education Fund	\$ 5,357.31	
Hans Schneider Prize Fund	\$ 29,659.11	
ELA Fund	\$ 342.54	
ILAS/LAA Fund	\$ 15,591.11	
		<u>\$ 187,985.24</u>

INCOME:

Dues	\$ 10,320.00	
Israel Gohberg ILAS-IWOTA Lecture		
Lecture Fund Donations	\$ 30.00	
General Fund Donations	\$ 3,160.00	
Conference Fund Donations	\$ 40.00	
Taussky-Todd Fund	\$ 240.00	
Hans Schneider Lecture Fund	\$ 30.00	
Uhlig Education Fund	–	
Hans Schneider Prize Fund	\$ 10.00	
ELA	\$ 100.00	
Elsevier Sponsorship (flow-through)	\$ 5,000.00	
LAMA for Speaker (T&F flow-through)	\$ 975.00	
Corporate dues Income	\$ 550.00	
Royalty Income	–	
Interest - Great Western	\$ 98.41	
Interest on Great Western Certificate of Deposit (#1)	\$ 420.54	
Vanguard - Dividend Income	\$ 1,552.08	
- Short Term Capital Gains	–	
- Long Term Capital Gains	\$ 50.29	
Variances in the Market	\$ (1,541.30)	
Total Income		\$ 21,035.02

EXPENSES:

Conference Expenses	\$ 2,304.74	
ELA	\$ 2,000.00	
Treasurer's Assistant	\$ 500.00	
General Expenses	\$ 759.58	
Credit Card Processing & Wire Transfer Fees	\$ 741.49	
Non-ILAS Conferences	\$ 4,000.00	
Hans Schneider Lecture	\$ 1,221.65	
Taylor & Frances flow-through (LAMA)	\$ 1,000.00	
2017 ILAS Conference - Elsevier flow-through	\$ 5,000.00	
TT Lecture	\$ 1,500.00	
Hans Schneider Prize	\$ 2,248.54	
IMAGE Costs	\$ 302.51	
Business License	\$ 0.00	
Ballot Costs	\$ 331.80	
Web Hosting & Online Membership Forms	\$ 819.14	
Misc Expenses	\$ 179.73	
Total Expenses		\$ 22,909.18

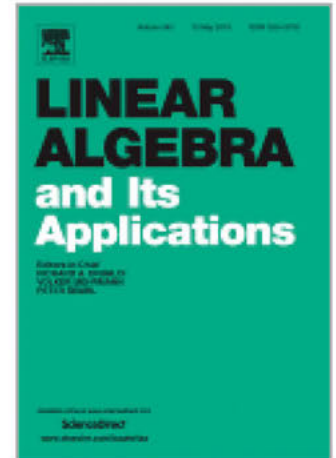
Net Account Balance on March 31, 2018

Vanguard (ST Fed Bond Fund Admiral 7998.604 Shares)	\$ 85,566.14	
Checking Account - Great Western	\$ 53,524.12	
Certificate of Deposit	\$ 46,325.81	
Accounts Payable (ISU cc)	\$ 720.00	
		<u>\$ 186,136.07</u>
General Fund	\$ 100,772.42	
Israel Gohberg ILAS-IWOTA Lecture	\$ 3,530.00	
Conference Fund	\$ 9,914.55	
Olga Taussky Todd/John Todd Fund	\$ 11,071.68	
Hans Schneider Lecture Fund	\$ 12,035.89	
Frank Uhlig Education Fund	\$ 5,357.31	
Hans Schneider Prize Fund	\$ 27,420.57	
ELA Fund	\$ 442.54	
ILAS/LAA Fund	\$ 15,591.11	
		<u>\$ 186,136.07</u>

Celebrating the Golden Anniversary of Linear Algebra and its Applications

A note from the Editors-in-Chief

The year 2018 marks the golden anniversary of *Linear Algebra and its Applications* (LAA). LAA was first published 50 years ago. It was the first journal devoted to linear algebra and indeed was instrumental in defining linear algebra as a serious field of study and promoting its development and application. Now, there are other journals devoted to linear algebra in its pure, applied, and numerical/computational forms.



A description of the early history of LAA can be found in the article "**LAA is 40 years old**" volume 428 (2008), 1–3), written by the then editors-in-chief, Richard A. Brualdi, Volker Mehrmann, and Hans Schneider.

Linear algebra touches almost all parts of mathematics: analysis, classical algebra, number theory, combinatorics, graph theory, information theory, geometry, operator theory, ... and, in turn, these different parts of mathematics have an impact on the development of linear algebra providing new directions of study.

Since its inception, LAA has been published by Elsevier which has been very supportive and very responsive to our editorial concerns. The breadth, significance, and applicability of linear algebra can be seen in every volume of LAA. We look forward to its continued growth and LAA's involvement in it.

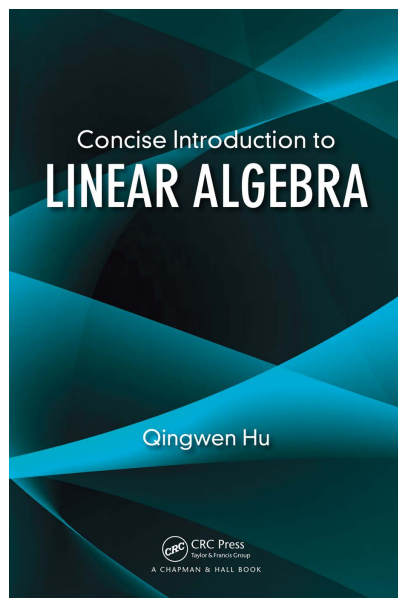
Richard A. Brualdi
Volker Mehrmann
Peter Semrl

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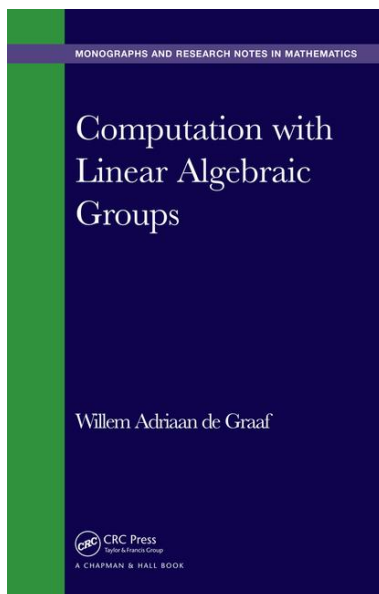
Linear Algebra and its Applications is affiliated with the International Linear Algebra Society.



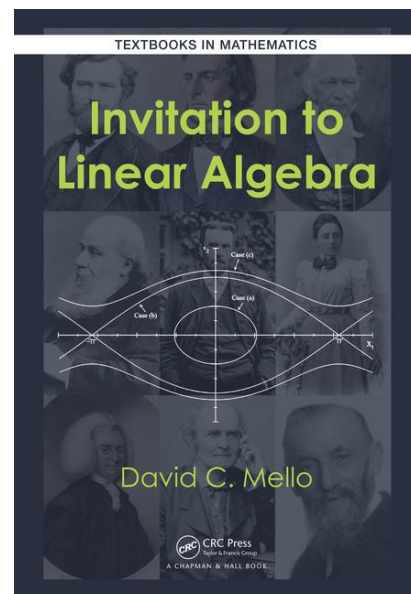
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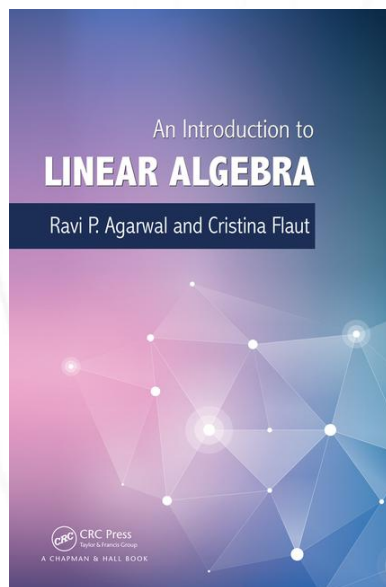
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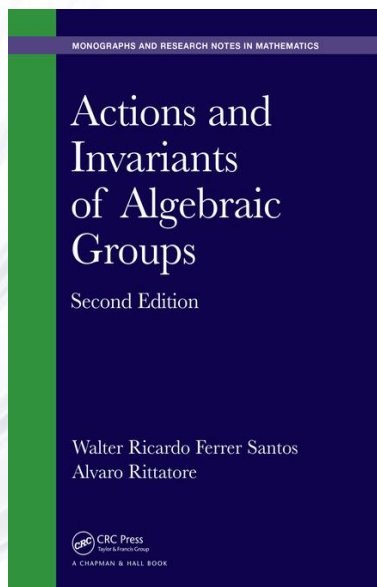
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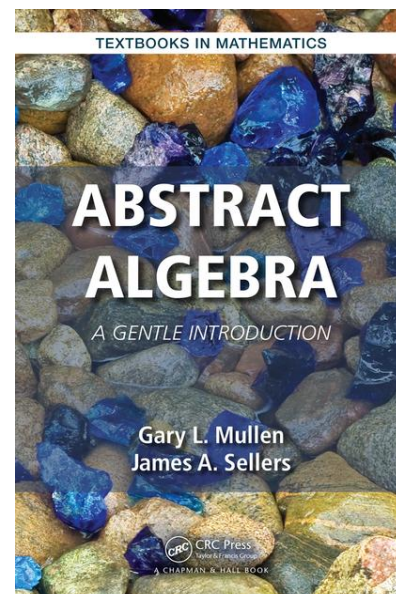
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IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 59-1, 59-2, 59-3, and 59-4. Solutions are invited to Problems 58-2 part (b), 58-4, 59-2 part (iv) and 59-5, and for all of the problems of issue 60.

Problem 59-1: Rationality of the Matrix Exponential

Proposed by Ovidiu FURDUI, *Technical University of Cluj-Napoca, Cluj-Napoca, Romania*, ofurdui@yahoo.com

Let $M_n(\mathbb{Q})$ denote the set of n by n matrices with rational entries. Let $n \geq 2$ be an integer and let $A \in M_n(\mathbb{Q})$. The matrix exponential e^A is defined using the usual power series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Prove that $e^A \in M_n(\mathbb{Q})$ if and only if A is a nilpotent matrix.

Solution 59-1 by Jeffrey STUART, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

Let $A \in M_n(\mathbb{Q})$. Suppose A is nilpotent. Then $A^k = 0_n$, the $n \times n$ zero matrix, for all $k \geq n$, and

$$e^A = \sum_{k=0}^n \frac{1}{k!} A^k,$$

which is clearly in $M_n(\mathbb{Q})$. Conversely, suppose that $e^A \in M_n(\mathbb{Q})$. Then the eigenvalues of e^A must be algebraic numbers over $\mathbb{C}$. Since $A \in M_n(\mathbb{Q})$, the eigenvalues of A must be algebraic numbers over $\mathbb{C}$. Observe that λ is an eigenvalue for A if and only if e^λ is an eigenvalue for e^A . By the Lindemann-Weierstrass Theorem, if a is a nonzero algebraic number, then e^a must be transcendental. Thus, it is impossible for both λ and e^λ to be algebraic unless $\lambda = 0$. That is, all eigenvalues of A must be zero, and hence, A must be nilpotent.

Also solved by Gérald BOURGEOIS and Eugene A. HERMAN.

Problem 59-2: Root of Cauchy-Bunyakovsky-Schwarz

Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si
and Tatjana PETEK, *University of Maribor, Maribor, Slovenia*, tatjana.petek@um.si

Let $A \in M_n(\mathbb{C})$, $n \geq 3$. Let $\|\cdot\|$ denote the Euclidean norm. Show that the following conditions are equivalent:

- (i) A is a scalar matrix.
- (ii) $\sqrt{\|At\|^2\|t\|^2 - |t^*At|^2} = 0$ for every $t \in \mathbb{C}^n$.
- (iii) $\sqrt{\|At\|^2\|t\|^2 - |t^*At|^2}$ is a polynomial in $2n$ real variables of $\operatorname{Re}(t)$ and $\operatorname{Im}(t)$, $t \in \mathbb{C}^n$.
- (iv) The map $t \mapsto \sqrt{\|At\|^2\|t\|^2 - |t^*At|^2}$, $t \in \mathbb{C}^n$, regarded as a function of $2n$ real variables of $\operatorname{Re}(t)$ and $\operatorname{Im}(t)$, is differentiable at every eigenvector t_0 of A and thus at every vector $t \in \mathbb{C}^n$.

What happens if $n = 2$?

Solution 59-2 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Solution (partial, with (iv) omitted): Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), we use the identity

$$\sqrt{\|u\|^2\|t\|^2 - |t^*u|^2} = \frac{1}{\|t\|} \left\| \|t\|^2 u - (t^*u)t \right\| \quad \text{for all } t, u \in \mathbb{C}^n, t \neq 0.$$

Let $t = pe_j + qe_k$, where p, q are real, $e_1, \dots, e_n$ are the standard unit basis vectors in $\mathbb{C}^n$, and $j \neq k$. Also let A_k denote the k th column of A and a_{jk} denote its (j, k) -entry. Then

$$\begin{aligned} \|t\|^2 At - (t^*At)t &= (p^2 + q^2)(pA_j + qA_k) - (pe_j + qe_k)^*(pA_j + qA_k)(pe_j + qe_k) \\ &= (p^2 + q^2)(pA_j + qA_k) - (p^2a_{jj} + pqa_{jk} + pqa_{kj} + q^2a_{kk})(pe_j + qe_k). \end{aligned}$$

For $m \neq j, k$, the m th term of this vector is $(p^2 + q^2)(pa_{mj} + qa_{mk})$. The j th and k th terms (t_j and t_k) are, respectively,

$$\begin{aligned} t_j &= (p^2 + q^2)(pa_{jj} + qa_{jk}) - p(p^2a_{jj} + pqa_{jk} + pqa_{kj} + q^2a_{kk}) \\ &= q(pqa_{jj} + q^2a_{jk} - p^2a_{kj} - pqa_{kk}) \quad \text{and} \\ t_k &= (p^2 + q^2)(pa_{kj} + qa_{kk}) - q(p^2a_{jj} + pqa_{jk} + pqa_{kj} + q^2a_{kk}) \\ &= -p(pqa_{jj} + q^2a_{jk} - p^2a_{kj} - pqa_{kk}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{\|At\|^2 \|t\|^2 - |t^*At|^2} &= \frac{1}{\|t\|} \left\| \|t\|^2 At - (t^*At)t \right\| \\ &= \frac{1}{\sqrt{p^2 + q^2}} \sqrt{\sum_{m \neq j, k} (p^2 + q^2)^2 |pa_{mj} + qa_{mk}|^2 + (p^2 + q^2) |pqa_{jj} + q^2a_{jk} - p^2a_{kj} - pqa_{kk}|^2} \\ &= \sqrt{\sum_{m \neq j, k} (p^2 + q^2) |pa_{mj} + qa_{mk}|^2 + |pqa_{jj} + q^2a_{jk} - p^2a_{kj} - pqa_{kk}|^2}. \end{aligned}$$

Since $n \geq 3$, the above sum over m is not vacuous. Thus the last displayed expression cannot be a polynomial in p and q unless $a_{mj} = a_{mk} = 0$ for all $m \neq j, k$. It follows that all the off-diagonal entries of A are zero. Therefore the last displayed expression is

$$|pqa_{jj} - pqa_{kk}| = |pq||a_{jj} - a_{kk}|$$

which cannot be a polynomial in p and q unless $a_{jj} = a_{kk}$. Hence A is a scalar matrix. $\square$

Problem 59-3: A Matrix Limit

Proposed by Ovidiu FURDUI, *Technical University of Cluj-Napoca, Cluj-Napoca, Romania*, ofurdui@yahoo.com

Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Evaluate the limit

$$\lim_{n \rightarrow \infty} \begin{bmatrix} \cos\left(\frac{a}{n}\right) & \frac{b}{n} \\ \frac{c}{n} & \cos\left(\frac{d}{n}\right) \end{bmatrix}^n.$$

Solution 59-3.1 by G  rald BOURGEOIS, *Universit   de la polyn  sie fran  aise, FAA'A, Tahiti, Polyn  sie Fran  aise*, bourgeois.gerald@gmail.com

The required result is

$$\lim_{n \rightarrow \infty} \begin{bmatrix} \cos\left(\frac{a}{n}\right) & \frac{b}{n} \\ \frac{c}{n} & \cos\left(\frac{d}{n}\right) \end{bmatrix}^n = \exp\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} \cosh(\sqrt{bc}) & \sqrt{\frac{b}{c}} \sinh(\sqrt{bc}) \\ \sqrt{\frac{c}{b}} \sinh(\sqrt{bc}) & \cosh(\sqrt{bc}) \end{bmatrix}.$$

The second equality follows directly from the MacLaurin series for e^x while the first equality is a special case of the following proposition:

Proposition. Let $P \in M_m(\mathbb{C})$ and let $Q_n \in M_m(\mathbb{C})$ be a matrix that depends on n s.t. $\|Q_n\| = O(1/n^\alpha)$ where $\alpha > 1$. Then $\lim_{n \rightarrow +\infty} (I_m + 1/nP + Q_n)^n = e^P$.

Proof. Whenever X is a matrix with $\|X\| < 1$, we may define $\log(I + X) = \sum_{k=1}^{\infty} (-1)^k \frac{X^k}{k}$. Let $A_n = I + 1/nP + Q_n$; for n large enough, $\|A_n - I\| < 1$. Then $\log(A_n)$ is defined.

Part 1. We show that $\log(A_n) = 1/nP + R_n$ where $\|R_n\| = O(1/n^2) + O(1/n^\alpha)$. Indeed $\log(A_n) = (1/nP + Q_n) - 1/2(1/nP + Q_n)^2 + \dots$ implies that $\|\log(A_n) - (1/nP + Q_n)\| = O(\|1/nP + Q_n\|^2) = O(1/n^2) + O(1/n^{2\alpha})$ and $\|\log(A_n) - 1/nP\| \leq \|Q_n\| + O(1/n^2) + O(1/n^{2\alpha})$. $\square$

Part 2. We show that $\|A_n^n - \exp(P)\| = O(\|nR_n\|)$. Indeed $A_n = \exp(1/nP + R_n)$ and therefore $A_n^n = \exp(P + nR_n)$. $\|\exp(P + nR_n) - \exp(P)\| \leq \exp(\|P\|)(\exp(\|nR_n\|) - 1) = O(\|nR_n\|)$. We are done because $\|nR_n\|$ tends to 0.

If we take $P = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ and $Q_n = \begin{bmatrix} \cos(\frac{a}{n}) - 1 & 0 \\ 0 & \cos(\frac{d}{n}) - 1 \end{bmatrix}$ in the proposition, we get the required equality above.

Solution 59-3.2 by Jeffrey STUART, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

A scalar or matrix expression $f(n)$ defined on the positive integers is said to be $O(n^{-k})$ for some positive integer k if $n^k f(n)$ is bounded as $n \rightarrow \infty$. Note that if $f(n)$ is $O(n^{-k})$ for some positive integer k , and if p is a positive integer, then $[f(n)]^p$ is $O(n^{-kp})$.

Theorem. Let $a(n)$, $\widehat{b}(n)$, $\widehat{c}(n)$, and $d(n)$ be real-valued functions defined for all positive integers n such that each of the functions is $O(n^{-2})$. Let $\beta, \gamma \in \mathbb{R}$ with $\beta\gamma \neq 0$. Let $b(n) = \frac{\beta}{n} + \widehat{b}(n)$ and $c(n) = \frac{\gamma}{n} + \widehat{c}(n)$. When $\beta\gamma > 0$,

$$\lim_{n \rightarrow \infty} \begin{bmatrix} 1 + a(n) & b(n) \\ c(n) & 1 + d(n) \end{bmatrix}^n = \begin{bmatrix} \cosh(\sqrt{\beta\gamma}) & \frac{\beta}{\sqrt{\beta\gamma}} \sinh(\sqrt{\beta\gamma}) \\ \frac{\gamma}{\sqrt{\beta\gamma}} \sinh(\sqrt{\beta\gamma}) & \cosh(\sqrt{\beta\gamma}) \end{bmatrix};$$

and when $\beta\gamma < 0$,

$$\lim_{n \rightarrow \infty} \begin{bmatrix} 1 + a(n) & b(n) \\ c(n) & 1 + d(n) \end{bmatrix}^n = \begin{bmatrix} \cos(\sqrt{-\beta\gamma}) & \frac{\beta}{\sqrt{-\beta\gamma}} \sin(\sqrt{-\beta\gamma}) \\ \frac{\gamma}{\sqrt{-\beta\gamma}} \sin(\sqrt{-\beta\gamma}) & \cos(\sqrt{-\beta\gamma}) \end{bmatrix}.$$

Corollary. Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Then

$$\lim_{n \rightarrow \infty} \begin{bmatrix} \cos(\frac{a}{n}) & \frac{b}{n} \\ \frac{c}{n} & \cos(\frac{d}{n}) \end{bmatrix}^n = \begin{bmatrix} \cosh(\sqrt{bc}) & \frac{b}{\sqrt{bc}} \sinh(\sqrt{bc}) \\ \frac{c}{\sqrt{bc}} \sinh(\sqrt{bc}) & \cosh(\sqrt{bc}) \end{bmatrix}.$$

Proof of Theorem. Let I_2 denote the 2×2 identity matrix. Let n be a positive integer. Define the 2×2 real matrices B and $\widehat{B}(n)$ by

$$B = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \text{ and } \widehat{B}(n) = \begin{bmatrix} a(n) & \widehat{b}(n) \\ \widehat{c}(n) & d(n) \end{bmatrix}.$$

Then each entry of $\widehat{B}(n)$ is $O(n^{-2})$, and

$$\begin{bmatrix} 1 + a(n) & b(n) \\ c(n) & 1 + d(n) \end{bmatrix} = I_2 + \frac{1}{n}B + \widehat{B}(n).$$

Since the entries of $\frac{1}{n}B + \widehat{B}(n)$ are all $O(n^{-1})$, the eigenvalues of $\frac{1}{n}B + \widehat{B}(n)$ are the roots of a polynomial $\lambda^2 + p\lambda + q$ where p and q are real, p is $O(n^{-1})$, and q is $O(n^{-2})$. Consequently, the eigenvalues of $\frac{1}{n}B + \widehat{B}(n)$ are real or complex numbers that are $O(n^{-1})$.

When n is sufficiently large, the eigenvalues of $\frac{1}{n}B + \widehat{B}(n)$ have magnitude less than 1. Consequently, when n is sufficiently large, the matrix Maclaurin series for the natural logarithm may be applied to $I_2 + \frac{1}{n}B + \widehat{B}(n)$:

$$\begin{aligned} \log \left(I_2 + \frac{1}{n}B + \widehat{B}(n) \right) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{1}{n}B + \widehat{B}(n) \right)^k = \frac{1}{n}B + \widehat{B}(n) - \frac{1}{2} \left(B + \widehat{B}(n) \right)^2 + \cdots \\ &= \frac{1}{n}B + O(n^{-2}) \end{aligned}$$

Then

$$n \cdot \log \left(I_2 + \frac{1}{n}B + \widehat{B}(n) \right) = B + O(n^{-1}),$$

and

$$\lim_{n \rightarrow \infty} \left[n \cdot \log \left(I_2 + \frac{1}{n}B + \widehat{B}(n) \right) \right] = B.$$

It follows from the continuity of the logarithm that

$$\lim_{n \rightarrow \infty} \left(I_2 + \frac{1}{n} B + \widehat{B}(n) \right)^n = e^B.$$

Observing that $B^2 = \beta\gamma I_2$,

$$\begin{aligned} e^B &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} B^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} B^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (\beta\gamma)^k I_2 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (\beta\gamma)^k B. \end{aligned}$$

When $\beta\gamma > 0$,

$$\begin{aligned} e^M &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (\sqrt{bc})^{2k} I_2 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (\sqrt{bc})^{2k} B \\ &= \cosh(\sqrt{bc}) I_2 + \frac{1}{\sqrt{bc}} \sinh(\sqrt{bc}) B \\ &= \begin{bmatrix} \cosh(\sqrt{bc}) & \frac{b}{\sqrt{bc}} \sinh(\sqrt{bc}) \\ \frac{c}{\sqrt{bc}} \sinh(\sqrt{bc}) & \cosh(\sqrt{bc}) \end{bmatrix}. \end{aligned}$$

When $\beta\gamma < 0$,

$$\begin{aligned} e^B &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k (-\beta\gamma)^k I_2 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k (-\beta\gamma)^k B \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\sqrt{-\beta\gamma})^{2k} I_2 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\sqrt{-\beta\gamma})^{2k} B \\ &= \cos(\sqrt{-\beta\gamma}) I_2 + \frac{1}{\sqrt{-\beta\gamma}} \sin(\sqrt{-\beta\gamma}) B \\ &= \begin{bmatrix} \cos(\sqrt{-\beta\gamma}) & \frac{b}{\sqrt{-\beta\gamma}} \sin(\sqrt{-\beta\gamma}) \\ \frac{c}{\sqrt{-\beta\gamma}} \sin(\sqrt{-\beta\gamma}) & \cos(\sqrt{-\beta\gamma}) \end{bmatrix}. \end{aligned}$$

□

Proof of Corollary. Observe that for any real number r and any sufficiently large positive integer n ,

$$\cos\left(\frac{r}{n}\right) = 1 - \frac{1}{2} \left(\frac{r}{n}\right)^2 + O(n^{-4}) = 1 + O(n^{-2}).$$

Let $a(n) = \cos\left(\frac{a}{n}\right) - 1$, and let $d(n) = \cos\left(\frac{d}{n}\right) - 1$. Then $a(n)$ and $d(n)$ are $O(n^{-2})$. Let $\beta = b$ and $\gamma = c$, and let $\widehat{b}(n) = \widehat{c}(n) = 0$ for all n . Then $b(n) = \frac{\beta}{n}$ and $c(n) = \frac{\gamma}{n}$. Apply the theorem. □

Remark 1. Suppose that $f(x)$ and $g(x)$ are analytic in a neighborhood of 0, that $f(x)$ and $g(x)$ are even functions, and that $f(0) = g(0) = 1$. If $bc \neq 0$, then the theorem can be used to evaluate

$$\lim_{n \rightarrow \infty} \begin{bmatrix} f\left(\frac{1}{n}\right) & \frac{b}{n} \\ \frac{c}{n} & g\left(\frac{1}{n}\right) \end{bmatrix}^n.$$

Examples of choices for $f(x)$ and $g(x)$ include $\cosh(rx)$ and $\cos(rx)$ for any nonzero real number r , $(1 \pm rx^2)^{-k}$ for any nonzero real number r and any positive integer k , and $\exp(rx^2)$ for any nonzero real number r .

Solution 59-3.3 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

We replace the given matrix by the more general form

$$A = \begin{bmatrix} f\left(\frac{a}{n}\right) & \frac{b}{n} \\ \frac{c}{n} & f\left(\frac{d}{n}\right) \end{bmatrix}.$$

where f is any function defined in a neighborhood of the origin for which $f(0) = 1$, $f'(0) = 0$, and the remainder, $R_2(x) = f(x) - (f(0) + f'(0)x) = f(x) - 1$, satisfies $R_2(x) = O(x^2)$ as $x \rightarrow 0$. (The cosine function satisfies these conditions.) The eigenvalues of A are

$$\lambda_{\pm} = \frac{1}{2} \left(f\left(\frac{a}{n}\right) + f\left(\frac{d}{n}\right) \pm \sqrt{\left(f\left(\frac{a}{n}\right) - f\left(\frac{d}{n}\right) \right)^2 + 4\frac{bc}{n^2}} \right)$$

with corresponding eigenvectors

$$v_{\pm} = \begin{bmatrix} b \\ n(\lambda_{\pm} - f\left(\frac{a}{n}\right)) \end{bmatrix}.$$

Let P be the matrix whose columns are v_+ and v_- . Then $P^{-1}AP = \text{diag}(\lambda_+, \lambda_-)$, and so $A^n = P \text{diag}(\lambda_+^n, \lambda_-^n)P^{-1}$. Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} A^n &= \lim_{n \rightarrow \infty} \begin{bmatrix} b & b \\ n(\lambda_+ - f\left(\frac{a}{n}\right)) & n(\lambda_- - f\left(\frac{a}{n}\right)) \end{bmatrix} \cdot \lim_{n \rightarrow \infty} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \\ &\cdot \lim_{n \rightarrow \infty} \frac{1}{bn(\lambda_- - \lambda_+)} \begin{bmatrix} n(\lambda_- - f\left(\frac{a}{n}\right)) & -b \\ -n(\lambda_+ - f\left(\frac{a}{n}\right)) & b \end{bmatrix}. \end{aligned}$$

Consider the several individual terms

$$\begin{aligned} n(\lambda_{\pm} - f\left(\frac{a}{n}\right)) &= \frac{n}{2} \left(f\left(\frac{d}{n}\right) - f\left(\frac{a}{n}\right) \pm \sqrt{\left(f\left(\frac{a}{n}\right) - f\left(\frac{d}{n}\right) \right)^2 + 4\frac{bc}{n^2}} \right) \\ &= \frac{n}{2} \left(O(1/n^2) \pm \sqrt{(O(1/n^2))^2 + 4\frac{bc}{n^2}} \right) \\ &\rightarrow \pm\sqrt{bc} \text{ as } n \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} \lambda_{\pm}^n &= \left(\frac{1}{2} \left(2 + O(1/n^2) \pm \sqrt{(O(1/n^2))^2 + 4\frac{bc}{n^2}} \right) \right)^n \\ &= \left(1 + O(1/n^2) \pm \frac{1}{n} \sqrt{(O(1/n))^2 + bc} \right)^n \\ &\rightarrow e^{\pm\sqrt{bc}} \text{ as } n \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} n(\lambda_+ - \lambda_-) &= n\sqrt{(O(1/n^2))^2 + 4\frac{bc}{n^2}} \\ &= \sqrt{(O(1/n))^2 + 4bc} \rightarrow 2\sqrt{bc} \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n \rightarrow \infty} A^n &= \begin{bmatrix} b & b \\ \sqrt{bc} & -\sqrt{bc} \end{bmatrix} \begin{bmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}} \end{bmatrix} \frac{1}{2b\sqrt{bc}} \begin{bmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{bmatrix} \\ &= \frac{1}{2b\sqrt{bc}} \begin{bmatrix} b\sqrt{bc}(e^{\sqrt{bc}} + e^{-\sqrt{bc}}) & b^2(e^{\sqrt{bc}} + e^{-\sqrt{bc}}) \\ bc(e^{\sqrt{bc}} - e^{-\sqrt{bc}}) & b\sqrt{bc}(e^{\sqrt{bc}} - e^{-\sqrt{bc}}) \end{bmatrix} \\ &= \begin{bmatrix} \cosh \sqrt{bc} & \sqrt{b/c} \cosh \sqrt{bc} \\ \sqrt{c/b} \sinh \sqrt{bc} & \sinh \sqrt{bc} \end{bmatrix}.\end{aligned}$$

Aside: The same method shows that $\lim_{n \rightarrow \infty} A^n$ also exists when $f'(0) = k \neq 0$, but the entries of the limit matrix have many more terms and these involve the parameters a , d , and k .

Problem 59-4: Path Connectedness

Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Show that the set of complex n by n matrices with n distinct eigenvalues is path connected. Is the set of real n by n matrices with n distinct eigenvalues path connected?

Solution 59-4 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

We show that both the given set and the set of $n \times n$ diagonalizable matrices are path connected. Let A belong to either set. Then A is diagonalizable and so there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Since P has only finitely many eigenvalues, there exists $z \in \mathbb{C}$ with $|z| = 1$ such that the ray $\mathbb{R}^-z$ consisting of all negative scalar multiples of z contains no eigenvalue of P . Let

$$f(t) = (Q_t)^{-1}AQ_t, \quad \text{where } Q_t = tP + z(1-t)I \text{ and } 0 \leq t \leq 1.$$

The matrices Q_t are invertible, since otherwise there would exist $t \in (0, 1)$ and a nonzero vector $u \in \mathbb{C}^n$ such that $Q_t u = 0$. Then $Pu = -z(1-t)u/t$, and so u would be an eigenvector whose eigenvalue lies on the ray $\mathbb{R}^-z$. Furthermore, each matrix $f(t)$ is diagonalizable if A is diagonalizable, and has n distinct eigenvalues if A does. Hence, $f(1) = P^{-1}AP = D$ and $f(0) = z^{-1}Az = A$ are path connected in each of the two sets. It remains to show that any two diagonal matrices, D and E , are path connected. Within the set of diagonalizable matrices, $tD + (1-t)E$, $0 \leq t \leq 1$, is a suitable path. Within the set of matrices with n distinct eigenvalues, that path might go outside the set. Instead, we choose another diagonal matrix F in that set with the property that none of the diagonal entries of F is equal to any of the diagonal entries of D or E . It suffices to show that D and F are path connected and so are F and E . Let $D = \text{diag}(d_1, \dots, d_n)$ and $F = \text{diag}(f_1, \dots, f_n)$. Since $\{d_1, \dots, d_n\}$ and $\{f_1, \dots, f_n\}$ are disjoint and finite, there is a continuous path in the complex plane, $c(t)$, $0 \leq t \leq 1$, such that $c(0) = d_1$, $c(1) = f_1$, and $c(t)$ never touches any of the elements of $\{d_2, \dots, d_n\}$ or $\{f_2, \dots, f_n\}$. Hence $\text{diag}(c(t), d_2, \dots, d_n)$ is a legitimate path between $\text{diag}(d_1, d_2, \dots, d_n)$ and $\text{diag}(f_1, d_2, \dots, d_n)$. Continuing in this manner through all n diagonal entries, we see that D and F are path connected. Similarly, F and E are path connected.

Now, when paths are limited to real matrices, the set is not path connected. Here is a counterexample when $n = 2$: Suppose there is a continuous path $C(t)$, $0 \leq t \leq 1$, within the set of real 2×2 matrices with 2 distinct eigenvalues, such that $C(0)$ has a pair of complex conjugate eigenvalues and $C(1)$ has two distinct real eigenvalues. Let $C(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$, and define $f(t) = (a(t) - d(t))^2 + 4b(t)c(t)$, $0 \leq t \leq 1$. Thus f is continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$. By the Intermediate Value Theorem, $f(t_0) = 0$ for some $t_0 \in (0, 1)$. Thus $C(t_0)$ has a repeated real eigenvalue, which is a contradiction.

Similar counterexamples can be constructed for all larger values of n . Here one may use the fact that, for polynomials with no repeated roots, the roots of the polynomials are continuous functions of the coefficients.

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Problems: We introduce 4 new problems in this issue and invite readers to submit solutions for publication in *IMAGE*. **Submissions:** Please submit proposed problems and solutions in macro-free L^AT_EX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 60-1: Skew-Symmetric Forms and Determinants

Proposed by Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $n \geq 2$, let $A \in M_n(\mathbb{R})$ be a rank r skew-symmetric matrix, and let $x, y \in \mathbb{R}^n$. Express $x^T A y$ as a weighted sum of $\frac{r}{2}$ determinants of matrices, with each matrix having x and y as two of its columns. Show that $\frac{r}{2}$ is the smallest number of determinants that suffice to represent $x^T A y$. (Examples: If $n = 2$ and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then

$$x^T A y = a \det \begin{pmatrix} x & y \end{pmatrix}. \quad (1)$$

If $n = 3$ and $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, then

$$x^T A y = \det \begin{pmatrix} x & y & v \end{pmatrix}. \quad (2)$$

where $v = (c, -b, a)^T$.

Problem 60-2: A System of Matrix Equations

Proposed by G  rald BOURGEOIS, *Universit   de la polyn  sie fran  aise, FAA'A, Tahiti, Polyn  sie Fran  aise*, bourgeois.gerald@gmail.com

We consider the system of matrix equations

$$\mathcal{S}: \quad A^3 + B^3 = 0_3, \quad AB - B^2 A^2 = I_3$$

where the unknowns A and B are 3×3 complex matrices.

1. Find an explicit solution (A, B) of $\mathcal{S}$ such that A and B have no common invariant proper subspace.
2. Find an explicit solution (A, B) of $\mathcal{S}$ such that A and B have at least one common invariant proper subspace and A^6 is not a scalar matrix.

Problem 60-3: Distance in the Commuting Graph

Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Let $\mathbb{F}$ be an arbitrary field and let A and B be nonscalar matrices in $M_n(\mathbb{F})$ with $A^2 = A$ and $B^2 = 0$. Show that there exists a nonscalar matrix which commutes with both A and B .

Problem 60-4: An Agreeable Permutation Problem

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let M be an n by n matrix all of whose entries are either 1 or -1 . Let σ be a permutation in S_n . We say that σ is an agreeable permutation if $\prod_{k=1}^n m_{k\sigma(k)}$ is equal to the sign of σ . For all $1 \leq i, j \leq n$, let $A(i, j)$ be the number of agreeable permutations which map i to j . Show that the number $A(i, j)$ is independent of i and j if and only if the two smallest singular values of M are either both equal to $\sqrt{n}$ or both equal to zero.

Solutions to Problems 59-1, 59-2, 59-3 and 59-4 are on pages 41–46.