The Bulletin of the International Linear Algebra Society

IMAGE

Serving the International Linear Algebra Community

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About *IMAGE*

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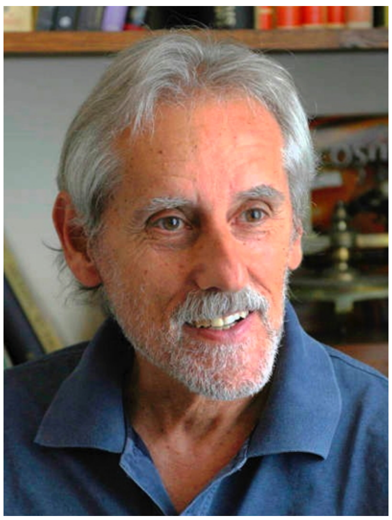
FEATURE INTERVIEWS

"Fantasy, creativity, and imagination are fundamental ingredients for research in mathematics"

Dario Bini Interviewed by Raf Vandebril[1]

R.V. - How did you get interested in mathematics?

D.B. - Actually, it happened very early. Since when I was in primary school, I was attracted by puzzling issues and by problem solving. I liked games like writing and pronouncing words counter-wise from bottom to top or creating "secret" alphabets by replacing each letter with new strange symbols and using it like a cryptographic tool to communicate with friends. I remember that when my teacher proposed to the whole class some arithmetic or geometric problem, for me it was a great pleasure to provide a solution. I think that this gratification, received in the first years of my student life, probably has been one of the motivations of my interest in mathematics. I believe that even now, when I analyze a research problem and try to solve it, it is like I try to repeat the same gratifying experience felt in the first years of primary school.



Dario Bini

R.V. - And then, after primary school?

D.B. - My interest for math continued in the middle and high schools and was fostered by very good teachers. I was quite eager to learn new things. At the end of high school I had no doubts to enroll in mathematics at the university. At the university I discovered an unexpected beauty and elegance of fine mathematics. I was fascinated by the structure of things, and by the almost endless creative power of math where you may give form to whatever you like in a rigorous way just by fixing your axioms and applying logical inference. Somehow I fell in love with the Bourbaki approach where formalism and generality go beyond concepts and intuition.

R.V. - And what about creativity and fantasy?

D.B. - The pleasure to create new things started very early. When I was a child there were not many toys available. We had to use our fantasy and the few things available around to create our toys. The imagination was the main engine. The pleasure to create something new together with friends, relying on our minds and on our hands was great. It is the same process and the same pleasure that I encounter now when I play with theory and algorithms together with my collaborators. Indeed, fantasy, creativity, and imagination are fundamental ingredients for research in mathematics.

R.V. - And when did you encounter matrices?

D.B. - I encountered matrices during my thesis. The subject was the theory of M-matrices. Only theory, no algorithms, no numerical linear algebra. In that time, the scholars who influenced me much were Miroslav Fiedler and Vlastimil Pták. I read many papers of theirs. I personally met Fiedler many years after I got my degree. This was an encounter with great emotion.

R.V. - Which other persons were very influential in your career?

Milvio Capovani. He was a man having strong intuitions and a strategic view, somehow visionary. In 1977 he had the courage to propose to me work on a 10-year standing problem: the complexity of matrix multiplication. After the paper by Volker Strassen of 1969, most people thought that Strassen's algorithm, with complexity $O(n^{\log_2 7})$ for $n \times n$ matrix multiplication, was optimal and could not be improved. Milvio created a small group of Grazia Lotti and Francesco Romani, two young computer scientists, and myself, with the goal of analyzing matrix structures hidden in the tensor of matrix multiplication with the hope to arrive at an improvement of Strassen's algorithm and at a decrease of the exponent of matrix multiplication complexity. He was firmly convinced that this was possible.

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R.V. - I know that you achieved this goal.

Yes, we worked with great enthusiasm by mixing theoretical and experimental analysis. Eventually, we discovered an unexpected property, i.e., unlike matrices, sequences of low rank tensors can converge to a tensor of higher rank. Thus we introduced the concept of border rank and of approximate complexity. Then we started analyzing the border rank of some small matrix multiplication tensors and we found that the product of a 2×2 triangular and a 2×2 full matrix can be approximated in 5 instead of 6 multiplications. This led to the exponent $\log_{12} 1000$, which improved both Strassen's and Pan's exponents. Then we showed that from an approximate algorithm it is possible to recover an exact algorithm with almost the same asymptotic complexity. I still remember our excitement when we discovered these properties and obtained these achievements. My gratitude to Milvio is great.

R.V. - I believe that this is the research result of which you are the most proud.

Indeed, matrix multiplication complexity is my first love and is the most important to me. But there are several other results which I like much. They belong to different parts of my scientific life and involve other collaborators. I have to cite the results on the analysis of matrix structures and their interplay with polynomial computations which characterized my long collaboration with Victor Pan. Here it is not a single result but the whole set of results synthesized in a joint book with Victor. Concerning structured matrices and polynomials, the results obtained in collaboration with Luca Gemignani and with Beatrice Meini on matrix analysis and algorithms for structured Markov chains have left an important trace in me. The most exciting and unexpected achievement in this area was to discover that the cyclic reduction algorithm, introduced by Gene Golub at the end of the 1960s for solving certain block tridiagonal systems, could be used for solving polynomial matrix equations and is strictly related to the Graeffe iteration applied to matrix polynomials. This iteration was introduced independently by Lobachevsky, Dandelin and Graeffe many years before, between 1826 and 1837.

Another result that I like much concerns the concrete and practical achievements in polynomial root-finding, namely the software package MPSolve for computing zeros of polynomials to any guaranteed precision, designed and implemented with Giuseppe Fiorentino and recently improved with Leonardo Robol.

R.V. - Which collaborations do you cherish the most?

All the collaborations that I had have been relevant and left an important scientific and a human trace in my life, even those which lasted only one paper. Having a collaborator and writing a joint paper is like having a companion to play with and enjoying together the friendship and the camaraderie of the research game. I believe I have been very lucky in this regard since all my collaborators have been fantastic game companions and at the same time brilliant and creative scholars, endowed of a rich humanity. In the collaboration relationships I have had so far, I could share ideas in a continuous and completely free way with no competition at all. I would like to tell something about all my collaborators, but they are so many.

As a sample, I just wish to recall the nice time spent with Victor Pan. When we were young, each year we used to spend a couple of weeks together in "full immersion" with math. Our conversations concerned mathematics, algorithms, structured matrices, polynomials and more. When tired with math, we switched to football, to the Italian/international political situation, to movies, poetry, literature, like old friends usually do. I remember our pleasant conversations on Toeplitz matrices after swimming in the Tyrrhenian Sea or during hiking in the Apuan Alps.

R.V. - Why have you written so many books?

Probably it is the need that I feel to put order in some area where the results are scattered or not well organized. It is also a way to have a wide and general view on some topic.

R.V. - Which places did you visit for research stays?

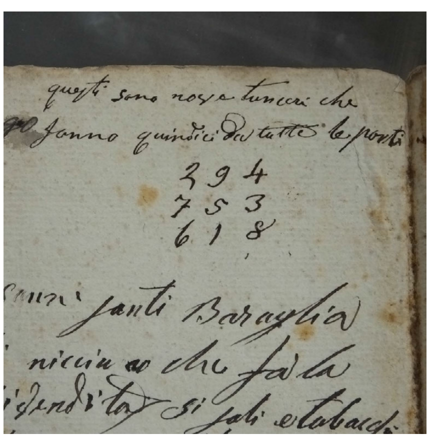
I visited several places but I think the most enriching experience was the time I spent in Stanford from September to December 1979 at the Department of Math and Computer Science. I met Gene Golub at a conference in Plzen in 1978 (my first conference). He invited me to visit Stanford University. There were many young and promising students in the department, among whom were Marsha Berger, Dan Boley, Randy LeVeque, Stephen Nash, Nick Trefethen, and Paul Van Dooren. I remember that I shared the office with Paul for a couple of weeks. In Stanford I also met Donald Knuth, another exciting experience.

R.V. - What was the most surprising thing that happened to you, research related?

Well, I can tell you the funny story of border rank, that is, the way we arrived at discovering this property. As I already told you, Milvio Capovani proposed we investigate the properties of the tensor associated with matrix multiplication. Francesco and Grazia designed and implemented a sophisticated algorithm for testing if a given integer value could be an upper bound to the rank of a given tensor. On my side I gave some theoretical results concerning lower bounds to the tensor rank. The program implemented by Francesco and Grazia was a sort of black box which took as input the tensor, a bound k to the tensor rank, and as output it gave a decreasing sequence. If the limit of this sequence was 0, then the tensor had rank less than or equal to k. It was based on the minimization of a non-negative functional. We tested the program with a simple tensor of known rank and it seemed ok. I asked Francesco to try a small tensor which, according to my lower bounds, could not have rank 2. So, we tried this tensor with $k = 2$. We expected that, if everything was consistent and correct, the output sequence provided by the program should have been bounded from below by a positive constant. Unexpectedly, the sequence was clearly converging, even though slowly, to 0.

This apparent contradiction puzzled us for a couple days, where we were looking either for some flaw in the proof of the lower bound, or for some bug in the implementation of the program. Then we discovered that everything was right – in fact the functional did not have minimum zero but its infimum was zero, perfectly in accordance with the theory. We understood that this tensor of rank 3 was the limit of a sequence of tensors of rank 2. That is the concept of border rank.

Notebook dated 1875.



Magic square on first page of book.

R.V. - Is linear algebra in the tradition of your family?

In my family there are no mathematicians, even though I remember that my grandmother was able to make complicated computations with numbers in a very natural way. Also, both my parents, as I remember, were quite good in arithmetic and in solving puzzles. My father was particularly creative. However, it was a surprise to me to discover that my grand-grandfather was fascinated by magic squares. In fact, some time ago, when the last uncle in the family of my mother passed away, I came into possession of the book where the oldest member of the family used to take note of important events. In the first page of this book, dated 1875, there is a 3×3 matrix, a magic square with it written: "These are nine numbers which make 15 in all directions."

R.V. - Do you still enjoy research and is there a particular kind of research you prefer?

I guess that research is one of the most exciting things in the life of a mathematician. It is a way where your mind moves with total freedom and the curiosity and the fantasy play fundamental roles. The production of your research is just yours, possibly shared with collaborators. Differently from the work in a factory or in an office, you are the owner of your product and you receive the benefits of what you produce. I am firmly convinced that research must be completely free. I am more excited when a theoretical result finds some important application, or when it leads to the design of an algorithm which turns out to be very effective. It is the moment where you can touch with your hands the concrete outcome of your research.

R.V. - What suggestions do you have for young people?

Be intellectually independent, enjoy research and avoid boring subjects, but at the same time listen to the suggestions and advice of more experienced and older people. If something does not work properly, say a flaw in the proof of a theorem or some bug in an algorithm, don't give up! It means that the problem is more challenging, interesting and worth being investigated!

“The field is continuously evolving in part due to emerging applications”

Daniel Szyld Interviewed by Froilán Dopico[1]

In the exciting and creative atmosphere of the 2017 Householder Symposium on Numerical Linear Algebra, Daniel Szyld was interviewed on different aspects of his career, his current work, and his relationship with ILAS.

F.D. - Since *IMAGE* is “The Bulletin of the International Linear Algebra Society,” my first question is about your relationship with ILAS. How has ILAS influenced you and how have you influenced ILAS?

D.S. - First of all, ILAS organizes its conferences, which are important venues to learn about the field, to learn from and interact with colleagues, and to know what people are working on. I have participated in many ILAS conferences, I have enjoyed them, and they have been very rewarding to me.

Second, ILAS has provided a place where many of us feel part of a large, active, and vibrant community. This has been a very important factor in shaping the linear algebra community.

Daniel Szyld

Lastly, the *Electronic Journal of Linear Algebra*, which is the research journal of ILAS, has become a premiere journal of the field. I played a small part in launching this journal, since Hans Schneider asked me to attend a workshop at MSRI on electronic journals, and with that information we started *ELA*. Moreover, I was the first managing editor of *ELA*, when Daniel Hershkowitz and Volker Mehrmann were co-editors-in-chief. I feel particularly proud of this contribution to ILAS.

F.D. - You mentioned *ELA* in your previous answer. One of the distinctive features of your career is the intense editorial work you have done and are still doing, serving, for instance, as Editor-in-Chief of *SIMAX* (the *SIAM Journal on Matrix Analysis and Applications*), Senior Editor of *LAA* (*Linear Algebra and its Applications*), Advisory Editor of *ELA*, as well as serving on the editorial boards of several other top journals. Therefore, you are in a privileged position to provide an informed opinion to *IMAGE* readers on the interplay and past and future evolution of journals specializing in linear algebra.

D.S. - As everyone knows, *LAA* has been associated with ILAS from the beginning of the society, publishing special issues devoted to ILAS conferences and sponsoring speakers. This has had a positive influence on the society. I am an editor of both *LAA* and *ELA*, which have very similar scopes, but *ELA* is the journal of our society, which is nonprofit, published by and for the members of the society, and I encourage everyone to support it with very strong research submissions. *SIMAX* has a different scope, more geared towards applications and numerical methods. Thus, in my view, *ELA* and *SIMAX* complement each other well, two strong journals marching towards the future together.

In the last few years, there has been a mushrooming of for-profit journals charging authors publication fees. I consider them mostly to be fraudulent scams. We do not need such journals and I have consistently refused to serve on their editorial boards.

F.D. - In the first interview of this series, Olga Holtz asked Hans Schneider, “How has linear algebra changed in the 60 years that you have been involved?” Among several other things, Hans said, “I have a regret about one development. In the 1960s, numerical and theoretical linear algebra formed a continuum and one group; now there is a split, at least in the U.S. Perhaps this change was inevitable.” Do you share Hans’ point of view?

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D.S. - I was present when Hans Schneider gave an after-dinner lecture titled something like "When did linear algebra become applied?". The gist of his argument was that if someone uses your linear algebra result, then it is applied. As a very young scientist at that time, I thought it was a great argument. There are many mathematicians who do not see the difference between numerical, applied, and theoretical linear algebra, and I include myself in that group. There are a lot of theoretical problems inspired by real applications. Numerical experiments are often essential to inspire and/or illustrate theoretical results. I have observed repeatedly how numerical examples lead to ideas for theoretical results and, of course, vice versa, which is more common but equally important.

F.D. - In the next questions, I would like to focus on your research career. When and how did you get interested in mathematics in general and in linear algebra in particular?

D.S. - I attended a technical high school in my native Buenos Aires (Argentina), where I studied electronics and telecommunications, and, of course, there was a lot of mathematics in the program. For example, we studied complex analysis for the course called "Wave Propagation and Antennas". About 80% of my classmates went to industry and 20% went to universities to study engineering. I liked mathematics very much and, as you see, I was destined to go to an engineering school. However, hiking in the mountains I met a biologist who explained to me that I could be equally or more useful to society being a mathematician. This gave me permission to follow my desires and go to the School of Exact Sciences (which hosts the program in mathematics). A small detail is that the technical school was six years compared to the five years of regular baccalaureate, and that after the fourth year I was itching to go to the University. So, over the course of a year I prepared all the humanities subjects that I was lacking, took the exams, and received my baccalaureate. In this way, I started the university while at the same time finishing my six years of the technical high school degree.

Thus, as it happened, I have two high school degrees, and no undergraduate degree. The Graduate School at New York University (NYU) admitted me to the Ph.D. program before I completed my undergraduate studies, and they forgot to tell me that they expected me to graduate before I started. I showed up in New York, and they asked me for my diploma. To make a long story short, eventually they accepted their mistake, and I was able to continue my graduate studies.

My interest in linear algebra arose later. In my first semester as a graduate student at the Courant Institute (at NYU), I took a numerical linear algebra course with Olof Widlund. I did not have funding. Then, Wassily Leontief, a Nobel Prize winner in economics, needed a research assistant to help with what turned out to be a large sparse matrix problem, an input-output model of the world economy, and Olof recommended me for that position. In addition to the numerical methods and the FORTRAN code – 16,000 lines of code – I had to deal with questions about the mathematical aspects of dynamic input-output models. I taught myself nonnegative matrices, M-matrices, Perron-Frobenius Theory, splittings, and fell in love with linear algebra.

F.D. - From your previous answer, it is clear that Olof Widlund and Wassily Leontief had an important impact on your career. Which other colleagues have influenced you most as a researcher?

D.S. - At the senior level, I learned a lot from Ivo Marek, with whom I have coauthored many papers. It was great to visit him in Prague about once a year in the late eighties. Richard Varga was a wonderful mentor. In 1991, I started to work with Andreas Frommer, who became my most frequent collaborator (I have more joint papers with him than with anyone else). Two other researchers from whom I learned a lot are Michele Benzi and Valeria Simoncini.

Perhaps this is a good moment to mention colleagues who supported me throughout my active career, such as Dianne O'Leary, Hans Schneider, Richard Brualdi, Susanne Brenner, and Howard Elman. A short anecdote in this context is the following. When I was fired from Duke University (my three-year contract was not renewed), Marvin Knopp from Temple University phoned his friend Gene Golub to see if I was as good as the letters were saying, and Gene told Marvin, "Yes, hire him!" I have been at Temple for 27 years. Temple has been a great place in many respects, and has allowed me to grow professionally over the years.

F.D. - Do you prefer to do research on classical problems or on the latest fashionable topics?

D.S. - I like to prove theorems and they are not always fashionable. Sometimes, if I am lucky, my work has some influence.

F.D. - Among the many papers you have published in your career, which do you like most and why?

D.S. - This is very hard to answer. One theorem I like very much in part because it is simple and very useful is: "Given a nonsingular matrix A and a matrix T with spectral radius less than one, there exists a unique splitting $A = M - N$ with $T = M^{-1}N$." With Michele Benzi, we tackled the case of A singular and proved that the existence still holds, but not the uniqueness. We completely characterized all possible such splittings. I also very much like the theory of inexact Krylov methods that I developed with Valeria Simoncini in the early 2000s. I am also proud of my work on asynchronous methods. I was introduced to this topic by Andreas Frommer, and together we published several papers in this area, including a much-cited survey in 2000.

F.D. - How has research in linear algebra changed during your career?

D.S. - The field is continuously evolving in part due to emerging applications. There are many examples. I will name just a few: tensor analysis, network science, random matrix theory, nonlinear eigenvalue problems, nonnegative matrix factorizations. None of these were in vogue say 30 years ago.

F.D. - What advice would you like to give to young researchers in linear algebra?

D.S. - First and foremost, work on what excites you the most. It is wonderful when you have the enthusiasm for a topic, and then you can be most productive and creative. At the same time, you want to choose topics with potential for impact and/or for growth. You should resist the temptation of writing papers which are small extensions of previous ones. Always try to make the work complete before you submit your paper. I will repeat one piece of advice I was given: "When you finish a paper, put it aside for a week, then read it again, and when you are sure that you do not want to change anything, submit the paper."

F.D. - Any additional thoughts you want to share with *IMAGE* readers?

D.S. - One of the things I appreciate very much about the linear algebra community is the wonderful camaraderie which exists among all of us. In fact, we are conducting this interview at the Householder Symposium XX on Numerical Linear Algebra at Virginia Tech, where this fellowship is very palpable. In other fields, there is fierce cut-throat competition. In our field there is more collaboration, and these collaborations can sometimes turn into lifelong friendships.

F.D. - Thank you very much for this interview.

Send News for *IMAGE* Issue 62

Issue 62 of *IMAGE* is due to appear online on June 1, 2019. Send your news for this issue to the appropriate editor by April 15, 2019. *IMAGE* seeks to publish all news of interest to the linear algebra community. Photos are always welcome, as well as suggestions for improving the newsletter. Send your items directly to the appropriate person:

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FEATURE ARTICLE

Spectra of Simplicial Complexes

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1. Spectra of graphs and simplicial complexes. The study of spectra of graphs is a classical and well-developed part of graph theory. One of the starting points is a fundamental paper by Kirchhoff [38] studying graph Laplacians that contains the celebrated *Matrix-Tree Theorem* for the number of spanning trees of a graph, which includes as a special case Cayley's [12] famous formula n^{n-2} for the number of labeled trees on n vertices.

The Laplacian eigenvalues of a graph G are related to many important properties of G, in particular regarding its connectivity and expansion properties. They can also be used to bound the independence number and the chromatic number. See, e.g., [11, 13, 35, 41] for background and further references.

In the present article, we consider eigenvalues of higher-dimensional simplicial complexes, a special class of hypergraphs originally stemming from the world of algebraic topology. In recent years, considerable work has been devoted to generalizing the classical theory of graphs to the setting of higher-dimensional simplicial complexes. Many results have focused on the notion of expansion; see, e.g., [19, 26, 32, 49], but also other natural concepts such as random walks [48, 45], trees [22, 37], planarity [43], independence and chromatic numbers [25, 31] have been extended to higher dimensions. Some of these notions were introduced and studied previously in the context of hypergraphs. Pure k-dimensional simplicial complexes are essentially $(k+1)$-uniform hypergraphs, but the topological point of view brings the machinery of algebraic topology such as homology theory to the subject.

Eckmann [24] introduced higher-dimensional Laplacians to study discrete boundary value problems on simplicial complexes. Subsequently, combinatorial Laplacians were applied in a variety of contexts. Dodziuk [15] and Dodziuk and Patodi [17] showed how the continuous Laplacian of a Riemannian manifold can be approximated by the combinatorial Laplacians of a suitable sequence of successively finer triangulations of the manifold.

Kalai [37] used combinatorial Laplacians to prove a higher-dimensional generalization of Cayley's formula for the number of labeled trees, and further results in this direction, including a generalization of the Matrix-Tree Theorem, were obtained in [1, 20, 22]. Spectra of Laplacians can be used to compute Betti numbers of simplicial complexes [28, 29] and are associated to random walks on simplicial complexes [48, 45]. Classes of complexes with integral spectra are, e.g., studied in [23, 18, 40]. The eigenvalues of random simplicial complexes have been studied in [34, 39]. For further background and references regarding combinatorial Laplacians, see also [36].

In this article we will focus on two further combinatorial properties of graphs and simplicial complexes that are connected to spectral information: expansion and discrepancy in Sections 4 and 5, and independence numbers and chromatic numbers in Sections 6 and 7. We first begin with a very quick introduction to simplicial complexes and higher-dimensional Laplacians. As this is a bit technical, the reader should feel free to skip the next two sections and return to them when needed.

2. Simplicial complexes. A (finite abstract) simplicial complex X on a finite set V is a family of subsets of V, which are called *faces*, that is closed under taking subsets. The *dimension* of a face F is $\dim(F) = |F| - 1$. The dimension of X is the maximal dimension of any face. For short, we say *k-complex* for a k-dimensional simplicial complex. A very basic example of a k-dimensional simplicial complex is the *complete k-complex* K_n^k that has vertex set $V = [n]$ and contains all possible faces of dimension i for all $i \leq k$.

In this article, we focus on 2-dimensional simplicial complexes. They consist of a vertex set X_0, a set X_1 of 1-dimensional faces, *edges*, and a set X_2 of 2-dimensional faces. By convention $X_{-1} = \{\emptyset\}$. A 2-complex is *pure* if all maximal simplices in X have dimension 2. Pure 2-dimensional simplicial complexes are thus essentially the same as 3-uniform hypergraphs. The 1-*skeleton* of X is the simplicial complex $X_{-1} \cup X_0 \cup X_1$, which, like any 1-dimensional simplicial complex, can be considered as the graph $G = (X_0, X_1)$. The *degree* of an edge e is the number of 2-faces that contain e.

3. Laplacians of graphs and simplicial complexes. For a graph $G = (V, E)$ with $|V| = n$ vertices, the *combinatorial Laplacian* is the matrix $L = L(G) := D - A \in \mathbb{R}^{n \times n}$, where $D = D(G) \in \mathbb{R}^{n \times n}$ is the diagonal matrix with entries $D_{v,v} = \deg_G(v)$, the *degrees* of the vertices, and A is the adjacency matrix of G. If we fix an orientation for each edge, we can also write $L = B^\top B$, where $B = B(G) \in \mathbb{R}^{|E| \times n}$ is the signed incidence matrix of G defined by

$$(B)_{e,v} = \begin{cases} 1 & \text{if } e = (u,v) \text{ for some } u \in V, \\ -1 & \text{if } e = (v,u) \text{ for some } u \in V, \\ 0 & \text{if } v \notin e. \end{cases}$$

It is an easy observation that the all-ones vector $\mathbb{1} = (1,\dots,1)^T$ satisfies $L\mathbb{1} = 0$. As a positive semidefinite matrix, L has only non-negative eigenvalues. These are typically indexed in increasing order, i.e.,

$$0 = \lambda_1(L) \leq \cdots \leq \lambda_n(L).$$

The following generalization of the graph Laplacian was introduced by Eckmann [24]. The definition of these higher-dimensional Laplacians is based on the notion of real (co-)homology.

Let X be a simplicial complex. We again assume that every face of X is endowed with an *orientation*, i.e., a local ordering of its vertices. Then we can define *oriented incidence numbers*. If X is 2-dimensional, we consider the following cases: For $v \in V$ and $e \in X_1$ we define $[e:v] = B_{e,v}$ as above. Analogously, for $t \in X_2$ and $e \in X_1$ we have $[t:e] \in \{0, \pm 1\}$, where the sign depends on whether the local ordering of e agrees with the one induced by the orientation of t, e.g., $[(u,v,w):(u,v)] = 1$ and $[(u,v,w):(v,u)] = -1$. For $v \in V$ and the unique empty face $\emptyset \in X_{-1}$, we set $[v:\emptyset] = 1$.

Just as above in the case of graphs, these incidence numbers define signed incidence matrices $\delta_i \in \mathbb{R}^{|X_{i+1}|\times|X_i|}$. The i-th *up-Laplacian* is the matrix $L_i^{\uparrow}(X) := \delta_i^{\top}\delta_i$. We also define the i-th *down-Laplacian* $L_i^{\downarrow}(X) := \delta_{i-1}\delta_{i-1}^{\top}$. For a 2-complex X, the incidence matrix δ_1 in dimension 1 is defined by $(\delta_1)_{t,e} = [t:e]$. We have $\delta_0 = B = B(G)$ for the graph $G = (V,E)$ with $E = X_1$, and $\delta_{-1} = \mathbb{1}$. For $i = 0$, we have $L_0^{\uparrow}(X) = L(G)$, while $L_0^{\downarrow}(X) = J$ is the all-ones matrix.

In the context of real (co-)homology, we consider the corresponding maps on the vector space $\mathbb{C}^i(X;\mathbb{R})$ of functions from X_i to $\mathbb{R}$. The characteristic functions e_F of faces $F \in X_i$ form an orthonormal basis of $\mathbb{C}^i(X;\mathbb{R})$. The matrix δ_i is the representation of the *coboundary map* $\delta_i : \mathbb{C}^i(X;\mathbb{R}) \to \mathbb{C}^{i+1}(X;\mathbb{R})$ with respect to this basis.

A simple but essential property of the coboundary maps is that $\delta_i \circ \delta_{i-1} = 0$. Thus, we have $B^i(X;\mathbb{R}) \subseteq Z^i(X;\mathbb{R})$, where $B^i(X;\mathbb{R}) = \operatorname{im}\delta_{i-1}$ is the space of *i-dimensional coboundaries* and $Z^i(X;\mathbb{R}) = \ker\delta_i$ is the space of *i-dimensional cocycles*. The quotient group $H^i(X;\mathbb{R}) := Z^i(X;\mathbb{R})/B^i(X;\mathbb{R})$ is then called the *i-th cohomology group of X with coefficients in* $\mathbb{R}$.

Analogously, we can define the homology groups of a simplicial complex. For this, the spaces $\mathbb{C}^i(X;\mathbb{R})$ are endowed with the standard inner product $\langle f,g\rangle = \sum_{F\in X_i} f(F)g(F)$ and the *boundary map* $\partial_{i+1} = \delta_i^* : \mathbb{C}^{i+1}(X;\mathbb{R}) \to \mathbb{C}^i(X;\mathbb{R})$ is defined as the adjoint of the coboundary map δ_i. Thus, as a matrix in the basis $\{e_F : F \in X_i\}$, we have $\partial_{i+1} = \delta_i^{\top}$.

The spaces of *boundaries* $B_i(X;\mathbb{R}) := \operatorname{im}\partial_{i+1}$ and of *cycles* $Z_i(X;\mathbb{R}) := \ker\partial_i$ are subspaces of $\mathbb{C}^i(X;\mathbb{R})$ satisfying $B_i(X;\mathbb{R}) \subseteq Z_i(X;\mathbb{R})$ and thus define $H_i(X;\mathbb{R}) := Z_i(X;\mathbb{R})/B_i(X;\mathbb{R})$, the *$i$-th reduced homology group* of X.

The following diagram summarizes these linear maps:

$$\mathbb{C}^2(X;\mathbb{R}) \underset{\partial_2}{\overset{\delta_1}{\rightleftarrows}} \mathbb{C}^1(X;\mathbb{R}) \underset{\partial_1}{\overset{\delta_0}{\rightleftarrows}} \mathbb{C}^0(X;\mathbb{R}) \underset{\partial_0}{\overset{\delta_{-1}}{\rightleftarrows}} \mathbb{C}^{-1}(X;\mathbb{R})$$

The Laplacians of X can also be considered as operators on $\mathbb{C}^i(X;\mathbb{R})$: $L_i^{\downarrow}(X) := \delta_{i-1}\partial_i$ and $L_i^{\uparrow}(X) := \partial_{i+1}\delta_i$. Both Laplacians are positive semidefinite. For the up-Laplacian we have $\ker L_i^{\uparrow}(X) = Z^i(X;\mathbb{R})$ and $B^i(X;\mathbb{R})$ corresponds to the trivial kernel $\langle\mathbb{1}\rangle$ in the graph case. For the down-Laplacian we have $\ker L_i^{\downarrow}(X) = Z_i(X;\mathbb{R})$. It is not hard to see that $Z_i(X;\mathbb{R}) = B^i(X;\mathbb{R})^{\perp}$ and $Z^i(X;\mathbb{R}) = B_i(X;\mathbb{R})^{\perp}$.

For $\mathcal{H}_i(X;\mathbb{R}) := Z_i(X;\mathbb{R}) \cap Z^i(X;\mathbb{R})$, we have the *Hodge decomposition* of $\mathbb{C}^i(X;\mathbb{R})$ into pairwise orthogonal subspaces

$$\mathbb{C}^i(X;\mathbb{R}) = \mathcal{H}_i(X;\mathbb{R}) \oplus B^i(X;\mathbb{R}) \oplus B_i(X;\mathbb{R}).$$

In particular, $\mathcal{H}_i(X;\mathbb{R}) \simeq H^i(X;\mathbb{R}) \simeq H_i(X;\mathbb{R})$.

As an example, we consider the complete k-complex K_n^k and $0 \leq i \leq k-1$. We denote the i-th *up-Laplacian* $L_i^{\uparrow}(K_n^k)$ by $L_i^{\uparrow}$ and the i-th *down-Laplacian* $L_i^{\downarrow}(K_n^k)$ by $L_i^{\downarrow}$. It is easy to verify that $L_i^{\uparrow} + L_i^{\downarrow} = nI$. Thus, $\mathcal{H}_i(K_n^k;\mathbb{R}) = \ker L_i^{\uparrow} \cap \ker L_i^{\downarrow} = \{0\}$ and $C^i(K_n^k;\mathbb{R}) = B^i(K_n^k;\mathbb{R}) \oplus B_i(K_n^k;\mathbb{R})$. The eigenvalues of $L_i^{\uparrow}$ are hence n on $B_i(K_n^k;\mathbb{R})$ and 0 on $B^i(K_n^k;\mathbb{R})$, the trivial kernel. For $L_i^{\downarrow}$ we get n on $B^i(K_n^k;\mathbb{R})$ and 0 on $B_i(K_n^k;\mathbb{R})$.

4. Expansion and discrepancy in graphs. It is a simple observation that a graph is connected if and only if the second smallest eigenvalue of its Laplacian is non-zero. There are several results that show that this connection between spectral and expansion properties in graphs is in fact deeper. The eigenvalues of a graph can help to measure how well-connected the graph is.

One way to measure the (edge) expansion of a graph $G = (V, E)$ is the *Cheeger constant* of G defined by

$$h(G) := \min_{\substack{A\subseteq V\\ 0<|A|<|V|}} \frac{|V||E(A, V\setminus A)|}{|A||V\setminus A|}.$$

Here, $E(A, V \setminus A)$ is the set of edges with one endpoint in A and the other in $V \setminus A$. A famous result connecting the smallest non-trivial Laplacian eigenvalue and the Cheeger constant is the *discrete Cheeger inequality*:

Theorem 1 (Discrete Cheeger inequality [2, 4, 16, 51])**.** *Let G be a graph with second smallest eigenvalue $\lambda(G)$ of the Laplacian $L(G)$ and maximum degree d_{max}. Then: $\lambda(G) \le h(G) \le \sqrt{8d_{max}\lambda(G)}$.*

Two other results relating the eigenvalues of a graph to its expansion properties are the *Expander Mixing Lemma* and the *Inverse Mixing Lemma*. They state a connection between the eigenvalues and the *discrepancy* of G, expressing for each pair of vertex subsets $S, T \subset V$ how far the number of edges $E(S, T)$ between them diverges from the expected number in a random graph of corresponding density.

Theorem 2 (Expander Mixing Lemma [3])**.** *Let G be a graph with n vertices and average degree d. Let $0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ be the eigenvalues of $L(G)$ and set $\mu(G) := \max_{i\ge 2}\{|\lambda_i - d|\}$. Then, for all $S, T \subset V$,*

$$\left||E(S,T)| - \frac{d}{n}|S||T|\right| \le \mu(G)\cdot\sqrt{\frac{|S||T||V\setminus S||V\setminus T|}{n^2}}.$$

Note that the usual formulation of the Expander Mixing Lemma is for d-regular graphs, where $\mu(G)$ is the largest (in absolute value) non-trivial eigenvalue of the adjacency matrix $A(G)$.

Theorem 3 (Inverse Mixing Lemma [8])**.** *Let G be a d-regular graph with n vertices. Suppose that for all $S, T \subset V$ with $S \cap T = \emptyset$,*

$$\left||E(S,T)| - \frac{d|S||T|}{n}\right| \le \mu\cdot\sqrt{|S||T|}.$$

Then $\mu(G) = O\left(\mu\left(\log(\frac{d}{\mu}) + 1\right)\right)$.

We will focus on the Expander Mixing Lemma and the lower bound of the Cheeger inequality, which have very similar simple proofs that can be adapted for simplicial complexes. The proofs of the upper bound of the Cheeger inequality (see, e.g., [5] for a short proof) and the Inverse Mixing Lemma [8] are more involved.

A crucial observation is that the number of edges connecting two vertex sets can be expressed in terms of the Laplacian and the characteristic vectors of these vertex sets: for $S, T \subset V$ with $S \cap T = \emptyset$ we have $\mathbb{1}_S^\top L\mathbb{1}_T = (\delta\mathbb{1}_S)^\top\delta\mathbb{1}_T = -|E(S,T)|$ and, similarly, $\mathbb{1}_S^\top L\mathbb{1}_S = |E(S, V\setminus S)|$. The second ingredient consists of tools from linear algebra connecting values of the bilinear form defined by L with its spectrum.

To prove the Expander Mixing Lemma for sets S, T with $S \cap T = \emptyset$, we consider the matrix $M = L(G) - \frac{d}{n}L(K_n)$. The Laplacian $L(K_n)$ of the complete graph has eigenvalues 0 with eigenvector $\mathbb{1}$ and n with eigenspace $\mathbb{1}^\perp$. Hence, the non-trivial eigenvalues of M are $\lambda_i(L) - d$ for $i \ge 2$, and M has spectral radius $\rho(M) = \mu(G)$. As M is symmetric with $M\mathbb{1} = 0$, we have

$$|x^\top My| \le \rho(M)\cdot\|\mathrm{pr}_{\mathbb{1}^\perp}(x)\|\,\|\mathrm{pr}_{\mathbb{1}^\perp}(y)\|, \tag{1}$$

where $\mathrm{pr}_{\mathbb{1}^\perp}(x)$ is the projection of x onto $\mathbb{1}^\perp$. We have $\mathrm{pr}_{\mathbb{1}^\perp}(\mathbb{1}_S) = \mathbb{1}_S - \frac{|S|}{|V|}\mathbb{1}$ and $\|\mathrm{pr}_{\mathbb{1}^\perp}(x)\| = \sqrt{\frac{|S||V\setminus S|}{|V|}}$. Hence by (1),

$$\left||E(S,T)| - \frac{d}{n}|S||T|\right| = |\mathbb{1}_S^\top M\mathbb{1}_T| \le \mu(G)\cdot\sqrt{\frac{|S||T||V\setminus S||V\setminus T|}{|V|^2}}.$$

The lower bound of the Cheeger inequality can be shown similarly using the variational characterization of eigenvalues.

5. Expansion and discrepancy in complexes. We now want to see how much of this connection carries over to higher dimensions. For simplicity, we will focus on simplicial complexes of dimension 2, but the presented results and their proofs also work for higher dimensions. A further assumption we make is that the complex has all possible faces of non-maximal dimension, i.e., that the complex has a complete 1-skeleton. This is a more serious restriction but is needed for technical reasons.

Now, let X be a 2-dimensional simplicial complex with n vertices and a complete 1-skeleton. In this section, we will write $L(X)$ for the up-Laplacian $L_1^{\uparrow}(X)$ and δ for δ_1. Because X has a complete 1-skeleton, we know that $B := B^1(X;\mathbb{R}) = B^1(K_n^2;\mathbb{R})$ and $B^\perp = Z_1(K_n^2;\mathbb{R})$. The analogue of the second-smallest eigenvalue of the graph Laplacian is the smallest eigenvalue of $L(X)$ on $B^\perp$. If this is non-zero, we know that $H^1(X;\mathbb{R}) = 0$. Let $d = 3|X_2|/\binom{|V|}{2}$ be the average degree of an edge.

In order to prove a generalization of the Expander Mixing Lemma using higher-dimensional Laplacians, we can try to follow the strategy presented above. The matrix $M = L(X) - \frac{d}{n}L(K_n^2)$ maps B to 0. Its non-trivial eigenvalues are of the form $\lambda - d$, where λ is an eigenvalue of $L(X)$ on $B^\perp$. If we set $\mu(X) = \max\{|\lambda - d| : \lambda \text{ eigenvalue of } L(X) \text{ on } B^\perp\}$, then we have, just as in (1):

$$\left|x^\top L(X)y - \frac{d}{n}x^\top L(K_n^2)y\right| \leq \mu(X)\cdot \|\mathrm{pr}_{B^\perp}(x)\|\,\|\mathrm{pr}_{B^\perp}(y)\| \tag{2}$$

for all $x, y \in \mathbb{R}^{|X_1|}$. All that is left to do is to find vectors x and y for which $x^\top L(X)y$ expresses a combinatorial quantity of interest. What this quantity is depends on our notion of a "connected" complex and how we want to measure "well-connectedness."

A straightforward approach is to consider subsets $A_0, A_1, A_2 \subset V$ of the vertex set and study the set

$$T(A_0, A_1, A_2) := \{t \in X_2 : t = \{v_1, v_2, v_3\}, v_i \in A_i\}.$$

For disjoint sets $A_0, A_1, A_2 \subset V$, choose the orientations of edges and triangles in circular order as depicted in Figure 1. Now we can consider characteristic vectors of the type $\mathbb{1}_{E(A_0,A_1)}$ and indeed get $(\mathbb{1}_{E(A_0,A_1)})^\top L\mathbb{1}_{E(A_1,A_2)} = (\delta\mathbb{1}_{E(A_0,A_1)})^\top\delta\mathbb{1}_{E(A_1,A_2)} = |T(A_0,A_1,A_2)|$, as $(\delta f)(t) = \sum_{e\in X_1}[t:e]f(e)$.

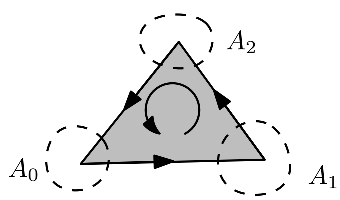

Figure 1: Orientations

What is $\mathrm{pr}_{B^\perp}(\mathbb{1}_{E(A_0,A_1)})$? We need to find the unique $x \in Z^1(K_n^2;\mathbb{R})$, i.e., x with $\partial x = 0$, such that $\delta x = \delta\mathbb{1}_{E(A_0,A_1)}$. It turns out that $x = \mathbb{1}_{E(A_0,A_1)} - \frac{|A_1|}{|V|}\delta\mathbb{1}_{A_0} + \frac{|A_0|}{|V|}\delta\mathbb{1}_{A_1}$ and we get:

Theorem 4 (Mixing Lemma for Complexes [49])**.** *Let X be a 2-dimensional simplicial complex with n vertices and complete 1-skeleton. Let $d = \frac{3|X_2|}{\binom{|V|}{2}}$ be the average degree of an edge. For disjoint subsets $A_0, A_1, A_2 \subset V$,*

$$\left||T(A_0,A_1,A_2)| - \frac{d}{n}|A_0||A_1||A_2|\right| \leq \mu(X)\cdot\sqrt{\frac{|A_0||A_1||V\setminus(A_0\cup A_1)||A_1||A_2||V\setminus(A_1\cup A_2)|}{|V|^2}}. \tag{3}$$

With a similar strategy, employing the variational characterization of eigenvalues, one can prove higher-dimensional versions of the lower bound of the Cheeger inequality; see [33, 49]. It is also possible to prove a mixing lemma for complexes with non-complete lower skeleton; see [47]. A higher-dimensional analogue of the Inverse Mixing Lemma is presented in [14]. Currently, no generalizations of the upper bound of the Cheeger inequality are known.

More generally, we can consider a map $\phi : V \to S^1$ and study the set

$$T(\phi) = \left\{t \in \binom{V}{3} : \mathbf{0} \in \phi(t)\right\},$$

where $\phi(t) = \mathrm{conv}\{\phi(v) : v \in t\}$. Given such a map ϕ, we can consider x_ϕ defined by $x_\phi(e) = \frac{1}{n}|\{v \in V : \mathbf{0} \in \phi(\{v\} \cup e)\}|$. It is not hard to see that $\delta x_\phi = \mathbb{1}_{T(\phi)}$ if we choose orientations such that each edge is oriented clockwise in the circular order given by ϕ. Furthermore, $\partial x_\phi = 0$ and $\|x_\phi\| = \sqrt{\frac{|T(\phi)|}{|V|}}$. Hence, for ϕ and ψ inducing the same circular order,

$$\left| |T(\phi) \cap T(\psi) \cap X_2| - \frac{d}{n}|T(\phi) \cap T(\psi)| \right| \leq \mu(X) \cdot \sqrt{\frac{|T(\phi)||T(\psi)|}{|V|^2}}. \tag{4}$$

This relates the Laplacian spectrum to the concept of *geometric overlap*, another possible measure of expansion. For a k-dimensional simplicial complex X, its geometric overlap $c(X)$ is the maximal c such that for every map $f : V \to \mathbb{R}^k$ there is a point $p \in \mathbb{R}^k$ that is hit by at least $c \cdot |X_k|$ many images of k-faces of X:

$$|\{F \in X_k : p \in \mathrm{conv}\{f(v) : v \in F\}\}| \geq c \cdot |X_k|.$$

For a graph G it is easy to see that large expansion implies large overlap: For any map $f : V(G) \to \mathbb{R}$, the median of the image points is contained in at least $\epsilon(G) \cdot |E|$ edges, where $\epsilon(G) = \frac{h(G)}{4}\frac{|V|}{|E|}$. The geometric overlap can hence be interpreted as a measure of higher-dimensional expansion; see, e.g., [27].

For complete complexes $X = K_n^k$, this question, phrased in terms of point sets in $\mathbb{R}^k$ and the simplices spanned by them, has received a lot of attention. It is known that $c(K_n^k)$ asymptotically does not depend on n: Boros and Füredi [10] showed that $c(K_n^2) = \frac{2}{9} - o(1)$. This was extended to complete complexes in arbitrary dimension by Bárány [7]. While for $k = 2$ the factor $\frac{2}{9}$ is known to be asymptotically tight, the determination of the best value for larger k is the subject of ongoing research; see, e.g., [32, 44].

Let us see how our result above yields a bound for geometric overlap: For a 2-complex X and $f : V \to \mathbb{R}^2$, we can argue as follows. Since $c(K_n^2) = \frac{2}{9} - o(1)$, we can find a point $p \in \mathbb{R}^2$ such that $|\{t \in \binom{V}{3} : p \in \mathrm{conv}\{f(v) : v \in t\}\}| \geq c_2\binom{n}{3}$ for a fixed $c_2 \in \mathbb{R}$, independent of f and n. Now consider the map $\phi : V \to S^1$ defined by $\phi(V) = \frac{f(v)-p}{\|f(v)-p\|}$. It is not hard to see that $|\{t \in X_2 : p \in \mathrm{conv}\{f(v) : v \in t\}\}| = |T(\phi) \cap X_2|$. Hence, (4) (with $\psi = \phi$) implies that

$$|\{t \in X_2 : p \in \mathrm{conv}\{f(v) : v \in t\}\}| \geq c_2 \cdot \frac{d - \mu(X)}{|V|}\binom{n}{3}.$$

We thus get a lower bound of $c(X) \geq c_2 \cdot \dfrac{d - \mu(X)}{|V|}\dfrac{\binom{|V|}{3}}{|X_2|}$.

A natural generalization of the notion of geometric overlap is the *topological overlap* of a simplicial complex, where instead of the simplexwise linear maps considered above we allow arbitrary continuous maps from $\|X\|$ to $\mathbb{R}^k$. The best general bounds for geometric overlap of complete complexes are by Gromov [32], who employed a topological proof method that also applies to continuous maps. This method actually relates the topological overlap of a complex to its *coboundary expansion*, a further generalization of edge expansion in graphs; see also [19]. We will not consider the definition of this notion here but only mention that the coboundary expansion of a simplicial complex measures how far it is from having non-trivial $\mathbb{F}_2$-cohomology. This implies that the underlying notion of "connectedness" is stricter than the ones considered above, as $H^1(X; \mathbb{F}_2) = 0$ implies $T(\phi) \cap X_2 \neq \emptyset$ for any $\phi : V \to S^1$.

One might wonder whether coboundary expansion can also be bounded in terms of the Laplacian spectrum. This, however, is not possible. One counterexample is presented in [34], a probabilistic construction of simplicial complexes with very good spectral behaviour that are not coboundary expanders. In this article we only want to consider the method used to bound the spectrum of this counterexample. The essential tool is a fundamental estimate due to Garland [30] that relates the eigenvalues of higher-dimensional Laplacians to those of the graphs that arise as *links* of lower-dimensional faces. Again, we consider only 2-dimensional complexes, but the result holds in higher dimensions as well. For a vertex $v \in V$ in a 2-dimensional simplicial complex X, the *link* of v is the graph $\mathrm{lk}(v) := (V \setminus \{v\}, \{\{u, w\} : \{v, u, w\} \in X_2\})$.

Theorem 5 ([30]; see also [9, Theorems 1.5, 1.6])**.** *Let X be a pure 2-dimensional complex with minimal edge degree $d_{\min}$ and maximal edge degree $d_{\max}$. Assume that $\lambda_{\min} \leq \lambda_2(L(\mathrm{lk}(v))) \leq \lambda_{n-1}(L(\mathrm{lk}(v))) \leq \lambda_{\max}$ for all $v \in V$. Then all eigenvalues of $L_1^\uparrow(X)$ on $(B^1X)^\perp$ lie in* $[2\lambda_{\min} - d_{\max}, 2\lambda_{\max} - d_{\min}]$.

A second class of counterexamples to a higher-dimensional Cheeger inequality involving coboundary expansion is presented in [50] – an explicit construction for an infinite family of simplicial k-balls whose spectral expansion is bounded away from zero, while the coboundary expansion tends to zero. In the same article, the authors also present a counterexample to simple higher-dimensional generalizations of the other part of the Cheeger inequality.

The search for higher-dimensional coboundary (or topological) expanders − families of simplicial complexes that have bounded face degrees and coboundary expansion (or topological overlap) bounded away from zero − has been a very active field of research; see, e.g., [19, 26].

6. Independent sets and colorings of graphs. The second topic we want to look at in more detail is the relation of spectral properties with the independence number and the chromatic number. Again, we first consider the situation for graphs and then adapt these ideas to higher-dimensional simplicial complexes.

For a graph $G = (V, E)$, a classical result relating the independence number $\alpha(G)$ to the spectrum of G is the so-called *ratio bound*[2] attributed to Hoffman (see, e.g., [11, Theorem 3.5.2]):

$$\alpha(G) \le |V| \cdot \left(1 - \frac{d_{\min}}{\lambda_{\max}(L)}\right), \tag{5}$$

where $d_{\min}$ is the minimum degree of G, and $\lambda_{\max}(L)$ is the largest eigenvalue of the Laplacian of G. As $|V| \le \chi(G)\alpha(G)$, any upper bound on α yields a lower bound on the chromatic number $\chi(G)$. Here, we get $\chi(G) \ge \frac{\lambda_{\max}(L)}{\lambda_{\max}(L) - d_{\min}}$.

The theta number $\vartheta(G)$, introduced by Lovász in his seminal paper [42], provides bounds for the independence number of G and the chromatic number of $\overline{G}$, the complement of G. It satisfies

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G}), \tag{6}$$

the so called "sandwich theorem." The inequality $\vartheta(\overline{G}) \le \chi(G)$ is always at least as strong as the inequality $n/\vartheta(G) \le \chi(G)$, as $n \le \vartheta(G)\vartheta(\overline{G})$ [42, Corollary 2].

The parameter $\vartheta(G)$ is the optimal value of a semidefinite program, and as such is computationally easy; in contrast, the independence number and the chromatic number are difficult to compute. As we will see, the ratio bound (5) can be proven using the theta number.

The theta number is defined as

$$\vartheta(G) = \sup\{\langle J, Y\rangle \ : \ Y \in \mathbb{R}^{V\times V},\ Y \succeq 0,\ \langle I, Y\rangle = 1,\ Y_{v,v'} = 0 \ \text{ if } \{v, v'\} \in E\}, \tag{7}$$

where I and J are the identity matrix and the all-ones matrix.

What is the motivation for this definition? For an independent set $S \subset V$ we can consider the positive semidefinite matrix $Y^S := \mathbb{1}_S\mathbb{1}_S^T$. Since S does not contain any edges, $Y^S_{v,v'} = 0$ if $\{v, v'\} \in E$. We have $\langle I, Y^S\rangle = |S|$, so $|S|^{-1}Y^S$ is feasible for (7). As $\langle J, Y^S\rangle = |S|^2$, we see that $|S| \le \vartheta(G)$.

To get upper bounds for $\vartheta(G)$, the *dual formulation* of (7) is often convenient:

$$\vartheta(G) = \inf\{\lambda_{\max}(Z) \ : \ Z \in \mathbb{R}^{V\times V},\ Z = J + T,\ T_{v,v'} = 0 \ \text{ if } \{v, v'\} \notin E\}. \tag{8}$$

Here, $\lambda_{\max}(Z)$ denotes the largest eigenvalue of Z.

We can prove the ratio bound (5) using this dual formulation: If we choose $T = tA = t(D - L)$, a multiple of the adjacency matrix of G, we get $\alpha(G) \le \lambda_{\max}(J + t(D - L)) \le \lambda_{\max}(J - tL) + \max_{v\in V} t \cdot \deg(v)$. Now we can minimize over t. The matrices J and L commute, so the eigenvalues of $J - tL$ are easy to analyze. The optimal choice for t turns out to be $t = -\frac{n}{\lambda_{\max}(L)}$, which leads to the ratio bound (5).

To see the connection between the theta number and chromatic numbers, we note that $\chi(G)$ is the smallest number ℓ such that there is a homomorphism from G to the complete graph K_ℓ. Since $\vartheta(\overline{K_\ell}) = \ell$, and one can prove that $\vartheta(\overline{G}) \le \vartheta(\overline{G'})$ if there is a homomorphism from G to G', the second inequality of (6) follows immediately.

7. Independent sets and colorings of complexes. Let us now see how these ideas transfer to the setting of higher-dimensional simplicial complexes. Again, we restrict ourselves to 2-dimensional complexes for simplicity, but the presented ideas can easily be adapted to higher dimensions. The notion of independence number extends in a natural way: For a 2-dimensional simplicial complex X, an independent set is a set of vertices that does not contain any 2-dimensional face of X, and the *independence number* $\alpha(X)$ is the maximum cardinality of an independent set.

In [31], Golubev proves a generalization of the ratio bound (5), an upper bound for $\alpha(X)$ in terms of the Laplacian spectrum of X. This upper bound involves the largest eigenvalues λ_i of the i-th up-Laplacians of X and the minimal

[2] For a regular graph, and in terms of the adjacency matrix instead of the Laplacian L of G, this formulation reduces to the usual one.

degrees d_i of the i-faces of X for $i \in \{0,1\}$:

$$\alpha(X) \leq n\left(1 - \frac{(d_0+1)d_1}{\lambda_0\lambda_1}\right). \tag{9}$$

For complexes X with complete 1-skeleton the inequality simplifies to

$$\alpha(X) \leq n\left(1 - \frac{d_1}{\lambda_1}\right). \tag{10}$$

A generalization of the theta number is presented in [6]. To motivate its definition we follow the strategy for graphs presented above. For an independent set $S \subset V$ of X, we consider, as a generalization of the characteristic vector of S, the matrix $\delta_{\binom{S}{2}}$ defined by

$$\left(\delta_{\binom{S}{2}}\right)_{e,v} = \begin{cases} 0 & \text{if } e \nsubseteq S \\ \delta_{e,v} & \text{otherwise,} \end{cases}$$

where $e \in \binom{V}{2}$, $v \in V$ and δ is the matrix of the coboundary operator δ_0 with respect to the basis of elementary cochains. We then consider the positive semidefinite matrix $Y^S = \delta_{\binom{S}{2}}\delta^T_{\binom{S}{2}}$, indexed by $\binom{V}{2}$. We have

$$(Y^S)_{e,e'} = \begin{cases} 0 & \text{if } e \cup e' \nsubseteq S \\ (L_1^{\downarrow})_{e,e'} & \text{otherwise,} \end{cases} \tag{11}$$

where $L_1^{\downarrow}$ is the down-Laplacian of the complete complex K_n^2. Hence,

$$\langle Y^S, I\rangle = 2\binom{|S|}{2}$$

and

$$\langle Y^S, L_1^{\downarrow}\rangle = 4\binom{|S|}{2} + \sum_{\substack{|e\cup e'|=3 \\ e\cup e' \subseteq S}} 1 = 4\binom{|S|}{2} + 6\binom{|S|}{3} = 2\binom{|S|}{2}|S|.$$

The properties of Y^S lead to the following definition of $\vartheta_2(X)$.

$$\begin{aligned} \vartheta_2(X) := \sup\big\{\langle L_1^{\downarrow}, Y\rangle \;:\; & Y \in \mathbb{R}^{\binom{V}{2}\times\binom{V}{2}},\ Y \succeq 0,\ \langle I, Y\rangle = 1, \\ & Y_{e,e'} = 0 \text{ if } e \cup e' \in X_2, \\ & Y_{e,e'} = 0 \text{ if } |e \cup e'| \geq 4, \\ & \epsilon_{e,e'}Y_{e,e'} = \epsilon_{e'',e^\dagger}Y_{e'',e^\dagger} \text{ if } e\cup e' = e'' \cup e^\dagger\big\}. \end{aligned} \tag{12}$$

Here, $\epsilon_{e,e'} := [e : e \cap e'][e' : e \cap e']$. We saw above that the matrix $\frac{1}{2\binom{|S|}{2}}Y^S$ is feasible for $\vartheta_2(X)$ with objective value $|S|$, which shows that $\alpha(X) \leq \vartheta_2(X)$.

Note that also for a complex X with non-complete 1-skeleton the program defining $\vartheta_2(X)$ ranges over matrices in $\mathbb{R}^{\binom{V}{2}\times\binom{V}{2}}$ and uses the down-Laplacian $L_1^{\downarrow}$ of the complete complex. At first glance, it might seem more natural to use matrices in $\mathbb{R}^{X_1\times X_1}$ and the down-Laplacian of the complex itself, $L_1^{\downarrow}(X)$, but this does not always lead to an upper bound for the independence number of the complex; see [6].

We can again consider the dual program of (12), in order to obtain another formulation of $\vartheta_2(X)$, similar to (8):

$$\begin{aligned} \vartheta_2(X) = \inf\big\{\ \lambda_{\max}(Z) \;:\; & Z = L_1^{\downarrow} + T,\ T \text{ symmetric}, \\ & T_{e,e} = 0 \text{ for all } e \in \tbinom{V}{2} \\ & \textstyle\sum_{e\cup e'=t} \epsilon_{e,e'}T_{e,e'} = 0 \ \text{ if } t \in \tbinom{V}{3} \setminus X_2\big\}. \end{aligned} \tag{13}$$

Just as for graphs, any matrix T that is feasible for (13) yields an upper bound on $\vartheta_2(X)$ and therefore an upper bound of the independence number of X. We can apply this principle to prove the higher-dimensional ratio bound proved by Golubev [31] in the case of a 2-dimensional simplicial complex X with complete 1-skeleton.

As in the graph case, we take $T = \gamma(L_1^{\uparrow}(X) - D_1(X))$ for some $\gamma \in \mathbb{R}$ that will be chosen later. Here, $D_1(X)$ is the diagonal matrix with entries $D_{e,e} = \deg(e)$. Then

$$\lambda_{\max}(L_1^{\downarrow} + T) \leq \lambda_{\max}(L_1^{\downarrow} + \gamma L_1^{\uparrow}(X)) + \max_{e \in X_1}(-\gamma \deg(e)).$$

We assume that X has a complete 1-skeleton, so we have $L_1^{\downarrow} = L_1^{\downarrow}(X)$ and $L_1^{\downarrow} L_1^{\uparrow}(X) = 0$. Let us denote by Λ the set of non-zero eigenvalues of $L_1^{\uparrow}(X)$. The non-zero eigenvalues of the matrix $L_1^{\downarrow} + \gamma L_1^{\uparrow}(X)$ are: n, associated to the eigenspace $B^1(X;\mathbb{R}) = B^1(K_n^2;\mathbb{R})$, and $\gamma\lambda$, for $\lambda \in \Lambda$, corresponding to eigenvectors in $B_1(X;\mathbb{R})$. For $\gamma = \frac{n}{\lambda_{\max}(L_1^{\uparrow}(X))}$, we have $\lambda_{\max}(L_1^{\downarrow} + \gamma L_1^{\uparrow}(X)) = n$ and we get:

$$\alpha(X) \leq \vartheta_2(X) \leq n\left(1 - \frac{d_{\min}(X)}{\lambda_{\max}(L_1^{\uparrow}(X))}\right).$$

Note that the assumption of having a complete 1-skeleton is essential here. For complexes with non-complete 1-skeleton the two bounds are incomparable; see [6].

The *chromatic number* $\chi(X)$ is the least number of colors needed to color the vertices so that no maximal face of X is monochromatic; in other words, it is the smallest number of parts of a partition of the vertices into independent sets. Note that in the study of hypergraphs, this is also known as the *weak chromatic number*, while $\chi(X_1)$, the chromatic number of the 1-skeleton, is known as the *strong chromatic number*.

Any upper bound on the independence number yields a lower bound on the chromatic number, as $n \leq \chi(X)\alpha(X)$. So we have $\chi(X) \geq n/\vartheta_2(X)$, and the higher-dimensional ratio bound gives

$$\chi(X) \geq \frac{\lambda_0\lambda_1}{\lambda_0\lambda_1 - (d_0+1)d_1}.$$

What would be the natural analog of the upper bound in the sandwich theorem (6) in the setting of 2-complexes? For a pure 2-dimensional complex X, let us define the *complementary complex* $\overline{X}$ as the pure simplicial complex of dimension 2 whose 2-dimensional faces are the 3-subsets of V that do not belong to X_2. For the complete 2-complex K_n^2, the color classes of an admissible coloring have at most 2 elements. Hence, $\chi(K_n^2) = \lceil n/2 \rceil$. So, for all 2-dimensional complexes X, we have $\alpha(\overline{X}) \leq 2\chi(X)$, and it is a natural question whether $\vartheta_2(\overline{X}) \leq 2\chi(X)$. This inequality, however, does not hold in general; see [6].

In [6] an ad hoc notion of chromatic number for a simplicial complex X, denoted $\chi_2(X)$, is introduced that does satisfy the inequality $\vartheta_2(\overline{X}) \leq \chi_2(X)$. While $\chi(X)$ is defined using vertex colorings, the definition of $\chi_2(X)$ is based on colorings of edges respecting orientations. For this, a notion of homomorphisms for simplicial complexes generalizing graph homomorphisms is defined. It is constructed such that $\vartheta_2(\overline{X}) \leq \vartheta_2(\overline{X'})$ for two 2-complexes X and X' whenever there is a homomorphism from X to X'. If we then denote by $\chi_2(X)$ the smallest number ℓ such that there exists a homomorphism from X to the complete 2-complex K_ℓ^2, we see that $\vartheta_2(\overline{X}) \leq \chi_2(X)$. Furthermore, $\chi_2(X) \leq \chi(X_1)$, as a vertex coloring with ℓ colors that is a proper graph coloring for X_1 gives rise to a homomorphism from X to K_ℓ^2.

8. Open questions. While there is an abundance of results from spectral graph theory that have yet to be generalized to higher dimensions, the following are some questions directly related to the topics discussed in this article.

- The sets $T(A_0, A_1, A_2)$ and $T(\phi)$ considered in the higher-dimensional Expander Mixing Lemmas (3) and (4) are special cases of hypercuts, a generalization of the notion of graph cuts that are studied in [21, 46]. It would be interesting to interpret (2) for other classes of hypercuts.
- While a higher-dimensional version of the Inverse Mixing Lemma is known [14], currently there are no known generalizations of the upper bound of the Cheeger inequality.
- In the general case, Golubev's bound (9) involves spectral information from the Laplacians in all dimensions, while the definition of $\vartheta_2(X)$ is based on the Laplacians in dimension 1 only. It is an interesting question whether a strengthening of the theta number involving all dimensions is possible. This might lead to a bound that is as good as Golubev's bound in all cases.

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LINEAR ALGEBRA HISTORY

On the origins, history and state of linear algebra in Greek universities

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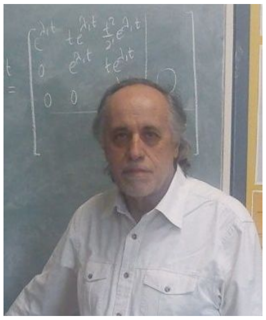
Until the 1970s, linear algebra was taught in Greek universities as part of abstract algebra. The material was restricted to basic matrix algebra, determinants, and linear systems. In polytechnic schools, basic elements of linear algebra were taught as needed in other mathematics courses. For example, since 1972, in the mechanical engineering program of the National Technical University of Athens, ten weeks of lectures were devoted to linear algebra in the course called Advanced Mathematics. Since the mid 1970s, Linear Algebra has been taught in the mathematics departments of all Greek universities in the form of two semester-long courses, typically offered during the first or second semester. Linear Algebra is presently an independent one-semester course in most engineering and science programs. A brief review of personnel and research activities by institution follows.

National Technical University of Athens (NTUA). In 1972, Professor Dimitrios Daskalopoulos published the first comprehensive textbook in linear algebra in Greek. It was titled *Applied Linear Algebra*, had 320 pages, and contained complete proofs in vector space theory, matrix algebra, determinants, linear systems, and spectral analysis, as well as bilinear and quadratic forms.

The latter part of the lectures on eigenvalues and quadratic forms was originally presented by Ioannis (John) Maroulas, as a newly-appointed assistant lecturer in the chair (unit) of Advanced Mathematics. Maroulas moved on to study the existence and computation of solutions to Lyapunov and Riccati equations, which composed his doctoral dissertation under S. Barnett at the University of Bradford (UK), *Theory of Generalized Polynomials with Applications to Linear Systems Theory.* Since 2010, Professor Maroulas has been an Emeritus Professor at NTUA. He supervised the theses of Panayiotis (Panos) Psarrakos (1997), now Professor and Dean of the School of Applied Mathematics and Physical Sciences at NTUA; Maria Adam (2001), now Assistant Professor in the Department of Computer Science and Biomedical Informatics at the University of Thessaly; Aik. Aretaki (2011); and G. Katsouleas (2012).

Left to right: Aretaki, Adam, Maroulas, Tsatsomeros, Psarrakos, Katsouleas, 2016.

In the spring of 1985, Professors John Maroulas and Dimitrios Kravvaririts organized in Athens a *Symposium on Operator Theory*; distinguished lecturers included Hans Schneider, Peter Lancaster, Harm Bart, Thomas Laffey and Vlastimil Pták. Volume 84 (1986) of *Linear Algebra and its Applications* was dedicated to this symposium. In the summer of 2000, Maroulas and Tsatsomeros organized in Nafplio the 5$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii, with Chandler Davis as the featured invited speaker.



Grigoris Kalogeropoulos

National and Kapodistrian University of Athens. Linear Algebra was taught for the first time as an independent course in 1973 by Stylianos Andreadakis. The textbook *Lectures in Linear Algebra* (Μαθήματα Γραμμικής Άλγεβρας) was the second Greek book on the subject. Notably, the book used the analysis-based notation for vector space transformations f, that is, af for $f(a)$. Consequently, an m-by-n matrix A was viewed as a linear map $x \mapsto xA$.

In 1985, Grigoris Kalogeropoulos introduced the course *Matrix Calculus and Applications*, in which special attention was paid to canonical forms of matrices, in particular the Jordan canonical form via Ferrers diagrams. This course was a prerequisite for the course in *Linear Control Theory*.

Currently on staff at the University of Athens is Marilena Mitrouli, who is active in linear algebra research. She recently organized the conference on Numerical Analysis and Scientific Computation with Applications (NASCA '18) in Kalamata, Greece; international invited speakers included Michele Benzi, Dario Bini, Paul Van Dooren, Petros Drineas, Yousef Saad, and Hassane Sadok.

Marilena Mitrouli

Aristotle University of Thessaloniki. Linear Algebra was introduced as an independent course in the Department of Mathematics in 1970–71 by Konstantinos Lakkis, who also commenced teaching the same course in the Department of Physics in the following year. Presently, Linear Algebra is a 5-credit-hour core course with a lab hour, and a second, elective course, Advanced Linear Algebra, has been established.

Left to right: Vologiannidis, Antoniou, Karampetakis, Tzekis, Vardoulakis, 2008.

In 1979, Professor Antonios Vardoulakis, now Emeritus, chaired a newly-formed division in Control Theory. He led an active group of doctoral students, including G. Fragulis (1990), now Professor at the Technological Educational Institute of West Macedonia; N. P. Karampetakis (1993), now Professor at AUT; E. Antoniou (2000) and P. Tzekis (2001), now associate professors at the Technological Educational Institute of Thessaloniki; S. Vologiannidis (2005), now an Assistant Professor at the Technological Educational Institute of Central Macedonia; and Ch. Kazantzidou (2013).

University of Patras. Linear Algebra was introduced as an independent first-year course in the Department of Mathematics in 1973–74 by Dimitrios Stratigopoulos. The material taught was of abstract nature and a second course included group and ring theory. Prior, linear algebra was taught as general background material by Grigorios Tsangas. A course in linear algebra also has been taught in the Department of Computer Engineering and Informatics since it was established in 1980.

The latter department hosts the High Performance Information Systems Laboratory, whose software division is directed by Stratis Gallopoulos. He supervised the doctoral studies of C. Bekas (2003), now Distinguished Research Staff Member, Manager Foundations of Cognitive Solutions, IBM Research – Zurich; G. Kollias (2009); and D. Zeimpekis (2009).

Stratis Gallopoulos

University of Ioannina. Starting in 1971, Linear Algebra became an independent course, first taught by Georgios Kazantzidis, and was split into two semester-long courses in 1985. The second course was first taught by Simeon Bozabalidis. In 1972, Apostolos Hadjidimos was the first professor in Greece to be designated as Chair of Numerical Analysis. His research is largely in numerical linear algebra, and his student Dimitrios (Takis) Noutsos is currently a linear algebra contributor and professor at Ioannina. Other students of Hadjidimos are G. Avdelas (1975), S. Galanis (1979), A. Yeyios (1979), N. Krimnianiotis (1979), A. Psimarni (1982), M. Lapidakis (2008), N. Gaïtanos (1983), and M. Alanelli (2007), who is now Special Scientist at the Cyprus University of Technology. Noutsos's students are M. Tzoumas (1994) and P. Vassalos (2003), now Assistant Professor in the Department of Informatics at Athens University of Economics and Business.

Noutsos (left) and Hadjidimos, 2010.

In conclusion, we would be remiss not to mention in passing two mainstays of modern Greek mathematics with ties to linear algebra: Cyparissos Stephanos and Constantinos Carathéodory.

Cyparissos Stephanos

Cyparissos Stephanos (1857–1917). He received his Ph.D. in 1878 from the National and Kapodistrian University of Athens. In the early 1880s he studied mathematics in Paris and returned to a professorship and leadership role at the University of Athens. He also served as a professor at the National Technical University of Athens. Stephanos is mostly known for his contributions in algebra and projective geometry (desmic systems). Some of the early theorems on determinantal identities have been traced to the work of Stephanos, e.g., in "Sur une extension du calcul des substitutions linéaires", *Journal de Mathématiques Pures et Appliquées*, 5^{e} série, tome 6 (1900), pp. 73–128.

Constantinos Carathéodory (1873–1950). Born in Berlin, he earned a doctorate in mathematics at the University of Göttingen (under Hermann Minkowski) in 1904. He served as Professor at the universities of Hanover, Breslau, Göttingen, Berlin, Smyrna, at the National and Kapodistrian University of Athens, and at the National Technical University of Athens, and he concluded his career in Munich. He contributed or provided the foundations to many fields of mathematics, including the calculus of variations, point set measure theory, function theory, and convex analysis, as well as thermodynamics. His legacy includes at least two dozen theorems or theories that bear his name. Some of his work advanced linear algebraic concepts, e.g., his 1935 study of the properties of bordered Hessian matrices.

Constantinos Carathéodory

Ackowledgments. We thank Professors E. Gallopoulos, A. Hadjidimos, N. Karampetakis, J. Maroulas, M. Mitrouli, D. Noutsos, and A. Papistas for information they provided. Our sources included Wikipedia entries, as well as the article "Carathéodory's Theorem on Constrained Optimization and Comparative Statics" by E. Drandakis.

LINEAR ALGEBRA EDUCATION

National Pedagogical Initiatives on Linear Algebra

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Linear Algebra workshop group photo, October 6, 2018.

Linear Algebra is an important course for many STEM students. Research shows that the transition into an abstract and formal presentation of concepts creates difficulties for many students. In the early 90s, the Linear Algebra Curriculum Study Group (LACSG) suggested a set of recommendations [1] for the first course in linear algebra at the undergraduate level. A quarter of century later, it is timely to revise the existing recommendations in order to move the field forward. A two-day NSF-funded (DUE 1822247) workshop, titled *National Pedagogical Initiatives on Linear Algebra*, was held at the University of Oklahoma, October 4–6, 2018. The goals of the workshop included: (a) working toward revising the recommendations for the teaching and learning of linear algebra; (b) generating new research questions and encouraging collaboration between research mathematicians and mathematics educators.

The plenary talks at the workshop were:

1. *The Linear Algebra Curriculum Study Group (LACSG) Recommendations: A Cognitive-Pedagogical Examination*
   Guershon Harel, Professor of Mathematics, University of California, San Diego
   (Guershon Harel also participated in the LACSG in 1990.)

2. *Teaching and Learning: The How, What, and Why, a Holistic Approach*
   Frank Uhlig, Professor Emeritus, Auburn University

The workshop comprised nine sessions, each designed to unravel a specific facet of teaching and learning linear algebra. These sessions consisted of presentations followed by group discussions. All sessions were recorded in the sense that each group wrote their ideas and suggestions in shared Google docs during the discussions. The workshop concluded with an open discussion on why a revised document was necessary. The attendees discussed some short-term as well as some long-term goals.

The Sessions: Titles and Presenters

1. Applications in Linear Algebra: Amanda Harsy (Lewis University) and Marie Snipes (Kenyon College)
2. Research on Education of Linear Algebra: Sepideh Stewart (University of Oklahoma), Christine Andrews-Larson (Florida State University) and Michelle Zandieh (Arizona State University)
3. Proof in Linear Algebra: Guershon Harel (University of California, San Diego)
4. Understanding the Mathematics of Linear Algebra through Interactive Visualization (GeoGebra): James Factor (Alverno College)

5. STEM Panel (University of Oklahoma): Nathan Goodman (Electrical and Computer Engineering), S. Lakshmivarahan (Computer Science) and Wayne Stewart (Statistics); Moderator: Sepideh Stewart
6. Inquiry-Oriented Linear Algebra – IOLA: Megan Wawro (Virginia Tech)
7. Technology in Introductory Linear Algebra: Robert Beezer (University of Puget Sound)
8. The Second Course in Linear Algebra: Sheldon Axler (San Francisco State University)
9. Final Session: An open discussion.

Linear Algebra workshop session on proof.

The list of workshop participants is as follows.

Aditya Adiredja (University of Arizona); Christy Andrews-Larson (Florida State University); Sheldon Axler (San Francisco State University); Robert Beezer (University of Puget Sound); Peter Blomgren (San Diego State University); Eugene Boman (Penn State, Harrisburg); Gunhan Caglaya (New Jersey City University); Minerva Catral, (Xavier University); John Eggers (University of California, San Diego); James Factor (Alverno College); Mohammad Haider (Florida State University); Amanda Harsy (Lewis University); Guershon Harel (University of California, San Diego); Gulden Karakok (University of Northern Colorado); Steve Leon (University of Massachusetts Dartmouth); Judi McDonald (Washington State University); Jeff Meyer (California State University San Bernardino); Spencer Payton (Lewis-Clark State College); David Plaxco (Clayton State University); Chris Rasmussen (San Diego State University); Janet Sipes (Arizona State University); Marie Snipes (Kenyon College); David Strong (Pepperdine University); Maria Trigueous (Instituto Tecnológico Autónomo de México); Frank Uhlig (Auburn University); Megan Wawro (Virginia Tech); Michelle Zandieh (Arizona State University); and Sepideh Stewart (University of Oklahoma), the organizer.

In addition, a group of faculty, postdoctoral fellows, and students from the mathematics department of the University of Oklahoma were invited to participate at this workshop. Faculty: Noel Brady (department chair), Catherine Hall (academic advisor), Keri Kornelson, Alan Roche (chair of the undergraduate committee), Milos Savic, and Wayne Stewart. Postdocs: Nick Davidson, Jonathan Epstein, Stephanie Schnake, and Jonathan Troup. Math Instructors: Julia Maddox. Graduate Students: Dustin Gaskins, Jessi Lajos, Thomas Lane, Manami Roi, and Cory Wilson. Undergraduate Students: David McKnight and Katherine Simmons.

Going forward, we have formed a working committee, LACSG 2.0, in order to continue our work on this project. The committee members are: Sheldon Axler, Robert Beezer, Minerva Catral, Guershon Harel, Judi McDonald, Richard Penney, Sepideh Stewart, David Strong, and Megan Wawro.

References.

[1] D. Carlson, C. R. Johnson, D. C. Lay, and A. D. Porter. The linear algebra curriculum study group recommendations for the first course in linear algebra. *The College Mathematics Journal*, 24(1):41–46, 1993.

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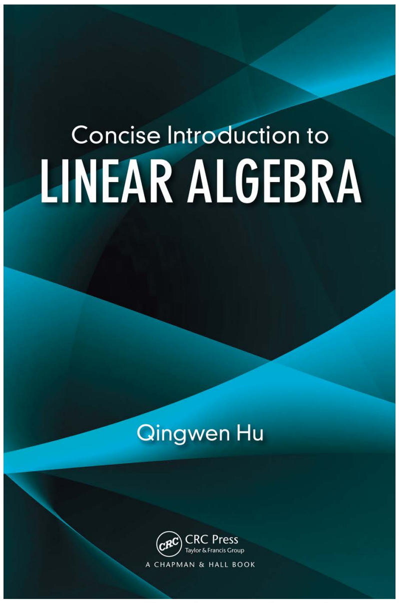

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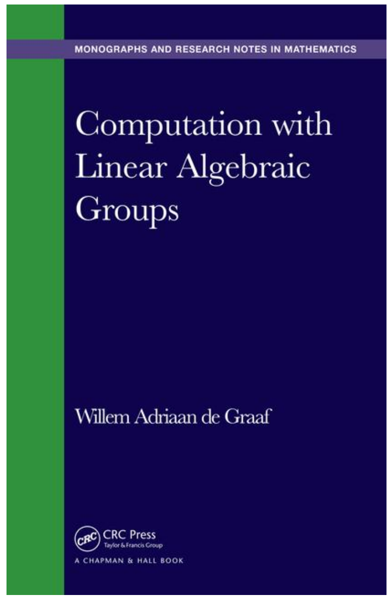

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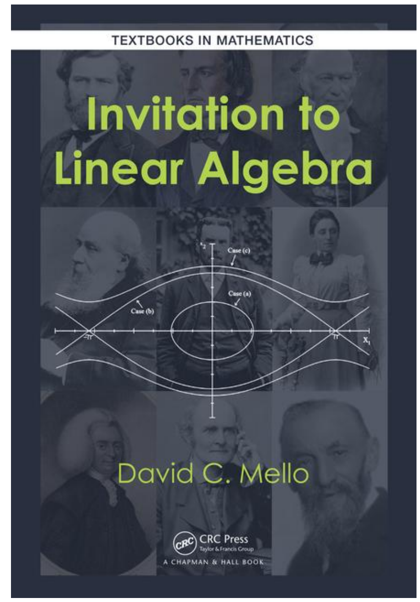

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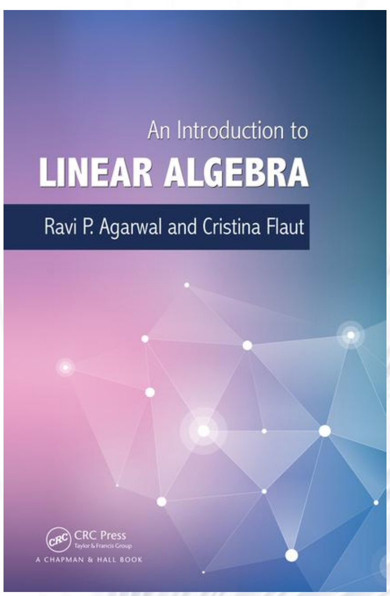

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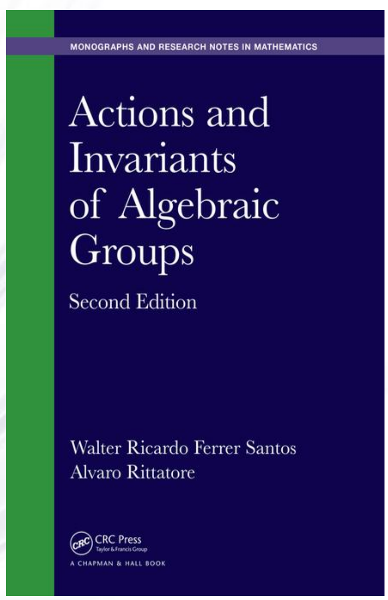

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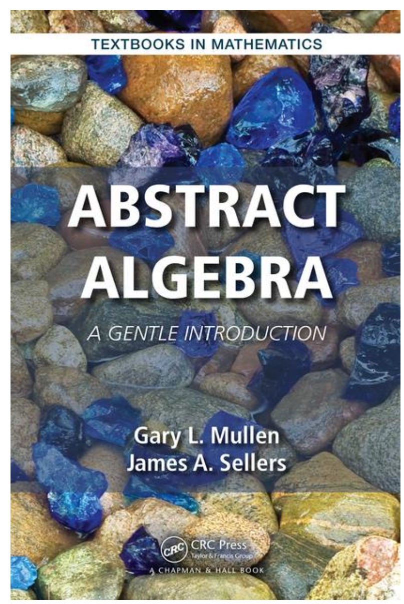

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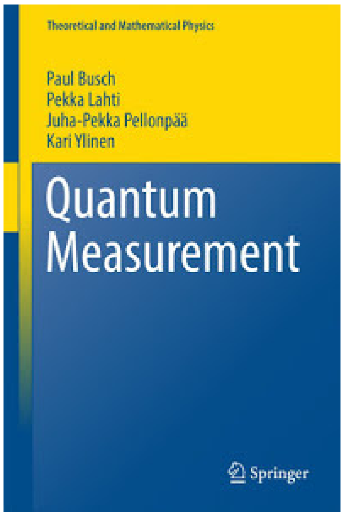
BOOK REVIEW

Quantum Measurement
by Paul Busch, Pekka Lahti, Juha-Pekka Pellonpää, and Kari Ylinen

Theoretical and Mathematical Physics, Springer, 2016, ISBN 978-3-319-43387-5, 542+xii pages.
Reviewed by Douglas Farenick, University of Regina, douglas.farenick@uregina.ca



This book's preface opens with: "*Quantum Measurement* is a book on the mathematical and conceptual foundations of quantum mechanics, with a focus on measurement theory." Why, then, is this book of interest to linear algebraists? Does it concern vector spaces? Matrices and linear transformations? Spectral theory? Are there interesting applications? I would suggest that the answer is "yes" to all of these questions. Further, my informal observation is that traditional linear algebraists are now finding – perhaps more so than in the past – that their expertise aligns very well with the current needs of theoretical and mathematical physics, which makes a mathematics book concerning an aspect of quantum theory to be of potential interest and value to readers of *IMAGE*.

Let me begin with the vector spaces. They are almost always complex (although there is a theory of quaternionic quantum mechanics, not treated in this book, that has yet to receive mainstream acceptance) and they are equipped with inner products. Therefore, the vector spaces of interest are complex Hilbert spaces. Already this may sound more like functional analysis than linear algebra; however, a very substantial volume of modern research in quantum theory concerns finite-dimensional Hilbert space. Consequently, Hermitian matrices, positive semidefinite matrices, (orthogonal) projections, and unitary matrices play prominent roles, as do "superoperators" on matrix spaces, such as completely positive linear maps.

The objects of study in the book under review are, in the most simple of descriptions, k-tuples ν of positive semidefinite $d \times d$ matrices that sum to the identity matrix. Such tuples ν are called *quantum measurements*, for reasons I shall later explain. If each matrix in the k-tuple ν is an orthogonal projection, then the fact that the matrices sum to the identity implies that the ranges of the projections are pairwise orthogonal; hence, such measurements ν are called projective measurements. There is a beautiful theorem due to Naimark that relates arbitrary quantum measurements with projective measurements: it says that for every quantum measurement $\nu = (A_1, \dots, A_k)$ in $M_d(\mathbb{C})$ there exists an $s \geq d$, a projective measurement $\pi = (P_1, \dots, P_k)$ in $M_s(\mathbb{C})$, and an isometry $W : \mathbb{C}^d \to \mathbb{C}^s$ such that $A_j = W^* P_j W$ for each j. This is a very nice mathematical proposition, regardless of the potential application or interpretation in theoretical physics, and *Quantum Measurement* contains many such gems that are of interest to the everyday mathematician.

So why are these tuples ν of matrices called quantum measurements? In quantum theory, physical systems are represented mathematically by a Hilbert space, let's say of dimension d and which we take to be $\mathbb{C}^d$. The possible states of the system are represented by density matrices, which are positive semidefinite $d \times d$ matrices ρ of trace $\operatorname{Tr} \rho = 1$. If the rank of a density matrix ρ is 1, then ρ is considered to be a physically pure state because it is associated with a unit vector $\xi \in \mathbb{C}^d$ satisfying $\xi\xi^* = \rho$ (any two unit vectors ξ that give rise to ρ in this way are related by a phase factor $e^{i\theta}$, which linear algebraists recognise as a fancy way of describing linear dependence). A quantum measurement $\nu = (A_1, \dots, A_k)$ of the system is a mathematical model for the act of making physical measurements or observations of the system with k possible outcomes. The matrices A_j represent quantum effects associated with measurements (such as the emitting of a photon which is then registered by the environment). The underlying statistical principle is that $\operatorname{Tr}(\rho A_j)$ represents the probability that the measurement will result in outcome j when the system is in state ρ. These matrix-theoretic ideas can be dressed more formally in the language of operator-valued measures, which is what *Quantum Measurement* sets out to do.

The authors are recognised experts in the area. Indeed, in addition to their extensive research results, Busch and Lahti, along with Peter Mittelstaedt, co-authored the widely-read monograph titled *The Quantum Theory of Measurement* in 1991. *Quantum Measurement*, appearing 25 years later, is an up-to-date and, at 542 pages, fuller account of the field.

The book is organised by sections, the first of which is titled "Mathematics" and is devoted to preparatory material from operator theory. These first eight chapters run close to 190 pages and are especially excellent for students who desire to become mathematical physicists. Readers who are primarily familiar with linear algebra will find this section to be a good self-contained reference for their understanding of the remainder of the book.

Part II of *Quantum Measurement* is titled "Elements," and it is here that one is introduced to notions of quantum theory, including the idea of measurement. Measure-theoretic and probabilistic ideas and concepts are especially relevant in these chapters. Here, as throughout the book, the authors provide a rigorous mathematical treatment of uncertainty issues that are inherent in quantum theory.

Part III, titled "Realisations," discusses some of the most important examples and touches upon ideas in quantum information theory. The first chapter in this third part is Chapter 14, which contains some very lovely linear algebra and geometry involving 2×2 matrices.

The final part, Part IV, is titled "Foundations." The discussion begins with the famous Bell inequalities and continues with the foundations of quantum mechanics, which is in many ways fascinating for a mathematician to encounter. We, in pure mathematics, long ago rejected constructivism – to take one example of an opinion on an issue that was once central in mathematics (namely, what is mathematics about and how ought it be developed?). However, the variety of approaches to the axioms of quantum theory are still hotly debated in the theoretical physics community.

To be honest, *Quantum Measurement* is a book written by mathematical physicists for mathematical physicists. It is not a book of the "gentle introduction" variety. However, as I hope I have made clear above, there is a great deal in this book that is useful to the linear algebraist who has research interests in quantum theory and its applications (quantum computing, quantum information). Counting myself as one of those, I have found this well-written book to be an excellent source of reference material and to be an entryway into a body of literature that is remarkably vast and rich.

ILAS NEWS

2018 ILAS Elections: Nominations

Contributed announcement from Peter Šemrl, ILAS President

The Nominating Committee for the 2018 ILAS elections has completed its work. Nominated for a three-year term, beginning March 1, 2019, as ILAS Vice-President is: Hugo J. Woerdeman.

Nominated for the two open three-year terms, beginning March 1, 2019, as "at-large" members of the ILAS Board of Directors are: Michael Tsatsomeros, Fernando De Terán, João Quieró, and Valeria Simoncini.

Many thanks to the Nominating Committee: Chi-Kwong Li (Chair), Wayne Barrett, Heike Faßbender, Judi McDonald, and Christian Mehl.

Three ILAS Members Selected as 2019 AMS Fellows

The American Mathematical Society has announced its 2019 class of AMS Fellows, and among these are three members of ILAS, namely:

- Shmuel Friedland (University of Illinois at Chicago) – For contributions to the theory of matrices, tensors, and their applications to other areas.
- Stephan Ramon Garcia (Pomona College) – For contributions to operator theory and leadership in undergraduate research and mentoring.
- Olof B. Widlund (New York University, Courant Institute) – For contributions to numerical analysis of domain decompositions within computational mathematics and for incubation through his writing and mentorship of a broad international, creative community of practice applied to highly resolved systems simulations.

For more details, see `http://www.ams.org/profession/ams-fellows/new-fellows`.

Reduced rates for *LAA* and *LAMA* for ILAS members

Contributed announcement from Peter Šemrl, ILAS President

The special subscription rate for the electronic version of *Linear Algebra and its Applications* for individual members of ILAS is $100 and the reduced subscription rate for *Linear and Multilinear Algebra* for individual members of ILAS is $125. Subscriptions purchased at the personal rate are strictly for personal, non-commercial use only.

Volker Mehrmann Awarded W. T. and Idalia Reid Prize by SIAM

Volker Mehrmann, Professor at Technische Universität Berlin, was selected by the Society for Industrial and Applied Mathematics (SIAM) to receive the 2018 W. T. and Idalia Reid Prize. This prize is awarded annually to one member of the scientific community for outstanding work in, or other contributions to, the broadly defined areas of differential equations and control theory.

Professor Mehrmann is a longtime member of ILAS and an Editor-in-Chief of *Linear Algebra and its Applications*.

For more details, see:

`https://sinews.siam.org/Details-Page/prize-spotlight-volker-mehrmann`

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Volker Mehrmann Elected President of EMS

Volker Mehrmann, Professor at Technische Universität Berlin, was elected at the meeting of the EMS Council in Prague (June 23–24, 2018) to serve as president of the European Mathematical Society.

Professor Mehrmann is a longtime member of ILAS and an Editor-in-Chief of *Linear Algebra and its Applications*.

He will begin his 3-year term as EMS President on January 1, 2019.

For more details, see the press release issued by TU Berlin at `https://idw-online.de/en/news698758`

OBITUARY NOTICE

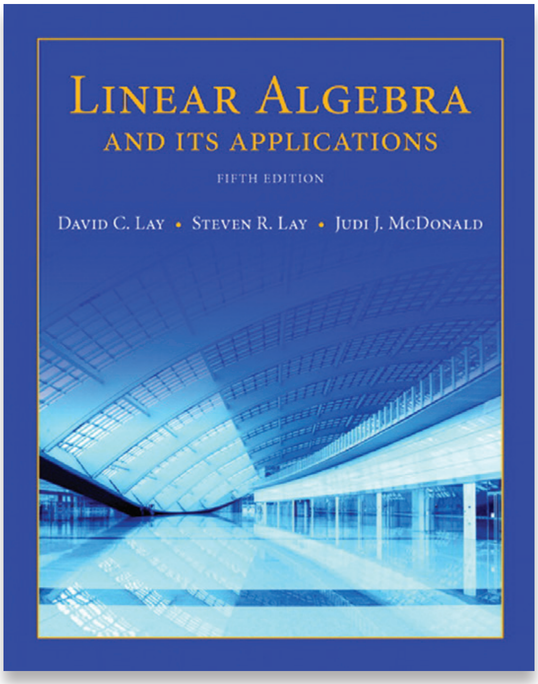
David Lay 1941–2018

Submitted by Judith McDonald

The linear algebra community will greatly miss David Lay, who recently passed away. David earned a B.A. from Aurora University in Illinois in 1962, and went on to graduate studies at UCLA, where he pursued his passion and gift for mathematics. After completing the Ph.D. program at UCLA in 3 ½ years, Dr. Lay moved across the country to begin a nearly 43-year professorship at the University of Maryland, College Park. He published more than 30 research articles in functional analysis and linear algebra, and earned four university awards for teaching excellence, including the 1996 title of Distinguished Scholar-Teacher of the University of Maryland. He dedicated much of his career to the art of teaching mathematics effectively and in a way that would inspire even those previously apathetic about or hostile to math. His father had also been a math professor, as is his brother, Steven. David had a tremendous influence on the teaching and learning of linear algebra, being one of the original members of the Linear Algebra Curriculum Study Group, and authoring the top-selling textbook in this area. Many of us have had the pleasure of teaching from his linear algebra, calculus or functional analysis textbooks, and those of us who worked with him directly benefitted greatly from his passion for mathematics and his insights into mathematics education. He is remembered by hundreds of students as a most excellent, kind, sometimes silly, and enthusiastic math professor. His leadership and influence will be greatly missed. Donations can be made in David Lay's honor to the "Lay Family Math Assistantship" at Lee University Central Gifts, P.O. Box 3450, Cleveland, TN 37320-3450.

David Lay's *Linear Algebra,* Fifth Edition

Click here to watch author Judi McDonald present on Interactive Figures for Linear Algebra



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Lay • Lay • McDonald © 2016 ISBN 0-13-402269-6

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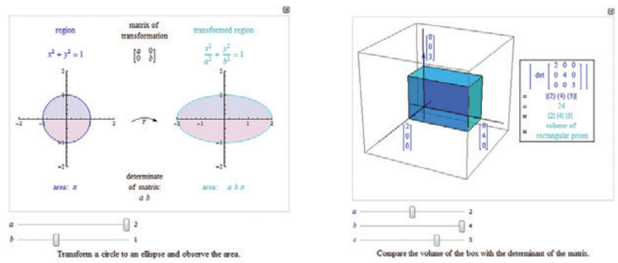

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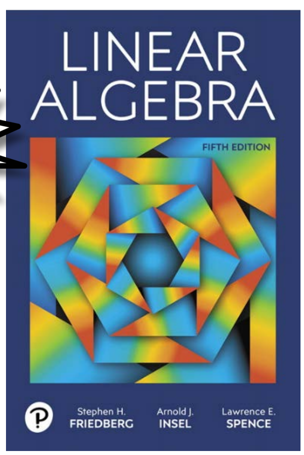

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CONFERENCE REPORTS

14th Western Canada Linear Algebra Meeting (WCLAM) Washington State University, Pullman, WA, USA, May 26–27, 2018

Report by Michael Tsatsomeros

The 14th Western Canada Linear Algebra Meeting (WCLAM) took place on the Washington State University (WSU) Pullman campus, May 26–27, 2018. WCLAM is a biennial meeting that has been been held at various sites throughout Western Canada, beginning with an inaugural meeting at the University of Regina in 1993. This was the first time WCLAM was held in the US, recognizing the strong participation and contribution to linear algebra by mathematicians in the Pacific Northwest.

Participants of the 2018 WCLAM

The meeting provided an opportunity for researchers to present accounts of their recent efforts, and to hold informal discussions. Graduate students and early career faculty were featured prominently. There were 38 participants, two invited speakers, and 16 additional presentations in linear algebra and related fields. The meeting was generously supported by the Pacific Institute for the Mathematical Sciences, the International Linear Algebra Society, the College of Arts and Sciences at WSU, as well as the WSU Department of Mathematics and Statistics.

26th International Workshop on Matrices and Statistics (IWMS-2018) Dawson College, Montréal, Canada, June 5–7, 2018

Report by Ka Lok Chu, Simo Puntanen & George P. H. Styan

The 26th International Workshop on Matrices and Statistics (IWMS-2018) was held at Dawson College, Montréal, Canada, June 5–7, 2018.

The purpose of this workshop was to bring together researchers sharing an interest in a variety of aspects of statistics and its applications, as well as matrix analysis and its applications to statistics, and to offer them an opportunity to discuss current developments in these subjects. The workshop helped bridge the gap among statisticians, computer scientists and mathematicians in understanding each other's tools.

The International Organizing Committee (IOC) was chaired by Ka Lok Chu (Montréal, Canada), with help from Simo Puntanen (Tampere, Finland) and George P. H. Styan (Montréal, Canada). The International Scientific Committee comprised S. Ejaz Ahmed (Co-chair; St. Catharines, Ontario, Canada), Hans Joachim Werner (Co-chair; Bonn, Germany), and George P. H. Styan (Honorary Chair) with help from Francisco Carvalho (Tomar, Portugal), Katarzyna Filipiak (Poznań, Poland), Jeffrey J. Hunter (Auckland, New Zealand), Daniel Klein (Košice, Slovakia), Augustyn Markiewicz (Poznań, Poland), Simo Puntanen, Júlia Volaufová (New Orleans, Louisiana, USA), and Dietrich von Rosen (Uppsala, Sweden).

The IOC of IWMS-2018 is most grateful to Karina D'Elmo and Myriam Dimanche for their help with local arrangements at Dawson College.

The invited speakers were:

- Oskar Maria Baksalary (Poznań, Poland): "Representations of the Moore–Penrose inverses of matrices"
- Adi Ben-Israel (Piscataway, NJ, USA): "Matrix volume and its applications"
- Dennis S. Bernstein (Ann Arbor, Michigan, USA): "Input estimation for linear discrete-time dynamical systems"
- S. W. Drury (Montréal, Canada): "Some old conjectures of Bapat and Sunder"
- Yonghui Liu (Shanghai, China): "Estimation and influence diagnostics for an autoregressive model under skew-normal distributions"
- Mika Mattila (Tampere, Finland): "The arithmetic Jacobian matrix and determinant"
- Christopher C. Paige (Montréal, Canada): "The effects of loss of orthogonality on large sparse matrix computations"
- Yongge Tian (Beijing, China): "Multilevel statistical models, least-squares estimators, and reverse-order laws for generalized inverses"

Contributed speakers:

- Jorge Delgado Gracia & Juan Manuel Pẽna (Zarazoga, Spain): "Schoenmakers–Coffey matrices"
- Simo Puntanen: "Linear prediction sufficiency in the misspecified linear model"

On Tuesday, June 5th, we held a *Mini-symposium celebrating the 100th birth anniversary of Theodore Wilbur Anderson, Jr.*, who was born on June 5th, 1918, and passed away on September 17th, 2016; the speakers were Simo Puntanen and George P. H. Styan. Related links:

- `http://www.sis.uta.fi/tilasto/iwms/PunSty-Tomar-handout-13july08T.pdf` (Bibliography for TWA, 2008)
- `http://www.sis.uta.fi/tilasto/iwms/TWA-NYT-10oct16.jpg` (*New York Times* Obituary, 10 Oct 2016)

On Wednesday, June 6th, several talks by invited speakers were presented, as well as some contributed talks. On Thursday, June 7th, the *Third mini-symposium on magic squares, prime numbers and postage stamps* was held, including speakers Ka Lok Chu and George P. H. Styan, and a poster by Richard William Farebrother (Manchester, UK). The first two of these mini-symposia were held at IWMS-2016 (`http://www.iwms.ipt.pt/?page=home`) in Madeira, Portugal and at Dawson College, Montréal (2017), celebrating George P. H. Styan's 80th birthday; see the program of G80 at `https://people.uta.fi/~simo.puntanen/M2-G80-7sep17-red.pdf`.

Participants of IWMS-2018

For more about the IWMS-2018 Workshop, see the web site (`https://profchu.wixsite.com/mysite`) and the final program (`http://www.sis.uta.fi/tilasto/iwms/program-IWMS-2018-Montreal.pdf`).

For previous IWMS-workshops, see the IWMS-website, where, for example, "A short history of the International Workshop on Matrices and Statistics" can be downloaded (`http://www.sis.uta.fi/tilasto/iwms/IWMS-history.pdf`). The 27th International Workshop on Matrices and Statistics will be held in Shanghai, Shanghai University of International Business and Economics, June 6–9, 2019; see `http://www.suibe.edu.cn/txxy/iwms2019`.

UPCOMING CONFERENCES AND WORKSHOPS

Special session on Innovative and Effective Ways to Teach Linear Algebra *at the* 2019 Joint Mathematics Meetings Baltimore, MD, USA, January 16–19, 2019

The upcoming 2019 Joint Mathematics Meetings of the American Mathematical Society (AMS) and the Mathematical Association of America (MAA) will include an MAA Contributed Paper Session on Innovative and Effective Ways to Teach Linear Algebra.

This session will serve as a forum in which to share and discuss new or improved teaching ideas and approaches. These innovative and effective ways to teach linear algebra include, but are not necessarily limited to: (1) hands-on, in-class demos; (2) effective use of technology, such as *Matlab*, *Maple*, *Mathematica*, Java applets or Flash; (3) interesting and enlightening connections between ideas that arise in linear algebra and ideas in other mathematical branches; (4) interesting and compelling examples and problems involving particular ideas being taught; (5) comparing and contrasting visual (geometric) and more abstract (algebraic) explanations of specific ideas; (6) other novel and useful approaches or pedagogical tools.

These sessions, now in their twelfth year, are organized by Sepideh Stewart (University of Oklahoma), Gil Strang (Massachusetts Institute of Technology), David Strong (Pepperdine University), and Megan Wawro (Virginia Tech). There will be fifteen talks given during the morning and afternoon sessions on Thursday, January 17. The schedule of talks is available at:

- `http://jointmathematicsmeetings.org/meetings/national/jmm2019/2217_program_thursday.html#2217:MCPSTRF5`
- `http://jointmathematicsmeetings.org/meetings/national/jmm2019/2217_program_thursday.html#2217:MCPSTRF6`

Special session on Combinatorial Matrix Theory *at the* 50th Southeastern International Conference on Combinatorics, Graph Theory & Computing Florida Atlantic University, Boca Raton, FL, USA, March 4–8, 2019

There will be a Special Session on Combinatorial Matrix Theory at the 50th Southeastern International Conference on Combinatorics, Graph Theory and Computing (SEICCGTC), held March 4–8, 2019 at Florida Atlantic University in Boca Raton, FL. The presentations in this session will explore the the symbiotic relationship between matrices and graphs, which has benefited both linear algebra and combinatorics, as well as many applications. The spectral analysis of various matrices associated with graphs, such as the Laplacian, signless Laplacian, normalized Laplacian, adjacency matrix, and distance matrix, has proved oddly useful in uncovering combinatorial properties of graphs. Furthermore, a matrix can be viewed from a variety of combinatorial perspectives, for example, as a pattern that retains only discrete information from the matrix, such as the signs of the entries, or by using a graph or directed graph to describe patterns of nonzero entries in the matrix. One may then ask what information such a combinatorial description carries about properties of the matrix such as its rank or spectrum.

The organizers of the session are David Brown (Utah State University) and Leslie Hogben (Iowa State University). About 20 talks are anticipated. If your research uses the tools of graph theory and combinatorics to study matrices or uses the tools of linear algebra and matrix theory to study graphs and combinatorial structures, consider submitting an abstract to present in this session.

The main campus of FAU is located three miles from the Atlantic Ocean, on an 850-acre site in Boca Raton, south of Palm Beach and north of Fort Lauderdale and Miami. The climate is subtropical with an average temperature of 75°F (24°C). More information can be found at `http://www.math.fau.edu/combinatorics2019/index.php`

Special session on Combinatorial Matrix Theory *at the* 2019 AMS Spring Southeastern Sectional Meeting Auburn University, Auburn, AL, USA, March 15–17, 2019

This session aims to bring interested students and researchers together to foster more research in combinatorial matrix theory. Combinatorial matrix theory exploits the symbiotic relationship between matrix theory and combinatorics, as techniques from one area continue to yield rich theorems in the other. One of the ways this relationship is exploited is by associating a graph or a digraph with a matrix, and then looking at the matrix through a combinatorial lens by ignoring the values of the matrix entries and focusing on the pattern of the entries. For example, sometimes it is desirable to only focus on the sign or zero-nonzero pattern of the entries (as in the study of sign patterns or zero-nonzero patterns that require or allow certain matrix properties, and the study of minimum rank and inverse eigenvalue problems, for example). Another way to exploit the aforementioned relationship is to study a graph by associating a particular matrix to it, which leads to an area that resides within combinatorial matrix theory: spectral graph theory. An embryonic area in graph theory is the study of the zero forcing number of a graph, which has been shown to have an intimate connection with matrix theory. The topics featured in the session may include, but are not limited to, those listed above.

The session organizers are Zhongshan Li (Georgia State University) and Xavier Martínez-Rivera (Auburn University). For more details, contact Xavier Martínez-Rivera at `xaviermr@auburn.edu`, or see `http://www.ams.org/meetings/sectional/2261_program_ss2.html`.

27th International Workshop on Matrices and Statistics (IWMS2019) Shanghai, China, June 6–9, 2019

The 27th International Workshop on Matrices and Statistics, IWMS-2019, will be held June 6–9, 2019 at the Shanghai University of International Business and Economics, Shanghai, China.

This series of workshops has a long history and we welcome the opportunity to hold the workshop once again in China. The 19th workshop was held in Shanghai in 2010 and was a resounding success, as was the 24th workshop in Haikou. We are sure that this 27th workshop will continue in the same tradition.

The purpose of the workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. The workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in linear algebra and matrix theory and may exchange ideas with researchers from a wide variety of countries.

Plenary Speakers include: Oskar Maria Baksalary (Poland), Kai-Tai Fang (Hong Kong, China), Arjun K. Gupta (USA), Steve Kirkland (Canada), Jianxin Pan (UK), K. Manjunatha Prasad (India), Yongge Tian (China), Fuzhen Zhang (USA), and Lixing Zhu (Hong Kong, China).

Planned mini-symposia (with organizers) include: Decompositions of tensor spaces with applications to multilinear models (Dietrich van Rosen); Linear statistical models (Simo Puntanen); Magic squares, prime numbers and postage stamps (George P. H. Styan); Inference in parametric models (Julia Volaufova); Statistical diagnosis (Shuangzhe Liu); and Experimental design (Kai-Tai Fang).

The International Organizing Committee of IWMS-2019 consists of Jeffrey J. Hunter (New Zealand) (Chair), Dietrich von Rosen (Sweden) (Vice-Chair), George P. H. Styan (Canada) (Honorary Chair), S. Ejaz Ahmed (Canada), Francisco Carvalho (Portugal), Katarzyna Filipiak (Poland), Daniel Klein (Slovakia), Augustyn Markiewicz (Poland), Simo Puntanen (Finland), Julia Volaufova (USA), and Hans Joachim Werner (Germany).

The Local Organizing Committee consists of Yonghui Liu (Chair), Hui Liu (Vice-Chair), Chengcheng Hao, and Cihai Sun. For further information, please visit `http://www.suibe.edu.cn/txxy/iwms2019`.

Workshop on Numerical Ranges and Numerical Radii (WONRA) Kawagoe, Japan, June 21–24, 2019

The 15th Workshop on Numerical Ranges and Numerical Radii (WONRA) will be held at Toyo University, Kawagoe Campus, Japan, June 21 (Fri.)–24 (Mon.), 2019.

This is one of the special workshops for the celebrated Toeplitz-Hausdorff Theorem. The year 2018 was the 100th anniversary of this celebrated theorem asserting that the numerical range of an operator is always convex. There has been a lot of research activity on the topic since this fundamental result was established. The high level of research activity is due to the many connections the subject has to different pure and applied areas. The purpose of the workshop is to stimulate research and foster interactions between researchers interested in the subject. The informal workshop atmosphere will facilitate the exchange of ideas from different research areas and, hopefully, the participants will leave informed of the latest developments and newest ideas. Another special workshop was held in Germany in the summer of 2018, and the 2020 workshop following the usual schedule will take place in Coimbra in the summer of 2020.

The conference organizers are: Chi-Kwong Li, College of William and Mary; Hiroshi Nakazato, Hirosaki University; Hiroyuki Osaka, Ritsumeikan University; and Takeaki Yamazaki, Toyo University.

Please notify Takeaki Yamazaki (`t-yamazaki@toyo.jp`) by May 19th, 2019 if you plan to attend the workshop. Provide your name, affiliation, and e-mail. If you are giving a talk, please also submit your title and abstract to Takeaki Yamazaki by May 19th, 2019. For more details, please see `http://www2.toyo.ac.jp/~t-yamazaki/WONRA/WONRA19.html`.

ILAS 2019: Linear Algebra Without Borders
Rio de Janeiro, Brazil, July 8–12, 2019

The 22nd conference of the International Linear Algebra Society, ILAS 2019: Linear Algebra Without Borders, will be held July 8–12, 2019 in Rio de Janeiro, Brazil, at the main campus of Fundação Getúlio Vargas (FGV), a Brazilian think tank and higher education institution founded in 1944 with the aim of promoting Brazil's economic and social development.

The theme of the conference is "Linear Algebra Without Borders" and refers primarily to the fact that linear algebra and its myriad of applications are interwoven in a borderless unit. In the conference, the organizers plan to illustrate this with a program whose plenary talks and symposia represent the many scientific "countries" of linear algebra, and which invites participants to "visit" them. This theme also refers to the openness and inclusiveness of linear algebra to researchers of different backgrounds.

Plenary speakers:

- **David Bindel**, Cornell University, USA, *SIAG-LA Lecturer*
- **Christoph Helmberg**, Technische Universität Chemnitz, Germany
- **Leslie Hogben**, Iowa State University, USA
- **Apoorva Khare**, Indian Institute of Science, Bangalore, India
- **Igor Klep**, University of Auckland, New Zealand
- **Gitta Kutyniok**, Technische Universität Berlin, Germany
- **Joseph Landsberg**, Texas A&M University, USA, *LAA Lecturer supported by Elsevier*
- **Federico Poloni**, Pisa University, Italy
- **Nikhil Srivastava**, University of California, Berkeley, USA
- **Yuan Jin Yun**, Universidade Federal do Paraná, Brazil

Invited mini-symposia (with organizers): Matrix Analysis (James Pascoe, Miklos Palfia), Frames (Gitta Kutyniok, Deanna Needell), Matrix Equations and Matrix Inequalities (Fuzhen Zhang, Qing-Wen Wang), Algebra and Tensor Spaces (David Gleich, Yang Qi), Linear Algebra and Quantum Information Science (Yiu-Tung Poon, Raymond Nung-Sing Sze, Sarah Plosker), Combinatorial Matrix Theory (Byan Shader, Shaun Fallat, Steve Butler, Kevin Vander Meulen), Matrix Techniques in Operator Theory and Operator Algebras (Hugo Woerdeman), Spectral Graph Theory (Sebastian Cioabă, Jack Koolen, Leonardo Lima), Linear Algebra Education (Sipedeh Stewart, Rachel Quinlan), and Nonnegative Inverse Spectral Problems (Raphael Loewy, Ricardo L. Soto).

The scientific organizing committee is: Nair Abreu (Brazil), Ravindra Bapat (India), Leslie Hogben (USA), Alfredo Iusem (Brazil), Steve Kirkland (Canada), Chi-Kwong Li (USA), Volker Mehrmann (Germany), Beatrice Meini (Italy), Rubens Sampaio (Brazil), Peter Šemrl (Slovenia), Ricardo Soto (Chile), and Vilmar Trevisan (Brazil).

Linear Algebra and its Applications plans to publish a special issue dedicated to this ILAS conference: papers corresponding to talks given at the conference should be submitted by December 15th, 2019 via the Elsevier Editorial System (EES) , choosing the special issue SI:ILAS2019 Conference. Special editors for the ILAS 2019 issue are: Nair Abreu, Christoph Helmberg, Gitta Kutyniok, and Vilmar Trevisan.

Ongoing updates and more information about the conference will be found at `http://ilas2019.org`.

8th International Conference on Matrix Analysis and Applications (ICMAA 2019) Reno, NV, USA, July 15–18, 2019

This meeting aims to stimulate research and interaction between mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications, and to provide an opportunity for researchers to exchange ideas and developments on these subjects. The previous conferences were held in China (Beijing, Hangzhou), United States (Nova Southeastern University), Turkey (Selçuk University, Konya), Vietnam (Duy Tân University, Da Nang), and Japan (Shinshu University, Nagano Prefecture).

The organizers include: Tin-Yau Tam, University of Nevada, Nevada, USA; Qing-Wen Wang, Shanghai University, Shanghai, China; and Fuzhen Zhang, Nova Southeastern University, Florida, USA.

The registration fee is $80 for participants and the registration fee is waived for students. Participants need to confirm with Pan-Shun Lau (`panlau@connect.hku.hk`) by April 15th, 2019. Further information about the conference is available online at: `https://wolfweb.unr.edu/homepage/ttam/2019-ICMAA.html`.

9th International Congress on Industrial and Applied Mathematics (ICIAM 2019) Universitat de València, Spain, July 15-19, 2019

Organized by the Spanish Society of Applied Mathematics (SEMA) under the auspices of the International Council for Industrial and Applied Mathematics (ICIAM), ICIAM 2019 will showcase the most recent advances in industrial and applied mathematics.

Submission deadlines:

- Mini-symposia: December 10th, 2018 (extended)
- Contributed Papers: January 7th, 2019
- Financial support program to attend ICIAM 2019: February 25th, 2019
- Posters: April 1st, 2019

Further information may be found at `https://www.iciam2019.com`.

International Workshop on Operator Theory and its Applications (IWOTA 2019) Instituto Superior Técnico, University of Lisbon, Portugal, July 22–26, 2019

IWOTA 2019 will be focused on the latest developments in functional analysis, operator theory and related fields, and intends to bring together mathematicians and engineers working in operator theory and its applications, namely mathematical physics, control theory and signal processing.

Invited speakers include: Pera Ara (Spain), Joseph Ball (USA), Serban Belinschi (France), Gordon Blower (UK), Albrecht Böttcher (Germany), António Caetano (Portugal), Ana Bela Cruzeiro (Portugal), Ken Davidson (Canada), Roland Duduchava (Georgia), Ruy Exel (Brazil), Pedro Freitas (Portugal), Eva Gallardo (Spain), J. William Helton (USA), Rien Kaashoek (Netherlands), Yuri Karlovich (Mexico), Igor Klep (New Zealand), Stephanie Petermichl (France), Lars-Erik Persson (Sweden), Steffen Roch (Germany), Peter Šemrl (Slovenia), Bernd Silbermann (Germany), Orr Shalit (Israel), Frank Speck (Portugal), Ilya Spitkovsky (UAE), Christiane Tretter (Switzerland), Nikolai Vasilevski (Mexico), and Nina Zorboska (Canada).

The organizing committee consists of M. Amélia Bastos (IST, UL), António Bravo (IST, UL), Catarina Carvalho (IST, UL), Luís Castro (U. Aveiro), Alexei Karlovich (FCT, UNL), and Helena Mascarenhas (IST, UL).

Abstracts for contributed talks should be submitted before May 24th, 2019. Further information is available at `https://iwota2019.math.tecnico.ulisboa.pt/home`.

International Conference and PhD-Master Summer School on "Groups and Graphs, Designs and Dynamics" (G2D2) Three Gorges Mathematical Research Center, Yichang, China, August 12–25, 2019

The main goal of this event is to bring together researchers from different fields of mathematics and its applications mainly based on group theory, graph theory, design theory, and the theory of dynamical systems. All scientific activities will take place in the Three Gorges Mathematical Research Center at China Three Gorges University (Yichang, China) from August 12–25, 2019.

G2D2 is concerned with various aspects of mathematics, especially those relating to simple structures and simple processes. Four minicourses and four colloquium talks will let participants see order and simplicity from possibly new perspectives and share insights with experts. We will also schedule invited talks (45 minutes) and contributed talks (20 minutes) with topics ranging from coding theory, design theory, ergodic theory, graph theory, group theory, matrix theory, optimization theory, and quantum information theory to symbolic dynamics.

The scientific committee consists of: Peter Cameron, University of St. Andrews; Genghua Fan, Fuzhou University; Tatsuro Ito, Anhui University; Alexander Ivanov, Imperial College London; Ilia Ponomarenko, St. Petersburg Department of Steklov Institute of Mathematics; Zhiying Wen, Tsinghua University; Qing Xiang, University of Delaware; Mingyao Xu, Peking University; and Xiangdong Ye, University of Science and Technology of China.

Selected papers based on talks from G2D2 will be published in a special issue of *The Art of Discrete and Applied Mathematics*. For further information, see `http://math.sjtu.edu.cn/conference/G2D2`.

International Conference on Matrix Analysis and its Applications (MAT-TRIAD 2019)
Liblice, Czech Republic, September 8–13, 2019

MAT-TRIAD provides an opportunity to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications to other areas of science. Researchers and graduate students interested in recent developments in matrix and operator theory and computation, spectral problems, applications of linear algebra in statistics, statistical models, matrices and graphs, as well as combinatorial matrix theory are particularly encouraged to attend.

The invited speakers are: Dario Bini, University of Pisa, Italy; Mirjam Dür, University of Augsburg, Germany; Shmuel Friedland, University of Illinois, Chicago, USA (The Hans Schneider ILAS Lecturer); Arnold Neumaier, University of Vienna, Austria; Martin Stoll, Technical University of Chemnitz, Germany; and Zdeněk Strakoš, Charles University, Prague, Czech Republic. There will also be two invited talks by recipients of Young Scientists Awards from MatTriad'2017: Álvaro Barreras, Universidad Internacional de La Rioja, Spain; and Ryo Tabata, National Institute of Technology, Fukuoka, Japan.

The scientific committee consists of Tomasz Szulc, chair (Poland); Natália Bebiano (Portugal); Ljiljana Cvetkovič (Serbia); Heike Faßbender (Germany); and Simo Puntanen (Finland).

Important dates:

- Special session proposal deadline: January 31, 2019
- Registration and abstract submission deadline: May 31, 2019

Contact the organizers at: `mattriad@math.cas.cz` or get further information at: `http://mattriad.math.cas.cz`.

Householder Symposium XXI on Numerical Linear Algebra
Selva di Fasano, Italy, June 14–19, 2020

The next Householder Symposium will be held from June 14–19, 2020 at Hotel Sierra Silvana, Selva di Fasano (Br), Italy.

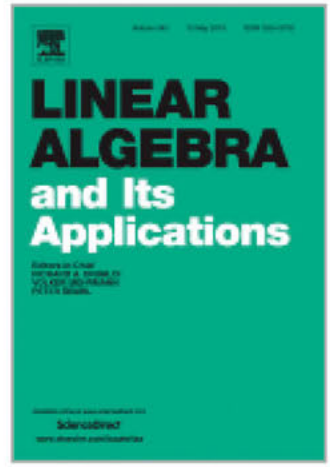
Attendance at the meeting is by invitation, and participants are expected to attend the entire meeting. Applications are solicited from researchers in numerical linear and multi-linear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control. Each attendee is given the opportunity to present a talk or a poster. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. Applications are due by October 31, 2019.

This meeting is the twenty-first in a series, previously called the Gatlinburg Symposia, but now named in honor of its founder, Alston S. Householder, a pioneer of numerical linear algebra. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1, 2017 will be presented. Nominations are due by January 31, 2020.

Journal Birthday Celebration

Celebrating the Golden Anniversary of Linear Algebra and its Applications

A note from the Editors-in-Chief



The year 2018 marks the golden anniversary of *Linear Algebra and its Applications* (LAA). LAA was first published 50 years ago. It was the first journal devoted to linear algebra and indeed was instrumental in defining linear algebra as a serious field of study and promoting its development and application. Now, there are other journals devoted to linear algebra in its pure, applied, and numerical/computational forms.

A description of the early history of LAA can be found in the article "**LAA is 40 years old**" volume 428 (2008), 1–3), written by the then editors-in-chief, Richard A. Brualdi, Volker Mehrmann, and Hans Schneider.

Linear algebra touches almost all parts of mathematics: analysis, classical algebra, number theory, combinatorics, graph theory, information theory, geometry, operator theory, … and, in turn, these different parts of mathematics have an impact on the development of linear algebra providing new directions of study.

Since its inception, LAA has been published by Elsevier which has been very supportive and very responsive to our editorial concerns. The breadth, significance, and applicability of linear algebra can be seen in every volume of LAA. We look forward to its continued growth and LAA's involvement in it.

Richard A. Brualdi
Volker Mehrmann
Peter Semrl

Read the most downloaded articles from *Linear Algebra and its Applications* in the last 90 days by visiting: **www.elsevier.com/locate/laa**.

Linear Algebra and its Applications is affiliated with the International Linear Algebra Society.

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 58-2(b), 58-4 and 60-3. Solutions are invited to Problems 59-2(iv), 59-5, 60-1, 60-2, 60-4, and for all of the problems of issue 61.

Problem 58-2: A Matrix Exponential Problem
Proposed by Moubinool Omarjee, *Lycée Henri IV, Paris, France*, ommou@yahoo.com

Let A be an n by n matrix with complex entries. The matrix exponential e^A is defined using the usual power series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

We say that an n by n matrix A has the entrywise exponential property if the (i, j)-entry of e^A is $e^{a_{ij}}$ for all $1 \leq i, j \leq n$.

(a) Show that an n by n skew-Hermitian matrix has the entrywise exponential property only if $n = 1$.

(b) Show that an n by n Hermitian matrix has the entrywise exponential property only if $n = 1$.

(c) Show that no two by two matrix has the entrywise exponential property.

Editor's note: The conjecture that an n by n matrix has the entrywise exponential property only if $n = 1$ is an open problem. The solutions to parts (a) and (c) by Eugene Herman were published in issue 59.

Solution 58-2b by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Our result uses the Hadamard determinant inequality which states that if M is an n by n positive definite matrix, then $\det(M) \leq \prod_{k=1}^{n} m_{kk}$, with equality if and only if M is a diagonal matrix. Now suppose A is an n by n Hermitian matrix with the entrywise exponential property. Then e^A is a positive definite matrix which (if $n \geq 2$) is not a diagonal matrix since all its entries are nonzero. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A. Then

$$e^{\mathrm{Tr}(A)} = e^{\Sigma_{k=1}^{n} \lambda_k} = \prod_{k=1}^{n} e^{\lambda_k} = \det(e^A) \leq \prod_{k=1}^{n} e^{a_{kk}} = e^{\Sigma_{k=1}^{n} a_{kk}} = e^{\mathrm{Tr}(A)}.$$

Since the left-hand and right-hand sides are equal, Hadamard's determinant inequality applied to e^A must give an equality which only occurs when $n = 1$.

Problem 58-4: Characteristic Polynomials of 3 by 3 Real Correlation Matrices
Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

A real correlation matrix is a positive semidefinite matrix all of whose entries are real and all of whose main diagonal entries are equal to one. Let A be a three by three real correlation matrix and let $p(x)$ be the characteristic polynomial of A. Show that if A is a convex combination of real rank one correlation matrices then there exists $a \geq 0$ such that the polynomial $q(x) = \int_a^x p(t)dt$ has all four of its roots real and nonnegative.

Solution 58-4 by the proposer

The four rank one real three by three correlation matrices are $\{v_k v_k^T\}_{k=1}^4$ where $v_1 = (1, 1, 1)^T$, $v_2 = (1, 1, -1)^T$, $v_3 = (1, -1, 1)^T$, and $v_4 = (1, -1, -1)^T$. Note that the three by three identity matrix I_3 satisfies $I_3 = \frac{1}{4}\sum_{k=1}^4 v_k v_k^T$. Let A be a convex combination of real rank one correlation matrices. Then there exist nonnegative constants $\{\lambda_k\}_{k=1}^4$ such that $A = \sum_{k=1}^4 \lambda_k v_k v_k^T$. Then for all real x, $xI_3 - A = \sum_{k=1}^4 (\frac{1}{4}x - \lambda_k) v_k v_k^T$. Let M be the three by four matrix whose columns are v_1, v_2, v_3 and v_4 and let $D_x = \mathrm{diag}(\frac{1}{4}x - \lambda_1, \frac{1}{4}x - \lambda_2, \frac{1}{4}x - \lambda_3, \frac{1}{4}x - \lambda_4)$. Then $xI_3 - A = MD_xM^T$. Let M_k be the three by three matrix formed from deleting the kth column of M, and note that the absolute value of the determinant of M_k is always four. (We can obtain this either by direct calculation or by using the determinant formula for the area of a triangle in terms of its vertices.) Taking the determinant of both sides and applying the Cauchy-Binet Theorem, we get $p(x) = \det(xI_3 - A) = \sum_{k=1}^4 \det(M_k)^2 \prod_{j \neq k}(\frac{1}{4}x - \lambda_j) = \frac{1}{4}\sum_{k=1}^4 \prod_{j \neq k}(x - 4\lambda_j)$. Let $q(x) = \frac{1}{4}\prod_{j=1}^4 (x - 4\lambda_j)$. Then $q(x)$ has all roots real and nonnegative and $q'(x) = p(x)$ so $q(x) = \int_a^x p(t)dt$ where a is the smallest root of $q(x)$.

Problem 60-3: Distance in the Commuting Graph

Proposed by Bojan Kuzma, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Let $n \geq 3$ and $\mathbb{F}$ be an arbitrary field and let A and B be nonscalar matrices in $M_n(\mathbb{F})$ with $A^2 = A$ and $B^2 = 0$. Show that there exists a nonscalar matrix which commutes with both A and B.

Editor's Note: It was pointed out by Roger Horn that $n \geq 3$ needs to be added for this result to be true. I regret the oversight. We present Dr. Horn's counterexample for $n = 2$, and then a solution we received from Eugene Herman with the $n \geq 3$ condition added is presented. This does provide another interesting example where the 2 by 2 case is special in matrix and operator theory results. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A^2 = A$ and $B^2 = 0$. A computation reveals that

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

commutes with B if and only if $z = 0$ and $w = x$. Another computation reveals that

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$$

commutes with A if and only if $y = 0$. Therefore, the only matrices that commute with both A and B have the form

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = xI,$$

that is, they can only be scalar matrices.

Solution 60-3 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

In what follows, the symbol $\oplus$ denotes the direct sum of subspaces. For a linear transformation M, denote its kernel and range by $K(M)$ and $R(M)$, respectively. A subspace S is said to be an invariant subspace of M if $x \in S$ implies that $Mx \in S$.

We claim that $A^2 = A$ implies

$$\mathbb{F}^n = K(A) \oplus K(I - A).$$

This is true since $K(A) \cap K(I - A) = \{0\}$ and, for any $v \in \mathbb{F}^n$, $v = (I - A)v + Av$ where $(I - A)v \in K(A)$ and $Av \in K(I - A)$. Next, $B^2 = 0$ implies that $R(B) \subseteq K(B)$. We also claim that if $\mathbb{F}^n = K(B) \oplus X$ for some subspace X of $\mathbb{F}^n$, then $\dim(X) = \dim(R(B))$ and the restriction of B to X is a linear isomorphism from X onto $R(B)$. The first part of the claim follows from the rank-nullity theorem and the fact that the dimensions of complementary subspaces sum to the dimension of the whole space. That the restriction of B to X is injective follows from the fact that $K(B) \cap X = \{0\}$. The restriction is surjective since the dimensions of its domain and codomain are equal.

We will use the following sufficient conditions (which are also necessary).

Lemma. (i) If $C \in M_n(\mathbb{F})$ has $K(A)$ and $K(I - A)$ as invariant subspaces, then $AC = CA$.
(ii) Let $\mathbb{F}^n = K(B) \oplus X$ where X is a subspace of $\mathbb{F}^n$ and let B' denote the restriction of the transformation B to X. If $C \in M_n(\mathbb{F})$ has $K(B)$ and $R(B)$ as invariant subspaces and if the restriction C' of C to X satisfies $BC' = CB'$, then $BC = CB$.

Proof. (i) By the above displayed formula, it suffices to prove that $AC = CA$ on each of the subspaces $K(A)$ and $K(I - A)$. If $v \in K(A)$, then $Av = 0$, and so $CAv = 0$. Also $Cv \in K(A)$, and so $ACv = 0$. If $v \in K(I - A)$, then $Av = v$ and so $CAv = Cv$. Also $Cv \in K(I - A)$, and so $Cv = ACv$.
(ii) It suffices to prove $BC = CB$ on each of the subspaces $K(B)$ and X. If $v \in K(B)$, then $Bv = 0$ and so $CBv = 0$. Also $Cv \in K(B)$ and so $BCv = 0$. If $v \in X$, then $BC'v = CB'v$ and so $BCv = CBv$.

Using the lemma, we can restate the problem as:
Given (i) nonzero subspaces U and V of $\mathbb{F}^n$ such that $\mathbb{F}^n = U \oplus V$; (ii) nonzero subspaces W, W', and X of $\mathbb{F}^n$ such that

$$W' \subseteq W, \quad \mathbb{F}^n = W \oplus X, \quad \text{and} \quad \dim(W') = \dim(X);$$

and a linear isomorphism $M : X \to W'$, prove that there exists a nonscalar linear transformation C on $\mathbb{F}^n$ with invariant subspaces U, V, W, and W' which satisfies

$$C' = M^{-1}CM,$$

where C' is the restriction of C to X.

We can show that this is equivalent to the requirement that $W' \cap U$, $W' \cap V$, $W \cap U$, and $W \cap V$ are invariant subspaces of C, while $X \cap U$ and $X \cap V$ are invariant subspaces of C'.

Since $C' = M^{-1}CM$, $X \cap U$ and $X \cap V$ being invariant subspaces of C' is equivalent to $M(X \cap U)$ and $M(X \cap V)$ being invariant subspaces of C.

We can restate the requirements on C as having the following invariant subspaces: $S \cap U$, $T \cap U$, $S \cap V$, $T \cap V$, $W \cap U$, and $W \cap V$, where $S \oplus T = W' \subseteq W$ and $\mathbb{F}^n = U \oplus V$. We now have two cases:

Case (1). Suppose $W' = W$. Then $\dim(W') = n/2 \geq 2$. If $W' \cap U$ and $W' \cap V$ are both nonzero, then at least one of $S \cap U$ and $T \cap U$ is nonzero and at least one of $S \cap V$ and $T \cap V$ is nonzero. Define C to be the identity on one of the nonzero subspaces and zero on the other three subspaces. If one of $W' \cap U$ and $W' \cap V$ is zero, we may assume by symmetry that $W' \cap V$ is zero. Hence,

$$W' = (W' \cap U) \oplus (W' \cap V) = W' \cap U = (S \cap U) \oplus (T \cap U)$$

If both $S \cap U$ and $T \cap U$ are nonzero, define C as above. If one of these two subspaces is zero, then the other (say $S \cap U$) has dimension at least two. So we can define C to be the identity on a nonzero proper subspace of $S \cap U$ and zero on a complementary subspace.

Case (2). Suppose $W' \subsetneq W$. Then $W = W' \oplus Z$ for some nonzero subspace Z of W. At least one of $W' \cap U$ and $W' \cap V$ is nonzero. Hence, at least one of $S \cap U$, $T \cap U$, $S \cap V$, and $T \cap V$ is nonzero. Define C to be the identity on one of the nonzero subspaces and zero on the other three subspaces. Also, at least one of $Z \cap U$ and $Z \cap V$ is nonzero. Define C to be zero on both subspaces.

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce four new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 61-1: Ranks of Stacked Permutation Matrices
Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let $\{P_j\}_{j=1}^{m!}$ be the $m \times m$ permutation matrices listed in any order. Furthermore, let $\text{vec}(P_j)$ denote the $m^2 \times 1$ matrix obtained by stacking the m columns of P_j into a single column with the leftmost column of P_j being at the top of $\text{vec}(P_j)$ and the rightmost column being at the bottom. Moreover, let Q be the $m^2 \times m!$ matrix whose jth column is $\text{vec}(P_j)$, and let R be the submatrix of Q consisting only of the columns of Q formed from the permutation matrices which happen to be symmetric. Find the ranks of Q and R.

Problem 61-2: Circulant Matrices over a Finite Field
Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let p be a prime number and C be a $p \times p$ circulant matrix over the finite field with p elements. Show that the determinant of C is also an eigenvalue of C. (Recall: A $p \times p$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \mod p$.)

Problem 61-3: Square Roots of Matrix Conjugation
Proposed by Dijana ILIŠEVIĆ, *University of Zagreb, Croatia*, ilisevic@math.hr
and Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Entrywise complex conjugation on $M_n(\mathbb{C})$ is a real-linear operator. Show it has a real-linear square root if and only if n is even. Find an explicit formula for at least one of its real-linear square roots.

Problem 61-4: The Sarrus Number of a Positive Semidefinite Matrix
Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Sarrus' Rule is a quick way of calculating the determinant of a 3×3 matrix:

$$\det(A) = \left(\sum_{k=0}^{2} \prod_{i=1}^{3} a_{i,k+i} \right) - \left(\sum_{k=0}^{2} \prod_{i=1}^{3} a_{i,k-i} \right),$$

where A is a 3×3 matrix and $k + i$ and $k - i$ are taken mod 3.

For general $n \times n$ matrices, we can define the Sarrus number analogously:

$$\text{Sar}(A) = \left(\sum_{k=0}^{n-1} \prod_{i=1}^{n} a_{i,k+i} \right) - \left(\sum_{k=0}^{n-1} \prod_{i=1}^{n} a_{i,k-i} \right),$$

where A is an $n \times n$ matrix and $k + i$ and $k - i$ are taken mod n. When $n \neq 3$, the Sarrus number and the determinant of a matrix may no longer be equal. Show, however, that the Sarrus number of a positive semidefinite matrix is always nonnegative.

Solutions to Problems 58-2(b), 58-4 and 60-3 are on pages 40–42.