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FEATURE INTERVIEW

“Mathematics is beautiful and one of the great human achievements”

Helene Shapiro Interviewed by Frank Uhlig¹

F.U. - What about your math “history”? How did you get into math? Why Caltech? How did you come to be Olga’s Ph.D. student?

H.S. - In ninth grade geometry I loved the idea of a proof, with each step justified by previously established results or basic logical reasoning. My first-year classes at Kenyon College were much more abstract and theoretical than anything I had seen before and really drew me into mathematics. I think the Moore method abstract algebra course in my second year showed me I wanted to be a mathematician.



Helene Shapiro

I started graduate school at Princeton, but that program was more for students who had already taken basic graduate courses and were ready to specialize. After a few months, I wanted to transfer and contacted Caltech, where I had been accepted before. I talked to Robert Dilworth, the graduate student chair. He suggested I finish the year at Princeton and then transfer, but I wanted to get on with things, so in February I started at Caltech. The second year, I took a matrix theory course with Olga Taussky-Todd (OTT). It was not standard core linear algebra, but topics Olga had worked on, often with interesting links between matrix theory and other fields. I asked OTT if she would take me as a thesis student. She later told me this was a surprise – she said I looked sullen in her class and she thought I was not enjoying it.

OTT took good care of me. She could always refer me to a relevant paper – I was amazed at her knowledge of the research literature going back for decades. She told me to read *LAA* and *LAMA*. I read a paper [8] by Watters on simultaneous quasi-diagonalization of matrices, related to Frobenius’s theorem on simultaneous triangularization of commuting matrices (OTT had included this in her course) and realized the Watters result could be generalized. Then I was interested in the fact that the

field of values of a 2×2 matrix was an ellipse and the foci corresponded to the eigenvalues. OTT told me to look up the papers of Kippenhahn [1] and Murnaghan [4]. These involved the characteristic polynomial of a pencil generated by a pair of Hermitian matrices; OTT had worked on this with Motzkin in their work [2, 3] on the L property. Eventually I focused on a conjecture in Kippenhahn’s paper. My results could settle the conjecture up to the 4×4 case, and OTT wanted me to do 5×5 . I did a messy calculation for that, hoping it might lead to a counterexample. But it did not. Several years later, Waterhouse looked at this problem and showed Kippenhahn’s conjecture is not true [7].

I had loved my four years at Kenyon and hoped for a job at an undergraduate liberal arts college. Swarthmore offered me one, but the timing was too late; I had already accepted a postdoc at the University of Wisconsin–Madison. (I have OTT and Hans Schneider to thank for this.) Hans Schneider was very good to me and we taught a graduate matrix theory course together. From Hans Schneider I learned about the Weyr characteristic and about the connections between directed graphs and the Perron-Frobenius Theorem for nonnegative matrices. Then I heard from Swarthmore; they were hiring again. I figured most of my second year at Wisconsin would be eaten up with another job search, so I applied again to Swarthmore and started there the next year, 1980, and stayed until retiring in 2014.

F.U. - What do you see as the main value (if there is one) of a mathematical education today?

H.S. - First, I would give the same reasons we study art, music, literature, philosophy, science, or any branch of learning and human creation: Mathematics is beautiful and one of the great human achievements. In mathematics one learns to construct a chain of logical reasoning and express it in exact language and notation. There is the abstractness and generality – mathematical tools are so powerful because they apply to so many situations. Problems are stripped down to their essential features and a single mathematical equation or abstract structure can apply to a variety of applications. For example, the same differential equation can model a variety of situations. The rule for matrix multiplication that represents function composition is also meaningful when you have a zero-one matrix representing a graph.

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Of course, most students take math because it is indispensable for science, engineering, or any endeavor that depends on quantitative analysis. Many take math courses to satisfy requirements, and seem more concerned about getting credentials (a math major or minor) than enjoying the subject. They may view math as a bunch of computational procedures, and don't want to spend time and effort on something hard unless they know how they will use it. But one cannot predict exactly what will be useful.

F.U. - What kind of math are/were you most interested in? Which classes did you teach, which did you like most in teaching? What was fun and what was dreadful?

H.S. - At first, what I wanted to teach most were the abstract algebra courses. I got to do this, along with lots of calculus and elementary linear algebra classes. I also liked topics courses in coding theory, combinatorial matrix theory, and advanced linear algebra. After a while, I started teaching courses I didn't know much about, such as ordinary and partial differential equations. I was at an undergraduate college, and sometimes the people better suited to teach these courses were away on leave, or did not want the course. I liked teaching courses where I was learning something – the truth is that I prefer being a student to teaching. What I enjoyed most about teaching was learning new things and preparing lectures, trying to be clear and well organized, and to strike the appropriate balance between theory and example. Grading exams and homework papers could be painful; most classes had some students who wrote up their work carefully, but sometimes I felt I was spending a lot of time writing comments on a paper that had been hastily dashed off and that the student was not going to look at anyway.

What I came to dread in my job as a faculty member were the job searches. We hired a steady stream of leave replacements; almost every year we did a search with hundreds of applicants, dozens of interviews at the Meetings, and then on-campus interviews which were all-day affairs. I deeply resented this enormous time drain. The institution saves money by hiring non-tenure-track people at junior-level salaries rather than adding tenure-track positions, but Swarthmore is a very well endowed college. Then there are the committees – I usually avoided the really time-consuming ones, so I should not complain. But I felt faculty had little say in decisions about the number of tenure-track positions, amount of classroom and office space, etc. The board and administration control funding. Growth in the size of the administration is another pet peeve of mine – more vice presidents, more deans, more consultants. And the growth of the assessment-industrial complex – these assessment exercises are often another waste of faculty time. I'm glad to be retired and not have to deal with this anymore.

The combination of teaching and administrative stuff seemed to eat up my time and energy. Many at small colleges do keep their research going, but this was a struggle for me. I did publish a few things, including with coauthors. I am pleased with two expository papers [5, 6] I wrote, and have Roger Horn to thank for them. He asked me to contribute to a special issue of *LAA* on canonical forms and I did that survey paper on unitary similarity. The Weyr characteristic was part of the story. Several years later he asked me for something for the *Monthly*, so I suggested the Weyr characteristic and canonical form. It seemed to me that Weyr's work was little known, yet the key ideas had been re-found and used by people in numerical linear algebra.

F.U. - Have you adjusted your teaching over the years? Failures and successes? Advice?

H.S. - I gained a better sense of where the students were and which things confused them. In most classes, I lectured; I felt I could organize a coherent lecture, but was not good at directing productive class discussion or encouraging group work. I felt like a fraud asking students to work in groups, because I want to be left alone to figure things out by myself. In the abstract algebra class, I eventually tried the Moore method, because I had loved my two Moore method classes as a student. I worried there would be times when no one would be ready to present anything, so I assigned each section to a group of students and told them they were responsible. Overall, it worked out well. Some students clearly worked only on specific things assigned to them, but I figured the really interested ones would work on everything.

Fads pass through the teaching business and people can feel pressured to adopt the latest one, but I think different approaches work for different students. Some like lectures, some figure it out on their own from the text and others want to talk things out with teachers and classmates. People have different teaching styles and it is good to have variety. Some give beautiful, well-organized lectures, others are very skilled at interacting with the class, and some people are effective and inspiring to their students even if the actual classes don't seem that great. I had a lot of good teachers who had varied teaching styles, but they were all passionately interested in their subject and wanted to communicate that.

F.U. - How do you feel about online teaching?

H.S. - It's hard to say. It's great that excellent presentations and lectures are easily available online. But what about the work the student needs to do? In a serious teaching situation, students submit work and get feedback and consultation from an expert. That one-on-one attention is a key part of the process. I also am suspicious that academic institutions

hope to use online teaching to cut back on faculty (and then use the money to hire more administrators).

F.U. - Were you formed, maybe de-formed, in some way by being a math teacher and researcher? How has math changed your outlook on life in general? Is there/was there a “social burden” for you being a professor?

H.S. - Well, outside of mathematics, things are rarely settled with logical argument. It can be maddening to see things done in inefficient, stupid ways because decisions are based on personal preference or political considerations rather than logic and evidence. On the other hand, mathematics can provide a great escape from the insanity of the “real world.”

Being a woman in mathematics was often uncomfortable – but things were much easier for me than for my predecessors. By the time I finished college in 1975, the women’s movement had achieved a lot. I was the only female math graduate student at Caltech, but was not made to feel that I did not belong there. I think faculty wanted to be more supportive. When I was applying for jobs, institutions were under pressure from “affirmative action” to get some women into their math and science departments. Without “affirmative action,” I think departments would have continued to dismiss applications from women, so I owe a great deal to the women before me who fought so hard for women to be accepted in fields that were traditionally dominated by men.

F.U. - What do you want to communicate to the younger and older readers of this interview?

H.S. - Oh, I want to let them know about my book! I had not expected to write a book, but wrote up notes for two upper-level courses – a second course in linear algebra and a course in combinatorial matrix theory. I had planned to use Horn and Johnson for the linear algebra course, and the book by Brualdi and Ryser for the other, but these texts didn’t fit well with the topics I wanted to cover. So I wrote up notes and then decided to put it all together and add chapters on things I liked that had not made it into the courses. I didn’t know if anyone would publish a book with this collection of topics, but decided to write what I wanted and see what happened. I was pleased the AMS accepted it as it was.

[*Editor’s note:* Helene’s book, *Linear Algebra and Matrices: Topics for a Second Course*, was reviewed in the Fall 2017 issue of *IMAGE*.]

I haven’t had a big research career, so am in no position to give advice there. I would like to pass along something OTT hammered into me: to look at the original papers. She said the great things were in the papers, rather than the later textbooks which have everything nicely tidied up. (On the other hand, I love a really well-done, elegant exposition, and this is often only possible with hindsight.) I would say to work on things you care about. But I have never taught graduate students and had to worry about their careers, so maybe this is not the right advice – I don’t know. There is so much pressure to publish and have enough when you are up for reappointment or tenure. I wonder how much of what is published will be of interest in the long run. For younger readers, I would encourage them to go after what they really want. But that’s easy for me to say, because things worked out well for me.

I have been a bit disturbed when untenured colleagues are timid about speaking up in department meetings – they seem to feel that “junior” faculty should be quiet. I did not feel that as a “junior” person; I felt that my department colleagues wanted to hear from everyone and that it was okay to speak out and disagree. But I was fortunate to be in a department that was like that.

References.

- [1] R. Kippenhahn. Über den Wertevorrat einer Matrix. *Math. Nachr.*, 6:193–228, 1951.
- [2] T. S. Motzkin and O. Taussky. Pairs of matrices with property L. *Trans. Amer. Math. Soc.*, 73:108–114, 1952.
- [3] T. S. Motzkin and O. Taussky. Pairs of matrices with property L. II. *Trans. Amer. Math. Soc.*, 80:387–401, 1955.
- [4] F. D. Murnaghan. On the field of values of a square matrix. *Proc. Natl. Acad. Sci. USA*, 18(3):246–248, 1932.
- [5] H. Shapiro. A survey of canonical forms and invariants for unitary similarity. *Linear Algebra Appl.*, 147:101–167, 1991.
- [6] H. Shapiro. The Weyr characteristic. *Amer. Math. Monthly*, 106(10):919–929, 1999.
- [7] W. C. Waterhouse. A conjectured property of Hermitian pencils. *Linear Algebra Appl.*, 51:173–177, 1983.
- [8] J. F. Watters. Simultaneous quasi-diagonalization of normal matrices. *Linear Algebra and Appl.*, 9:103–117, 1974.

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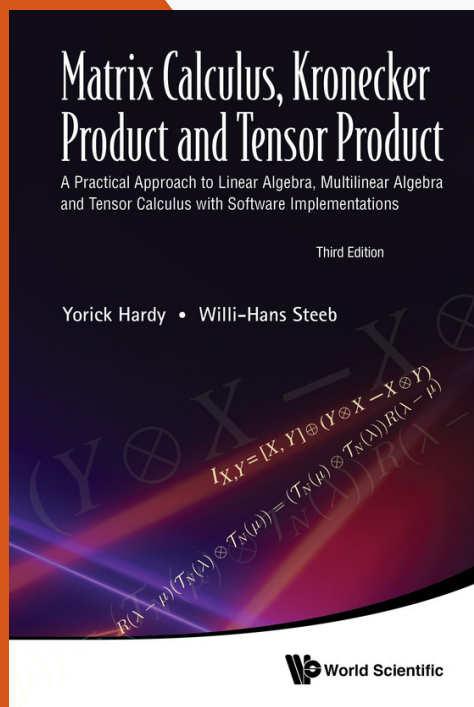
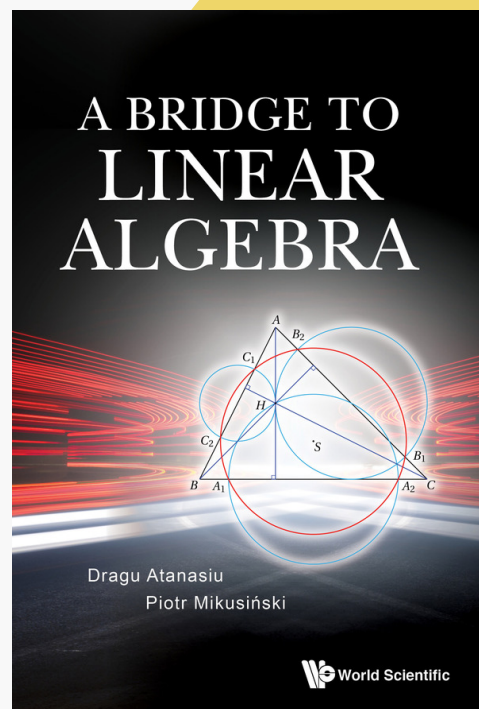
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FEATURE ARTICLE

Adjacency Spectral Theory for Uniform Hypergraphs

Gregory J. Clark, University of South Carolina, Columbia, SC, USA, gjclark@email.sc.edu
and

Joshua N. Cooper, University of South Carolina, Columbia, SC, USA, cooper@math.sc.edu

1. Adjacency spectrum of a hypergraph. A graph G is an ordered pair $G = ([n], E)$ where $E \subseteq \binom{[n]}{2}$. The *adjacency matrix* of the graph, $A(G)$, is the $n \times n$ $\{0, 1\}$ -matrix where $A(G)_{i,j} = 1$ if $ij \in E(G)$ and is zero otherwise. In this way the *adjacency spectrum* of G , denoted $\sigma(G)$, is the multiset of eigenvalues of $A(G)$; moreover, the adjacency spectrum is precisely the multiset of roots of the characteristic polynomial of $A(G)$. Henceforth we say *spectrum* to mean the adjacency spectrum, but note that there are numerous matrices associated to a graph (e.g., the adjacency matrix, Laplacian, signed Laplacian, distance, etc.). The enterprise of spectral graph theory is to associate the spectral structure of a matrix associated with a graph to combinatorial properties of the underlying graph and vice versa. Spectral hypergraph theory seeks to do the same but for hypermatrices and their underlying hypergraphs. Formally, a *k-uniform hypergraph*, abbreviated *k-graph*, is an ordered pair $\mathcal{H} = ([n], E)$ where $E \subseteq \binom{[n]}{k}$. As we will see, the spectrum of a hypergraph exhibits some phenomena different from the spectral theory of ordinary graphs. When we say “hypergraph”, we specifically mean a *k-graph* for $k > 2$, but maintain the nomenclature of *subgraph* in place of “subhypergraph”.

It is not obvious *a priori* how to define a hypergraph and its spectrum so that it analogizes the graph case. We summarize various approaches below.

1. **Matrices and linear spectral theory.** A natural approach is to encode the incidence structure of a hypergraph into a matrix and relate the eigenvalues thereof back to the structure of the hypergraph. This approach has been widely used in applied sciences (e.g., social networks, reaction and metabolic networks, protein complex networks, etc. [19]). A benefit of this approach is that one can immediately apply linear algebra to study hypergraphs, but considerable loss of information about the structure may be incurred when shoehorning the hypergraph into a two-dimensional array.
2. **Hypermatrices and low-rank decompositions.** One can define a hypermatrix to be a higher-dimensional array which maintains desirable properties of matrices. In particular, one might seek to analogize the property that the array can be decomposed into rank one hypermatrices (i.e., outer products of vectors with themselves) as in the classical Spectral Theorem from linear algebra. Important contributions to this approach include a generalization of the expander mixing lemma [14] and the considerable literature on rank-1 decomposition of tensors (e.g., [24]).
3. **Hypermatrices and polynomial maps.** Similar to the previous approach, one considers the entries of a hypermatrix as coefficients of multilinear forms. In the case where the underlying hypermatrix is *symmetric* (i.e., the entries are invariant under permutation of the indices) these multilinear forms define homogeneous polynomials. The study of symmetric tensors has led to a useful definition of the spectrum of a hypergraph as the roots of a characteristic polynomial which is the resultant of a certain polynomial system whose coefficients are drawn from the adjacency hypermatrix, as described below.

Presently, we address spectral hypergraph theory from the perspective of hypermatrices and polynomial maps. We further note that, for simplicity, several of the results herein are presented as a specific case of a more general result concerning (a certain family of) tensors. In spectral hypergraph theory literature, the nomenclature of tensor and hypermatrix are used interchangeably (as we do so in this paper) although that it is not the case in multilinear algebra.

The (*normalized*) *adjacency hypermatrix* of a *k-graph* $\mathcal{H}$, as defined in [16], is an *order k and dimension n hypermatrix*, denoted $\mathcal{A}(\mathcal{H})$, that is a collection of n^k elements $a_{i_1, i_2, \dots, i_k}$, where

$$a_{i_1, i_2, \dots, i_k} = \begin{cases} 1/(k-1)! & : \{i_1, i_2, \dots, i_k\} \in E(\mathcal{H}) \\ 0 & : \text{otherwise.} \end{cases}$$

Note that such hypermatrices are *cubical symmetric* where “cubical” refers to the property that each index has n possible values (i.e., length n in each of k dimensions). An order k dimension n symmetric hypermatrix $\mathcal{A}$ uniquely defines a

homogeneous degree k polynomial in n variables (a.k.a. a “ k -form”) by

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1, i_2, \dots, i_k=1}^n a_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

If we write $\mathbf{x}^{\otimes r}$ for the order r dimension n hypermatrix with $i_1, i_2, \dots, i_r$ entry $x_{i_1} x_{i_2} \dots x_{i_r}$ and $\mathbf{x}^r \in \mathbb{C}^r$ for the vector with i th entry $\mathbf{x}_i^r$, then the above expression can be written as

$$\mathcal{A} : \mathbf{x}^{\otimes k-1} = \lambda \mathbf{x}^{\circ(k-1)},$$

where “ $:$ ” denotes tensor contraction, “ $\otimes$ ” denotes tensor product, and “ $\circ$ ” denotes the product of hypermatrices entrywise (a.k.a the Hadamard product). For a k -graph $\mathcal{H}$ with n vertices, call $\lambda \in \mathbb{C}$ an *eigenvalue* of $\mathcal{A}(\mathcal{H})$ if there is a nonzero vector $\mathbf{x} \in \mathbb{C}^n$, which we call an *eigenvector*, satisfying

$$\sum_{i_2, i_3, \dots, i_k=1}^n a_{j i_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k} = \lambda x_j^{k-1}, \quad j \in [n].$$

Definition 1 ([41]). *The symmetric hyperdeterminant of $\mathcal{A}$, denoted $\det(\mathcal{A})$ is the resultant of the polynomials which comprise the coordinates of $\mathcal{A} : \mathbf{x}^{\otimes k-1}$, i.e., the unique irreducible monic polynomial in the entries of $\mathcal{A}$ whose vanishing detects the existence of nontrivial solutions to $\mathcal{A} : \mathbf{x}^{\otimes k-1} = 0$ (see [26]). Let λ be an indeterminate. The characteristic polynomial $\phi_{\mathcal{A}}(\lambda)$ of a hypermatrix $\mathcal{A}$ is $\phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I} - \mathcal{A})$.*

In particular, the characteristic polynomial of a k -graph with n vertices is the univariate (in λ) polynomial obtained by the resultant of the *eigenequations*, $\{p_i\}_{i \in [n]}$, denoted $\text{RES}(p_1, \dots, p_n)$, where

$$p_i = \lambda x_i^{k-1} - \sum_{\{i, j_1, j_2, \dots, j_{k-1}\} \in E(H)} x_{j_1} x_{j_2} \dots x_{j_{k-1}}.$$

For those interested in an algebraic treatment of the resultant and its generalizations we suggest Gelfand, Kapranov, and Zelevinsky [26]; and for those looking for an algorithmic approach we recommend Cox, Little, and O’Shea [18]. For simplicity, we denote the adjacency characteristic polynomial of a k -graph $\mathcal{H}$ by $\phi(\mathcal{H}) = \phi_{\mathcal{A}_{\mathcal{H}}}(\lambda)$ and write

$$\phi(\mathcal{H}) = \sum_{i=0}^t c_i \lambda^{t-i},$$

where c_i is the *codegree- i* coefficient. In this way the adjacency spectrum of $\sigma(\mathcal{H})$ of $\mathcal{H}$ is the multiset of roots of $\phi(\mathcal{H})$. We remark that $\sigma(\mathcal{H}) \subseteq \mathbb{C}$, whereas $\sigma(G) \subseteq \mathbb{R}$, since $A(G)$ is symmetric and real-valued.

We next consider an analogue of the multiplicity bound for matrices and in doing so compute the algebraic and geometric multiplicities of the eigenvalues of the 3-uniform single-edge.

2. Algebraic and geometric multiplicity. There are two different notions of the multiplicity of an eigenvalue λ of a matrix M from classical linear algebra. The *algebraic multiplicity* of λ , denoted $\text{am}(\lambda)$, is the multiplicity of λ as a root of the characteristic polynomial of M . The *geometric multiplicity* of λ , denoted $\text{gm}(\lambda)$, is the dimension of the eigenspace of λ . Famously, $\text{am}(\lambda) \geq \text{gm}(\lambda)$ for $\lambda \in \sigma(M)$. The algebraic and geometric multiplicity of a tensor (or, in our case, the normalized adjacency symmetric hypermatrix of a hypergraph $\mathcal{H}$) are similarly defined. More precisely, for $\lambda \in \sigma(\mathcal{H})$, $\text{am}(\lambda)$ is the multiplicity of λ as a root of $\phi(\mathcal{H})$ and $\text{gm}(\lambda)$ is the dimension of the affine variety $V(\lambda) = (\mathbf{x} : \mathcal{A} : \mathbf{x}^{\otimes k-1}) \subseteq \mathbb{C}^n$ (a.k.a. the *eigenvariety*). Note that eigenvarieties are not linear subspaces in general, as the eigenequations are multilinear. An analogue of the multiplicity bound for hypermatrices remains open, but the following conjecture was given in [30].

Conjecture 1 ([30]). *Let $\lambda \in \sigma(\mathcal{H})$. Then $\text{am}(\lambda) \geq \text{gm}(\lambda)(k-1)^{\text{gm}(\lambda)-1}$.*

In [30], Conjecture 1 is proved for low-rank symmetric tensors and tensors whose eigenvarieties contain certain linear subspaces. The latter result implies the well-known multiplicity bound for matrices (i.e., when $k = 2$).

As an example, we show that the 3-uniform single-edge, $\mathcal{E} = K_3^{(3)} = ([3], \{[3]\})$ satisfies Conjecture 1. The eigenequations

for $\mathcal{E}$ are

$$p_i = \lambda x_i^2 - x_j x_k, \{i, j, k\} = [3].$$

One can easily verify that $\sigma(\mathcal{E}) = \{0, \zeta_3^0 = 1, \zeta_3^1, \zeta_3^2\}$, where ζ_3 is the principal third root of unity. Observe that

$$V(0) = \text{Span}_{\mathbb{C}}\{(1, 0, 0)\} \cup \text{Span}_{\mathbb{C}}\{(0, 1, 0)\} \cup \text{Span}_{\mathbb{C}}\{(0, 0, 1)\},$$

which is clear from substituting $\lambda = 0$ into the eigenequations as

$$0 = x_2 x_3 = x_1 x_3 = x_1 x_2.$$

We further claim $V(1) = \text{Span}_{\mathbb{C}}(1, 1, 1)$ and note

$$x_1^2 = x_2 x_3 \implies x_1^4 = x_2^2 x_3^2 = (x_1 x_3) x_3^2 \implies x_1^3 = x_3^3 \implies x_1 = x_3.$$

By a similar computation we can show $x_1 = x_2 = x_3$ and in general $V(\zeta_3^t) = \text{Span}_{\mathbb{C}}(1, 1, 1)$. We have then that $\text{gm}(0) = \text{gm}(\zeta_3^t) = 1$, which satisfies Conjecture 1. From [16] we have that $\phi(\mathcal{E}) = \lambda^3(\lambda^3 - 1)^3$ so that Conjecture 1 is satisfied.

We next consider the computation of the characteristic polynomial of a hypergraph.

3. A Harary-Sachs Theorem for hypergraphs. An early, seminal result of Harary [28] (and later, more explicitly, Sachs [44]) in spectral graph theory expressed the coefficients of a graph's characteristic polynomial as certain weighted sums of the counts of various subgraphs of G .

Theorem 1 (Harary-Sachs Theorem). *Let G be a labeled simple graph on n vertices. If H_i denotes the collection of i -vertex graphs whose components are edges or cycles, and c_i denotes the codegree- i coefficient of the characteristic polynomial of G (i.e., the coefficient of λ^{n-i}), then*

$$c_i = \sum_{H \in H_i} (-1)^{c(H)} 2^{z(H)} [\#H \subseteq G],$$

where $c(H)$ is the number of components of H , $z(H)$ is the number of components which are cycles, and $[\#H \subseteq G]$ denotes the number of (labeled) subgraphs of G which are isomorphic to H .

In particular, for a graph G the Harary-Sachs Theorem yields

$$c_1 = 0, \quad c_2 = -|E(G)|, \quad \text{and} \quad c_3 = -2(\# \text{ of triangles in } G).$$

Determining the aforementioned combinatorial formulas is often given as an exercise to students in graph theory and can be done using the permutation expansion of the determinant. An analogous formula was given for the first $k+1$ coefficients of $\phi(\mathcal{H})$ for a k -graph $\mathcal{H}$ in [16].

Theorem 2 ([16]). *Let $\mathcal{H}$ be a k -graph. Then, for $1 < i < k$, $c_i = 0$,*

$$c_k = -k^{k-2}(k-1)^{n-k}|E(\mathcal{H})|,$$

and

$$c_{k+1} = -C_k(k-1)^{n-k}(\# \text{ of } K_{k+1}^{(k)} \text{ in } \mathcal{H}),$$

where C_k is some constant depending on the k , and $K_{k+1}^{(k)} = ([k+1], \binom{k+1}{k})$ is the k -uniform simplex.

The authors then computed C_k for $k \leq 5$ (n.b., $C_2 = 2$ is given by the Harary-Sachs Theorem). The present authors, in [11], show how to compute C_k explicitly, do so for $k \in [100]$, and give the asymptotic formula $C_k = \exp(k \log k(2 + o(1)))$.

Observe that the Harary-Sachs Theorem relates the spectrum of a graph and its *elementary subgraphs* (i.e., disjoint unions of edges and cycles). The present authors have generalized this theorem to hypergraphs and provided an analogous description of elementary subgraphs which we refer to as *Veblen graphs*. A Veblen graph is a k -valent (i.e., the degree of each vertex is divisible by k) k -graph for $k \geq 2$. This result allows us to compute partial information about the

characteristic polynomial of a hypergraph, even though computing the whole polynomial is computationally costly. In particular, we have generalized the Harary-Sachs Theorem to hypergraphs in the following way.

Theorem 3 ([11]). *Let $\mathcal{H}$ be a k -uniform hypergraph on n vertices. If $\mathcal{V}_i$ denotes the set of k -uniform Veblen multi-hypergraphs (i.e., all vertices have degree divisible by k), and c_i denotes the codegree- i coefficient in the characteristic polynomial of $\mathcal{H}$, then*

$$c_i = \sum_{H \in \mathcal{V}_i(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H(\#H \subseteq \mathcal{H}),$$

where $c(H)$ is the number of components of H , C_H is a certain computable coefficient of H , and $(\#H \subseteq \mathcal{H})$ is the number of particular maps of H to subgraphs of $\mathcal{H}$.

To demonstrate this theorem, we provide the first fifteen leading coefficients of the Fano Plane and the subgraphs formed by removing one and two edges, respectively, in Figure 1. We further argue that Theorem 3 is a faithful generalization of the Harary-Sachs Theorem, as Theorem 3 simplifies to the Harary-Sachs Theorem when $\mathcal{H}$ is a graph (i.e., a 2-graph). We point out that this result implies that the spectrum of a hypergraph is in principle computable from the counts of Veblen subgraphs, just as the counts of elementary subgraphs determine the spectrum of a graph. While this situation for hypergraphs is predictably more complicated than the graph case, it does directly connect the spectrum of a hypergraph to its structure.

We suspect that one can provide a sufficient spectral condition for quasirandom hypergraphs using Theorem 3. Intuitively, a hypergraph is *quasirandom* if it has the same number of copies of any particular subgraph as one would expect in a random graph (where, in its simplest form, each edge is taken with probability $1/2$). This idea was first introduced for graphs in [10] and was later extended to hypergraphs in [9], which shows that a hypergraph is quasirandom if it has approximately the expected number of even partial octahedra (as described therein). One can restate this condition in terms of the coefficients, and perhaps the spectrum itself, by showing that the linear combinations of subgraph counts appearing in the result are indeed “forcing sets” for quasirandomness.

	FP-2	FP-1	FP
c_0	1	1	1
c_1	0	0	0
c_2	0	0	0
c_3	-240	-288	-336
c_4	0	0	0
c_5	0	0	0
c_6	28320	40788	55524
c_7	0	0	-696
c_8	0	0	0
c_9	-2190860	-3788016	-6017746
c_{10}	0	0	220038
c_{11}	0	0	0
c_{12}	125012034	259553826	481293561
c_{13}	0	0	-34237560
c_{14}	0	0	-122004
c_{15}	5612445168	-13997317932	-30303162330

Figure 1: Leading coefficients of the Fano plane and its subgraphs

4. Computing the Characteristic Polynomial of a Hypergraph. Computing the resultant in general is known to be NP-hard over any field [27]. Thus, computing the characteristic polynomial of a hypergraph using traditional tools from commutative algebra is usually intractable. However, we can try to determine the characteristic polynomial of a hypergraph another way. Given the set of roots of a polynomial without multiplicity and an appropriate number of leading coefficients, one can determine the multiplicity of its roots using the Faddeev-LeVerrier Algorithm, a matrix form of the Newton Identities. In [13], the present authors provide a numerically stable algorithm for computing the multiplicity of the roots of a polynomial where the roots (without multiplicity) and some leading coefficients are known. The algorithm is stable in the sense that if an eigenvalue is approximated by an ε -disk, where ε depends “reasonably” on the parameters of the problem, then the resulting disk approximating its multiplicity contains exactly one integer. We have applied this

algorithm to compute the adjacency characteristic polynomial of various hypergraphs when traditional resultant solvers have failed to do so. In order to apply our algorithm to compute $\phi(\mathcal{H})$ we need a sufficient number of leading coefficients and the set spectrum of $\mathcal{H}$. We use Theorem 3 to determine the leading coefficients of the characteristic polynomial and appeal to the *Lu-Man Method* to determine the set of eigenvalues of $\mathcal{H}$.

The Lu-Man Method was introduced in [36] to study the spectral radius of a hypergraph and was further developed in [2], [49], etc. The method is based on the concept of an “ α -consistent labeling”, which is an assignment of complex values to every vertex-edge pair which satisfies certain conditions. We refer to the process of determining all α -consistent labelings of a hypergraph as the Lu-Man Method, and it allows us to determine the totally nonzero eigenvalues of a hypergraph. A *totally nonzero eigenvalue* of a hypergraph is a nonzero eigenvalue which corresponds to an eigenvector with all nonzero entries. We denote the set of all totally nonzero eigenvalues of $\mathcal{H}$ as $\sigma^+(\mathcal{H}) \subseteq \sigma(\mathcal{H})$. In general, using the Lu-Man Method to determine the set of all totally nonzero eigenvalues of a hypergraph can be difficult (as we have witnessed for hypergraphs with overlapping cycles). It is not hard to see that if $(\lambda, \mathbf{x})$ is a nonzero eigenpair of $\mathcal{H}$, then λ is an eigenvalue of $\mathcal{H}[\text{supp}(\mathbf{x})] \subseteq \mathcal{H}$, the subgraph of $\mathcal{H}$ induced by the support of $\mathbf{x}$ [12]. This implies that the nonzero set spectrum of a hypergraph is contained in the union of the totally nonzero eigenvalues of all of its subgraphs (c.f. Cauchy’s Interlacing Theorem).

We now compute the spectrum of the *Rowling Hypergraph* $\mathcal{R}$ (as shown in Figure 2, where a disk is centered at each root with area proportional to the root’s multiplicity), whose spectrum can be written as a union of the totally nonzero eigenvalues of isolated subgraphs. Note that not *all* induced subgraphs of $\mathcal{R}$ are isolated (e.g., the loose path on two edges) so that we have to verify that the totally nonzero eigenvalues of the two-path are not extendable in $\mathcal{R}$. Observe that the only isolated induced subgraphs of $\mathcal{R}$ are the single-edge and $\mathcal{R}$. From our discussion in a previous section, we have that the totally nonzero eigenvalues of the single-edge are the roots of $x^3 - 1$. Furthermore, by the Lu-Man Method, the totally nonzero eigenvalues of $\mathcal{R}$ are the roots of the minimal polynomials

$$m_1(x) = x^{15} - 13x^{12} + 65x^9 - 147x^6 + 157x^3 - 64, \quad m_2(x) = x^6 - x^3 + 2, \quad \text{and} \quad m_3(x) = x^6 - 17x^3 + 64.$$

We present the leading coefficients of $\mathcal{R}$ given in Figure 1, as the Rowling hypergraph is isomorphic to the Fano Plane with two edges removed. The aforementioned algorithms allow us to compute

$$\phi(\mathcal{R}) = x^{133}(x^3 - 1)^{27}(x^{15} - 13x^{12} + 65x^9 - 147x^6 + 157x^3 - 64)^{12}(x^6 - x^3 + 2)^6(x^6 - 17x^3 + 64)^3.$$

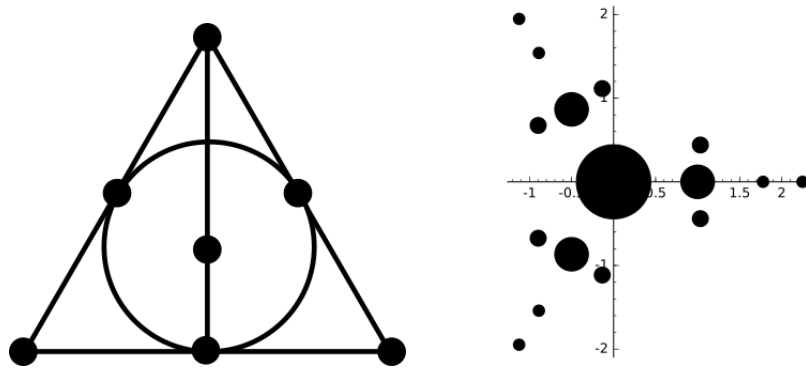


Figure 2: The Rowling hypergraph and its spectrum

In summary, we can stably compute the characteristic polynomial of a hypergraph $\mathcal{H}$ if we know its set spectrum and sufficiently many leading coefficients. We determine the set spectrum of a hypergraph via the Lu-Man Method by first considering the totally nonzero eigenvalues of its subgraphs. In particular, if an induced subgraph is not isolated, we have to determine which (if any) of its totally nonzero eigenpairs are extendable in $\mathcal{H}$. We then compute an appropriate number of leading coefficients of the characteristic polynomial by way of Theorem 3.

In the following section, we present the set spectrum of various hypergraph families.

5. The Spectra of Various Hypergraph Families.

Hypertrees

A hypertree is a hypergraph which is cycle-free in the sense that one cannot follow distinct edges and visit the same vertex twice. The following characterization of totally nonzero eigenvalues of hypertrees is provided in [49].

Theorem 4 ([49]). λ is a totally nonzero eigenvalue of a hypertree H if and only if it is a root of the polynomial

$$\varphi(H) = \sum_{i=0}^m (-1)^i |\mathcal{M}_i| x^{(m-i)r},$$

where $\mathcal{M}_i$ is the collection of all i -matchings of H .

Using this result, we characterize the spectrum of a hypertree as follows.

Theorem 5. Let $\mathcal{H}$ be a k -uniform hypertree with $k \geq 3$. Then λ is a nonzero eigenvalue of $\mathcal{H}$ if and only if there exists an induced subtree $H \subseteq \mathcal{H}$ such that λ is a root of the polynomial $\varphi(H)$.

It turns out that Theorem 5 characterizes hypertrees among 3-graphs. We further remark that Theorem 5 is not true for (2-uniform) trees (c.f. Cauchy's Interlacing Theorem), making it an unusual example of a result in spectral hypergraph theory which fails in the graph case. This peculiarity is demonstrated by *power trees*, hypertrees formed by *blowing up* the edges of a tree with new vertices.

Power Trees

A *power tree* is a k -uniform hypergraph created by adding $k - 2$ new vertices to each edge of a tree. Given a tree T , we denote its k -uniform power tree by T^k . Power trees form an interesting family for study since their eigenvalues can be determined from the underlying tree. From [50] we have that if $\lambda \neq 0$ is an eigenvalue of any subgraph of a tree T , then $\lambda^{2/k}$ is an eigenvalue of T^k for $k \geq 3$; moreover, by Theorem 5 this characterizes the eigenvalues of a power tree. In [12], the present authors extend this characterization to say that a tree is a power tree if and only if its spectrum is a subset of $\mathbb{C}[\zeta_k]$, where ζ_k is the principal k th root of unity. As an example, consider the *hummingbird hypergraph* $\mathcal{B} = ([13], E)$, where

$$E = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 8, 9\}, \{3, 10, 11\}, \{3, 12, 13\}\}.$$

Notice that $\mathcal{B}$ is not a power tree, so that its spectrum is not a subset of $\mathbb{C}[\zeta_3]$. Using the aforementioned algorithms, we have

$$\begin{aligned} \phi(\mathcal{B}) = & x^{20983}(x^9 - 6x^3 + 8x^3 - 4)^{729}(x^9 - 5x^6 + 5x^3 - 2)^{972} \\ & \cdot (x^3 - 1)^{1782}(x^6 - 4x^3 + 2)^{486}(x^9 - 4x^6 + 3x^3 - 1)^{324} \\ & \cdot (x^6 - 3x^3 + 1)^{216}(x^3 - 3)^{54}(x^3 - 2)^{119}. \end{aligned}$$

In Figure 3 we provide a plot of $\sigma(\phi_{\mathcal{B}})$ drawn in the complex plane, where a disk is centered at each eigenvalue with an area proportional to the algebraic multiplicity of its underlying root in $\phi_{\mathcal{B}}$.

Notably, starlike hypergraphs (e.g., paths and stars) are power trees. Let P_n^k denote the k -uniform (loose) hyperpath with n edges and let S_n^k denote the k -uniform hyperstar formed with n edges (here, all edges share a common vertex). In [17], the authors vastly generalize the methods of [16] by using the Poisson Product Formula to determine the multiset spectrum of S_n^k .

Theorem 6 ([17]). The characteristic polynomial for S_n^3 is

$$\phi_{S_n^3}(\lambda) = \lambda^{(2n-2)4^n} \prod_{r=0}^n (\lambda^3 - r)^{\binom{n}{r} 3^r}.$$

We briefly remark that [17] considers S_n^k as a particular case of the *sunflower*. A *sunflower* $\mathcal{S}(m, q, k)$ hypergraph is defined as follows. For $m > 0$, and q and k which satisfy $0 < q < k$, let S be a set of q vertices ("seeds") and m disjoint sets $\{E_i\}_{i=1}^m$ of $k - q$ vertices each ("petals"). The edges of the sunflower are then $S \cup E_i$ for $1 \leq i \leq m$. In this way, a single-seed sunflower is a star. The family of sunflower hypergraphs is important in extremal hypergraph theory.

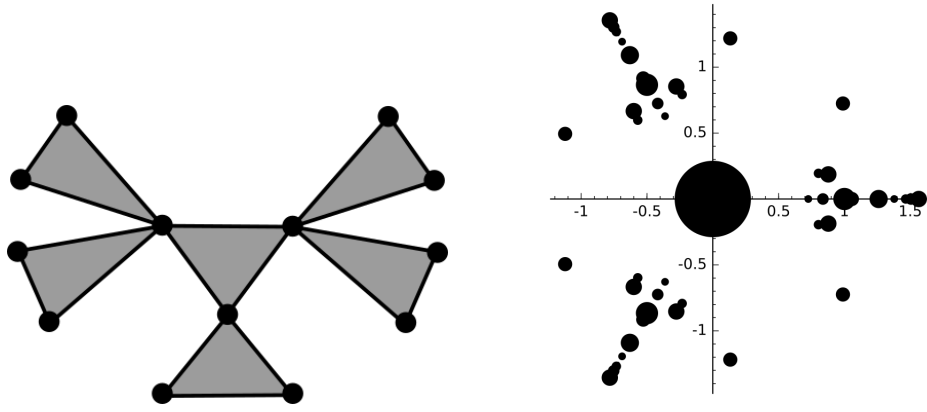


Figure 3: The hummingbird hypergraph and its spectrum

Recently, Bao et. al. [4] applied a variant of a chip-firing process to compute the resultant for starlike hypergraphs. The authors were able to provide a recursive formula for the characteristic polynomial of hyperpaths as well as k -uniform stars. Let

$$\mu_{n,k}(s) = \begin{cases} k^{s(k-2)}((k-1)^{k-1} - k^{k-2})(k-1)^{(n-s-1)(k-1)} & : s \in [0, n-1] \\ k^{n(k-2)} & : s = n. \end{cases}$$

Further define a function $g^i(x)$ in the indeterminate λ recursively by

$$g^{-1}(x) = 0, \quad g^0(x) = 1, \quad g^1(x) = g(x) = \frac{1}{1 - \frac{x}{\lambda^k}}, \quad \text{and} \quad g^i(x) = g^{i-1}(g(x)), \quad \text{for } i \geq 2.$$

For the k -uniform path with n edges we have the following.

Theorem 7 ([4]).

$$\phi(P_n^k) = \lambda^{(k-2)(k-1)^{n(k-1)}} \prod_{s=0}^n \left(\lambda - \frac{g^{s-1}(1)}{\lambda^{k-1}} \right)^{\mu_{n,k}(s)} \phi(P_{n-1}^k)^{(k-1)^{k-1}}.$$

In [4] a formula is also provided for the starlike k -uniform hypergraph formed by connecting paths at a single vertex.

Theorem 8 ([4]). *Let $S_{n_1, \dots, n_m}^k$ be the k -uniform starlike hypergraph and suppose exactly t hyperpaths length 1. Then*

$$\phi(S_{n_1, \dots, n_m}^k) = \lambda^{(m(k-2)+t)(k-1)} \sum_{j=1}^m r_j \prod_{\substack{i \in [m] \\ n_i > 1}} \phi(P_{n_i-1}^k)^{(k-1)^{k-1} + \sum_{j \neq i} r_j} \prod_{\substack{s_i \in [0, n_i] \\ i \in [m]}} \left(\lambda - \sum_{i=1}^m \frac{g^{s_i-1}(1)}{\lambda^{k-1}} \right)^{\prod_{i=1}^m \mu_{n_i, k}(s_i)}.$$

In particular, this implies that

$$\phi(S_n^k) = \lambda^{r(k-1)^r} \prod_{p=0}^m \left(\lambda - \frac{p}{\lambda^{k-1}} \right)^{\binom{m}{p} k^{(k-2)p} ((k-1)^{k-1} - k^{k-2})^{n-p}},$$

where $r = m(k-1)$.

Random and Complete Hypergraphs

A central topic in spectral graph theory is the spectra of random graphs. This area addresses questions concerning quasirandomness, graph expansion, and mixing time for Markov chains. We discuss Turán-type problems in the following section. The spectrum of an (Erdős-Rényi) random graph is well understood; however, understanding the spectrum of a random hypergraph is a more subtle matter. A simpler boundary case of this question is when each edge is chosen with probability 1. Here the resulting hypergraph is the complete k -uniform hypergraph on n vertices, $K_n^{(k)}$. One approach to examining the spectrum of a hypergraph is to show that if two hypermatrices are “close,” then they will have spectra which are “close.” Consider the normalized all-ones order n dimension k hypermatrix, $(k-1)!^{-1} \mathbb{J}_n^k$, and observe that this hypermatrix agrees with $\mathcal{A}(K_n^{(k)})$ except for tuples of indices which are not all distinct. It is shown in [15] that

there is an approximate bijection between the set of eigenvalues of $K_n^{(k)}$ and the set of eigenvalues of $(k-1)!^{-1}\mathbb{J}_n^k$ for the case of $k=2,3$. More precisely, the set of eigenvalues L of $K_n^{(k)}$ and the set of eigenvalues M of $(k-1)!^{-1}\mathbb{J}_n^k$ satisfy $\delta(L, M) = o(n^{k-1})$, where δ is the Minkowski distance. Further, [15] put forward the following conjectured analogue of the Weyl Inequality for hypergraphs, a result which would imply that the eigenvalues of $K_n^{(k)}$ and $(k-1)!^{-1}\mathbb{J}_n^k$ are asymptotically the same.

Conjecture 2 ([15]). *Suppose A and B are hypermatrices such that $\|A-B\| \leq \varepsilon$ for some norm $\|\cdot\|$ (or spectral radius) and $\varepsilon > 0$. Then there is a bijection ρ between the eigenvalues (with multiplicity) of A and B and a function f with $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$ so that $|\lambda - \rho(\lambda)| < f(\varepsilon)$ for each eigenvalue λ .*

We now present results concerning the spectrum of $K_n^{(3)}$ and $\mathbb{J}_3^n$. From [16] we have that $K_n^{(3)}$ has eigenvalues $0, 1, \binom{n-1}{2}$, and at most $2n$ others, which can be found by substituting the roots of one of the $n/2$ univariate quartic polynomials given by

$$P_t(c) = \binom{t}{2}c^4 + t(n-t-1)c^3 + \left(\binom{n-t-1}{2} - \binom{t-1}{2}\right)c^2 - (t-1)(n-t)c - \binom{n-t}{2}$$

for $t \in [n/2]$ into the quadratic polynomial

$$f_t(c) = \binom{t}{2}c^2 + t(n-t-1)c + \binom{n-t-1}{2}.$$

Furthermore, from [17] we have

$$\phi(\mathbb{J}) = \lambda^{(n-1)(k-1)^{n-1}} \prod_{\substack{\mathbf{r} \in \mathbb{N}^{k-1} \\ \mathbf{r} \cdot \hat{\mathbf{1}} = n}} \left(\lambda - (\mathbf{r} \cdot \hat{\zeta})^{k-1} \right)^{\mu(\mathbf{r})/(k-1)},$$

where $\hat{\zeta}$ denotes the vector $(1, \zeta_{k-1}, \dots, \zeta_{k-1}^{k-2})$ for some primitive k th root of unity ζ_{k-1} , $\hat{\mathbf{1}}$ denotes the all-ones vector, and $\mu(\mathbf{r})$ is the number of ways to choose $s_0, \dots, s_{m-2}$ so that

$$(\mathbf{s} \cdot \hat{\zeta})^{k-1} = (\mathbf{r} \cdot \hat{\zeta})^{k-1}.$$

More generally, let $D(n, k, p)$ denote the difference between the appropriately scaled all-ones hypermatrix and the adjacency matrix of the random k -uniform hypergraph with probability $p \in (0, 1)$. That is,

$$D(n, k, p) = p\mathbb{J}_n^k - (k-1)!\mathcal{A}(G_k(n, p)).$$

The case of $p = 1/2$ is of interest since this is the natural analogue of the Erdős-Rényi $G(n, 1/2)$ random graph. Famously, Füredi and Komlós [25] showed that all eigenvalues of a random graph are $\Theta(\sqrt{n})$ with high probability, except the largest eigenvalue, which is $O(\sqrt{n})$. A similar statement for k -uniform hypergraphs is presented in [15].

Theorem 9 ([15]). *For $D = D(n, k, p)$, $\rho(D) < Bn^{(k-1)/2}\sqrt{\log n}$ with high probability for some constant B which depends on k .*

Note that for the graph case (i.e., $k=2$) Füredi-Komlós gives $O(\sqrt{n})$, whereas the previous result gives $O(\sqrt{n} \log n)$. The question as to whether Theorem 9 is tight for $k > 2$ remains open.

Cospectral Hypergraphs

Two graphs are *cospectral*, or *isospectral*, if they have the same multiset spectrum (i.e., they have the same characteristic polynomial). In 1973, Schwenk showed that almost all trees have a cospectral mate [45]. Thirty years later, van Dam and Haemers published a survey paper on graphs which are uniquely determined by their spectrum, abbreviated *DS* [47]. In it, they suggest that it is conceivable that almost all graphs are DS. In [6], Bu, Zhou, and Wei show that complete k -uniform hypergraphs have this property (so the complete graph is DS for any uniformity). Determining which hypergraphs are DS appears to be an interesting question.

To demonstrate this, recall that the smallest pair of cospectral simple graphs is the star on five vertices and the four-cycle plus an isolated vertex. We show that the power graphs of these two graphs are no longer cospectral. Let $\mathcal{S}$ and $\mathcal{C}$ be the 3-graphs formed by adding a unique vertex to each edge in S_5 and in $C_4 \cup \{v\}$, respectively. We will show that $\mathcal{S}$

and $\mathcal{C}$ are not cospectral despite their base graphs being so. From Theorems 4 and 5 we have

$$\phi(\mathcal{S}) = x^{m_0}(x^3 - 1)^{m_1}(x^3 - 2)^{m_2}(x^3 - 3)^{m_3}(x^3 - 4)^{m_4},$$

where $m_i > 0$. Using the Lu-Man Method we obtain

$$\phi(\mathcal{C}) = x^{n_0}(x^3 - 1)^{n_1}(x^3 - 2)^{n_2}(x^3 - 4)^{n_3}$$

where $n_i > 0$. Observe that $\sigma(\mathcal{S})$ and $\sigma(\mathcal{C})$ are not equal as *sets*. Indeed $\mathcal{S}$ and $\mathcal{C}$ are not cospectral mates. We ask then, what are smallest cospectral k -graphs for $k \geq 3$?

Consider the following variants of cospectral (hyper)graphs. Two (hyper)graphs are *weakly cospectral* if their spectra are equal as sets. Moreover, when H_1 and H_2 are hypergraphs, we say that H_1 is *subspectral* to H_2 if $\phi(H_1) \mid \phi(H_2)$. Under these definitions we can restate a conjecture of [12] as follows.

Conjecture 3 ([12]). *A hypertree is subspectral to any hypertree which contains it as a subtree.*

6. The Spectral Radius. The spectral radius of a graph, namely the largest modulus of an eigenvalue of the graph, has been extensively studied and is well understood. As an example, the Perron-Frobenius Theorem implies that the spectral radius of a graph is positive, unique, and has a corresponding eigenvector which has all positive entries. The Perron-Frobenius Theorem for nonnegative tensors has been developed over the past several years.

Definition 2 ([23]). *A tensor $\mathcal{A} = [f_{i_1, \dots, i_d}] \in \mathbb{R}^{m_1 \times \dots \times m_d}$ is associated with an undirected d -partite graph $G(\mathcal{A}) = (V, E)$ where V is the disjoint union $V = \bigcup_{j=1}^d V_j$, with $V_j = [m_j]$, $j \in [d]$. The edge $(i_k, i_l) \in V_k \times V_l$, $k \neq l$ belongs to E if and only if $f_{i_1, i_2, \dots, i_d} > 0$ for some $d-2$ indices $\{i_1, \dots, i_d\} \setminus \{i_k, i_l\}$. The tensor $\mathcal{A}$ is called *weakly irreducible* if G is connected. We call $\mathcal{A}$ *irreducible* if for each proper nonempty subset $I \subset V$, the following condition holds. Let $J = V \setminus I$. Then there exist $k \in [d]$, $i_k \in I \cap V_k$, and $i_j \in J \cap V_j$ for each $j \in [d] \setminus \{k\}$ such that $f_{i_1, \dots, i_d} > 0$.*

Theorem 10. The Perron-Frobenius Theorem for Nonnegative Tensors.

1. ([48]) *If $\mathcal{A}$ is a nonnegative tensor of order k and dimension n , then $\rho(\mathcal{A})$ is an eigenvalue associated with a nonnegative eigenvector.*
2. ([23]) *Furthermore, if $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the only eigenvalue of $\mathcal{A}$ associated with a positive eigenvector and this eigenvector is unique up to a positive scalar.*
3. ([8]) *Moreover, if $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A})$ is the only eigenvalue of $\mathcal{A}$ which is associated with a nonnegative eigenvector and this eigenvector is unique up to a positive scalar.*

In [33], Keevash, Lenz and Mubayi introduce the α -spectral radius of a hypergraph to address Turán-type problems on hypergraphs. This concept was extended to the p -spectral radius by Nikiforov in [39] as follows.

Definition 3 ([39]). *For $k \geq 2$ and $p \geq 1$, the p -spectral radius of a k -graph $\mathcal{H} = (V, E)$ on n vertices is defined to be*

$$\rho_p(\mathcal{H}) = \max_{\mathbf{x} \in \mathbb{R}^n: \|\mathbf{x}\|_p = 1} k \cdot \sum_{\{i_1, i_2, \dots, i_k\} \in E} x_{i_1} x_{i_2} \cdots x_{i_k},$$

where the maximum is taken over all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$.

Note that the p -spectral radius associates with each hypergraph $\mathcal{H}$ a real-valued function ρ that is a function of p . The ‘fine-tuning’ of p can be used to tease out important structure associated with $\mathcal{H}$. For example, when $p = 1$, $\rho_1(\mathcal{H})$ is the Lagrangian as defined in [37] to address Turán-type problems, in particular the question of Turán densities in hypergraphs (e.g., [32]). A notable example is the result of Motzkin and Straus [37] wherein the authors give a spectral proof of the relationship between the Lagrangian of a graph and its clique number. The Motzkin-Straus theorem has been extended to k -graphs by Bulò and Pelillo [42] and to non-uniform hypergraphs (e.g., [40], [29]). When $p = 2$, $\rho_2(\mathcal{H})$ is the largest Z -eigenvalue (i.e., eigenvalue of an even-order tensor which has length one under the two-norm). For additional information about Z -eigenvalues, we refer the reader to [41].

From a broader perspective, the p -spectral radius is fundamental to Nikiforov’s development of an analytic theory of hypergraphs [39]. In particular, the study of the multilinear functions which arise from the eigenequations of uniform

hypergraphs appear to be amenable to analytic methods. In the following subsections we present results and open questions concerning the p -spectral radius of hypergraphs.

The Maximum p -Spectral Radius of a Hypergraph with m Edges

As in the graph case, the enterprise of providing bounds on the largest spectral radius of hypergraphs within various families has been fruitful. One natural family to consider is the collection of all graphs with m edges. In [35], Lu provides a historical account of the problem of determining the maximum of $\rho_p(\mathcal{H})$ over all k -graphs with m edges for a given p . Below we summarize some of the high points of this project.

In 1985, Brualdi and Hoffman [5] conjectured that the maximum spectral radius of a graph G with $m = \binom{s}{2} + t$ edges is attained by the graph H_m consisting of a complete graph K_s and an additional vertex with t edges. Two years later, Stanley [46] affirmed the conjecture for the case when $m = \binom{s}{2}$ and Rowlinson [43] resolved the conjecture completely in the following year. Shortly thereafter, this question was extended to hypergraphs by the so-called *Lagrangian of $\mathcal{H}$* , denoted by $\mu(\mathcal{H}) = \frac{1}{k} \rho_1(\mathcal{H})$. Frankl and Füredi [21] conjectured that the maximum Lagrangian of a k -uniform hypergraph with m edges is realized by a k -uniform hypergraph consisting of the first m sets in $\binom{[N]}{k}$ under the colexicograph order (i.e., $A < B$ if $\max(A \Delta B) \in B$). Nearly twenty years later, Nikiforov conjectured that for a k -graph $\mathcal{H}$ with $m = \binom{s}{k}$ edges, $\mu(\mathcal{H}) \leq ms^{-k}$, which agrees with the Frankl-Füredi Conjecture in the case where s is an integer. Similar to the Brualdi-Hoffman Conjecture, the diagonal case (i.e., $k = p \geq 2$) was resolved by Bai and Lu [2] where they prove a generalization of Stanley's theorem to hypergraphs. The Frankl-Füredi Conjecture has since been resolved with the following theorem.

Theorem 11 ([35]). *Let $k \geq 2$ and $\mathcal{H}$ be a k -graph with m edges. Write $m = \binom{s}{k}$ for some real $s \geq k - 1$. We have*

$$\mu(\mathcal{H}) \leq ms^{-k}.$$

The equality holds if and only if s is an integer and $\mathcal{H}$ is the complete k -uniform hypergraph on s vertices, possibly with isolated vertices.

This theorem can further be used to prove the following (see the use of the power-mean inequality in [39]).

Theorem 12 ([35]). *For any integer $k \geq 2$, real $p \geq 1$, real $s \geq k - 1$, and k -uniform hypergraph $\mathcal{H}$ with $m = \binom{s}{k}$ edges,*

$$\rho_p(\mathcal{H}) \leq \frac{km}{s^{k/p}}.$$

Moreover, this bound is sharp and equality holds if and only if s is an integer and $\mathcal{H}$ is the complete k -uniform hypergraph possibly with isolated vertices.

We conclude our discussion by considering the case where the underlying hypmermatrix does not necessarily correspond to a hypergraph, but to an arbitrary $\{0, 1\}$ -tensor. Although we began with a discussion of the spectral radius of a graph, determining the spectral radius of an arbitrary $\{0, 1\}$ -matrix is of interest in linear algebra – and directed graph theory, particularly including its overlap with Markov chain theory. In [5], Brualdi and Hoffman also considered the question of determining the largest spectral radius over all $\{0, 1\}$ -matrices with m ones. Progress has been made on this problem for $\{0, 1\}$ -matrices (notably by Friedman [22]) and was recently extended to $\{0, 1\}$ -tensors by the following theorem of Bai and Lu.

Theorem 13 ([3]). *For any k -order $\{0, 1\}$ -tensor $\mathcal{A}$ with m ones, the spectral radius $\rho(\mathcal{A})$ (i.e., the eigenvalue of $\mathcal{A}$ with greatest absolute modulus) satisfies*

$$\rho(\mathcal{A}) \leq m^{\frac{r-1}{r}}$$

with equality if and only if $m = k^r$ for some positive integer k and $\mathcal{A}$ is equivalent to $\mathbb{J}_k^r$.

On the Computation and Application of the p -Spectral Radius

We refer to the work of Chang, Ding, Qi and Yan [7]. As spectral graph theory has become important to the study of networks, so too has spectral hypergraph theory. Hypergraphs allow one to model complex relationships among interacting agents, for example. However, trying to use tensor/hypergraph methods to maintain sensitivity to multi-party interactions comes at a great cost in computational efficiency. In [7], the authors provide a convergent numerical method to determine $\rho_p(\mathcal{H})$ for $p > 1$. In the case when $p = 1$, the algorithm provides some approximation to the Lagrangian of

hypergraphs. Their algorithm for computing the p -spectral radii of hypergraphs (abbreviated to the “CSRH” algorithm) is able to compute the p -spectral radii of hypergraphs with millions of vertices efficiently. Moreover, the algorithm gives a ranking of the vertices based on an approximate Perron-Frobenius eigenvector. Interestingly, different values of p can yield different rankings of the vertices. In particular, low values of p favor groups of vertices while higher values of p elevate the importance of individual vertices. Chang et al. apply the CSRH algorithm to rank 10,305 authors based on their publication information using $p = 2$ and $p = 12$. They further provide an analysis of the rankings and compare them to the MultiRank algorithm [38], another hypergraph ranking.

7. Spectral Symmetry. Classical Markov chain theory implies that when a nonnegative irreducible square matrix M has k eigenvalues with modulus equal to $\rho(M)$, those eigenvalues are equally distributed on the spectral circle. By the Perron-Frobenius Theorem, then, they are $\rho(M)e^{i\frac{2\pi j}{k}}$ for $0 \leq j \leq k-1$. Furthermore, the spectrum of M remains invariant under rotation by an angle of $\frac{2\pi}{k}$ in the complex plane. An elementary result of spectral graph theory is that a graph is bipartite if and only if its spectrum is symmetric about the origin. Since the spectrum of a graph is real-valued, this is equivalent to saying that the spectrum is invariant under multiplication by any second root of unity. This notion is generalized by the definition of an ℓ -symmetric spectrum as given in [8].

Definition 4. Let $\mathcal{A}$ be an k th order n -dimensional tensor, and let ℓ be a positive integer. Then $\mathcal{A}$ is called *spectral ℓ -symmetric* if

$$\sigma(\mathcal{A}) = e^{i\frac{2\pi}{\ell}} \sigma(\mathcal{A}).$$

The greatest ℓ for which $\mathcal{A}$ is ℓ -symmetric is referred to as the *cyclic index* of $\mathcal{A}$, denoted $c(\mathcal{A})$, and $\mathcal{A}$ is called *spectral $c(\mathcal{A})$ -cyclic*. In the case where $c(\mathcal{A}) = 1$, $\mathcal{A}$ is spectral 1-cyclic and is referred to as *spectral nonsymmetric*.

Definition 5. Let $k \geq 2$ and $\ell \geq 2$ be integers such that $\ell \mid k$. An k th order n -dimensional tensor $\mathcal{A}$ is called *(k, ℓ) -colorable* if there exists a map $\chi : [n] \rightarrow [k]$ such that if $a_{i_1, \dots, i_m} \neq 0$, then

$$\chi(i_1) + \dots + \chi(i_m) \equiv \frac{k}{\ell} \pmod{k}.$$

In this case, χ is called a *(k, ℓ) -coloring* of $\mathcal{A}$.

Note that when $\ell = k$, the adjacency hypermatrix of a k -graph has a (k, ℓ) -coloring if it is k -partite (a.k.a. a “ k -cylinder”) because one can take $\chi \equiv 1$ on one color class and $\chi \equiv 0$ on its complement. The following result gives an elegant combinatorial characterization of spectral ℓ -symmetric hypergraphs.

Theorem 14 ([20]). Let $\mathcal{H}$ be a connected k -uniform hypergraph. Then $\mathcal{H}$ is spectral ℓ -symmetric if and only if $\mathcal{H}$ is (k, ℓ) -colorable.

Corollary 15 ([20]). Let $\mathcal{H}$ be a k -uniform hypergraph on n vertices. If $\mathcal{H}$ is spectral ℓ -symmetric, then $\ell \mid k$. Moreover, if $\mathcal{H}$ contains a k -uniform simplex, then $\mathcal{H}$ is spectral nonsymmetric.

Let $\mathcal{H} = (V, E)$ be a k -uniform hypergraph. A *generalized power* of $\mathcal{H}$ was defined in [31]. For any integers m and s such that $m > k$ and $1 \leq s \leq \frac{m}{k}$, the *generalized power* of $\mathcal{H}$, denoted $\mathcal{H}^{m,s}$, is defined as the m -uniform hypergraph with vertex set $(\bigcup_{v \in V} \mathbf{v}) \cup (\bigcup_{e \in E} \mathbf{e})$, and edge set $\{\mathbf{u}_1 \cup \dots \cup \mathbf{u}_t \cup \mathbf{e} : e = \{u_1, \dots, u_t\} \in E\}$, where $\mathbf{v}$ denotes an s -set corresponding to v and $\mathbf{e}$ denotes an $(m - ks)$ -set corresponding to e and all those sets are pairwise disjoint.

Conjecture 4 ([20]). $c(\mathcal{H}^{m,s}) = s \cdot c(\mathcal{H})$.

8. Our Favorite Open Problems. We conclude with a list of some of our favorite open problems. Some problems are repeated from above for the convenience of the reader.

1. Prove the multiplicity bound given in Conjecture 1: For $\lambda \in \sigma(\mathcal{H})$, $\text{am}(\lambda) \geq \text{gm}(\lambda)(k-1)^{\text{gm}(\lambda)-1}$.
2. Provide an interpretation of or bounds on the multiplicity of the zero eigenvalue. In the case of a tree, one can show that the multiplicity of the zero eigenvalue is equal to the size of the largest matching, by the Harary-Sachs Theorem. Can a similar statement for hypertrees be proven using Theorem 3?

3. Provide a spectral condition equivalent to quasirandomness for hypergraphs. In particular, show that the linear combinations of subgraph counts provided by Theorem 3 are a *forcing set* for quasirandom hypergraphs, implying that the characteristic polynomial being random-like is equivalent to other formulations of quasirandomness. We refer the interested reader to [1], [20] and [34].
4. Can Theorem 3 be used to develop a permutation-like expansion for the resultant, even in specific cases (c.f. the A -resultant [26])?
5. Determine the totally nonzero eigenvalues of the Fano Plane via the Lu-Man Method (n.b. this would allow for the computation of the characteristic polynomial of the Fano Plane).
6. Provide a spectral approach to hypergraph Turán numbers. This has the potential to resolve, or at least shed light on, the famous open question of the Turán-density of $K_4^{(3)}$.
7. Prove a hypermatrix analogue of the Weyl Inequality, as in Conjecture 2, and use it to completely describe the eigenvalues of complete and random hypergraphs.
8. Determine the smallest cospectral k -graphs for $k \geq 3$.
9. What is the relationship between the automorphism group of $\mathcal{H}$, the set spectrum of $\mathcal{H}$, and the multiplicities of its eigenvalues? When the hypergraph is highly symmetric, are there relatively few distinct eigenvalues as is the case for strongly regular graphs?
10. Provide (multi)linear algebraic conditions for hypergraph embeddability into Euclidean space (for example, embedding a 3-uniform hypergraph with non-crossing flat triangles in $\mathbb{R}^3$), as in the Colin de Verdière invariant, Tutte's Spring Theorem, and Fiedler vectors.
11. What is the spectrum of the ultracube, the Cartesian power of a hyperedge? See [16].

References.

- [1] E. Aigner-Horev, D. Conlon, H. Hàn, Y. Person, and M. Schacht. Quasirandomness in hypergraphs. *Electron. J. Combin.*, 25(3):Paper 3.34, 22, 2018.
- [2] S. Bai and L. Lu. A bound on the spectral radius of hypergraphs with e edges. *Linear Algebra Appl.*, 549:203–218, 2018.
- [3] S. Bai and L. Lu. Spectral radius of $\{0, 1\}$ -tensor with prescribed number of ones. *Linear Algebra Appl.*, 558:205–235, 2018.
- [4] Y.-H. Bao, Y.-Z. Fan, Y. Wang, and M. Zhu. A Combinatorial Method for Computing Characteristic Polynomials of Starlike Hypergraphs. *arXiv e-prints*, June 2018. <https://arxiv.org/abs/1806.06199>.
- [5] R. A. Brualdi and A. J. Hoffman. On the spectral radius of $(0, 1)$ -matrices. *Linear Algebra Appl.*, 65:133–146, 1985.
- [6] C. Bu, J. Zhou, and Y. Wei. E-cospectral hypergraphs and some hypergraphs determined by their spectra. *Linear Algebra Appl.*, 459:397–403, 2014.
- [7] J. Chang, W. Ding, L. Qi, and H. Yan. Computing the p -spectral radii of uniform hypergraphs with applications. *J. Sci. Comput.*, 75(1):1–25, 2018.
- [8] K. C. Chang, K. Pearson, and T. Zhang. On eigenvalue problems of real symmetric tensors. *J. Math. Anal. Appl.*, 350(1):416–422, 2009.
- [9] F. Chung. Quasi-random hypergraphs revisited. *Random Structures & Algorithms*, 40(1):39–48, 2011.
- [10] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.
- [11] G. J. Clark and J. Cooper. A Harary-Sachs Theorem for Hypergraphs. *Submitted*.
- [12] G. J. Clark and J. N. Cooper. On the adjacency spectra of hypertrees. *Electron. J. Combin.*, 25(2):Paper 2.48, 8, 2018.

- [13] G. J. Clark and J. Cooper. Leading Coefficients and the Multiplicity of Known Roots. *arXiv e-prints*, June 2018. <https://arxiv.org/abs/1806.05222>.
- [14] E. Cohen, D. Mubayi, P. Ralli, and P. Tetali. Inverse expander mixing for hypergraphs. *Electron. J. Combin.*, 23(2):Paper 2.20, 24, 2016.
- [15] J. Cooper. Adjacency Spectra of Random and Uniform Hypergraphs. *arXiv e-prints*, July 2015. <https://arxiv.org/abs/1507.07118>.
- [16] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436(9):3268–3292, 2012.
- [17] J. Cooper and A. Dutle. Computing hypermatrix spectra with the Poisson product formula. *Linear Multilinear Algebra*, 63(5):956–970, 2015.
- [18] D. A. Cox, J. Little, and D. O’Shea. *Using Algebraic Geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2005.
- [19] E. Estrada and J. A. Rodríguez-Velázquez. Subgraph centrality and clustering in complex hyper-networks. *Phys. A*, 364:581–594, 2006.
- [20] Y.-Z. Fan, T. Huang, Y.-H. Bao, C.-L. Zhuang-Sun, and Y.-P. Li. The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs. *arXiv e-prints*, Apr. 2017. <https://arxiv.org/abs/1704.08799>.
- [21] P. Frankl and Z. Füredi. Extremal problems whose solutions are the blowups of the small Witt-designs. *J. Combin. Theory Ser. A*, 52(1):129–147, 1989.
- [22] S. Friedland. The maximal eigenvalue of 0-1 matrices with prescribed number of ones. *Linear Algebra Appl.*, 69:33–69, 1985.
- [23] S. Friedland, S. Gaubert, and L. Han. Perron-Frobenius theorem for nonnegative multilinear forms and extensions. *Linear Algebra Appl.*, 438(2):738–749, 2013.
- [24] S. Friedland, V. Mehrmann, R. Pajarola, and S. K. Suter. On best rank one approximation of tensors. *Numer. Linear Algebra Appl.*, 20(6):942–955, 2013.
- [25] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.
- [26] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.
- [27] B. Grenet, P. Koiran, and N. Portier. The multivariate resultant is NP-hard in any characteristic. In *Mathematical Foundations of Computer Science 2010*, volume 6281 of *Lecture Notes in Comput. Sci.*, pages 477–488. Springer, Berlin, 2010.
- [28] F. Harary. The determinant of the adjacency matrix of a graph. *SIAM Rev.*, 4:202–210, 1962.
- [29] Y. Hou, A. Chang, and L. Zhang. A homogeneous polynomial associated with general hypergraphs and its applications. *arXiv e-prints*, Nov. 2018. <https://arxiv.org/abs/1811.01735>.
- [30] S. Hu and K. Ye. Multiplicities of tensor eigenvalues. *Commun. Math. Sci.*, 14(4):1049–1071, 2016.
- [31] L. Kang, L. Liu, L. Qi, and X. Yuan. Some results on the spectral radii of uniform hypergraphs. *arXiv e-prints*, May 2016. <https://arxiv.org/abs/1605.01750>.
- [32] P. Keevash. Hypergraph Turán problems. In *Surveys in Combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.
- [33] P. Keevash, J. Lenz, and D. Mubayi. Spectral extremal problems for hypergraphs. *SIAM J. Discrete Math.*, 28(4):1838–1854, 2014.
- [34] L. Lovász and B. Szegedy. Finitely forcible graphons. *J. Combin. Theory Ser. B*, 101(5):269–301, 2011.

- [35] L. Lu. The maximum p -Spectral Radius of Hypergraphs with m Edges. *arXiv e-prints*, Mar. 2018. <https://arxiv.org/abs/1803.08653>.
- [36] L. Lu and S. Man. Connected hypergraphs with small spectral radius. *Linear Algebra Appl.*, 509:206–227, 2016.
- [37] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canad. J. Math.*, 17:533–540, 1965.
- [38] M. K.-P. Ng, X. Li, and Y. Ye. Multirank: Co-ranking for objects and relations in multi-relational data. In *Proceedings of the 17th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '11, pages 1217–1225, New York, NY, USA, 2011. ACM.
- [39] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457:455–535, 2014.
- [40] Y. Peng, H. Peng, Q. Tang, and C. Zhao. An extension of the Motzkin-Straus theorem to non-uniform hypergraphs and its applications. *Discrete Appl. Math.*, 200:170–175, 2016.
- [41] L. Qi. Eigenvalues of a real supersymmetric tensor. *J. Symbolic Comput.*, 40(6):1302–1324, 2005.
- [42] S. Rota Bulò and M. Pelillo. A generalization of the Motzkin-Straus theorem to hypergraphs. *Optim. Lett.*, 3(2):287–295, 2009.
- [43] P. Rowlinson. On the maximal index of graphs with a prescribed number of edges. *Linear Algebra Appl.*, 110:43–53, 1988.
- [44] H. Sachs. Über Teiler, Faktoren und charakteristische Polynome von Graphen. *Teil I*, 12:7–12, 1966.
- [45] A. J. Schwenk. Almost all trees are cospectral. In *New Directions in the Theory of Graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*, pages 275–307. Academic Press, New York, 1973.
- [46] R. P. Stanley. A bound on the spectral radius of graphs with e edges. *Linear Algebra Appl.*, 87:267–269, 1987.
- [47] E. R. van Dam and W. H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 373:241–272, 2003. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002).
- [48] Y. Yang and Q. Yang. Further results for Perron-Frobenius theorem for nonnegative tensors. *SIAM J. Matrix Anal. Appl.*, 31(5):2517–2530, 2010.
- [49] W. Zhang, L. Kang, E. Shan, and Y. Bai. The spectra of uniform hypertrees. *Linear Algebra Appl.*, 533:84–94, 2017.
- [50] J. Zhou, L. Sun, W. Wang, and C. Bu. Some spectral properties of uniform hypergraphs. *Electron. J. Combin.*, 21(4):Paper 4.24, 14, 2014.

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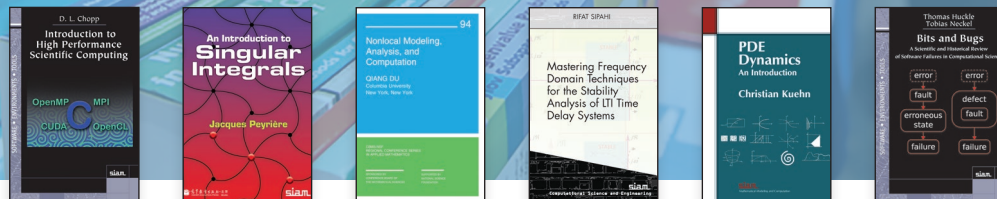
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LINEAR ALGEBRA EDUCATION

Teaching Linear Algebra Using Primary Historical Sources

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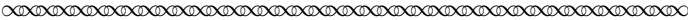
1. Primary source projects. Linear Algebra is a course that sits at the crossroads of many undergraduate STEM programs. As instructors, we often teach linear algebra oriented toward the future, using technology and real-world applications to assist in understanding, and demonstrate linear algebra’s potential utility for future scientific work. However, we may be missing an opportunity if we fail to connect the future of linear algebra to its past. Offering a glimpse of the origins of linear algebra allows students who will never take a history of math class to discover mathematics as a living, creative endeavor which is constantly developing new ideas and new methods. Sharing the evolution of an algorithm opens the door to invite students to create new and better ways to solve old problems.

A Primary Source Project (PSP) is a curriculum module designed to introduce a core topic in a standard mathematics course through the use of guided reading of original historical sources. Ideally an instructor would replace the modern textbook section on the topic with the PSP. The modules are designed to be implemented using active learning strategies, yet they are highly adaptable to any style of instruction. Questions are distributed throughout the text to help guide students through the reading and related procedures. Notes to the instructors are included to aide in lesson planning.

The purpose of this note is to make the linear algebra community aware of the PSP *Solving a System of Linear Equations Using Ancient Chinese Methods* [3]. This PSP introduces Gaussian elimination through readings from the ancient Chinese text *The Nine Chapters on the Mathematical Art* [5]. The following discussion shares the goals, content and student outcomes of this PSP as an invitation to adapt it for your own instruction.

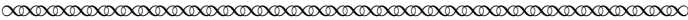
2. Introducing elimination through the Fangcheng Rule. Content of the PSP. Students may naturally assume that Gaussian elimination was invented by Carl Friedrich Gauss (1777–1855). However, Gaussian elimination has a much longer history. Using elimination to solve a system of linear equations first appears in the ancient Chinese text *The Nine Chapters on the Mathematical Art*, hereafter denoted the *Nine Chapters* for brevity. The *Nine Chapters* is an anonymous text, compiled across generations of mathematicians, that is believed to have been compiled in the first century BCE. It comprises a series of 246 problems and their solutions, organized into nine chapters by topic.

China developed a place value system in which the digits 1–9 were represented by counting rods (small bamboo sticks) placed next to each other and digits placed in columns to indicate place value. Computation was performed by physically manipulating the counting rods on a flat surface, also referred to as a counting board. The arrangement of the numerals on the counting board naturally extends to techniques of using arrays of numbers to organize the information in a problem. Chapter 8 of the *Nine Chapters* is titled *Fangcheng* which translates as *Rectangular Arrays*. It consists of 18 problems which are systems of linear equations. The problems are solved by the “Fangcheng Rule.” Below is a translation of Problem 1 and the first half of the Fangcheng Rule [5]. The part of the Fangcheng Rule printed here is equivalent to the instructions for reducing the augmented matrix for a system of equations to echelon form.



Problem 1: Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* [per bundle]; medium grade paddy $4\frac{1}{4}$ *dou*; [and] low grade paddy $2\frac{3}{4}$ *dou*.

The Array (*Fangcheng*) Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot...



The Chinese array for Problem 1 would be constructed as shown below.

1	2	3
2	3	2
3	1	1
26	34	39

Ancient Chinese was read from top to bottom and then right to left. Rotating to modern notation read from left to right then top to bottom results in the modern augmented matrix for Problem 1. The Fangcheng Rule instructions become elementary row operations in the modern orientation.

The PSP is arranged in five sections. Section 1 gives the students an overview of the history of Gaussian elimination as well as background information about the *Nine Chapters*. Section 2 discusses counting rod arithmetic, including an example of the ancient Chinese algorithm for multiplication and arithmetic with negative numbers. Section 3 guides the students through solving a system of linear equations using the Fangcheng Rule, using the Chinese array notation with modern numerals. Section 4 introduces the modern algorithm, and finally Section 5 asks students to compare the Fangcheng Rule with the modern algorithm.

Goals and student outcomes of the PSP. The main purpose of creating a curriculum module from the ancient Chinese material is to share the remarkable achievement of the ancient Chinese and introduce Gaussian elimination in a creative and interactive way. The pedagogical advantage of introducing Gaussian elimination using ancient Chinese notation is that it levels the playing field. All are presented with difficult language and unfamiliar notation, and students with less mathematical background are not disadvantaged. The PSP is also constructed for student-centered learning. One student commented that students needed to be “creative” and “open-minded” when working with the PSP.

The second aim is to provide a vehicle to discuss the importance of efficient notation and the challenges of efficient computation. The first half of the Chinese algorithm transforms the array into triangular form, which is equivalent to modern echelon form. The second half of the Fangcheng Rule explains how to solve this triangular array in a manner that is not equivalent to back substitution. However, as Hart [3] points out, the Chinese method delays the use of fractions until the very last step. Since the problems posed had nontrivial rational solutions, this aided in efficient computation. Even with modern technology, efficient numerical methods for solving large systems are still very much our concern. The PSP highlights computational differences that allow this issue to arise naturally. The visible result in the author’s class was that students were more accepting of changes to the order of row operations to ease the arithmetic.

Student learning outcomes directly attributed to the PSP in the author’s class in Fall 2017 were encouraging. Students seemed more engaged during class and reported on the end-of-course survey that their group work on the PSP had been a highlight. Another instructor using this PSP in his linear algebra class in Fall 2017 reported that his students felt very satisfied that they could solve the ancient problem and relate it to math with which they were familiar. Students were also amazed in the realization that Gaussian elimination was first developed by the ancient Chinese.

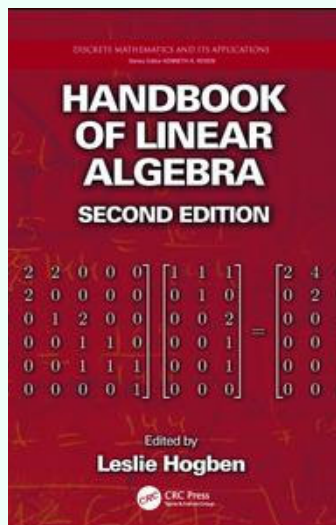
3. Conclusions. This PSP introduces Gaussian elimination via the ancient Chinese Fangcheng Rule and is suitable for interactive learning techniques. Implementation of the full PSP requires 2.5–3 weeks of instructional time. However, the essential techniques for reducing a matrix to echelon form in both ancient Chinese and modern notation may be covered in about 3 hours of instruction. The “Notes to Instructor” provided with the PSP provide a suggested syllabus schedule for both a full and a focused implementation, as well as suggestions for use in other contexts. The PSP has also been vetted by a thorough review process examining pedagogy and historical accuracy. It is freely available at https://digitalcommons.ursinus.edu/triumphs_linear/1/ for your consideration.

4. Acknowledgments. The development of this student project has been partially supported by the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) program with funding from the National Science Foundation’s Improving Undergraduate STEM Education Program under grant number 1523494. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation. For more information about TRIUMPHS, visit <https://blogs.ursinus.edu/triumphs/>.

References.

- [1] M. Flagg. *Solving a System of Linear Equations Using Ancient Chinese Methods*, available at https://digitalcommons.ursinus.edu/triumphs_linear/1/.
- [2] J. F. Gracar. Mathematicians of Gaussian elimination. *Notices Amer. Math. Soc.*, 58(6):782–792, 2011.
- [3] R. Hart. *The Chinese roots of linear algebra*. Johns Hopkins University Press, Baltimore, MD, 2011.
- [4] V. J. Katz. *A history of mathematics*. Pearson Education, New York, NY, third edition, 2009.
- [5] K. Shen, J. N. Crossley, and A. W.-C. Lun. *The Nine Chapters on the Mathematical Art*. Oxford University Press, New York; Science Press Beijing, Beijing, 1999.

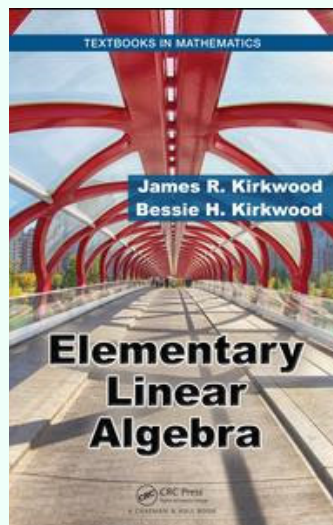
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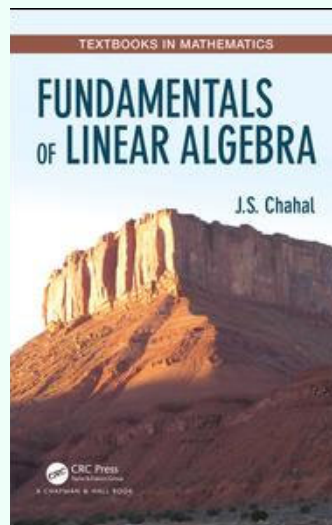
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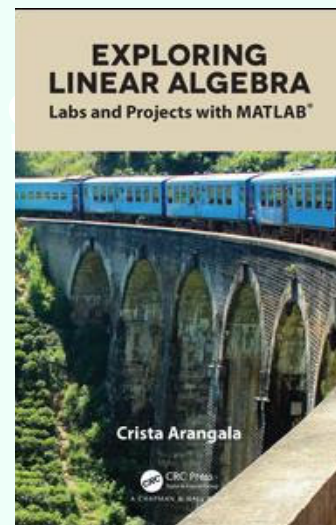
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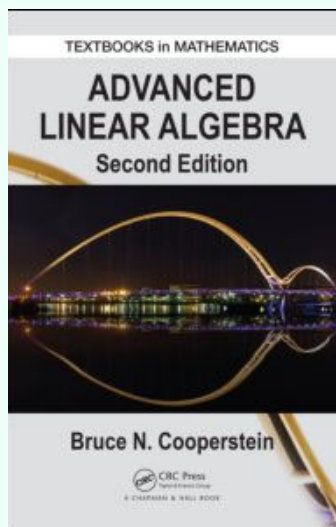
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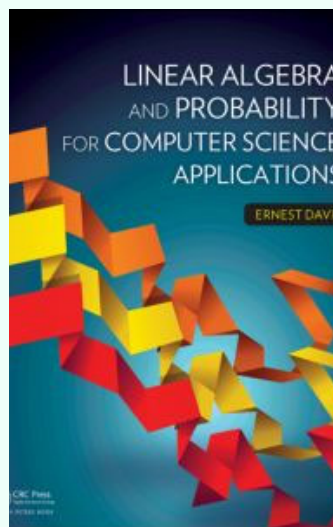
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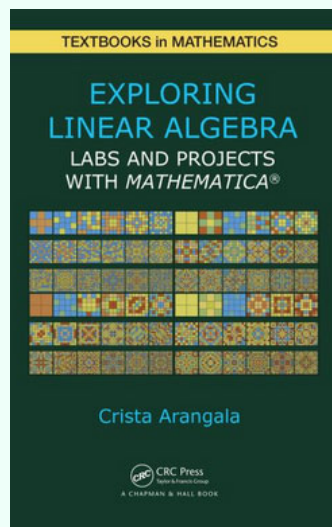
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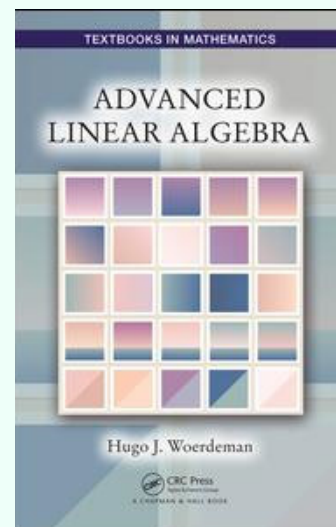
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Learning Determinants from Cramer and Cauchy: A TRIUMPHS Primary Source Project

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1. Introduction. Determinants have long been included as a central topic in the linear algebra curriculum for undergraduate mathematics, science and engineering students, and for good reason. The useful ways in which determinants appear in results both theoretical and practical, or in criteria for determining when problems are appropriately solvable are legion. But the coordinate approach to defining and working with determinants is always complicated – necessarily and irredeemably so, as it makes use of the action of the full n th order symmetric group in its formulation. Sheldon Axler’s provocative *Down with Determinants!* campaign of the 1990s rested on this lamentable fact [1]. Indeed, frustration with the complexity of the concept of the determinant is no modern phenomenon. No less a luminary than Felix Klein complained – in the 1920s! – of having to lead students through the morass of determinants:

I find that in my own elementary lectures, I have, for pedagogical reasons, pushed determinants more and more into the background. Too often have I had the experience that, while the students acquired facility with the formulas, which are so useful in abbreviating long expressions, they often failed to gain familiarity with their *meaning*, and skill in manipulation prevented the student from going into all the details of the subject and so gaining a mastery. [4, p. 166]

One way to approach the pedagogical problems presented by complicated theoretical ideas is adopted by many textbook authors: motivate the theory through engaging problems, then move carefully from the concrete to the abstract through graded examples that highlight the central properties under concern; identify useful jargon through the adoption of key definitions and notational conventions chosen for their representational power; finally, provide exercises in sufficient number for students both to practice using the notation and jargon and to explore the concepts on their own.

But there is another approach. It is encapsulated in the following famous quotation attributed to Niels Henrik Abel: “It appears to me that if one wants to make progress in mathematics one should study the masters and not the pupils.” [5, p. 138] This is the approach adopted by the present author, who has designed a classroom module for the first linear algebra course designed to teach the determinant and nearly all of its standard properties. The approach is based on student engagement with excerpts selected from two important works in the history of the development of the determinant: Gabriel Cramer’s *Introduction to the Analysis of Algebraic Curved Lines*, which contains Cramer’s Rule in his own words, and Augustin-Louis Cauchy’s masterful 1815 memoir presenting a fully developed theory of determinants for the first time, *On functions which take only two values, equal but of opposite sign, by means of transpositions performed among the variables which are contained therein*. (Both are used, of course, in English translations from the French originals.)

This classroom module, titled *Determining the Determinant*, is a Primary Source Project (PSP), one of more than 50 similar projects developed under the aegis of *TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (TRIUMPHS), a five-year NSF-funded grant initiative to design, write, test, disseminate and evaluate classroom modules that teach topics across the undergraduate mathematics curriculum based on primary historical sources (in English translation) and which incorporate active learning principles together with carefully designed student tasks to help students understand the texts they are reading. (All TRIUMPHS PSPs are freely available for download at the group’s website, <https://blogs.ursinus.edu/triumphs>.)

This note will describe the goals of the PSP *Determining the Determinant*, and how it attempts to meet those goals. Another TRIUMPHS PSP, by Mary Flagg, is described in this same issue of IMAGE; it teaches Gaussian elimination methods using readings taken from an ancient Chinese work, the *Nine Chapters on the Mathematical Art* [3].

2. Determining the Determinant: Cramer. Cramer’s *Introduction to the Analysis of Algebraic Curved Lines* [2] was one of the earliest works in algebraic geometry; in it, the author developed a theory that studied the number and types of intersections among collections of algebraic curves. In the course of determining the equation of the conic section that passed through five given points in the plane, Cramer considered the equation of a general conic,

$$A + By + Cx + Dyy + Exy + xx = 0.$$

The five pairs of coordinates of the points gave values for x and y that presented five equations, linear in the parameters A, B, C, D , and E . Thus, finding the desired equation meant solving a system of five linear equations in five unknowns. The task appeared to intimidate Cramer; he commented that “The calculation of it truly will be quite long”! He relegated the details to an appendix, wherein he considered the more general problem of finding the unique solution to a system of n linear equations in n unknowns (assuming there was one), for $n = 1, 2, 3$. His system of equations had the form

$$\begin{aligned} A^1 &= Z^1 z + Y^1 y + X^1 x + V^1 v + \& c. \\ A^2 &= Z^2 z + Y^2 y + X^2 x + V^2 v + \& c. \\ A^3 &= Z^3 z + Y^3 y + X^3 x + V^3 v + \& c. \\ A^4 &= Z^4 z + Y^4 y + X^4 x + V^4 v + \& c. \\ &\& c. \end{aligned}$$

(with superscripts instead of subscripts enumerating the constants), whence his formulas for the case $n = 3$, in the original orthography, appeared thus [2]:

$$\begin{aligned} z &= \frac{A^1 Y^2 X^3 - A^1 Y^3 X^2 - A^2 Y^1 X^3 + A^2 Y^3 X^1 + A^3 Y^1 X^2 - A^3 Y^2 X^1}{Z^1 Y^2 X^3 - Z^1 Y^3 X^2 - Z^2 Y^1 X^3 + Z^2 Y^3 X^1 + Z^3 Y^1 X^2 - Z^3 Y^2 X^1} \\ y &= \frac{Z^1 A^2 X^3 - Z^1 A^3 X^2 - Z^2 A^1 X^3 + Z^2 A^3 X^1 + Z^3 A^1 X^2 - Z^3 A^2 X^1}{Z^1 Y^2 X^3 - Z^1 Y^3 X^2 - Z^2 Y^1 X^3 + Z^2 Y^3 X^1 + Z^3 Y^1 X^2 - Z^3 Y^2 X^1} \\ x &= \frac{Z^1 Y^2 A^3 - Z^1 Y^3 A^2 - Z^2 Y^1 A^3 + Z^2 Y^3 A^1 + Z^3 Y^1 A^2 - Z^3 Y^2 A^1}{Z^1 Y^2 X^3 - Z^1 Y^3 X^2 - Z^2 Y^1 X^3 + Z^2 Y^3 X^1 + Z^3 Y^1 X^2 - Z^3 Y^2 X^1} \end{aligned}$$

He then stated (without proof) a theorem for the general case – what we now know as Cramer’s Rule:

The number of equations & of unknowns being n , we will find the value of each unknown by forming n fractions of which the common denominator has as many terms as there are diverse arrangements of n different things...

The form of the terms appearing in these denominator expressions was then described: They are products of coefficients, one for each of the variables $z, y, x, v, \dots$ of the system, and one for each of the equations of the system, which he numbered $1, 2, 3, \dots$, thereby indexing the coefficients $Z, Y, X, V, \dots$ of the respective variables. He stated that the numbers that indexed the equations corresponded precisely to “the first n numbers arranged in all possible ways.” To each term corresponded a sign, which, Cramer observed, depended on the parity of the number of derangements in the permuted listing of the index numbers. Having described the algebraic structure of the denominator expressions in his formulas, all that was left was to describe their numerators:

we will have the value of z by giving to this denominator the numerator which is formed by changing, in all its terms, Z into A [the letter used to represent the constant terms in each equation]. And the value of y is the fraction which has the same denominator & for numerator the quantity which results when we change Y into A , in all the terms of the denominator. And we find in a similar manner the value of the other unknowns.

Students see from this reading that determinants arise organically from the solutions of linear systems, and that Cramer’s work to clarify the *form* of the formulas he derived, and not just the formulas themselves, constituted the real mathematical work. (In particular, Cramer never did provide formulas for the coefficients of the conic equation problem that launched this analysis, and naturally so, as the determinants in these formulas would have involved $5! = 120$ terms each!)

3. Determining the Determinant: Cauchy.

A good deal more attention in this PSP is then given to reading selections from Cauchy’s memoir, a real *tour de force* of mathematical depth and farsighted analysis, written when the author was just 23 years of age. In the first part of this memoir, he presented the first full treatment of the structure of the symmetric group in the literature, including proofs of theorems stating that *the sign of a permutation of the first n positive integers is given by $(-1)^{n-g}$, where g counts the*

number of distinct cycles into which the permutation can be factored. But group theory is not the object of concern of this PSP. Instead, most of the excerpts selected for students to read come from the second part of the memoir, devoted to the study of alternating symmetric functions.

Cauchy defined an alternating symmetric function to be one “which may change in sign but not in value, by virtue of . . . transpositions” of the variables involved. (Recall the full title of the memoir.) For instance, he gives as two examples of alternating symmetric functions

$$f(a_1, a_2, a_3) = \sin(a_1 - a_2) \sin(a_1 - a_3) \sin(a_2 - a_3), \text{ and}$$

$$g(a_1, b_1; a_2, b_2; a_3, b_3) = a_1 b_2 + a_2 b_3 + a_3 b_1 - a_2 b_1 - a_1 b_3 - a_3 b_2.$$

He stated that he was studying what Gauss had earlier called *determinants*, which interested him partly because of their connection to Cramer’s work:

a certain kind of alternating symmetric function which present themselves in a great number of analytic investigations. It is by means of these functions that we express the general values of unknowns that many equations of the first degree contain.

After exercises in which students using the PSP are asked to check that certain expressions are indeed alternating symmetric, and to create some alternating symmetric functions of their own, we turn to Cauchy’s main objectives.

To define and describe the n th-order determinant, Cauchy used the notation $\mathbf{S}(\pm a_{1,1} a_{2,2} \cdots a_{n,n})$ to represent the anti-symmetric sum that we would write as

$$\sum_{\sigma} \text{sgn}(\sigma) \cdot a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n},$$

where the summation is over all permutations σ of the elements of the set $\{1, 2, \dots, n\}$. (For Cauchy, the symbol $\mathbf{S}(\cdot)$ was used as a symmetric summation operator when applied to algebraic terms with doubled subscripts, and the insertion of the $\pm$ sign was meant to denote the corresponding antisymmetric sum.) He recognized that the determinant involved all n^2 possible expressions a_{ij} among the $n!$ terms, and he then arranged these a_{ij} into a square array which he termed a *symmetric system*, identical to what we now call the matrix (a_{ij}) . Recall that this was written in 1815, about 35 years before Sylvester first used the word *matrix* to describe such an array!

The PSP steps back briefly from Cauchy’s memoir to take care of some topics that were not considered by Cauchy but which are standard for beginning students to master. First, after being shown the well-known “basketweave” methods for computing 2×2 and 3×3 determinants, students verify that these methods produce the same expressions that they saw in Cramer’s appendix and Cauchy’s memoir. Next, using Cauchy’s formula for the determinant, students show that the determinant of an upper or lower triangular matrix is equal to the product of its diagonal entries. Finally, after defining the elementary matrix $E[i, j : c]$ ($i \neq j$) as the matrix obtained from the identity matrix by replacing its (i, j) -entry with the value c , students check that the determinant of this matrix is c when $i = j$, and 1 otherwise.

Returning to the memoir, we pick up where we left off: Cauchy next defined what we now call the transpose of a matrix and observed that “the symmetric products of the systems are clearly equal to each other,” i.e., $\det A^T = \det A$. (Students are asked to give their own explanation of this fact.)

Cauchy then tackled the procedure we now call Laplace expansion of a determinant along a row or column:

The determinant of order $n - 1$, or D_{n-1} , will be given by the equation

$$D_{n-1} = \mathbf{S}(\pm a_{1,1} a_{2,2} \cdots a_{n-1,n-1}).$$

If we multiply the latter by $a_{n,n}$, we will have the algebraic sum of the symmetric products in the determinant D_n which have $a_{n,n}$ as a factor, [an expression that] will not be, in general, an alternating symmetric function. But if, in this same function, we transpose in all possible ways the indices in question, . . . then we will have

an alternating symmetric function that we can denote by

$$\mathbf{S}[\pm a_{n,n} \mathbf{S}(\pm a_{1,1} a_{2,2} \dots a_{n-1,n-1})].$$

That is, he used his antisymmetrizing operator a second time; by comparing the terms of this latter sum with those of the full determinant D_n of order n , he then argued that the two expressions are identical. This showed that one can evaluate an n th order determinant by expanding by cofactors along the n th column of the system/matrix. He then argued, in a similar way, that the formula

$$D_n = \mathbf{S}[\pm a_{\mu,\mu} \mathbf{S}(\pm a_{1,1} a_{2,2} \dots a_{\mu-1,\mu-1} a_{\mu+1,\mu+1} \dots a_{n,n})]$$

holds for any index $\mu = 1, 2, \dots, n$, from which it follows that the determinant can be expanded along any column (or row). Indeed, he defined the values $b_{\nu,\mu}$ (what we call the cofactors of the matrix entries) in order that

we will obtain the following equations:

$$\begin{cases} D_n = a_{1,1}b_{1,1} + a_{2,1}b_{2,1} + \dots + a_{n,1}b_{n,1}, \\ D_n = a_{1,2}b_{1,2} + a_{2,2}b_{2,2} + \dots + a_{n,2}b_{n,2}, \\ \vdots \\ D_n = a_{1,n}b_{1,n} + a_{2,n}b_{2,n} + \dots + a_{n,n}b_{n,n}, \end{cases}$$

in which we must suppose, in general, that

$$\begin{cases} b_{\mu,\mu} = \mathbf{S}(\pm a_{1,1} a_{2,2} \dots a_{\mu-1,\mu-1} a_{\mu+1,\mu+1} \dots a_{n,n}), \\ b_{\nu,\mu} = \mathbf{S}(\mp a_{1,1} a_{2,2} \dots a_{\mu-1,\mu-1} a_{\mu,\mu+1} \dots a_{\nu-1,\nu} a_{\nu+1,\nu+1} \dots a_{n,n}). \end{cases}$$

In another brief pause from reading Cauchy's memoir, students are asked to think about the effect on a determinant of switching a pair of columns (or rows) in the system/matrix (a_{ij}) to obtain the system/matrix (a'_{ij}) ; Cauchy did the same in his memoir, but his argument is rather long-winded. With some careful assistance, students can now show that $\det(a'_{ij}) = -\det(a_{ij})$. This completes their knowledge about determinants of elementary matrices, which is exploited at the end of the PSP. As another corollary to this formula, they prove that the determinant vanishes for any matrix containing a pair of equal rows or columns.

Next, Cauchy studied the system (b_{ij}) of cofactors, which he called the *adjoint* of (a_{ij}) . The previous corollary is used to verify the second of the formulas below:

by the preceding,

$$D_n = a_{1,\mu}b_{1,\mu} + a_{2,\mu}b_{2,\mu} + \dots + a_{n,\mu}b_{n,\mu};$$

we will have therefore generally

$$0 = a_{1,\nu}b_{1,\mu} + a_{2,\nu}b_{2,\mu} + \dots + a_{n,\nu}b_{n,\mu}.$$

This last equation will be satisfied whenever ν and μ are two numbers different from one another.

As the indices ν and μ range over all values $1, 2, \dots, n$, these equations now combine to produce Laplace's Adjoint Formula:

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = (\det A) \cdot I.$$

Using this result, Cauchy was able to give a full proof of Cramer's Rule (using a now well-known argument), closing a loop of rigor for the students working on this PSP.

In the final stroke of genius executed by Cauchy in the portions of his memoir taken up here, students read how he prefigured the concept of matrix multiplication:

We now consider a system of equations of the form

$$\left\{ \begin{array}{l} \alpha_{1,1}a_{1,1} + \alpha_{1,2}a_{1,2} + \cdots + \alpha_{1,n}a_{1,n} = m_{1,1}, \\ \alpha_{2,1}a_{1,1} + \alpha_{2,2}a_{1,2} + \cdots + \alpha_{2,n}a_{1,n} = m_{1,2}, \\ \dots\dots\dots, \\ \alpha_{n,1}a_{1,1} + \alpha_{n,2}a_{1,2} + \cdots + \alpha_{n,n}a_{1,n} = m_{1,n}; \\ \\ \alpha_{1,1}a_{2,1} + \alpha_{1,2}a_{2,2} + \cdots + \alpha_{1,n}a_{2,n} = m_{2,1}, \\ \alpha_{2,1}a_{2,1} + \alpha_{2,2}a_{2,2} + \cdots + \alpha_{2,n}a_{2,n} = m_{2,2}, \\ \dots\dots\dots, \\ \alpha_{n,1}a_{2,1} + \alpha_{n,2}a_{2,2} + \cdots + \alpha_{n,n}a_{2,n} = m_{2,n}; \\ \\ \dots\dots\dots; \\ \\ \alpha_{1,1}a_{n,1} + \alpha_{1,2}a_{n,2} + \cdots + \alpha_{1,n}a_{n,n} = m_{n,1}, \\ \alpha_{2,1}a_{n,1} + \alpha_{2,2}a_{n,2} + \cdots + \alpha_{2,n}a_{n,n} = m_{n,2}, \\ \dots\dots\dots, \\ \alpha_{n,1}a_{n,1} + \alpha_{n,2}a_{n,2} + \cdots + \alpha_{n,n}a_{n,n} = m_{n,n}. \end{array} \right.$$

This system of equations contains three systems of symmetric quantities.

illustrating what we would interpret today as the multiplication of the matrices (α_{ik}) and (a_{jk}) (in that order) to produce the product matrix (m_{ij}) . By comparing the product

$$\mathbf{S}(\pm\alpha_{1,\mu}\alpha_{2,\nu}\dots\alpha_{n,\pi})\mathbf{S}(\pm a_{1,\mu}a_{2,\nu}\dots a_{n,\pi}),$$

in which “we assume that the indices $\mu, \nu, \dots \pi$ are all different from each other,” to the determinant $\mathbf{S}(\pm m_{1,1} m_{2,2} \dots m_{n,n})$ by means of the above equations, Cauchy concluded that the quantities $\det(\alpha_{ik}) \cdot \det(a_{jk})$ and $\det(m_{ij})$ agreed, at least up to sign; by considering the case where the factor “matrices” were both equal to the “identity matrix,” wherein the product would also have to be the same, he finally arrived at the Determinant Product Formula: $\det AB = \det A \cdot \det B$.

In the last section of the PSP, students are shown how Gaussian elimination of a square matrix can be represented as a factorization of the matrix as a product of elementary matrices and an echelon form of the original matrix. This leads at last to a much more computationally efficient method of computing a determinant than Cramer, Cauchy, or Laplace were able to provide. Armed with the Product Formula, students can see that calculating a determinant will devolve into the multiplication of the determinants of these elementary matrix factors with the determinant of the final echelon form matrix, and all the latter calculations they know to perform. This procedure for computing an arbitrary determinant ends the project.

While it may not cover all the topics concerning determinants that it would be good for students of a one-semester linear algebra course to know – the most glaring omission being the interpretation of a determinant as the signed volume in n -space of the parallelepiped formed by the matrix rows (or columns) – an impressive number of powerful ideas are presented in this memoir, and are exploited in the PSP in a way that students are able to process and make their own.

References.

- [1] S. Axler. Down with determinants! *Amer. Math. Monthly*, 102(2):139–154, 1995.
- [2] G. Cramer. *Introduction à l’Analyse des Lignes Courbes Algébriques*. Cramer & Philibert, Geneva, 1750.
- [3] M. Flagg, Teaching linear algebra using primary historical sources, *IMAGE*, 62:23–24, 2019.
- [4] F. Klein. *Elementary Mathematics from a Higher Standpoint. Volume II: Geometry*. Springer-Verlag, Berlin, 2016. Translated by Gert Schubring.
- [5] O. Ore. *Niels Henrik Abel: Mathematician Extraordinary*. University of Minnesota Press, Minneapolis, MN, 1957.



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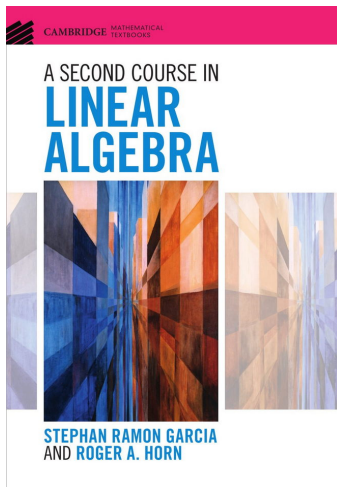


BOOK REVIEW

A Second Course in Linear Algebra by Stephan Ramon Garcia and Roger A. Horn

Cambridge University Press, 2017, ISBN 978-1-107-10381-8, 426+xix pages.

Reviewed by Rajesh Pereira, University of Guelph, pereirar@uoguelph.ca



It is often said that you cannot judge a book by its title. In this case the title is apt since this is a very good textbook for the second Linear Algebra course. I would suggest that an even more fitting title for this book would be “How to think like a matrix theorist”. The reader will learn how to approach problems in linear algebra the way a matrix or operator theorist would. There are many examples of this, of which I give one that is representative. Block matrix techniques are explained and used throughout this textbook. I never taught block matrix techniques (at least not intentionally; I once accidentally used them in response to a student’s question), but after reading this book I am convinced that they are a natural and useful topic for a second linear algebra course. Thinking back on that incident, it is clear that there is often a gap between how we would explain a problem to a student and how we would approach the problem ourselves. This book closes that gap and introduces the reader to the methods and ideas that researchers in matrix theory use.

The text covers the theory behind a number of applied topics. As an example, while not a numerical linear algebra textbook, it covers Gershgorin disks, Householder matrices, the Bauer-Fike Theorem, the defect from normality, least squares and minimal solutions, as well as all the major matrix decompositions (SVD, Schur Triangular, Cholesky, Polar and QR). The coverage here is so extensive that it is easier to list the topics *not* included: the Perron-Frobenius Theorem and combinatorial matrix theory are the only things that come to mind here. Almost every other topic that you might consider in a second linear algebra course is covered.

One topic deserves special mention. Singular values are often omitted from the second linear algebra course, and hence from many second course linear algebra textbooks, despite the importance of the subject. (One example of its importance: The Statistical Society of Canada mentions the Singular Value Decomposition in its accreditation guidelines as a topic that statisticians should know.) This textbook correctly covers singular values and the singular value decomposition and uses them in interesting ways such as to give a simple definition of the Moore-Penrose inverse from which its existence and uniqueness are readily apparent.

The examples in the text have clearly been chosen so as to anticipate their coverage in other courses. The Pauli spin matrices and the Hadamard gate from quantum computation are given as examples of unitary matrices. Differential, integral and shift operators are given as examples of linear operators. Readers get an introduction to linear functionals and the Riesz Representation Theorem in a gentle finite-dimensional setting. Fourier series is given as an example of an orthonormal basis and enough Fourier series theory is covered for the reader to be able to solve the Basel Problem and prove the Riemann-Lebesgue Lemma, both of which are given as exercises. The latter illustrates another feature of the book: Readers will be asked to do real mathematics in the exercises, such as proving Hadamard’s Inequality or the existence of unitary dilations for strict contractions. These are mixed in with easier exercises and are not used to prove later results in the book, so it can be read without doing the exercises.

Every chapter ends in a section entitled, “Some important concepts,” a checklist of a half dozen to a dozen important concepts which can help the reader to see if they have grasped the key concepts in the chapter. This makes the text useful for self-study. The index is also notable. The subheadings in the index are an extremely useful feature which other authors should copy. Take an index item like “complex symmetric matrices”. It has a number of subheadings, one of which is “definition”, which leads you to the definition, while the others are properties such as: “every matrix similar to a ”; “need not be diagonalizable”; “pseudoinverse is symmetric”; and so on. The index tells you what the important results on complex symmetric matrices (or any other topic) are and where to find them. The fact that you can find results in the index even if they are not named theorems is extremely useful.

At this point, it will come as no surprise that I highly recommend this book. It is an excellent textbook for a second course in linear algebra. It is also useful for self-study. And it is particularly valuable for anyone beginning research in any area that heavily uses linear algebra, not only because it covers all of the important results and techniques, but also because it teaches you to think like a matrix theorist.

ILAS NEWS

ILAS Election Results

Hugo Woerdeman was re-elected to the position of Vice President for a three-year term beginning on March 1, 2019. Valeria Simoncini and Michael Tsatsomeros were elected to three-year terms as members of the ILAS Board, beginning on March 1, 2019.

ILAS Member Selected as 2019 SIAM Fellow



Froilán Dopico

The Society for Industrial and Applied Mathematics (SIAM) recently announced its 2019 SIAM Fellows, and among those named is ILAS member Froilán Dopico from the Universidad Carlos III de Madrid. Professor Dopico is recognized for contributions in numerical linear algebra and the solution of polynomial and rational eigenvalue problems via linearizations. For details, see <https://sinews.siam.org/Details-Page/siam-announces-class-of-2019-fellows>.

ILAS Member Selected as Inaugural CMS Fellow

Peter Lancaster, of the University of Calgary, has been selected by the Canadian Mathematical Society (CMS) as among its inaugural class of CMS fellows. For details, see <https://cms.math.ca/MediaReleases/2018/Fellows>.

2019 Hans Schneider Prize Awarded to Lek-Heng Lim and Volker Mehrmann

Lek-Heng Lim from the University of Chicago and Volker Mehrmann from Technische Universität Berlin are the 2019 Hans Schneider prize recipients.

Lek-Heng Lim, along with collaborators, has made several fundamental contributions to linear and multilinear algebra, matrix theory, and their applications. These include his work on eigen- and singular values of tensors, on tensor ranks, on a Perron-Frobenius Theorem for nonnegative tensors, and on the ill-posedness and NP-hardness of some tensor problems. A striking result of his shows that every $n \times n$ matrix is a product of at most $2n + 5$ Toeplitz matrices. Another lays the theoretical foundation for measuring the distance between subspaces of different dimensions. This work finds applications in statistics, data analysis, optimization and computational mathematics. At the same time, it makes use of diverse ideas and techniques from several areas of mathematics.

Volker Mehrmann was one of the early contributors to the development of efficient and reliable numerical algorithms for systems and control theory. He has made fundamental contributions to the solution of algebraic and differential algebraic equations arising in optimal control, and to efficient numerical codes for implementing these solutions. He has also made very significant contributions to numerical methods for linear algebra problems with special (such as Hamiltonian or symplectic) structure and developed techniques that preserve this structure when computing spectra. He has played an important role in the development of software which implements these algorithmic ideas. He has also made sure that his theoretical results and numerical software are being used in industry. In addition, he has played a very important role in mentoring younger mathematicians, in the workings of many mathematical societies, and as an editor of several major journals.

Volker Mehrman will be awarded the prize at the ILAS Conference in Rio de Janeiro in 2019, and Lek-Heng Lim will be awarded the prize at the ILAS Conference in Galway in 2020.

The 2019 Hans Schneider Prize Selection Committee consisted of Rajendra Bhatia (chair), Richard Brualdi, Shmuel Friedland, Thomas Laffey, Paul Van Dooren, and Peter Šemrl (ex officio).

ILAS President/Vice President Annual Report: March 31, 2019

Respectfully submitted by Peter Šemrl, ILAS President, peter.semrl@mf.uni-lj.si
and Hugo Woerdeman, ILAS Vice President, hugo@math.drexel.edu

1. Board-approved actions since the last report include:

- Michael Tsatsomeros was reappointed as Editor-in-Chief of *ELA* for another three-year term (March 1, 2019 – February 28, 2022).
- Minerva Catral was appointed to the position of Assistant Secretary/Treasurer starting September 2018.

2. ILAS elections ran from November 27, 2018 – January 27, 2019, and proceeded via electronic voting. The following were (re-)elected to offices with three-year terms that began on March 1, 2019:

- Vice President: Hugo Woerdeman
- Board of Directors: Valeria Simoncini and Michael Tsatsomeros

The following continue in the ILAS offices which they currently hold:

- President: Peter Šemrl (term ends February 29, 2020)
- Secretary/Treasurer: Leslie Hogben (term ends February 28, 2021)
- Second Vice President (for ILAS conferences): Steve Kirkland (term ends February 29, 2020)
- Board of Directors: Maria Isabel Bueno (term ends February 28, 2021), James Nagy (term ends February 29, 2020), Rachel Quinlan (term ends February 29, 2020), and Vilmar Trevisan (term ends February 28, 2021)

Ravindra Bapat and Helena Šmigoc completed their terms on the ILAS Board of Directors on February 28, 2019. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated. We also thank the members of the Nominating Committee – Chi-Kwong Li (chair), Wayne Barrett, Heike Fassbender, Judi McDonald, and Christian Mehl – for their work on behalf of ILAS, and also extend gratitude to all candidates that agreed to have their names stand for the elections.

3. The following ILAS-endorsed meetings have taken place since our last report:

- SIAM Conference on Applied Linear Algebra (SIAM-ALA18) Hong Kong Baptist University, Hong Kong, China, May 4–8, 2018. Valeria Simoncini was a Hans Schneider ILAS Lecturer, and Mark Embree was an ILAS Lecturer. <http://www.math.hkbu.edu.hk/siam-ala18/>
- Western Canada Linear Algebra Meeting (WCLAM 2018) Washington State University, Pullman, Washington, USA, May 26–27, 2018 <http://www.math.wsu.edu/math/faculty/tsat/W-CLAM18.html>
- Linear Algebra, Matrix Analysis and Applications (ALAMA2018) Complejo San Juan, Sant Joan d'Alacant, Spain, May 30 – June 1, 2018 <https://web.ua.es/en/alama2018/home.html>
- Riordan Arrays and Related Topics (5RART 2018) Busan, Republic of Korea, June 25–29, 2018 <https://sites.google.com/view/5rart/home>
- International Conference on Algebra and Related Topics (ICART 2018) Rabat, Morocco, July 2–5, 2018 <http://www.fsr.ac.ma/icart2018/>
- Numerical Analysis and Scientific Computation with Applications (NASCA 2018) Kalamata, Greece, July 2–6, 2018. Michael Tsatsomeros was an ILAS Lecturer. <http://nasca18.math.uoa.gr>

4. ILAS has endorsed the following conferences of interest to ILAS members:

- 27th International Workshop on Matrices and Statistics (IWMS2019) Shanghai, China, June 6–9, 2019 <http://www.suibe.edu.cn/ttxy/iwms2019/>

- 8th International Conference on Matrix Analysis and Applications (ICMAA 2019) Reno, NV, USA, July 15–18, 2019 <https://wolfweb.unr.edu/homepage/ttam/2019-ICMAA.html>
- International Workshop on Operator Theory and its Applications (IWOTA 2019) Lisbon, Portugal, July 22–26, 2019. Albrecht Böttcher will be an ILAS Lecturer. <https://iwota2019.math.tecnico.ulisboa.pt/home>
- International Conference and PhD-Master Summer School on “Groups and Graphs, Designs and Dynamics” (G2D2) Yichang, China, August 12–25, 2019. Michael Boyle will be an ILAS Lecturer. <http://math.sjtu.edu.cn/conference/G2D2/>
- International Conference on Matrix Analysis and its Applications (MAT-TRIAD 2019) Liblice, Czech Republic, September 8–13, 2019. Shmuel Friedland will be a Hans Schneider ILAS Lecturer. <http://mattriad.math.cas.cz>

5. The following ILAS conferences are scheduled:

- 22nd ILAS Conference: Linear Algebra Without Borders, July 8–12, 2019, Rio de Janeiro, Brazil. The co-chairs of the organizing committee are Nair Abreu and Vilmar Trevisan. Hans Schneider Prize winner Volker Mehrmann will be recognized at this meeting. Joseph Landsberg will be an *LAA* Lecturer, Apoorva Khare a *LAMA* Lecturer, and David Bindel a *SIAG-LA* Lecturer. <http://ilas2019.org/>
- 23rd ILAS Conference: ILAS 2020 will be held at National University of Ireland, Galway, June 22–26, 2020. The chair of the organizing committee is Rachel Quinlan. Hans Schneider Prize winner Lek-Heng Lim will be recognized at this meeting. <http://www.maths.nuigalway.ie/ILAS2020/>

6. The Electronic Journal of Linear Algebra (*ELA*) is now in its 35th volume. *ELA*’s URL is <http://repository.uwyo.edu/ela/>.

Volume 34 was published in 2018 and contains 51 papers.

Bryan Shader (University of Wyoming) and Michael Tsatsomeros (Washington State University) continue as the Editors-in-Chief.

7. *IMAGE* is the semi-annual bulletin for ILAS available online at <http://ilasic.org/IMAGE/>. The Editor-in-Chief is Louis Deaett (Quinnipiac University).

8. ILAS-NET is a moderated newsletter for mathematicians worldwide, with a focus on linear algebra; it is managed by Sarah Carnochan Naqvi.

An archive of ILAS-NET messages is available at <http://www.ilasic.org/ilas-net/>. To send a message to ILAS-NET, please send the message (preferably in text format) in an email to ilasic@uregina.ca indicating that you would like it to be posted on ILAS-NET. If the message is approved, it will be posted soon afterwards. To subscribe to ILAS-NET, please complete the form at <http://ilasic.us10.list-manage.com/subscribe?u=6f8674f5d780d2dc591d397c9&id=dbda1af1a5>.

9. ILAS’ website, known as the ILAS Information Centre (IIC), is located at <http://www.ilasic.org> and provides general information about ILAS (e.g., ILAS officers, bylaws, special lecturers), as well as links to pages of interest to the ILAS community.

Respectfully submitted,

Peter Šemrl, ILAS President (peter.semrl@fmf.uni-lj.si); and
Hugo J. Woerdeman, ILAS Vice-President (hugo@math.drexel.edu).

ILAS 2018–2019 Treasurer's Report
April 1, 2018 – March 31, 2019
by Leslie Hogben, ILAS Treasurer

Net Account Balance on March 31, 2018

Vanguard (ST Fed Bond Fund Admiral 7876.686 Shares)	\$ 85,566.14	
Checking Account - Great Western	\$ 53,524.12	
Certificate of Deposit	\$ 46,325.81	
Accounts Payable	\$ 720.00	
		\$ 186,136.07
General Fund	\$ 100,772.42	
Israel Gohberg ILAS-IWOTA Lecture	\$ 3,530	
Conference Fund	\$ 9,914.55	
Olga Taussky Todd/John Todd Fund	\$ 11,071.68	
Hans Schneider Lecture Fund	\$ 12,035.89	
Frank Uhlig Education Fund	\$ 5,357.31	
Hans Schneider Prize Fund	\$ 27,420.57	
ELA Fund	\$ 442.54	
ILAS/LAA Fund	\$ 15,591.11	
		\$ 186,136.07

INCOME:

Dues	\$ 5,360.00	
Israel Gohberg ILAS-IWOTA Lecture Fund		
Donations	\$ 90.00	
General Fund Donations	\$ 3,070.00	
Conference Fund Donations	\$ 100.00	
Taussky-Todd Fund Donations	\$ 20.00	
Hans Schneider Lecture Fund Donations	\$ 100.00	
Uhlig Education Fund Donations	\$ 10.00	
Schneider Prize Fund Donations	\$ 180.00	
ELA Fund Donations	\$ 150.00	
Corporate Dues Income	\$ 350.00	
Interest – Great Western	\$ 102.82	
Interest on Great Western Certificate of Deposit	\$ 254.79	
Vanguard – Income	\$ 2,632.63	
Misc Income	\$ 78.00	
Total Income		\$ 12,498.24

EXPENSES:

ILAS Conference Expenses	\$ 0.00	
ELA	\$ 0.00	
Treasurer's Assistant	\$ 500.00	
General Expenses	\$ 0.00	
Credit Card Processing & Wire Transfer Fees	\$ 378.00	
Non-ILAS Conferences	\$ 3,753.86	
Hans Schneider Lecture	\$ 0.00	
Taussky-Todd Lecture	\$ 0.00	
Hans Schneider Prize	\$ 0.00	
IMAGE Costs	\$ 297.00	
Business License	\$ 61.25	
Ballot Costs	\$ 272.60	
Web Hosting & Online Membership Forms	\$ 119.98	
Misc Expenses	\$ 0.00	
Total Expenses		\$ 5,382.69

Net Account Balance on March 31, 2019

Vanguard (ST Fed Bond Fund Admiral 7998.604 Shares)	\$ 88,198.77	
Checking Account – Great Western	\$ 58,472.25	
Certificate of Deposit	\$ 46,580.60	
Accounts Payable (ISU cc)	\$ 330.00	
		\$ 193,581.62
General Fund	\$ 107,567.97	
Israel Gohberg ILAS-IWOTA Lecture	\$ 3,620.00	
Conference Fund	\$ 10,014.55	
Olga Taussky Todd/John Todd Fund	\$ 11,091.68	
Hans Schneider Lecture Fund	\$ 12,135.89	
Frank Uhlig Education Fund	\$ 5,367.31	
Hans Schneider Prize Fund	\$ 27,600.57	
ELA Fund	\$ 592.54	
ILAS/LAA Fund	\$ 15,591.11	
		\$ 193,581.62

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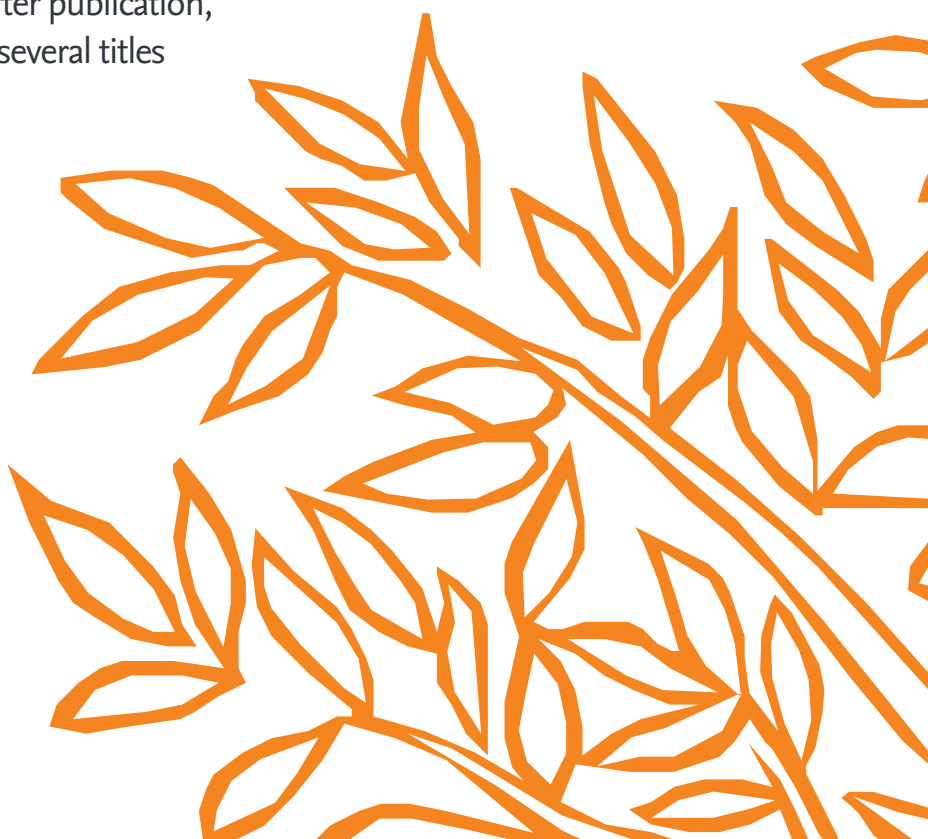
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OBITUARY

Thank you, sir. Thank you, my friend.

in memory of David Clark Lay

March 1, 1941 – October 12, 2018

Submitted by Harm Bart¹

Sometimes MathSciNet reflects that something special is going on. This was the case with the review Harro Heuser wrote about the paper “Equivalence, linearization and decomposition of holomorphic operator functions” by Israel Gohberg, Marinus A. Kaashoek and David C. Lay. The review begins with this sentence: *The wealth of interesting and intricate results of this paper is best summarized by quoting from the introduction...* A “wealth of interesting and intricate results” indeed. Among them one that is a landmark in the study of analytic operator functions. I’ll come back to the result in question later.

David Clark Lay is probably most widely known because of his world famous textbook on linear algebra. There is no need to elaborate on that here. He also was a coauthor of the well-known book [8] on functional analysis, an update of an older edition written by Angus E. Taylor. In these works he proves himself a master of exposition. Yes, David was a tremendous educator of mathematics. Not only in writing. As a hands-on teacher in the classroom or as a lecturer in seminars and at conferences he was formidable as well.

But David was an able researcher too. Details can of course be found on MathSciNet. Here I mention that David has been active over a period of about thirty years, a period which can be divided roughly into three parts of approximately ten years. Each part had its own mathematical flavor which I will now briefly indicate.

In his early years, David worked on the spectral theory of linear operators, both bounded and unbounded. Topics that interested him in particular were Fredholm operators, Riesz operators, essential spectra, and the analysis of poles of the resolvent of an operator using the notions of ascent and descent. This stage of his career resulted in about six publications, some of them with others, like Marinus (Rien) Kaashoek and Bernhard Gramsch.

Then came another time span in which the focus of David’s research was on analytic or, more generally, meromorphic operator-valued functions. Much effort went into the investigation of meromorphic relative inverses of such functions. Meromorphy has two aspects here. It can be considered locally, at a single point, but also globally, on a given domain. Due to the non-uniqueness of relative inverses, the latter is a tricky issue which requires, so to speak, gluing together local results. About ten papers carrying David’s name were published in this period. Cooperation was mainly with Israel Gohberg, Rien Kaashoek and myself. From the work with the first two came the fundamental paper [4], referred to in the first paragraph of this note. More about it in a moment.

In the last ten years of David’s active involvement in research activities, he worked mostly with Israel Gohberg and Robert L. Ellis. Among the topics dealt with were band extensions, Toeplitz matrices, Hankel matrices, maximum distance problems, and orthogonal polynomials. Some fourteen papers were the result.

At this point, let me elaborate on the paper [4]. In 1969, I became a Ph.D. student with Rien Kaashoek as supervisor. The topic he suggested was “analytic operator-valued functions.” More specifically, the aim was to investigate under what circumstances results on the resolvent of Riesz operators have counterparts for generally non-linear analytic operator-valued functions. Meromorphic behavior was a special focus here. An important technique under development at the time was to relate to a given operator function $A(\lambda)$ a pencil $\lambda E - F$ of operators in such a way that relevant local spectral properties of A were reflected by certain local spectral properties of $\lambda E - F$. In my thesis, I polished an approach taken by Karl-Heinz Förster and was able to characterize poles of the resolvent $A(\lambda)^{-1}$ in terms of generalized ascent and descent. In [4], instead of a connection with a pencil $\lambda E - F$ involving a possibly non-invertible operator S , a relationship was exhibited with a spectral pencil $\lambda I - T$, with I the identity operator on the



David Clark Lay

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underlying Banach space on which T is defined. In this way – amazingly enough – many issues for analytic operator-valued functions could be directly related to standard results from ordinary spectral theory. For instance, the results on poles of the resolvent in my thesis could now almost immediately be read off from the corresponding theorems in the spectral case as obtained earlier by David Lay (cf. [5]). It should be mentioned that the Gohberg-Kaashoek-Lay result goes beyond local validity and is of a global scope. An essential ingredient of it is analytic equivalence after extension, meaning that a direct sum $A(\lambda) \oplus J$ of $A(\lambda)$ with an appropriate identity operator J can be written in the form $L(\lambda)(\lambda I - T)R(\lambda)$, where the operator-valued functions $L(\lambda)$ and $R(\lambda)$ are analytic and have only invertible values. Meanwhile, analytic equivalence after extension plays a role in many publications – quite recent ones too; see, for example, [2], [3], [7] and [9].

The paper [4], central in the preceding paragraph, drew considerable attention. It seems this has not been the case with [6], a paper by David Lay entitled “Subspaces and echelon forms”. Perhaps the reason is that the article is of an educational character, was published in a corresponding journal, and was not reviewed for MathSciNet. However that may be, there is a story connected to it worthy of being told here.

In the fall of 2003, in the context of my work with Torsten Ehrhardt and Bernd Silbermann, I came across the following problem: Given a linear subspace N in $\mathbb{C}^m$, does there exist an idempotent upper triangular $m \times m$ matrix U having N as its image? The answer turns out to be positive. This appears from the following observation.

Let M be any $m \times m$ matrix. Via row operations, bring M into reduced row echelon form E . Then permute the rows of E in such a way that the pivots appear on the diagonal. The resulting matrix E_M , uniquely determined by M , is clearly upper triangular, and it is easily seen to be idempotent. Also the null spaces of M and E_M coincide. Now put $U = I_m - E_M$, where I_m is the $m \times m$ identity matrix. Then E_M is upper triangular, idempotent, and its image coincides with the null space of M . One gets a matrix U with the properties mentioned above by starting out with an M (which can be taken to be idempotent) having N as its null space.

As a byproduct, one has this: Suppose we want to solve a homogenous system of linear equations, without loss of generality taken to be square, so of the type $Mx = 0$. As we tell our students, with E as above, one can just as well solve $Ex = 0$, or what amounts to the same, $E_M x = 0$, and this can be done via back substitution. However, a basis for the solution space can (generally) be obtained much more quickly from the matrix $I_m - E_M$. Indeed, the non-vanishing columns of that matrix form a basis for the solution space of $Mx = 0$. Thus, the procedure is: Bring M into reduced row echelon form, permute the rows such that the pivots appear on the diagonal, subtract the resulting matrix E_M from the identity matrix, and write down the non-vanishing columns.

I could not believe we were the first people to see this. So I started a search. When this remained fruitless, I approached several colleagues and asked them if they were aware of this simple algorithm. None of them was. David also answered that he hadn't seen it before. But still I found it highly unlikely that we were first. So I went hunting on the Internet. And... I found it – to my major surprise in a paper by David! Indeed, it is in the discussion on Algorithm III in [6]. Clearly David had forgotten about it. Perhaps therefore the observation, though educationally useful, is not in his Linear Algebra book. When I brought this up in one of our conversations, he was receptive to the thought that it should have a place there. It did not materialize, however, perhaps because his health problems started to interfere. I suggest to call the upper triangular idempotent E_M described above, coming from the reduced row echelon form of a matrix M , the *Lay idempotent of M* .

Besides being a highly appreciated colleague and a co-author, David Lay was a friend. So let me now turn to reminiscences of a more personal nature.

It was in 1973 when I met David Lay for the first time. He and his wife Lillian had come to Amsterdam to visit my Ph.D. advisor Rien Kaashoek. David knew him from their time at UCLA in the academic year 1965–1966. They had formed a mathematical gang of four there, with Lothrop Mittenhal and Trevor West as the other members. Rien had told me about these good times, and I was anxious to meet David. When I arrived at the university he was not in Rien's office but in the library. So I went there and we met. It was a case of sympathy at first sight. We immediately liked each other. A couple of days later we met again in the home of Rien and his wife Wiesje. This time Lillian was also present, and the budding friendship broadened, so that my wife Greetje became involved too.

David and Lillian left for Kaiserslautern in Germany, but then they came back and stayed in Amsterdam for a somewhat longer period. It was during this time that an intense cooperation between Rien, David and me developed. In the context of the long-lasting weekly seminar on Operator Theory and Analysis run by Rien, we studied certain papers by Russian colleagues containing work done jointly with, or under the strong influence of, the great mathematician Israel Gohberg. The articles were translated into German by a non-mathematician, which here and there led to amusingly bizarre formulations, but we worked ourselves through them – with success: A couple of papers came out as a result. They were finished when David was already back at what had become his home base, College Park in Maryland. The

communication went by what is now sometimes jokingly called “snail mail.” A letter took almost a week to get from Amsterdam to College Park. Then David had to think about an answer, and so it would not be less than three weeks before we had that answer. How much elasticity that gave!

David was a terrific companion to write papers with. We often profited from his extraordinary talent for clear exposition. That talent also helped him later when he wrote his excellent linear algebra book. Countless students have learned matrix theory from it. As already noted, he became one of the leading educators in this area.

I happened to be a witness to the early steps toward this influential book. Indeed, in 1988 I was in Maryland for about six weeks and stayed in the home of David and Lillian. The book was one of the topics of conversation, though there were many more. In the academic year 1979–1980, David and his family had lived for several months in the same small village in The Netherlands where my wife and I had our home. During this time, our friendship had far transcended mathematics. It had developed into what Aristotle in his book on ethics identifies as the highest form of friendship, friendship not based on just having fun with each other or on the possibility for mutual profit, but on appreciation of each other’s personality. Our wives and our children had become an inseparable part of it too.

So many good memories! But then came our visit to Annapolis in 2008. The moment David greeted us at the door, Greetje and I felt that something was different. In retrospect, it was – sadly enough – the first stage of his illness. We saw him several times in the years that followed. Deeper and deeper became his withdrawal from world and time.

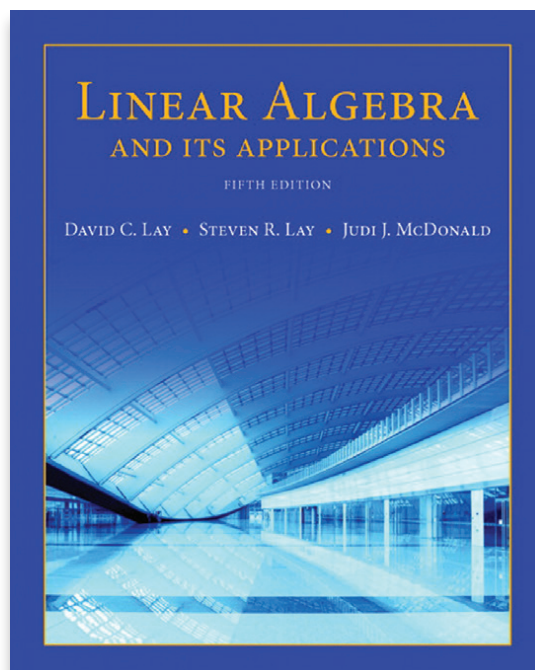
But still, for quite a while, there was the connection with his old world. In 2013, my wife and I stayed with David and Lillian in their new home in Dublin, Ohio. When I woke up after the first night, I asked myself how to approach my longtime friend. I decided to ignore his memory problems, and just tell him about my recent mathematical work with Torsten Ehrhardt and Bernd Silbermann. It had, after all, a relationship with the paper [1] we had written a long time ago, joint with Rien Kaashoek. The effect was amazing! There was resonance. His face lit up, his eyes were glittering. It was as if he heard an old beloved song. He even reacted with some sensible questions. It did not stick, almost immediately washed away by failing memory, but it did show his deep love of mathematics, the background of our friendship of so many years.

The last time my wife Greetje and I visited David was in the first days of June last summer, almost forty-five years after I met him for the first time in Amsterdam. He did not recognize us. We tried to give him the feeling that we were friends. When we left, I said goodbye and patted him gently on the shoulder. He looked up and said, with a faint hint of a smile: *Thank you, sir.* I felt that I would never see him again. I close, mirroring what he said, with a heartfelt *Thank you, my friend.*

References.

- [1] H. Bart, M. A. Kaashoek, and D. C. Lay. The integral formula for the reduced algebraic multiplicity of meromorphic operator functions. *Proc. Edinburgh Math. Soc.* (2), 21(1):65–71, 1978/79.
- [2] L. P. Castro, R. Duduchava, and F.-O. Speck. Diffraction from polygonal-conical screens, and operator approach. In *Operator theory, operator algebras and applications*, volume 242 of *Oper. Theory Adv. Appl.*, pages 113–137. Birkhäuser/Springer, Basel, 2014.
- [3] C. Engström and A. Torshage. On equivalence and linearization of operator matrix functions with unbounded entries. *Integral Equations Operator Theory*, 89(4):465–492, 2017.
- [4] I. C. Gohberg, M. A. Kaashoek, and D. C. Lay. Equivalence, linearization, and decomposition of holomorphic operator functions. *J. Functional Analysis*, 28(1):102–144, 1978.
- [5] D. C. Lay. Spectral analysis using ascent, descent, nullity and defect. *Math. Ann.*, 184:197–214, 1969/1970.
- [6] D. C. Lay. Subspaces and echelon forms. *The College Mathematics Journal*, 24(1):57–62, 1993.
- [7] F.-O. Speck. Paired operators in asymmetric space setting. In *Large truncated Toeplitz matrices, Toeplitz operators, and related topics*, volume 259 of *Oper. Theory Adv. Appl.*, pages 681–702. Birkhäuser/Springer, Cham, 2017.
- [8] A. E. Taylor and D. C. Lay. *Introduction to functional analysis*. John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.
- [9] S. ter Horst, M. Messerschmidt, A. C. M. Ran, M. Roelands, and M. Wortel. Equivalence after extension and Schur coupling coincide for inessential operators. *Indag. Math. (N.S.)*, 29(5):1350–1361, 2018.

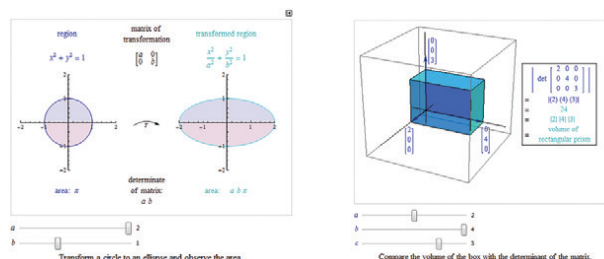
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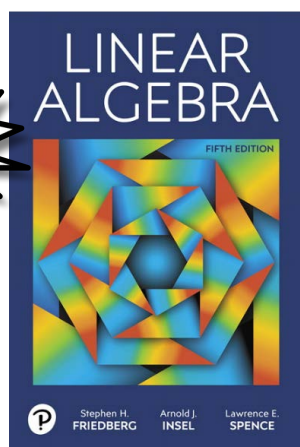
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OBITUARY NOTICE

Roy Westwick, 1933–2019

Submitted by Tin-Yau Tam

Professor Roy Westwick passed away peacefully on April 15, 2019 in Richmond, Canada. He was 86 years old. Professor Westwick received his Ph.D. from the Department of Mathematics at the University of British Columbia (UBC) in 1959, under the supervision of the well-known linear algebraist Professor Marvin Marcus. He published in the area of linear and multilinear algebra. During his 35 years at UBC, Professor Westwick produced eight Ph.D. graduates, according to the Mathematics Genealogy Project. He and his kindness will be greatly missed.

Professor Westwick's obituary can be found online at <http://vancouver.sunandprovince.remembering.ca/obituary/roy-westwick-1074099169>

CONFERENCE REPORTS

International Conference on Algebra and Related Topics (ICART 2018) Rabat, Morocco, July 2–5, 2018

The International Conference on Algebra and Related Topics (ICART 2018) was held from the 2nd to the 5th of July 2018 at the Faculty of Sciences, Mohammed V University in Rabat, Morocco, and consisted of three sessions:

- Applied and Computational Homology in Topology, Algebra and Geometry
- Homological Algebra, Modules, Rings and Categories
- Linear and Multilinear Algebra and Function Spaces

The purpose of the conference was to create a knowledge exchange platform between Moroccan experts, international experts, and local Ph.D. students. There were about 180 participants from different countries. The presentations had a high scientific level. They were followed by fruitful discussions and led to several collaborations. The proceedings of ICART2018 will be published in two volumes:

- (1) As a volume in the “Contemporary Mathematics” series of the American Mathematical Society.
- (2) As a special issue in the journal *Algebra Colloquium*.

For more details, please visit the ICART2018 website: <http://fsr.um5.ac.ma/icart2018/index.html>



Participants of ICART2018

UPCOMING CONFERENCES AND WORKSHOPS

27th International Workshop on Matrices and Statistics (IWMS2019) **Shanghai, China, June 6–9, 2019**

The 27th International Workshop on Matrices and Statistics, IWMS-2019, will be held June 6–9, 2019 at the Shanghai University of International Business and Economics, Shanghai, China.

This series of workshops has a long history and the organizers welcome the opportunity to hold the workshop once again in China. The 19th workshop was held in Shanghai in 2010, and the 24th workshop was in Haikou, so this 27th workshop will continue in the same tradition.

The purpose of the workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. The workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in linear algebra and matrix theory and may exchange ideas with researchers from a wide variety of countries.

Plenary Speakers include: Oskar Maria Baksalary (Poland), Kai-Tai Fang (Hong Kong, China), Arjun K. Gupta (USA), Steve Kirkland (Canada), Jianxin Pan (UK), K. Manjunatha Prasad (India), Yongge Tian (China), Fuzhen Zhang (USA), and Lixing Zhu (Hong Kong, China).

Planned mini-symposia (with organizers) include: Decompositions of tensor spaces with applications to multilinear models (Dietrich van Rosen); Linear statistical models (Simo Puntanen); Magic squares, prime numbers and postage stamps (George P. H. Styan); Inference in parametric models (Julia Volaufova); Statistical diagnosis (Shuangzhe Liu); and Experimental design (Kai-Tai Fang).

The International Organizing Committee of IWMS-2019 consists of Jeffrey J. Hunter (New Zealand) (Chair), Dietrich von Rosen (Sweden) (Vice-Chair), George P. H. Styan (Canada) (Honorary Chair), S. Ejaz Ahmed (Canada), Francisco Carvalho (Portugal), Katarzyna Filipiak (Poland), Daniel Klein (Slovakia), Augustyn Markiewicz (Poland), Simo Puntanen (Finland), Julia Volaufova (USA), and Hans Joachim Werner (Germany).

The Local Organizing Committee consists of Yonghui Liu (Chair), Hui Liu (Vice-Chair), Chengcheng Hao, and Cihai Sun. For further information, please visit <http://www.suibe.edu.cn/ttxy/iwms2019>.

5th International Conference on Matrix Inequalities and Matrix Equations (MIME 2019) **Shanghai, China, June 7–10, 2019**

The 5th International Conference on Matrix Inequalities and Matrix Equations (MIME 2019) will be held June 7–10, 2019, hosted by Guilin University of Electronic Technology; Shanghai University.

The purpose of the conference is to stimulate research and foster the interaction of researchers interested in matrix inequalities and matrix equations. The conference hopes to provide a convenient platform for the exchange of research experiences and ideas from different research areas related to matrix inequalities and matrix equations.

The keynote speaker will be Rajendra Bhatia, Ashoka University, India.

The Scientific Organizing Committee consists of Delin Chu (National University of Singapore); Dragana Cvetkovic-Ilic (University of Nis, Serbia); Chi-Kwong Li (College of William and Mary, USA); Zhongshan Li (Georgia State University, USA); Yiu-Tung Poon (Iowa State University, USA); Tin-Yau Tam (University of Nevada, Reno, USA); Qing-Wen Wang (Shanghai University, China); Yang Zhang (University of Manitoba, Canada).

The Local Organizing Committee (LOC) consists of Xuefeng Duan (Chair); Nan Gao; Zhuoheng He; Xiaomei Jia; Chunmei Li; Lingji Lou; Shuling Pan; Jingjing Peng; Zhenyun Peng; Yonghui Qin; Jiancai Sun; Fuping Tan; Qing-Wen Wang (Co-Chair); Xiying Yuan; Jianjun Zhang; Yunfei Zhou.

For detailed information and updates, please visit the conference website at <http://wwwmath.shu.edu.cn/mime2019>. Contact for questions: Dr. Zhuo-Heng He (hzh19871126@126.com).

5th ALAMA Workshop on Numerical Linear Algebra València, Spain, June 17–18, 2019

The Spanish network on Linear Algebra, Matrix Analysis, and Applications (ALAMA) has announced the 5th ALAMA Workshop on Numerical Linear Algebra (NLA).

The workshop will take place at the Universitat Politècnica de València, June 17–18, 2019, and will consist of 6 talks (of 2 hours each) on different topics and applications of NLA. These topics include preconditioning and some of its applications, matrix functions for graphs and networks, software for NLA, nonlinear eigenvalue and eigenvector problems, together with some applications to mining and learning with graphs, and strictly sign regular matrices.

For more information and to access the registration form, visit http://red-alama.es/jornadas_ALN2019.

ILAS 2019: Linear Algebra Without Borders Rio de Janeiro, Brazil, July 8–12, 2019



The 22nd conference of the International Linear Algebra Society, ILAS 2019: Linear Algebra Without Borders, will be held July 8–12, 2019 in Rio de Janeiro, Brazil, at the main campus of Fundação Getúlio Vargas (FGV), a Brazilian think tank and higher education institution founded in 1944 with the aim of promoting Brazil's economic and social development.

The theme of the conference is “Linear Algebra Without Borders” and refers primarily to the fact that linear algebra and its myriad of applications are interwoven in a borderless unit. In the conference, the organizers plan to illustrate this with a program whose plenary talks and symposia represent the many scientific “countries” of linear algebra, and which invites participants to “visit” them. This theme also refers to the openness and inclusiveness of linear algebra to researchers of different backgrounds.

Plenary speakers:

- **David Bindel**, Cornell University, USA, *SIAG-LA Lecturer*
- **Christoph Helmberg**, Technische Universität Chemnitz, Germany
- **Leslie Hogben**, Iowa State University, USA
- **Apoorva Khare**, Indian Institute of Science, Bangalore, India
- **Igor Klep**, University of Auckland, New Zealand
- **Gitta Kutyniok**, Technische Universität Berlin, Germany
- **Joseph Landsberg**, Texas A&M University, USA, *LAA Lecturer supported by Elsevier*
- **Federico Poloni**, Pisa University, Italy
- **Nikhil Srivastava**, University of California, Berkeley, USA
- **Yuan Jin Yun**, Universidade Federal do Paraná, Brazil

Invited mini-symposia (with organizers): Matrix Analysis (James Pascoe, Miklos Palfia), Frames (Gitta Kutyniok, Deanna Needell), Matrix Equations and Matrix Inequalities (Fuzhen Zhang, Qing-Wen Wang), Algebra and Tensor Spaces (David Gleich, Yang Qi), Linear Algebra and Quantum Information Science (Yiu-Tung Poon, Raymond Nung-Sing Sze, Sarah Plosker), Combinatorial Matrix Theory (Byan Shader, Shaun Fallat, Steve Butler, Kevin Vander Meulen), Matrix Techniques in Operator Theory and Operator Algebras (Hugo Woerdeman), Spectral Graph Theory (Sebastian Cioabă, Jack Koolen, Leonardo Lima), Linear Algebra Education (Sipedeh Stewart, Rachel Quinlan), and Nonnegative Inverse Spectral Problems (Raphael Loewy, Ricardo L. Soto).

The scientific organizing committee is: Nair Abreu (Brazil), Ravindra Bapat (India), Leslie Hogben (USA), Alfredo Iusem (Brazil), Steve Kirkland (Canada), Chi-Kwong Li (USA), Volker Mehrmann (Germany), Beatrice Meini (Italy), Rubens Sampaio (Brazil), Peter Šemrl (Slovenia), Ricardo Soto (Chile), and Vilmar Trevisan (Brazil).

Linear Algebra and its Applications plans to publish a special issue dedicated to this ILAS conference: papers corresponding to talks given at the conference should be submitted by December 15th, 2019 via the Elsevier Editorial System (EES) at <http://ees.elsevier.com/laa>, choosing the special issue SI:ILAS2019 Conference. Special editors for the ILAS 2019 issue are: Nair Abreu, Christoph Helmberg, Gitta Kutyniok, and Vilmar Trevisan.

Ongoing updates and more information about the conference will be found at <http://ilas2019.org>.

9th International Congress on Industrial and Applied Mathematics (ICIAM 2019) **Universitat de València, Spain, July 15-19, 2019**

Organized by the Spanish Society of Applied Mathematics (SEMA) under the auspices of the International Council for Industrial and Applied Mathematics (ICIAM), ICIAM 2019 will showcase the most recent advances in industrial and applied mathematics.

Submission deadlines have passed, but details of the conference can be found at <https://www.iciam2019.com>.

International Workshop on Operator Theory and its Applications (IWOTA 2019) **Instituto Superior Técnico, University of Lisbon, Portugal, July 22–26, 2019**

IWOTA 2019 will be focused on the latest developments in functional analysis, operator theory and related fields, and intends to bring together mathematicians and engineers working in operator theory and its applications, namely mathematical physics, control theory and signal processing.

Invited speakers include: Pera Ara (Spain), Joseph Ball (USA), Serban Belinschi (France), Gordon Blower (UK), Albrecht Böttcher (Germany), António Caetano (Portugal), Ana Bela Cruzeiro (Portugal), Ken Davidson (Canada), Roland Duduchava (Georgia), Ruy Exel (Brazil), Pedro Freitas (Portugal), Eva Gallardo (Spain), J. William Helton (USA), Rien Kaashoek (Netherlands), Yuri Karlovich (Mexico), Igor Klep (New Zealand), Stephanie Petermichl (France), Lars-Erik Persson (Sweden), Steffen Roch (Germany), Peter Šemrl (Slovenia), Bernd Silbermann (Germany), Orr Shalit (Israel), Frank Speck (Portugal), Ilya Spitkovsky (UAE), Christiane Tretter (Switzerland), Nikolai Vasilevski (Mexico), and Nina Zorboska (Canada).

The organizing committee consists of M. Amélia Bastos (IST, UL), António Bravo (IST, UL), Catarina Carvalho (IST, UL), Luís Castro (U. Aveiro), Alexei Karlovich (FCT, UNL), and Helena Mascarenhas (IST, UL).

Registration is available now. The fee for registration is 220€ (110€ for students and young researchers not older than 30) and should be paid by bank transfer before June 14, 2019. After this date, registration is still available until July 12, 2019, but the fees increase to 280€ and 140€, respectively.

Further information is available at <https://iwota2019.math.tecnico.ulisboa.pt/home>.

International Conference and Ph.D.-Master Summer School on **“Groups and Graphs, Designs and Dynamics” (G2D2)** **Three Gorges Mathematical Research Center, Yichang, China, August 12–25, 2019**

The main goal of this event is to bring together researchers from different fields of mathematics and its applications mainly based on group theory, graph theory, design theory, and the theory of dynamical systems. All scientific activities will take place in the Three Gorges Mathematical Research Center at China Three Gorges University (Yichang, China) from August 12–25, 2019.

G2D2 is concerned with various aspects of mathematics, especially those relating to simple structures and simple processes. Four minicourses and four colloquium talks will let participants see order and simplicity from possibly new perspectives and share insights with experts. We will also schedule invited talks (45 minutes) and contributed talks (20 minutes) with topics ranging from coding theory, design theory, ergodic theory, graph theory, group theory, matrix theory, optimization theory, and quantum information theory to symbolic dynamics.

The scientific committee consists of: Peter Cameron, University of St. Andrews; Genghua Fan, Fuzhou University; Tatsuro Ito, Anhui University; Alexander Ivanov, Imperial College London; Ilia Ponomarenko, St. Petersburg Department of Steklov Institute of Mathematics; Zhiying Wen, Tsinghua University; Qing Xiang, University of Delaware; Mingyao Xu, Peking University; and Xiangdong Ye, University of Science and Technology of China.

Selected papers based on talks from G2D2 will be published in a special issue of *The Art of Discrete and Applied Mathematics*. For further information, see <http://math.sjtu.edu.cn/conference/G2D2>.

International Conference on Matrix Analysis and its Applications (MAT-TRIAD 2019) Liblice, Czech Republic, September 8–13, 2019

MAT-TRIAD provides an opportunity to bring together researchers sharing an interest in a variety of aspects of matrix analysis and its applications to other areas of science. Researchers and graduate students interested in recent developments in matrix and operator theory and computation, spectral problems, applications of linear algebra in statistics, statistical models, matrices and graphs, as well as combinatorial matrix theory are particularly encouraged to attend.

The invited speakers are: Dario Bini, University of Pisa, Italy; Mirjam Dür, University of Augsburg, Germany; Shmuel Friedland, University of Illinois, Chicago, USA (The Hans Schneider ILAS Lecturer); Arnold Neumaier, University of Vienna, Austria; Martin Stoll, Technical University of Chemnitz, Germany; and Zdeněk Strakoš, Charles University, Prague, Czech Republic. There will also be two invited talks by recipients of Young Scientists Awards from MatTriad'2017: Álvaro Barreras, Universidad Internacional de La Rioja, Spain; and Ryo Tabata, National Institute of Technology, Fukuoka, Japan.

The scientific committee consists of Tomasz Szulc, chair (Poland); Natália Bebiano (Portugal); Ljiljana Cvetković (Serbia); Heike Faßbender (Germany); and Simo Puntanen (Finland). Contact the organizers at mattriad@math.cas.cz or get further information at: <http://mattriad.math.cas.cz>.

Householder Symposium XXI on Numerical Linear Algebra Selva di Fasano, Italy, June 14–19, 2020

The next Householder Symposium will be held from June 14–19, 2020 at Hotel Sierra Silvana, Selva di Fasano (Br), Italy. Attendance is by invitation, and participants are expected to attend the entire meeting. Applications are solicited from researchers in numerical linear and multilinear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control. Each attendee is given the opportunity to present a talk or poster. Some talks will be plenary lectures, while others will be shorter presentations arranged in parallel sessions. Applications are due by October 31, 2019.

This meeting is the twenty-first in a series, previously called the Gatlinburg Symposia, but now named in honor of its founder, Alston S. Householder, a pioneer of numerical linear algebra. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1, 2017 will be presented. Nominations are due by January 31, 2020.

ILAS 2020: Classical Connections Galway, Ireland, June 22–26, 2020

The 23rd meeting of the International Linear Algebra Society, ILAS 2020: Classical Connections, will be hosted by the School of Mathematics at the National University of Ireland, Galway, June 22–26, 2020. For an impression of the venue, see <http://www.nuigalway.ie/visitors>.

JOURNAL ANNOUNCEMENT

Special Matrices Special Issue for Charles R. Johnson

The De Gruyter journal *Special Matrices* is pleased to announce a special issue in honor of Charles R. Johnson in recognition of his numerous and profound contributions to the fields of matrix theory and linear algebra. Submissions which cover one or more of the following topics are particularly welcome: matrix completion problems; possible eigenvalue multiplicities for matrices with a given graph; the nonnegative inverse eigenvalue problem; positive definite matrices; the field of values; and matrix stability.

Prior to submission, authors should carefully read over the journal guidelines, which can be located online at <https://www.degruyter.com/view/j/spma>. All manuscripts are subject to the standard peer-review process, and should be submitted online via <http://www.editorialmanager.com/spma>. When entering your submission, please choose the option “Special Issue Dedicated to Charles R. Johnson”. The deadline for submissions is August 31st, 2019.

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IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 60-1, 61-1, 61-2 and 61-4. Solutions are invited to Problems 59-5, 60-2, 60-4, 61-3 and for all of the problems of issue 62.

Problem 60-1: Skew-Symmetric Forms and Determinants

Proposed by Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $n \geq 2$, let $A \in M_n(\mathbb{R})$ be a rank r skew-symmetric matrix, and let $x, y \in \mathbb{R}^n$. Express $x^T A y$ as a weighted sum of $\frac{r}{2}$ determinants of matrices, with each matrix having x and y as two of its columns. Show that $\frac{r}{2}$ is the smallest number of determinants that suffice to represent $x^T A y$. (Examples: If $n = 2$ and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then

$$x^T A y = a \det \begin{pmatrix} x & y \end{pmatrix}. \quad (1)$$

If $n = 3$ and $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, then

$$x^T A y = \det \begin{pmatrix} x & y & v \end{pmatrix}. \quad (2)$$

where $v = (c, -b, a)^T$.

Solution 60-1 by Omran KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

Since A is skew-symmetric, we know that its rank r is even so let $p = r/2$. Further, by the Spectral Theorem we know that there exists an orthonormal basis $(\epsilon_1, \dots, \epsilon_n)$ of $\mathbb{R}^n$ and real numbers $\lambda_1, \dots, \lambda_p$ such that for $1 \leq k \leq p$ we have

$$A\epsilon_{2k-1} = -\lambda_k \epsilon_{2k}, \text{ and } A\epsilon_{2k} = \lambda_k \epsilon_{2k-1}.$$

and $A\epsilon_k = 0$ for $2p < k \leq n$.

Now, if $x, y \in \mathbb{R}^n$, then $x = \sum_{i=1}^n \alpha_i \epsilon_i$ and similarly $y = \sum_{i=1}^n \beta_i \epsilon_i$. Thus,

$$\begin{aligned} x^T A y &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \epsilon_i^T A \epsilon_j = \sum_{i=1}^{2p} \sum_{j=1}^p (\alpha_i \beta_{2j-1} \epsilon_i^T A \epsilon_{2j-1} + \alpha_i \beta_{2j} \epsilon_i^T A \epsilon_{2j}) \\ &= \sum_{j=1}^p \lambda_j (-\alpha_{2j} \beta_{2j-1} + \alpha_{2j-1} \beta_{2j}) = \sum_{j=1}^p \lambda_j \det \begin{pmatrix} \alpha_{2j-1} & \beta_{2j-1} \\ \alpha_{2j} & \beta_{2j} \end{pmatrix} \\ &= \sum_{j=1}^p \lambda_j \det(\epsilon_1, \dots, \epsilon_{2j-2}, x, y, \epsilon_{2j+1}, \dots, \epsilon_n). \end{aligned}$$

Thus, $x^T A y$ can be expressed as a weighted sum of $r/2$ determinants of matrices, with each matrix having x and y as two of its columns.

Now we show that $r/2$ is the smallest number of such determinants that can be used in such a representation.

Let $\{v_3^{(k)}, \dots, v_n^{(k)} : 1 \leq k \leq m\}$ be a set of vectors in $\mathbb{R}^n$ such that for all $x, y \in \mathbb{R}^n$ we have

$$x^T A y = \sum_{k=1}^m \alpha_k \det(x, y, v_3^{(k)}, \dots, v_n^{(k)}).$$

If $\mathbf{e} = (e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{R}^n$ and we consider the matrix $A_k = (a_{ij}^{(k)}) \in M_n(\mathbb{R})$, defined by, $a_{ij}^{(k)} = \alpha_k \det(e_i, e_j, v_3^{(k)}, \dots, v_n^{(k)})$, then we see immediately that $A = A_1 + \dots + A_m$. Further, if $v_3^{(k)}, \dots, v_n^{(k)}$ are linearly independent, then $A_k v_j^{(k)} = 0$, for $j = 3, \dots, n$, so $\dim \ker(A_k) \geq n - 2$, and consequently, $\text{rank}(A_k) \leq 2$. Meanwhile if

$v_3^{(k)}, \dots, v_n^{(k)}$ are linearly dependent then $A_k = 0$ and again $\text{rank}(A_k) = 0 \leq 2$. Thus

$$\text{rank}(A) \leq \sum_{k=1}^m \text{rank}(A_k) \leq 2m.$$

We conclude that $m \geq r/2$.

Problem 61-1: Ranks of Stacked Permutation Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let $\{P_j\}_{j=1}^{m!}$ be the $m \times m$ permutation matrices listed in any order. Furthermore, let $\text{vec}(P_j)$ denote the $m^2 \times 1$ matrix obtained by stacking the m columns of P_j into a single column with the leftmost column of P_j being at the top of $\text{vec}(P_j)$ and the rightmost column being at the bottom. Moreover, let Q be the $m^2 \times m!$ matrix whose j th column is $\text{vec}(P_j)$, and let R be the submatrix of Q consisting only of the columns of Q formed from the permutation matrices which happen to be symmetric. Find the ranks of Q and R .

Solution 61-1 by Ravindra BAPAT, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in

We will show that the rank of Q is $(m-1)^2 + 1$ and the rank of R is $\frac{m(m-1)}{2} + 1$. The problem clearly admits the following reformulation. Let $\mathcal{V}$ be the subspace of $\mathcal{M}_m$, the space of $m \times m$ matrices, spanned by the permutation matrices of order m and let $\mathcal{W}$ be the subspace of $\mathcal{M}_m$ spanned by the symmetric permutation matrices of order m . Then find the dimensions of $\mathcal{V}$ and $\mathcal{W}$.

It is well-known that the dimension of $\mathcal{V}$ is $(m-1)^2 + 1$, see, for example, [1, Theorem 2.1] or [2, Theorem 6]. However we sketch a proof. Let $\mathcal{U}$ be the vector space of $m \times m$ matrices with equal row and column sums. Clearly $\mathcal{V} \subset \mathcal{U}$. We claim that $\mathcal{V} = \mathcal{U}$. If $X \in \mathcal{U}$, then there exist $\alpha > 0$ and $\beta > 0$ such that $\beta(X + \alpha J)$ is doubly stochastic, where J denotes the matrix of all ones. By the Birkhoff-von Neumann Theorem, $\beta(X + \alpha J)$ is a convex combination of permutation matrices. Since J is clearly a positive linear combination of permutation matrices, it follows that X is a linear combination of permutation matrices and hence $\mathcal{V} = \mathcal{U}$.

We now construct a basis for $\mathcal{U}$. For $i, j = 1, \dots, m-1$, let E_{ij} be the $m \times m$ matrix with 1 at positions (i, j) and (m, m) , -1 at positions (i, m) and (m, j) , and zeros elsewhere. Let $X \in \mathcal{U}$ and suppose the row and column sums of X are all equal to α . Then

$$X = \frac{\alpha}{m}J + \sum_{i,j=1}^{m-1} \left(x_{ij} - \frac{\alpha}{m}\right) E_{ij}.$$

Since the $E_{ij}, i, j = 1, \dots, m-1$ and J are linearly independent (as may be seen by noting that for any $1 \leq k, l \leq m-1$, E_{kl} is the only matrix in the set of the E_{ij} s that has a nonzero (k, l) -entry) they form a basis for $\mathcal{U}$ and it follows that $\mathcal{U}$ has dimension $(m-1)^2 + 1$.

We now turn to the dimension of $\mathcal{W}$. Let $\mathcal{Z}$ be the vector space of symmetric $m \times m$ matrices with equal row and column sums. Clearly $\mathcal{W} \subset \mathcal{Z}$.

Now suppose $A \in \mathcal{Z}$. For $i \neq j$, let P_{ij} be the symmetric permutation matrix with ones at the positions (i, j) and (j, i) as well as at the positions (k, k) whenever $k \neq i, j$. Let $B = \sum_{1 \leq i < j \leq m} a_{ij} P_{ij}$. Then $B \in \mathcal{W} \subset \mathcal{Z}$. Since A and B have the same off-diagonal entries, $A - B$ is a diagonal matrix in $\mathcal{Z}$ and hence must be a multiple of the identity. Since both B and $A - B$ are in the subspace $\mathcal{W}$, we must have $A \in \mathcal{W}$. Therefore $\mathcal{W} = \mathcal{Z}$.

We now construct a basis for $\mathcal{Z}$. For $i = 1, \dots, m-1$, let F_i be the $m \times m$ matrix with 1 at positions (i, i) and (m, m) , -1 at positions (i, m) and (m, i) , and zeros elsewhere. For $i, j = 1, \dots, m-1, i < j$, let F_{ij} be the $m \times m$ matrix with 1 at positions (i, j) and (j, i) , -1 at positions $(i, m), (j, m), (m, i), (m, j)$, 2 at position (m, m) and zeros elsewhere. Let $X \in \mathcal{Z}$ and suppose the row and column sums of X are all equal to α . Then it can be verified that

$$X = \frac{\alpha}{m}J + \sum_{i=1}^{m-1} \left(x_{ii} - \frac{\alpha}{m}\right) F_i + \sum_{1 \leq i < j \leq m-1} \left(x_{ij} - \frac{\alpha}{m}\right) F_{ij}.$$

Thus if $X \in \mathcal{Z}$, then X can be expressed as a linear combination of $F_i, i = 1, \dots, m-1; F_{ij}, i, j = 1, \dots, m-1, i < j$ and J . Since $F_i, i = 1, \dots, m-1; F_{ij}, i, j = 1, \dots, m-1, i < j$, and J are linearly independent (This may be seen by noting

that for any $1 \leq k, l \leq m-1$, there is only one matrix in the union of the set of all F_i 's and the set of all F_{ij} 's that has a nonzero (k, l) -entry. This matrix is F_k when $k = l$ and F_{kl} when $k < l$. It follows that $\mathcal{Z}$ has dimension $\frac{m(m-1)}{2} + 1$.

References

- [1] P. M. Gibson. Generalized doubly stochastic and permutation matrices over a ring. *Linear Algebra Appl.*, 30:101–107, 1980.
- [2] D. B. Leep and G. Myerson. Marriage, magic, and solitaire. *Amer. Math. Monthly*, 106(5):419–429, 1999.

Problem 61-2: Circulant Matrices over a Finite Field

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let p be a prime number and C be a $p \times p$ circulant matrix over the finite field with p elements. Show that the determinant of C is also an eigenvalue of C . (Recall: A $p \times p$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod{p}$.)

Solution 61-2 by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

We will use the following three facts:

- (i) With prime p , the binomial symbols satisfy $\binom{p}{m} \equiv 0 \pmod{p}$ when $1 \leq m \leq p-1$. Hence, if A and B are commuting matrices over the finite field with p elements, then $(A+B)^p = A^p + B^p$ by the Binomial Theorem.
- (ii) A circulant matrix $A \in \mathcal{M}_p(\mathbb{Z}_p)$ equals $A = \sum_{k=0}^{p-1} c_k P^k$, where P is the permutation matrix corresponding to the long cycle $(1, 2, \dots, p-1, p)$.
- (iii) A scalar λ is an eigenvalue of a matrix A if and only if $\det(A - \lambda I) = 0$.

By repeated application of fact (i),

$$A^p = \left(\sum_{k=0}^{p-1} c_k P^k \right)^p = \sum_{k=0}^{p-1} (c_k)^p P^{kp} = \sum_{k=0}^{p-1} c_k I,$$

where in the last equality we used Fermat's Little Theorem, which states that in a finite field with p elements, we have $x^p = x$. Hence,

$$\det A = (\det A)^p = \det(A^p) = \det \left(\sum_{k=0}^{p-1} c_k I \right) = \sum_{k=0}^{p-1} c_k.$$

It follows that $A - (\det A)I = -(c_1 + \dots + c_{p-1})I + \sum_{k=1}^{p-1} c_k P^k$, so by repeating the arguments above,

$$\det(A - (\det A)I) = -(c_1 + \dots + c_{p-1}) + \sum_{k=1}^{p-1} c_k = 0.$$

Problem 61-4: The Sarrus Number of a Positive Semidefinite Matrix

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Sarrus' Rule is a quick way of calculating the determinant of a 3×3 matrix:

$$\det(A) = \left(\sum_{k=0}^2 \prod_{i=1}^3 a_{i,k+i} \right) - \left(\sum_{k=0}^2 \prod_{i=1}^3 a_{i,k-i} \right),$$

where A is a 3×3 matrix and $k+i$ and $k-i$ are taken mod 3.

For general $n \times n$ matrices, we can define the Sarrus number analogously:

$$\text{Sar}(A) = \left(\sum_{k=0}^{n-1} \prod_{i=1}^n a_{i,k+i} \right) - \left(\sum_{k=0}^{n-1} \prod_{i=1}^n a_{i,k-i} \right),$$

where A is an $n \times n$ matrix and $k+i$ and $k-i$ are taken mod n . When $n \neq 3$, the Sarrus number and the determinant of a matrix may no longer be equal. Show, however, that the Sarrus number of a positive semidefinite matrix is always nonnegative.

Solution 61-4 by Ravindra BAPAT, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in

Let A be an $n \times n$ matrix. If G is a subgroup of S_n and χ a real-valued character on G , then

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

is called a generalized matrix function. It follows from a well-known theorem of Schur that if A is positive semidefinite, then $d_\chi(A) \geq 0$ [1, page 214].

Let μ_k and η_k be the permutations of $\{1, \dots, n\}$ given by $\mu_k(i) = k + i$ and $\eta_k(i) = k - i$ for $k = 0, 1, \dots, n-1$, where $k + i$ and $k - i$ are taken mod n . Note that $\mu_0 = \text{id}$, the identity permutation.

The following facts are easily verified: (i) $\mu_1^k = \mu_k$ (ii) $\eta_k^2 = \text{id}$ (iii) $\mu_1 \eta_k = \eta_{k-1}$ (iv) $\eta_k \mu_1 = \eta_{k+1}$, for $k = 0, 1, \dots, n-1$.

It follows from these facts that if $G = \{\mu_k, \eta_k : k = 0, 1, \dots, n-1\}$, then G is a subgroup of S_n . Furthermore, the function χ given by $\chi(\mu_k) = 1$ and $\chi(\eta_k) = -1$ for $k = 0, 1, \dots, n-1$ is a character on G .

Now note that

$$\text{Sar}(A) = \left(\sum_{k=0}^{n-1} \prod_{i=1}^n a_{i, k+i} \right) - \left(\sum_{k=0}^{n-1} \prod_{i=1}^n a_{i, k-i} \right)$$

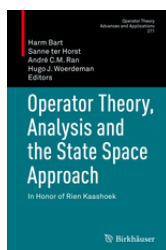
is a generalized matrix function and hence is nonnegative if A is positive semidefinite.

Reference

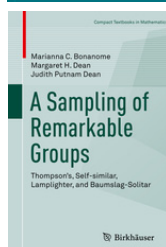
- [1] Russell Merris, *Multilinear Algebra*, Gordon and Breach Science Publishers, Amsterdam, 1997.

Editor's Note: In the above solution, the group G is the Dihedral group D_n . The reason why Sarrus' rule works for 3×3 matrices is that $D_3 = S_3$.

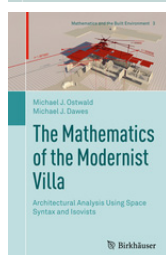
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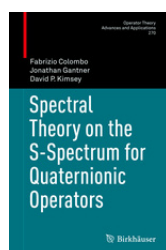
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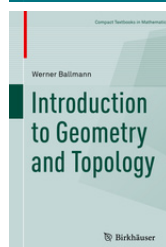
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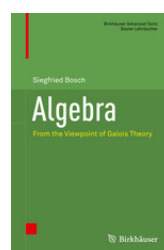
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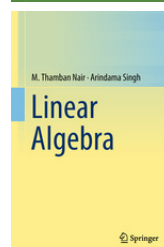
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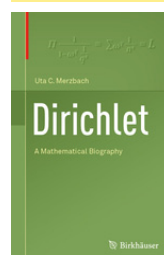
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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce three new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.

Submissions: Please submit proposed problems and solutions in macro-free L^AT_EX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 62-1: Permuted Circulants with Distinct Diagonal Elements

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let n be an even number. Show that if C is an $n \times n$ circulant matrix and P and Q are two $n \times n$ permutation matrices, then the matrix product PCQ never has n distinct entries on its main diagonal. (Recall: An $n \times n$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod n$.)

Problem 62-2: Vector Spaces over Finite Fields

Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Let V be a d -dimensional vector space over the finite field with q elements. A subset S of V is said to be *hyperplane complete* if every element of V can be written as a linear combination of at most $d - 1$ elements from S . Show that there is a hyperplane complete subset of V having at most $q + d - 1$ elements. If $d \geq q$, show that there is a hyperplane complete subset of V which has $d + 1$ elements.

Problem 62-3: A Convex Rank Sequence

Proposed by Sneha SANJEEVINI, *University of Michigan, Ann Arbor, MI, USA*, snehasnj@umich.edu

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and Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $T_0, T_1, T_2, \dots$ be $m \times n$ matrices with entries in a field $\mathbb{F}$. For each positive integer k , define the block-Toeplitz matrix $\mathcal{T}_k$ by

$$\begin{bmatrix} T_0 & 0 & 0 & 0 & 0 \\ T_1 & T_0 & \cdots & 0 & 0 \\ T_2 & T_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & T_0 & 0 \\ T_{k-1} & T_{k-2} & \cdots & T_1 & T_0 \end{bmatrix}.$$

Show that the sequence $(\text{rank } \mathcal{T}_k)_{k \geq 1}$ is convex; that is,

$$2 \text{rank } \mathcal{T}_{k+1} \leq \text{rank } \mathcal{T}_{k+2} + \text{rank } \mathcal{T}_k, \quad \text{for } k = 1, 2, \dots$$

Example: For $k = 2$, the inequality becomes

$$2 \text{rank} \begin{bmatrix} T_0 & 0 & 0 \\ T_1 & T_0 & 0 \\ T_2 & T_1 & T_0 \end{bmatrix} \leq \text{rank} \begin{bmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ T_2 & T_1 & T_0 & 0 \\ T_3 & T_2 & T_1 & T_0 \end{bmatrix} + \text{rank} \begin{bmatrix} T_0 & 0 \\ T_1 & T_0 \end{bmatrix}.$$

Solutions to Problems 60-1, 61-1, 61-2 and 61-4 are on pages 47–50.