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FEATURE INTERVIEW

“New coauthors bring new friends and travels”

Vladimir Sergeichuk Interviewed by Roger Horn¹

R.H. - You were born in 1949 north of Kiev, near Ukraine’s current border with Belarus. During your lifetime, Ukraine has experienced famine, population transfers, a nuclear reactor disaster, the dissolution of the Soviet Union, independence, several revolutions, the occupation of Crimea, and an ongoing armed insurgency in Eastern Ukraine. How did these events affect you, your family, and the scientific community in Ukraine?

V.S. - I was born 32 years after the October 1917 revolution led by the Bolshevik Party of Vladimir Lenin. These years were very hard for the population of the new country. Many events could have prevented me from being born:

My grandfather had a water mill. It was hard work; he had to raise sacks of grain to the top of his mill (which strengthened his health; he lived for 94 years). More than 1.8 million such “prosperous” peasants were deported, mainly to Siberia from 1930–1931, as part of the Soviet program of the collectivization of the agricultural sector. A third of them died from 1929 to 1933. My grandfather was not deported, just because he regularly fed and treated with moonshine the members of the collectivization commission and gave a bribe to its chairman.

My parents got married in 1940, when they studied at the Leningrad Agricultural Academy. My father was drafted into the army at the end of the year. After six months, Nazi Germany attacked the USSR. The Soviet army was not ready for war and only 36% of its soldiers survived the first four months of the war.



Vladimir Sergeichuk (left) with Roger Horn on the shore of Great Salt Lake, Utah, 2002

In the early days of war, the Soviet army retreated erratically. My father was a clerk at the headquarters of the regiment on the border with Germany. A car with staff documents could not pass over a destroyed bridge and was abandoned. The colonel who led the headquarters was shot for this. My father was arrested and also sentenced to be shot. By the third day, he did not care whether he was shot or not. An officer recruited sappers and my father was offered to become a sapper instead of execution. After that, he was not afraid of anything in the war. He believed that he would be killed sooner or later.

At the beginning of the war, my mother studied at the Leningrad Agricultural Academy. A stranger (with whom they later became friends) helped my mother get on the train and leave Leningrad. She saved my mom’s life. The Siege of Leningrad (September 8, 1941 to January 27, 1944) resulted in the deaths of over a million Soviet soldiers and civilians. This is more than the loss of US and British troops throughout the war.

The economy of the USSR was developing rapidly until the 1960s, and then growth slowed. The negative trends intensified in 1985, when Gorbachev started reforms to restructure society and the economy, which led to the dissolution of the Soviet Union into 15 independent republics in 1991.

The economy of independent Ukraine was in deep depression until 1999; its gross domestic product had fallen to 40% of the 1991 level. My dentist told me that some of her patients broke their teeth in their sleep, which had not happened before. From 2000–2013, the economy of Ukraine grew every year, excluding its fall by 15% in 2008. My student Andrii Dmytryshyn reminded me recently that in 2007 I said in my lectures that students do not have to leave Ukraine since the quality of life increases each year.

But the economic situation worsened after the Euromaidan (Nov. 2013 – Feb. 2014). On the wave of patriotism, non-specialists came to the leadership of the country. In the first three years, more than half of commercial banks were liquidated; the national currency rate collapsed three times. Many factories were closed because of the embargo on trade

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with Russia. Many Ukrainians left the country. The number of labor migrants from Ukraine to Poland increased fivefold, from 320,000 in 2014 to about 1,500,000. During the years of independence, the number of scientists in Ukraine decreased by a factor of 5.

The first 30 years of the Soviet Union were terrible because of the Stalinist repressions. Nearly one million were executed or exiled from 1937–1938. However, one decision of Joseph Stalin dramatically increased the effectiveness of Soviet science. At the end of World War II, he summoned the Minister of Education and said, “Pay attention to the scientists. They are modest people and will not ask for themselves.” He ordered a raise in the salary of scientists by 5–10 times. Academician Alexander Sheindlin wrote: “One fine morning, I woke up rich.” Then the salary of scientists gradually decreased. My father headed a department in the Ukraine Ministry of Agriculture. In 1967, he defended his Ph.D. thesis and started to head a department at the Institute of Agriculture. His salary doubled exactly. The last Soviet president, Mikhail Gorbachev, resented that his salary was half the salary of the President of the Academy of Sciences.

I am happy that I studied and became a mathematician in the post-Stalin era, when life in the USSR was most quiet and comfortable. The quality of life of the Soviet people by the 1980s was comparable to that of the Americans. (André Weil said that the Soviet Union and the USA were quantitative civilizations.) The country was socially oriented: free medicine, the best school education system in the world, scholarships for university students, there were no homeless. Full employment, no fear of not being able to provide for the family. Public safety, I was not afraid to walk in Kiev at night. People helped each other. However, labor productivity was low. Heart surgeon Nikolai Amosov believed that socialism contradicts human nature. He wrote that a person is lazy by nature; most people need both positive and negative incentives to work hard, but socialism does not give negative incentives.

R.H. - As a young man, you were interested in relativity and physics. How and when did you get seriously interested in mathematics? What made you decide to become a mathematician?

When I was 12 and 13 years old, I did not do school homework because I went to the chess club every day. I did the right thing. Playing chess, and especially solving chess problems, is almost abstract algebra; they are more useful for the future mathematician than the primitive mathematics which is taught for teens of this age. I played in a qualifying tournament, won first place, and left chess since science is more interesting.

It was the time of the first space flights; interest in science was very high. I read several popular books on the special and general theory of relativity. It was very unlike the physics that was taught in school. I read a textbook on physics for technical engineering and I was amazed at how effectively derivatives and integrals work in mechanics.

At that time, it was difficult to enter Kiev State University. When I entered the mathematics department, only one of seven applicants became a student. They were selected by the results of exams in mathematics and physics that were difficult. For more than a year, I was prepared for the examinations in mathematics by Yuri Prizva, who was a Ph.D. student in the mathematics department and the son of my father’s friend. Under Yuri’s influence, I entered the mathematics department of the university.

R.H. - You were a student of the great algebraist Andrei Vladimirovich Roiter, who was a student of Dmitry Konstantinovich Faddeev. Roiter had 13 Ph.D. students; please tell us something about how he worked with them, and his influence on your career. Did you have any memorable experiences participating in Roiter’s seminar?

I am very happy that I started to work with Roiter when I was a student. Andrei Roiter and his wife Liudmila Nazarova graduated from Leningrad State University in 1960 and began their Ph.D. studies under the direction of Dmitry Faddeev. When Roiter wrote his doctoral dissertation on integral representations of rings, he solved many problems about reduction of matrices, which he considered as secondary. However, after defending his thesis, his opinion completely changed. He started to develop the theory of matrix problems. He is one of the founders of the modern theory of representations of finite dimensional algebras, which is in fact the theory of systems of linear mappings satisfying some polynomial relations; a part of linear algebra.

Crucial to me was Roiter’s seminar on representation theory at the Institute of Mathematics in Kiev. Professor Claus Ringel from Bielefeld University called it fantastic. Modern representation theory had just been created and every week the seminar participants told about their new results. It was very instructive to learn from Roiter about his research, the research of other mathematicians, and about his interpretation of new mathematical results. Roiter developed a new direction in mathematics, and so he always had many problems. He generously shared them. In the summer, we all went outside of town to play volleyball. Roiter was a very honest and principled man.

R.H. - Your published work has more than 40 coauthors so far. What do you think of George Mackey's comment that "the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism"?

I agree with Mackey's justification of his comments: "research in pure mathematics is a very cooperative activity in which everyone builds on the work of someone else and in turn has his own work built upon. On the other hand, mathematicians tend to work alone (and occasionally in pairs) and to be intensely individualistic." Every mathematician solves problems by his own methods. I could not solve some problems without my collaborators. I learn from each of them. Without them, my life would be much duller. New coauthors bring new friends and travels. Immediately after the collapse of the USSR, I was three times in Zurich thanks to collaboration with Peter Gabriel. (He introduced quiver representations. As my supervisor A.V. Roiter said, definitions are more important than theorems.) I lived for about a year in Salt Lake City in Roger Horn's house. (I am grateful to Roger and Susan Horn for their fantastic hospitality.) I was six times in Beer-Sheva, Israel due to Genrich Belitskii. I lived for more than a year in Al Ain (UAE) due to Victor Bovdi. Now I am in São Paulo until May 2020 due to Vyacheslav Futorny. By his invitation, I have been living in São Paulo for more than three years during several visits. I really like São Paulo for its wonderful weather, fruits, and people.

I am also very grateful to my students and coauthors Lena Klimenko, Nadya Shvai (Zharko), Tatiana Gerasimova, Andrii Dmyrshun, and Tatiana Klymchuk. They activate me and add new colors to my life.

R.H. - Camille Jordan's canonical form is a standard textbook topic. The canonical form of his Czech contemporary Eduard Weyr is less well known, but you played a role in its rediscovery and use. Tell us something about that. What are the relative advantages of these two canonical forms?

In 1984, I published an algorithm for reducing a square complex matrix to canonical form under transformations of unitary similarity. From Helene Shapiro's survey [5], I learned that Dudley Littlewood published this algorithm in 1953. I tried unsuccessfully to construct a similar algorithm for a pair of matrices under similarity. Then I found it in Genrich Belitskii's paper [1] published in 1983 in the collection of works of the Kharkov Institute of Low Temperatures. Belitskii's algorithm reduces each matrix pair by simultaneous similarity transformations to a "canonical" pair such that two pairs are simultaneously similar if and only if their canonical pairs coincide. In his algorithm, Belitskii uses a "modified Jordan matrix"; it is permutation similar to a Jordan matrix and has the following property: The set of matrices commuting with it consists of all block triangular matrices, with fixed partition into blocks, in which some blocks must be zero or must be equal.

During 1988–1992, P. Gabriel, L.A. Nazarova, A.V. Roiter, D. Vossieck and I proved the geometric form of Yuriy Drozd's "Tame-Wild Theorem." (I never worked so hard!) It was published in [2]. Still working on the proof of this statement, I began to think about a new proof based on Belitskii's algorithm. I published it in [4]. Presenting Belitskii's algorithm, I used the term "Weyr matrices" instead of Belitskii's "modified Jordan matrices," since their partition into blocks is determined by Weyr characteristics. My article is very complicated; I worked on it for many years. I twice e-mailed preliminary versions of my article to Helene Shapiro and she found these matrices in Eduard Weyr's articles.

R.H. - Please tell us about some of your results that you like best.

My supervisor Andrei Roiter taught me how to reduce matrices. Almost all my articles are on the reduction of matrices or sets of matrices to canonical or simple forms with respect to some set of admissible transformations.

I am proud of the method that was developed by Andrei Roiter and me. It reduces the problem of classifying systems of linear mappings and bilinear or sesquilinear forms to the problem of classifying systems of linear mappings. I applied it to many classification problems, including the canonical form problems for bilinear and sesquilinear forms, pairs of symmetric or skew-symmetric forms, pairs of Hermitian forms, and isometric or self-adjoint operators on a space with indefinite scalar product. In [3], I solved them over any field of characteristic not 2 up to classification of Hermitian forms over finite extensions of the field. Roger Horn and I gave simple canonical forms of real, complex, and quaternion matrices under congruence and $*$ -congruence.

I like my articles in which Belitskii's and Littlewood's algorithms are generalized to representations of quivers and unitary representations of quivers; that is, to finite sets of vector spaces or inner product spaces and linear mappings between them. I study canonical forms of their matrices.

I also like the articles in which my coauthors and I construct miniversal deformations of matrix pencils, contragredient matrix pencils, and matrices under congruence and $*$ -congruence.

R.H. - What suggestions do you have for young mathematicians beginning their careers?

Actively seek a good supervisor. Choose among those who are actively engaged in research and have articles in leading journals.

Limit yourself. The first chief designer of the Soviet space program, Sergei Korolev, wrote, “Do only the most important, otherwise the secondary will fill your life, take away all your energy, and you will never proceed to the main.”

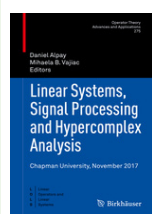
Work hard while you are young. The best age for doing math is under 30; then you can relax. G. H. Hardy wrote, “No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man’s game.”

Have an exercise-related hobby such as sports or dancing. You need positive emotions when you work hard on a difficult problem for a long time.

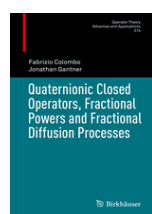
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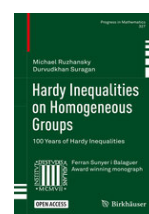
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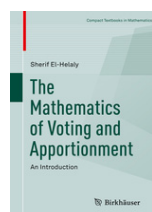
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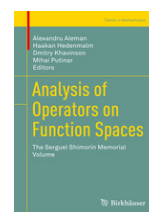
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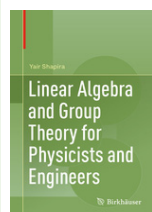
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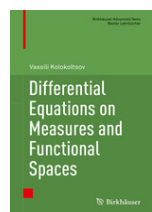
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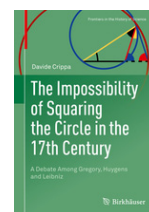
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FEATURE ARTICLE

The Early Development of the Quaternionic Numerical Range

Wasin So, San Jose State University, San Jose, CA, USA, wasin.so@sjsu.edu

1. Introduction. Let $\mathbb{H}$ be the skew algebra of quaternions generated by $\{1, i, j, k\}$ over the reals $\mathbb{R}$:

$$\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Let $q_0 = \text{Re}(q)$, $q_0 + q_1i = \text{Co}(q)$, and $q_1i + q_2j + q_3k = \text{Im}(q)$. Hence, the complex algebra can be identified as $\mathbb{C} = \{q_0 + q_1i : q_0, q_1 \in \mathbb{R}\}$, and we have $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. Define the conjugate of q as

$$\bar{q} = q_0 - q_1i - q_2j - q_3k,$$

and the length of q as

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Note that $q\bar{q} = \bar{q}q = |q|^2$. Given two quaternions α and β , we say α is *similar* to β if there is a quaternion q of unit length such that $\alpha = \bar{q}\beta q$, or equivalently, $\text{Re}(\alpha) = \text{Re}(\beta)$ and $|\text{Im}(\alpha)| = |\text{Im}(\beta)|$. In particular, α and $\bar{\alpha}$ are similar. Note that the similarity of quaternions is an equivalence relation. Let $[\lambda] = \{\bar{q}\lambda q : |q| = 1\}$ denote the equivalence class with representative λ . Explicitly,

$$[\lambda] = \text{Re}(\lambda) + \{q_1i + q_2j + q_3k : q_1^2 + q_2^2 + q_3^2 = |\text{Im}(\lambda)|^2\},$$

and so

$$\text{conv}([\lambda]) = \text{Re}(\lambda) + \{q_1i + q_2j + q_3k : q_1^2 + q_2^2 + q_3^2 \leq |\text{Im}(\lambda)|^2\}$$

where $\text{conv}(\cdot)$ denotes the convex hull of a set.

Definition 1. Let A be an $n \times n$ quaternionic matrix. The quaternionic numerical range (QNR) of A is defined as

$$W(A) = \{x^*Ax : x^*x = 1, x \in \mathbb{H}^n\} \subset \mathbb{H},$$

where $*$ denotes the conjugate transpose of a vector or matrix.

Similarly, we have the real numerical range (RNR) when everything is real, and the complex numerical range (CNR) when everything is complex. It is known that the RNR is a closed interval (maybe a degenerate one, i.e., a singleton). It is also known that the CNR is always convex, a result due to Toeplitz and Hausdorff. In general, the QNR is hard to compute due to the non-commutativity of quaternionic multiplication, and is even harder to display because it is a 4-dimensional object over the reals. In view of the RNR and CNR, a natural question about the QNR is:

Is the QNR always convex?

The answer is: *No!* And counterexamples are not hard to find. Indeed, it is interesting to observe that the equivalence class $[\lambda]$ is actually the QNR of a 1×1 matrix with the single entry λ . Moreover, we have:

Theorem 1. Let A be 1×1 . Then $W(A)$ is convex if and only if A is a real matrix.

Therefore, a counterexample can be obtained by taking A to be the 1×1 matrix with the single entry i . Then $W(A) = [i]$ is *not* convex, because $i, -i \in [i]$ but their midpoint $\frac{1}{2}(i + (-i)) = 0 \notin [i]$. Nonetheless, there are still many interesting questions worth asking about the QNR:

Q1: Which part of a QNR is convex?

Q2: What are the necessary and sufficient conditions on a matrix to guarantee a convex QNR?

Q3: Which classes of matrices guarantee a convex QNR?

In the rest of this article, we trace the early history of the QNR by following chronologically a dozen documents and a talk, which together record how these questions have been tackled. Along the way, we try to shed some light on the following mysteries.

M1: Why was the simple counterexample of a 1×1 matrix not mentioned by Kippenhahn?

M2: Why did Jamison not publish his manuscript?

M3: Why did Thompson send Jamison's manuscript to Au-Yeung?

2. Kippenhahn's paper (1951). The first paper on the QNR was written in German by Kippenhahn:

R. Kippenhahn. Über den wertevorrat einer matrix. *Mathematische Nachrichten*, 6(34):193–228, 1951.

In this paper, Kippenhahn discussed both the CNR and QNR. In the QNR part, he gave the definitions of the QNR and the bild, which is defined and denoted as $B(A) = W(A) \cap \mathbb{C}$. He rightly observed that $W(A)$ is uniquely determined by $B(A)$: $W(A) = \bigcup_{\lambda \in B(A)} [\lambda]$. Then he claimed that $W(A)$ is convex based on this property of $B(A)$:

$$\text{If } \alpha \in B(A) \text{ and } 0 \leq t \leq 1 \text{ then } t\alpha + (1-t)\bar{\alpha} \in B(A).$$

But this turns out to be *false!* He had made a fatal mistake in his proof, as was first pointed out by So, Thompson and Zhang (1994) and then fully documented in the English translation of Kippenhahn's paper by Zachlin and Hochsenbach in *Linear and Multilinear Algebra* 56 (2008) 185–225. We now know with hindsight that what Kippenhahn did prove is:

Theorem 2.

$$\text{conv}(B(A)) = W_{\mathbb{C}} \left(\begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix} \right),$$

where $W_{\mathbb{C}}(\cdot)$ denotes the complex numerical range of a complex matrix, and $A = A_1 + jA_2$ with both A_1 and A_2 being complex matrices.

As a consequence, we have a characterization of the convexity of the QNR:

Theorem 3. $W(A)$ is convex if and only if $B(A)$ is convex.

Rudolf Kippenhahn is an interesting figure. The results in his paper were part of his Ph.D. thesis, which was under the supervision of Wilhem Specht (as in Specht's Theorem). This paper is the only paper in mathematics published by Kippenhahn, as he changed his career path to astrophysics, which was his first love. He even became the director of The Max Planck Institute for astrophysics in Germany with a prize named after him. In a 1978 interview (available at <https://www.aip.org/history-programs/niels-bohr-library/oral-histories/5091>), Kippenhahn explained his decision of changing career paths: "I never had the drive really for pure research in mathematics."

It is a mystery why Kippenhahn did not think of the easy counterexample of the 1×1 matrix we mentioned earlier. Possibly he was trying to find examples for all sizes, which is not so straightforward, as the following theorem shows.

Theorem 4. For $n \geq 2$, $W(\lambda I_n)$ is always convex for any quaternion λ (even non-real λ).

3. Jamison's manuscript (1972). James Edward Jamison (1943–2014) wrote his Ph.D. thesis "Extension of some theorems of complex functional analysis to linear spaces over the quaternions and Cayley numbers" in 1970 under Anthony Joseph Penico. In his thesis, Jamison did not study the QNR, but he finished the following manuscript unaware of Kippenhahn's paper.

J. E. Jamison, Numerical range and numerical radius in quaternionic Hilbert space. Unpublished manuscript (1972).

In this manuscript, he observed that the QNR is not convex in general, with the 2×2 counterexample

$$W \left(\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \right) = \{ \overline{x_1} i x_1 + |x_2|^2 : |x_1|^2 + |x_2|^2 = 1 \},$$

which is not convex because it contains both $\frac{1}{2}(1+i)$ and $\frac{1}{2}(1-i)$, but not the midpoint $\frac{1}{2}$. Moreover, he showed:

- $W(A) \cap \mathbb{R}$ is convex.
- If A is Hermitian, then $W(A)$ is convex.

He also proposed the problem of characterizing those matrices with convex QNR.

Another mystery in QNR is why Jamison did not publish his manuscript. According to MathSciNet, Jamison never published a paper on quaternions other than his thesis. In the fall of 1971, he and Richard Fleming started to work together on problems of complex Banach spaces. This collaboration continued until Jamison's death in 2014, and resulted in 21 papers and 2 books. In the article "From helgermites to Lipschitz: remembering Jim Jamison" in *Problems and Recent Methods in Operator Theory, Contemp. Math.*, 687:1–9. Amer. Math. Soc., Providence, RI, 2017, Fleming recalled:

"Throughout the years of collaboration, I often had to remind Jim that we had enough trouble just working on complex spaces."

This may explain why Jamison never had a chance to publish his manuscript on the QNR.

4. Thompson's talk (1982). Robert Charles Thompson (1931–1995) was one of the seven invited speakers at the first SIAM conference on linear algebra. He gave the following talk:

Robert C. Thompson, "Core Linear Algebra," The First SIAM Conference on Linear Algebra, Rayleigh, NC, April 26–29, 1982.

In this talk, he discussed a list of 14 topics that were near and dear to his heart, and the 7th one was numerical range. He mentioned Kippenhahn's claim about the convexity of the QNR. After the talk, Jamison mentioned to Thompson about his work on the QNR. Later, in 1983, Jamison sent Thompson the unpublished manuscript, which was forwarded by Thompson to Yik Hoi Au-Yeung.

A third mystery in QNR is *why* Thompson sent Jamison's manuscript to Au-Yeung. In 1973, Au-Yeung had spent a half-year sabbatical at the University of California, Santa Barbara, where Thompson was a faculty member. This may have been the time when Thompson learned about Au-Yeung's interest in quaternionic matrices. Hence, Thompson sent Jamison's manuscript to Au-Yeung.

5. Au-Yeung's paper (1984). Yik Hoi Au-Yeung's interest in quaternions started when he was a graduate student working on an eigenvalue problem of quaternionic matrices in 1964. In 1983, he received Jamison's manuscript from Thompson. After a few weeks, Au-Yeung was able to finish a paper:

Y. H. Au-Yeung. On the convexity of numerical range in quaternionic Hilbert spaces. *Linear and Multilinear Algebra*, 16(1-4):93–100, 1984.

In this paper, he proved some fundamental results.

- $W(A)$ is convex if and only if $W(A) \cap \mathbb{R} = \{\operatorname{Re}(q) : q \in W(A)\}$.
- $W(A)$ is convex if and only if $W(A) \cap \mathbb{C} = \{\operatorname{Co}(q) : q \in W(A)\}$.
- When A is normal, the convexity of $W(A)$ can be characterized by an explicit criterion on the eigenvalues $h_t + k_t i$ of A where $h_1 \leq h_2 \leq \dots \leq h_n$ and $k_t \geq 0$:

$$W(A) \text{ is convex if and only if } (h_1 - h_2)k_1 = 0 = (h_{n-1} - h_n)k_n.$$

- If A is skew-Hermitian, then $W(A)$ is convex.
- The first published counterexample to the convexity of the QNR:

$$W \left(\begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \text{ is not convex.}$$

6. So, Thompson and Zhang's paper (1994). In the late 1980s, Thompson was so intrigued by the dichotomy between the convexity claim by Kippenhahn and the counterexample by Au-Yeung that he decided to investigate the convexity claim above further. Thompson devised a computer program to visualize the bild for 2×2 matrices. Examples of non-convex bild, and hence non-convex QNR, were easily found. More importantly, after seeing many 2×2 examples, Thompson conjectured that:

- The upper bild $B^+(A) = W(A) \cap \{q_0 + q_1 i : q_0 \in \mathbb{R}, q_1 \geq 0\}$ is convex.

This conjecture was verified for the case when A is a normal matrix by Thompson and two of his students, So and Zhang, in the paper

W. So, R. C. Thompson, and F. Z. Zhang. The numerical range of normal matrices with quaternion entries. *Linear and Multilinear Algebra*, 37(1-3):175–195, 1994.

Moreover, they proved that the shape of $B^+(A)$ is a polygon determined by the eigenvalues of a normal matrix A .

7. Zhang’s paper (1995). Fuzhen Zhang was able to simplify the proof of the convexity of the upper bild for a normal matrix using a limiting argument in the paper

F. Zhang. On numerical range of normal matrices of quaternions. *J. Math. Phys. Sci.*, 29(6):235–251, 1995.

Zhang was a student of Thompson, having written the dissertation

F. Zhang, Permanent inequalities and quaternion matrices. Ph.D. Dissertation, University of California, Santa Barbara, 1993.

He also wrote the following paper in which part of the history of the QNR was told.

F. Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57, 1997.

8. Au-Yeung’s paper (1995). Au-Yeung provided another proof in the following paper using algebra and geometry for the convexity of $B^+(A)$ when A is a normal matrix:

Y. H. Au-Yeung. A short proof of a theorem on the numerical range of a normal quaternionic matrix. *Linear and Multilinear Algebra*, 39(3):279–284, 1995.

9. Au-Yeung’s article (1995). In the following paper, the earlier 1×1 counterexample was mentioned for the first time in the literature.

Y.-H. Au-Yeung. On the eigenvalues and numerical range of a quaternionic matrix. In *Five Decades as a Mathematician and Educator: On the 80th Birthday of Professor Yung-Chow Wong*, pages 19–30. World Sci. Publ., River Edge, NJ, 1995.

Moreover, Au-Yeung proved that:

- For a fixed $b \geq 0$, $W(A) \cap \{t + bi : t \in \mathbb{R}\}$ is convex (possibly empty).

Note that $W(A) \cap \{b + ti : t \in \mathbb{R}\}$ may not be convex.

10. So and Thompson’s paper (1996). In 1992, So attempted to prove the convexity of the upper bild for a general matrix. Although So made some progress, he did not finish. For help he turned to his Ph.D. advisor, Thompson. In 1995, right before he died, Thompson completed the proof of his own conjecture.

W. So and R. C. Thompson. Convexity of the upper complex plane part of the numerical range of a quaternionic matrix. *Linear and Multilinear Algebra*, 41(4):303–365, 1996.

The proof of the main result is very long and computational; it is definitely not an elegant argument, nor does it offer any conceptual understanding.

11. Thompson’s paper (1997). It would be nice if the upper bild could be realised as a CNR, as then its convexity would follow from the Toeplitz-Hausdorff Theorem. Unfortunately, this approach does not work, as shown in the following paper of Thompson:

R. C. Thompson. The upper numerical range of a quaternionic matrix is not a complex numerical range. *Linear Algebra Appl.*, 254:19–28, 1997.

12. So's paper (1998). The QNR convexity criterion in terms of the eigenvalues of a normal matrix is generalized in the following paper:

W. So. An explicit criterion for the convexity of quaternionic numerical range. *Canad. Math. Bull.*, 41(1):105–108, 1998.

In that paper, So proved that:

- $W(A)$ is convex if and only if $(h_1 - h_2)k_1 = 0 = (h_{n-1} - h_n)k_n$,

where $h_1 + k_1i, h_2 + k_2i, \dots, h_n + k_ni$ are called the quasi-diagonal elements of A and are defined as follows. Write $A = H + K$ where $H = \frac{1}{2}(A + A^*)$ and $K = \frac{1}{2}(A - A^*)$. Take a unitary U such that $U^*HU = \text{Diag}(h_1 \leq h_2 \leq \dots \leq h_n)$, and a diagonal unitary D such that the diagonal of D^*U^*KUD is $\text{Diag}(k_1i, k_2i, \dots, k_ni)$ with $k_t \geq 0$ for all t .

Some consequences of this characterization are:

- The QNR of a real matrix is convex because the quasi-diagonal elements of a real matrix are real (i.e., $k_t = 0$ for all t).
- The QNR of a Hermitian matrix is convex because the quasi-diagonal elements of a Hermitian matrix are real (i.e., $k_t = 0$ for all t).
- The QNR of a skew-Hermitian matrix is convex because the quasi-diagonal elements of a skew-Hermitian matrix are imaginary (i.e., $h_t = 0$ for all t).

13. Au-Yeung and Siu's paper (1999). The last paper to discuss is:

Y. H. Au-Yeung and L. S. Siu. Quaternionic numerical range and real subspaces. *Linear and Multilinear Algebra*, 45(4):317–327, 1999.

In this paper, the authors proved:

- If $\mathbb{V}$ is a real subspace of $\mathbb{H}$ such that $\dim \mathbb{V} \geq 2$ and $\mathbb{R} \subset \mathbb{V}$ then $W(A)$ is convex if and only if $W(A) \cap \mathbb{V}$ is convex.
- If $\mathbb{V}$ is a real proper subspace of $\mathbb{H}$ such that $\mathbb{R} \subseteq \mathbb{V}$ then $W(A)$ is convex if and only if $W(A) \cap \mathbb{V} = \{\text{proj}_{\mathbb{V}}(\lambda) : \lambda \in W(A)\}$ is convex, where $\text{proj}_{\mathbb{V}}(\cdot)$ is the orthogonal projection onto $\mathbb{V}$.

Lok Shun Siu was a student of Au-Yeung. He finished a master's thesis:

L. S. Siu, A study of polynomials, determinants, eigenvalues and numerical ranges over quaternions. M.Phil. thesis, University of Hong Kong, 1997.

14. Conclusion. There are lessons to learn from history. Sometimes, a “mistake” has its place in the development of mathematics, as the one made by Kippenhahn shows. Giving talks can be crucial in the development of mathematics as Thompson's talk demonstrates. The significance of Thompson's talk in the development of the QNR cannot be ignored. Because of this talk and the activities that followed, study of the QNR was re-initiated 30 years after Kippenhahn's paper.

There are a few more papers on the QNR that appeared after 2000. It will take another article, and maybe another author, to document the latest development of the QNR.

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LINEAR ALGEBRA EDUCATION

Motivating Student Learning in Linear Algebra Through Applications and Inquiry

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1. Background. In this article we describe the content and the instructive approach of our free open-source linear algebra text titled *Linear Algebra and Applications: An Inquiry-Based Approach*. The textbook is freely available in PDF format from <https://scholarworks.gvsu.edu/books/21>. We also will happily share source files. (If interested, contact the authors.) The text contains a large variety of applications of linear algebra, and the method through which the applications and content are delivered is heavily learner-focused and inquiry-based. We begin with some background as to how and why we have authored this text.

Professors Alayont and Schlicker approached this project from different directions. Professor Schlicker's training was in pure mathematics. He was captivated by the beautiful ideas he saw in his courses and in his research, and the creative thought processes that he found there. While the (very) few applications that were presented in his undergraduate mathematics classes were intriguing, they were tangential in his mind and did not really contribute to his interest in the subject. When he started teaching mathematics, he wanted to convince his students to see mathematics as he did, and so he taught much as he was taught. For the most part he mentioned applications only in a very general way, or the applications were the standard textbook ones of a form that students probably did not consider to be real applications (e.g., how fast is the bottom of a ladder moving if the top of the ladder is sliding down a wall). At one point he came to the realization that, no matter how good a teacher he might be, it is very unlikely that he will convince the majority of his students to view mathematics in the same way he does. This changed part of his approach to teaching. To try to help motivate students to learn material, he now actively searches for and collects real-life applications of mathematics to incorporate into his courses with the goal of helping students see that the material he is asking them to learn is very useful beyond just the mathematics curriculum.

Professor Alayont knows, from her own learning and teaching experiences, and from reading the education literature [1, 5], how, in order to learn, it is important to be motivated and actively involved, activating previous knowledge, testing ideas, making connections and learning from mistakes. She is motivated to learn pretty much anything math-y, but realizes that for many, if not most, of her students the motivation might require a real-life application of the mathematics. Also, being able to apply a theory in different contexts improves a learner's future ability to apply the theory to yet another context. Therefore, she believes that combining applications and active inquiry-based learning (IBL) techniques will help students develop both stronger knowledge of content and also the ability to use that content in the future.

2. The book's format and philosophy. Each section of the text has a similar format – see Section 7 of the book for an example. A section begins with a list of 'focus questions' that are intended to help students identify the main topics of the section. Then comes a short description of an application that uses the main concepts of the material in that section in a significant way (the application in Section 7 is computer graphics.) This is followed by a preview activity that is intended to be completed before class to set the stage for the material in the section. This activity introduces the concepts of that section in a way that only requires what has been covered in class, and possibly assumed prerequisite course knowledge. It might involve working with concrete examples that serve as a foundation for the theoretical knowledge, or applying some basic definitions to become familiar with them, or experimenting with the goal of conjecturing some results. Student work on the preview activities can then be collected before class via online submissions or discussed briefly in small groups at the beginning of class. In the preview activity in Section 7, students apply the definition of a matrix transformation acting on $\mathbb{R}^2$ and its visualization on a few specific examples to gain an initial understanding.

The material for the section is discussed afterwards in class in an inquiry-based format. IBL, according to the definition from [3], is "a form of active learning in which students are given a carefully scaffolded sequence of mathematical tasks and are asked to solve and make sense of them, working individually or in groups." We strive for our students to be active learners, so much of the material in a section is learned or discovered through activities that students complete in class. After students finish the activities, as a whole class we summarize the results and the main ideas of the activities to make sure every student is following the topic. In Section 7, students work with the idea of linearity of a transformation in the first in-class activity. Then, in the next two activities, they work with the concepts of one-to-one and onto and the connections to matrix-vector equation representations. We incorporate relevant theory in summary form in the text in between the activities as well.

Each section concludes with two thoroughly worked examples, some computational, some theoretical. We also include a summary of the main ideas from the section, and then exercises. Following the section is a project that dives deeper into the application introduced at the beginning. This application project is connected to the material in that section in an

integral way. The projects are also actively completed by students through answering guiding questions. By assigning pre-class readings that include the introductions to the applications on a regular basis, our students are exposed throughout the semester to a wide variety of applications of the material they are learning.

The projects provide a deeper immersion into an application and can be used in a number of different ways. The intent of the projects is for students to wrestle with the applications and how linear algebra is used in them. These projects can be assigned as written projects for students to complete outside of class in groups. In this way students can get hands-on experience with significant applications. We typically allow students two (for the shorter projects) to three weeks (for more involved projects) to complete the project activities and write a project report. The variety of applications in the text makes it possible to give students a choice of which projects to do. For example, for the first project of the semester students might be allowed to work on any one of the projects from the first three sections. In this approach, students will be exposed to three projects, at least briefly, while deciding which one to choose. Also, with different options, students should be able to find a project that interests them. Our use of application projects is consistent with recommendations from professional organizations. For example, the report [2] from the Linear Algebra Curriculum Study Group (LACSG) recommends that some applications should be included in a linear algebra course to give an indication of the “pervasive use of linear algebra in many client disciplines.” Also, the most recent Committee on the Undergraduate Program in Mathematics (CUPM) report [4] from the Mathematical Association of America recommends that linear algebra courses should include applications “both to highlight the broad usefulness of linear algebra and to help students see the role of the theory in the subject as it is applied.”

The application projects are connected to the material in the sections, but the sections themselves are independent of the projects. An instructor can pick and choose which projects to do without loss of continuity in the flow of the text. In addition, the focus questions and the introductory project description are separate from the preview activity and can be skipped before completing the preview activity if desired. In the next section we describe three of the projects in more detail.

3. Examples of applications. All but two of the thirty-seven sections, those focused on proving the equivalence of all the various parts of the Invertible Matrix Theorem and the properties of the determinant, have application projects. As mentioned in the previous section, each application is tied to the material in its section. As a first example, in Section 4 when we introduce linear combinations of vectors in $\mathbb{R}^n$, students learn about analyzing how a knight moves on a chessboard. For this project, students only need to know about vectors and operations on vectors. To start, the eight moves that a knight can make are broken down into four moves (which are represented as vectors) and their additive inverses. From there, students use linear combinations of these vectors to determine how to move a knight from its original position on the board to any other position on the board. Within the project, students have to make judicious choices of free variables in order to correctly analyze cases. This application comes with a GeoGebra applet that we have written that allows students to implement the linear combinations of moves to see how their combinations actually move a knight. They can use this applet to experiment, and to check their work.

When we study the dot product of vectors in Section 27, students are introduced to back-face culling, a computer science application in which a computer can render an object more quickly by eliminating the parts of the object that are not visible from the viewer’s perspective. The background necessary for this project is an understanding of vectors, vector operations, the dot product, and determinants. Within the project students learn about normal vectors (to indicate direction), which naturally brings up the cross product. They then use the dot product to discover how to determine which parts of an object face the viewer and which do not.

As students become more adept at reading and writing about applications, and as they learn more linear algebra, we can challenge them with more sophisticated projects. After students have been introduced to abstract vector spaces (function spaces in particular) and bases for vector spaces in Section 23, they are presented with the application of using wavelets to compress images. Students work with the Haar wavelets as dilations and translations of the mother Haar wavelet. They learn how these wavelets form bases for different vector spaces, understand the effect of applying the wavelets to an image, and then apply these ideas to compress an image that they create. Students work with wavelets as functions, and have to master quite a bit of notation. This is a more complicated project, one that we would allow students three weeks to complete in stages. For this application students can use a Sage worksheet that we have written to create a 16×16 grayscale image matrix of their own, apply wavelets to the image matrix, perform hard thresholding, and then reconstruct their compressed image.

In all of the application projects, students are expected to read and comprehend the material on their own, and then fill in many important details through the project activities. We also expect our students to submit formal written project reports, which helps them develop their communication skills. Many of the applications are accompanied by either GeoGebra applets or Sage worksheets that we have written that help the students to perform computations and to visualize the applications at work.

The titles of the application projects by section are given in Table 1 below. In the table, a (G) next to the name of an application indicates one or more accompanying GeoGebra applets and an (S) indicates one or more Sage worksheets for the application.

1. Electrical Circuits and the Wheatstone Bridge	19. Leslie Matrices and Population Modeling (G)
2. A Polynomial Fitting Application: Simpson's Rule	20. The Gershgorin Disk Theorem
3. Modeling a Chemical Reaction	21. <i>None</i>
4. Analyzing Knight Moves (G)	22. Hamming Codes and the Hat Puzzle
5. An Input-Output Model in Economics	23. Image Compression with Wavelets (S)
6. Generating Bézier Curves (G)	24. Principal Component Analysis
7. The Geometry of Matrix Transformations (G)	25. Finding Formulas for Sums of Powers
8. Strassen's Algorithm	26. Describing Orbits of Planets
9. The Google PageRank Algorithm (G)	27. Back-Face Culling
10. The Richardson Arms Race Model	28. Rotations in 3-Space (G)
11. <i>None</i>	29. Fourier Series and Musical Tones
12. Connecting GDP and Consumption in Economics	30. Gaussian Quadrature and Legendre Polynomials
13. Solving the Lights Out Game (G)	31. The Multivariable Second Derivative Test
14. Modeling Population Migration	32. The Tennis Racket Effect
15. Lattice Based Cryptography	33. Latent Semantic Indexing
16. Area and Volume using Determinants	34. The Global Positioning System
17. The Ehrenfest Model of the Second Law of Thermodynamics	35. Fractals and Iterated Function Systems (S)
18. Binet's Formula for the Fibonacci Numbers	36. Shamir's Secret Sharing and Lagrange Polynomials
	37. Second Order Linear Differential Equations

Table 1: Section number and application.

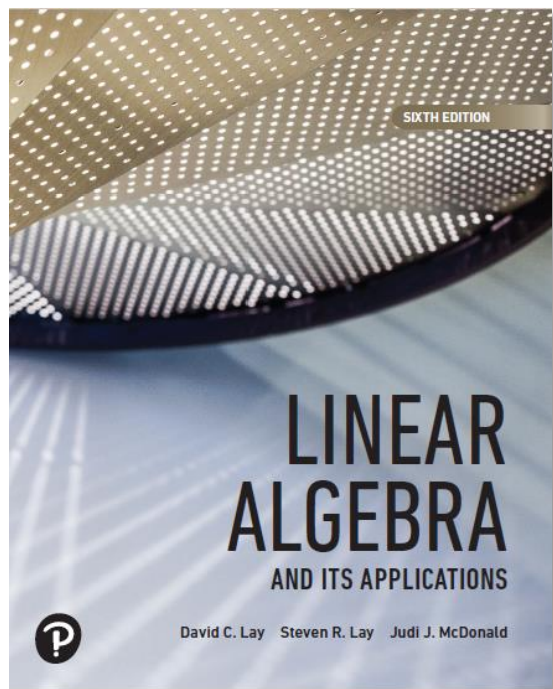
4. The big picture. Our open-source linear algebra text *Linear Algebra and Applications: An Inquiry-Based Approach* uses a discovery-based approach along with a large variety of applications to teach linear algebra while showing students how linear algebra is used in real life. Students are actively involved in their learning and the instructor is available to guide students when they grapple with understanding the concepts and working with these concepts. We believe this approach helps students develop a deeper understanding and an appreciation of linear algebra.

An interested reader can find a free PDF version of the text at <https://scholarworks.gvsu.edu/books/21>, from which the reader can find the table of contents as well as the above-referenced parts of the text. A hard copy is available from Amazon at <https://www.amazon.com/dp/1687348693>. Persons interested in obtaining the source files for the text can do so by making a request to the authors.

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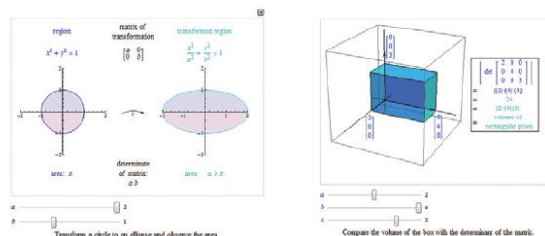
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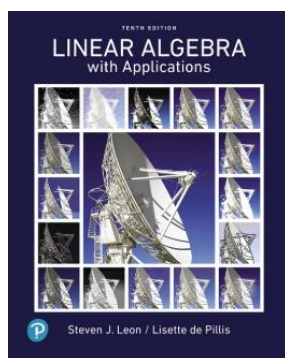
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Report on the Education Session at the 22nd ILAS Conference

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At the 2019 ILAS meeting in Brazil, five speakers shared some useful and timely thoughts on teaching linear algebra.

Jane Breen spoke on ideas to improve basic learning in Linear Algebra. Specifically, Jane described her experience with active learning in a summer class of 60 students at the University of Manitoba. She described some of the struggles faced in this class, including: “huge dependence on tedious procedures;” “not enough time;” “no control over content;” and “majority of students motivated mostly by wanting to pass the class rather than by actually wanting to learn linear algebra.” Jane discussed possible teaching techniques to combat these difficulties, such as group work, voting/polling methods, and flipped classroom/inquiry-based learning. She gave caveats of active learning styles of teaching, shared ideas that she has tried (e.g., clickers, worksheets, etc.), and evaluated the pros and cons of using these techniques.

Mary Flagg discussed her project “Solving a System of Linear Equations Using Ancient Chinese Methods,” part of the broader “TRransforming Instruction in Undergraduate Mathematics via Primary Historical Sources” (TRIUMPHS) project. This Primary Source Project (PSP) introduces Gaussian Elimination using “Fangcheng” (Chapter 8 of the first-century text *The Nine Chapters on the Mathematical Art*. [Editor’s note: For a complete description of this project, see the article “Teaching Linear Algebra Using Primary Historical Sources” by Mary Flagg in the Spring 2019 issue of *IMAGE*.] Mary reported that the PSP has been formally site tested in six other classrooms at other universities and has been downloaded 1600+ times from the Digital Commons Online Library. Her main goal is to “share the amazing accomplishment of the ancient Chinese and introduce the material in an interactive way.”

Statistician Wayne Stewart spoke on teaching multivariate statistics to students with mixed mathematical backgrounds. In his view, “rather than limit students to those with a strong mathematical exposure, it may be best to take those who have a strong desire to learn data analytic methods and teach prerequisite mathematics as the course unfolds.” His approach is more intuitive and introduces principal component analysis (PCA) methodology and related ideas in 2 dimensions before moving on to a more elegant eigenvalue/eigenvector approach. This tactic promotes better geometric appreciation and also helps justify the methods usually employed with PCA in such packages as FactoMineR, an R package commonly used when analyzing data via PCA. He demonstrated an example of a simple rotation of axes in conjunction with an R Shiny package that calculates the covariance between rotated variables to visually and dynamically illustrate the idea behind PCA.

Rachel Quinlan spoke about proof evaluation as a learning activity. She emphasized that to fully understand a discipline, it is not adequate only to know its products and their uses, but rather to understand the genesis of its knowledge, and how that knowledge is warranted. Dating back at least to ancient Greek scholars, mathematics has evolved through deductive proof. Rachel discussed both proof as a theoretical concept and proving as a fundamental practice of the disciplinary community. She observed that “proof evaluation is a constant feature of our disciplinary activity in wide contexts. Does its explicit inclusion in the curriculum give an opportunity to novice members of our community for authentic engagement?” Employing the situated learning theory of Lave and Wenger [2], which holds that “learning takes place in a community of practice and that novice members become expert by a process of authentic participation in the practices at increasingly advanced levels,” she examined excerpts from her students’ work in evaluating proofs.

Sepideh Stewart gave a report on the 2018 workshop on linear algebra education held at the University of Oklahoma with the goal to (1) work toward writing research-based recommendations for the teaching and learning of linear algebra; and (2) generate new research questions and ideas that encourage collaboration between research mathematicians and mathematics educators. [Editor’s note: For a full report from Sepideh Stewart on this workshop, see the Fall 2018 issue of *IMAGE*.] Sepideh also shared two recent international endeavors and resources in linear algebra education: [3] and [4].

Overall, the session was very successful and popular. Indeed, for a couple of the talks, not all interested persons were able to fit in the room. We look forward to another fruitful education session at the June 2020 ILAS meeting.

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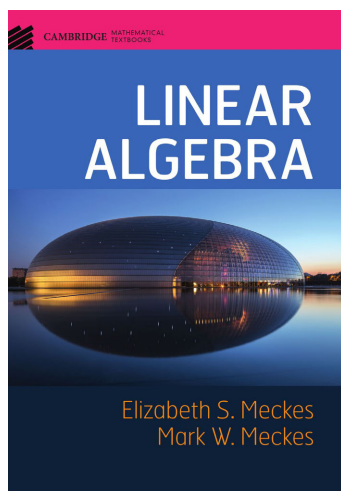
BOOK REVIEW

Linear Algebra

by Elizabeth S. Meckes and Mark W. Meckes

Cambridge University Press, 2018, ISBN 978-1-107-17790-1, 427+xvi pages.

Reviewed by Colin Garnett, Black Hills State University, colin.garnett@bhsu.edu



This book takes a novel approach to teaching a first course in linear algebra. The authors introduce topics in computational and abstract theory side by side in each section. The book does not introduce determinants until the very last chapter, which forces the students to work a little harder to calculate things like the eigenvalues. This could force the students to gain experience and hopefully insight into the eigenvalues of different matrices. In the course that I taught from this book, many of the students had prior experience with the determinant and so I had a difficult time convincing them to buy into the textbook's way of calculating the eigenvalues.

The book is divided into six chapters: Linear Systems and Vector Spaces; Linear Maps and Matrices; Linear Independence; Bases and Coordinates; Inner Products; Singular Value Decomposition and the Spectral Theorem; and Determinants. It is probably clear from that list of topics that the book culminates in a discussion of determinants. Since determinants are often viewed as a requisite topic for a first course in linear algebra, you may have to tighten the timing for each of the other topics leading up to the final chapter. It is also possible that you may want to leave out at least one earlier topic to make time for determinants. Each section in each chapter culminates in exercises,

often including portions of the proofs of the theorems from the section. There are also color-coded boxes throughout the book with "Definitions," "Theorems," "Quick Exercises," "Key Ideas," and a feature called "Perspectives" at the end of most chapters. The authors introduce key concepts early and often and give many opportunities for the instructor to emphasize them.

Chapter 1 contains a discussion of linear systems and vector spaces. It is a great introduction to linear algebra, especially for a student with experience in algebra as well as vector calculus. The interplay between these two seemingly unrelated topics allows students to see the link between them.

Chapter 2 discusses linear maps and matrices. This natural pairing of topics allows the reader to make connections between vector spaces and the very concrete and practical familiarity of matrices.

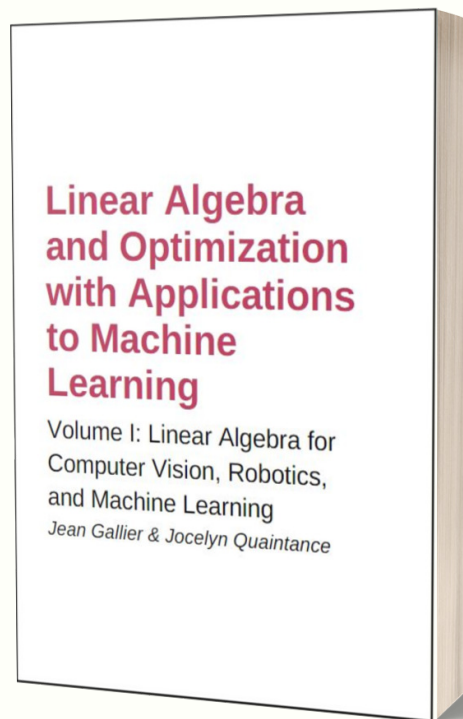
Chapter 3 discusses linear independence, bases and coordinates. This idea of coordinatizing the vector spaces allows for a discussion of geometry.

Chapter 4 discusses inner products, moving very quickly to orthonormal bases, orthogonal projections and normed spaces. It finishes up with a discussion of isometries and the QR decomposition.

Chapter 5 discusses the singular value decomposition and the Spectral Theorem. It begins with a discussion of singular value decomposition (SVD) and gives several versions of the decomposition before moving on to discuss the process of calculating the SVD through the adjoint.

The final chapter discusses determinants. As this chapter comes last, the authors are able to use all of the topics in the book to give a full and comprehensive description of the determinant. This is nice because it allows for a more natural development of the topic. There is also a downside, namely that with determinants coming so late they are not available in the discussion of any topics that occur earlier in the book.

This book, by the authors' own admission, is an ambitious attempt to wade into the fray of linear algebra textbooks. The authors do a nice job of pairing abstract topics with more concrete ones and are mostly successful in getting the topics to reinforce one another. The book leaves determinants to the end and in my experience with using the book in my own course the students were somewhat resistant to this idea. It is my opinion that if one adopts this book one must really sell the ideas that the authors lay out in the preface. With that said, I found the book a pleasure to use and read. It has lots of very nice features and was the basis for a course that my students enjoyed. I would recommend this book for consideration by anyone looking at adopting a textbook for a first course in linear algebra.



Linear Algebra and Optimization with Applications to Machine Learning

Volume I: Linear Algebra for Computer Vision, Robotics, and Machine Learning
By Jean Gallier & Jocelyn Quaintance (University of Pennsylvania, USA)

This book provides the mathematical fundamentals of linear algebra to practicers in computer vision, machine learning, robotics, applied mathematics, and electrical engineering. By only assuming a knowledge of calculus, the authors develop, in a rigorous yet down to earth manner, the mathematical theory behind concepts such as: vectors spaces, bases, linear maps, duality, Hermitian spaces, the spectral theorems, SVD, and the primary decomposition theorem. At all times, pertinent real-world applications are provided. This book includes the mathematical explanations for the tools used which we believe that is adequate for computer scientists, engineers and mathematicians who really want to do serious research and make significant contributions in their respective fields.

Readership: Undergraduate and graduate students interested in mathematical fundamentals of linear algebra in computer vision, machine learning, robotics, applied mathematics, and electrical engineering.

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ILAS NEWS

Pauline van den Driessche awarded 2019 CAIMS-SCMAI Research Prize

The Canadian Applied and Industrial Mathematical Society has chosen long-time ILAS member Pauline van den Driessche of the University of Victoria as the recipient of the 2019 CAIMS-SCMAI Research Prize for exceptional research contributions in applied or industrial mathematics. The award was given in recognition of her contributions to mathematical epidemiology and matrix analysis, and the high impact of this work in many areas of applied mathematics more generally.

For further details, see: <https://caims.ca/award/caims-scmair-research-prize>

2020 Taussky-Todd Lecturer

Contributed announcement from Peter Šemrl

Raf Vandebril will be the Taussky-Todd Lecturer at the 2020 ILAS Conference in Galway.

The 2020 Taussky-Todd Lecture Selection Committee consisted of Anne Greenbaum (chair), Ravi Bapat, Michele Benzi, Chi-Kwong Li, and Peter Šemrl.

2020 ILAS Elections: Nominations

Contributed announcement from Peter Šemrl

The Nominating Committee for the 2020 ILAS elections has completed its work. Nominated for a three-year term, beginning March 1, 2020, as ILAS President are: Daniel Szyld and Tin-Yau Tam.

Nominated for the two open three-year terms, beginning March 1, 2020, as “at-large” members of the ILAS Board of Directors are: Enide Andrade, Sebastian Cioabă, Dragana Cvetković-Ilić, and Carlos Marijuan.

Many thanks to the Nominating Committee: Richard Brualdi (Chair), Nair Abreu, Ravi Bapat, Steve Kirkland, and Naomi Shaked-Monderer.

JOURNAL ANNOUNCEMENTS

Electronic Journal of Linear Algebra (ELA) milestone: 1,000 papers!

Contributed announcement from Michael Tsatsomeros

Since 1996, the *Electronic Journal of Linear Algebra (ELA)* has been the research publication of the International Linear Algebra Society (ILAS). Now *ELA* has reached the milestone of one thousand papers published since its inaugural year.

The journal is freely accessible around the world via

<https://repository.uwo.edu/ela>

and is made possible by the selfless contributions of its authors, referees and editors. Thanks are due to all of them and to its great many loyal readers.

As members and friends of ILAS, please continue to promote and support the goals of *ELA*.

New co-Editor-in-Chief of *ELA*

Contributed announcement from Peter Šemrl

The ILAS Journals Committee nominated Froilán Dopico as a new co-Editor-In-Chief of *ELA*, to replace Bryan Shader, who stepped down at the end of his term. The ILAS Board approved the nomination, and the appointment became effective on August 1, 2019.

Special Issue of *LAMA* on Numerical Ranges and Numerical Radii

Contributed announcement from Chi-Kwong Li

The study of the numerical range and numerical radii has a long history. In particular, a fundamental result on the subject is the Toeplitz-Hausdorff Theorem concerning the convexity of the numerical range, which was proved in 1918–19. There were workshops on the topic held in Germany in 2018, and in Japan in 2019 to celebrate the 100th anniversary of the Toeplitz-Hausdorff Theorem.

In connection with these activities, *Linear and Multilinear Algebra* will publish a special issue on the subject: <https://think.taylorandfrancis.com/linear-and-multilinear-algebra-numerical-ranges-and-numerical-radii>

The deadline for submissions is December 31, 2019. Papers will go through the usual editorial procedure and may be submitted through the journal website: <https://mc.manuscriptcentral.com/glma>.

The editors of the special issue are:

Yiu-Tung Poon, Iowa State University, USA (ytpoon@iastate.edu); Tin-Yau Tam, Auburn University & University of Nevada, USA (ttam@unr.edu); Takeaki Yamazaki, Toyo University, Japan (t-yamazaki@toyo.jp)

Send News for *IMAGE* Issue 64

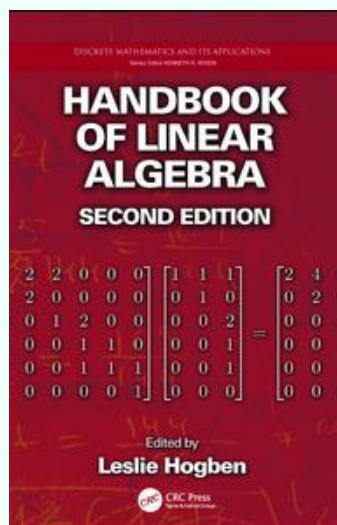
IMAGE seeks to publish all news of interest to the linear algebra community. Issue 64 of *IMAGE* is due to appear online on June 1, 2020. Send your news for this issue to the appropriate editor by April 15, 2020. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- feature articles to Sebastian Cioabă (cioaba@udel.edu)
- interviews of senior linear algebraists to Carlos Fonseca (cmdafonseca@hotmail.com)
- book reviews to Colin Garnett (Colin.Garnett@bhsu.edu)
- problems and solutions to Rajesh Pereira (pereirar@uoguelph.ca)
- linear algebra education news to David Strong (David.Strong@pepperdine.edu)
- advertisements to Amy Wehe (awehe@fitchburgstate.edu)
- announcements and reports of conferences/workshops/etc. to Jephian C.K. Lin (jephianlin@gmail.com)

Send all other correspondence to the editor-in-chief, Louis Deaett (louis.deaett@quinnipiac.edu).

For past issues of *IMAGE*, please visit <http://www.ilasic.org/IMAGE>.

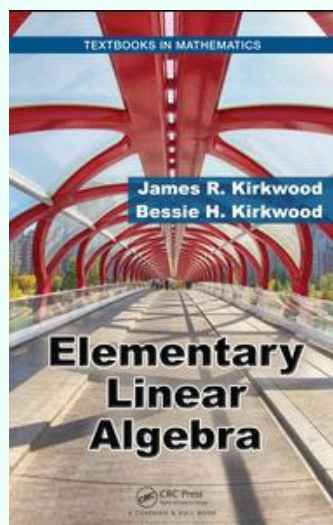
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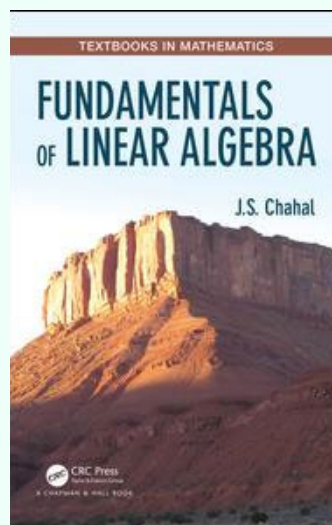
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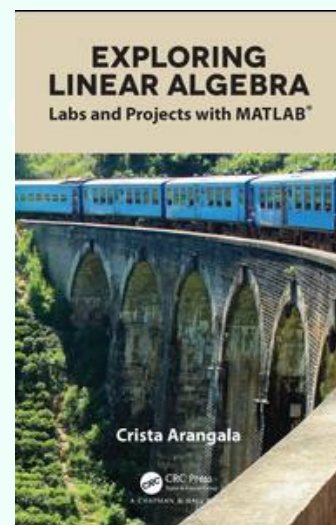
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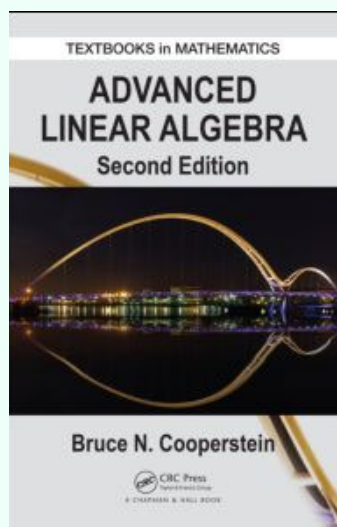
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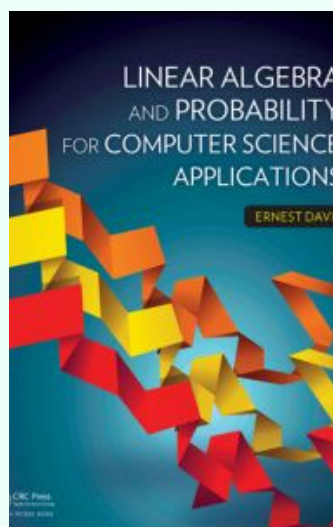
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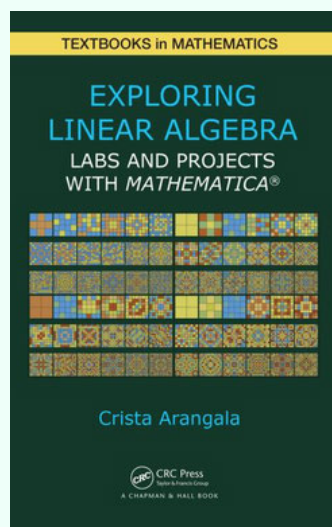
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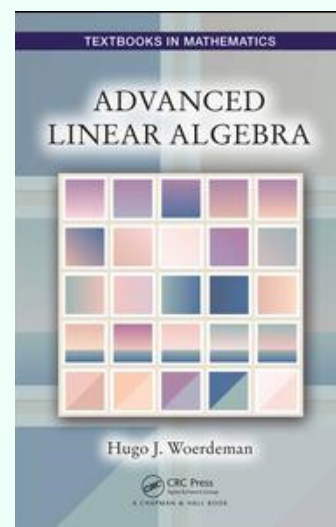
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CONFERENCE REPORTS

27th International Workshop on Matrices and Statistics Shanghai, China, June 6–9, 2019

Report by Jeffrey J. Hunter

The 27th International Workshop on Matrices and Statistics (IWMS-2019) was held at the Shanghai University of International Business and Economics (SUIBE) Gubei Campus, Shanghai, China, over the period of June 6–9, 2019. The International Organising Committee (IOC) was chaired by Jeffrey J. Hunter (New Zealand), and included Dietrich von Rosen (Sweden) as Vice-Chair, George P.H. Styan (Canada) as Honorary Chair, S. Ejaz Ahmed (Canada), Francisco Carvalho (Portugal), Katarzyna Filipiak (Poland), Daniel Klein (Slovakia), Augustyn Markiewicz (Poland), Simo Puntanen (Finland), Julia Volaufova (USA) and Hans Joachim Werner (Germany). The Local Organising Committee was chaired by Yonghui Liu (Chair) and assisted by Hui Liu (Vice-Chair), Chengcheng Hao and Cihai Sun.



Participants of the 27th International Workshop on Matrices and Statistics

The invited plenary presentations (in alphabetical order) were: Oskar Maria Baksalary (Adam Mickiewicz University in Poznan, Poland) *A gaze at recent applications and characterizations of the Moore-Penrose inverse*; Rajendra Bhatia (Ashoka University, New Delhi, India) *Geometry and means of positive definite matrices*; Kai-Tai Fang (BNU-HKBU United International College, China) *Representative points of elliptically symmetric distributions*; Shuangzhe Liu (University of Canberra, Australia) *Professor Heinz Neudecker and matrix differential calculus*; Jianxin Pan (Manchester University, United Kingdom) *Calibration for non-positive definite covariance matrix*; K. Manjunatha Prasad (Manipal Academy of Higher Education, Manipal, India) *Inverse complimentary matrix method and its applications to general linear model*; Yongge Tian (Central University of Finance and Economics, Beijing, China) *Identifying conditions for multilinear matrix equations to always hold with applications*; Fuzhen Zhang (Nova Southeastern University, Fort Lauderdale, USA) *Inequalities for selected eigenvalues of the product of matrices*; Shurong Zheng (Northeast Normal University, Changchun, Jilin, China) *Statistical inference on high-dimensional covariance matrices*; Lixing Zhu (Hong Kong Baptist University, Hong Kong, China) *Order determination for large-dimensional matrices*.

Besides the plenary sessions, the workshop program contained six mini-symposia and two contributed sessions. The mini-symposia and their organisers were as follows: MS1. Experimental Design (Kai-Tai Fang) MS2. Inference in Parametric Models (Julia Volaufova) MS3. Linear Models and Multivariate Analysis (Simo Puntanen) MS4. Predictive Modelling and Diagnostics (Shuangzhe Liu) MS5. Decompositions of Tensor Spaces with Applications to Multilinear Models (Dietrich von Rosen) MS6. Statistical Modelling for Complex Data (Rui Li).

In addition, the Workshop incorporated the 4th International Mini-Symposium on “Magic squares, prime numbers and postage stamps,” organised by George Styan. The main contribution was a poster “An introduction to some magic squares by Paul Daniels and by Steve Martin, and to the Kostabi/Leigh Bereshit bara Elohim drawing Nova Ratio for Pope Emeritus Benedict XVI, all illustrated philatelically” by Ka Lok Chu, Simo Puntanen and George Styan.

The workshop had confirmed participation from 12 countries, with 26 participants from overseas and 43 from mainland China, for a total of 69 participants. Registration for the workshop was held on the afternoon of Wednesday, June 5

at the Lobby of the Brawway Hotel, followed by a Welcome Dinner at the Zhuang Yuan Lou Restaurant. Additional registrations were handled the morning of Thursday, June 6 at the workshop venue on the campus of SUIBE, prior to the formal opening ceremony which included a speech from Yonglin Xu (Vice President, SUIBE). This was followed by a photo session. Lunch and dinner were provided for all participants each day at the workshop venue. The workshop banquet was held on Friday evening at the Zhuang Yuan Lou Restaurant. The workshop closed just before lunch on Sunday, June 9.

The website for IWMS-2019 is: <http://www.suibe.edu.cn/ttxy/iwms2019>

8th International Conference on Matrix Analysis and Applications (ICMAA 2019) University of Nevada, Reno, USA, July 15–18, 2019

Report by Tin-Yau Tam, Qing-Wen Wang, and Fuzhen Zhang

The 8th International Conference on Matrix Analysis and Applications (ICMAA 2019) took place July 15–18, 2019 at the University of Nevada, Reno (UNR), Nevada, USA. Organized by Tin-Yau Tam, Qing-Wen Wang and Fuzhen Zhang, the meeting aimed to stimulate research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications and to provide an opportunity for researchers to exchange ideas. There were 60 participants from 10 countries, and 38 talks. The keynote speaker was Lek-Heng Lim of the University of Chicago, USA and the two plenary speakers were Shaun Fallat of the University of Regina, Canada and Shmuel Friedland of the University of Illinois at Chicago, USA. The meeting was generously supported by the International Linear Algebra Society, as well as by the Office of Research and Innovation, the College of Science, and the Department of Mathematics and Statistics of UNR.



Group photo from ICMAA 2019

The previous ICMAA conferences were held in China (Beijing, Hangzhou), the United States (Nova Southeastern University), Turkey (Selçuk University, Konya), Vietnam (Duy Tan University, Da Nang), and Japan (Shinshu University, Nagano Prefecture). The next two conferences will be held at the University of Aveiro, Portugal (June 18–20, 2020) and Yunnan University, Yunnan, China (July 23–26, 2021). When the time comes, more information will be announced via ILAS-NET.

International Conference on Matrix Analysis and its Applications (MAT-TRIAD 2019) Liblice, Czech Republic, September 8–13, 2019

Report by Milan Hladík and Miroslav Rozložník

The 8th international conference on Matrix Analysis and its Applications, MAT-TRIAD 2019, was held September 8–13, 2019 in Liblice (about 40 km from Prague) at the Conference Center of the Czech Academy of Sciences. MAT-TRIAD

2019 was an ILAS-endorsed meeting and a registered satellite meeting of the International Congress on Industrial and Applied Mathematics (ICIAM) held in Valencia, Spain, July 15–19, 2019. MAT-TRIAD 2019 brought together researchers sharing an interest in a variety of aspects of matrix analysis and its applications to many areas of science. It gave an opportunity to introduce advanced researchers, younger colleagues, and students to recent developments in matrix and operator theory, spectral problems, numerical linear algebra, combinatorial matrix theory, and applications of linear algebra to statistics, optimization or graph theory.



Group photo from MAT-TRIAD 2019

The conference was co-organized by the Institute of Mathematics of the Czech Academy of Sciences, the Faculty of Mathematics and Physics of Charles University, and by the Institute of Computer Science of the Czech Academy of Sciences, all located in Prague. The Scientific Committee of MAT-TRIAD consisted of Natália Bebiano (University of Coimbra, Portugal), Ljiljana Cvetković (University of Novi Sad, Serbia), Heike Faßbender (Technical University Braunschweig, Germany), Simo Puntanen (University of Tampere, Finland) and Tomasz Szulc (Adam Mickiewicz University, Poznań, Poland). The Local Organizing Committee consisted of Miro Rozložník, Milan Hladík, Hana Bílková, Jan Bok, David Hartman, Jaroslav Horáček, Miroslav Tůma and Petr Tichý (all from the Czech Academy of Sciences or Charles University, Prague, Czech Republic).

The scientific program of the conference consisted of 2 invited lectures, 6 invited talks and 63 contributed talks, including 35 talks in 4 special sessions: Total positivity (organisers: Mohammad Adm and Jürgen Garloff), Tropical matrix algebra and its applications (organisers: Aljoša Peperko and Sergei Sergeev), Recent developments of verified numerical computations (organizers: Takeshi Ogita and Siegfried M. Rump) and Interval matrices (organiser: Milan Hladík). The invited lectures were delivered by Shmuel Friedland (University of Illinois at Chicago) as the Hans Schneider ILAS Lecturer, with the lecture “The Collatz-Wielandt quotient for pairs of nonnegative operators” and by Zdeněk Strakoš (Charles University, Prague) with the lecture “Operator preconditioning, spectral information and convergence behavior of Krylov subspace methods”. The invited speakers were Dario Bini (University of Pisa), Mirjam Dür (University of Augsburg), Arnold Neumaier (University of Vienna) and Martin Stoll (Technical University of Chemnitz). The remaining two invited talks were given by the two recipients of the Young Scientist Award from MAT-TRIAD 2017: Álvaro Barreras (Universidad Internacional de La Rioja) and Ryo Tabata (National Institute of Technology, Fukuoka).



Winners of the MAT-TRIAD 2019 Young Scientist Award: Marie Kubínová (left) and Yuki Nishida.

Following long-term tradition, during the conference closing the members of the Young Scientist Award Committee announced the winners of the award for the best talks presented by Ph.D. students and young scientists: Marie Kubínová (Czech Republic) and Yuki Nishida (Japan). These winners will be invited speakers at the MAT-TRIAD conference to be held the second or third week of September 2021 in Curia, Portugal.

A special issue of *Applications of Mathematics* will be devoted to MAT-TRIAD 2019, featuring selected papers related to the presentations given at the meeting. The conference participants represented a well-balanced combination of high-calibre speakers and new faces in the field. They shared the common opinion that the conference was professionally organized, with a friendly and warm atmosphere.

The scientific program and book of abstracts from MAT-TRIAD 2019 and other information about the conference can be found at <https://mattriad.math.cas.cz>.

UPCOMING CONFERENCES AND WORKSHOPS

Special session on Matrices and Graphs at the 2020 Joint Mathematics Meetings Denver, CO, USA, January 16, 2020

A special session on “Matrices and Graphs” will be held at the Joint Mathematics Meetings in Denver, Colorado on January 16, 2020. The session is organized by Leslie Hogben (Iowa State University and the American Institute of Mathematics) and Bryan Shader (University of Wyoming).

The keynote presenter will be Steve Kirkland (University of Manitoba), who will be opening the session with a talk entitled “Directed Forests and the Constancy of Kemeny’s Constant.”

Other talks are:

“Kemeny’s constant and random walks on graphs” by Jane Breen (Ontario Tech University); “The normalized distance Laplacian” by Carolyn Reinhart (Iowa State University); “Colin de Verdiere invariants and antipodal mappings” by Hein van der Holst (Georgia State University); “The Strong Inner Product Property” by Bryan A. Curtis (University of Wyoming); “Some graphs whose maximum nullity and zero forcing number are the same” by Derek D. Young (Mount Holyoke College); “On the maximum multiplicity of the k -th largest eigenvalue of a tree” by Shaun Fallat (University of Regina); “Tree covers and the positive semidefinite maximum nullity of a graph” by Chassidy Bozeman (Mount Holyoke College); “Redundant Power Dominating Sets” by Daniela Ferrero (Texas State University, San Marcos); “Opinion and Spreading Models on Social Networks” by Mason A. Porter (University of California, Los Angeles); “Infectious Power Domination for Hypergraphs” by Beth Bjorkman (Iowa State University); “Propagation time for probabilistic zero forcing” by Jesse Geneson (Iowa State University); and “Extreme Throttling Numbers” by Joshua Carlson (Williams College).

Special session on Innovative and Effective Ways to Teach Linear Algebra at the 2020 Joint Mathematics Meetings Denver, CO, USA, January 17, 2020

The upcoming 2020 Joint Mathematics Meetings of the American Mathematical Society (AMS) and the Mathematical Association of America (MAA) will include an MAA Contributed Paper Session on Innovative and Effective Ways to Teach Linear Algebra.

This session will serve as a forum in which to share and discuss new or improved teaching ideas and approaches. These innovative and effective ways to teach linear algebra include, but are not necessarily limited to: (1) hands-on, in-class demos; (2) effective use of technology, such as *Matlab*, *Maple*, *Mathematica*, Java applets or Flash; (3) interesting and enlightening connections between ideas that arise in linear algebra and ideas in other mathematical branches; (4) interesting and compelling examples and problems involving particular ideas being taught; (5) comparing and contrasting visual (geometric) and more abstract (algebraic) explanations of specific ideas; (6) other novel and useful approaches or pedagogical tools.

These sessions, now in their 13th year, are organized by Sepideh Stewart (University of Oklahoma), Gil Strang (Massachusetts Institute of Technology), David Strong (Pepperdine University), and Megan Wawro (Virginia Tech). There will be five talks given during the afternoon sessions on Thursday, January 17. The schedule of talks is available at: http://jointmathematicsm meetings.org/meetings/national/jmm2020/2245_program_friday.html#2245:MCPSTRH1

Western Canada Linear Algebra Meeting (WCLAM 2020) Brandon University, Manitoba, Canada, May 23–24, 2020

The Western Canada Linear Algebra Meeting (WCLAM) provides an opportunity for mathematicians in western Canada and the USA working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Anyone working in linear algebra or a related field, including combinatorics, graph theory, matrix analysis, and applied mathematics, is encouraged to submit an abstract for a contributed talk or poster.

The participation fee of CAD\$30 will be waived for participating students and postdoctoral fellows. Subject to funding, students and postdoctoral fellows will receive free accommodation in residence and potentially additional travel support.

WCLAM 2020 will have three distinguished invited speakers: Ada Chan (York University), Doug Farenick (University of Regina), and Judi McDonald (Washington State University).

The organisers of WCLAM 2020 are: Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Manitoba), Sarah Plosker (Brandon University), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria). The local organisers are: Steve Kirkland (University of Manitoba) and Sarah Plosker (Brandon University) (ploskers@brandonu.ca).

Further information is available at <https://www.brandonu.ca/wclam>.

Linear Algebra, Matrix Analysis, and Applications (ALAMA2020) Alcalá de Henares, Spain, June 3–5, 2020

The thematic network ALAMA (Linear Algebra, Matrix Analysis, and Applications) will hold its seventh biennial meeting June 3–5, 2020 in Alcalá de Henares, after the previous editions held in Vitoria-Gasteiz (2008), Valencia (2010), Leganés (2012), Barcelona (2014), León (2016) and Alicante (2018).

This 2020 edition is a celebration in honour of Ion Zaballa, professor at Universidad del País Vasco, and of Dario A. Bini, professor at Università di Pisa, so that this ALAMA meeting is celebrated jointly with the seventeenth edition of ALN2gg (Due giorni di Algebra Lineare Numerica). The meeting ALN2gg periodically gathers the Italian community of numerical linear algebra, this time for three days.

The joint meeting will be celebrated in a singular environment: the historical Colegio de San Ildefonso of the University of Alcalá, in the city of Alcalá de Henares (province of Madrid), whose historical centre was in 1998 declared a UNESCO World Heritage Site.

The scientific committee includes: Ana Marco [President] (Universidad de Alcalá), Raymond Honfu Chan (City University of Hong-Kong), Froilán M. Dopico (Universidad Carlos III de Madrid), Christian Mehl (TU Berlin), Juan Manuel Peña (Universidad de Zaragoza), Lothar Reichel (Kent State University), Stefano Serra-Capizzano (Insubria University and Uppsala University), Ana M. Urbano (Universitat Politècnica de València), and Marc Van Barel (KU Leuven).

Abstract submission is due no later than March 1, 2020, and early registration ends April 12, 2020. Further information can be found at: <https://congresosalcala.fgua.es/alama2020>

Workshop on Numerical Ranges and Numerical Radii (WONRA 2020) Curia, Portugal, June 13–16, 2020

The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2020) will take place June 13–16, 2020, at the Hotel Termas da Curia, Portugal.

The purpose of the workshop is to stimulate research and foster interaction between researchers interested in the subject. The informal workshop atmosphere will facilitate the exchange of ideas from different scientific areas and, hopefully, the participants will be informed on the latest developments and new ideas. Background on the subject and on previous meetings may be found at the WONRA website (<http://www.mat.uc.pt/~wonra2020>) and also on the Wikipedia page for the history of the workshop and related meetings: https://en.wikipedia.org/wiki/Workshop_on_Numerical_Ranges_and_Numerical_Radii.

The organizing committee consists of Natália Bebiano (CMUC, University of Coimbra), Chi-Kwong Li (College of William and Mary, Williamsburg, Virginia, USA), Susana Furtado (CEAFEL, University of Porto), and Ana Nata (CMUC, Polytechnic Institute of Tomar).

The workshop is endorsed and sponsored by the International Linear Algebra Society (ILAS), Compete (Programa operacional factores de competitividade), CMUC - Centre for Mathematics, University of Coimbra, and CEAFEL - Centre for Functional Analysis, Linear Structures and Applications.

One may visit <http://www.mat.uc.pt/~wonra2020/> for further information, and send e-mail to wonra2020@mat.uc.pt with any questions.

XXI Householder Symposium on Numerical Linear Algebra Selva di Fasano (Br), Italy, June 14–19, 2020

The next Householder Symposium will be held June 14–19, 2020 at the hotel Sierra Silvana, Selva di Fasano (Br), Italy.

This meeting is the twenty-first in a series, previously called the Gatlinburg Symposia, but now named in honor of its founder, Alston S. Householder, a pioneer of numerical linear algebra. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. Topics include numerical linear and multi-linear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control.

The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1, 2017 will be presented.

The Householder Committee includes Zhaojun Bai (University of California, Davis, USA), David Bindel (Cornell University, USA), James Demmel (University of California, Berkeley, USA), Zlatko Drmac (University of Zagreb, Croatia), Heike Faßbender [chair and SIAM representative] (Technical University of Braunschweig, Germany), Sherry Li (Lawrence Berkeley National Laboratory, USA), Volker Mehrmann (Technical University of Berlin, Germany), James Nagy (Emory University, USA), Valeria Simoncini (University of Bologna, Italy), and Andrew Wathen (Oxford University, UK).

This meeting is sponsored by ILAS, MathWorks, and SIAM. Attendance at the meeting is by invitation. Further information can be found at <https://users.ba.cnr.it/iac/irmanm21/HHXXI>.

Workshop on “Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra” West Greenwich, Rhode Island, USA, June 14–20, 2020

An intensive, one-week, hands-on summer research conference focusing on the inverse eigenvalue problem for graphs, zero forcing and applications will be held June 14–20, 2020 at the Whispering Pines Conference Center in West Greenwich, Rhode Island. The conference is part of the American Mathematical Society’s Mathematics Research Communities, which is its professional development program. The goal of these communities is to develop collaborative research skills, build a network focused on the selected mathematical topic, and provide mentoring from leaders in the chosen area. The conference is organized by Shaun Fallat (University of Regina), H. Tracy Hall (NewVistas LLC), Leslie Hogben (Iowa State University and the American Institute of Mathematics), Bryan Shader (University of Wyoming) and Michael Young (Iowa State University).

The conference is supported by the National Science Foundation and the American Mathematical Society.

Early-career mathematicians—those who are close to finishing their doctorates or have recently finished—are invited to apply for this conference. Each selected participant will be provided support for travel to the summer conference site and all accommodations and meals there, support for travel to the 2021 Joint Mathematics Meetings, and support for follow-up collaboration travel during the following year.

Funding for participants who are not U.S. residents or enrolled in U.S. graduate programs is very limited. Applications are being accepted via MathPrograms.org (<https://www.mathprograms.org/db/programs/829>) until the deadline of 11:59 p.m. Eastern Time, February 15, 2020.

For more details, visit <http://www.ams.org/programs/research-communities/mrc>.

9th International Conference on Matrix Analysis and Applications (ICMAA 2020) Aveiro, Portugal, June 18–20, 2020

The 9th International Conference on Matrix Analysis and Applications (ICMAA 2020) will be held at the University of Aveiro in Aveiro, Portugal, June 18–20, 2020. This meeting aims to stimulate the research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications and to provide an opportunity for researchers to exchange ideas and discuss developments on these subjects. The previous ICMAA conferences were held in China (Beijing, Hangzhou), the United States (Nova Southeastern University), Turkey (Selçuk University, Konya), Vietnam (Duy Tan University, Da Nang), Japan (Shinshu University, Nagano Prefecture) and the

United States (University of Nevada, Reno). Former keynote speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland, Alexander A. Klyachko (ILAS guest speaker), Shmuel Friedland, Man-Duen Choi, Tsuyoshi Ando, Fumio Hiai and Lek-Heng Lim.

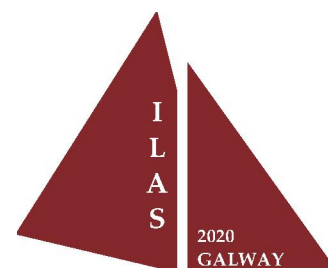
The keynote speaker of ICMAA 2020 is Peter Šemrl, University of Ljubljana, Slovenia, and the two invited speakers are Natália Bebiano, University of Coimbra, Portugal and Chi-Kwong Li, College of William and Mary, USA.

The organizers are Enide Andrade (Organizing Committee Chair), University of Aveiro, Aveiro, Portugal; Rute Lemos, University of Aveiro, Aveiro, Portugal; Tin-Yau Tam (Organizing Committee co-Chair), University of Nevada, Reno, USA; Qing-Wen Wang, Shanghai University, Shanghai, China; and Fuzhen Zhang, Nova Southeastern University, Florida, USA. The workshop is endorsed and sponsored by the International Linear Algebra Society (ILAS); the Center for Research and Development in Mathematics and Applications (CIDMA); the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia) through the Center for Research and Development in Mathematics and Applications (CIDMA) within project UID/MAT/04106/2019; and the Mathematics Department (DMat-UA), University of Aveiro (UA), Portugal.

Please visit <http://icmaa2020.web.ua.pt> for detailed information and updates. Contact Rute Lemos (rute@ua.pt) or Enide Andrade (enide@ua.pt) with any questions.

ILAS 2020: Classical Connections Galway, Ireland, June 22–26, 2020

The 23rd meeting of the International Linear Algebra Society, ILAS 2020: Classical Connections, will be hosted by the School of Mathematics at the National University of Ireland, Galway, June 22–26, 2020. The venue will be the beautiful riverside campus of the National University of Ireland, Galway.



The conference theme is “Classical Connections.” This will be reflected in the plenary programme and minisymposia, and all participants are encouraged to think about relating their themes to their historical roots. Contributions on all aspects of linear algebra and its applications are welcome.

The conference will feature 10 plenary talks, by the following speakers:

- Shmuel Friedland, University of Illinois at Chicago, USA
- Nicolas Gillis, Université de Mons, France
- Rod Gow, NUI Dublin, Ireland
- Misha Kilmer (SIAG-LA Lecture), Tufts University, USA
- Monique Laurent, Centrum Wiskunde & Informatica, the Netherlands
- Lek-Heng Lim (Hans Schneider Prize Lecture), University of Chicago, USA
- Clément de Seguins Pazzis, Lycée privé Sainte-Geneviève, France
- Christiane Tretter, University in Bern, Switzerland
- Vilmar Trevisan, Instituto de Matemática, UFRGS, Brazil
- Raf Vandebril (Tausky-Todd Lecture), KU Leuven, Belgium

Invited mini-symposia (with organizers): Graph spectra (D. Cardoso, C. Justel, R. del Vecchio), Spectral properties of non-negative matrices (C. Marijuán, P. Paparella), Copositive and completely positive matrices (A. Berman, M. Dür, N. Shaked-Monderer), Mathematics of quantum information (R. Levene, I. Todorov), Combinatorial matrix theory (J. Breen, R. Canogar), The inverse eigenvalue problem for graphs (J. Lin, P. Oblak), General preservers (L. Molnár, G. Geher), Distance matrices of graphs (P.N. Choudhury, A. Khare), Linear algebra education (A. Cronin, S. Stewart), Numerical linear algebra for PDEs (S. MacLachlan, N. Madden), The research and legacy of Richard A. Brualdi (A. Berliner, L. Deaett, S. Meyer), Matrix positivity: theory and applications (A. Belton, D. Guillot).

The Scientific Organising Committee includes: Nair Abreu (Brazil), Peter Cameron (Scotland), Mirjam Dür (Germany), Ernesto Estrada (Scotland), Vyacheslav Futorny (Brazil), Stephen Kirkland (Canada), Yongdo Lim (Korea), Rachel Quinlan (Ireland), Peter Šemrl (Slovenia), Helena Šmigoc (Ireland), Françoise Tisseur (England), Paul Van Dooren (Belgium).

The conference proceedings will be published as a special issue of *Linear Algebra and its Applications*. The editors for this special issue are: Nicolas Gillis, Rachel Quinlan, Clément de Seguins Pazzis, and Helena Šmigoc. Peter Šemrl is the Editor-in-Chief of *LAA* responsible for this special issue.

Ongoing updates and more information about the conference can be found at <http://ilas2020.ie>.

6th Workshop on Algebraic Designs, Hadamard Matrices & Quanta Kraków, Poland, June 29 – July 3, 2020

The 6th Workshop on Algebraic Designs, Hadamard Matrices & Quanta will be held at Jagiellonian University, as well as at the Institute of Mathematics, in Kraków, Poland.

The list of confirmed invited speakers includes: Ingemar Bengtsson (Stockholm, Sweden), Robert Craigen (Winnipeg, Canada), Ilias Kotsireas (Waterloo, Canada), Hadi Kharaghani (Lethbridge, Canada), Mate Matolcsi (Budapest, Hungary), and Padraig Ó Catháin (Worcester, USA).

Early conference registration is due by March 1, 2020. Further information can be found at <https://chaos.if.uj.edu.pl/hadamard2020>.

9th Linear Algebra Workshop (LAW'20) Portorož, Slovenia, June 29 – July 4, 2020

The 9th Linear Algebra Workshop (LAW'20) will continue the tradition of the previous meetings. Its main theme will be the interplay between operator theory and linear algebra on one side and algebra with a variety of algebraic and geometric structures on the other.

The organizers believe in including time to think about the problems, not only to listen to each other's formal talks. So, the workshop will be organized in a less formal way. Following tradition, only a few hours of talks will be scheduled for the morning sessions, while afternoons will be reserved for work in smaller groups. Participants are welcome to suggest their own topics of interest for these working groups. All topics from linear algebra within the broad scope outlined above are welcome. Many past working groups have produced multi-author papers.



Portorož, Slovenia

The workshop will be held at the Faculty of Maritime Studies and Transport, University of Ljubljana, Portorož. Portorož (Italian: Portorose, literally “of roses”) is a Slovenian Adriatic seaside resort and spa town located in the Municipality of Piran in southwestern Slovenia. Piran is an ancient Venetian city built as an exporting port for salt, which is still being produced in the vicinity in the traditional way. The workshop will be held following the 23rd Conference of the International Linear Algebra Society in Galway, Ireland, and it is a satellite conference of the 8th European Congress of Mathematics to be held July 5–11, 2020 in Portorož, Slovenia. Those interested in attending should register by April 15, 2020 (early bird), or by June 10, 2020 (regular). More information can be found at <http://www.law05.si/law20>.

28th International Workshop on Matrices and Statistics (IWMS 2020)

Manipal, India, December 15–17, 2020

The 28th International Workshop on Matrices and Statistics (IWMS 2020), will be held December 15–17, 2020 at the Center for Advanced Research for Applied Mathematics and Statistics, Manipal Academy of Higher Education (MAHE), Manipal, Karnataka, India.

The purpose of the Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. The Workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in linear algebra and matrix theory and may exchange ideas with researchers from a wide variety of countries. As well as a range of plenary speakers, the meeting will strengthen the interactions between participants through a range of mini-symposia in various areas of specialization.

Themes of the workshop will include: Matrix Analysis, Projectors in Linear Models & Multivariate Analysis, Growth Curve Models, Linear Regression Models, Linear Statistical Inference, Modelling Covariance Structures, Multivariate and Mixed Linear Models, and Statistics in Big Data Analysis.

The Scientific Committee consists of Ravindra B. Bapat, Manjunatha Prasad Karantha, Steve Kirkland, and Simo Puntanen. The Organizing Committee consists of Narayana Sabhahit (Chairman, Registrar, MAHE) and Manjunatha Prasad Karantha (Organizing Secretary, Coordinator, CARAMS, MAHE)

CRR Day on December 17, 2020: The present 28th IWMS will be held alongside ICLAA 2020 (December 17–19, 2020) and CRR Day will be held on December 17, the common day of events, to celebrate 100 years of C. R. Rao, who is among greatest statisticians and matrix theorists India has ever produced.

Please visit <https://carams.in/events/international-workshop-on-matrices-and-statistics> for more details and registration.

International Conference on Linear Algebra and its Applications (ICLAA 2020)

Manipal, India, December 17–19, 2020

In a sequel to the ICLAA series, the International Conference on Linear Algebra and Its Applications will be held December 17–19, 2020 at the Center for Advanced Research for Applied Mathematics and Statistics, Manipal Academy of Higher Education (MAHE), Manipal, Karnataka, India.

The themes of the conference shall focus on classical matrix theory, nonnegative matrices and special matrices, matrices and graphs, combinatorial matrix theory, matrix and graph methods in statistics and biological science, and matrices in error analysis and its applications.

The scientific committee consists of Ravindra B. Bapat, Manjunatha Prasad Karantha, Steve Kirkland, and Simo Puntanen. The Organizing Committee consists of Narayana Sabhahit (Chairman, Registrar, MAHE) and Manjunatha Prasad Karantha (Organizing Secretary, Coordinator, CARAMS, MAHE),

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Please visit <https://carams.in/events/iclaa2020> for more details and registration.

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- operator theory

$$\zeta^\alpha = \sum_{\pi} \operatorname{sgn} \pi \zeta^{\alpha - \operatorname{id} + \pi}$$

$$\zeta^\alpha \otimes \zeta^\beta = \sum_{\pi} \operatorname{sgn} \pi \zeta^{\alpha - \operatorname{id} + \pi} \otimes \zeta^\beta$$

$$\zeta^\alpha \otimes \zeta^\beta = \sum_{\pi} \operatorname{sgn} \pi (\zeta^\beta \downarrow S_{\alpha - \operatorname{id} + \pi} \uparrow S_{\pi})$$

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IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problem 60-4 and to all of the problems of issue 62. Solutions are invited to Problems 59-5, 60-2, 61-3 and for all of the problems of issue 63.

Problem 60-4: An Agreeable Permutation Problem

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let M be an n by n matrix all of whose entries are either 1 or -1 . Let σ be a permutation in S_n . We say that σ is an agreeable permutation if $\prod_{k=1}^n m_{k\sigma(k)}$ is equal to the sign of σ . For all $1 \leq i, j \leq n$, let $A(i, j)$ be the number of agreeable permutations which map i to j . Show that the number $A(i, j)$ is independent of i and j if and only if the two smallest singular values of M are either both equal to $\sqrt{n}$ or both equal to zero.

Solution 60-4 by Ravindra BAPAT, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in

Let M be an $n \times n$ matrix with entries ± 1 and suppose the two smallest singular values of M are both equal to 0. Then $\text{rank}(M) < n - 1$. Let M_{ij} be the submatrix of M obtained by deleting row i and column j . Then each M_{ij} is singular and hence $\det(M_{ij}) = 0$ for each i, j . Let $C_{ij} = \{\sigma \in S_n : \sigma(i) = j\}$. Then

$$(-1)^{i+j} m_{ij} \det(M_{ij}) = \sum_{\sigma \in C_{ij}} \prod_{k=1}^n m_{k\sigma(k)} = 2A(i, j) - (n-1)!. \quad (1)$$

It follows that $A(i, j)$ is independent of i and j .

Now suppose the smallest singular value of M is $\sqrt{n}$. Then the smallest eigenvalue of MM^T is n . The diagonal entries of MM^T are all n . It follows from the equality case in Raleigh's variational inequality that e_i , the i th unit vector, is an eigenvector of MM^T for the eigenvalue n for $i = 1, \dots, n$. Thus $MM^T = nI_n$ and M is a Hadamard matrix. Therefore $M^{-1} = \frac{1}{n}M^T$ and hence

$$(-1)^{i+j} \frac{\det(M_{ij})}{\det M} = \frac{m_{ij}}{n}. \quad (2)$$

It follows from (1) and (2) that

$$2A(i, j) - (n-1)! = \frac{\det M}{n}$$

and hence $A(i, j)$ is independent of i and j .

We now turn to the converse. Suppose $A(i, j)$ is independent of i and j . If M is nonsingular, then it follows from (1) that $M^{-1} = \gamma M^T$ for some γ . Then $MM^T = \frac{1}{\gamma}I_n$ and all eigenvalues of MM^T are $\frac{1}{\gamma}$. Since $\text{Tr}(MM^T) = n^2$, it follows that $\gamma = \frac{1}{n}$. Thus all eigenvalues of MM^T are n and hence all singular values of M are $\sqrt{n}$.

If M is singular, then letting $A(i, j) = \theta$ for all i, j , we see from (1) that

$$\det(M_{ij}) = (-1)^{i+j} m_{ij} (2\theta - (n-1)!). \quad (3)$$

It follows from (3) that for $i = 1, \dots, n$,

$$\det M = \sum_{j=1}^n (-1)^{i+j} m_{ij} \det(M_{ij}) = n(2\theta - (n-1)!). \quad (4)$$

Since M is singular, it follows from (3) and (4) that $\det(M_{ij}) = 0$ for $i, j = 1, \dots, n$. Thus $\text{rank}(M) < n - 1$ and the two smallest singular values of M are both equal to zero.

Problem 62-1: Permuted Circulants with Distinct Diagonal Elements

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let n be an even number. Show that if C is an $n \times n$ circulant matrix and P and Q are two $n \times n$ permutation matrices, then the matrix product PCQ never has n distinct entries on its main diagonal. (Recall: An $n \times n$ matrix C is said to

be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod n$.)

Solution 62-1.1 by Ravindra BAPAT, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in

Let C be an $n \times n$ circulant matrix. It will be sufficient to show that if σ is a permutation of $1, \dots, n$, then the entries $c_{1\sigma(1)}, \dots, c_{n\sigma(n)}$ are not all distinct. This will be proved if we show that the numbers $(\sigma(i) - i) \pmod n, i = 1, \dots, n$ are not all distinct. Let, if possible, σ be such that $(\sigma(i) - i) \pmod n$ for $i = 1, \dots, n$ are all distinct. Then these numbers must be $0, 1, \dots, n - 1$ (in some order). Adding $(\sigma(i) - i) \pmod n$ for $i = 1, \dots, n$, we see that $1 + 2 + \dots + (n - 1)$ must be $0 \pmod n$. Thus $(n - 1)n/2 = 0 \pmod n$. This is a contradiction as n is even. Thus there exist $i \neq j$ such that $\sigma(i) - i = \sigma(j) - j \pmod n$. It follows that $c_{i\sigma(i)} = c_{j\sigma(j)}$ and the result is proved.

Solution 62-1.2 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

The conclusion is true for all $n \neq 1, 3$. It is trivial for $n = 2$, so we may assume $n \geq 4$. Denote the first row of C by $a_0, a_1, \dots, a_{n-1}$ and assume these entries are all distinct. Since C is circulant,

$$C = [c_{ij}]_{1 \leq i, j \leq n}, \quad \text{where } c_{ij} = a_{j-i}$$

and the subscript $j - i$ is interpreted modulo n . Any permutation can be expressed as a product of transpositions, and so the product $B = PCQ$ may be thought of as the result of a sequence of permutations of any two rows and any two columns of C . Under such permutations, certain aspects of the structure of C are invariant. Specifically, each row and each column of $B = [b_{ij}]$ contains the entries $a_0, a_1, \dots, a_{n-1}$ in some order. Furthermore, if $b_{ij} = b_{kl} = a_m$, where $i \neq k$ and $j \neq l$, then $\{b_{il}, b_{kj}\} = \{a_{m-1}, a_{m+1}\}$. We refer to this as the *four corners property*. Suppose the diagonal entries of B are all distinct. By relabeling if necessary, denote them by $a_0, a_1, \dots, a_{n-1}$. Column 2 of B contains a_0 but it is not in rows 1 or 2. That is, $a_0 = b_{i2}$ for some $i \geq 3$. By the four corners property applied to $b_{11} = b_{i2} = a_0$, we have $\{b_{12}, b_{i1}\} = \{a_1, a_{n-1}\}$. Since $b_{22} = a_1$, we must have $b_{i1} = a_1$. By the four corners property applied to $b_{i1} = b_{22} = a_1$, we have $b_{21} = a_2$ since $b_{21} \neq a_0$. And, by the four corners property applied to $b_{21} = b_{33} = a_2$, we have $b_{23} = a_3$ and $b_{31} = a_1$, which implies $i = 3$. By the four corners applied to $b_{23} = b_{44} = a_3$, we have a contradiction, since a_2 cannot be in row 2 or column 3.

Problem 62-2: Vector Spaces over Finite Fields

Proposed by Bojan KUZMA, *University of Primorska, Slovenia*, bojan.kuzma@famnit.upr.si

Let V be a d -dimensional vector space over the finite field with q elements. A subset S of V is said to be *hyperplane complete* if every element of V can be written as a linear combination of at most $d - 1$ elements from S . Show that there is a hyperplane complete subset of V having at most $q + d - 1$ elements. If $d \geq q$, show that there is a hyperplane complete subset of V which has $d + 1$ elements.

Solution 62-2 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Let $\{v_1, v_2, \dots, v_d\}$ be a basis of V . For the first part of the problem, let

$$S = \{v_1, v_2, \dots, v_{d-1}\} \cup \{\alpha v_1 + v_d \mid \alpha \in \mathbb{F}_q\}.$$

Then $|S| = q + d - 1$, and if $v \in V$ then $v = \sum_{k=1}^d a_k v_k$ for some $a_1, a_2, \dots, a_d \in \mathbb{F}_q$. If $a_d = 0$, then v is a linear combination of $d - 1$ elements of S . If $a_d \neq 0$, let $\alpha = a_1 a_d^{-1}$. Then

$$\sum_{k=2}^{d-1} a_k v_k + a_d(\alpha v_1 + v_d) = \sum_{k=1}^d a_k v_k = v,$$

and so v is again a linear combination of $d - 1$ elements of S .

For the second part of the problem, let

$$S = \left\{ v_1, v_2, \dots, v_d, \sum_{k=1}^d v_k \right\}.$$

Then $|S| = d + 1$, and if $v \in V$, then $v = \sum_{k=1}^d a_k v_k$ for some $a_1, a_2, \dots, a_d \in \mathbb{F}_q$. If $a_k = 0$ for some $k \in \{1, 2, \dots, d\}$, then v is a linear combination of $d - 1$ elements of S . Otherwise, since $\mathbb{F}_q$ has q elements and none of the a_k is zero and

the number of coefficients a_k is at least q , two of the a_k must be equal. By relabeling if necessary, assume $a_1 = a_2$. Then

$$\sum_{k=3}^d (a_k - a_1)v_k + a_1 \sum_{k=1}^d v_k = a_1 v_1 + a_1 v_2 + \sum_{k=3}^d a_k v_k = v,$$

and so v is again a linear combination of $d - 1$ elements of S .

Problem 62-3: A Convex Rank Sequence

Proposed by Sneha SANJEEVINI, *University of Michigan, Ann Arbor, MI, USA*, snehasnj@umich.edu

and Omran KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*, omran_kouba@hiast.edu.sy

and Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $T_0, T_1, T_2, \dots$ be $m \times n$ matrices with entries in a field $\mathbb{F}$. For each positive integer k , define the block-Toeplitz matrix $\mathcal{T}_k$ by

$$\begin{bmatrix} T_0 & 0 & 0 & 0 & 0 \\ T_1 & T_0 & \cdots & 0 & 0 \\ T_2 & T_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & T_0 & 0 \\ T_{k-1} & T_{k-2} & \cdots & T_1 & T_0 \end{bmatrix}.$$

Show that the sequence $(\text{rank } \mathcal{T}_k)_{k \geq 1}$ is convex; that is,

$$2 \text{rank } \mathcal{T}_{k+1} \leq \text{rank } \mathcal{T}_{k+2} + \text{rank } \mathcal{T}_k, \quad \text{for } k = 1, 2, \dots$$

Example: For $k = 2$, the inequality becomes

$$2 \text{rank} \begin{bmatrix} T_0 & 0 & 0 \\ T_1 & T_0 & 0 \\ T_2 & T_1 & T_0 \end{bmatrix} \leq \text{rank} \begin{bmatrix} T_0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 \\ T_2 & T_1 & T_0 & 0 \\ T_3 & T_2 & T_1 & T_0 \end{bmatrix} + \text{rank} \begin{bmatrix} T_0 & 0 \\ T_1 & T_0 \end{bmatrix}.$$

Solution 62-3.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

We use the fact that if U and V are subspaces of a vector space such that $U \cap V = \{0\}$, then $\dim(U + V) = \dim U + \dim V$. Decompose $\mathcal{T}_{k+1}$ as a sum of two matrices whose ranges have intersection $\{0\}$:

$$\mathcal{T}_{k+1} = \begin{bmatrix} \mathcal{T}_k & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & O & \cdots & O & O \\ T_k & T_{k-1} & \cdots & T_1 & T_0 \end{bmatrix}.$$

Since the rank of a matrix equals the dimension of its range, the rank of the above sum equals the sum of the ranks of the two summands. Thus,

$$2 \text{rank } \mathcal{T}_{k+1} = \text{rank } \mathcal{T}_{k+1} + \text{rank } \mathcal{T}_k + \text{rank} \begin{bmatrix} T_k & T_{k-1} & \cdots & T_1 & T_0 \end{bmatrix}$$

and

$$\text{rank } \mathcal{T}_{k+2} = \text{rank } \mathcal{T}_{k+1} + \text{rank} \begin{bmatrix} T_{k+1} & T_k & \cdots & T_1 & T_0 \end{bmatrix}.$$

Since it is clear that

$$\text{rank} \begin{bmatrix} T_k & T_{k-1} & \cdots & T_1 & T_0 \end{bmatrix} \leq \text{rank} \begin{bmatrix} T_{k+1} & T_k & \cdots & T_1 & T_0 \end{bmatrix},$$

we have

$$\begin{aligned} 2 \operatorname{rank} \mathcal{T}_{k+1} &= \operatorname{rank} \mathcal{T}_{k+1} + \operatorname{rank} \mathcal{T}_k + \operatorname{rank} \begin{bmatrix} T_k & T_{k-1} & \cdots & T_1 & T_0 \end{bmatrix} \\ &\leq \operatorname{rank} \mathcal{T}_{k+1} + \operatorname{rank} \mathcal{T}_k + \operatorname{rank} \begin{bmatrix} T_{k+1} & T_k & \cdots & T_1 & T_0 \end{bmatrix} \\ &= \operatorname{rank} \mathcal{T}_{k+2} + \operatorname{rank} \mathcal{T}_k. \end{aligned}$$

Solution 62-3.2 by Nir COHEN, *UFRN-Natal, Brazil*, nir@ccet.ufrn.br

We prove the required inequality and show that it is tight. Namely, assuming the matrices $T_0, T_1, \dots, T_{k-1}, T_k$ and hence also $\operatorname{rank}(\mathcal{T}_{k-1})$ and $\operatorname{rank}(\mathcal{T}_k)$ are fixed and that the matrix T_{k+1} is a variable, we show that

$$\min_{T_{k+1}} \operatorname{rank}(\mathcal{T}_{k+2}) = 2 \operatorname{rank}(\mathcal{T}_{k+1}) - \operatorname{rank}(\mathcal{T}_k). \quad (5)$$

Indeed, as part of the minimal rank completion problem we have the identity

$$\min_X \operatorname{rank} \begin{pmatrix} A & B \\ X & C \end{pmatrix} = \operatorname{rank} \begin{pmatrix} A & B \end{pmatrix} - \operatorname{rank} \begin{pmatrix} B \\ C \end{pmatrix} + \operatorname{rank} \begin{pmatrix} B \\ C \end{pmatrix} \quad (6)$$

(a special case of formula (0.2) of [2], also shown in [1]). Choosing

$$A = \begin{bmatrix} T_0 & T_1 & \cdots & T_{k-1} & T_k \end{bmatrix}^T, \quad B = \begin{pmatrix} 0 & 0 \\ \tau_k & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{bmatrix} T_k & T_{k-1} & \cdots & T_1 & T_0 \end{bmatrix},$$

we observe that with

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix} = \mathcal{T}_{k+2}, \quad \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} \tau_{k+1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \tau_k & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \tau_{k+1} \end{pmatrix}, \quad (7)$$

the identity (6) reduces to the required identity (5).

References

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- [2] H. J. Woerdeman. The lower order of lower triangular operators and minimal rank extensions. *Integral Equations Operator Theory*, 10(6):859–879, 1987.

Solution 62-3.3 by Ravindra BAPAT, *Indian Statistical Institute, New Delhi, India*, rbb@isid.ac.in

Let $\mathcal{T}_1 = T_0$ and let $\Gamma_j = [\mathcal{T}_j, 0]$ be of order $m_j \times nk$ for $j = 1, \dots, k$. Note that Γ_j is the submatrix of $\mathcal{T}_k$ consisting of the top m_j rows of $\mathcal{T}_k$. We also let $\Delta_j = [T_j, T_{j-1}, \dots, T_0, 0]$ for $j = 1, \dots, k$, which is of order $m \times nk$. Let $\operatorname{rank} \Gamma_1 = \alpha_0$ and choose α_0 rows of Γ_1 that form a basis for its row space. The corresponding α_0 rows in Δ_1 must also be linearly independent and thus we have a total of $2\alpha_0$ linearly independent rows in Γ_2 . Augment these $2\alpha_0$ rows by another α_1 rows of Γ_2 , coming necessarily from Δ_1 , so that the $2\alpha_0 + \alpha_1$ rows form a basis for the row space of Γ_2 . Out of these $2\alpha_0 + \alpha_1$ rows, $\alpha_0 + \alpha_1$ rows are from Δ_1 , and if we augment the set by the corresponding $\alpha_0 + \alpha_1$ rows of Δ_2 , then we get a total of $3\alpha_0 + 2\alpha_1$ linearly independent rows of Γ_3 . We may augment the set by α_2 rows of Γ_3 , coming necessarily from Δ_2 , so that we have $3\alpha_0 + 2\alpha_1 + \alpha_2$ rows of Γ_3 which form a basis for its row space.

Continuing in this manner, we may choose $j\alpha_0 + (j-1)\alpha_1 + \cdots + \alpha_{j-1}$ rows of Γ_j which form a basis for its row space, for $j = 1, \dots, k$. Thus we have $\operatorname{rank} \Gamma_k = k\alpha_0 + (k-1)\alpha_1 + \cdots + \alpha_{k-1}$, $\operatorname{rank} \Gamma_{k+1} = (k+1)\alpha_0 + k\alpha_1 + \cdots + \alpha_k$ and $\operatorname{rank} \Gamma_{k+2} = (k+2)\alpha_0 + (k+1)\alpha_1 + \cdots + \alpha_{k+1}$. It follows that

$$\operatorname{rank} \Gamma_{k+2} + \operatorname{rank} \Gamma_k - 2 \operatorname{rank} \Gamma_{k+1} = \alpha_{k+1} \geq 0.$$

Since $\operatorname{rank} \Gamma_k = \operatorname{rank} \mathcal{T}_k$ for $k = 1, \dots, k$, the proof is complete.

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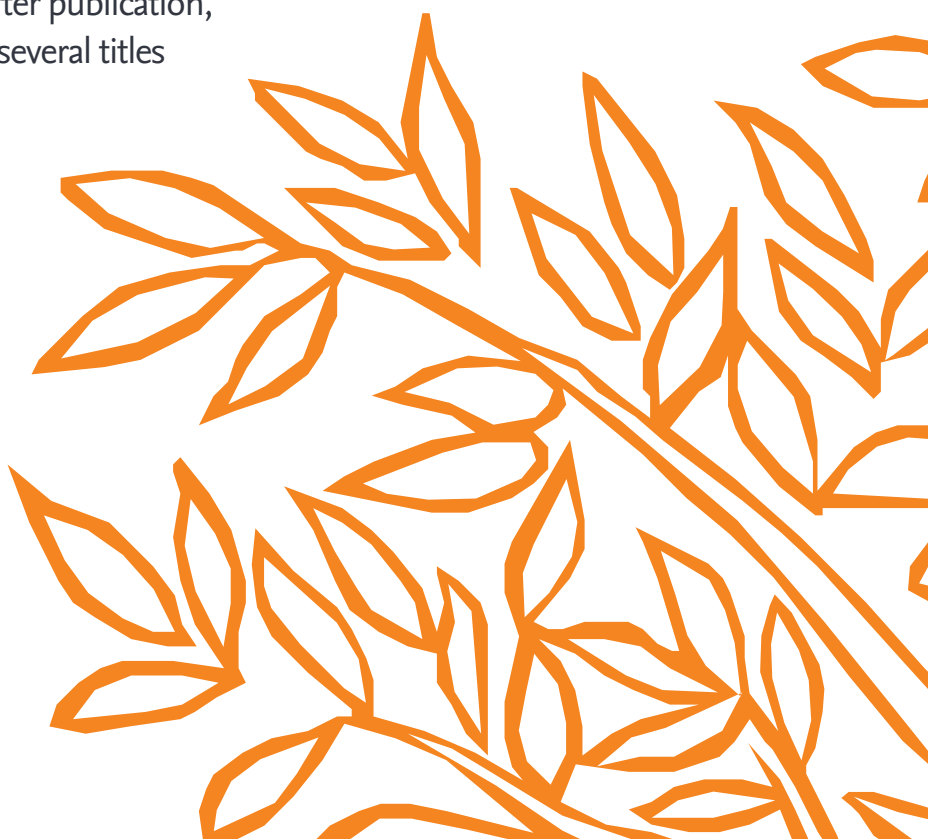


IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce three new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.

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NEW PROBLEMS:

Problem 63-1: Special Cases of a 100-Euro Conjecture.

Proposed by Siegfried M. RUMP, *Hamburg University of Technology, Hamburg, Germany*, rump@tuhh.de

When $v = (v_1, v_2, \dots, v_n)^T$ and $w = (w_1, w_2, \dots, w_n)^T$ are real n -vectors, we write $|v| \geq |w|$ to signify that $|v_k| \geq |w_k|$ for all k such that $1 \leq k \leq n$. If $A = [a_{ij}]$ is an $n \times n$ real matrix, we let $|A|$ denote the $n \times n$ matrix $[|a_{ij}|]$. The author has offered (see <http://www.ti3.tuhh.de/rump/100EuroProblem.pdf>) a 100-euro prize for the first proof or counterexample for the following conjecture: If A is an $n \times n$ real matrix such that all row sums of $|A|$ are equal to n , then there exists a nonzero $x \in \mathbb{R}^n$ such that $|Ax| \geq |x|$.

- (a) Show that the conjecture is true if A is symmetric, $\text{rank}(A) = 1$, or $\text{rank}(|A|) \leq 2$.
- (b) More generally, show that the conjecture holds if $|A|$ has all of its eigenvalues real.

Problem 63-2: Normal Principal Submatrices of a Normal Matrix

Proposed by Achiya DAX, *Hydrological Institute, Jerusalem, Israel*, dax20@water.gov.il

Let $E = \{z \in \mathbb{C} : |z - z_0| > r\}$ be the exterior of a circle of radius r and centre z_0 in the complex plane, where r and z_0 are arbitrary fixed positive and complex numbers, respectively. Let M be a normal matrix and N be a normal principal submatrix of M of any order. Show that the number of eigenvalues (counted with multiplicity) of the submatrix N which lie in E is always less than or equal to the number of eigenvalues of M which lie in E .

Problem 63-3: Products of Rectangular Circulant Matrices

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

An $n \times n$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod n$. Let m , n and q be positive integers. We can define a rectangular circulant matrix as follows: An $m \times n$ matrix C is said to be a rectangular circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod{\gcd(m, n)}$. As an example, a 6×9 rectangular circulant looks like

$$\begin{bmatrix} a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \\ a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \end{bmatrix}.$$

Let A be an $m \times n$ rectangular circulant and B be an $n \times q$ rectangular circulant.

- (a) Show that AB is also a rectangular circulant.
- (b) What is the maximum rank that AB can have?
- (c) Show that the Moore-Penrose inverse of a rectangular circulant matrix is a rectangular circulant matrix.

Solutions to Problems 60-4, 62-1, 62-2 and 62-3 are on pages 33-36.