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IMAGE

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Contents continued on page 2.

About *IMAGE*

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We thank Carlos for serving as a contributing editor of *IMAGE* since the beginning of 2012! Carlos was responsible for introducing and overseeing the interviews section, which has become a favorite of *IMAGE* readers.

We also thank David, who has served as a contributing editor of *IMAGE* since mid-2013! David has encouraged conversations about linear algebra education at conferences and via the education column in *IMAGE*.

We anticipate appointing new contributing editors with these responsibilities in the near future. Nominations are welcome!

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FEATURE INTERVIEW

"Never a dull moment"

Beresford N. Parlett interviewed by Ming Gu[1]

Beresford Parlett contemplating the source of the Columbia River (British Columbia)

M.G. - How did you come to study math?

B.P. - We were at war with Germany (World War II), and my little prep school which took boys from 7 to 13 was evacuated from its regular place on the South Coast (Brighton) to a city called Litchfield more or less in the middle of England. We were trying to prepare for invasion from across the channel. My prep school was owned and run by a Mr. Burr, who loved mathematics. He was a gifted, if eccentric, teacher. In addition to running the school, he wrote a witty play every year to be acted by the boys. In math his way was that we had no textbooks and we were not allowed to take any notes. All the math had to be in our heads. A few of the boys found we could do that. We just worked problems in class under his guidance. Occasionally, while we were working hard on a problem, he would shoot a rabbit from the classroom window. Never a dull moment.

With his methods, by the age of 12 I knew well the various ways to write down the equation for a tangent at a point on a curve. This was made easier because the only functions we ever dealt with were polynomials, no objects from trigonometry or analysis. We would write down the equation for a secant and then let one point of intersection approach the other. Things worked out fine. I got very good at that. We all did. And the word "calculus" was never mentioned. I could handle derivatives to polynomials very well. He got us very advanced in what's called synthetic geometry (plane geometry). So I knew a lot about circles associated with triangles and so on.

At age 13, I managed to win a competitive exam to get a scholarship to my public school (taking boys from 13 to 18). Only there did I realize how advanced I was compared to all my peers. After a year, at age 14 (nearly 15) I passed the national exams (O-levels) and I had to choose a focus and that was to either be in the classics or in mathematics. Although I liked Latin as much as math, I had only one year of Greek and with that handicap mathematics would be a better bet. So that's how I ended up doing nothing but math and physics after the age of 15. I am so grateful now that I did focus on mathematics, but I did not appreciate it at the time.

Next, I was very lucky that a young maths master arrived, fresh from Cambridge, for my last year. Thanks to his coaching I won a competitive exam and got a scholarship to New College, Oxford. That was in 1950. This was the first scholarship to an Oxford or Cambridge college that the school had won since the war ended and the headmaster was so excited that he gave a half holiday for the whole school of about 400 boys.

I must add one more item. A week after leaving school, I was in my basic training for the Army. There they found out that I was severely myopic (nearsighted) and discharged me on that account. Then one week after coming out of the Army I was up at Oxford. I was not pleased, because I was two years younger than my peers and I had expected to come up to Oxford two years later in 1953.

At Oxford I had an active social life. I was in the first rowing crew for my college (in the off-season) and we won our oars. I then turned my attention to acting, but despite all this frivolity managed to get my B.A. in 1955. So that's really the end of my school days.

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M.G. - Where would you have gone if you had not been discharged?

B.P. - I had a choice of going to either Korea or Kenya (to fight the Mau Mau). Both of those places were quite exciting. I'd actually signed up for either of them. And I had a full expectation to be an officer. I wanted adventure, not theorems.

M.G. - You could have gone to Korea.

B.P. - Yes, yes. Some people think I was lucky to be discharged.

M.G. - I think so.

B.P. - Um. I would like to carry on about Stanford.

M.G. - Okay, yes. Could you tell me more about your experience there?

B.P. - I'd like to. I started in a Ph.D. program in September of 1958. I mention that because the previous three years were spent by me either in manual labor in sawmills or as a lumber salesman based in London. When I arrived and was told that there was a placement exam in about a week's time, I was scared stiff. To my great surprise, having done no mathematics at all for three years. I placed second in order of the 23 of us in the program. But I was a long way behind Jimmy Ax, who was first, and he's now a celebrated number theorist. (Actually, he left our Ph.D. program at the end of that first year in order to get his Ph.D. at UC Berkeley.)

I quickly adapted to the American system of university education and I liked it very much. Many of the faculty at Stanford in math were fugitives from Hitler's Europe. I was also a teaching assistant in the Math department and what was really nice at Stanford, compared to Berkeley, was that I had complete control over one section (of first-year Calculus) out of nine that Stanford offered its freshmen, and this gave me the right to an office, which I shared with another graduate student who was a year ahead of me. This was a man called Jim Cochran who, when he got his Ph.D., went to Bell Labs for a while, and ended up being a dean at Washington State University in Pullman, WA. He was a great help to me.

I can't resist the temptation of talking about my first summer at Stanford. That was in 1959. The American Rhodes scholar friend who persuaded me to come to the USA paid for my first year in the graduate dormitory on campus called Crothers Memorial. I learned, early in June 1959, that I had to vacate my room there. What should I do? I moved my clothes to my office desk and I moved my math papers onto the open shelves in the office. Also I found that I could store a breakfast yogurt and also luncheon meats in the International House refrigerator, which I did. My routine was to make a sandwich for myself at breakfast time. I took that sandwich to the swimming pool at lunchtime every day, and after swimming a few laps I ate my sandwich and sunbathed. In the evening I walked off campus to the nearest mall and enjoyed the one hot meal of the day. A pleasant and healthy routine. At all other moments I studied.

The extra benefit was that I found that the key to my office in the math building fitted the lock of the door to the ladies' restroom, which happened to be just opposite the office I shared with Jim Cochran. After a little experimentation, I developed a routine. Every night, the campus police came through the math building at about a quarter to midnight, and they found me studying diligently. As soon as they left the building, I opened the window to my office and then I wedged open the door and then I opened the door to the ladies' restroom and spread my sleeping bag on the sofa that was always there in those days. I put on my pajamas and slept well until about six or 6:30 in the morning. The department secretary came in to work punctually at 7:45, by which time the ladies' restroom was back in proper order.

M.G. - What you are saying means that you slept less than eight hours every night.

B.P. - Yes, but they were comfortable and I felt well-rested. The reason I did all this was that I was very keen to spend as much time as possible preparing for the Ph.D. qualifying exams, which were coming up in April of 1960, and I felt I was behind hand through having prepared to be a businessman for three years prior to 1958 and I had a lot to catch up. I did indeed do very little else but study, but I did take a quarter-time job in the computing center, which was mainly handling punched cards for the IBM 650 Stanford had at that time. And I'm extremely glad that I did study so hard, because 23 of us took the written exams and only seven of us passed. A week later, those seven had orals, and five of us passed. Luckily, I was one of the five. So I was extremely pleased with the way I spent the summer of 1959.

M.G. - So what happened to the rest of the cohort?

B.P. - Good question. Everyone was allowed two shots at the exams. So all the people who had not passed could take the quals the next year. However some decided to go elsewhere for a Ph.D., including a powerful researcher in numerical linear algebra named Bill Gragg who wrote his dissertation under Peter Henrici at UCLA.

There is another twist to my story. Three days before my oral exam, I was in the changing rooms at the swimming pool and there I met my brilliant friend Al Novikov who ended up as a professor of mathematics at NYU. I said to him, "Al, I've got an oral coming up in three days. One of the special subjects I've chosen is group representations and I feel I know all the basic material very well, but I just don't have a feel for the subject as a whole, I don't really see why anyone would study group representations. Can you help me?" There, on the spot in the changing rooms, Al gave me a beautiful five-minute talk on why one would want to study group representations, using very simple examples like sines and cosines and so on. I was, of course, both impressed and delighted. I was even more delighted when the first question I was asked on my oral, three days later, was: Why would you study group representations? Al's five-minute lecture was very fresh in my head and I reproduced it pretty well. And I could tell that the members of the committee were impressed. That got me off to a wonderful start and I think the confidence that ensued carried me through the rest of the exam. I did well just because I got the right question at the start. Nine bows to Al Novikov.

M.G. - Changing topics, let me ask what are your favorite research results so far.

B.P. - I would like to talk about the Mismatch Theorem. This concerns the Lanczos algorithm which was presented in a paper, way back in 1950, by Cornelius Lanczos. Most attention has been paid to the symmetric case. However I was interested in the reduction to tridiagonal form of large unsymmetric matrices. At each step, the wanted tridiagonal grows by one row and one column, just three new entries. It was quite well known, not just to me, that the algorithm can break down in the unsymmetric case. Breakdown means that a crucial new entry of the tridiagonal vanishes. Usually that breakdown can be circumvented if one is willing to go to a block version of the tridiagonal. Nevertheless, what was known, but not well, was that increasing block size does not cure all the bad cases that can occur. One can have 'incurable' breakdown. By the way, I am assuming exact arithmetic for the whole discussion. No roundoff effects.

Let me just say a word about the unsymmetric algorithm. The main difference from the symmetric case is that you build up two, not one, sets of Lanczos vectors, which I call the left and the right. Each starts from an (arbitrary) starting vector. And of course, at each step the tridiagonal Lanczos matrix grows by one row and column. And one vital property is that the left and right Lanczos vectors are what is often called bi-orthogonal. I don't really like that word. I think it's much better to use a more old fashioned phrase which is that the two sets of vectors are self-dual. What that means in practice is: If the procedure goes to the full number of steps, those two matrices, Left and Right, will be inverses of each other. In addition, at each step you still have a diagonal matrix from their product, Left times Right. I think it was early in the 1970s, perhaps 1971, that I set one of my Ph.D. students, Derek Taylor, to study incurable breakdown.

He came into my office one day somewhat later and said: At incurable breakdown every eigenvalue of the tridiagonal Lanczos matrix is an exact eigenvalue of the big matrix. I said: Oh, that's impossible. It's ridiculous and no invariant subspaces are in sight. Nevertheless, he was correct. When I grasped it, I said: Explain that and you have a thesis. Well, Derek Taylor chose to expand each of the two starting vectors, left and right, in the basis of eigenvectors of the big matrix. And what he found was that if the eigenvectors that supported the left initial vector and the set of eigenvectors that supported the right starting vector were mismatched (i.e., the sets were not the same) then incurable breakdown was certain, before the final step. Several years later I found a more geometric proof, quite complicated, only to learn from Jim Demmel that I had rediscovered the Kalman Gilbert Canonical Structure Theorem from linear systems theory. Not original, but perhaps the deepest result I have discovered for myself.

M.G. - So what are your favorite results in numerical linear algebra?

B.P. - I think my most significant work has all been joint with others. And the first one I should mention came in 1993 when K. V. Fernando was visiting Berkeley, and he and I produced the dqds algorithm. That acronym is pronounced "the differential qd algorithm with shifts" and its function is the accurate calculation of the singular values of a bi-diagonal matrix, however small they may be. This daunting goal was not even attempted in earlier days and is not always, or even often, possible. Demmel and Kahan had recently shown that the bi-diagonal QR algorithm could also give high relative accuracy, but only if no shifts were used.

I do want to say something about this funny word "differential". dqd is a little-known variant of the qd algorithm that Rutishauser introduced, in 1954 I think. Historically, the qd algorithm gave rise to the LR algorithm for tridiagonals and then for full matrices. Then the LR algorithm was replaced by the QR algorithm. Next, Demmel and Kahan used the bidiagonal QR algorithm, without shifts, to compute singular values of bidiagonals. Finally, Fernando saw that what they were doing was equivalent in exact arithmetic to Rutishauser's original qd but in a variant form which uses an extra variable. Rutishauser's notes reveal that he knew this variant, gave it the name differential, but showed little interest and never published it. He did not use shifts, but it is not difficult to bring them in. I felt that it was rather nice to circle back to qd after the detour via LR and QR. I was pleased to find a special error analysis for our dqds algorithm. The result can be expressed as follows. If you execute one dqds transform and there is no overflow or underflow, and if there are no NaN's (Not a Number), then, in fact, tiny special perturbations in the last two bits of each number in the input array and the output array yield an exact dqds transform, which of course preserves the singular values exactly. This holds whether or not there is any element growth. It is a little bit hard to know what to make of that result. But it has been vital in subsequent work on the MRRR algorithm. It is called the "high mixed relative accuracy" of dqds. Here, "mixed" refers to the need to perturb the output as well as the input.

My collaboration with my last Ph.D. student at UC Berkeley, Inderjit Dhillon, uses the dqds transform in various ways, which I will not go into since it is rather technical, to produce a procedure that can compute k eigenpairs of a symmetric tridiagonal matrix in $O(nk)$ operations. That was a very pleasing result, but much too technical to try to talk about here. No Gram-Schmidt orthogonalization is needed.

The algorithm has been realized in LAPACK. That is all I want to say about MRRR.

M.G. - That's really nice. And so now do you want to tell me some stories about your book *The Symmetric Eigenvalue Problem*?

B.P. - Yes, I think the story to tell is that, with a bit of encouragement from George Forsythe, I wrote a whole book on the unsymmetric eigenvalue problem. I put together everything I knew and had discovered in my dissertation. By the time I'd finished the book, my advisor, George Forsythe, had died from stomach cancer. Cleve Moler had taken his place as consulting editor at Prentice Hall, and he reviewed my book. He said, "I don't really think we want to publish it. It's just a bag of tricks. But if you add a chapter on the symmetric case, then maybe we will publish it." I accepted his criticisms and went to work and started to develop that chapter on the symmetric case. The timing was such that there were several recent results in the symmetric case that I was very pleased to put in the book. One was that I found a simple way of presenting Chris Paige's wonderful surprising results on the Lanczos algorithm run in finite precision without stopping. He showed that, despite departing substantially from what it would give in exact arithmetic, the procedure does not lose any information on the true eigenvalues however long the algorithm runs, perhaps many more than n steps, with n the order of the big matrix. The eigenvalues of the original big matrix appear as multiple eigenvalues of the Lanczos matrix. My simplification of his analysis was to assume that all arithmetic is exact except for the multiplication of the most recent Lanczos vector by the big matrix. It is that roundoff error that drives the finite precision output. So I wanted to put that in that extra chapter. Then Walter Hoffman and I had found an elegant proof of the global, not just local, convergence of the QR algorithm on a tridiagonal symmetric matrix using Wilkinson's shift.

M.G. - This is not in finite precision.

B.P. - Indeed, we are back to the comfort of exact arithmetic. Wilkinson had recently proved that, with Wilkinson's shift, global convergence is assured, but he couldn't say how fast initially. His key observation was that the product of the last two off-diagonal entries always converges to zero. But it could initially be slow. What Hoffman and I found was that if you look at a slightly different function, namely the product of the second from last off-diagonal entry by the square of the last off-diagonal entry, then that function goes to zero geometrically with ratio $1/\sqrt{2}$ right from the start. Our proof is more elementary than Wilkinson's. I was sad that this result has been ignored. It is clearly the "right" proof.

And thirdly I wanted to include unpublished work by my friend, colleague, and mentor Velvel Kahan who had some powerful but complicated error bounds on eigenvalue approximations from eigenvalues of submatrices.

So anyway, my chapter grew rather large and I published it as a whole book. My original version on the unsymmetric eigenvalue problem was just a rather painful learning experience (and, of course, is hopelessly out of date now).

M.G. - It never got published?

B.P. - It never got published. My book on the symmetric case has been well received and I think I was wise to include code only at a high level, and that sparingly. Two things I neglected were sparsity and parallel computation.

M.G. - Yeah, that was not the right time.

B.P. - I actually had finished the symmetric book by 1978.

M.G. - Okay, so do you want to tell me some stories about your collaboration with the other people in your field?

B.P. - I cannot think of any good stories. I have been a minor author on software projects with Jim Demmel and Osni Marques. I enjoyed working with John Reid during my sabbatical leave at Harwell in England and enjoyed collaborating with Martin Gutknecht, Osni Marques, and Carla Ferreira on different projects. No more stories.

Beresford Parlett at home with his late wife Fredrica and cat Lily

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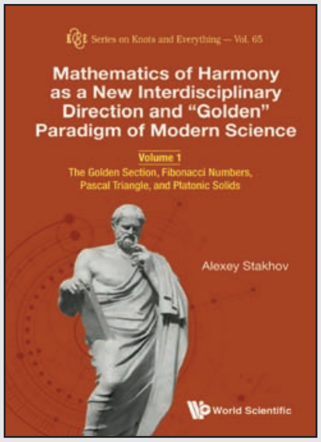

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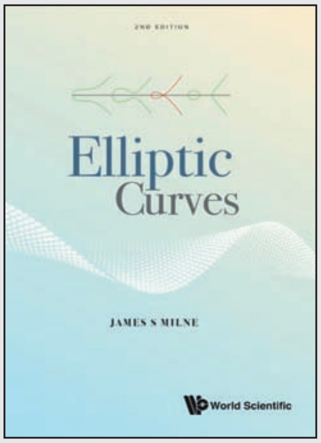

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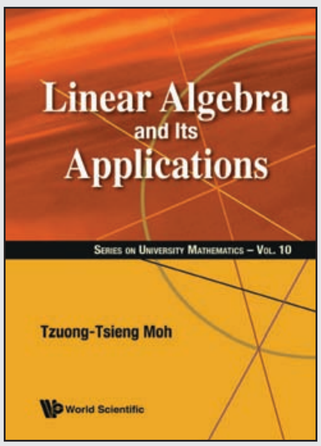

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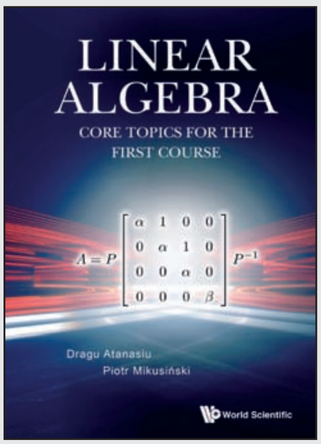

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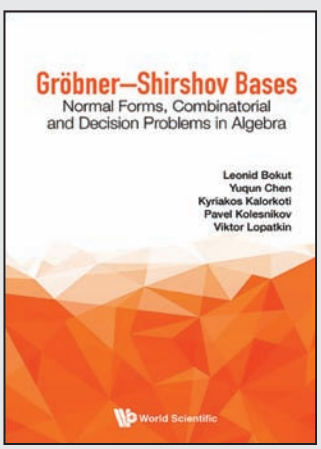

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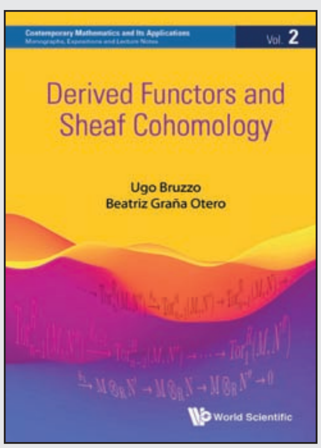

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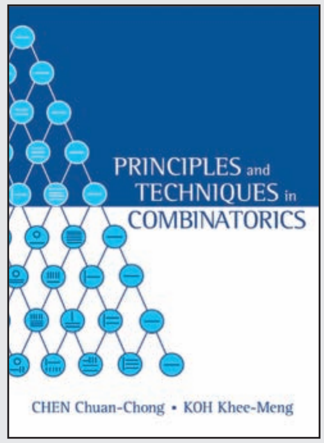

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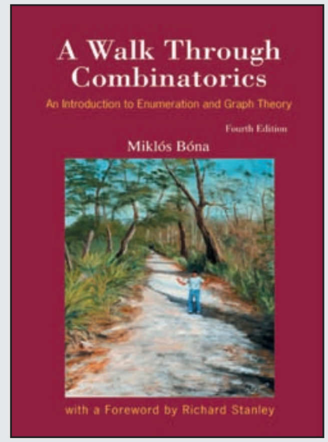

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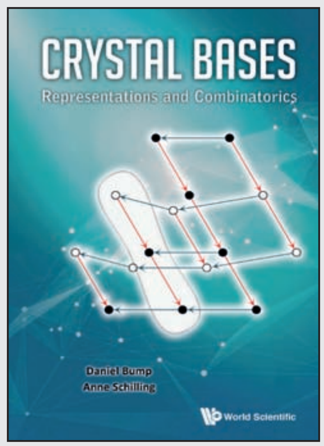

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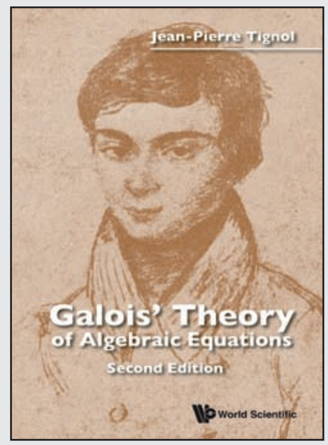

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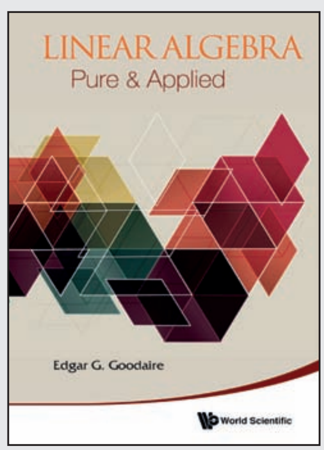

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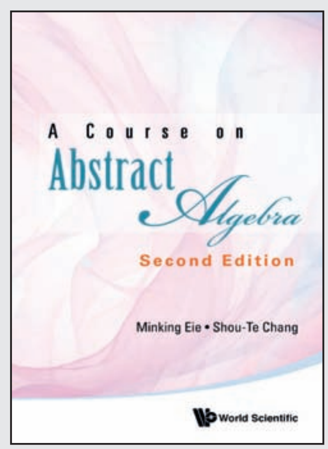

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FEATURE ARTICLE

The Eigenvalues of Random Matrices

Elizabeth Meckes[1], Case Western Reserve University, Cleveland, Ohio, USA, elizabeth.meckes@case.edu

1. Introduction. Although one can see earlier glimmers (e.g., [29]), the study of random matrices as such originated in statistics in Wishart's 1928 consideration of random sample covariance matrices [54], in numerical analysis in von Neumann and collaborators' work in the 1940s on numerical methods for solving linear systems [49, 23], and in nuclear physics in Wigner's 1955 introduction of random matrices as models for atoms with heavy nucleii [52, 53]. Since then, random matrix theory has found countless applications both within mathematics and in science and engineering. While using random matrices as statistical models in the presence of uncertainty is perhaps the most obvious way to go, there is a more fundamental reason for those who study and use matrices to know something about random matrices: *recognizing what's typical*. First mathematical questions tend to be about what is possible: How large or small can the eigenvalues be? How long might the algorithm take? How many edges can a graph have and still contain no triangles? But for some purposes, what is possible is less relevant than what is typical: How large or small do the eigenvalues *tend to be*? How long does the algorithm *usually take*? For what size graphs is it *fairly common* to have no triangles?

The single problem in random matrix theory which has received the most attention is that of understanding the eigenvalues of random matrices. Of course, this is not actually a single problem, but a huge class of problems: There are many ways to build a random matrix (leading to many drastically different eigenvalue distributions), and there are many different things to understand about any given random matrix model. In this note, I will give a broad overview of some random matrix models and some of what is known about their eigenvalues.

2. Random eigenvalues. A random matrix is a measurable function from a probability space into a set of matrices. Perhaps more concretely, a random matrix is a matrix whose entries are random variables with some joint distribution. If M is an $n \times n$ random matrix, the eigenvalues of M are a collection of n random points (not necessarily distinct, although in most of the cases we'll discuss, they are distinct with probability one).

In many cases, there is an explicit density formula for the eigenvalues of a particular random matrix. For example, the (complex) Ginibre ensemble consists of random matrices with independent, identically distributed standard complex Gaussian entries (i.e., entries distributed as $\frac{1}{\sqrt{2}}Z_1 + \frac{i}{\sqrt{2}}Z_2$ for Z_1 and Z_2 i.i.d. standard Gaussian variables). In this case, there is an explicit formula for the density of the eigenvalues, given by

$$\varphi_n(z_1, \dots, z_n) = \frac{1}{\pi^n \prod_{k=1}^n k!} \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{1 \le j < k \le n} |z_j - z_k|^2.$$

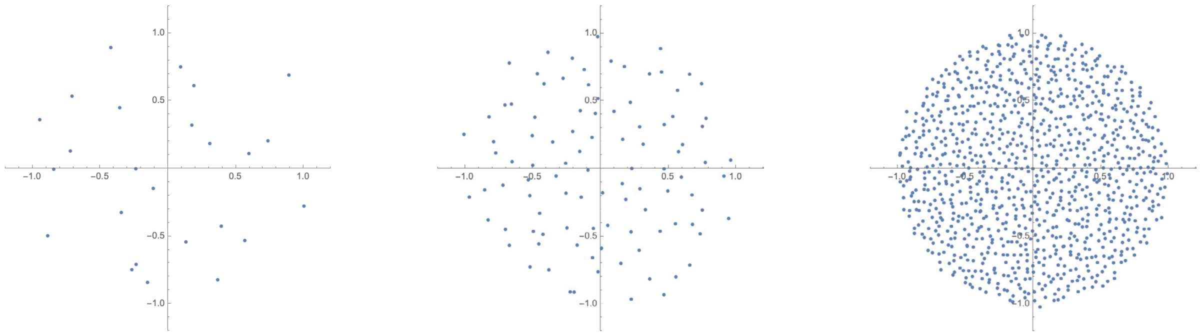
In a sense, then, we know everything about the eigenvalues of the Ginibre ensemble: We know the rules which determine where the eigenvalues are and with what probabilities. And having such an explicit eigenvalue density is indeed a powerful tool. But just as knowing the rules of chess is a far cry from being ready to play a Grandmaster, knowing this explicit formula still leaves a lot to discover. In fact, the eigenvalue density for the Ginibre ensemble is one of the simpler eigenvalue density formulae one sees, but it is complicated enough that even relatively simple questions about the typical behavior of the eigenvalues are not immediately accessible.

Most early applications of random matrix theory were about the limiting behavior of the spectra as the size of the matrix tends to infinity, either to develop a statistical understanding of "very large" matrices, or to use large matrices to approximate infinite-dimensional operators. An important tool for encoding the eigenvalues of a random matrix is the *empirical spectral measure*: If M has eigenvalues $\lambda_1, \dots, \lambda_n$, then its empirical spectral measure μ_M is the random probability measure putting equal mass at each of the eigenvalues:

$$\mu_M = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j}.$$

[1]Supported by ERC Advanced Grant 740900 (LogCorRM) and Simons Fellowship 678148

A key reason for considering the empirical spectral measure is that it does provide a natural framework in which to talk about limiting behavior, as the size of the matrix tends to infinity. Consider the following collections of eigenvalues from random matrices of increasing size:



Visually, it is clear what is happening: Whatever random procedure we are using to generate these matrices, as we generate larger and larger random matrices, the eigenvalues seem to be filling out the disc. In other words, the empirical spectral measures seem to be converging to the uniform probability measure on the disc, in a phenomenon known as *the circular law* (see Section 5 below).

Of course, since the empirical spectral measures are random probability measures, there are various senses of convergence. The simplest is *convergence in expectation*. Given a random matrix M, we define the expectation $\mathbb{E}\mu_M$ of its empirical spectral measure (a non-random probability measure) as follows: For a test function f (defined on whatever space the eigenvalues of M lie in),

$$\int f d(\mathbb{E}\mu_M) := \mathbb{E}\left(\int f d\mu_M\right),$$

assuming that the right-hand side is defined. That is, to integrate with respect to the expected spectral measure, we integrate with respect to the (random) empirical spectral measure to get a random variable, and then take its expectation. If we have a sequence $\{M_n\}$ of random matrices, we say that their spectral measures converge in expectation to a limiting measure μ if the sequence $\{\mathbb{E}\mu_{M_n}\}$ converges weakly to μ; i.e., if

$$\int f d\left(\mathbb{E}\mu_{M_n}\right) \xrightarrow{n\to\infty} \int f d\mu$$

for bounded continuous test functions f. Roughly, convergence in expectation means that, on average, the distribution of the eigenvalues resembles the mass distribution of the limiting measure, for large n.

Stronger notions of convergence include convergence *weakly in probability* and *weakly almost surely*. For a sequence $\{M_n\}$ of random matrices, we say that their spectral measures converge weakly in probability to μ if for each bounded continuous test function f, the random variables $\left\{\int f d\mu_{M_n}\right\}$ converge to $\int f d\mu$ in probability; i.e., for each f and each $\epsilon > 0$,

$$\mathbb{P}\left[\left|\int f d\mu_{M_n} - \int f d\mu\right| > \epsilon\right] \xrightarrow{n\to\infty} 0.$$

We say that the sequence $\{\mu_{M_n}\}$ converges weakly almost surely to μ if for each test function f, the random variables $\left\{\int f d\mu_{M_n}\right\}$ converge to $\int f d\mu$ with probability one. Convergence weakly almost surely is stronger than convergence weakly in probability, but both can be thought of as saying that, using any fixed test function f to compare measures, for large n one will typically not be able to see much difference between the random spectral measure μ_{M_n} and the reference measure μ. Thought of in terms of simulation, the notions of convergence weakly in probability and convergence weakly almost surely both justify the idea that a *single simulation* of the eigenvalues of M_n for large n is likely to produce a collection of dots whose spatial distribution resembles the mass distribution described by μ, whereas convergence in expectation only means that if you do such a simulation many times and look at the results in aggregate, you will see a spatial distribution of dots which resembles μ.

While the notions of convergence discussed above have been formulated in terms of bounded, continuous test functions, there are also other classes of functions that are frequently used. One possibility is indicator functions of measurable sets: If M is a random matrix whose eigenvalues $\lambda_1, \ldots, \lambda_n$ necessarily lie in some set $S \subseteq \mathbb{C}$, and A is a measurable subset of S, we define

$$\mathcal{N}_M(A) := \#\{j : \lambda_j \in A\}.$$

The set function $\mathcal{N}_M$ (or just $\mathcal{N}$ if this creates no confusion) is called the *eigenvalue counting function* and the notions of convergence above can also be formulated in terms of this function; e.g., $\{\mu_{M_n}\}$ converges weakly almost surely to μ if for all measurable A for which $\mu(\partial A) = 0$,

$$\mathcal{N}_{M_n}(A) \xrightarrow{n\to\infty} \mu(A)$$

with probability one.

Beyond considering fixed sets A, one can also consider families of sets A_n which are, e.g., shrinking with n. This is what is meant by *local laws*. Within the realm of local laws, there are two qualitatively distinct regimes, which are thought of as observing eigenvalues on either *microscopic* or *mesoscopic* scales. When we consider eigenvalues on a microscopic scale, we are identifying limits of the eigenvalue counting function on sets shrinking quickly enough that we expect to see a bounded number of eigenvalues. The mesoscopic scale refers to the broad range of sets shrinking with n quickly enough that we only expect to see a negligible percentage of the total number of eigenvalues, but that still the expected number of eigenvalues tends to infinity; these are perhaps only semi-local laws. The circular law illustrated above holds at all scales: the microscopic, mesoscopic, and macroscopic (sets independent of n). That is, an approximation of the counting function by the appropriately scaled uniform measure of the same set is valid for discs whose radius is as small as $n^{-\frac{1}{2}+\epsilon}$ for any ϵ, so that the expected number of eigenvalues in such a disc is of the order $n^{2\epsilon}$; see [9].

Another natural class of test functions is polynomials, for which it suffices to consider monomials. Note that if M is a random matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\int z^k d\mu_M = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^k = \frac{1}{n}\operatorname{Tr}(M^k).$$

This observation underlies the moment method in random matrix theory and is the reason that many important results on the eigenvalue distributions of random matrices are formulated as convergence of traces of powers.

As with non-random measures, the empirical spectral measure can be studied via transform methods. One transform which has proved to be particularly valuable in random matrix theory is the *Stieltjes transform*: If μ is a probability measure on the real line, its Stieltjes transform $S_\mu(z)$ is the function

$$S_\mu(z) = \int \frac{1}{x-z} d\mu(x)$$

defined for $z \in \mathbb{C}\setminus\mathbb{R}$. If M is a Hermitian matrix and $R(z) = (M - zI)^{-1}$ is the resolvent of M, then, for μ_M the spectral measure of M,

$$S_{\mu_M}(z) = \frac{1}{n}\operatorname{Tr}(R(z)).$$

Stieltjes transforms can be inverted; that is, one can recover the measure from its Stieltjes transform. Moreover, there are continuity results relating the convergence of (deterministic or random) Stieltjes transforms to convergence of the corresponding measures. See, e.g., [3].

A drawback of the empirical spectral measure and the various spectral statistics that can be formulated in terms of it is that it puts all the eigenvalues on equal footing. This means that it is well-suited to studying the collection of eigenvalues as a whole, but less well-suited to studying very fine properties of the spectrum. For example, consider a basic model of a random Hermitian matrix, using independent Gaussian entries subject to the Hermitian requirement. (With appropriate normalization, this is called the GUE; see Section 3 below.) Since the eigenvalues are necessarily real, they can be ordered, e.g., as $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. The limiting spectral measure is known, and from it, one can identify a predicted location for, say, $\lambda_{\frac{n}{2}}$. Gustavsson [27] showed that the fluctuations of a single eigenvalue (as long as it is not too close to the edge of the spectrum) about this predicted location are Gaussian. Studying this kind of fine property of the spectrum really necessitates looking at the n-dimensional random vector of eigenvalues, rather than encoding the collection as a single random measure.

3. Wigner matrices. One of the most thoroughly studied ensembles of random matrices are Wigner matrices, first introduced in [52, 53] in the context of nuclear physics. From the perspective of a specialist in probabilty and linear algebra, Wigner matrices are some of the most natural random matrices, because they combine a natural probabilistic assumption (independence) with a natural linear algebraic assumption (symmetry). An $n \times n$ random matrix $X = [x_{ij}]_{1\le i,j\le n}$ is a (real or complex) *Wigner matrix* if

- $\mathbb{E}x_{ij} = 0$ for all i, j,
- $\{x_{ij}\}_{1 \le i \le j \le n}$ are independent, and
- either $x_{ji} = x_{ij}$ for all $i < j$ (the real case) or $x_{ij} = \overline{x_{ji}}$ for all $i < j$ (the complex case).

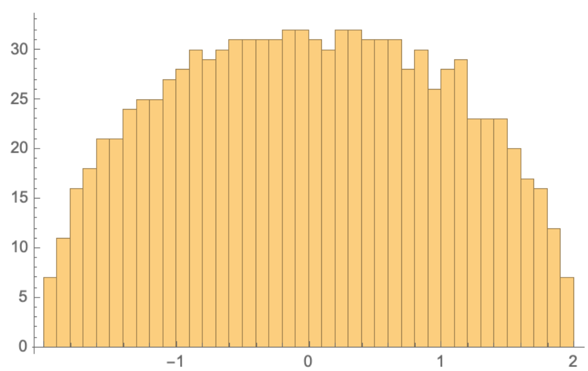
Important special cases are the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE): A GOE matrix is a real Wigner matrix in which the entries above the diagonal are i.i.d. standard Gaussian variables and the diagonal entries are i.i.d. centered Gaussians with variance 2. With these normalizations, the distribution of a GOE matrix is invariant under conjugation by a fixed orthogonal matrix: If $A \in \mathbb{O}(n)$ and X is a GOE matrix, then AXA^T is also a GOE matrix. Similarly, a GUE matrix is a complex Wigner matrix in which the entries above the diagonal are i.i.d. standard *complex* Gaussian variables (i.e., of the form $Z_1 + iZ_2$ for Z_1, Z_2 real centered Gaussians with variance $\frac{1}{2}$) and the diagonal entries are i.i.d. standard (real) Gaussians. A GUE matrix is distributionally invariant under conjugation by a fixed unitary matrix.

In his seminal paper [53], Wigner showed that the empirical spectral measure of a (suitably normalized) random symmetric matrix with i.i.d. entries on and above the diagonal converges in expectation to the semi-circular distribution σ, with density

$$d\sigma(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbb{1}_{[-2,2]}(x)dx.$$

The following improvement to almost sure convergence is due to Arnold [1].

Theorem 1 (The strong semi-circle law)**.** *Let $\{x_{ij}\}_{1 \le i \le j}$ be an infinite collection of independent real random variables such that $\{x_{ij}\}_{1 \le i < j}$ are i.i.d. with variance 1 and finite fourth moment, and $\{x_{ii}\}_{1 \le i}$ are i.i.d. Let X_n be the $n \times n$ real Wigner matrix formed from $\{x_{ij}\}_{1 \le i \le j \le n}$, and let μ_n denote the empirical spectral measure of $\frac{1}{\sqrt{n}}X_n$. Then as n tends to infinity, μ_n tends weakly almost surely to the semi-circular distribution σ.*



Histogram of the eigenvalues of a 1000×1000 *real Wigner matrix*

The semi-circle law is an example of *universality* in random matrix theory: A large Wigner matrix has eigenvalues distributed approximately according to the semi-circle law, independent of the distribution of the entries. Proving universality for various aspects of random matrices dominated much of random matrix theory in the beginning of the 21st century, culminating in such results as the Tao–Vu Four Moment Theorem (discussed below). Aside from the intrinsic interest, the practical value of the universality phenomenon is that computations can be done using explicit formulas available for, e.g., the GOE or GUE, and then the conclusions are known by universality to hold very broadly, including in cases in which no explicit formulas exist.

Results like the semi-circle law are about the *bulk* of the eigenvalues of a random matrix; they describe the macroscopic behavior of the entire spectrum, and are not sensitive to behaviors involving only a vanishing proportion of the eigenvalues. In particular, it would be consistent with the semi-circle law for Wigner matrices to frequently have a small number of very large eigenvalues, as long as that number was negligible compared to the total number of eigenvalues. This is, however, not the case: With probability one, the largest eigenvalue tends to 2 and the smallest to -2, as the size of the matrix tends to infinity. Necessary and sufficient conditions on the distribution of the entries in order to obtain this convergence were found by Bai and Yin [5], and are as follows.

Theorem 2 (Bai–Yin)**.** *Let $\{x_{ij}\}_{1\leq i\leq j}$ be an infinite collection of independent real random variables such that*

- $\{x_{ij}\}_{1\leq i<j}$ *are i.i.d.,*
- $\mathbb{E}x_{12} \leq 0, \qquad \mathbb{E}x_{12}^2 = 1, \qquad \mathbb{E}x_{12}^4 < \infty,$
- $\{x_{ii}\}_{1\leq i}$ *are i.i.d., and*
- $\mathbb{E}(x_{11}^+)^2 < \infty.$

Let X_n be the $n\times n$ real Wigner matrix formed from $\{x_{ij}\}_{1\leq i\leq j\leq n}$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then with probability one,

$$\lim_{n\to\infty} \frac{\lambda_n}{\sqrt{n}} = 2.$$

Knowing that the largest eigenvalue of a large Wigner matrix is about 2, the next natural question is to consider the fluctuations of the largest eigenvalue about 2: both their size and shape. The semi-circle law itself predicts that the size of the fluctuations should be of the order $n^{-2/3}$: the area under the semi-circular density from $2-\epsilon$ to 2 is of the order $\frac{1}{n}$ when $\epsilon \sim n^{-2/3}$. This turns out to be correct. The exact shape of the fluctuations was first identified for the GUE (and then the GOE and GSE – Gaussian Symplectic Ensemble) by Tracy and Widom [47, 48] to be governed by the probability distributions now known as the Tracy–Widom laws. These distributions are not so straightforward to characterize: e.g., in the unitary case, the limiting eigenvalue distribution has cumulative distribution function

$$F_2(t) = \exp\left(-\int_t^\infty (x-t)q(x)^2\,dx\right),$$

where q solves the differential equation

$$q''(x) = xq(x) + 2q(x)^3 \qquad\qquad q(x) \sim \mathrm{Ai}(x)\,\mathrm{as}\,x\to\infty,$$

and $\mathrm{Ai}(x)$ is the *Airy function* (which is itself defined implicitly by a contour integral). The result in the GUE case is then as follows.

Theorem 3 (Tracy–Widom)**.** *For each n, let X_n be an $n\times n$ GUE random matrix, and let λ_n denote its largest eigenvalue. Then for all $t\in\mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left[n^{2/3}\left(\frac{\lambda_n}{\sqrt{n}} - 2\right) \leq t\right] = F_2(t).$$

As in the case of the semi-circle law, the Tracy–Widom fluctuations of the largest eigenvalue are universal; they do not really depend on the distribution of the entries of the Wigner matrix, just on the basic Wigner-type structure. Such universality was first proved (under moment and symmetry assumptions) by Soshnikov in [43].

Considerable effort was spent in the early part of the 21st century on universality in general, and in particular as it applied to local statistics. Many papers were written, in particular by Tao and Vu (e.g., [45, 44, 46]), Erdős, Yao, and collaborators (e.g., [17, 14, 9, 18]) and the two groups together [15]. One of the crowning achievements of this line of research is the Tao–Vu Four Moment Theorem. The statement involves a matching condition on moments; two complex random variables X and Y are said to have *matching moments up to order k* if

$$\mathbb{E}\Re(X)^a\Im(X)^b = \mathbb{E}\Re(Y)^a\Im(Y)^b$$

for all integers $a, b \geq 0$ such that $a + b \leq k$.

A random variable X satisfies condition **C0** if there are constants C, C' such that for all $t \geq C'$,

$$\mathbb{P}[|X| \geq t^C] \leq e^{-t}.$$

Theorem 4 (Tao–Vu Four Moment Theorem)**.** *There is a constant $c_0 > 0$ such that the following holds. Let $X_n = [x_{ij}]_{1\leq i,j\leq n}$ and $X_n' = [x_{ij}']_{1\leq i,j\leq n}$ be Wigner matrices, all of whose entries satisfy condition* **CO** *(with the same constants). Assume that for all $i<j$, x_{ij} and x_{ij}' match to order 4 and for all i, x_{ii} and x_{ii}' match to order 2. Let $k\in\{1,\ldots,n^{c_0}\}$ and let $G:\mathbb{R}^k\to\mathbb{R}$ be a smooth function such that*

$$\sup_{x\in\mathbb{R}^k} |\nabla^j G(x)| \leq n^{c_0}$$

for all $0\leq j\leq 5$.

Denote the eigenvalues of $\sqrt{n}X_n$ *and* $\sqrt{n}X_n'$ *by*

$$\lambda_1 \leq \cdots \leq \lambda_n \qquad \text{and} \qquad \lambda_1' \leq \cdots \leq \lambda_n',$$

respectively. Then for any $1 \leq i_1 \leq \cdots \leq i_k \leq n$ *and* n *large enough,*

$$\left|\mathbb{E}G(\lambda_{i_1}, \ldots, \lambda_{i_k}) - \mathbb{E}G(\lambda_{i_1}', \ldots, \lambda_{i_k}')\right| \leq n^{-c_0}.$$

The theorem says that, even at this very fine scale (we are looking at intervals of fixed width when the eigenvalues have been blown up to be spread over an interval of size approximately $4n$), the joint distributions of collections of individual eigenvalues of Wigner matrices are all about the same, as long as the moments of the off-diagonal entries match to order 4 and the moments of the diagonal entries match to order 2.

Having gone so far to describe the very fine structure of the eigenvalue distributions of Wigner matrices, much attention has shifted to the eigenvectors. Very vaguely, one would expect the frame of orthonormal eigenvectors of a Wigner matrix to behave as though it were distributed according to the orthogonally invariant probability measure on the set of all orthonormal frames of n vectors in $\mathbb{R}^n$ (what is called Haar measure on the Stiefel manifold). Such a conjecture is essentially another universality conjecture, since it is known to be true for the GOE/GUE because of invariance properties of those ensembles. There are a few results pointing in this direction, chiefly results on *eigenvector delocalization* (see [17, 16]), which say that the size of the components of unit eigenvectors of Wigner matrices are about $\frac{1}{\sqrt{n}}$ with high probability, a fact which is also true of random vectors chosen uniformly on the unit sphere. The recent paper [36] surveys the general state of the art on eigenvectors of random matrices.

4. Wishart matrices. Wishart random matrices were introduced and studied in [54] as a model for random sample covariance matrices. Let Σ be a symmetric positive definite $p \times p$ matrix and let $X_1, \ldots, X_n$ be independent, identically distributed centered Gaussian random vectors with covariance matrix Σ. Let X denote the matrix whose columns are given by the X_j. The random matrix

$$XX^T = \sum_{j=1}^{n} X_j X_j^T$$

is said to be a $p \times p$ Wishart random matrix with scale matrix Σ and n degrees of freedom; this distribution is abbreviated $W(p, n; \Sigma)$, with the Σ often ommitted when $\Sigma = I_p$. In the case that the X_j are i.i.d. draws from an unknown underlying multivariate Gaussian distribution, the matrix $\frac{1}{n}XX^T$ is the maximum likelihood estimator for Σ.

There are multiple limiting regimes determined by the relationship between p and n. If the X_j are i.i.d. samples (i.e., p-dimensional data vectors), then the most classical regime is $n \gg p$; i.e., lots of samples of low- or moderate-dimensional data. A more modern context is to consider p and n of similar size; i.e., the context of either a rather limited number of samples or high-dimensional data. In this regime, there is a further distinction between $p \leq n$ (at least as many data points as the dimension) and $p > n$.

In terms of the study of eigenvalues, the interest has been largely in the context of $\frac{p}{n}$ tending to some limiting value in $(0, \infty)$. In [31], Marčenko and Pastur found the limiting distribution of the eigenvalues in this context, in what is now known as the Marčenko–Pastur law. The most straightforward (and best-known) version treats Wishart matrices with identity scale matrix, as follows.

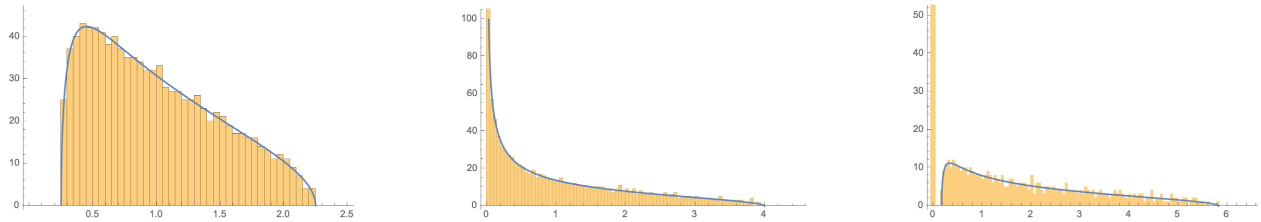
Theorem 5 (Marčenko–Pastur law for Wishart matrices)**.** *Suppose that* $p = p(n)$ *is such that*

$$\lim_{n \to \infty} \frac{p(n)}{n} = \alpha \in (0, \infty).$$

For each p*, let* μ_p *be the empirical spectral measure of the random matrix* $W_p := \frac{1}{n}X_pX_p^T$*, where* X_p *is a* $p \times n$ *random matrix of i.i.d. standard Gaussian random variables. Then the sequence* $\{\mu_p\}$ *converges weakly almost surely to the Marčenko–Pastur law* μ_α*, defined by*

$$d\mu_\alpha(x) = \left(1 - \frac{1}{\alpha}\right)_+ \delta_0 + \frac{1}{\alpha 2\pi x}\sqrt{(b-x)(x-a)}\mathbb{1}_{[a,b]}(x)dx,$$

where $a = (1 - \sqrt{\alpha})^2$ *and* $b = (1 + \sqrt{\alpha})^2$.



Histograms of the eigenvalues of 1000×1000 *Wishart matrices with* $\alpha = \frac{1}{4}$*,* $\alpha = 1$*, and* $\alpha = 2$*, with corresponding Marčenko–Pastur densities*

When $\alpha = 1$, then $\frac{1}{n} X_p X_p^T$ is asymptotically square with eigenvalues supported in $[0, 4]$; a change of variables argument then gives the convergence of the empirical spectral measure of $\sqrt{\frac{1}{n} X_p X_p^T}$, which can be thought of as the empirical *singular value* measure of $\frac{1}{\sqrt{n}} X_p$. This is the Marčenko–Pastur quarter-circle law:

Theorem 6 (Quarter-circle law)**.** *Suppose that* $p = p(n)$ *is such that*

$$\lim_{n \to \infty} \frac{p(n)}{n} = 1.$$

For each p*, let* σ_p *be the empirical spectral measure of the random matrix* $Y_p := \sqrt{\frac{1}{n} X_p X_p^T}$*, where* X_p *is a* $p \times n$ *random matrix of i.i.d. standard Gaussian random variables. Then the sequence* $\{\sigma_p\}$ *converges weakly almost surely to the quarter-circle law* q *on* $[0, 2]$*, given by*

$$dq(x) = \frac{1}{\pi} \sqrt{4 - x^2} \mathbb{1}_{[0,2]}(x) dx.$$

Marčenko and Pastur actually considered a rather more general context than the one described above: They found the limiting eigenvalue distribution for random matrices of the form $A + XTX^*$, where X is a $p \times n$ matrix of i.i.d. random variables (subject to some moment assumptions), T is a diagonal matrix with (possibly random) real entries, and A is a Hermitian (possibly random) matrix; in the case that A and T are random, it is assumed that (X, A, T) are independent. They considered the limiting eigenvalue distribution of $A + XTX^*$ in the case that A and T themselves had limiting eigenvalue distributions. This setting includes in particular a limiting version of the motivating case described above, taking T to be the matrix of eigenvalues of Σ and assuming that those eigenvalues are approximately distributed according to some fixed reference distribution. Their approach in [31] was via the Stieltjes transform, and the limiting eigenvalue distribution for general random matrices of the form $A + XTX^*$ was characterized in terms of this transform, with the direct characterization as in Theorem 5 above available only when $T = I$.

As mentioned above, random covariance matrices are another context of universality in random matrix theory, in that if the entries of the matrix X are independent, the results are the same as in the Gaussian case. The original paper [31] already required only independence of the entries of X and moment assumptions, and through the work of a series of authors (see [38, 50, 55]), the conditions were ultimately weakened to require only that the entries of X be independent with mean 0 and variance 1.

The more classical context from the statistical point of view, namely $p \ll n$, has also been studied; the limiting spectral distribution in this case is governed by the semi-circle law:

Theorem 7 (Bai–Yin [4])**.** *Let* $n \in \mathbb{N}$ *and* $p = p(n)$ *be such that* $\lim_{n\to\infty} p(n) = \infty$ *and* $\lim_{n\to\infty} \frac{p(n)}{n} = 0$*. Let* $\{X_{ij}\}_{i,j \geq 1}$ *be a collection of i.i.d. random variables with mean 0, variance 1, and finite fourth moment. For each pair* (n, p)*, let* X_p *denote the* $p \times n$ *matrix with entries* $\{X_{ij}\}_{\substack{1 \leq i \leq p \\ 1 \leq j \leq n}}$*, and let* μ_p *be the empirical spectral measure of the matrix*

$$A_p = \frac{1}{2\sqrt{np}} (X_p X_p^T - n I_p).$$

Then as $n \to \infty$*, the sequence* $\{\mu_p\}$ *tends weakly almost surely to the semi-circle law* σ *on* $[-1, 1]$*, with density*

$$d\sigma(x) = \frac{2}{\pi} \sqrt{1 - x^2} \mathbb{1}_{[-1,1]}(x).$$

The Marčenko–Pastur law and the semi-circle law for Wishart matrices are results about the bulk of the spectrum, but as in the case of Wigner matrices, there is also significant interest in the edge of the spectrum. Note that there is a difference in behavior between the cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$ in Theorem 5. Most obviously, if $\alpha > 1$ (i.e., $p \gg n$), then the $p \times p$ matrix $X_p X_p^T$ is singular, hence the presence of the mass at 0 in the limiting spectral measure. For all p, $X_p X_p^T$ is nonnegative definite, and for $p \leq n$, $X_p X_p^T$ is positive definite with probability one. If $p = n$, the spectrum of the random matrix $W_p = \frac{1}{n} X_p X_p^T$ is said to have a hard left edge: The support of the limiting measure of W_p in that case is $[0, 4]$, and it is impossible for W_p to have eigenvalues below the lower limit of this limiting support. If $\alpha \neq 1$, this is no longer the case, and a given random matrix may have eigenvalues smaller than the left endpoint of the limiting support. This raises the immediate question of how likely one is to see such behavior (at either edge of the limiting support): It would be consistent with the Marčenko–Pastur law that there are typically some eigenvalues of W_p between 0 and $a = (1 - \sqrt{\alpha})^2$, as long as it were a negligible fraction of the total collection of eigenvalues. However, this turns out not to be the case:

Theorem 8 (Bai–Yin [6], Yin–Bai–Krishnaiah [56])**.** *Let $n \in \mathbb{N}$ and $p = p(n)$ be such that $\lim_{n\to\infty} \frac{p(n)}{n} = \alpha > 0$. Let $\{X_{ij}\}_{i,j\geq 1}$ be a collection of i.i.d. random variables with mean 0, variance 1, and finite fourth moment. For each pair (n, p), let X_p denote the $p \times n$ matrix with entries $\{X_{ij}\}_{\substack{1\leq i\leq p\\ 1\leq j\leq n}}$, and let $W_p := \frac{1}{n} X_p X_p^T$. Denote the eigenvalues of W_p by*

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_p.$$

Then with probability one, as n tends to infinity,

$$\lambda_{\max\{1,p-n+1\}} \to (1 - \sqrt{\alpha})^2 \qquad \text{and} \qquad \lambda_p \to (1 + \sqrt{\alpha})^2.$$

As in the Wigner case, the next step is to consider the fluctuations of the extreme eigenvalues about their limits, and once again, they are governed by the Tracy–Widom laws; see [39]. There are also local laws and universality results in the Wishart case, analogous to those in the Wigner case; see, e.g., [46].

5. The Ginibre ensemble. The Ginibre ensembles are in a sense the most naive of the random matrix ensembles: They are formed by filling square matrices with i.i.d. Gaussian variables: The real Ginibre ensemble has i.i.d. standard Gaussian entries and the complex Ginibre ensemble has i.i.d. standard complex Gaussian entries; i.e., the real and imaginary parts are i.i.d. real Gaussians with variance $\frac{1}{2}$. From a technical standpoint, the complex Ginibre ensemble is considerably easier to study than its real counterpart; for example, in the real case, there is a nonzero probability that all of the eigenvalues are real, and so there is no eigenvalue density formula. A common difficulty with both ensembles, though, is that the matrices are nonnormal with probability one. While they are almost surely diagonalizable, the eigenvalues are not such nice functions of the matrix in the nonnormal case.

As discussed in Section 2, the limiting eigenvalue distribution of the (suitably normalized) Ginibre ensemble is known to be the uniform measure on the unit disc; this was first proved in expectation by Mehta [34] and almost surely by Silverstein (then unpublished but subsequently appearing in [30]) in the complex case and by Edelman [13] in the real case.

Theorem 9 (The strong circular law for the Ginibre ensemble)**.** *Let μ_n be the empirical spectral measure of $\frac{1}{\sqrt{n}} G_n$, where G_n is a (real or complex) $n \times n$ Ginibre random matrix. Then $\{\mu_n\}$ converges weakly almost surely to the uniform probability measure on $\{|z| \leq 1\} \subseteq \mathbb{C}$.*

It is interesting to note that the Marchenko–Pastur quarter-circle law for Wishart matrices can be reinterpreted as a limiting distribution for the singular values of a Ginibre matrix. Indeed, the singular values of $\frac{1}{\sqrt{n}} G_n$ are the eigenvalues of $\sqrt{\frac{1}{n} G_n G_n^*}$, which is a (normalized) Wishart $W(n, n; I_n)$ matrix. Theorem 6 can thus be reinterpreted as follows.

Theorem 10. *Let G_n be an $n \times n$ (real or complex) Ginibre matrix with singular values $s_1, \ldots, s_n$, and let*

$$\nu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{\frac{s_j}{\sqrt{n}}}$$

be the empirical singular value measure of $\frac{1}{\sqrt{n}} G_n$. Then the measures ν_n converge weakly almost surely to the quarter-circle law on $[0, 2]$, with density

$$\rho_{qc}(x) = \frac{1}{\pi} \sqrt{4 - x^2}\, \mathbb{1}_{[0,2]}(x).$$

As in the case of Wigner and Wishart matrices, there is a distinction between bulk and edge eigenvalues of the Ginibre ensemble, although in this case, the edge is that of the unit disc. Let G_n be an $n \times n$ Ginibre random matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and let

$$\rho(G_n) := \max_{1 \le j \le n} |\lambda_j|$$

denote the spectral radius of G. The asymptotic behavior of the spectral edge of G_n is characterized in the following theorem.

Theorem 11 (Rider [40])**.** *For G_n and $\rho(G_n)$ as above, with probability one,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \rho(G_n) = 1.$$

Moreover, if $\gamma_n = \log\left(\frac{n}{2\pi}\right) - 2\log(\log(n))$ and

$$Y_n := \sqrt{4n\gamma_n}\left(\frac{1}{\sqrt{n}}\rho(G_n) - 1 - \sqrt{\frac{\gamma_n}{4n}}\right),$$

then Y_n converges in distribution to the Gumbel law, i.e., the probability measure on $\mathbb{R}$ with cumulative distribution function $F_{\mathrm{Gum}}(x) = e^{-e^{-x}}$.

That is, the eigenvalues of $\frac{1}{\sqrt{n}} G_n$ almost surely lie in the unit disc, and the spectral radius of $\frac{1}{\sqrt{n}} G_n$ fluctuates about $1 - \sqrt{\frac{\gamma_n}{4n}}$ on a scale of $\frac{1}{\sqrt{4n\gamma_n}}$ according to the Gumbel law.

The results above again hold in a universal way. The following definitive result on the spectral measures of random matrices with i.i.d. complex entries is due to Tao and Vu [44], concluding a line of research with many contributions; e.g., [20, 2, 24, 37].

Theorem 12 (The strong circular law)**.** *Let $\{X_n\}$ be a sequence of $n \times n$ random matrices with i.i.d. entries having mean zero and variance one, and for each n, let μ_n denote the empirical spectral measure of $\frac{1}{\sqrt{n}} X_n$. Then $\{\mu_n\}$ converges weakly almost surely to the uniform probability measure on $\{|z| \le 1\} \subseteq \mathbb{C}$.*

A different direction of generalization from the Ginibre ensemble involves considering random matrices in which the entries are not independent. Beginning in the work of Girko (see [21, 22]), various researchers have considered extensions of the i.i.d. case to cases in which the pairs (x_{ij}, x_{ji}) may be correlated (distinct such pairs are still assumed to be independent of each other and to have a common distribution in $\mathbb{C}^2$). In this setting, one obtains not the circular law, but an "elliptical law," in which the limiting eigenvalue distribution is uniform on an ellipse whose major and minor axes are related to the correlation between x_{ij} and x_{ji}. The following theorem describes the real case.

Theorem 13 (Nguyen–O'Rourke [35])**.** *Let (ξ_1, ξ_2) be a random vector in $\mathbb{R}^2$ with mean zero, variance one components, such that $\mathbb{E}\xi_1\xi_2 = \rho \in (-1, 1)$. Suppose that $\{x_{ij}\}_{i,j\ge1}$ is a collection of random variables such that*

1. *$\{x_{ii}\}_{i\ge1} \cup \{(x_{ij}, x_{ji})\}_{1\le i<j}$ are independent;*
2. *for each $i < j$, (x_{ij}, x_{ji}) has the same distribution as (ξ_1, ξ_2); and*
3. *$\{x_{ii}\}_{i\ge1}$ are i.i.d. with mean zero and finite variance.*

If X_n is the random matrix with entries $\{x_{ij}\}_{1\le i,j\le n}$ and μ_n denotes the empirical spectral measure of $\frac{1}{\sqrt{n}} X_n$, then the sequence μ_n converges weakly almost surely to the uniform probability measure on the ellipse

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \frac{(\Re(z))^2}{(1+\rho)^2} + \frac{(\Im(z))^2}{(1-\rho)^2} \le 1 \right\}.$$

In [35], the authors prove a corresponding result in the complex case which allows for limited types of correlations between the complex entries x_{ij} and x_{ji}. They conjecture that these restrictions can be dropped to obtain the full complex analog of Theorem 13.

6. Random unitary matrices. On the classical compact matrix groups $\mathbb{O}(n)$ (the orthogonal group), $\mathbb{SO}(n)$ (the special orthogonal group), $\mathbb{U}(n)$ (the unitary group), $\mathbb{SU}(n)$ (the special unitary group), and $\mathbb{Sp}(2n)$ (the symplectic group), there is a natural way to choose a random element; the corresponding probability measure is called Haar measure. The defining property of Haar measure is its *translation invariance*: If U is a random element distributed according to Haar measure on one of the matrix groups above, and A is a fixed matrix in the same group, then AU and UA both have the same distribution as U itself. Geometrically, Haar measure can be seen as simply the volume measure on each of these groups, viewed as manifolds embedded in Euclidean space. More computationally, Haar measure on, say, $\mathbb{O}(n)$ is the result of taking a random matrix with i.i.d. Gaussian entries (i.e., a real Ginibre matrix) and performing the Gram–Schmidt process. See [32] for many other constructions of Haar measure.

The eigenvalues of a matrix in any of the groups above lie on the unit circle in the complex plane. The translation invariance (sometimes called rotation invariance in this context) of Haar measure on $\mathbb{U}(n)$ shows that the set of eigenvalues $\{e^{i\theta_1}, \dots, e^{i\theta_n}\}$ of a random unitary matrix U has a rotationally-invariant distribution: The distribution of the eigenvalues of U is the same as the distribution of the eigenvalues of $e^{i\theta}U$ for any fixed θ, since $e^{i\theta}I$ is a fixed element of $\mathbb{U}(n)$. This invariance property does not hold for the other groups; e.g., the eigenvalues of a real orthogonal matrix come in complex conjugate pairs.

The limiting eigenvalue distribution for random matrices on the classical compact groups is the same in all cases: The uniform measure ν on the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$. This was proved first for the unitary group by Diaconis–Shahshahani in [12] via the moment method and Fourier analysis, with the basic idea as follows. Let μ_n denote the empirical spectral measure of a random matrix $U \in \mathbb{U}(n)$. Then μ_n is a random measure on the circle, and its Fourier transform is given by

$$\hat{\mu}_n(j) = \int_{\mathbb{S}^1} z^j \, d\mu_n(z) = \frac{1}{n} \operatorname{Tr}(U^j).$$

Diaconis and Shahshahani found exact formulae for expected values of products of traces of powers of U, from which one can show that, with probability one, for each $j \in \mathbb{Z}$,

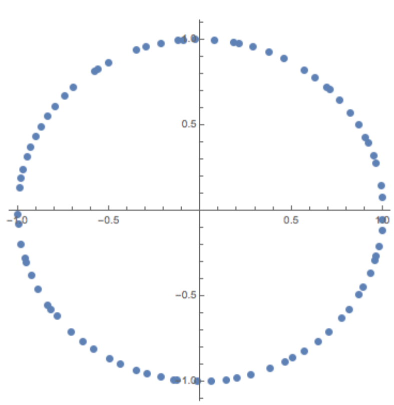
$$\hat{\mu}_n(j) \xrightarrow{n\to\infty} \hat{\nu}(j),$$

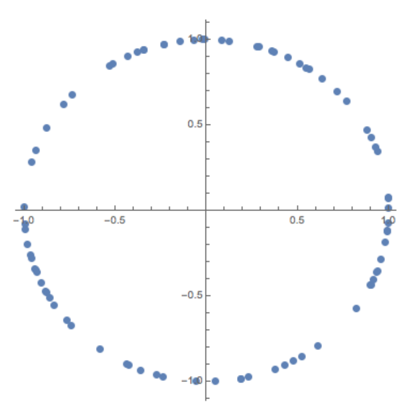
and thus $\mu_n \to \nu$ weakly almost surely.

It is not terribly surprising that the limiting eigenvalue distributions in these cases are the uniform measure on the circle, but what is less obvious is just how uniform they are. Explicit eigenvalue densities were identified by Hermann Weyl in the cases of all the classical compact groups, and are known as the Weyl integration formulae. In the case of $\mathbb{U}(n)$, the eigenvalue density is

$$\frac{1}{n!(2\pi)^n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2,$$

with respect to $d\theta_1 \cdots d\theta_n$ on $[0, 2\pi)^n$. Observe that for any given pair (j,k), $|e^{i\theta_k} - e^{i\theta_j}|^2$ is zero if $\theta_j = \theta_k$ (and small if they are close), but $|e^{i\theta_k} - e^{i\theta_j}|^2$ is 4 if $\theta_j = \theta_k + \pi$. This has the effect that the eigenvalues repel each other and therefore tend to be very evenly spaced around the circle, in a phenomenon known as "eigenvalue rigidity." This effect is clearly visible in simulations, even for relatively small matrices:





In the picture on the right, 80 points were dropped uniformly and independently (thus there is no repulsion); there are several large clumps of points close together, and some largish gaps. The picture on the left is of the eigenvalues of a random 80×80 unitary matrix; one can clearly see that they are more regularly spaced around the circle.

There are various ways one could quantify this phenomenon; one such theorem is the following.

Theorem 14 (E. Meckes–M. Meckes [33]). *Let μ_n be the empirical spectral measure of a random matrix in $\mathbb{U}(n)$ and let ν denote the uniform measure on the circle. Let*

$$W_1(\mu_n,\nu) := \sup_{f:|f(z)-f(w)|\leq|z-w|}\left|\int f d\mu_n - \int f d\nu\right|$$

denote the L_1 Wasserstein (or L_1 Kantorovich) distance from μ_n to ν. There is an absolute constant C such that with probability one, for n large enough,

$$W_1(\mu_n,\nu) \leq C\frac{\sqrt{\log(n)}}{n}.$$

For context, the empirical measure corresponding to n i.i.d. uniform points on the circle is typically at a distance about $\frac{1}{\sqrt{n}}$ to uniform, while a measure supported on n points spaced exactly evenly about the circle is a distance $\frac{\pi}{n}$ from uniform. The empirical spectral measure of a random unitary matrix is thus much closer to uniform than that of a collection of i.i.d. random points, and nearly as close as n perfectly spaced points on the circle.

Theorem 14, together with the subsequent comparisons with the case of i.i.d. uniform points and evenly spaced points, is an example of *non-asymptotic* random matrix theory; that is, the quantitative theory of random matrices with large but fixed dimension, as opposed to the more classical limit theory one sees in results like the semi-circle law. There has been a significant move in this quantitative direction in recent years (see in particular [41]), partly because such dimension-dependent estimates are crucial in many applications.

It should be noted that the phenomenon of eigenvalue rigidity/repulsion is present in many random matrix ensembles, including those discussed in the previous sections (see, e.g., [11, 10]), but for simplicity we have discussed it only in the context of random unitary matrices, where the results are a bit more straightforward to state.

7. Random matrices with prescribed singular values. In [28], Horn raised (and answered) the question of what the possible eigenvalues of a square matrix were if the singular values were specified. In [51], Weyl had shown that if A is an $n \times n$ matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$, then

$$\prod_{j=1}^{k}|\lambda_j| < \prod_{j=1}^{k}\sigma_j,\ \forall k<n \qquad \text{and} \qquad \prod_{j=1}^{n}|\lambda_j| = \prod_{j=1}^{n}\sigma_j.$$

Horn showed that these relationships are necessary and sufficient: Any sequences of real numbers σ_j and λ_j satisfying the above conditions are the singular values and eigenvalues of some $n \times n$ matrix.

The natural probabilistic version of Horn's question is as follows. Suppose that A is an $n \times n$ matrix with prescribed singular values $\sigma_1 \geq \cdots \geq \sigma_n$, but is "otherwise random." What are the eigenvalues of A typically like?

More carefully, given $\sigma_1 \geq \cdots \geq \sigma_n$, let Σ be the diagonal matrix with the σ_j on the diagonal, and choose U and V according to Haar measure on the unitary group $\mathbb{U}(n)$, as in Section 6. Let $A = U\Sigma V^*$. Then A is a random matrix with singular values $\sigma_1, \ldots, \sigma_n$, and this random matrix model is very natural, in that it involves choosing the left- and right-singular vectors independently, with each set chosen uniformly among orthonormal bases of $\mathbb{C}^n$. (Note: V^* could be replaced with V, which has the same distribution; it is written in this way only to make the connection with the singular vectors obvious.)

One way to treat the question of the typical behavior of the eigenvalues of A is via the empirical spectral measure. Let

$$\nu_A := \frac{1}{n}\sum_{j=1}^{n}\delta_{\sigma_j} \qquad \text{and} \qquad \mu_A := \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$$

denote the empirical singuar value measure and the empirical spectral measure of A, respectively. Suppose that A_n is a sequence of random matrices as described above, such that the corresponding singular value measures ν_{A_n} converge weakly to a probability measure ν on $\mathbb{R}_+$. A probabilistic analog of the Horn–Weyl theorem would be to show that the corresponding empirical spectral measures μ_{A_n} converge to a determistic limiting measure, and to describe it in terms of ν. This was the content of the Feinberg–Zee Single Ring Theorem [19]. Their proof glossed over some technicalities, but a fully rigorous version was subsequently proved by Guionnet, Krishnapur and Zeitouni [25]. The following improvement, due to Rudelson and Vershynin [42], removes an unpleasant technical condition on the sequence of prescribed singular values.

Theorem 15 (The Single Ring Theorem). *Let $\Sigma_n = \operatorname{diag}(\sigma_1^{(n)}, \ldots, \sigma_n^{(n)})$, where $\sigma_1^{(n)} \geq \cdots \geq \sigma_n^{(n)} \geq 0$, and for each n, let U_n and V_n be independent Haar-distributed unitary matrices. Let ν_n and μ_n denote the singular value measure and spectral measure of A_n, respectively. Suppose that the sequence $\{\nu_n\}$ of singular value measures converges weakly to a probability measure ν which is compactly supported on $\mathbb{R}_+$. Assume further that*

- *there exists an $M > 0$ such that $\mathbb{P}[\sigma_1^{(n)} \geq M] \to 0$ as $n \to \infty$; and*
- *there exist constants $\kappa_1, \kappa_2 > 0$ such that, for any $z \in \mathbb{C}$ with $\Im(z) > n^{-\kappa_1}$,*

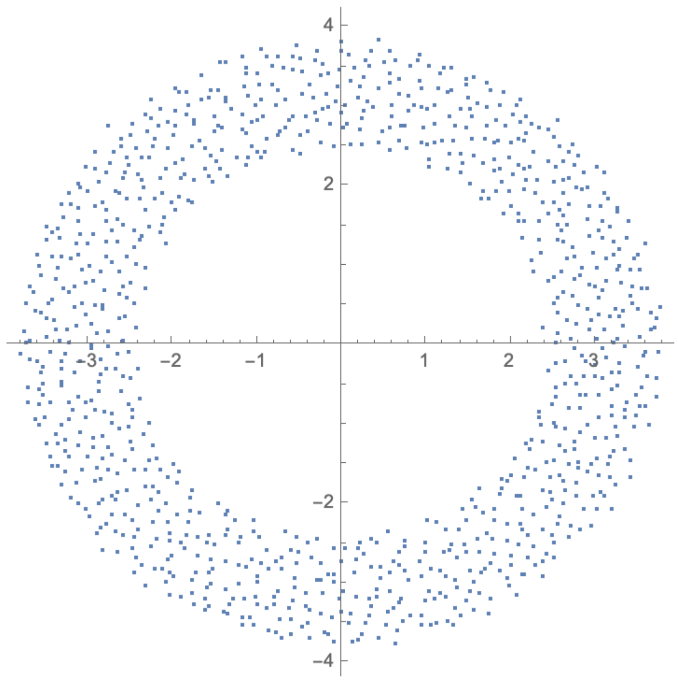
$$|\Im(S_{\nu_n}(z))| \leq \kappa_2,$$

where $S_\nu(z)$ is the Stieltjes transform of ν.

Then the sequence $\{\mu_n\}$ of empirical spectral measures converges weakly in probability to a probability measure μ. This limiting measure has a density which can be explicitly calculated in terms of ν, and has support equal to the single ring $\{z \in \mathbb{C} : a \leq |z| \leq b\}$, where

$$a = \left(\int_0^\infty x^{-2} d\nu(x) \right)^{-1/2} \qquad \text{and} \qquad b = \left(\int_0^\infty x^2 d\nu(x) \right)^{1/2}.$$

It is interesting that even if the support of ν consists of disconnected pieces, so that there are some forbidden regions for the singular values, the support of the eigenvalues is still, in the limit, this single annulus with no gaps.



The eigenvalues of a 1000×1000 random matrix of the form $U\Sigma V^*$, with U and V independent random unitary matrices and $\Sigma = \operatorname{diag}(\sigma_{11}, \ldots, \sigma_{nn})$ with $\sigma_{kk} = 1 + \frac{5k}{n}$.

Work on the Single Ring Theorem has spurred further work on this random matrix model, including results on the convergence of the largest- and smallest-modulus eigenvalues (see [26, 7]) and a local version of the theorem [8].

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LINEAR ALGEBRA EDUCATION

Outgoing message from David Strong

After seven years of serving as the *IMAGE* Contributing Editor for Education, I'll be "passing the baton" to another ILAS member who will bring to this position his or her own unique expertise and passion for teaching and mentoring our next generation of mathematicians. (We are currently in the process of determining who that will be.) While I will of course remain actively involved in ILAS and actively involved in teaching and studying and doing linear algebra for many years to come, this issue of *IMAGE* will be my last in which to oversee the education column. Consequently, let me use this final opportunity to share a few parting thoughts. While the thoughts below may be nothing new, it's nice to be reminded of them, especially right now with the tumultuous times we live in.

Keep loving what you do. The world has been immensely blessed in a variety of ways by our ongoing discovery and development of mathematical ideas and their applications. Life has been made better for us individually and collectively in so many ways due to our increased understanding of the world around us. All of us have experienced the simple yet profound beauties of mathematics, especially those related to linear algebra. Remember the almost magical feelings you experienced as a student discovering ideas for the first time, or in later years discovering the deeper layers of ideas you thought you already fully understood. And transmit that joy and excitement on to your students. Keep your students at the center of your teaching. Think back to your own college/university days and the professors that changed your life, both academically and personally. You have done the same for many students, and you have made more of a difference than you realize in so many lives at a time when many of them most need it. In short, keep up the good work.

My best wishes to you all as this semester draws to a close. Here's to a time when we can all interact with our students and with each other at mathematics conferences and elsewhere in more natural and meaningful ways.

David Strong
IMAGE Education Editor
Chair, ILAS Education Committee

JOURNAL ANNOUNCEMENTS

Special issue of *Computational and Mathematical Methods* on Linear Algebra, Matrix Analysis and Applications

Contributed announcement from Fernando De Terán

The journal *Computational and Mathematical Methods (CMM)* will publish a special issue on on Linear Algebra, Matrix Analysis and Applications.

From the journal web site (`https://onlinelibrary.wiley.com/journal/25777408`): "*Computational and Mathematical Methods* is an interdisciplinary journal dedicated to publishing the world's top research in the expanding area of computational mathematics, science and engineering. The journal connects methods in business, economics, engineering, mathematics and computer science in both academia and industry."

The description of this special issue is as follows.

> "Mathematical modeling of problems arising in engineering, physics, mechanics, etc., leads in many cases, directly or after a discretization process, to solving systems of linear equations of finite dimension. Besides, other problems of numerical analysis, such as approximation, interpolation, nonlinear systems, computation of eigenvalues, etc., lead to the resolution of large systems. Thus, it is essential to develop efficient numerical linear algebra methods, in both sequential and parallel environments, to solve them. The main purpose of this special issue is to collect the most recent methods for solving such problems and applications from a multidisciplinary approach."

The guest editors responsible for this special issue are:

- Luca Bergamaschi, Department of Civil Environmental and Architectural Engineering, University of Padua, Italy (`berga@dmsa.unipd.it`)
- Fernando de Terán, Departamento de Matemáticas, Universidad Carlos III, Spain (`fteran@math.uc3m.es`)
- Pedro Alonso, Departamento de Matemáticas, Universidad de Oviedo, Spain (`palonso@uniovi.es`)

All papers must be submitted via the online system `https://mc.manuscriptcentral.com/compandmathmethods`. Authors need to make sure that they specify that the paper is a contribution for "Special Issue on Linear Algebra, Matrix Analysis and Applications" and select the article type, when prompted. All papers will be peer reviewed according to the high standards of *CMM*.

The deadline for submissions is June 30, 2021.

ILAS NEWS

Raf Vandebril is the new ILAS Second Vice President for ILAS Conferences

Contributed announcement from Daniel Szyld, ILAS President

Raf Vandebril (KU Leuven) is the new Second Vice President for ILAS Conferences. He replaces Steve Kirkland, who served in the position for over five years.

ILAS Members Selected as 2021 American Mathematical Society Fellows

The American Mathematical Society (AMS) has recently announced its 2021 AMS Fellows, recognizing individuals who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among those selected are two ILAS members:

Raymond Chan (City University of Hong Kong) was cited for "contributions to computational mathematics, in particular to numerical linear algebra and applications to imaging sciences."

Valeria Simoncini (Università di Bologna) was cited for "contributions to computational mathematics, in particular to numerical linear algebra."

For more information and details on the AMS Fellows program, see `http://www.ams.org/profession/ams-fellows/ams-fellows`.

ILAS Member Receives London Mathematical Society Fröhlich Prize

The London Mathematical Society (LMS) has awarded the 2020 Fröhlich Prize to Françoise Tisseur for her "important and highly innovative contributions to the analysis, perturbation theory, and numerical solution of nonlinear eigenvalue problems."

Françoise has been a member of ILAS for many years and served on the ILAS Board of Directors from 2011 until 2014.

In memory of Albrecht Fröhlich, the Fröhlich Prize is awarded in even-numbered years for "original and extremely innovative work in any branch of mathematics." See `https://www.lms.ac.uk/prizes/lms-prize-regulations` for details on the prize.

See `https://www.lms.ac.uk/sites/lms.ac.uk/files/files/Tisseur_Frohlich_citation.pdf` for the full award citation for Françoise; see `https://www.lms.ac.uk/news-entry/26062020-1657/lms-prize-winners-2020` for the full list of 2020 LMS prize winners.

Pietro Paparella appointed Assistant Manager of ILAS-NET and the ILAS Information Center (IIC)

Contributed announcement from Daniel Szyld, ILAS President

Pietro Paparella (University of Washington Bothell) has been appointed as the new Assistant Manager for ILAS-NET and the ILAS Information Center (IIC).

Pietro will be working with Sarah Carnochan Naqvi to complement her work which serves the ILAS community taking care of this very important and valuable resource.

ILAS to Partner with the American Mathematical Society for the Annual Joint Mathematics Meetings

Contributed announcement from Daniel Szyld, ILAS President

A new partnership between ILAS and the American Mathematical Society (AMS) will see ILAS become a partner with the AMS for the annual Joint Mathematics Meetings (JMM), starting in January 2022. The meeting will feature an ILAS Lecture, and at least four ILAS Special Sessions.

ILAS Secretary/Treasurer Leslie Hogben realized we had this opportunity, and made the initial contact with the AMS last January.

For the first JMM that will take place under this agreement, in Seattle, Washington, USA, 5–8 January 2022, both the speaker for the "ILAS Lecture at the JMM" as well as the Special Sessions will be selected by the ILAS Board. For future meetings, the Board will develop appropriate mechanisms for selection.

If you have suggestions for the first speaker in this series, please write directly to the ILAS President, Daniel Szyld, at `szyld@temple.edu`.

If you would want to organize a Special Session, please send a short proposal with the title of the session (and possible speakers) to our Vice President for Conferences, Raf Vandebril, as soon as you can, but no later than December 15, 2020.

Nominations for Upcoming ILAS Elections

Contributed announcement from Daniel Szyld, ILAS President

The Nominating Committee for the 2021 ILAS elections has prepared a slate of candidates as follows.

Nominated for the two open three-year terms, beginning March 1, 2021, as at-large members of the ILAS Board of Directors are: Geir Dahl, Melina Freitag, Apoorva Khare, and Fernando de Terán.

Nominated for a three-year term, also beginning March 1, 2021, as ILAS Secretary/Treasurer is Minerva Catral.

The Nominating Committee consisted of Michael Overton (chair), Raphi Loewy, Jennifer Pestana, Rachel Quinlan, and Alison Ramage.

ILAS Social Media Presence

Contributed announcement from David Watkins, chair, ILAS Outreach and Membership Committee

ILAS now has a social media presence. Please follow us on Twitter (`@ILAS_news`) and like and follow us on Facebook (`@International.Linear.Algebra.Society`). We're just getting started, but we plan to provide timely and useful information to the linear algebra community. Please join us – and tell your friends!

ILAS Members Advise Winners of Householder Prize XXI

The prestigious Alston S. Householder Prize is awarded every three years to the best thesis in numerical linear algebra over the previous three-year period. Most recently, the Householder Prize XXI (2020) recognized the best such dissertation submitted between January 1, 2017 and December 31, 2019. The two winners were both advised by ILAS members.

Estelle Massart, while a student at Université catholique de Louvain, was co-advised by ILAS member Pierre-Antoine Absil, together with Julien Hendrickx, on her thesis "Exploiting rank structures in the numerical solution of Markov chains and matrix functions," submitted in 2019.

Stefano Massei also received the prize, for his thesis on "Exploiting rank structures in the numerical solution of Markov chains and matrix functions," completed in 2017 at the Scuola Normale Superiore di Pisa, under the supervision of ILAS member Dario Bini.

In addition, an Honorable Mention was received by Pawan Goyal for his thesis, "System-theoretic model order reduction for bilinear and quadratic-bilinear systems," which was co-advised by ILAS member Peter Benner, together with Serkan Gugercin, at the Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany, and submitted in 2018.

For more details, see `https://users.ba.cnr.it/iac/irmanm21/HHXXI/hhprize.html`.

Send News for *IMAGE* Issue 66

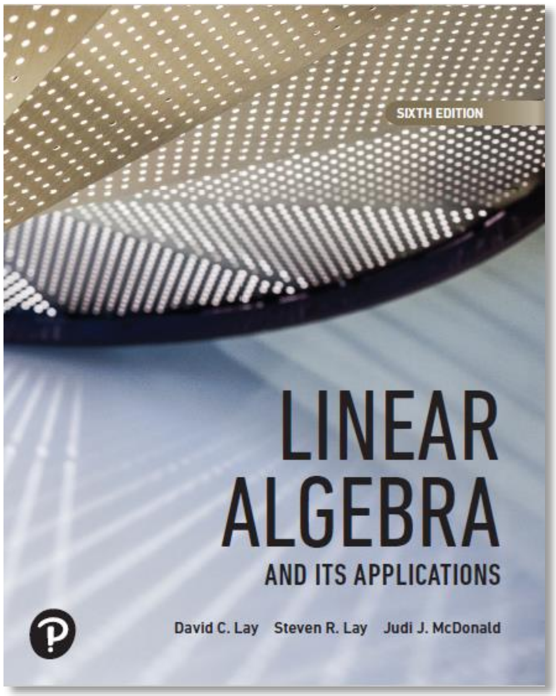
IMAGE seeks to publish all news of interest to the linear algebra community. Issue 66 of *IMAGE* is due to appear online on June 1, 2021. Send your news for this issue to the appropriate editor by April 15, 2021. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- feature articles to Sebastian Cioabă (`cioaba@udel.edu`)
- book reviews to Colin Garnett (`Colin.Garnett@bhsu.edu`)
- problems and solutions to Rajesh Pereira (`pereirar@uoguelph.ca`)
- advertisements to Amy Wehe (`awehe@fitchburgstate.edu`)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (`jephianlin@gmail.com`)
- interviews and linear algebra education news to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)

Send all other correspondence to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`).

For past issues of *IMAGE*, please visit `http://www.ilasic.org/IMAGE`.

New! Lay's *Linear Algebra*, Sixth Edition



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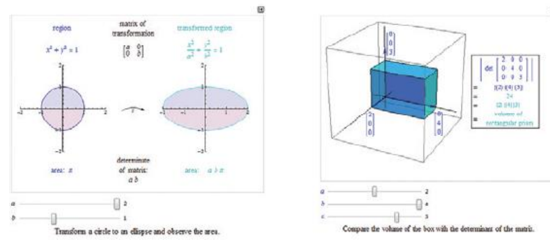

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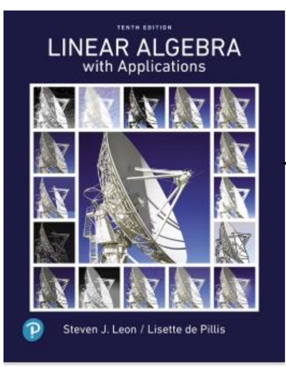

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UPCOMING CONFERENCES AND WORKSHOPS

International Conference on Applied Linear Algebra, Probability and Statistics (ALAPS 2020) (to be held online) December 17–18, 2020

The virtual conference ALAPS 2020 will be organized by the Center for Advanced Research in Applied Mathematics and Statistics (CARAMS), Manipal Academy of Higher Education (MAHE), Manipal, Karnataka, India, in honor of the great Indian statistician Professor C. R. Rao (Calyampudi Radhakrishna Rao) in celebration of the completion of his 100^{th} year.

The themes of the conference are focused on, but not limited to: theory and applications of linear algebra and matrix theory; mathematical statistics such as probability theory, multivariate analysis and statistical inference; applied statistics such as econometrics, biometrics, bioinformatics, biostatistics, mathematical genetics, etc.; network theory and applications.

The conference consists of invited talks from several mathematicians and statisticians associated with Professor C. R. Rao and working in the areas to which Rao contributed. The organizers welcome contributed talks (10+2 min each) from participants, with topics from the theme areas.

Speakers and participants may join through the web link appearing on the individual page (the "My Page") of CARAMS that becomes available after registration for the event. Alternatively, a limited number of delegates may join the event from any of the following locations, with due consent from the organizers:

- Indian Statistical Institute, Delhi
- Department of Mathematics, SMIT, Sikkim
- Department of Data Science, PSPH, MAHE, Manipal
- Department of Mathematics, MIT, MAHE, Manipal

Participants seeking a certificate of participation/contribution are requested to register for the event with due payment of the registration fee. Please visit `https://carams.in/events/alaps2020/` for more details and registration.

Special session on **The Inverse Eigenvalue Problem for Graphs, Zero Forcing, and Related Topics** ***at the*** **2021 Joint Mathematics Meetings (to be held online) January 7, 2021**

A special session on "The Inverse Eigenvalue Problem for Graphs, Zero Forcing, and Related Topics" will be held (virtually) from 8–11:45AM and 1–3:50PM US Mountain Standard Time (UTC-07:00) on January 7, 2021. The session is organized by Leslie Hogben (Iowa State University and the American Institute of Mathematics) and Bryan Shader (University of Wyoming). The speakers are:

- Carolyn Reinhart (Iowa State University)
- Boris Brimkov (Slippery Rock University)
- Sean English (University of Illinois at Urbana-Champaign)
- Juergen Kritschgau (Iowa State University)
- Joseph Alameda (Iowa State University)
- Mary K. Flagg (University of St. Thomas)
- Daniela Ferrero (Texas State University)
- Hein van der Holst (Georgia State University)
- H. Tracy Hall (Hall Labs, LLC)
- Jephian C.-H. Lin (National Sun Yat-sen University)
- Shaun M. Fallat (University of Regina) and
- Bryan A. Curtis (Sungkyunkwon University).

The web page at `https://jointmathematicsmeetings.org/meetings/national/jmm2021/2247_program_ss37.html` includes titles, coauthors, and abstracts.

SIAM Conference on Applied Linear Algebra (LA21) with the embedded 23rd Conference of the International Linear Algebra Society New Orleans, USA (and online), May 17–21, 2021

The SIAM Conference on Applied Linear Algebra is the meeting of the SIAM Activity Group on Linear Algebra. These conferences, organized by SIAM every three years, are the premier international conferences on applied linear algebra, bringing together diverse researchers and practitioners from academia, research laboratories, and industries all over the world to present and discuss their latest work and results on applied linear algebra.

In 2021, the 23rd Conference of the International Linear Algebra Society (ILAS 2021) will be embedded in LA21.

Invited speakers: (* indicates a speaker supported in cooperation with ILAS)

- Allison Baker (National Center for Atmospheric Research, USA)
- David Bindel (Cornell University, USA)
- *Paola Boito (Università di Pisa, Italy)
- Julianne Chung (Virginia Tech, USA)
- Inderjit Dhillon (University of Texas at Austin, USA)
- Yuji Nakatsukasa (Oxford University, United Kingdom)
- Deanna Needell (University of California, Los Angeles, USA)
- Catherine Powell (University of Manchester, United Kingdom)
- *Bryan Shader (University of Wyoming, USA)
- Umesh Vazirani (University of California, Berkeley, USA)

Prizes and Special Lectures:

- ILAS Hans Schneider Prize Lecture
  Lek-Heng Lim (University of Chicago, USA)
- ILAS Olga Tausky-Todd Lecture
  Raf Vandebril (Katholieke Universiteit Leuven, Belgium)
- SIAM Activity Group on Linear Algebra Early Career Prize
  Recipient to be announced
- SIAM Activity Group on Linear Algebra Best Paper Prize
  Recipient to be announced

SIAM intends to hold LA21 as either a hybrid or a 100% virtual conference. Visit the FAQ on COVID-19 contingency plans for LA21 at `https://sinews.siam.org/Details-Page/faq-on-covid-19-contingency-plans-for-la21` for details and up-to-date information.

For more information on LA21, visit `https://www.siam.org/conferences/cm/conference/la21`, or contact the co-chairs of the organizing Organizing Committee, Misha Kilmer (`misha.kilmer@tufts.edu`) and Andreas Stathopoulos (`andreas@cs.wm.edu`).

Western Canada Linear Algebra Meeting (WCLAM 2021) Brandon, Canada, May 29–30, 2021

The Western Canada Linear Algebra Meeting (WCLAM) provides an opportunity for mathematicians in western Canada and the USA working in linear algebra and related fields to meet, present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Anyone working in linear algebra or a related field, including combinatorics, graph theory, matrix analysis, and applied mathematics, is encouraged to submit an abstract for a contributed talk or poster.

The participation fee of CAD$30 will be waived for participating students and postdoctoral fellows. Subject to funding, students and postdoctoral fellows will receive free accommodation in residence and potentially additional travel support. WCLAM 2021 will have three distinguished invited speakers: Ada Chan (York University), Doug Farenick (University of Regina), and Judi McDonald (Washington State University).

The organisers of WCLAM 2021 are: Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Manitoba), Sarah Plosker (Brandon University), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria). The local organisers are: Steve Kirkland (University of Manitoba) and Sarah Plosker (Brandon University) (`ploskers@brandonu.ca`). Further information is available at `https://www.brandonu.ca/wclam`.

The organisers hope that the meeting will take place in person, however, depending on the evolving COVID-19 situation, the meeting may need to take place virtually. Please check the meeting website for the latest information.

6th Workshop on Algebraic Designs, Hadamard Matrices & Quanta
Kraków, Poland, June 28 – July 2, 2021

The 6th Workshop on Algebraic Designs, Hadamard Matrices & Quanta will be held at Jagiellonian University, as well as at the Institute of Mathematics, in Kraków, Poland.

The list of confirmed invited speakers includes:

- Ingemar Bengtsson (Stockholm, Sweden)
- Robert Craigen (Winnipeg, Canada)
- Dane Flannery (Galway, Ireland)
- Shmuel Friedland (Chicago, USA)
- Dardo Goyeneche (Antofagasta, Chile)
- Markus Grassl (Gdańsk, Poland)
- Hadi Kharaghani (Lethbridge, Canada)
- Ilias Kotsireas (Waterloo, Canada)
- Máté Matolcsi (Budapest, Hungary)
- Koji Momihara (Kumamoto, Japan)
- Akihiro Munemasa (Tōhoku, Japan)
- Ion Nechita (Toulouse, France)
- Padraig Ó Catháin (Worcester, USA)
- Eric Swartz (William & Mary, USA)
- Behruz Tayfeh-Rezaie (Tehran, Iran)
- Mihály Weiner (Budapest, Hungary)
- Qing Xiang (Newark, USA)
- Danylo Yakymenko (Kyiv, Ukraine)

Early conference registration is due by March 1, 2021. Further information will be available at the conference web site, `https://chaos.if.uj.edu.pl/hadamard2020`.

International Workshop on Operator Theory and its Applications (IWOTA 2021)
Lancaster, UK, August 16–20, 2021

The International Workshop on Operator Theory and its Applications will be hosted at Lancaster University in the UK, on 16–20th August 2021. The meeting will present several plenary and semiplenary lectures by leading international researchers in operator theory and related areas of analysis. In conjunction with the International Linear Algebra Society, the meeting will feature the inaugural Israel Gohberg Lecture, by Vern Paulsen (University of Waterloo). The lecture is named in honour of the scientific contributions of Israel Gohberg, who was a visionary and driving force of IWOTA. There will be several special sessions, covering many aspects of operator theory and its applications. Early career researchers are welcome to attend IWOTA, and are eligible to apply to the organizers for financial support, which has been generously provided by the Engineering and Physical Sciences Research Council (EPSRC) and the US National Science Foundation. The IWOTA meeting will be preceded by the Young Functional Analysts' Workshop at Lancaster, 12–14th August 2021, and followed by MTNS at Cambridge, 23–27th August 2021.

For more details on the meeting, see `https://www.lancaster.ac.uk/maths/iwotauk2021`.

28th International Workshop on Matrices and Statistics (IWMS 2021)
Manipal, India, December 13–15, 2021

The 28th International Workshop on Matrices and Statistics (IWMS 2021) will be held December 13–15, 2021 at the Center for Advanced Research in Applied Mathematics and Statistics (CARAMS), Manipal Academy of Higher Education (MAHE), Manipal, Karnataka, India.

The purpose of the Workshop is to stimulate research and, in an informal setting, to foster the interaction of researchers in the interface between statistics and matrix theory. The Workshop will provide a forum through which statisticians may be better informed of the latest developments and newest techniques in linear algebra and matrix theory and may exchange ideas with researchers from a wide variety of countries. In addition to featuring a range of plenary speakers, the meeting will strengthen the interactions between participants through a range of mini-symposia in various areas of specialization.

Themes of the workshop will include: matrix analysis, projectors in linear models & multivariate analysis, growth curve models, linear regression models, linear statistical inference, modelling covariance structures, multivariate and mixed linear models, and statistics in big data analysis.

The Scientific Committee consists of Ravindra B. Bapat, Manjunatha Prasad Karantha, Steve Kirkland, and Simo Puntanen. The Organizing Committee consists of Narayana Sabhahit (Chairman, Registrar, MAHE) and Manjunatha Prasad Karantha (Organizing Secretary, Coordinator, CARAMS, MAHE).

CRR Day on December 15th, 2021: IWMS 2021 will be held alongside ICLAA 2021 (December 15–17, 2021) and CRR Day will be held on December 15th, the day common to these events, to celebrate 100 years of C. R. Rao, who is among the greatest statisticians and matrix theorists India has ever produced.

Please visit `https://carams.in/events/international-workshop-on-matrices-and-statistics` for more details and registration.

International Conference on Linear Algebra and its Applications (ICLAA 2021)
Manipal, India, December 15–17, 2021

The next meeting of the International Conference on Linear Algebra and its Applications (ICLAA) conference series, previously scheduled for December 17–19, 2020, has been moved to December 2021 due to the COVID-19 pandemic. ICLAA 2021 will be held at the Center for Advanced Research in Applied Mathematics and Statistics (CARAMS), Manipal Academy of Higher Education (MAHE), Manipal, Karnataka, India.

The themes of the conference shall focus on classical matrix theory, nonnegative matrices and special matrices, matrices and graphs, combinatorial matrix theory, matrix and graph methods in statistics and biological science, and matrices in error analysis and its applications.

The Scientific Committee consists of Ravindra B. Bapat, Manjunatha Prasad Karantha, Steve Kirkland, and Simo Puntanen. The Organizing Committee consists of Narayana Sabhahit (Chairman, Registrar, MAHE) and Manjunatha Prasad Karantha (Organizing Secretary, Coordinator, CARAMS, MAHE).

ICLAA 2021 has been endorsed by ILAS, which will support Steve Kirkland to deliver the Hans-Schneider Lecture at the meeting. In addition, the *Electronic Journal of Linear Algebra (ELA)* will publish a special issue dedicated to ICLAA 2021, in honor of Professor C. R. Rao.

CRR Day on December 15th, 2021: ICLAA 2021 will be held alongside IWMS 2021 (December 13–15, 2021) and CRR Day will be held on December 15th, the day common to these events, to celebrate 100 years of C. R. Rao, who is among the greatest statisticians and matrix theorists India has ever produced.

Please visit `https://carams.in/events/iclaa2021` for more details and registration.

ILAS 2022: Classical Connections
Galway, Ireland, June 20–25, 2022

The 24th meeting of the International Linear Algebra Society, ILAS 2022: Classical Connections, will be hosted by the School of Mathematics at the National University of Ireland, Galway, June 20–25, 2022. The venue will be the beautiful riverside campus of the National University of Ireland, Galway.

This meeting, initially scheduled for 2020, was cancelled due to the COVID-19 pandemic. The scientific programme will be refreshed from the one planned for 2020, and contributions on all aspects of linear algebra and its applications are welcome. The conference theme is “Classical Connections.” This will be reflected in the plenary programme and mini-symposia, and all participants will be encouraged to think about relating their themes to their historical roots.

Ongoing updates and more information about the conference will be posted at `http://ilas2020.ie`.

Society for Industrial and Applied Mathematics
3600 Market Street, 6th Floor, Philadelphia, PA 19104-2688 USA
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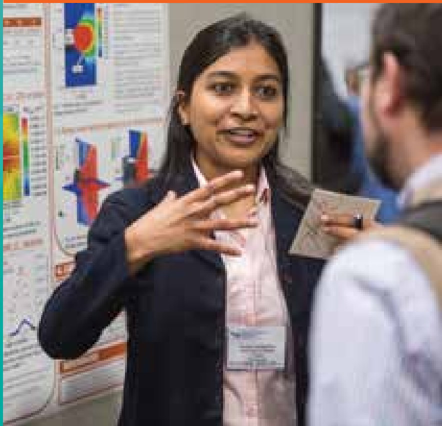

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 64-1, 64-3, 64-4 and 64-5. We also explain the flaw in Problem 64-2. Solutions are invited to Problems 60-2, 61-3, 63-1, 63-3 and for all of the problems of issue 65.

Problem 64-1: A Matrix Limit

Proposed by Dennis S. Bernstein, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu
and Ankit Goel, *University of Michigan, Ann Arbor, MI, USA*, ankgoel@umich.edu
and Syed Aseem Ul Islam, *University of Michigan, Ann Arbor, MI, USA*, aseemisl@umich.edu
and Tam Nguyen, *University of Michigan, Ann Arbor, MI, USA*, Tam.Nguyen@ieee.org

Suppose P is a real $n \times n$ (symmetric) positive semidefinite matrix and M is a real $n \times m$ matrix with the range of M being equal to the range of P. Find a closed-form expression for $\lim_{\lambda \to 0^+} \frac{1}{\lambda}[P - PM(\lambda I_m + M^T PM)^{-1} M^T P]$.

Solution 64-1.1 by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, rhorn@math.utah.edu

We pose and solve a generalization of the stated problem. The Moore-Penrose inverse of a complex matrix A is $A^\dagger$; the conjugate transpose of A is A^*; A is an *EP-matrix* if A and A^* have the same range.

Problem 64-1′: Let P be an $n \times n$ complex EP-matrix, and let M be an $n \times m$ complex matrix. If M and P have the same range, then

$$\lim_{\lambda \to 0} \big(P - PM(\lambda I + M^* PM)^{-1} M^* P\big) = (MM^*)^\dagger.$$

Let $r = \operatorname{rank} P = \operatorname{rank} M$, and let

$$M = V \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} W^*$$

be a singular value decomposition, in which V and W are unitary and S is $r \times r$ real, positive diagonal, and nonsingular. Partition V as $V = [V_1 \ V_2]$, in which V_1 has r columns. Then $M = V_1[S \ 0]W^*$, so the range of M is the range of V_1. Let $P = U(B \oplus 0)U^*$, in which U is unitary and B is $r \times r$ and nonsingular [1, 2.6.P28a, p. 158]. Partition $U = [U_1 \ U_2]$ conformally to V. Then $P = U_1[B \ 0]U^*$, so the range of P is the range of U_1. The hypothesis ensures that U_1 and V_1 have the same range. Thus, there is an $r \times r$ unitary matrix Z such that $U_1 = V_1 Z$ [1, 2.1.P22, p. 93]. Then

$$\begin{aligned} P &= V_1 Z[B \ 0][V_1 Z \ U_2]^* = V_1[ZBZ^* \ 0][V_1 \ U_2]^* \\ &= [V_1 \ V_2] \begin{bmatrix} ZBZ^* & 0 \\ 0 & 0 \end{bmatrix} [V_1 \ V_2]^* = V \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} V^*, \end{aligned}$$

in which $Q = ZBZ^*$ is $r \times r$ and nonsingular. We have

$$MM^* = V \begin{bmatrix} S^2 & 0 \\ 0 & 0 \end{bmatrix} V^* \quad \text{and} \quad (MM^*)^\dagger = V \begin{bmatrix} S^{-2} & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

as well as

$$PM = V \begin{bmatrix} QS & 0 \\ 0 & 0 \end{bmatrix} W^* \quad \text{and} \quad M^* PM = W \begin{bmatrix} SQS & 0 \\ 0 & 0 \end{bmatrix} W^*.$$

Let $|\lambda|$ be nonzero and less than the smallest singular value of SQS. Then $\lambda I + SQS$ is nonsingular and a calculation reveals that

$$P - PM(\lambda I + M^* PM)^{-1} M^* P = V \begin{bmatrix} \lambda^{-1}\big(Q - QS(\lambda I + SQS)^{-1} SQ\big) & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

The computation

$$\begin{aligned} &\lambda^{-1}\big(Q - QS(\lambda I + SQS)^{-1} SQ\big) \\ &= \lambda^{-1}\big(Q - QS(I + \lambda(SQS)^{-1})^{-1}(SQS)^{-1} SQ\big) \\ &= \lambda^{-1}\big(Q - QS(I + \lambda(SQS)^{-1})^{-1} S^{-1}\big) \end{aligned}$$

$$
\begin{aligned}
&= \lambda^{-1}\left(Q - QS(I - \lambda(SQS)^{-1} + O(\lambda^2))S^{-1}\right) \\
&= S^{-2} + O(\lambda) \\
&\to S^{-2} \text{ as } \lambda \to 0
\end{aligned}
$$

verifies the asserted value of the limit.

Reference

[1] R.A. Horn and C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

Solution 64-1.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Assume at first that P is nonsingular. Since M^TPM is positive semidefinite, there is an orthogonal matrix U and a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_m)$ such that $U^TM^TPMU = D$ and $d_i \geq 0$ for $i = 1, \dots, m$. Then

$$
\left(U^T\left(\lambda I + M^TPM\right)U\right)^{-1} = (\lambda I + D)^{-1} = \operatorname{diag}\left(\frac{1}{\lambda + d_1}, \dots, \frac{1}{\lambda + d_m}\right)
$$

and so $\left(\lambda I + M^TPM\right)^{-1} = U\operatorname{diag}\left(\frac{1}{\lambda+d_1}, \dots, \frac{1}{\lambda+d_m}\right)U^T$. Let Q be the positive definite square root of P and $N = QMU$.

Hence,

$$
E := P - PM(\lambda I + M^TPM)^{-1}M^TP = Q^2 - QN\operatorname{diag}\left(\frac{1}{\lambda + d_1}, \dots, \frac{1}{\lambda + d_m}\right)N^TQ,
$$

and so, since $N^TN = U^TM^TPMU = \operatorname{diag}(d_1, \dots, d_m)$,

$$
\begin{aligned}
N^TQ^{-1}EQ^{-1}N &= N^TN - N^TN\operatorname{diag}\left(\frac{1}{\lambda + d_1}, \dots, \frac{1}{\lambda + d_m}\right)N^TN \\
&= \operatorname{diag}\left(d_1 - \frac{d_1^2}{\lambda + d_1}, \dots, d_m - \frac{d_m^2}{\lambda + d_m}\right) = \operatorname{diag}\left(\frac{\lambda d_1}{\lambda + d_1}, \dots, \frac{\lambda d_m}{\lambda + d_m}\right).
\end{aligned}
$$

Therefore,

$$
\lim_{\lambda \to 0^+} N^TQ^{-1}\frac{1}{\lambda}EQ^{-1}N = I \quad \text{and so} \quad \lim_{\lambda \to 0^+} Q^{-1}NN^TQ^{-1}\frac{1}{\lambda}EQ^{-1}NN^TQ^{-1} = Q^{-1}NN^TQ^{-1}.
$$

Note that $Q^{-1}NN^TQ^{-1} = MUU^TM^T = MM^T$, which is nonsingular for the following reason. Since MM^T is $n \times n$, we need only show that its range is $\mathbb{R}^n$. We use the orthogonal direct sum decomposition $\mathbb{R}^n = \mathcal{R}(M) \oplus \mathcal{K}(M^T)$ (the range of M plus the kernel of M^T). Since $\mathcal{R}(M) = \mathcal{R}(P) = \mathbb{R}^n$, we have $\mathcal{K}(M^T) = \{0\}$. However, if some nonzero vector v is orthogonal to $\mathcal{R}(MM^T)$, then $0 = \langle v, MM^Tv\rangle = \|M^Tv\|^2$, which is a contradiction. In the last limit above, multiply on the left and right by $\left(MM^T\right)^{-1}$ to obtain

$$
\lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[P - PM(\lambda I + M^TPM)^{-1}M^TP\right] = \left(MM^T\right)^{-1}.
$$

Now assume P is singular. Let P' be the restriction of P to $\mathcal{R}(P)$ and M' be the restriction of M to $\mathcal{R}(M^T)$. Let E again denote the given expression in square brackets. We use the orthogonal direct sum decompositions

$$
\mathcal{K}(P) \oplus \mathcal{R}(P) = \mathbb{R}^n \quad \text{and} \quad \mathcal{K}(M) \oplus \mathcal{R}(M^T) = \mathbb{R}^m.
$$

The second of these implies that M' maps $\mathcal{R}(M^T)$ onto $\mathcal{R}(P')$, since $\mathcal{R}(M) = \mathcal{R}(P)$. Consider the action of E on each of the complementary subspaces $\mathcal{K}(P)$ and $\mathcal{R}(P)$. In particular, E maps $\mathcal{K}(P)$ to $\{0\}$. On $\mathcal{R}(P)$, the product M^TP maps $\mathcal{R}(P)$ to $\mathcal{R}(M^T)$. The matrix $\lambda I_m + M^TPM$ maps $\mathcal{R}(M^T)$ to itself, and so its inverse does that as well. Finally, PM maps $\mathcal{R}(M^T)$ to $\mathcal{R}(P)$, and so we may write

$$
E = P' - P'M'(\lambda I_r + (M')^TP'M')^{-1}(M')^TP'
$$

on $\mathcal{R}(P)$, where r is the dimension of $\mathcal{R}(M^T)$. Therefore, on $\mathcal{R}(P)$,

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} E = \left(M'(M')^T\right)^{-1}.$$

Note: The same proof works for complex matrices. Simply replace the transpose by the transpose-conjugate, real n-space by complex n-space, and "orthogonal" by "unitary".

Problem 64-2: Simple Neo-Pythagorean Means

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

This question explores when a perfect square is the arithmetic or harmonic mean of two distinct perfect squares.

(a) Suppose that $x, y, z \in \mathbb{N}$ with $x < y < z$. Show that y^2 is the arithmetic mean of x^2 and z^2 if and only if there exists a right-angle triangle whose hypotenuse has length y and whose other two sides have lengths $\frac{z-x}{2}$ and $\frac{z+x}{2}$.

(b) Suppose that $x, y, z \in \mathbb{N}$ with $x < y < z$. Show that y^2 can never be the harmonic mean of x^2 and z^2 (i.e., $2/y^2 = 1/x^2 + 1/z^2$ has no integer solutions except when $x = y = z$).

Editor's note: Eugene Herman has pointed out that $(x, y, z) = (5, 7, 35)$ is a solution to the equation in Problem 64-2b. In fact he notes that if we substitute $z = xy$ into $2/y^2 = 1/x^2 + 1/z^2$, we get the Pell's equation $2x^2 - y^2 = 1$. This means that there is an infinite sequence of solutions with $(5, 7, 35)$ and $(29, 41, 1189)$ being the first two solutions in the set obtained from the Pell's equation. We withdraw this problem and replace it with Problem 65-2 whose part b has the added hypothesis that x and z be relatively prime. We regret the error.

Problem 64-3: 2×2 Matrix Diagonalization

Proposed by Fuzhen ZHANG, *Nova Southeastern University, Fort Lauderdale, Florida, USA*, zhang@nova.edu

Let F be a field and let $M_n(F)$ be the set of all $n \times n$ matrices with entries in F. A matrix $M \in M_n(F)$ is said to be *diagonalizable* over F if there exist D a diagonal matrix in $M_n(F)$ and S an invertible matrix in $M_n(F)$ such that $M = SDS^{-1}$. Let $A = (a_{ij})$ be a 2×2 symmetric matrix over F whose off-diagonal entries are nonzero. It is well-known that if $F = \mathbb{R}$, then the real symmetric matrix A will always be diagonalizable. Now let $\Theta_A = \frac{a_{11} - a_{22}}{a_{12}}$.

(a) Show that if $F = \mathbb{C}$, then A is diagonalizable over $\mathbb{C}$ if and only if $\Theta_A \neq \pm 2i$.

(b) Show that if $F = \mathbb{Q}$, then A is diagonalizable over $\mathbb{Q}$ if and only if there exist $m, n \in \mathbb{N}$ such that $\Theta_A = \frac{m^2 - n^2}{mn}$.

Solution 64-3

by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

The eigenvalues of A are

$$\begin{aligned}\lambda_+, \lambda_- &= \frac{1}{2}\left(a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}\right) \\ &= \frac{1}{2}\left(a_{11} + a_{22} \pm \frac{1}{a_{12}}\sqrt{\Theta_A^2 + 4}\right).\end{aligned}$$

(a) If $\Theta_A \neq \pm 2i$, then $\Theta_A^2 + 4 \neq 0$. The eigenvalues of A are therefore distinct, and so A is diagonalizable over $\mathbb{C}$. If A is diagonalizable over $\mathbb{C}$ and $\Theta_A = \pm 2i$, then the eigenvalues of A are equal and so A is similar to a multiple of the identity matrix. But then A is the same multiple of the identity matrix, which contradicts the hypothesis that $a_{12} \neq 0$.

(b) Since $\Theta_A^2 + 4 > 0$, the eigenvalues of A are distinct. Hence, A is diagonalizable over $\mathbb{Q}$ if and only if the eigenvalues are rational; and that is true if and only if $\Theta_A^2 + 4$ is the square of a rational number. If $\Theta_A = \frac{m^2 - n^2}{mn}$ for some $m, n \in \mathbb{N}$, then

$$\Theta_A^2 + 4 = \frac{(m^2 - n^2)^2 + 4m^2n^2}{m^2n^2} = \left(\frac{m^2 + n^2}{mn}\right)^2$$

and so A is diagonalizable over $\mathbb{Q}$. Now suppose $\Theta_A^2 + 4$ is the square of a rational number. We may assume $\Theta_A \neq 0$, since otherwise we can choose $m = n = 1$. Write $|\Theta_A| = p/q$, where p and q are relatively prime positive integers. Then

$\Theta_A^2+4=(p^2+4q^2)/q^2$ and so $p^2+4q^2=r^2$ for some $r\in\mathbb{N}$. That is, $(p,2q,r)$ is a Pythagorean triple. If p is odd, this triple is primitive, and so there exist $m,n\in\mathbb{N}$ such that $p=m^2-n^2$, $2q=2mn$, and $r=m^2+n^2$. Hence, $\Theta_A=\pm\frac{m^2-n^2}{mn}$. If p is even, then r is also even, and so $p=2p_1$, $r=2r_1$, where $p_1,r_1\in\mathbb{N}$. Hence, $p_1^2+q^2=r_1^2$ and (p_1,q,r_1) is primitive. Therefore, for some $m,n\in\mathbb{N}$, $p_1=m^2-n^2$, $q=2mn$, and $r_1=m^2+n^2$, and so $\Theta_A=\pm 2p_1/q=\pm\frac{m^2-n^2}{mn}$.

Part a also solved by Jeffrey Stuart.

Problem 64-4: A Matrix Equation

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let A be an $n\times n$ complex matrix with $\mathrm{Tr}(A)=0$ and with all nonzero eigenvalues having multiplicity one. Show that there exist invertible matrices P and Q such that $PAP^{-1}+QAQ^{-1}=A$.

Solution 64-4 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

We prove the equivalent statement that there exist matrices B and C similar to A such that $A=B+C$. Since A is similar to its Jordan form, we may assume that A is in that form. We write A, therefore, as a block-diagonal matrix with blocks $J_1(0),\dots,J_k(0),D$ in which the first k blocks are Jordan blocks with zeros on the diagonal and D is a diagonal trace-zero matrix whose diagonal entries are distinct and nonzero. It suffices to prove the result whenever A has one of these forms. If $A=O$, then $A=A+A$. If A has ones along its superdiagonal and zeros elsewhere, then $A=\frac{A}{2}+\frac{A}{2}$. The matrix $\frac{A}{2}$ is similar to A since $PAP^{-1}=\frac{A}{2}$ with $P=\mathrm{diag}(1,2,2^2,\dots,2^n)$.

Finally, assume $A=D=\mathrm{diag}(d_1,\dots,d_n)$ where $d_1,\dots,d_n$ are distinct, nonzero and sum to zero. We claim that there exists a matrix Z such that $\frac{D}{2}\pm Z$ are both similar to D. Since D is the sum of these two matrices, it remains only to prove our claim. Since $\mathrm{Tr}(D)=0$, $n\ge 2$. Define $Z=[z_{ij}]$ as the matrix whose first column and row have entries $z_{i1}=1$ for $i=2,\dots,n$,

$$z_{1j}=a_j:=\frac{\prod_{i=1}^n(d_j/2-d_i)}{\prod_{i\ne 1,j}(d_j/2-d_i/2)},\quad \text{for } j=2,\dots,n,$$

and $z_{ij}=0$ otherwise.

To prove our claim, it suffices to prove that D and $\frac{D}{2}\pm Z$ all have the same characteristic polynomial, since that polynomial is a product of distinct linear factors. Define $p(x)=\det(xI-D)=\prod_{i=1}^n(x-d_i)$ and

$$q(x)=\det(xI-(D/2-Z))=\prod_{i=1}^n(x-d_i/2)-\sum_{j=2}^n a_j\prod_{i\ne 1,j}(x-d_i/2).$$

Furthermore, $q(x)$ is also the characteristic polynomial of $\frac{D}{2}+Z$, since $\frac{D}{2}+Z$ is similar to $\frac{D}{2}-Z$. (In fact, $\frac{D}{2}+Z=S(\frac{D}{2}-Z)S^{-1}$ where $S=\mathrm{diag}(-1,1,1,\dots 1,1)$).

For $j\ge 2$, compute the difference

$$p(d_j/2)-q(d_j/2)=\prod_{i=1}^n(d_j/2-d_i)-\sum_{j=2}^n a_j\prod_{i\ne 1,j}(d_j/2-d_i/2)=0.$$

Since $\mathrm{Tr}(D)=0=\mathrm{Tr}(D/2-Z)$, the difference $p(x)-q(x)$ has degree at most $n-2$. Hence, since $p(x)-q(x)=0$ at $n-1$ distinct points, we have shown that $p(x)=q(x)$.

Problem 64-5: Symmetry of a Sum of Permutation Matrices

Proposed by Yue Liu, *Fuzhou University, Fujian, China*, liuyue@fzu.edu.cn
and Fuzhen Zhang, *Nova Southeastern University, Fort Lauderdale, Florida, USA*, zhang@nova.edu

Let $P=(p_{ij})$ be the basic circulant permutation matrix of order n; that is, $p_{i,i+1}=1$ for $i=1,2,\dots,n-1$, while $p_{n1}=1$, and all other entries are equal to 0. Let i and j be nonnegative integers. Show that P^i+P^j is a symmetric matrix if and only if either (a) n divides $i+j$, or (b) n is even and $\{i,j\}=\{0,\frac{n}{2}\}\pmod n$.

Solution 64-5.1 by Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

Since $P^{kn} = I_n$ for all nonnegative positive integers k, we can assume that $0 \le i < n$ and $0 \le j < n$. Without loss of generality, assume that $i \le j$.

For $0 < i < n$, P^i has $n - i$ ones on the ith band above the diagonal (hence it is an all-ones band), and i ones on the $(n-i)$th band below the diagonal (hence it is an all-ones band), with all other entries of P^i equal to zero. In particular, when n is even and $i = \frac{n}{2}$, P^i is itself symmetric; the only other case in which P^i is symmetric is when $i = 0$.

When n is even, $P^0 + P^{n/2} = I_n + P^{n/2}$ is symmetric, and $P^{n/2} + P^{n/2} = 2P^{n/2}$ is also symmetric. For both even and odd n, $P^0 + P^0$ is clearly symmetric.

Suppose that neither P^i nor P^j is symmetric. In order for $P^i + P^j$ to be symmetric, it is necessary that the sum of the entries above the diagonal in $P^i + P^j$ must equal the sum of the entries below the diagonal:

$$(n-i) + (n-j) = i + j.$$

That is, $i + j = n$. Note that when n is even, this would allow for $i = j = n/2$, except that we have required that P^i and P^j not be symmetric. It suffices to show that for these values of i and j, $P^i + P^j$ actually is symmetric. As noted above, each of P^i and P^j has two all-ones bands, so the all-ones bands for $P^i + P^j$ are the ith and jth $= (n-i)$th above the diagonal, and the $(n-i)$th and $(n-j)$th $= (n-(n-i))$th $= i$th below the diagonal. Thus, $P^i + P^j$ is symmetric.

Finally, lifting the restriction that i and j satisfy $i \le j < n$, suppose that $i = k_1 n + \widehat{i}$ and $j = k_2 n + \widehat{j}$ where k_1 and k_2 are nonnegative integers and where $0 \le \widehat{i} \le \widehat{j} < n$. When n is even, $\{i,j\} = \left\{0, \frac{n}{2}\right\} \pmod n$ is equivalent to $\widehat{i} = 0$ and $\widehat{j} = \frac{n}{2}$, and hence when $\{i,j\} = \left\{0, \frac{n}{2}\right\} \pmod n$, $P^i + P^j$ is symmetric. Since

$$i + j = \left(k_1 n + \widehat{i}\right) + \left(k_2 n + \widehat{j}\right) = \left(\widehat{i} + \widehat{j}\right) + (k_1 + k_2)\, n,$$

the condition that n divides $i+j$ is equivalent to the statement that $\widehat{i}+\widehat{j} = 0$ or $\widehat{i}+\widehat{j} = n$. In the former case, $P^i = P^j = I_n$, so $P^i + P^j$ is clearly symmetric. In the latter case, we have proven that this is equivalent to $P^i + P^j = P^{\widehat{i}} + P^{\widehat{j}}$ being symmetric with neither summand equal to the identity matrix.

Solution 64-5.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

If P is any permutation matrix, there exists a permutation σ of $(1, 2, \ldots, n)$ such that $Pe_j = e_{\sigma(j)}$ for $j = 1, \ldots, n$, where e_j is the jth standard unit basis vector of $\mathbb{R}^n$. We will allow P to be any permutation matrix whose corresponding permutation σ is an n-cycle. Such a permutation has the property (to be called the *identity-cycle property*) that if $\sigma^k(i) = i$ for some i, then σ^k is the identity map and so $n|k$. Therefore, if $\sigma^k(j) = i$ and $\sigma^l(i) = j$ for some pair (i,j), then $\sigma^{k+l}(i) = i$, which by the identity-cycle property implies that σ^{k+l} is the identity permutation and hence $n|(k+l)$. Let $p_{ij}(k)$ denote the (i,j)-entry of P^k. Then

$$p_{ij}(k) = \begin{cases} 1 & \text{if } \sigma^k(j) = i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Consider first the case where at least one of the two matrices P^k and P^l is symmetric. In this case $P^k + P^l$ is symmetric if and only if both P^k and P^l are symmetric. From the above formula for the entries of P^k, P^k is symmetric if and only if $\sigma^k(j) = i$ implies $\sigma^k(i) = j$, and that is true if and only if $\sigma^{2k}(i) = \sigma^k(j) = i$. So P^k is symmetric if and only if n divides $2k$. Similarly, P^l is symmetric if and only if n divides $2l$. Hence, in the case where at least one of the two matrices P^k and P^l is symmetric, the matrix $P^k + P^l$ is symmetric if and only if n divides both $2k$ and $2l$, which occurs if and only if $k, l \in \{0, \frac{n}{2}\} \pmod n$ when n is even and $k = l = 0 \pmod n$ when n is odd.

Finally, consider the case where neither P^k nor P^l is symmetric. Then if $P^k + P^l$ is symmetric, there must be a pair i, j such that the (i,j)-entry of P^k is one and the (j,i)-entry of P^l is one. This implies that $\sigma^k(j) = i$ and $\sigma^l(i) = j$, which by the above implies that n divides $k + l$. To show the converse, if n divides $k + l$, then since $P^T = P^{-1}$, we have $P^k = (P^{-k})^T = (P^l)^T$, which implies that $P^k + P^l$ is symmetric.

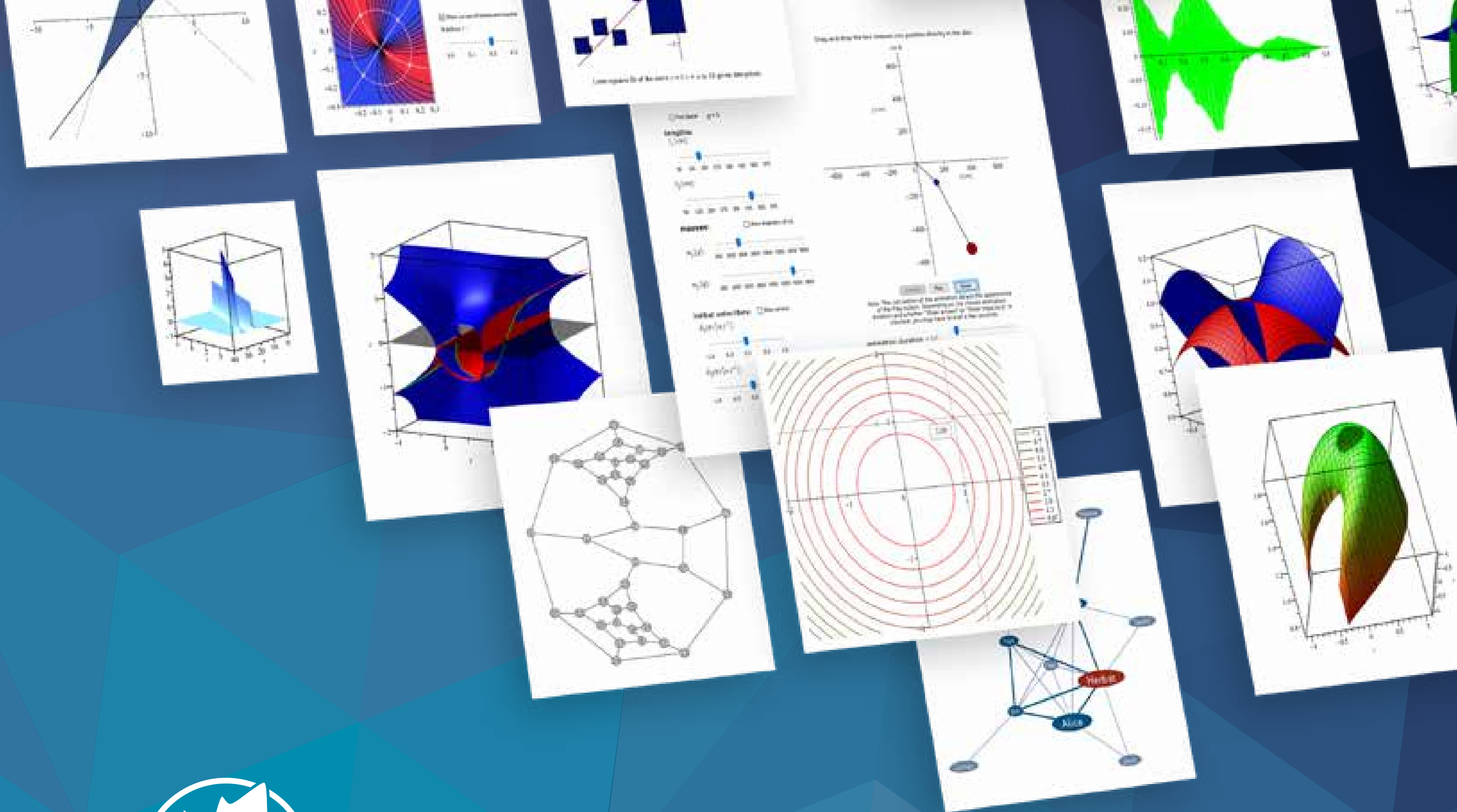

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce five new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 65-1: A Multivariable Recursion

Proposed by Sneha SANJEEVINI, *University of Michigan, Ann Arbor, MI, USA*, snehasnj@umich.edu
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and Dennis S. BERNSTEIN, *University of Michigan, Ann Arbor, MI, USA*, dsbaero@umich.edu

Let $D_0, \ldots, D_n$ be real $p \times p$ matrices such that $D_n \neq 0$ and $\det(z^n D_n + \cdots + zD_1 + D_0)$ is not the zero polynomial. Furthermore, let $\{y_j\}_{j=0}^{\infty}$ be an infinite sequence of vectors in $\mathbb{R}^p$ satisfying

$$D_n y_{k+n} + \cdots + D_1 y_{k+1} + D_0 y_k = 0$$

for all $k \geq 0$. Show that, if $y_0 = y_1 = \cdots = y_{n-1} = 0$, then, for all $k \geq n$, $y_k = 0$.

Problem 65-2: Simple Neo-Pythagorean Means (Corrected)

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

This question explores when a perfect square is the arithmetic or harmonic mean of two distinct perfect squares.

(a) Suppose that $x, y, z \in \mathbb{N}$ with $x < y < z$. Show that y^2 is the arithmetic mean of x^2 and z^2 if and only if there exists a right-angle triangle whose hypotenuse has length y and whose other two sides have lengths $\frac{z-x}{2}$ and $\frac{z+x}{2}$.

(b) Suppose that $x, y, z \in \mathbb{N}$ with $x < y < z$. Show that y^2 can never be the harmonic mean of x^2 and z^2 if x and z are relatively prime (i.e., $2/y^2 = 1/x^2 + 1/z^2$ has no positive integer solutions with $x < y < z$ and $\gcd(x, z) = 1$).

Editor's note: This is a corrected version of Problem 64-2. See the solutions section under Problem 64-2 for more details.

Problem 65-3: Number of Solutions of a Matricial System

Proposed by Gérald BOURGEOIS, *Université de la Polynésie française, FAA'A, Tahiti, Polynésie française*, bourgeois.gerald@gmail.com

Let $\mathbb{Z}/6\mathbb{Z}$ denote the finite ring of integers modulo six. Find the number of ordered pairs (A, B) of 3×3 matrices over $\mathbb{Z}/6\mathbb{Z}$ which simultaneously satisfy the two matrix equations $A^2 + B^2 = I_3$ and $AB = 0$.

Problem 65-4: An nth Derivative Inequality

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let $\|\cdot\|$ denote the usual operator norm on the space of $n \times n$ complex matrices. For any $n \times n$ complex matrix A, define ϕ_A to be the linear operator on the space of $n \times n$ matrices which maps any such matrix B to the commutator $AB - BA$. Let $p_A(z)$ be the monic characteristic polynomial of ϕ_A. Find the smallest positive number C_n such that $|p_A^{(n)}(0)| \leq C_n \|A\|^{n(n-1)}$ for all $n \times n$ complex matrices A.

Solutions to Problems 64-1, 64-3, 64-4, and 64-5 are on pages 35–39.