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#### Abstract

About IMAGE

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# Introducing Closure Under Linear Combinations: The One-way Hallways Task Sequence 

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1. Introduction. Linear combinations, paired with set theory and notions of closure, form the conceptual core of vector spaces, and thus also of linear algebra. Many scholars have written about the challenges students and instructors experience when learning about and teaching vector spaces. In this manuscript, we illustrate how the instructional design heuristics of Realistic Mathematics Education (RME) [5] can help provide students with a productive and experientially real first encounter with ideas about sets that are closed under linear combinations. The work presented here comes from a broader study funded by the National Science Foundation (1915156/1914793/1914841) focused on expanding researchbased curricula for inquiry-oriented linear algebra (IOLA: https://iola.math.vt.edu). This manuscript focuses on subspaces of $\mathbb{R}^{n}$, with particular emphasis on the idea of null spaces.
2. Background and Rationale for Task Sequence. The instructional design heuristics of RME emphasize the importance of experientially real starting points in which students can ground their mathematical activity, as well as the role of a more knowledgeable other in guiding students to reinvent key mathematical ideas. Beyond this, RME assumes that mathematical learning occurs as a process in which students' models of the mathematical relationships in one situation become their model for the mathematical relationships they attend to in a new situation. These shifts are somewhat akin to historical shifts such as those in which numbers like $\frac{3}{4}$ or $\sqrt{2}$ or $2+3 i$ were at one stage conceived of strictly as the results of mathematical processes but subsequently came to be considered as mathematical objects in their own right [3].
In this section we detail the process by which we developed an experientially real starting point that supports students in reasoning about sets that are closed under linear combinations, while developing their reasoning about null and column spaces through a context that involves mapping elements between two different vector spaces. Our process included narrowing our learning objectives, conducting a didactical phenomenology, piloting and refining our task sequence, and refining the core conceptualizations that we wanted to emerge.
Narrowed learning objectives: Our design team began by considering the overall objectives of the unit on vector spaces as they related to an introductory linear algebra course as a whole. The core idea we wanted to target was conceiving of vector spaces as sets that are closed under linear combinations. We decided it was not our aim for students to reinvent the vector space axioms. We considered possible levels of generality (e.g., abstract vector spaces, finite and infinite dimensional vector spaces, vector spaces other than $\mathbb{R}^{n}$, etc.) and possible corresponding contexts (e.g., Bezier curves, mathematical settings focused on questions such as whether linear combinations of two continuous functions would be continuous, etc.) Through consideration of possible experientially real starting points, paired with emerging recommendations that a first course not focus on an abstract treatment of vector spaces [4], we decided to focus on subspaces, particularly the null and column spaces of a matrix.

Conducted didactical phenomenology and identified experientially real context: Once we had narrowed to a focus on null and column spaces, we conducted a didactical phenomenology, considering what contexts could help "give rise" to a need for the notions of null and column spaces, following the process outlined in [2]. In exploring a number of resources, we identified networks with flow conservation as a phenomenon with multiple applications across contexts that could be mathematized in a way that is consistent with null spaces [1]. In order to avoid the need for extensive background knowledge in an applied discipline, we modified the context to focus on population movement in schools with one-way hallways during a pandemic to promote safe social distancing. This is the current experientially real starting point of our context. Further, we aimed to design this problem so that the inputs (vectors describing the number of students observed passing through each hallway) had different meanings in the context than the outputs (changes in room populations) so that the problem could be meaningfully represented through the matrix equation $A x=0$ for a matrix $A$ that was not square. This was intended to help students distinguish the meanings of the inputs from the outputs - a common conceptual challenge for students that we have witnessed in our prior semesters teaching linear algebra.

Developed, piloted, and refined task sequence: We developed an initial sequence of tasks, which we then piloted and refined several times - first with our project team, then with a pair of students who had previously taken linear algebra, then with two different intact linear algebra classes. Each pilot was video-recorded and analyzed to inform
refinements to the task sequence. Through these analyses, we refined our view of the core conceptualizations we wanted students to develop. Conception 1: Subspaces as (non-empty) sets that are closed under linear combinations. In the hallways context, this entails reasoning about individual elements and their contextual meanings, the contextual meanings of scaling and summing individual elements, and what it means for the resultant vector of a linear combination to be (or not be) in the same set as the original vectors. In piloting the task sequence, we found that students often described sets that were closed under linear combinations as spans. The view of subspaces as spans is a notable and valuable interpretation, one that allows instructors to help students make important conceptual connections with generating sets and bases. Conception 2: Closed loops are useful for generating subspaces, particularly null spaces. We found that students very intuitively identified what they often referred to as "loops" or "closed loops" (paths that start and end at the same place, thus leaving room populations unchanged). These are a valuable feature of our problem context that provide students with productive ways of reasoning about (prototypcial or basis) elements of what they will come to identify as null spaces. Further, this supports the notion that null spaces are in some way characteristic of "redundance" in matrix transformations. The idea of closed loops seems to be a productive entry point for describing null spaces and bases for null spaces.
3. Description of the Task Sequence. In its current form, the sequence consists of three core tasks organized around the scenario of one-way hallways in a school during a pandemic. The hallway scenario can be viewed as a coordination between two sets of quantities: the number of people who pass through each of the one-way hallways as observed by cameras and the change in the population of each classroom that occurs during a period of observation (Figure 1). For example, assuming people do not linger in the hallway, if three people pass by Camera 1 and Camera 2 during a class changeover, then the net change in the number of people in the Biology room is zero.


Figure 1: West Wing hallway diagram

This induces a mapping from the set of 5 -tuples encoding the number of people passing by the camera in each hallway ("camera data vectors"; in the West Wing of the school, these are vectors of the form $\left.\left(c_{1}, c_{2}, c_{3}, c_{4}, c_{5}\right)\right)$ to the set of 4 -tuples showing the net change in each room's population over the period of hallway observations ("room population change vectors"). Students come to find that these room population change vectors take the form

$$
\left[\begin{array}{c}
\Delta A \\
\Delta B \\
\Delta C \\
\Delta D
\end{array}\right]=\left[\begin{array}{c}
c_{4}+c_{5}-c_{1} \\
c_{1}-c_{2} \\
c_{2}-c_{3}-c_{4} \\
c_{3}-c_{5}
\end{array}\right]
$$

In Task 1, students are asked to use camera data vectors to represent different possible journeys that a student could make through the halls (e.g., all possible paths a single student in our group of five people could take walking from room $A$ to $C$, or $C$ to $C$ ) and reason about whether these sets are closed under linear combinations. (Note that the restrictions to positive integer-valued vector entries and scalars necessitate explicit conversations about the ambient space and field of scalars when discussing issues of closure; we view this as both important and valuable.) At the end of Task 1, instructors are intended to formalize a definition of subspaces and relate it to students' work in this task setting. In Task 2, students reason about the relationships between the camera data vectors and the room population change vectors and identify the usefulness of closed loops in describing sets of solutions to corresponding linear systems of equations or matrix equations. These activities are focused on supporting students to reason about the set of camera data vectors that do not change the populations of the rooms (which corresponds to the null space of the matrix that maps camera data vectors to room population change vectors). Task 2 culminates in students representing the problem context using a matrix transformation.

In Task 3, students are asked to find all paths that leave the room population fixed in a new wing of the school based on a given matrix (the East Wing; Figure 2), rather than a given hallway diagram. This task aims to develop a conceptual need to describe the sets of vectors that correspond to the null space. After students have worked on the task, instructors are intended to formalize a definition of null space, giving a name to this conceptually important entity that students have been exploring across the task sequence.

$$
\left(\begin{array}{ccccccc}
-1 & 0 & 0 & 0 & -1 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0
\end{array}\right)
$$

Figure 2: Matrix representing East Wing

One method for solving this problem that we have seen illustrated in student work draws on the idea of closed loops. Figure 3 shows the work of a student from our pilot data. This student interpreted the matrix in Figure 2 by drawing a map of the rooms, hallways, and closed loops corresponding to the matrix. More specifically, the student used closed loops to argue that the paths from $A$ to $A$ preserved room population, noting that the loops were also closed under vector addition and multiplication by positive integer valued scalars, since summing and scaling closed loops resulted in closed loops (illustrated with the green, blue, and red paths and the corresponding vectors listed at the bottom of Figure 3). The student related this conception of closed loops to a net change of zero in the classroom populations. We argue this intuition is helpful for students reasoning more broadly about the concept of a null space and its properties in higher dimensions.


Figure 3: Student diagram illustrating the East Wing with loops and corresponding vectors

In conclusion, this task sequence was designed to provide students insight into the concept of sets that are closed under linear combinations and null spaces. Closed loop reasoning provides students with a situationally grounded interpretation of what it means to be an element of a null space. We look forward to seeing how the ideas presented here can be further developed and leveraged in linear algebra classrooms.

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## BOOK REVIEW

Matrix Positivity<br>by Charles R. Johnson, Ronald L. Smith, and Michael J. Tsatsomeros

Cambridge University Press, September 2020, ISBN 978-1-1084-7871-7, xiv+208 pages<br>Reviewed by Ravindra Bapat, Indian Statistical Institute, rbb@isid.ac.in



When one mentions a "positive matrix," most mathematicians would think of positive definite (or semidefinite) matrices. However, for readers of IMAGE, a positive matrix would more likely mean an entrywise positive matrix. The theory of such matrices, starting with the pioneering work of Perron and Frobenius, is well known, thanks to the contributions of many researchers over the last hundred years. A positive matrix naturally generalizes the notion of a positive scalar, and there are several other generalizations that have sprung up over the years. This continues to be an active area of research, and the present book takes us on a delightful tour of the garden of a variety of matrix classes, all possessing some notion of positivity.

The book contains six chapters. The first chapter, titled "Background", describes very briefly some essential prerequisites, such as Gershgorin's Theorem, Perron's Theorem, and convexity.

The second chapter, titled "Positivity Classes", introduces several classes of matrices that may be seen as generalizing the positive scalars to matrices. These include the familiar entrywise positive matrices, positive definite matrices, $M$ matrices, totally positive matrices, $P$-matrices, and copositive matrices, as well as some less familiar classes, such as doubly nonnegative, monotone, and semipositive matrices. The emphasis is on giving the basic definitions and relationships among the various classes.

Each of the remaining four chapters deals with a particular class in detail. Those classes included are the semipositive matrices, $P$-matrices, inverse $M$-matrices, and copositive matrices. Each chapter contains recent contributions. Results by the authors are also included.

An $n \times n$ matrix is called semipositive if there exists a vector $x>0$ such that $A x>0$. The class of semipositive (SP) matrices includes all the other matrix classes discussed in the book. Equivalent definitions of SP matrices are given, and sums and products of SP matrices are discussed. One section is devoted to sign patterns and another one to spectral properties. It turns out that, except for nonpositive scalar matrices, every real $n \times n$ matrix is similar to an SP matrix. Thus, SP matrices do not enjoy any special spectral properties. (As an aside, I would mention that SP matrices are obviously related to matrices whose game theoretic value, in the sense of von Neumann, is positive. There are results from game theory, especially dealing with completely mixed games, which deserve to be better known in the context of SP matrices.)

The remaining sections in the chapter on SP matrices deal with related classes such as minimally SP (MSP) matrices and strictly semimonotone matrices. Linear preservers of SP and MSP classes are discussed. One section is devoted to cone-theoretic and Perron-Frobenius-type implications of semipositivity.

The chapter on $P$-matrices has sections on eigenvalues, special classes, detection of the $P$-property, and applications. It is interesting that the $P$-property is much harder to check than total positivity which has many more constraints.

The chapter on inverse $M$ (IM) matrices is quite substantial, containing 28 subsections filled with fresh, engaging ideas. As a sample we may mention subsections dealing with IM roots of IM matrices, partitioned IM matrices, almost principal minors, graph-theoretic properties, determinantal inequalities, and ultrametric matrices.

The last chapter is devoted to copositive matrices. An $n \times n$ matrix $A$ is said to be copositive if $x^{\prime} A x \geq 0$ for every vector $x \geq 0$. Copositive matrices have a long history, going back more than fifty years. This history is neatly summarized in one of the sections. The remaining sections deal with various preserver problems involving copositive matrices.

The book is a pleasure to read and deals with the subject matter in a clear, uncluttered fashion. It will be valuable to anyone interested in acquiring up-to-date knowledge about the various matrix positivity classes.

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## ILAS NEWS

## ILAS Welcoming and Inclusiveness Statement

The ILAS board has introduced an ILAS Welcoming and Inclusiveness Statement:

The International Linear Algebra Society (ILAS) is comprised of a diverse global membership. It is the policy of ILAS to offer a welcoming and inclusive environment to all participants in its activities, including all its meetings and conferences, regardless of gender, gender identity or expression, sexual orientation, race, color, national or ethnic origin, religion or religious belief, age, marital status, disabilities, and field of expertise.
ILAS strives to foster an atmosphere that encourages the free expression and exchange of ideas, free from all forms of discrimination, harassment, and retaliation, and that is welcoming and safe to all members and to those who participate in its activities.
If you feel harassed or unsafe in any way because of the actions, words, pictures, or expression of any other member or participant, we encourage you to bring this to the attention of the organizers, and/or ILAS executive officers. Alternatively, ILAS has designated three respected former officers, whom you may contact: Leslie Hogben, Stephen Kirkland, and Chi-Kwong Li (for contact emails, see the ILAS website).

## 2022 Hans Schneider Prize awarded to Pauline van den Driessche and Nicholas Higham

There are two recipients of the 2022 Hans Schneider Prize:

Pauline van den Driessche from the University of Victoria
 (Canada) is cited for her influential contributions in combinatorial matrix theory, in mathematical biology and the interaction between these two areas. She will present her lecture at the $24^{\text {th }}$ ILAS Conference in Galway, Ireland (http://ilas2020.ie), June 20-24, 2022.

Nicholas Higham from the University of Manchester (UK) is cited for his fundamental contributions in the analysis of a wide range of numerical linear algebra problems and matrix functions. He will present his lecture at the $25^{\text {th }}$
 ILAS Conference in Madrid, Spain, June 5-9, 2023.

Volker Mehrmann Selected as 2022 American Mathematical Society Fellow


The American Mathematical Society (AMS) has recently announced its 2022 AMS Fellows, recognizing individuals who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

Volker Mehrmann (Technische Universität Berlin) was cited for "contributions to scientific computing and numerical linear algebra, and service and leadership in the mathematical community."

For more information and details on the AMS Fellows program, see http://www.ams.org/profession/ams-fellows/ams-fellows.

## New ILAS website manager and ILAS-NET manager

## Contributed announcement from Daniel Szyld, ILAS President

Dominique Guillot, who recently helped set up the new ILAS website, agreed to take over the part of managing the website, effective immediately. To send him an ILAS website message, please write to ilasicweb@gmail.com.

Pietro Paparella, currently an assistant manager of ILAS-NET, will take over the position of ILAS-NET Manager starting January 1st, 2022. You can continue to send messages (for now) to ilasic@uregina.ca.

Sarah Carnochan Naqvi served as manager of ILAS-NET and the ILAS website for fourteen years. We are grateful for her valuable service to the Society. (A formal appreciation will take place at the banquet of the $25^{\text {th }}$ ILAS Conference.)

## 2022 ILAS Elections: Nominations

## Contributed announcement from Daniel Szyld, ILAS President

The Nominating Committee for the 2022 ILAS elections has completed its work.
Nominated for a three-year term, beginning March 1, 2022, as ILAS Vice-President are:

Froilán Dopico and Rachel Quinlan

Nominated for the two open positions on the ILAS Board of Directors are:

Joseph Landsberg, Lek-Heng Lim,
Lajos Molnar, and
Sarah Plosker

## Recently Published Books



Asymptotic Theory of Dynamic Boundary Value Problems in Irregular Domains Dmitrii Korikov, Boris Plamenevskii, Oleg Sarafanov
SBN 978-3-030-65371-2

## A Guide to Spectral

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## CONFERENCE REPORTS

## Spectral Graph Theory Online <br> (held online) April 28-29, 2021

## Report by Nair Abreu and Renata Del-Vecchio

The team of Brazilian researchers in Spectral Graph Theory organized a research event, called "SGT Online 2021". The purpose of this virtual meeting was to create an environment where renowned and young researchers could get together, listening to talks about current topics and learning about new results and current trends in spectral graph theory.
The event consisted of four sessions in two days. On the $28^{\text {th }}$, we had Session 1: Eigenvectors and Session 2: New Matrices. On the $29^{\text {th }}$, we had Session 3: Signed graphs and Session 4: Eigenvalue distribution and applications.
There were 11 invited talks in the two-day meeting. The speakers were:

- Jorge Alencar (Instituto Federal do Triângulo Mineiro, Brazil)
- Sebastian Cioabă (University of Delaware, USA)
- Alexander Farrugia (University of Malta Junior College, Malta)
- Vladimir Nikiforov (University of Memphis, USA)
- Rosário Fernandes (CMA/FCT/Universidade Nova de Lisboa, Portugal)
- Kauê Cardoso (Instituto Federal do Rio Grande do Sul ,Äì Campus Feliz, RS, Brazil)
- Francesco Belardo (Università degli studi di Napoli Federico II, Italy)
- Maurizio Brunetti (Università degli studi di Napoli Federico II, Italy)
- Zoran Stanic (University of Belgrade, Serbia)
- Steve Kirkland (University of Manitoba, Canada)
- Vilmar Trevisan (Universidade Federal do Rio Grande do Sul, Brazil)

The schedule of talks, including titles, abstracts, and YouTube recordings is available at:
http://spectralgraphtheory.org/sgt-online.

## The Western Canada Linear Algebra Meeting (WCLAM) (held online) May 29-30, 2021

## Report by Sarah Plosker

The Western Canada Linear Algebra Meeting (WCLAM) was originally scheduled for May of 2020, but, like so many other conferences, had to be postponed to 2021 due to COVID-19. WCLAM 2021 was hosted online by Brandon University May 29-30, 2021. The main objective of this meeting is to bring together researchers from across Western Canada and beyond with a focus on various areas within linear algebra and related fields.

The meeting organizing committee consisted of: Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Manitoba), Sarah Plosker (Brandon University), Michael Tsatsomeros (Washington State University), and Pauline van den Driessche (University of Victoria) and the local organizers were Steve Kirkland and Sarah Plosker.

WCLAM featured three invited speakers:

- Ada Chan (York University): Pretty Good Fractional Revival
- Doug Farenick (University of Regina): The space of the Toeplitz matrices as an operator system dual
- Judi McDonald (Washington State University): Orthogonality and Finite Fields

The meeting consisted of three one-hour invited lectures and thirty 30-minute contributed lectures. For what organizers believe to be the first time in WCLAM's history, the two-day meeting required parallel sessions. There were 82 participants from across Canada, USA, and strong international presence.

We would like to acknowledge our sponsors for the event: the Pacific Institute for the Mathematical Sciences (PIMS), Brandon University Faculty of Science, University of Manitoba Department of Mathematics, and the International Linear Algebra Society (ILAS).
In lieu of a group photo, below are screenshots of participants over Zoom.


For more information, see https://www.brandonu.ca/wclam/

# Mathematical Research Community (MRC) Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra <br> (held online) June 6-11, 2021 

## Report by Jane Breen and Jephian Lin

This report is a modified version of "MRC 2021 Report - Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra" that was submitted to the American Mathematical Society in June 2021.
This report provides information on the research activities of the 2021 Mathematical Research Community entitled Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra, organized by Shaun Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, Michael Young, Jane Breen, and Jephian C.-H. Lin. This community met virtually from 6th-11th June 2021, engaging in research on a wide range of problems relevant to the inverse eigenvalue problem for graphs, and zero forcing and related graph parameters.

This workshop was the culmination of over a year of pre-workshop research activities. Due to the global pandemic and the postponement of the MRC workshops in 2020, we pivoted to an online, virtual research community, and took advantage of the additional time to welcome the participants to the community, have them get to know each other, and explore the basic notions of the research topics. For three weeks in the summer of 2020 , the junior organizers created introductory videos on the research topics and provided reading assignments for each week, hosting a live tutorial session for questions and answers. Participants also met weekly in small groups with an organizer to work through examples. Following these three weeks, we arranged an 'intensive' week of activities, with two live sessions every day, giving introductions to advanced topics in the area, and to computational tools used in the research field (such as SageMath, computing Groebner bases, and so on). Following this week, we partitioned the participants into groups twice more, meeting weekly to discuss further reading and recorded lectures on either the topic of 'strong properties' in the context of the IEP-G, or on zero forcing.

Following the activities in Summer 2020, we kept the research community active in multiple ways, while remaining mindful of the increased workload of many of our participants during the academic year caused by the pandemic. Activities were spaced out throughout the year, and leaned towards mostly social events. In early 2021, the participants were encouraged to attend the $A M S$ Special Session on The Inverse Eigenvalue Problem for Graphs, Zero Forcing, and Related Topics (I and II) at the Joint Meetings; during the meeting, we provided extra events such as a virtual social gathering and Q\&A with presenters. We also recommended that participants attend similar sessions at the Southeastern International Conference on Combinatorics, Graph Theory \& Computing in March 2021 and the SIAM Applied Linear Algebra Conference (with an embedded meeting of the International Linear Algebra Society).
The workshop itself took place from the 6th-11th of June 2021, and began with an orientation by AMS staff on Sunday afternoon. This took place in order to acclimate participants to the virtual environment of Sococo. Many participants remained in the virtual environment after the orientation session to meet each other and catch up socially with those they had been in small groups with in the pre-workshop activities. Research problem statements were shared with participants the week before the workshop, and participants were asked to review these, and consider which projects they were interested in working on. The problems were divided into two categories, 'matrix problems' and 'graph problems', with the intention of each participant working on one problem from each category, giving them experience with research techniques on two distinct and equally important facets of this research field.
Each day of the workshop was divided into a morning session (11:00am to 1:30pm EDT) and an afternoon session (2:30pm to $5: 00 \mathrm{pm}$ EDT), with the morning reserved for matrix research groups and the afternoon for graph research groups. On Monday, at the start of each session, the organizers gave short summaries of the problem statements that participants had reviewed, to allow for any clarifying questions. Then participants voted on the projects they wished to take part in and were arranged into groups according to their choices. Group sizes ranged from 5-7 people, and every participant worked on research problems they had chosen. On the first day, we had one hour of research time in each session. On Tuesday-Thursday, the entire group met for an all-hands meeting at the start of both sessions to give one-minute updates on the progress in each group, then moved to separate research rooms for the remaining two hours of the session. Each research group used Zoom or Sococo for video conferencing, and used a collaborative whiteboard such as Miro, Scribble, or OpenBoard to work together. On Friday, each research group prepared a 5-10 minute presentation on current progress and future plans, and presented in a final all-hands meeting to the entire group.
In addition to the academic activities of the workshop, we arranged social events throughout the week, hosting game nights on Gather Town where participants and organizers chatted and played online games. The AMS also arranged several professional development events, such as an introduction to MathSciNet and an information session on AMS membership benefits.
Overall, it was a productive and rewarding week. The majority of research groups are now meeting regularly to continue working on the projects, and there will be an AMS Special Section at the Joint Math Meetings in 2022 on research resulting from this workshop and related work (there will also be an ILAS special session The Inverse Eigenvalue Problem for a Graph, Zero Forcing, Throttling and Related Topics).


Photo is courtesy of the American Mathematical Society

## Workshop on Applied Matrix Positivity <br> (held online) July 19-23, 2021

## Report by Alexander Belton

An online workshop on Applied Matrix Positivity was held from Monday 19th to Friday $23^{\text {rd }}$ July 2021. The meeting was organised by Alexander Belton (Lancaster University), Emilio Porcu (Khalifa University and Trinity College, Dublin) and Mihai Putinar (Newcastle University and University of California, Santa Barbara). It was facilitated by the International Centre for Mathematical Sciences, Edinburgh.

The workshop acted to encourage interactions between workers in positivity coming from different backgrounds, particularly those working on matrix analysis, positivity transforms, algebraic quantum theory and quantum probabilty, spatial statistics, and wavelet frames. Connections were made during the meeting, facilitated by Sococo, and new collaborations are expected as a result.

There were more than 40 registered participants. The full list of speakers is as follows.

- Tatiyana Apanasovich (George Washington University, USA)
- Christian Berg (University of Copenhagen)
- Rajendra Bhatia (Ashoka University, India)
- Maria Charina (University of Vienna, Austria)
- Projesh Nath Choudhury (Indian Institute of Science, Bengaluru, India)
- Francisco Cuevas (Federico Santa María Technical University, Chile)
- Xavier Emery (University of Chile, Chile)
- Franco Fagnola (Politecnico di Milano, Italy)
- Uwe Franz (University of Bourgogne Franche-Comté, France)
- Tanvi Jain (Indian Statistical Institute, Delhi Centre, India)
- Apoorva Kharee (Indian Institute of Science and Analysis \& Probability Research Group, Bangalore, India)
- Greg Knese (Washington University in St. Louis, USA)
- Claus Köstler (University College Cork, Ireland)
- Hemant Mishra (Indian Statistical Institute, Delhi Centre, India)
- James Pascoe (University of Florida, USA)
- Rajesh Pereira (University of Guelph, Canada)
- René Schilling (TU Dresden, Germany)
- Rajesh Sharma (Himachal Pradesh University, Shimla, India)
- Adam Skalski (Institute of Mathematics of the Polish Academy of Sciences, Poland)
- Joachim Stöckler (TU Dortmund, Germany)
- Lalit Vashisht (University of Delhi, India)
- Prateek Kumar Vishwakarma (Indian Institute of Science, Bangalore, India)

Many of the talks were recorded. A link to the recordings can be found on the workshop's webpage, which also contains a list of abstracts.

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https://www.icms.org.uk/events/workshops/amp
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Plans for a follow-up meeting are in development. This will hopefully take place in person and an announcement will follow in due course.

## International Workshop on Operator Theory and Applications Meeting (IWOTA) (held online) August 16-20, 2021

## Report by Gordon Blower

Originally, IWOTA Lancaster was scheduled for 2020, but was postponed until 2021 and was held as an online event. On Monday August 16, the Deputy Vice-Chancellor of Lancaster University Steve Bradley and the Vice President of IWOTA Igor Klepp opened the meeting by welcoming over 270 participants from 37 countries to IWOTA. These included over 107 early career researchers. The meeting featured Special Sessions on the following topics.

1. Operator spaces, quantized function theory and noncommutative $L_{p}$ spaces

Organizers: David Blecher and Christian Le Merdy
2. Operator algebras Organizer: Joachim Zacharias (Glasgow)
3. Operator algebras in quantum theory

Organizers: Jason Crann (Carleton University, Canada) and Ivan Todorov
4. Noncommutative probability and Random Matrices Organizers: Jani Virtanen (Reading) and Kenneth Dykema (TAMU)
5. Operator semigroups and functional calculus Organizers: Markus Haase (Kiel) and Yuri Tomilov (IMPAN, Warsaw)
6. Spectral theory and differential operators Organizers: Ian Wood (Kent), Malcolm Brown (Cardiff), Andrii Khrabustovskyi, (Graz University of Technology, Austria)
7. Operator theory and communications

Organizer: Lucinda Hadley (Lancaster)
8. Hilbert space operator theory and complex geometry Organizers: Tirthankar Bhattacharyya (IISc Bangalore) and Lucasz Kosinski (Jagiellonian University, Krakow)
9. Free algebraic geometry and free analysis Organizers: Victor Vinnikov (Ben Gurion) and Juric Volcic (TAMU)
10. Emerging topics: from Operator Algebras to Geometric Rigidity

A special session of IWOTA to mark the retirement of Professor Stephen C. Power from Lancaster University. Organizers Derek Kitson (Mary Immaculate College), Rupert Levene (University College Dublin)
11. Early Career Researchers Organizers: Elefteris Kastis and Maria Eugenia Celorrio (Lancaster).
12. Quantum groups and algebraic quantum field theory

Organizers: Uwe Franz (Besancon) and Robin Hillier (Lancaster)
13. Linear systems (cancelled)
14. Multivariable operator theory

Organizers: Greg Knese (Washington) and Michael Dritschel (Newcastle, UK)
15. Complex Analysis and Operator Theory

Organizers: Kehe Zhu (Albany) and Jani Virtanen (Reading)
16. Operator Ideals and Operators on Banach spaces

Organizers: Niels Laustsen, Tomasz Kania, Kevin Beanland

The main lectures of the conference were as follows.

## Monday August 16

Charles Batty: Bounded functional calculi for unbounded operators
Birgit Jacob: Controllability and Riesz bases of infinite-dimensional port-Hamiltonian systems
Victor Vinnikov: A Beurling-type theorem for indefinite Hardy spaces on finite bordered Riemann surfaces
Hakan Hedenmalm: Planar orthogonal polynomials and soft Riemann-Hilbert problems

Tuesday August 17
Benoit Collins: On the norm convergence of multi-matrix random matrix models
Magdalena Musat: Factorizable quantum channels, non-closure of quantum correlations and the Connes embedding problem
Zhengwei Liu: Quantum Fourier analysis
Kenneth Dykema: Spectral decompositions in finite von Neumann algebras

## Wednesday August 18

John E. McCarthy: Complete Pick spaces-what are they, and why are they interesting?
Raul Curto: The Beurling-Lax-Halmos theorem for infinite multiplicity
Mihai Putinar: Finding the cloud of a 2D point distribution
Zinaida Lykova: The $\mu$-synthesis interpolation problem and some associated domains

## Thursday August 19

The Israel Gohberg Lecture was founded to commemorate the scientific achievements of Israel Gohberg, who was a driving force behind IWOTA. The inaugural lecture was by Vern Paulsen: Cooperative games and entanglement.

Christian Le Merdy: $\ell^{1}$ bounded maps and $S^{1}$ bounded maps on noncommutative $L^{p}$ spaces
Juric Volcic: Positive noncommutative rational functions
Fritz Gesztesy: The limiting absorption principle and continuity properties of the spectral shift function for massless Dirac type operators

## Friday August 20

Marek Ptak: Invitation to IWOTA Krakow 2022. The main organizer of the meeting next year outlined the scientific agenda including the main speakers and the Special Sessions.
Daniel Alpay: Discrete analytic functions and Schur analysis
Omar El Fallah: Singular values of Hankel operators on Bergman spaces
J. W. Helton: Closing of IWOTA Lancaster 2021. The President of IWOTA asked participants to consider hosting future meetings.

The technical production of the talks was by Waggle Events. The UK Engineering and Physical Sciences Research Council supported the meeting via grant EP/T007524/1 IWOTA Lancaster UK 2021, principal investigator Gordon Blower, co investigators Stephen Power and Derek Kitson. The local administration was by Anna Barnett and Stephanie Kutschmann. The organizers intend to publish conference proceedings in the series Operator Theory: Advances and Applications, published by Birkhauser. Intending contributors are welcome to contact the main editor Yemon Choi (Lancaster) with proposals of original papers or surveys.

## Workshop on Matrix Equations and Tensor Techniques (METTIX) Perugia, Italy, September 9-10, 2021

## Report by Massimiliano Fasi



The ninth edition of the Matrix Equations and Tensor Techniques Workshop (METTIX) took place in hybrid format on Thursday and Friday September 9-10, 2021. Twenty-four participants gathered at the Department of Mathematics and

Computer Science of the University of Perugia, Italy, and over 60 attendees joined them virtually for two days of scientific talks and discussions. With 89 registered participants, 38 scheduled presentations, and 19 countries represented, this was the largest edition of the METT workshop to date.
The two days were opened by the keynote addresses of two invited speakers. On Thursday, Lieven de Lathauwer (KU Leuven, Belgium) discussed, in his seminar "From analysis to learning: Tensor-based assessment of latent similarity", the use of tensor decompositions as a tool for assessing similarity between tensors that represent data. On Friday, Dario Andrea Bini (University of Pisa, Italy) considered, in his lecture "Solving power series matrix equations encountered in stochastic processes by means of fixed point iterations", the computation of solutions to a nonlinear matrix equations arising in stochastic processes. In order to accommodate the large number of contributions, the remaining talks were distributed among "regular" and "blitz" sessions. Regular sessions featured traditional 15-minute talks each followed by 5 minutes for questions. The two blitz sessions were one hour long, and each consisted of 5 shorter presentations (5-minute talks followed by 5 minutes for questions) and a final 10-minute slot for further discussion. The alternation of remote and in-person speakers did not cause difficulties, and we are pleased to report that no major technical hiccups were experienced at any point during the conference.

The scientific programme was curated by the steering committee, which for this edition was constituted by

- Peter Benner (Max Planck Institute, Magdeburg, Germany),
- Heike Faßbender (Technische Universität Braunschweig, Germany),
- Lars Grasedyck (Aachen University, Germany),
- Daniel Kressner (École Polytechnique Fédérale de Lausanne, Switzerland),
- Beatrice Meini (Università di Pisa, Italy), and
- Valeria Simoncini (Università di Bologna, Italy).

Bruno Iannazzo (University of Perugia, Italy) was at the helm of the local organising committee, which comprised members from the University of Perugia, Italy (Elena Addis and Giulia Riganelli), the University of Pisa, Italy (Beatrice Meini, Federico Poloni, and Leonardo Robol), and Örebro University University, Sweden (Massimiliano Fasi).
The main sponsor of the workshop was the Istituto Nazionale di Alta Matematica (INdAM), which funded the travel expenses of the invited speakers as well as a travel grant for a doctoral student that attended the conference in person. The conference was endorsed by the International Linear Algebra Society (ILAS) and by the Activity Group on Linear Algebra of the Society for Industrial and Applied Mathematics (SIAG LA); support from the Department of Mathematics and Computer Science of the University of Perugia and from Department of Mathematics of the University of Pisa is gratefully acknowledged.
A brief account of the scientific content of the workshop can to be found in the Book of Abstracts, which is available on the conference website at https://indico.cs.dm.unipi.it/event/7/book-of-abstracts.pdf

## Send News for IMAGE Issue 68

$I M A G E$ seeks to publish all news of interest to the linear algebra community. Issue 68 of $I M A G E$ is due to appear online on June 1, 2022. Send your news for this issue to the appropriate editor by April 15, 2022. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- feature articles to Sebastian Cioabă (cioaba@udel.edu)
- interviews of senior linear algebraists to Adam Berliner (berliner@stolaf.edu)
- problems and solutions to Rajesh Pereira (pereirar@uoguelph.ca)
- linear algebra education news to Anthony Cronin (anthony.cronin@ucd.ie)
- advertisements to Amy Wehe (awehe@fitchburgstate.edu)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (jephianlin@gmail.com)
- book reviews to the editor-in-chief, Louis Deaett (louis.deaett@quinnipiac.edu)

Send all other correspondence to the editor-in-chief, Louis Deaett (louis.deaett@quinnipiac.edu).
For past issues of IMAGE, please visit http://www.ilasic.org/IMAGE.

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## UPCOMING CONFERENCES AND WORKSHOPS

## ILAS Partner Activities at the Joint Mathematics Meetings Seattle, WA, USA, January 5-8, 2022

Starting in 2022, ILAS is an official partner of the Joint Mathematics Meetings! The meeting will now include a plenary speaking, selected by ILAS to deliver the ILAS Invited Address; at the 2022 meeting, this will be given by Pauline van den Driessche of the University of Victoria, whose talk will be:

- Sign Patterns Meet Dynamical Systems. Wednesday, January 5, 2022, 9:00 - 9:50AM

The meeting will also feature three ILAS special sessions:

1. The Inverse Eigenvalue Problem for a Graph, Zero Forcing, Throttling and Related Topics, organized by Mary Flagg and Hein Van der Holst
2. Matrix Analysis and Applications, organized by Luyining Gan, Tin-Yau Tam, and Mohsen Aliabadi
3. The Interplay of Matrix Analysis and Operator Theory, organized by Kelly Bickel, Hugo J. Woerdeman, Ryan K. Tully-Doyle, and Meredith Sargent

For details on these sessions, see https://meetings.ams.org/math/jmm2022/meetingapp.cgi/Program/1051.
In addition, two of the AMS special sessions have organizers or co-organizers who are ILAS members:

1. Finding Needles in Haystacks: Approaches to Inverse Problems Using Combinatorics and Linear Algebra (an AMS Mathematics Research Communities Session), organized by Shahla Nasserasr, Sam Spiro, and Emily J Olson
2. Innovative and Effective Ways to Teach Linear Algebra, organized by David M. Strong, Sepideh Stewart, Megan Wawro, and Gil Strang

More information on the Invited Addresses and the ILAS Special Sessions can be found in the Scientific Program page of the JMM website at https://www.jointmathematicsmeetings.org/meetings/national/jmm2022/2268_intro.

For details on the ILAS partnership with the Joint Mathematics Meetings, see the official AMS news announcement at https://www.ams.org/news?news_id=6758.

## Linear Algebra, Matrix Analysis, and Applications Meeting and Due giorni di Algebra Lineare Numerica (ALAMA2022-ALN2gg) <br> Alcalá de Henares, Madrid, Spain, June 1-3, 2022

The $7^{\text {th }}$ ALAMA (Linear Algebra, Matrix Analysis, and Applications) meeting will be held jointly with the $17^{\text {th }}$ edition of ALN2gg (Due giorni di Algebra Lineare Numerica) and will be hosted at Alcalá de Henares (Madrid) in Spain June 1-3, 2022.

The meeting is celebrated in honour of Ion Zaballa, professor at Universidad del País Vasco (Spain), and of Dario A. Bini, professor at Università di Pisa (Italy).

The aim of the meeting is to gather researchers whose work is related to linear algebra, matrix analysis and their applications, so that it can be a place to exchange
 results, experiences and ideas. The program includes four plenary lectures by leading international researchers in the field, and communications distributed along three parallel sessions.


The Scientific Committee consists of Ana Marco, Raymond Honfu Chan, Froilán M. Dopico, Christian Mehl, Juan Manuel Peña, Lothar Reichel, Stefano Serra-Capizzano, Ana M. Urbano and Marc Van Barel. The Organizing Committee consists of Ana Marco, Fernando De Terán, Fabio Di Benedetto, Bruno Iannazzo, José Javier Martínez, Beatrice Meini, Federico Poloni and Raquel Viaña.

ALAMA2022-ALN2gg has been endorsed by ILAS, which will support Françoise Tisseur (ILAS speaker). In addition, the journal Linear and Multilinear Algebra will publish a special number of original articles presented in the meeting.

For more details and up-to-date information visit the conference web site at http://www.alama2022.com.

# The $16^{\text {th }}$ Workshop on Numerical Ranges and Numerical Radii (WONRA) Curia, Portugal, June 12-15, 2022 



The $16^{\text {th }}$ Workshop on Numerical Ranges and Numerical Radii (WONRA), whose purpose is to stimulate the interchange of ideas on this subject and its applications, will be held June 12-15, 2022. The venue is Hotel Termas da Curia, a thermal resort surrounded by its natural park, near Coimbra, in the center of Portugal. The previous meeting of this biennial workshop series celebrated the 100th anniversary of Toeplitz-Hausdorff Theorem. The high level of research activities on the topic, after this fundamental result was established, is due to the connections of the subject to many different branches of pure and applied mathematics, such as operator theory, functional analysis, $\mathrm{C}^{*}$-algebras, Banach algebras, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics, etc. Moreover, a wide range of tools, including algebra, analysis, geometry, combinatorics and computer programming, are useful in its study. An informal workshop atmosphere will facilitate the exchange of ideas from different research areas. For further details, please visit:

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http://www.mat.uc.pt/~wonra2022
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The organizing committee consists of Natália Bebiano (CMUC, University of Coimbra, Portugal), Graça Soares (CMATUTAD, University of Trás-os-Montes e Alto Douro, Portugal), Rute Lemos (CIDMA, University of Aveiro, Portugal) and Ana Nata (CMUC, Polytechnic Institute of Tomar, Portugal).

## The $9^{\text {th }}$ International Conference on Matrix Analysis and Applications (ICMAA) Aveiro, Portugal, June 15-17, 2022

The $9^{\text {th }}$ International Conference on Matrix Analysis and Applications, ICMAA 2022, will be held at the University of Aveiro, Aveiro, Portugal, on June 15-17, 2022.

This meeting aims to stimulate research and interaction of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications and to provide an opportunity for researchers to exchange ideas and developments on these subjects. The previous conferences were held in China (Beijing, Hangzhou), United States (Nova Southeastern University), Turkey (Selçuk University, Konya), Vietnam (Duy Tan University, Da Nang), Japan (Shinshu University, Nagano Prefecture) and United States (University of Nevada, Reno). Former keynote speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland, Alexander A. Klyachko (ILAS guest speaker), Shmuel Friedland, Man-Duen Choi, Tsuyoshi Ando, Fumio Hiai and Lek-Heng Lim.

The keynote speaker of ICMAA 2022 is Peter Šemrl, University of Ljubljana, Slovenia, and the two invited speakers are Natália Bebiano, University of Coimbra, Portugal and Chi-Kwong Li, College of William and Mary, USA.

The organizers are

- Enide Andrade (Organizing Committee Chair), University of Aveiro, Aveiro, Portugal; Rute Lemos, University of Aveiro, Aveiro, Portugal; Tin-Yau Tam (Organizing Committee co-Chair), University of Nevada, Reno, USA; Qing-Wen Wang, Shanghai University, Shanghai, China; Fuzhen Zhang, Nova Southeastern University, Florida, USA.

The workshop is endorsed and sponsored by:

- The International Linear Algebra Society (ILAS)
- The Center for Research and Development in Mathematics and Applications (CIDMA)
- The Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA, the Center for Research and Development in Mathematics and Applications, within project UIDB/MAT/04106/2020.
- The Department of Mathematics, University of Aveiro, Portugal.

For detailed information and updates contact Enide Andrade (enide@ua.pt) or Rute Lemos (rute@ua.pt) if you have any questions, and please visit the conference website for updates:

## The $24^{\text {th }}$ ILAS Conference: Classical Connections Galway, Ireland, June 20-24, 2022



The $24^{\text {th }}$ Conference of the International Linear Algebra Society will take place in Galway, Ireland, from June 20 to 24 , 2022. The conference theme is "Classical Connections." This will be reflected in the plenary programme and minisymposia, and all participants are encouraged to think about relating their themes to their historical roots. Contributions on all aspects of linear algebra and its applications are welcome. The conference proceedings will be published as a special issue of Linear Algebra and its Applications. The scientific programme will include more than 20 minisymposia on all areas of linear algebra, and plenary sessions including talks by the following speakers:

- Shmuel Friedland
- Nicolas Gillis
- Ilse Ipsen (LAA Lecture)
- Misha Kilmer (SIAG/LA Lecture)
- Monique Laurent
- Clément de Seguins Pazzis
- Christiane Tretter
- Vilmar Trevisan
- Pauline van den Driessche (Hans Schneider Prize Lecture)

The following minisymposia are planned:

1. Graph spectra - Domingos Cardoso, Claudia Justel and Renata del Vecchio
2. Spectral properties of non-negative matrices - Carlos Marijuán and Pietro Paparella
3. Copositive and completely positive matrices and related topics - Avi Berman, Mirjam Dür and Naomi ShakedMonderer
4. Mathematics of quantum information - Rupert Levene and Ivan Todorov
5. Combinatorial matrix theory - Jane Breen and Roberto Canogar
6. The inverse eigenvalue problem for graphs - Jephian Lin and Polona Oblak
7. General preservers - Lajos Molnár
8. Distance matrices of graphs - Projesh Nath Choudhury and Apoorva Khare
9. Linear algebra education - Anthony Cronin and Sepideh Stewart.
10. Numerical linear algebra for PDEs - Niall Madden
11. The research and legacy of Richard A. Brualdi - Adam Berliner, Louis Deaett and Seth Meyer
12. Matrix positivity: theory and applications - Alexander Belton and Dominique Guillot
13. Geometric rigidity theory - James Cruickshank
14. History of linear algebra - Kirk Soodhalter and Jörg Liesen
15. Companion matrix forms - Fernando de Terán and Kevin Vander Meulen
16. Riordan arrays and related topics - Paul Barry, Gi-Sang Cheon and Tian-Xiao He
17. Linear algebra for designs and codes - Ronan Egan, Ilias Kotsireas, Padraig Ó Catháin and Eric Swartz
18. Kemeny's constant on networks and its application - Ángeles Carmona, Maria Jose Jimenez and Margarida Mitjana.
19. Generalized inverses, operator matrices and tensor equations - Dragana Cvetkovic Ilic, Yimin Wei and Qing Wen Wang
20. Special matrices - Natália Bebiano, Susana Furtado and Mikail Tyaglov
21. Tensors for signals and systems - Kim Batselier, Philippe Dreesen and Bori Hunyadi
22. Coding theory and linear algebra over finite fields - Eimear Byrne, Alberto Ravagnani and John Sheekey

The scientific organising committee consists of: Nair Abreu, Peter Cameron, Mirjam Dür, Ernesto Estrada, Vyacheslav Futorny, Stephen Kirkland, Yongdo Lim, Rachel Quinlan, Peter Šemrl, Helena Šmigoc, Fran coise Tisseur, and Paul Van Dooren.

The local organizing committee consists of: Paul Barry, Jane Breen, Anthony Cronin, Richard Ellard, Kevin Jennings, Thomas Laffey, Niall Madden, Oliver Mason, Collette McLoughlin, Rachel Quinlan, Helena Šmigoc, and Kirk Soodhalter.

For further information and updates, please visit http://ilas2020.ie.
E-mail queries may be addressed to galway@ilas2020.ie.


We are looking forward to gathering the ILAS community in Galway in June!


## The $6^{\text {th }}$ Workshop on Algebraic Designs, Hadamard Matrices \& Quanta Kraków, Poland, June 27-July 1, 2022

The $6^{\text {th }}$ Workshop on Algebraic Designs, Hadamard Matrices \& Quanta will be held at Jagiellonian University, as well as at the Institute of Mathematics, in Kraków, Poland.
The list of confirmed invited speakers includes:


- Ingemar Bengtsson (Stockholm, Sweden)
- Robert Craigen (Winnipeg, Canada)
- Dane Flannery (Galway, Ireland)
- Shmuel Friedland (Chicago, USA)
- Dardo Goyeneche (Antofagasta, Chile)
- Markus Grassl (Gdańsk, Poland)
- Hadi Kharaghani (Lethbridge, Canada)
- Ilias Kotsireas (Waterloo, Canada)
- Máté Matolcsi (Budapest, Hungary)
- Koji Momihara (Kumamoto, Japan)
- Akihiro Munemasa (Tōhoku, Japan)
- Ion Nechita (Toulouse, France)
- Padraig Ó Catháin (Worcester, USA)
- Eric Swartz (William \& Mary, USA)
- Behruz Tayfeh-Rezaie (Tehran, Iran)
- Mihály Weiner (Budapest, Hungary)
- Qing Xiang (Newark, USA)
- Danylo Yakymenko (Kyiv, Ukraine).

Early conference registration is due by March 1, 2022. Further information can be found at https://chaos.if.uj.edu. pl/hadamard2020.

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## IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 66-1 and 66-3, and to part (c) of Problem 66-2. Solutions are invited to Problems 63-1 and $63-3$; to all of the problems from issue 65 ; to parts (a) and (b) of Problem 66-2; to Problem 66-4; and to all of the new problems from the present issue 67 .

## Problem 66-1: A Special Property of some Special Linear Groups

Proposed by Rajesh Pereira, University of Guelph, Guelph, Canada, pereirar@uoguelph.ca
Let $n \geq 2$ and $\mathbb{F}_{2}$ be the two-element field. Show that $S L_{n}\left(\mathbb{F}_{2}\right)$, the group of $n \times n$ invertible matrices with entries in $\mathbb{F}_{2}$, has a generating set all of whose elements lie in the same conjugacy class of $S L_{n}\left(\mathbb{F}_{2}\right)$.

Solution 66-1 by Hayden Julius, Kent State University, Kent, Ohio, USA, hjulius@kent.edu
First observe that $S L_{n}\left(\mathbb{F}_{2}\right)$ is generated by the elementary matrices

$$
E_{i j}=I_{n}+e_{i j}, \quad(i \neq j)
$$

where $I_{n}$ is the $n \times n$ identity matrix and $e_{i j}$ is the matrix with 1 in the $(i, j)$-entry and zeros elsewhere. Indeed, if $A \in S L_{n}\left(\mathbb{F}_{2}\right)$, then there exist $r$ elementary matrices $E_{1}, \ldots, E_{r}$ such that $E_{1} \cdots E_{r} A=I_{n}$. Hence, $A=E_{r}^{-1} \cdots E_{1}^{-1}$. Since we are working over $\mathbb{F}_{2}$, it is easy to verify that $E_{i j}^{-1}=E_{i j}$ for all $i \neq j$, so $A$ is a product of elementary matrices.

It remains to show that $E_{i j}$ and $E_{k l}$ are conjugate for all pairs $(i, j)$ and $(k, l)$ with $i \neq j$ and $k \neq l$. Suppose first that $n=2$. Then the matrices

$$
E_{12}=\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right] \quad \text { and } \quad E_{21}=\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]
$$

are similar via conjugation by the matrix $S=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]=e_{12}+e_{21}$. (We point out that, as a multiplicative group, $S L_{2}\left(\mathbb{F}_{2}\right)$ is isomorphic to the symmetric group on 3 letters, where $E_{12}$ and $E_{21}$ are two generating transpositions that are similar via conjugation by a third transposition.) This resolves the case $n=2$.

Suppose now that $n=3$. Throughout this paragraph, let $i, j, k \in\{1,2,3\}$ be distinct. Notice that $E_{i j}$ and its transpose $E_{j i}$ are similar via conjugation by the invertible matrix $e_{i j}+e_{j i}+e_{k k}$. Now suppose we have two elementary matrices $E_{i j}$ and $E_{i k}$ in the same row. Then $E_{i j}$ and $E_{i k}$ are similar via conjugation by the invertible matrix $e_{i i}+e_{j k}+e_{k j}$. This establishes the case $n=3$ since, to show that $E_{i j}$ and $E_{k j}$ (same column) are similar, we may note that $E_{i j}$ is similar to $E_{j i}$ (transpose), which is similar to $E_{j k}$ (same row), which is similar to $E_{k j}$ (transpose).

The previous two paragraphs have provided the conjugating matrices for $E_{i j}$ and $E_{k l}$ whenever the pairs $(i, j)$ and $(k, l)$ have at least one index in common; just add additional 1 s on the diagonal to ensure that the conjugating matrix is invertible. Thus, for $n \geq 4$, we need only determine a conjugating matrix when $i, j, k$, and $l$ are distinct. In this case,

$$
e_{i k}+e_{j l}+e_{k i}+e_{l j}+\sum_{q \notin\{i, j, k, l\}} e_{q q}
$$

is an invertible matrix that conjugates $E_{i j}$ to $E_{k l}$. Hence, the elementary matrices $\left\{E_{i j}\right\}_{i \neq j}$ form a generating set for $S L_{n}\left(\mathbb{F}_{2}\right)$ and all belong to the same conjugacy class.

Also solved by Eugene A. Herman, Grinnell College, Grinnell, Iowa, USA, eaherman@gmail.com

## Problem 66-2: The Cardinality of Hadamard Matrices

Proposed by Richard William Farebrother, Bayston Hill, Shrewsbury, England, R.W.Farebrother@hotmail.com
Let $n$ be an even number. An $n \times n$ matrix $H$ is called a Hadamard matrix if its entries lie in the set $\{-1,1\}$ and it satisfies $H H^{T}=n I_{n}$.
(a) Show that the number of $n \times n$ Hadamard matrices is divisible by $2^{2 n-1}(n-1)$ !.
(b) Show that the number of $n \times n$ Hadamard matrices is exactly $2^{2 n-1}(n-1)$ ! if and only if $n=2$ or 4 .
(c) Show that the number of $4 \times 4$ symmetric Hadamard matrices is exactly 64 .

Editor's note: This is a solution to part (c) of the problem only. The solver has noted that the number of $4 \times 4$ symmetric Hadamard matrices is 64 and not 32; the above statement of part (c) is the corrected version.

Solution 66-2c by Subhasish BeHERA, Indian Institute of Technology, Bhubaneswar, India, sb52@iitbbs.ac.in
We observe that the number of $4 \times 4$ symmetric Hadamard matrices is exactly 64 , not 32 . The proof follows.
Let $H=\left[h_{i j}\right]$ be a symmetric Hadamard matrix of order 4 and $R_{1}, R_{2}, R_{3}$, and let $R_{4}$ be the rows of $H$. We denote $t\left(R_{i}\right)$ the number of +1 entries present in row $R_{i}$. Observe that if $t\left(R_{i}\right)=0$ or 4 , then $t\left(R_{j}\right)=2$ for $i \neq j$, while if $t\left(R_{i}\right)=2$, then $t\left(R_{j}\right)=0,2$ or 4 for $i \neq j$. Similarly, if $t\left(R_{i}\right)=1$ or 3 , then $t\left(R_{j}\right)=1$ or 3 for $i \neq j$. By $\left|H_{u_{i}}\right|$, we let $t\left(R_{i}\right)$ denote the number of symmetric Hadamard matrices $H$ whose first row is $u_{i}$.

Case 1. $t\left(R_{1}\right)=0,2$ or 4 .
In this case there are $\binom{4}{4}+\binom{4}{2}+\binom{4}{0}=8$ choices for $R_{1}$ as arranged below:

$$
\left.\begin{array}{l}
u_{1}=\left(\begin{array}{llll}
1 & 1 & 1 & 1
\end{array}\right), u_{2}=\left(\begin{array}{llll}
1 & 1 & -1 & -1
\end{array}\right), u_{3}=\left(\begin{array}{llll}
1 & -1 & 1 & -1
\end{array}\right), u_{4}=\left(\begin{array}{lllll}
1 & -1 & -1 & 1
\end{array}\right) \\
-u_{2}=\left(\begin{array}{llll}
-1 & -1 & 1 & 1
\end{array}\right),-u_{3}=\left(\begin{array}{llll}
-1 & 1 & -1 & 1
\end{array}\right),-u_{4}=\left(\begin{array}{llll}
-1 & 1 & 1 & -1
\end{array}\right), \text { and } \\
-u_{1}=\left(\begin{array}{lll}
-1 & -1 & -1
\end{array}-1\right.
\end{array}\right) .
$$

Note that $u_{j}^{T} u_{i}=\left(-u_{j}\right)^{T} u_{i}=0$ for $i \neq j$.
Now, if $R_{1}=u_{1}$, then $h_{21}=h_{12}=1$, and hence there are 3 choices for $R_{2}$ and those are $u_{2}, u_{3}$, and $u_{4}$. Then we have the following cases.
(i) If $R_{2}=u_{2}$, then the first two components of $R_{3}$ must be 1 and -1 . Thus, for $R_{3}$ the two choices are $u_{3}$ and $u_{4}$. Further, if $R_{3}=u_{3}$, then $R_{4}=u_{4}$, and if $R_{3}=u_{4}$, then $R_{4}=u_{3}$.
(ii) If $R_{2}=u_{3}$, then the first two components of $R_{3}$ must both be 1 . So the only choice for $R_{3}$ is $u_{2}$ and then $R_{4}=u_{4}$.
(iii) Similarly, if $R_{2}=u_{4}$, then the only choice for $R_{3}$ is $u_{3}$ which implies that $R_{4}=u_{2}$.

If $R_{1}=u_{2}$, then $h_{21}=h_{12}=1$, and then there are exactly 3 choices for $R_{2}$, namely $u_{1}, u_{3}$, and $u_{4}$.
(i) If $R_{2}=u_{1}$, then the first two components of $R_{3}$ must be -1 and 1 . Hence, for $R_{3}$ there are two choices, $-u_{3}$ and $-u_{4}$. If $R_{3}$ is $-u_{3}$, then $R_{4}=-u_{4}$, while if $R_{3}$ is $-u_{4}$, then $R_{4}=-u_{3}$.
(ii) If $R_{2}=u_{3}$, then the first two components of $R_{3}$ must be -1 and 1 . So the only choice for $R_{3}$ is $-u_{4}$. In that case, $R_{4}$ must be $-u_{1}$.
(iii) Similarly, if $R_{2}=u_{4}$, then the only choice for $R_{3}$ is $-u_{1}$ which implies that $R_{4}=-u_{3}$.

If $R_{1}=u_{3}$, then $h_{21}=h_{12}=-1$ and hence there are 3 choices for $R_{2}$ and those are $-u_{1},-u_{2}$, and $-u_{4}$.
(i) If $R_{2}=-u_{1}$, then for $R_{3}$ the only choice is $u_{4}$ and in that case $R_{4}$ must be $-u_{2}$.
(ii) If $R_{2}=-u_{2}$, then the only choice for $R_{3}$ is $u_{1}$. Then $R_{4}$ must be $-u_{4}$.
(iii) Similarly, if $R_{2}=-u_{4}$, then the two choices for $R_{3}$ are $u_{1}$ and $u_{2}$. Now, if $R_{3}=u_{1}$, then $R_{4}=-u_{2}$, while if $R_{3}=u_{2}$, then $R_{4}=-u_{1}$.

If $R_{1}=u_{4}$, then $h_{21}=h_{12}=-1$ and hence there are exactly 3 choices for $R_{2}$, namely $-u_{1},-u_{2}$, and $-u_{3}$.
(i) If $R_{2}=-u_{1}$, then the first two components of $R_{3}$ must be -1 and -1 . So the only choice for $R_{3}$ is $-u_{2}$. In that case the unique choice for $R_{4}$ is $u_{3}$.
(ii) Similarly, if $R_{2}=-u_{2}$, then the only choice for $R_{3}$ is $-u_{3}$. This implies that $R_{4}=u_{1}$.
(iii) If $R_{2}=-u_{3}$, then there are two choices for $R_{3}$, namely $-u_{1}$ and $-u_{2}$. If $R_{3}=-u_{1}$, then $R_{4}=u_{2}$ and if $R_{3}=-u_{2}$, then $R_{4}=u_{1}$

Thus, $\left|H_{u_{i}}\right|=4$ for $i \in\{1,2,3,4\}$. By using arguments similar to those above, we can prove that $\left|H_{-u_{i}}\right|=4$ and that if $H_{1}, H_{2}, H_{3}$, and $H_{4}$ are the four symmetric Hadamard matrices of order 4 whose first row is $u_{i}$, then the choices for Hadamard matrices whose first row is $-u_{i}$ are $-H_{1},-H_{2},-H_{3}$, and $-H_{4}$. Thus, there are exactly 32 symmetric Hadamard matrices for the case $t\left(R_{1}\right)=0,2$ or 4 .

Case 2. $t\left(R_{1}\right)=1$ or 3 .
If $t\left(R_{1}\right)=1$ or 3 , then there are exactly $\binom{4}{1}+\binom{4}{3}=8$ choices for $R_{1}$, and those are given below:

$$
\begin{aligned}
& \left(\begin{array}{llll}
1 & -1 & -1 & -1
\end{array}\right),\left(\begin{array}{llll}
-1 & 1 & -1 & -1
\end{array}\right),\left(\begin{array}{llll}
-1 & -1 & 1 & -1
\end{array}\right),\left(\begin{array}{llll}
-1 & -1 & -1 & 1
\end{array}\right),\left(\begin{array}{llll}
-1 & 1 & 1 & 1
\end{array}\right), \\
& \left(\begin{array}{llll}
1 & -1 & 1 & 1
\end{array}\right),\left(\begin{array}{llll}
1 & 1 & -1 & 1
\end{array}\right),\left(\begin{array}{llll}
1 & 1 & 1 & -1
\end{array}\right) \text {. }
\end{aligned}
$$

In a similar way as in Case 1, it can be shown that there are exactly 32 symmetric Hadamard matrices for the case $t\left(R_{1}\right)=1$ or 3 . This gives a total of 64 distinct $4 \times 4$ symmetric Hadamard matrices in total.

## Problem 66-3: A Set of Circles that Form an Ellipse

Proposed by Rajesh Pereira, University of Guelph, Guelph, Canada, pereirar@uoguelph.ca
Let $k$ be a positive real number and, for any fixed $t \in[0,1]$, let $C_{t}$ be the circle $(x-t)^{2}+y^{2}=k t-k t^{2}$. Show that $\bigcup_{t \in[0,1]} C_{t}$ is a solid ellipse.

Solution 66-3 by Eugene A. Herman, Grinnell College, Grinnell, Iowa, USA, eaherman@gmail.com
The bounding ellipse is

$$
\frac{4}{k+1}\left(x-\frac{1}{2}\right)^{2}+\frac{4}{k} y^{2}=1
$$

Its center is $\left(\frac{1}{2}, 0\right)$, and its foci are $(0,0)$ and $(1,0)$. Since $C_{0}$ and $C_{1}$ are just the foci $(0,0)$ and $(1,0)$, we need only consider the circles $C_{t}$ for $0<t<1$. Let $\alpha$ be the unique angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $t=\frac{1}{2}+\frac{1}{2} \sin \alpha$. Then $t-t^{2}=\frac{1}{4}-\left(t-\frac{1}{2}\right)^{2}=\frac{1}{4} \cos ^{2} \alpha$, and so we can parametrize the collection of all $C_{t}$ for $0<t<1$ as

$$
x=\frac{1}{2}+\frac{1}{2} \sin \alpha+\frac{\sqrt{k}}{2} \cos \alpha \cos \theta, \quad y=\frac{\sqrt{k}}{2} \cos \alpha \sin \theta, \quad \text { where }-\frac{\pi}{2}<\alpha<\frac{\pi}{2}, 0 \leq \theta<2 \pi
$$

Also, we can parametrize the solid ellipse as

$$
x=\frac{1}{2}+\frac{\sqrt{k+1}}{2} r \cos \phi, \quad y=\frac{\sqrt{k}}{2} r \sin \phi, \quad \text { where } 0 \leq r \leq 1,0 \leq \phi<2 \pi
$$

We must show that these two parametrizations describe exactly the same set. However, if we let $u=\frac{2}{\sqrt{k+1}}\left(x-\frac{1}{2}\right)$ and $v=\frac{2}{\sqrt{k}}$, we can instead show the equivalence of

$$
u=\frac{\sin \alpha}{\sqrt{k+1}}+\sqrt{\frac{k}{k+1}} \cos \alpha \cos \theta, \quad v=\cos \alpha \sin \theta, \quad \text { where }-\frac{\pi}{2}<\alpha<\frac{\pi}{2}, 0 \leq \theta<2 \pi
$$

and the unit disk

$$
u=r \cos \theta, \quad v=r \sin \theta, \quad \text { where } 0 \leq r \leq 1,0 \leq \phi<2 \pi .
$$

Every point $(u, v)$ described by the first of these two parametrizations is also described by the second, since such a point $(u, v)$ lies in the unit disk:

$$
\begin{aligned}
1- & \left(u^{2}+v^{2}\right)=1-\left(\frac{\sin \alpha}{\sqrt{k+1}}+\sqrt{\frac{k}{k+1}} \cos \alpha \cos \theta\right)^{2}-(\cos \alpha \sin \theta)^{2} \\
& =\frac{1}{k+1}\left(k+1-\sin ^{2} \alpha-k \cos ^{2} \alpha \cos ^{2} \theta-2 \sqrt{k} \sin \alpha \cos \alpha \cos \theta-(k+1) \cos ^{2} \alpha \sin ^{2} \theta\right) \\
& =\frac{1}{k+1}\left(k+\cos ^{2} \alpha-k \cos ^{2} \alpha-\cos ^{2} \alpha \sin ^{2} \theta-2 \sqrt{k} \sin \alpha \cos \alpha \cos \theta\right) \\
& =\frac{1}{k+1}\left(k \sin ^{2} \alpha+\cos ^{2} \alpha \cos ^{2} \theta-2 \sqrt{k} \sin \alpha \cos \alpha \cos \theta\right) \\
& =\frac{1}{k+1}(\sqrt{k} \sin \alpha-\cos \alpha \cos \theta)^{2} \geq 0 .
\end{aligned}
$$

For the opposite direction, let $(u, v)$ be any point in the unit disk and let $r=\sqrt{u^{2}+v^{2}}$. Choose $\alpha \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$
\sin \alpha=\frac{u+\sqrt{k} \sqrt{1-r^{2}}}{\sqrt{k+1}}
$$

and choose $\theta \in[0,2 \pi)$ with $\sin \theta=\frac{v}{\sqrt{1-\sin ^{2} \alpha}}$. (Note: For $\alpha$ and $\theta$ to exist requires that $\left|u+\sqrt{k} \sqrt{1-r^{2}}\right| / \sqrt{k+1}<1$ and $v^{2} \leq 1-\sin ^{2} \alpha$. The solver has chosen to omit the tedious details, but claims to have confirmed this.) Since $\alpha \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\cos \alpha>0$, and so $v=\sqrt{1-\sin ^{2} \alpha} \sin \theta=\cos \alpha \sin \theta$. It remains to check the formula for $u$. First compute

$$
\begin{aligned}
\cos \alpha|\cos \theta| & =\cos \alpha \sqrt{1-\sin ^{2} \theta} \\
& =\cos \alpha \sqrt{1-\frac{v^{2}}{\cos ^{2} \alpha}} \\
& =\sqrt{1-\sin ^{2} \alpha-v^{2}} \\
& =\frac{1}{\sqrt{k+1}} \sqrt{(k+1)\left(1-v^{2}\right)-\left(u^{2}+k\left(1-r^{2}\right)+2 u \sqrt{k} \sqrt{1-r^{2}}\right)} \\
& =\frac{1}{\sqrt{k+1}} \sqrt{k u^{2}+1-r^{2}-2 u \sqrt{k} \sqrt{1-r^{2}}} \\
& =\frac{\left|\sqrt{k} u-\sqrt{1-r^{2}}\right|}{\sqrt{k+1}} .
\end{aligned}
$$

Our above definition of $\theta$ was not unique, since we had not yet specified the $\operatorname{sign}$ of $\cos \theta$. We now choose $\cos \theta \geq 0$ if $\sqrt{k} u \geq \sqrt{1-r^{2}}$ and $\cos \theta<0$ if $\sqrt{k} u<\sqrt{1-r^{2}}$. Therefore

$$
\cos \alpha \cos \theta=\frac{\sqrt{k} u-\sqrt{1-r^{2}}}{\sqrt{k+1}}
$$

and so

$$
\begin{aligned}
\frac{\sin \alpha}{\sqrt{k+1}+\sqrt{\frac{k}{k+1}} \cos \alpha \cos \theta} & =\frac{1}{k+1}\left(u+\sqrt{k} \sqrt{1-r^{2}}+\sqrt{k}\left(\sqrt{k} u-\sqrt{1-r^{2}}\right)\right) \\
& =\frac{(1+k) u}{k+1}=u
\end{aligned}
$$



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## IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce four new problems in this issue and invite readers to submit solutions for publication in IMAGE.
Submissions: Please submit proposed problems and solutions in macro-free $\mathrm{IAT}_{\mathrm{E}} \mathrm{X}$ along with the PDF file by e-mail to $I M A G E$ Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

New Problems:

Problem 67-1: Integer Solutions of a Matrix Equation
Proposed by Gérald Bourgeois, Université de la Polynésie française, FAA'A, Tahiti, Polynésie française, bourgeois. gerald@gmail.com

Let $n$ be a positive integer, and let $J_{n}$ be the $n \times n$ matrix all of whose entries are equal to 1 .
(a) Show that there exists a matrix $X \in M_{n}(\mathbb{Z})$ such that $X^{2}+X=J_{n}$ if and only if $n=m^{2}+m$ for some $m \in \mathbb{Z}$.
(b) When $n$ is of the form $m^{2}+m$, find the number of $n \times n$ zero-one matrices $X$ which solve $X^{2}+X=J_{n}$.

## Problem 67-2: A Solution to Two Matrix Equations

Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, obaksalary@gmail.com and Götz Trenkler, Dortmund University of Technology, Dortmund, Germany, trenkler@statistik.tu-dortmund.de Let $A$ be an $m \times n$ complex matrix. Find all matrices $X$ such that $A X=P_{A}$ and $X A=P_{A^{*}}$, where $P_{A}$ and $P_{A^{*}}$ denote the orthogonal projectors onto the column spaces (ranges) of $A$ and $A^{*}$, respectively.

## Problem 67-3: A Numerical Range Area Problem

Proposed by Rajesh Pereira, University of Guelph, Guelph, Canada, pereirar@uoguelph.ca
Let $M$ be a $2 \times 2$ submatrix of a $3 \times 3$ unitary matrix. Show that the area of the numerical range of $M$ is less than or equal to $\frac{\pi}{4}$.

Problem 67-4: The Schur Product of a Nilpotent Matrix and its Transpose
Proposed by Rajesh Pereira, University of Guelph, Guelph, Canada, pereirar@uoguelph.ca
Let $N$ be any nilpotent real matrix. Show that the Schur (entrywise) product of $N$ with its transpose can never be a positive definite matrix.

Solutions to Problems 66-1, 66-2c and 66-3 are on pages 25-28.

