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IMAGE

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Editor-in-chief: Louis Deaett (`louis.deaett@quinnipiac.edu`).

Contributing editors: Adam Berliner (`berliner@stolaf.edu`), Sebastian Cioabă (`cioaba@udel.edu`), Anthony Cronin (`anthony.cronin@ucd.ie`), Jephian C.-H. Lin (`jephianlin@gmail.com`), Rajesh Pereira (`pereirar@uoguelph.ca`), and Amy Wehe (`awehe@fitchburgstate.edu`).

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FEATURE INTERVIEW

"For me this is as good as it gets"

Leslie Hogben Interviewed by Shaun Fallat[1]

Leslie Hogben at Moraine Lake in Banff National Park, during the 2016 BIRS FRG The Inverse Eigenvalue Problem of a Graph.

S.F. - Can you tell me about your educational background and how you came to be interested in the subjects of linear algebra and graph theory?

L.H. - I had a very fortunate background and grew up in an academic family. The high school I attended offered lots of opportunities and my wonderful math teacher mentored me in a directed discovery project. This early research-type experience left a lasting impression on me. After high school, I went to Swarthmore College, which was a great experience as well, and during my undergraduate studies, I spent one year at the University of Warwick. For graduate school, I attended Yale University, which was a great choice for me, and worked under the supervision of Dr. Nathan Jacobsen. On a personal note, I met my future husband at Yale, which further added to the positive experience.

After receiving my doctorate, I accepted a tenure-track instructor position at Iowa State University. I was not interested in pursuing a postdoctoral position, even though there are benefits to a postdoc; today one normally has to take a postdoctoral position before moving into a tenure-track position at a doctoral university. My aspirations then were both research- and family-oriented and my plan was to get tenure, have a child, and stay at home and focus on my child, to which Iowa State was amenable. At the time I arrived, Anne Steiner was a female full professor who soon became department chair, and she served as a very important mentor for me and a key supporter of my personal/professional plans. Another important thing was that my husband, Mark, chose to give up a career in mathematics (because it was almost impossible for a husband and wife to be hired together) and pursue a law career. My daughter was born in 1985 and I took two years of leave without pay. I transitioned back to a full-time faculty position after a few years of working part time.

I did essentially no research for about 10 years. My doctoral dissertation was in the area of abstract algebra; however, my first love was graph theory, which was nurtured during my high school experience. I enjoyed studying abstract algebra and found the community to be pleasant. But I was happy doing my own thing, which, looking back on it now, was probably a mistake. Upon my return to research, I had to reinvent myself and, given my strict schedule for the sake of my family, it worked well for me to join the active Linear Algebra Seminar run by Bryan Cain. This seminar was extremely welcoming and had a flexible schedule, which was very important to me. It was during this seminar that I became very interested in a number of topics in linear algebra. Throughout my career, I have had a fabulous series of support networks that provided me with many opportunities and welcomed my choices, from high school through to becoming a research mathematician in multiple areas.

S.F. - I believe some of your early research in matrix theory was on the topic of matrix completion problems. What led you to conduct research on such problems?

L.H. - My interest in matrix completion problems was born in the Linear Algebra seminar at Iowa State University. I first joined a research group that grew from the seminar, where the topic considered was aspects of stability and

[1]Department of Mathematics and Statistics, University of Regina, Canada; shaun.fallat@uregina.ca

convergence, but at some point there was a lecture given on a matrix completion problem paper of Charlie Johnson and Brenda Kroschel. During this lecture, I realized that for me this is as good as it gets, namely an interplay between linear algebra and graph theory. I became very interested in this topic and managed to contribute some papers on my own initially.

About this time, I acquired a doctoral student, Amy Wangsness, and together with Luz DeAlba of Drake University (another participant in the linear algebra seminar) we began a research project involving certain types of completion problems. This group of three developed into a functioning research group, where others were invited to join and contribute, and later became known as the Combinatorial Matrix Theory Research Group at Iowa State University. This group met for years, but no official credit was offered to any of the participants. Around 2007, as part of a workforce grant proposal, we figured out a way to enable everyone (both faculty and students) to get credit and we called it Early Graduate Research, EGR, pronounced "eager".

Parallel to the research being performed locally, I also started to regularly attend research conferences in linear algebra. The first one was the SIAM-ALA conference in 2000. At this meeting, Luz, together with Jane Day, essentially adopted me. They were wonderful to interact with and helped me feel more comfortable talking with others in linear algebra. In 2001, I attended, on my own, a conference on combinatorial matrix theory in Pohong, Korea. I was somewhat shy and reserved around people I didn't know, but many of the participants at this conference were so nice and welcoming (including you, Richard Brualdi, Bryan Shader, and others) that it left a lasting impression with me. I should say that I always valued the research at conferences, but I did not like interacting with people there, I was just not that comfortable talking to people I didn't know. I learned how to deal with this reluctance by relying on my experience as a mother of a daughter in a figure skating club where I had to chat up the other mothers and begin managing ice shows, etc., so that my daughter's opportunities on the ice were not negatively impacted by my reserved nature. In fact, I later realized that the same skills I acquired managing ice shows nicely translated to organizing all types of mathematics conferences.

S.F. - You have been extremely prolific in both the graph theory and linear algebra research communities. Can you list some of what you feel are your most significant research accomplishments in linear algebra?

L.H. - This is difficult for me to answer... I guess helping shape the landscape of the current research enterprise that is minimum rank, zero forcing, and the inverse eigenvalue problem for graphs. Some of the collaborative projects that come to mind are the two recent contributions to the strong spectral and multiplicity properties of graphs and the original paper on zero forcing from the 2006 workshop at the American Institute of Mathematics. I suppose some hidden gems might be the team effort at the Banff International Research Station where we proved equality holds between the maximum nullity and zero forcing number of the complete edge subdivision of a graph, and an earlier paper on strong eventually nonnegative matrices. (**S.F. - I thought you might also mention the *Handbook of Linear Algebra* here...**) Well, since the question focussed on research, I was not sure if the *Handbook* applied. But yes, this is a significant accomplishment in linear algebra, but again it was not mine alone. This project was a wonderful synergy that I benefited from, and which benefits the community. Working on this project broadened my horizon and I learned a lot about the vast scope of linear algebra by editing this book.

S.F. - For a long while now you have been a trailblazer on inclusion and diversity, particularly in mathematics. Can you explain why this is important to you both as a mentor and as a researcher?

L.H. - Inclusion and Diversity were core values of the family that I was raised in... Early on, I worked on these matters locally, where impact was easily seen (that is, working with individual students and faculty at Iowa State). Later on, I became very involved in supporting students in our graduate program, and am now the Associate Dean in Graduate Studies and Faculty Development of the College of Liberal Arts and Sciences, where I continue to push my core values to support graduate students more broadly.

Concerning diversity, in my early days I was unaware of the differences in experience, education, and so on, and the impact those differences would have on individual success. I have had to learn and incorporate this in many things that I do each day. This also led me to focus on what I can control and not worry about other things. My core values mean that mentoring the next generation and broadening participation are interlaced. With the reinvention of my research career, as I mentioned, came a very nice and more collaborative model, working in teams and such. But I think it is very important to note that a lot of this expanded growth of collaboration within my research program occurred for me post-tenure, and extensive collaborative research was not the norm in the days before I got tenure. Nowadays, collaborative research is becoming more accepted (and sometimes expected) for young people. There is some concern that demonstration of independent research programs may be less clear, but still such collaborative research efforts are generally encouraged. I will say though that there is research out there that suggests subjects which institute alphabetizing of the authorship can in particular devalue the contributions from women and it is not a leap to suggest that the same holds for other marginalized groups as well. However, I will say that I like the alphabetized authorship model used in mathematics and how it de-emphasizes the need to keep score, for example.

It was my former student Amy that pointed out to me when she was preparing to graduate that there were very few other female graduate students in the program after her year, or choosing to enter our graduate program. From this moment on, I paid more attention to the importance and continual positive impact of recruiting a diverse group of students. This has helped me in my current position as Associate Director for Diversity at the American Institute of Mathematics. For me personally, it is wonderful and fun to mentor students and watch them blossom as researchers and teachers. It is exciting to see them go off, mature, and thrive. I find it so rewarding to see how much my students have grown. When it comes to recruiting students for the program (or taking on an advisee), the main question I ask myself is: Can our program (or can I as an advisor) meet the student's needs? Of course, capacity is an issue, and the individual merit of a student is always important. However, broadening your pool and building relationships that let you supervise qualified students with diverse backgrounds leads to a better understanding of the impact of the various experiences of students and helps me understand that life is not a level playing field.

S.F. - You edited two editions of the *Handbook of Linear Algebra*. How did this come about and what impact do you think this significant accomplishment has had on the linear algebra community?

L.H. - I was recruited to serve as editor for the *Handbook of Linear Algebra* (*HLA*) by Bob Stern of CRC Press, but I was concerned I did not have the appropriate level of visibility in linear algebra to make this successful. I contacted (by email) Richard Brualdi to join me as an Associate Editor for *HLA*. Two days later, I received a reply that began, "I am so sorry" and I assumed he was declining, but the reply continued, "for the slow reply"! He went on to explain that he had been stuck in an airport and would be happy to help with *HLA*. I was also fortunate to be able to recruit Anne Greenbaum and Roy Mathias as Associate Editors for the first edition and G. W. (Pete) Stewart as an Associate Editor for the second edition. I am very grateful to the Associate Editors, especially Richard, for their help on this project. I am also eternally grateful to the authors that contributed to *HLA*. It truly was a great experience for me and again I was fortunate to have been working with and supported by an excellent team.

S.F. - Can you talk about your role(s) in the Association for Women in Mathematics?

L.H. - I was a latecomer to the Association for Women in Mathematics (AWM). I was oblivious to such things as the effect of gender in mathematics early in my career. I joined the AWM sometime after 2000, after Amy pointed out to me their impact on others. I now realize I have been a beneficiary of the AWM's efforts and learned afterwards that I had been noticed early on (just after completing my degree) by the one of the leaders of the AWM – there were not many women in mathematics at the time! In the last 10 years, I have become very involved in the AWM, serving on a number of committees. Most recently, I am on the organizing committee for the 50$^{\text{th}}$ Anniversary Research Symposium, which was supposed to take place in 2021 but will now take place in June of 2022 at the Institute for Mathematics and its Applications.

S.F. - You are the Associate Director of Diversity at the American Institute of Mathematics (AIM). What is the purpose of this post and what about this job do you find most interesting and most rewarding?

L.H. - My involvement with AIM started with the 2006 AIM workshop on minimum rank. I was forwarded an email from my department chair at the time with a call for proposals for AIM workshops and, with me being largely unaware of NSF-funded institutes, a lightbulb went on. I was so excited about the prospect of organizing a workshop where you actually did research instead of just attending talks. I then contacted Bryan Shader who contacted Richard Brualdi and we submitted a proposal together. After that workshop in October of 2006, AIM was searching for a new deputy director and they were reaching out to former participants and organizers. They asked me about this position, but it required a move to California, which was something I could not do, so they asked if I would be interested in the part-time position as Associate Director for Diversity.

One of the things at AIM I find rewarding is the opportunity to help broaden participation in workshops. Working with the Deputy Director, we identify and encourage participation by contacting possible applicants, and sometimes we suggest names of possible participants to the organizers of workshops. I am really excited about being a part of creating the Research Experiences for Undergraduate Faculty (REUF) program at AIM, which is aimed at supporting faculty from universities and colleges in the U.S. that do not have doctoral programs. (Historically, research in the U.S. tended to be concentrated at universities with doctoral programs, although that is changing.) This program started with Roselyn Wilson of Florida A&M, when she mentioned to me that there are wonderful things happening for undergraduates and research (e.g., NSF REU programs), but there is essentially nothing for faculty at undergraduate institutions. I suggested she apply to AIM for a workshop, and she did. Now this program is a joint initiative of AIM and ICERM with funding provided by the NSF, and Brianna Donaldson of AIM and Ulrica Wilson of ICERM direct this program with me. Of course, REUF has evolved over time, but it still focuses on supporting and equipping faculty with the ability to conduct research with their undergraduates. Faculty are invited to AIM or ICERM for a week-long research-intensive workshop (we use an REU model, where research is learned by actually doing research) led by research-active faculty who have experience doing research with undergraduates. REUF also helps support research program renewals and research area transitions for faculty who have significant constraints imposed by their positions.

Finally, I must say it is so fun being a part of AIM and working with such great people!

S.F. - Can you tell us about your time as Secretary/Treasurer at ILAS? Volunteering for ILAS?

L.H. - My involvement in ILAS started differently than my involvement with AWM. Early on in my linear algebra career, I became aware of ILAS (probably from Bryan Cain) and joined it. I started attending ILAS conferences regularly and, at the 2005 conference in Regina (co-organized by you), I think it was either ILAS President Danny Hershkowitz or Secretary/Treasurer Jeff Stuart that announced a call for a volunteer to assist Jeff as Secretary/Treasurer of ILAS. I immediately walked up and volunteered. Later, in 2009, I became the Secretary/Treasurer of ILAS. I was on my own for a while, but that was okay, as I was in a position where I had access to significant professional funds through ISU and could hire help as needed from time to time. Not everyone in this post has that option, but I was happy to do it, as I loved ILAS! In 2018, Minerva Catral became the Assistant Secretary/Treasurer and was very helpful; she became Secretary/Treasurer in 2021. I should say, though, I do think in general having an Assistant Secretary/Treasurer of ILAS is very beneficial, and I am so happy with the current team of Minnie and Mike Tait.

Volunteering for ILAS has an obvious benefit to the Society since it supports such a wonderful organization. However, over time I think it was an important step for ILAS to recognize the need to help its volunteers in *ELA* by hiring professional typesetting services. Down the road, it may be reasonable to support the administrative functions of ILAS, which are currently carried out primarily by the President and the Secretary/Treasurer. There may also be benefits to the members volunteering as well. Depending on the role, value may be recognized by your department, which in turn may have implications on the roles you might choose to volunteer for before and after tenure. . .

S.F. - You have supervised many high-quality personnel, including as graduate students and postdocs. Can you tell us about some of your most positive experiences and what qualities you developed to mentor both strong research personnel and those that needed a bit more careful guidance?

L.H. - One of the great things about mentorship is that the learning goes both ways. I have been advised by students and I advise students. . . My mentorship is student-focused and is certainly impacted by their career goals. If a student is interested in academic pursuits, then I insist on a focus on research and teaching with at least one solo-authored paper from their thesis research. All of my students are involved with team-based research. However, if a student is more interested in a position in industry or government, then a team-based research experience is more important, including some demonstration of group research leadership. I am proud of all of my students and I think it is wonderful to see the variety of careers they have all undertaken. I think it is important to support their goals in the style that best supports their progress. You may need to adjust your mentoring style depending on the student so they move forward and you can help them succeed.

S.F. - I was part of a team that hosted the ILAS 2005 conference in Regina and, while it was a huge job, it was also so rewarding and satisfying and I was proud to host my friends and colleagues in Regina. In 2017, you were part of an organizing team that hosted an ILAS conference. Please tell us about how much work was done and if you found the overall experience to be worthwhile. Advice for others considering taking on such a task?

L.H. - Yes, I will say it was worthwhile and exciting and I enjoyed showcasing Iowa State... But the key for me was the ability to share the work required. It is important to recognize that access to resources (financial and other) and surrounding yourself with a competent team are essential when organizing such a conference. I had access to the wonderful services of the Ames convention bureau and relied on them a great deal. It turned out that ISU was a member of the Fields Institute in Canada, so we could apply to them for some funding (which went to good use), but for us it was access to their abstract submission system that we relied upon most heavily. (I learned about this system through your ILAS conference in 2005, incidentally.) I must say that organizing the ILAS conference for more than 300 people in 2017 was less work for me than organizing the small (less than 40 people) conference "Topics in Linear Algebra" held in 2002, because I did not have the resources in place in 2002.

Leslie Hogben with (former ISU postdoc) Minerva Catral at ILAS 2019.

S.F. - One versus the other... Research face-to-face, or virtually?

L.H. - I think both are important and each has its own advantages and disadvantages. If we ever return to some version of "normal," we should not strictly return to face-to-face research, but rather we should make use of both research models.

There were some individuals (perhaps those with disabilities, or those not able to travel because of their academic position or family situation) who were negatively impacted by using exclusively face-to-face research models, but with the pandemic and the switch to more virtual research models, they suddenly had access to such experiences. I know at AIM there have been suggestions to not give up on the virtual research experiences entirely as we return to in-person. For me, having an intense in-person collaboration to start things off, followed by a slower pace virtual collaboration is wonderful. The AMS MRC[2] was designed to have that in-person week-long experience. During a yearlong postponement, we did a fair amount of preliminary work virtually. Then in 2021 we got switched to virtual and started with an intense week-long virtual session, followed by a slower paced virtual collaboration. This can work, and indeed has worked for the research part of our MRC (papers have been submitted and more are in progress). But some things were lost, such as most of the networking and professional development. I know some of this did happen within research groups, but the online structure did not facilitate such activities very well.

One other thing to bear in mind is that some people (including me) can only handle reduced hours in front of a computer as opposed to an in-person activity, and when you are virtual it is impossible to get away from home obligations (both personal and professional), so you have less time. So, while we made it work, I would consider an entirely virtual MRC significantly less attractive than being able to meet in person for a week. In contrast, the AIM Research Community (ARC) model, which was designed as a virtual research community, has worked very well. The ARC, which was designed as a long-term less intense research model to accommodate the other demands on people's time, may be transformative down the road.

[2] *Editor's note:* AMS MRC is the Mathematics Research Communities program of the American Mathematical Society. The 2021 MRC community *Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra* was co-organized by Leslie, together with Shaun Fallat, Bryan Shader, H. Tracy Hall, and Michael Young.

FEATURE ARTICLE

Matrix Designs for COVID-19 Group Testing

Johannes J. Brust, University of California San Diego, San Diego, CA, USA, jjbrust@ucsd.edu

1. Introduction. The rapid development of vaccines for COVID-19 has constituted a major breakthrough for the return to pre-pandemic life. However, while vaccines are becoming globally widespread, high alert levels prevail. Even with vaccines, monitoring for the evolution of mutations and detecting new outbreaks call for continued vigilance. Therefore, *testing* is likely to prevail as a vital mechanism to inform decision-making in the near future. In order to conserve scarce testing resources, the Centers for Disease Control and Prevention (CDC) in the United States has, as of 2021, endorsed so-called *group* or *pooling* test methods. (Importantly, group testing is also being deployed worldwide, including in Central Europe, China, India, Israel and more nations.) Such methods date to Robert Dorfman's early work in 1943 [2], and rely on principles that can be expressed using linear algebra. Underlying pooling tests is the observation that to efficiently detect positive cases among a population with a very low occurrence prevalence, it can be advantageous to test groups of samples instead of testing each sample individually (since almost all samples are negative). Pooling test methods have close connections to various areas of mathematics, such as combinatorial design theory, combinatorial matrix theory, and the design of experiments.

This article describes matrix designs which contain all the needed information to represent group tests and are applicable for COVID-19 testing. Specifically, if the group sizes are prime numbers, then modular arithmetic effectively enables the construction of pooling tests. For non-primes, methods based on modular arithmetic cease to apply, even though it remains desirable to have similar constructions. Using an example, we show which property hinders a conventional construction for a non-prime instance, even though an effective design exists. Algorithms that generate binary matrices representing pooling test designs and corresponding decoding routines (which identify positive samples) are given. We also point to practical considerations when developing pooling methods, in particular regarding parameters for setups in a laboratory.

2. Connections to related mathematics. Group testing is closely related to *combinatorial design theory* (see, e.g., [6]), which studies collections of finite sets that exhibit various "balance" properties. Problems in this field are thought to have originated with *Kirkman's schoolgirl problem* (Kirkman 1850), which was formulated as: "Fifteen young ladies in a school walk in groups of three for seven days in succession; it is required to arrange them daily so that no two walk twice in a group" [5]. Solving this problem can be done in a variety of ways, including via algebraic number theory [11] or finite projective geometry and work inspired by this problem has many applications, including to coding theory, cryptography, network design, and computer science.

Solving the schoolgirl problem requires constructing a so-called Kirkman triple system, which is an arrangement of triples in which each pair of elements occurs exactly once. Denoting the 15 persons by letters A–O, one solution is given by [13, 20, 21]:

Sun	ABC	DEF	GHI	JKL	MNO
Mon	ADH	BEK	CIO	FLN	GJM
Tue	AEM	BHN	CGK	DIL	FJO
Wed	AFI	BLO	CHJ	DKM	EGN
Thu	AGL	BDJ	CFM	EHO	IKN
Fri	AJN	BIM	CEL	DOG	FHK
Sat	AKO	BFG	CDN	EIJ	HLM

Kirkman triple systems are special types of Steiner triple systems, which are known to exist when the number of elements $N \equiv 1$ or $3 \pmod 6$ [1]. Recently, Kirkman triple systems have been applied in the construction of pooling tests to detect COVID-19 [17].

3. One-round pooling tests. For a set $\mathcal{S} = \{1, 2, 3, \ldots, N\}$ of samples, a basic pooling test pools (mixes) all samples into one group, and subsequently uses a single test on this group. If the pool is negative, then it can be concluded that each individual sample is negative. If the pool tests positive, then at least one sample is positive, and all samples are retested individually. This strategy is an example of a 2-round (in general multi-stage) pooling test. Based on information from previous testing rounds, multi-stage tests update the samples to be tested in the next round. Such methods enable desirable reductions in the number of total needed tests, but they come at the expense of longer times to receive results (since each round adds approximately the same amount of time). An approach that overcomes potentially long wait times is a one-round scheme. In such schemes, multiple pools, each of size $m < N$, are formed and simultaneously tested. The distinct composition of the pools with respect to their individual samples is essential for one-round methods to exactly identify the true positive samples. The simultaneous identification of the positive samples can be represented by a linear system.

Since being positive (having the disease) or being negative (not having the disease) are binary states, binary digits, i.e., $x \in \{0, 1\}$, will be used throughout. Addition and multiplication are defined by bit-wise OR (+) and AND ($*$) operations, respectively. Most of the arithmetic carries over as expected, with the exception that $1 = 1 + 1$. Suppose now that for a pool size $m \geq 2$, the samples $\mathcal{S} = \{1, 2, 3, \ldots, N = m^2\}$ have been collected. We would like to infer the states of the elements of vector $\mathbf{x} \in \{0, 1\}^N$ based on $M < N$ observed test results, encoded in vector $\mathbf{y} \in \{0, 1\}^M$. Which elements of $\mathbf{x}$ are included in which tests of $\mathbf{y}$ is determined by the binary matrix $\mathbf{M} \in \{0, 1\}^{M \times N}$. Specifically, if test i contains sample c, then $\mathbf{M}_{i,c} = 1$. The Polymerase Chain Reaction (PCR) technology is used to generate the test outcomes in $\mathbf{y}$ (by amplifying the detectable viral load in each pool).

In an example scheme with $M = 6$ pools, each of size $m = 3$ and taken from a total of $N = 9$ samples, a linear system for doing M simultaneous tests (where rows are labeled a–f and columns 1–9) is given by:

$$\begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \\ \text{e} \\ \text{f} \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & & & & 1 & & & & 1 \\ & 1 & & & & 1 & 1 & & \\ & & 1 & 1 & & & & 1 & \\ 1 & & & & & 1 & & 1 & \\ & 1 & & 1 & & & & & 1 \\ & & 1 & & 1 & & 1 & & \end{bmatrix} \end{array} = \mathbf{M} \quad \text{and} \quad \mathbf{M}\mathbf{x} = \mathbf{y} \tag{1}$$

Here, the first row of $\mathbf{M}$ corresponds to pooling samples $(1, 5, 9)$ in test 1 (labeled a). Since the system in (1) is underdetermined, correctly identifying all positives in $\mathbf{x}$ is in general not feasible. However, under the condition that the number of positives is sufficiently low (which is typically the case with COVID-19 samples), certain properties of the pooling matrix $\mathbf{M}$ guarantee uniquely detecting up to $k - 1$ nonzeros, provided the number of positive samples is strictly less than k. (This example has $k = 2$.)

In particular, observe these three properties of $\mathbf{M}$:

P1. Each test contains exactly m samples (m nonzeros in each row).

P2. Each pair (x_i, x_j) of different samples occurs at most once (any pair of columns has at most one common nonzero).

P3. Each sample is contained in k tests (k nonzeros in each column).

When these conditions are met, $\mathbf{M}$ is called a *multipool* matrix, and k is the *multiplicity* of the matrix [15]. That the matrix has multiplicity k implies that the matrix is guaranteed to exactly identify the correct $\mathbf{x}$ when it contains up to $k - 1$ positive items.

A multipool matrix is a specific instance of a *block design* from combinatorial design theory [8, Part II]. Concretely, a balanced incomplete block design (BIBD) is a pair $(S, \mathcal{B})$ where S is a set of N elements and $\mathcal{B}$ is a collection of b m-subsets of S (called blocks) such that each element of S is contained in exactly r blocks and any 2-subset of S is contained in exactly λ blocks. Therefore, a multipool matrix corresponds to a BIBD with $\lambda = 1$ and $r = k$. When $\lambda = 1$, a block design with parameters (t, m, N), where $2 \leq t < m < N$ and every t-subset of S occurring in exactly one block is called a Steiner system and denoted $S(t, m, N)$. Thus, a multipool matrix is also a selection of blocks from an $S(2, m, N)$.

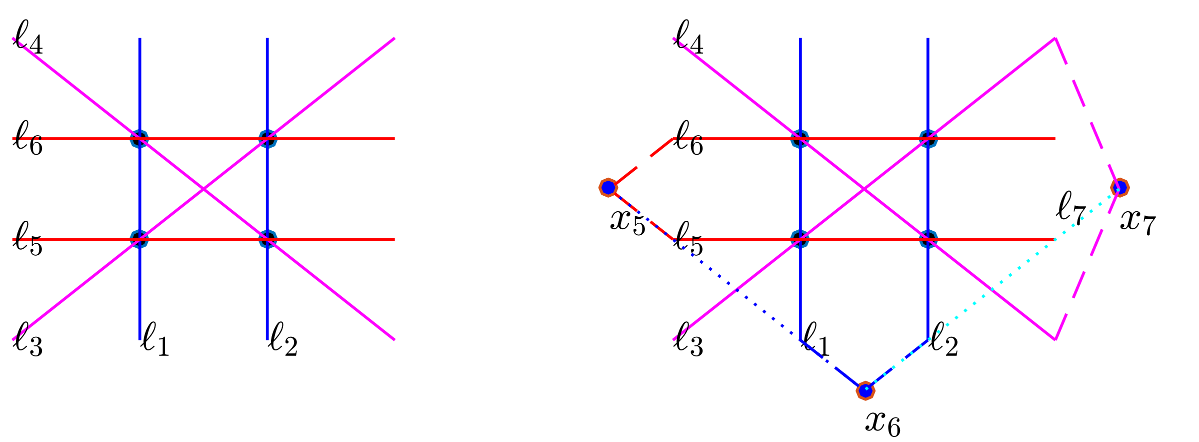
Moreover, a multipool matrix is a member of the family of *d-disjunct* matrices. The degree to which an $M \times N$ binary matrix is d-disjunct is related to the rank of a real matrix in $\mathbb{R}^{M \times N}$. Specifically, a d-disjunct matrix (also called a super-imposed code or d-cover-free family) is characterized by the property that no set of d columns has a boolean sum which is a superset of any other single column (see, e.g., [9]). Note that $\mathbf{M}$ from (1) has the value $d = k - 1 = 1$. In other words, when comparing each pair of columns, there is always at least one 1 in one column in where there is a 0 in the other. However, the matrix is not 2-disjunct, since the boolean sum $\mathbf{M}(:,2) + \mathbf{M}(:,3) = [\ 0\ 1\ 1\ 0\ 1\ 1\ ]^T$ contains every nonzero of $\mathbf{M}(:,4) = [\ 0\ 0\ 1\ 0\ 1\ 0\ ]^T$. Similar to multipool matrices, d-disjunct matrices are known to be able to exactly identify up to d nonzeros in $\mathbf{x}$ (see, e.g., [10, p. 1361]). A construction for forming d-disjunct matrices is given in [4], where the matrix is of size $\binom{n}{d} \times \binom{n}{j}$ based on parameters $d < j < n$. Because the number of columns in the resulting matrix is $\binom{n}{j}$, one may not form $\mathbf{M}$ in (1), for instance. One could, however, form a 1-disjunct matrix of size 6×9 with all column weights 2 by choosing $n = 6$, $j = 2$, and $d = 1$, and then deleting 9 columns. Even though this loses the constant row weight property of a multipool matrix, if one only desires a 1-disjunct matrix, then the construction in [4] does provide one.

The argument (formalized in the proof of [7, Corollary 1]) ensuring that a multipool matrix can correctly recover a solution (up to a certain number of positives) is described as follows. Specifically, suppose there are at most $k-1$ positives among all samples. Select and consider an arbitrary sample. If any of the first $k-1$ tests containing this sample are negative, then the sample must be negative. Otherwise, there are two possible cases. In the first case, the sample was unfortunately always paired with one of the $k-1$ positives, even though it is negative. The second case is that the sample is indeed positive. Hence, based on property P2, if the next test with this sample (i.e., test k) is negative, then the sample must be negative. If the test is positive, then the sample must be positive. By P3, each sample is contained in k tests, and therefore all samples can be identified. Note that P1 is not necessary for correctly decoding the state of a particular sample. However, having uniform pool sizes is a desirable property when assembling pools in laboratory settings, given that equipment and containers typically have uniform layouts.

An effective and fast algorithm to decode the observed $\mathbf{y}$ into $\mathbf{x}$ is called Combinatorial Orthogonal Matching Pursuit (COMP) [12], which declares $x_j = 1$ if all tests in which x_j is contained are positive. This algorithm correctly detects up to $k-1$ positives when $\mathbf{M}$ is a multipool matrix, and is straightforward to implement. The COMP decoding algorithm is, with "`.*`" representing element-wise multiplication:

$$x_j = 1 \quad \text{if} \quad \mathbf{M}_{:,j}\ .*\ \mathbf{y}_{:} = \mathbf{M}_{:,j} \quad 1 \le j \le N.$$

4. Affine and projective planes. Multipool matrices can be constructed based on the three properties P1, P2, and P3, which are closely related to structures from combinatorial design theory. In particular, *affine planes* are objects defined by a finite set of points, lines, and an incidence relation between the points and lines [8]. An affine plane is characterized by the relations that (1) any two distinct points are on precisely one shared line; (2) for any point x outside a line ℓ, there is one line through x that has no point in common with ℓ (such lines are considered parallel); and (3) there exist three points not on a common line. If the number of points in an affine plane is finite, and if one line of the plane contains m points, then four additional important relations hold: each line contains m points; each point is contained in $m+1$ lines; there are m^2 points in total; and there exist a total of $m^2 + m$ lines. The value m is called the *order* of the affine plane. To better understand the relations in an affine plane, panel (a) below illustrates one with order $m = 2$. Pairs of lines with no common points are considered parallel (and are displayed with the same color, e.g., ℓ_3 and ℓ_4 are considered parallel, as are ℓ_1 and ℓ_2).



(a) The affine plane of order $m = 2$ with $N = m^2 = 4$ points and $m^2 + m = 6$ lines. Line pairs $(\ell_1, \ell_2), (\ell_3, \ell_4), (\ell_5, \ell_6)$ are parallel.

(b) The projective plane of order $m = 2$ with $N = m^2 + m + 1 = 7$ points and $m^2 + m + 1 = 7$ lines. Parallel lines are extended (dashed).

A *projective plane* is related to an affine plane and can be derived from it by adding an additional point at “infinity” where each pair of parallel lines of the affine plane would meet. In panel (b), three additional points, x_5, x_6 and x_7 are added to the affine plane. The 6 lines of the affine plane are extended with dashes, so that the additional points are included on the respective lines. For instance, ℓ_1 in the projective plane from panel (b) contains all the points from ℓ_1 of the affine plane in panel (a) and in addition x_6, too. Moreover, one special line that connects all new points is added to the projective plane (the dotted line ℓ_7 that connects all new points in panel (b)). In a projective plane each line contains $m+1$ points, each point is contained in $m+1$ lines and there are m^2+m+1 total points and lines. The order of the projective plane is the same as the order of the corresponding affine plane.

Note that the structure of an affine plane can be used to construct a multipool matrix fulfilling properties P1, P2 and P3. First, the order of the affine plane is set to equal the number of samples per test, i.e., m. Since each line contains m points, each test in the multipool matrix, defined by a line, contains m samples (P1). Because each pair of points is contained in precisely one line, each pair of different samples in the multipool matrix occurs exactly once (P2). By including all lines from some $k \leq m+1$ parallel classes, each sample is contained in k tests (P3). Consequently, methods for constructing pooling matrices build on the concepts from finite affine and projective planes.

5. Pooling matrix designs when pool size is prime. Recent work on one-round pooling tests for COVID-19 in which k (the multiplicity) can vary as a parameter have been based on the assumption that m (the pool size) is a prime number. Täufer in [15] uses the Shifted Transversal Design from [7] to construct $\mathbf{M}$. In [14], matrix $\mathbf{M}$ is formed based on a bipartite graph via an algorithm called “Packing the Pencil of Lines” (PPoL). A reason for relying on prime numbers as pool sizes is that, when the samples are arranged in a square, primes make it possible to uniquely and straightforwardly modulate each sample of a pool to obtain the next sample. Specifically, the Shifted Transversal Design in [15, 7] is based on arranging all samples in a square, and then selecting each pool of m items based on slopes $s \in \{0, -1, -2, \ldots, -(m-1), -m\}$, starting with the element in the upper-left corner. The matrix from (1) (where $m = 3$, $N = 9$, and $k = 2$) can be generated in this way, by selecting the $mk = 6$ pools using $s = -1$ and $s = -2$.

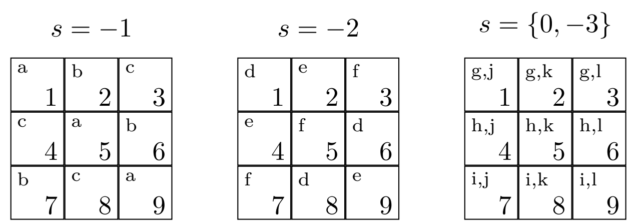


Figure 1: The Shifted Transversal Design is based on a square arrangement of samples and a prime pool size (here, $N = 9$ and $m = 3$). The 6 rows of $\mathbf{M}$ in (1) can be formed by using the indices from the pools generated by slopes $s = -1$ and $s = -2$. By using all slopes s with $-m \leq s \leq 0$, all different mk pools (for $k \leq m+1$) are found and labeled ((a,b,c),(d,e,f),(g,h,i),(j,k,l)). With $s = -1$, the three different pools (a,b,c) are constructed by starting in the top-left corner and moving (a total of m times) one unit down and one unit to the right. If a step to the right occurs at the edge of a row, the process is modulated to the beginning of the next row and continues until m samples are obtained. For $s = -2$, the three different pools (d,e,f) are obtained by moving one unit to the right and two units down. If during this process the edge of the square is reached, the remaining steps to the right (resp. down) are modulated to the beginning of the next row (resp. the top of the next column). For $s \in \{0, -3\}$, the pools (g,h,i) and (j,k,l) are obtained by pooling horizontally and vertically.

6. Pooling matrix designs when pool size is arbitrary. The constructions in Section 5 are applicable when the pool size is a prime number. To see how the situation changes when m is not prime, we illustrate the case of $N = 16 = 4^2 = m^2$. In particular, if one naively applies the same approach of pooling samples along different slopes and modulating, one can find that pairs of samples occur more than once. For illustration, in Figure 2 the first pools are generated using slopes $s \in \{0, -4\}$, with $s = -1$ and $s = -2$ following (the order in which pools are generated does not affect the outcome). When slope $s = -2$ is reached, pairs of samples occur in more than one different pool (marked by shaded squares). This implies that property P2 of a multipool matrix will not hold.

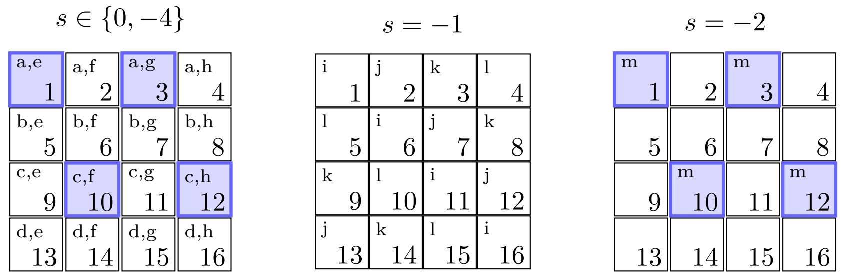


Figure 2: Pools of size $m = 4$ are collected from $N = 16$ samples. Similar to the Shifted Transversal Design (Figure 1) the samples are arranged in a square and pooled using slopes $s \in \{0, -4, -1, -2\}$. As the pool size is not prime, modulating pools at edges of the square eventually results in pairs of samples occurring more than once. (At slope $s = -2$, pairs of samples reappear, and thus $s = -3$ is not depicted.) For $s \in \{0, -4\}$, eight pools (a,b,c,d) and (e,f,g,h) are constructed by pooling horizontally and vertically. For $s = -1$, four different pools (i,j,k,l) are constructed by starting in the top-left corner and moving (m times for each pool) one unit to the right and one unit down. For $s = -2$, one additional pool, namely "m", is formed containing samples $\{1, 3, 10, 12\}$. Yet at that point the pairs $\{1, 3\}$ and $\{10, 12\}$ (marked by the shaded squares) are already contained in pools a $= \{1, 2, 3, 4\}$ and c $= \{9, 10, 11, 12\}$, respectively. Therefore, the resulting matrix based on this pooling scheme (with m not prime) will not be a multipool matrix.

To investigate the reason that pooling along slopes results in uniquely pairing samples when m is prime and not otherwise, we first arrange samples in a grid.

$$X = \begin{matrix} x_{0,0} & x_{0,1} & \cdots & x_{0,m-1} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m-1,0} & x_{m-1,1} & \cdots & x_{m-1,m-1} \end{matrix} \tag{2}$$

Define indices $0 \le i, l \le m - 1$ as well as $v = -s$ and $0 \le v \le m - 1$, and note that the pools for (2) can be computed as the sets $P(v, l; m) = \{X_{i,(l+iv) \bmod m} : 0 \le i \le m - 1\}$, with horizontal pools ultimately added as singular pools (see [15, Theorem 3]). The reason the selections in $P(v, l; m)$ form a multipool matrix when m is prime is that for any v and l there is exactly one solution for i in the equation $c = l + iv \pmod{m}$, for any arbitrary c with $0 \le c \le m - 1$. In particular, $c - l = iv \pmod{m}$, which has a unique solution for i when $\gcd(v, m) = 1$. When m is prime, this condition is automatically satisfied. An underlying concept is that the integers $0, 1, 2, \ldots, m - 1 \bmod m$ constitute a finite field when m is a prime number.

If one wishes to use arbitrary m (to improve efficiencies, or because of equipment constraints), a possible but seemingly not optimal approach is to find the closest larger prime to the desired pool size, i.e., $p > m$, construct $\mathbf{M}$ based on p, and then subsequently remove the last $p^2 - m^2$ columns of the pooling matrix. However, the resulting matrix will in general not possess the three properties of a multipool matrix, as some tests will contain fewer than m samples. Therefore, it is desirable to develop and investigate schemes that are applicable to as many values of m as possible.

7. Better matrix designs for COVID-19 pooling tests. Recently-proposed methods for constructing pooling matrices are based on assuming a prime number pool size [15, 7], or using a projective plane for the pooling matrix [14] (extended in [22]). The associated constraints of these constructions may limit their applicability in some cases. Nevertheless, it is known that affine planes can be constructed for pool sizes that are powers of primes (not just prime numbers) and that a projective plane can be obtained from an affine plane and vice-versa, with equal order. In [19, Figure 1.D], optimal pool sizes for a range of disease prevalences are proposed; these values, listed in increasing order, are $m = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 22, 32$, and are known to minimize the total number of tests in a 2-round pooling strategy. By developing an algorithm that constructs affine or projective planes of prime power orders, we can construct multipool matrices (and therefore a 1-round testing strategy) for almost all of the optimal pool sizes in the list with the exception of $m = 15$ and $m = 22$, which are not expressible as a prime number, prime power or any of the former two plus one.

In order to develop matrix designs using prime power orders, computations are defined over appropriate finite fields of prime power order. Returning to the example from Section 6 with $m = 4$ and $N = 16 = m^2$, we note that m is a prime power. One may recall that using lines modulus m did not result in a multipool matrix, since pairs of samples occurred more than once (cf. Figure 2). Instead, we employ $\mathbb{F}_4$, the finite field with 4 elements, to compute lines of pools. Arranging the samples in a grid, the pools are subsequently generated by $v, l \in \mathbb{F}_4$ and $P(v, l) = \{X_{i,(l+iv)} : i \in \mathbb{F}_4\}$. In this way, we can construct $(m+1)m$ pools with unique sample pairs when $m = 4 = 2^2$ is a prime power and $N = 16$. (Recall that this matrix cannot be constructed using pools based on the Shifted Transversal Design or similar methods, as illustrated in Figure 2.) Note that in this case, the construction gives the incidence structure of the affine plane of order 4.

$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
1 & & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & & & & 1 & \\
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\hline
1 & & & & & 1 & & & & & 1 & & & & & 1 \\
 & 1 & & & 1 & & & & & & & 1 & & & 1 & \\
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1 & & & & & & 1 & & & & & 1 & & 1 & & \\
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1 & & & & & & & 1 & & 1 & & & & & 1 & \\
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\hline
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1
\end{array}\right] = \mathbf{M}
$$

Observe that the matrix $\mathbf{M} \in \{0,1\}^{20\times 16}$ with multiplicity $k = 5$ possesses all three properties of a multipool matrix, even though $m = 4$ is not prime. In particular, each row contains exactly 4 nonzeros (P1), each pair of columns has exactly one nonzero in common (P2), and each sample is contained in exactly 5 tests (P3).

8. Implementation. In an illustrative implementation of a method using modular arithmetic (similar to the Shifted Transversal Design), suppose that the current pool corresponds to the ith test (row i in $\mathbf{M}$), and that the last added sample had index c, i.e., the last nonzero entry is $\mathbf{M}_{i,c} = 1$. Moreover, s is fixed (the slope) and the next index to be added is the $0 < l^{\text{th}} \leq m$ element of the current pool. MATLAB commands on how to compute the next sample index from the last one are shown in Table 1:

Table 1: Updating the sample index, when the previous index is c, the next sample is the lth element in the pool, and the slope is fixed by s.

```
 % c=last_index, m=pool_size, (m-(s+1))=adjusted_slope
d = c + (m-(s+1)); % Shift c by adj.  slope
if d <= l*m % Is d within largest sample index of row ?
  c = d;
else
  c = m*(l-1) + mod(d,l*m); % m*(l-1) prior samples and modulation
end
```

The pooling matrix is then updated by setting the (i,c)-entry to 1, i.e., by setting $\mathbf{M}_{i,c} = 1$.

9. Other pooling test methods. Besides the 1-round testing approaches described in this article, other pooling test methods are based a sequence of (multiple) rounds. For instance, in a *binary search* [3, Lemma], all samples are divided into two pools for each round. If a pool tests positive, then it is divided into two subsequent pools in the next round, and the process is continued. In any round in which a pool tests negative, all samples in it are declared negative as well. Strategies for how to use multi-round pooling tests for COVID-19 are proposed in [18]. Again, pooling methods have the greatest advantages when the prevalence of positive cases is low. If this is not the case, then serial (one-by-one) testing of each individual sample may be the optimal method.

10. Practical considerations. The design of appropriate binary matrices based on a parameter k enables the use of pooling tests which are *a priori* known to be able to correctly identify all samples, given the presence of up to $k-1$ positives. The number of tests required to do so is $M = km$, which can be significantly smaller than the $N = m^2$ tests that are needed when testing each sample individually. As long as "# of positive cases $\leq m-1$", one can expect pooling tests to allow for a smaller number of tests (and thus fewer testing reagents/resources) than individual testing. In reality this is typically the case, however the count of positive cases in a population is not exactly known, and may also change over time. For COVID-19, by monitoring the prevalence of the disease, the pooling matrix $\mathbf{M}$ can be augmented or compressed by varying the parameter k, when updated information about the prevalence is obtained. Moreover, small errors can accumulate in the process of assembling and amplifying the pooled samples in a PCR assay. This can be modelled by noise affecting the vector $\mathbf{y}$.

Subsequent continuous optimization formulations of the problem have been investigated by Shental et al [16]. However, the combinatorial COMP algorithm, coupled with appropriate matrix designs, is able to decode problems like in (1) (and much larger ones) in fractions of seconds. Implementations of the algorithms from this article are available at:

`https://github.com/johannesbrust/PT`

Finally, the sample size N is often confined to a small range of values due to the setup of PCR machines. The number of so-called wells (the spaces for samples in a PCR array) is typically $N \in \{24, 48, 54, 96, 155, 384\}$, and hence is often not a square. To determine pool sizes m for such sample sizes, one may take the square root and round-up to the nearest integer, i.e., take $m = \lceil \sqrt{N} \rceil$, then construct a pooling matrix using $\widehat{N} = m^2$. The last $\widehat{N} - N$ columns of this matrix can then be deleted. As an improvement to that approach, future work can investigate pooling designs when N is not square by considering factorizations $N = mr$, for appropriate values of r.

11. Acknowledgment. We thank Daniel Horsley for helpful comments and suggesting constructive design strategies for pool sizes that are not prime numbers.

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SPECIAL FEATURE

Twenty-Five Years of the *Electronic Journal of Linear Algebra*

Froilán Dopico[1] and Michael Tsatsomeros[2]

ELA
The Electronic Journal of
Linear Algebra

The *Electronic Journal of Linear Algebra* (*ELA*) is the research publication of the International Linear Algebra Society (ILAS). *ELA* is a refereed all-electronic journal that publishes mathematical articles in matrix analysis and the various aspects of linear algebra and its applications. *ELA* is completely free for authors and readers, and functions based on the contributions of its authors, referees and editors.

The year 2021 marked the quarter-century anniversary of *ELA*. To create a record and to preserve individual and institutional memories, we have collected some historical and anecdotal facts, as well as personal recollections by some of the founding members and initial officers of *ELA*. These memories of the early history of *ELA* are expanded with information on its subsequent development and with some of the latest news about the journal. We hope that this article will strengthen the general feeling that *ELA* is an important part of our society and that it exists as a collective effort of many members of ILAS, who have led *ELA* and ILAS to their current outstanding positions. We also hope that this article will provide perspective and bring *ELA* to the attention of new generations of ILAS members, since they are the future of ILAS and, thus, of *ELA*. Founding *ELA* and maintaining its growth have been possible thanks to the efforts of many historical figures of ILAS. Sustaining these efforts in the future will rest upon the shoulders of new generations of linear algebraists.

1. The beginning. We will see below that, according to Volker Mehrmann's and Daniel Szyld's recollections, Hans Schneider was the main ideological driving force behind the creation of *ELA*, strongly supported by considerable technical and logistical information collected by Daniel Szyld. Before diving into that part of the history, we start with the earliest documentary facts on the foundation of *ELA*. Included in the 1994–1996 ILAS annual reports to its membership, there are three items that record the inception and inauguration of *ELA*. They are quoted below.

In the minutes of the meeting of the ILAS subcommittee on "Electronic Publishing" (prepared by Paul Van Dooren, and held in Rotterdam, Aug. 30, 1994) we read that Richard Brualdi, Daniel Hershkowitz, Hans Schneider, Daniel Szyld and Paul Van Dooren were present, and that Graciano de Oliveira, Harm Bart and Michael Lundquist also attended. We also read that "The usefulness of the electronic service of ILAS-net was acknowledged but the discussions centered rather quickly around the possible need of a new electronic journal on Linear Algebra and Applications. After further discussions within the Board of Directors, it was decided to go ahead with this ILAS initiative. The journal would be called '*ELA* or *Electronic Journal on Linear Algebra*, an ILAS publication.' A tentative editorial board (during a trial phase) would consist of six people: Richard Brualdi, Daniel Hershkowitz, Roger Horn, Volker Mehrmann, Hans Schneider and Paul Van Dooren."

"A committee consisting of Richard A. Brualdi, Daniel Hershkowitz, Roger A. Horn, Steven J. Leon, Hans Schneider and James R. Weaver (chair) has been appointed by the President to propose changes in the ILAS bylaws in order to establish the publication of journals by ILAS. The proposed changes are to be discussed in the next meeting of the board

[1]Departamento de Matemáticas, Universidad Carlos III de Madrid, Madrid, Spain, dopico@math.uc3m.es
[2]Department of Mathematics & Statistics, Washington State University, Pullman, WA 99164, USA, tsat@wsu.edu.

Editor's note: This article was written in 2021 and was slated for the Fall 2021 issue of *IMAGE*, to commemorate the 25$^{\text{th}}$ anniversary of *ELA*. Due to an editorial oversight, it did not appear in that issue, which we regret.

of directors and the next ILAS business meeting, to be held in Atlanta, August 16–19, 1995." [*IMAGE* No. 15, p. 2, ILAS President/Vice-President Annual Report: June 1995]

"ILAS has launched *ELA – The Electronic Journal of Linear Algebra.* The structure of the editorial board of *ELA* is: Editors-in-Chief: Daniel Hershkowitz (also serves as Managing Editor) and Volker Mehrmann. Advisory Editors: Chandler Davis, Israel Gohberg, Tom Laffey and Richard Varga. Associate Editor: Daniel B. Szyld (also serves as Associate Managing Editor). As of June 17, 1996, *ELA* has published 2 papers. *ELA*'s primary site is at the Technion. Mirror sites are located in Temple University, in the University of Chemnitz and in the University of Lisbon." [*IMAGE* No. 17, p. 3, ILAS President/Vice-President Annual Report: June 1996]

It is worth emphasizing the hard work done by Daniel Hershkowitz, Volker Mehrmann and Daniel Szyld in setting up *ELA*'s sites at the Technion, TU Chemnitz and Temple University, respectively.

The original editorial board of *ELA* seems to have remained unchanged during the first three years, since the first notice that we have found after 1996 is in [*IMAGE* No. 22, p. 6, ILAS President/Vice-President's Annual Report: April 1999], where we read: "We are happy to announce the appointment of five new advisory editors and three new associate editors. The new advisory editors are: Richard A. Brualdi, Ludwig Elsner, Miroslav Fiedler, Shmuel Friedland, and Hans Schneider. The new associate editors are: Ravindra B. Bapat, Stephen J. Kirkland, and Bryan L. Shader."

2. Volker Mehrmann's recollections. Volker Mehrmann served as co-Editor-in-Chief of *ELA*, along with Daniel Hershkowitz, for less than three years. He left the position when he became co-Editor-in-Chief of *Linear Algebra and its Applications (LAA)* in 1999.

In an interview about *ELA*'s history, Volker confirms that Hans Schneider (president of ILAS at the time) was the main advocate for a new journal, with strong support and encouragement from Daniel Szyld. Hans was motivated by the recent and successful founding of the *Electronic Journal of Combinatorics (EJC)*, while Daniel Szyld had played a prominent role in the founding of the *Electronic Transactions of Numerical Analysis (ETNA)*. According to Volker, there were three main motivating factors for establishing *ELA*, which complemented each other:

(i) ILAS wanted to have a research journal, like other professional societies. SIAM was a primary example at that time. In particular, the SIAM Activity Group in Linear Algebra and the *SIAM Journal on Matrix Analysis and Applications (SIMAX)* were becoming increasingly focused on computational linear algebra. As a consequence, some core ILAS members felt the need for an additional journal that would be more conducive to theoretical linear algebra.

(ii) The decade of the 90s saw the first big surge of electronic, completely free-to-access journals. Hans Schneider was interested in exploring electronic journals and the impact of the Internet in scientific publication. As said earlier, *ETNA* (founded in 1993) and *EJC* (founded in 1994) were successful examples at that time that inspired the creation of *ELA*.

(iii) Around the time *ELA* was established, the number of submissions to *LAA* had grown significantly, forcing changes in their editorial policies and delays in the publication process (18–24 months; recall that *LAA* was then a mostly print publication). Similar delays were being experienced with the publication of papers in *Linear and Multilinear Algebra (LAMA)* and *SIMAX*. This led Hans Schneider and Richard Brualdi, who were the Editors-in-Chief of *LAA* at the time, to consider creating a new journal with high scientific standards and a more timely publication process. The goal of an electronic publication would be to publish each paper within 3–4 months, from submission to posting (a goal that *ETNA* had already achieved).

Volker expands on the initial vision for *ELA* by explaining that one of the original goals of *LAA* was to assist authors in writing and publishing good papers by providing scientific and editorial advice in the form of judicious referee reports and professional copy-editing. In the words of Volker, some initially poorly-prepared papers could be accepted at those times in *LAA*, as long as the results were substantial and the editorial process could be used to improve the exposition. With the boom in the number of submissions, however, this was no longer viable. The hope for *ELA* was that, as a journal of modest size, it could partially fulfill such a role.

The first two Editors-in-Chief of *ELA* (Volker Mehrmann and Daniel Hershkowitz) had no specific types of papers in mind. It was hoped that authors who wanted to publish a paper in a short period of time would be attracted to *ELA*. Volker also recalls that there was considerable uncertainty on the success of *ELA* because at that time "electronic-only journals" were viewed as second-rate and suspected to be of lower standards. This is still the case to some extent, according to Volker.

According to Volker's recollections, the original advisory editors of *ELA* (Chandler Davis, Israel Gohberg, Tom Laffey and Richard Varga) were essentially chosen to provide prestige, status and trust to the new journal. Initially, all of the editorial work was performed exclusively by Daniel Hershkowitz, Volker Mehrmann and Daniel Szyld.

3. Daniel Szyld's recollections. In 1993, Hans Schneider, as president of ILAS, asked Daniel Szyld to write a report to the ILAS Board of Directors on "the future of the Internet." The report was submitted in 1994 and circulated among the ILAS membership.

As we have seen above, at around the same time the idea of electronic journals was being discussed not only in ILAS but in much wider circles. In particular, a workshop on electronic journals was held in 1994 at the Mathematical Sciences Research Institute (MSRI) in Berkeley. Hans asked Daniel to attend in representation of ILAS. After the meeting, Daniel wrote a very detailed "how to" guide for creating electronic journals, presented in December 1994 to the "Electronic Subcommittee" of ILAS and used as a blueprint for *ELA*. That guide was entitled "Primer for the establishment of *ELA*, the Electronic Journal on Linear Algebra, an ILAS publication" and contains an amazing amount of technical and logistical information. Here, we quote only the first point: "1. Establish an Editorial Board. At the MSRI meeting, this point was emphasized as very important, fundamental, etc., to give credibility to the journal. This is not so much our problem, since: a. ILAS backing gives it plenty of credibility and prestige, and b. a tentative board has been already nominated." We mention this opening point because it is fully in agreement with Volker's comments about electronic-only-journals being perceived as lower-standard publications in the 1990s.

Daniel agreed to be Associate Editor and the first Associate Managing Editor of *ELA*, which meant he did the copy-editing of all the papers for eight years (from 1996 to 2003, included). Since then, Daniel has continued his editorial work for *ELA*, serving as an Advisory Editor from 2001 until today. Interestingly, Daniel is the only member of the original editorial board of *ELA* who is still serving on the board.

Another initial service that Daniel provided for *ELA* was acquiring the ISSN and having the journal indexed in *Mathematical Reviews*. Later on, he worked with the Institute for Scientific Information (ISI) of Thomson Reuters for *ELA* to be included in what was then called the Science Citation Index (SCI). Note that SCI has evolved into the current Journal Citation Reports (JCR) of the Web of Science (WoS), where *ELA* continues to be indexed.

4. Bryan Shader's recollections. Bryan Shader assumed the duties of co-Editor in Chief, with Ludwig Elsner, on July 15, 2010. Ludwig retired from *ELA* in 2011, after over a decade of service. In Fall 2010, in discussions with the ILAS Publications Committee, it was decided that a team led by Panos Psarrakos (along with Sarah Carnochan Naqvi, Michael Cavers and In-Jae Kim) would take over the production of *ELA* papers. Prior to this, for many years Michael Tsatsomeros (along with Panos Psarrakos since 2007) had shouldered this task after Daniel Szyld stopped being the Associate Managing Editor in 2003.

Under the leadership of Bryan, two prominent initiatives were taken in order to modernize *ELA* and make it more amenable to new internet tools and browsers that had been developed. First, an update to the *ELA* website was performed in 2012, and at that time the website and server for *ELA* were moved to the University of Wyoming. This gave *ELA* a more modern look, and allowed for *ELA*-specific Google searches.

Second, in September 2014, *ELA* moved to a new platform, part of Digital Commons (run by BePress) and supported by the University of Wyoming Library system. This move allowed online submission and tracking of papers, enhanced discoverability of papers via inclusion in more digital repositories (such as EBSCO), the ability to be notified of *ELA* papers when they are posted, and better digital curation through assignment of a digital object identifier (DOI) to each published paper; it also allowed MathSciNet and Web of Science to post an entry for an *ELA* paper immediately upon publication. All of these features contributed significantly to wider dissemination of *ELA* and to making *ELA* competitive with other journals, all of which (including the originally print-only ones) were now providing electronic publication.

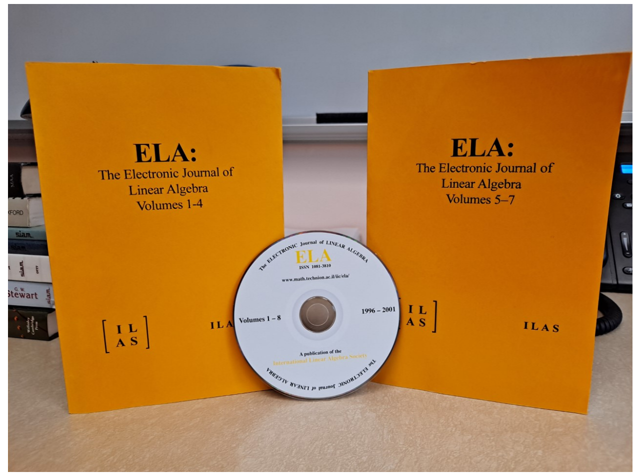
Due to his wife's illness, Bryan Shader resigned as *ELA* Editor-in-Chief in 2019, but he continues working hard for *ELA* as an Advisory Editor, Managing Editor, and as an intermediary between the current Editors-in-Chief and the staff of the University of Wyoming Library. In recognition of his many services to *ELA* and his scientific achievements, Bryan was invited to deliver the "25 years of *ELA* Plenary Lecture" at the SIAM Conference on Applied Linear Algebra 2021–23rd ILAS Conference held virtually May 17–21, 2021. The title of Bryan's lecture was "Recent Progress on the Inverse Eigenvalue Problem for Graphs: Matrices, Manifolds and Graphs".

In Fall 2019, when BePress chose to become a public company, the University of Wyoming decided to move its electronic journals to the Open Journal System (OJS) platform. *ELA* officially moved to OJS in January 2020. Since 2012 the University of Wyoming has contributed to *ELA* by providing the server, membership fees, and support for the *ELA* platform. This support is highly appreciated by *ELA* and, more generally, by ILAS.

5. Some anecdotal information.

***ELA* naming.** During a visit to the University of Regina in the early 90s, Hans Schneider canvassed the local linear algebra group as to their preferred abbreviation for the new electronic journal, one of the suggestions being "ELA". Michael Tsatsomeros recalls supporting "ELA", as in the Greek word έλα, which is an invitation to approach or to receive something. This positive connotation seemed appropriate for a new journal, according to Hans.

***ELA* CD/printed volumes.** It is interesting to look back at the apprehension and eagerness to promote an electronic/online journal in those early days. For that reason, ILAS decided to pursue additional, mainstream approaches to the dissemination of *ELA*. As a consequence, a printed collection of the first few volumes and a CD collection of *ELA*'s content were produced. Jim Weaver was in charge of the print version. Michael Tsatsomeros pursued the CD version, for which every paper up to that point had to be recompiled. A company in Vancouver, Canada, provided the CD duplication. These two versions of early *ELA* volumes were available for free to the attendees of ILAS conferences. (Michael still has a few CDs left in his office.)



***ELA*'s first paper** was published in June 1996. The paper, "Numerical ranges of an operator on an indefinite inner product space" by Chi-Kwong Li, Nam-Kiu Tsing and Frank Uhlig, is 17 pages long and dedicated to Chandler Davis.

The 1000$^{\text{th}}$ *ELA* paper was published on October 11th, 2019. It was posted by Michael Tsatsomeros and was prepared and copy-edited by Panos Psarrakos (it was the 515$^{\text{th}}$ paper prepared by Panos and his team). The paper is "Inequalities for sector matrices and positive linear maps" by Fuping Tan and Huimin Che.

6. *ELA* Editors-in-Chief.

Daniel Hershkowitz, Bar-Ilan University (1996–2010)
Volker Mehrmann, Technische Universität-Berlin (1996–1999)
Bryan Shader, University of Wyoming (2010–2019)
Ludwig Elsner, Bielefeld University (2010–2011)
Michael Tsatsomeros, Washington State University (2016–2022)
Froilán M. Dopico, Universidad Carlos III de Madrid (2019–)

7. Present and future of *ELA*. Times have changed a lot since *ELA* was founded, and some of the initial motivations for establishing *ELA* are clearly outdated. Nowadays, all journals are published electronically (even the ones with print versions), the time-lapse for publication of papers has been dramatically shortened since the 1990s, and the huge impact of the Internet on scientific research and publication is well understood. Among the original motivations that led to the launch of *ELA*, perhaps the only one that is still valid is that the strength and reputation of professional societies are often supported and demonstrated by the publication of strong and reputed research journals. In these 25 years, the strength and reputation of ILAS have reached very high levels through its ILAS conferences, prizes at different levels, programs of distinguished talks, mutual agreements with other societies, endorsements of other meetings, etc. In this scenario, we expect that the role of *ELA* within ILAS will become increasingly important and that this, in turn, will help maintain and improve *ELA*'s good standing for many years to come.

ELA is currently in its 38$^{\text{th}}$ volume and has published over one thousand papers. *ELA* remains committed to serving the linear algebra community by publishing high-quality research articles that contribute new information and insights to the various aspects of linear algebra and its applications. *ELA* relies on the voluntary efforts of its editors (and the support of their institutions) to remain freely available to authors and readers. *ELA* has recently switched to professional copy-editing, funded by ILAS membership fees, corporate contributions and donations. Moreover, some of the recent changes to the ILAS bylaws approved by the ILAS Board of Directors regard *ELA* directly. These include limitations to the terms that editors can serve on the board of *ELA*, and a more active involvement of the ILAS Journals Committee in *ELA*'s activities. We believe that these changes will help make the next 25 years of *ELA* very successful.

All contributors contacted for this article are acknowledged with thanks: Richard Brualdi, Danny Hershkowitz, Volker Mehrmann, Panos Psarrakos, Bryan Shader, Daniel Szyld, and Jim Weaver. Also all editors, managing editors, referees, authors and ILAS officers who have contributed to *ELA* during these first 25 years are sincerely acknowledged and thanked.

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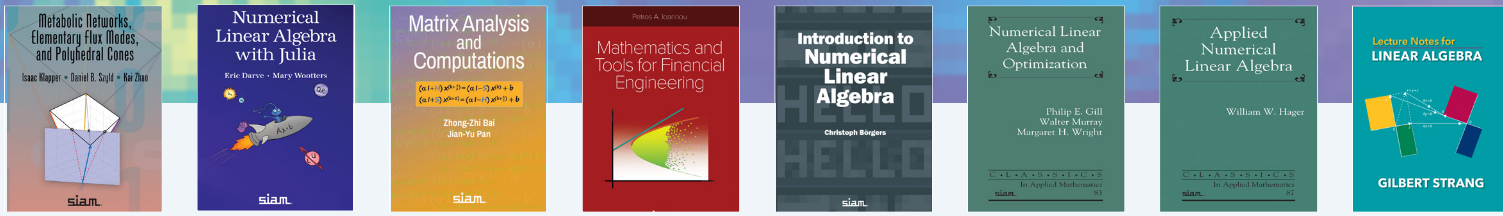

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LINEAR ALGEBRA EDUCATION

Using Real-Life Applications to Inspire a Deeper Appreciation of Eigenvectors

Heather A. Moon, Washington State University, Pullman, WA, USA, heather.moon@wsu.edu
Thomas J. Asaki, Washington State University, Pullman, WA, USA, tasaki@wsu.edu
Marie A. Snipes, Kenyon College, Gambier, OH, USA, snipesm@kenyon.edu

1. Introduction. The relevance of linear algebra to our world is abundant, but sharing this with students and meeting introductory course goals can pose a challenging balance. In this manuscript, we describe an approach we developed for students to learn linear algebra concepts through the lens of challenging, real-world data problems. Our goal in this project was to give students the experience of discovering linear algebra concepts in the process of investigating a compelling data problem. We found that while linear algebra textbooks almost always contain vignettes describing applications, those vignettes typically come *after* students have learned the topics. We wanted students to begin with a problem and encounter a need for tools to solve it. We introduce two main applications on the first day of our introductory linear algebra classes, and weave them throughout the course. Because our goal is to inspire mathematical tool development through applications, rather than just using new mathematics in applications, the path we chart through the material is different from what might be seen as typical. (See [1] for the complete work.)

2. Image Warping and Heat Diffusion. Here, we focus on the heat diffusion/image warping application, which motivates the development of vector spaces, coordinates, linear transformations, and, most dramatically, eigenvectors and diagonalization. We describe the application as presented to students, and then give some specific examples of activities that facilitate student discoveries.

We introduce the application by showing students an animated clip in which an image is warped into another image. A sequence of frames from such a video is shown in Figure 1. As students view the clip, we ask questions about what they notice in the image warping process: (a) Describe some features you see in the transition from the first image to the last. (b) Are there key features in the image that tend to disappear sooner than others? (c) How can you describe the features that “hang on” longer? (d) How can we begin to describe these mathematically?

Figure 1: Image warping sequence: small-scale features (e.g. lily pads) appear before the large-scale features (e.g. pond).

Students are often initially surprised that there is no “correct answer,” but the wide variety of creative answers leads to rich discussions that highlight the advantages of this open-endedness. Having whetted their appetites with two-dimensional images, we reduce the setting to 1D. We describe the process of heat diffusion on a rod, where we think of the heat as the intensity of the 1D “image,” and ask students to explore the connection between warping one 1D image into another and the heat diffusion of the difference between the images (heat states). (See Figure 2.)

Throughout the course, we revisit the heat diffusion application and relate it to linear algebra ideas. For example, we model heat states as vectors. Students visualize the vector space operations of addition and scalar multiplication, transitioning their understanding of vectors in $\mathbb{R}^n$ to a more general setting. We highlight how scalar multiplication changes the intensity values in a heat state while addition changes the shape. (See Figure 3.)

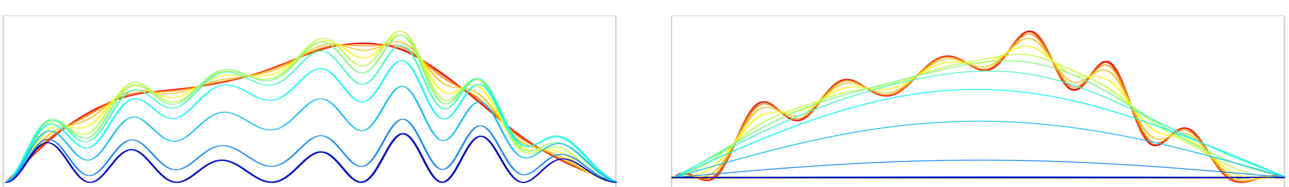

Figure 2: Left: Warping a 1D “image” (red to blue): high-frequency features appear early. Right: Diffusing a difference “image”: small-scale features leave early.

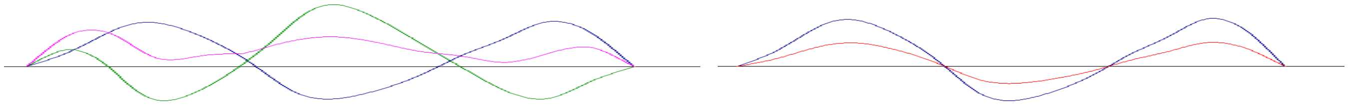

Figure 3: Addition (left) and scalar multiplication (right) of heat state vectors.

Later in the course, using calculus concepts, we help students recognize that the diffusion operator for heat states $u(t)$ is a linear transformation whose matrix representation E satisfies $Eu(t) = u(t + \delta t)$. So, to model the heat a long time into the future, they iterate. After performing computations, students recognize how computationally complex iterations are and that there needs to be a more efficient method. Therefore, as we lead students toward the usefulness of eigenvalues, eigenvectors, and diagonalizability, students are asked to brainstorm the form of a transformation that might be more efficient to iterate.

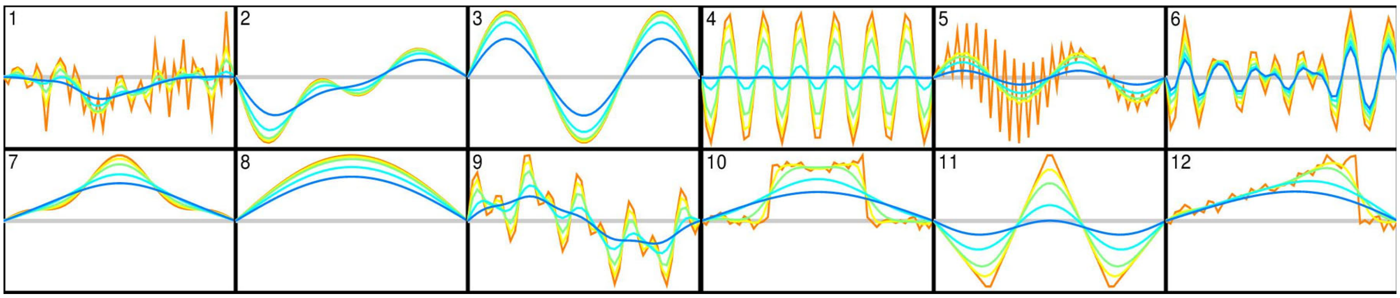

Figure 4: Students are asked to group the heat states according to properties common to their diffusion signatures.

A few classes later, we ask students to group the heat states according to properties common to their evolution. (See Figure 4.) We encourage students to make a large list with many different classifications – the responses are always creative and insightful. In the course of a discussion, the class recognizes that some "special vectors" exhibit a simple diffusion (only amplitude changes) while others show also shape changes during the diffusion process. Other variants of this observation arise, such as: "Some diffusions have the property that the heat at any location always moves monotonically toward zero, while others do not." We ask students how they would algebraically write the diffusion for these special vectors, and arrive at the (eigenvector) equation $Eu = cu$.

It is at this point that the puzzle pieces all come together for students. They recognize that for these special vectors, the diffusion operator is easy to iterate. And connecting this to properties of linear transformations, they find it easy to diffuse any linear combination of special vectors. From there, they see how a *basis* of eigenvectors would make diffusion of *any* vector simple! Once students see the utility, they are eager to investigate other properties of eigenvectors and to understand how to find them. Follow-up activities guide students toward diagonalization of the diffusion matrix E, and thinking about diagonalization as an appropriate change of basis.

Students also see how the relative magnitude of eigenvalues affects the long-term behavior of an iterative process. A final exploration connects the (sinusoidal) shape of eigenvectors of the diffusion matrix to the eigenvalues – realizing that the higher-frequency eigenvectors have smaller eigenvalues, resulting in faster diffusion. They then relate this to their qualitative observations about diffusion and image warping from early in the course.

3. Summary. Since the diffusion application is somewhat involved, we have found it most effective for students to see this application many times, at many different levels. Timing is key – students see a tangible need for eigenvectors, which otherwise can seem purely abstract and symbolic, long before they ever learn the terminology. We found this to promote student success in understanding the concepts beyond just algebraic computations. The activities also give students a taste of open-ended exploration and the process of creating mathematics. Complete materials for these activities, including MATLAB code, student handouts, and instructor notes, are freely available online (see [2]).

This work is part of the IMAGEMath project, funded by NSF-DUE 1503856, 1503929, 1504029, 1503870, and 1642095.

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[2] M.A. Snipes, T.J. Asaki, and H.A. Moon. IMAGEMath. `http://www.imagemath.org`, 2022.

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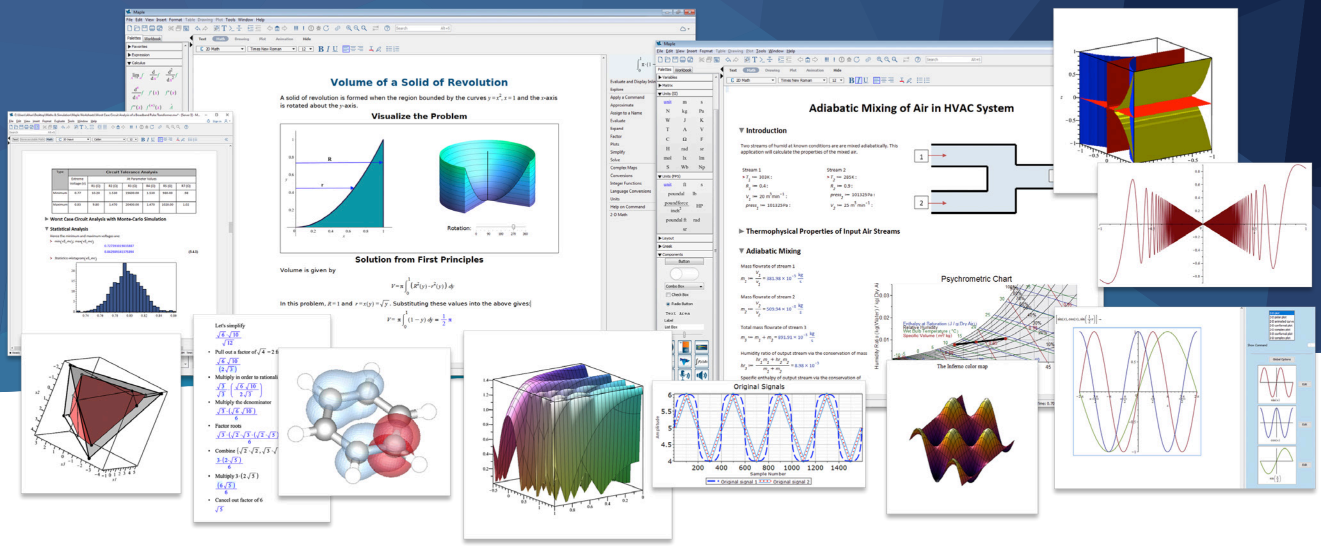

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ILAS President/Vice President Annual Report: May 1, 2022

Respectfully submitted by Daniel B. Szyld, ILAS President, szyld@temple.edu
and Froilán Dopico, ILAS Vice President, dopico@math.uc3m.es

The past year has been another momentous year for ILAS. Among the highlights is the first participation of ILAS as a partner in the Joint Mathematics Meetings (JMM) 2022, thanks to a partnership between ILAS and the American Mathematical Society approved by the ILAS Board in October 2020; a new Welcoming and Inclusiveness Statement; and a new grant program for mathematicians working in Linear Algebra fleeing situations of war or other similar circumstances.

1. Board-approved actions since the last report include:

    - The Board approved that ILAS signed an agreement with CNKI (Chinese Journal aggregator online) to post a list of *ELA* papers, as well as links to the abstracts and the papers themselves. The Board expects that this will increase the visibility of *ELA* in China.

    - The Board approved the "ILAS Welcoming and Inclusiveness Statement", which can be found at `https://ilasic.org/inclusiveness-statement`. If somebody feels harassed or unsafe in any way because of the actions, words, pictures, or expression of any member of ILAS or any participant in any ILAS conference or activity, the contact persons are the Executive Board, the Organizers of the conference or activity, and the following three respected former officers of ILAS (Leslie Hogben, Stephen Kirkland, and Chi-Kwong Li).

    - President Szyld informed the board of an impending major gift from Taylor and Francis, of approximately $60K which will endow the *LAMA* lecture, and possibly other ILAS activities and programs. This gift was negotiated by the ILAS member and co-Editor-in-Chief of *LAMA* Stephen Kirkland. We extend sincere thanks to Stephen and Taylor and Francis for making this donation possible.

    - The Board approved the institution of an investment oversight committee consisting of the ILAS President, the Secretary/Treasurer team, and an additional ILAS member. Michael Tsatsomeros will serve as the additional ILAS member until the end of President Szyld's term, which is February 28, 2023. Since then, an account with Vanguard has been opened, and initial investments in index funds were made. Funds needed to run the Society for six years are kept in cash.

    - The Board approved the new "ILAS Grant Program in support of Mathematicians working in Linear Algebra affected by conflicts" as a pilot program to be reassessed after two years, at which point the Board will decide to continue or adjust the program based on how it performed. The guidelines of the program can be found at `https://ilasic.org/grants-conflict`.

2. Other news:

    - There were two recipients of the 2022 Hans Schneider Prize. (1) Pauline van den Driessche from the University of Victoria (Canada) was cited for her influential contributions in combinatorial matrix theory, in mathematical biology and the interaction between these two areas. She will present her lecture at the 24$^{\text{th}}$ ILAS Conference in Galway, Ireland, June 20–24, 2022; (2) Nicholas J. Higham from the University of Manchester (UK) was cited for his fundamental contributions in the analysis of a wide range of numerical linear algebra problems and matrix functions. He will present his lecture at the 25$^{\text{th}}$ ILAS Conference in Madrid, Spain, June 12–16, 2023.

    - The Joint Mathematics Meetings 2022 (an ILAS partner conference) originally scheduled to take place in January 2022 in person in Seattle, was postponed until April 2022 due to the COVID pandemic and the modality was changed to virtual. The first ILAS Lecture at the JMM was delivered by Pauline van den Driessche, and there were three ILAS Special Sessions.

    - The Executive Board chose, by recommendation of the JMM Committee, Apoorva Khare (Indian Institute of Science, Bangalore) to be the ILAS Lecturer at the Joint Mathematics Meetings in Boston, January 4–7, 2023. In addition to this ILAS Lecture, there are four ILAS Special Sessions scheduled at JMM 2023, three on research topics and one on education.

- A Wikipedia page was created for *ELA*. The ILAS member Federico Poloni (Università di Pisa) did a great volunteer job in creating and attending this page. We express our sincere gratitude to Federico for his excellent work from which *ELA* and ILAS will benefit for many years to come.

3. ILAS elections ran January 5, 2022 – January 25, 2022, and proceeded via electronic voting. The following were elected to offices with three-year terms that began on March 1, 2022:

   - Vice President: Froilán M. Dopico
   - Board of Directors: Paola Boito and Lek-Heng Lim

   The following continue in the ILAS offices which they currently hold:

   - President: Daniel Szyld (term ends February 28, 2023)
   - Secretary/Treasurer: Minerva Catral (term ends February 29, 2024)
   - Second Vice President (for ILAS conferences): Raf Vandebril (term ends June 30, 2023)
   - Assistant Secretary/Treasurer: Michael Tait (term ends February 28, 2023)
   - Board of Directors: Sebastian Cioabă (term ends February 28, 2023), Dragana Cvetković Ilić (term ends February 28, 2023), Melina Freitag (term ends February 29, 2024), and Apoorva Khare (term ends February 29, 2024)

   On February 28, 2022, Hugo Woerdeman completed two consecutive terms as ILAS Vice President, for a total of six years on the ILAS Executive Board. Hugo Woerdeman will now serve a one-year term on the Executive Board and on the Board of Directors as Past Vice President until February 28, 2023. We extend sincere thanks to Hugo for his dedicated service to the Society.

   On February 28, 2022, Leslie Hogben completed her one-year term on the Executive Board and on the Board of Directors as Past Secretary/Treasurer. We extend sincere thanks to Leslie for her very long and dedicated service to the Society.

   Valeria Simoncini and Michael Tsatsomeros completed their terms on the ILAS Board of Directors on February 28, 2022. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated.

   We also thank the members of the Nominating Committee – Gianna Maria Del Corso, Stephen Kirkland, Michael Overton, Peter Šemrl (chair) and Vilmar Trevisan – for their efforts on behalf of ILAS, and all of the nominees for their participation in the elections.

4. New appointments:

   Assistant Secretary/Treasurer:

   Michael Tait

   ILAS Website Manager:

   Dominique Guillot

   ILAS-NET Manager:

   Pietro Paparella

   Investment Oversight Committee:

   Minerva Catral
   Michael Tait
   Daniel Szyld
   Michael Tsatsomeros

5. ILAS has endorsed the following conferences and seminar series of interest to ILAS members that have taken place since the last President/Vice President annual report:

   - Online seminar series on Numerical Linear Algebra, alternating Wednesdays at 4:00 p.m. CET (Central European Time). `https://sites.google.com/view/e-nla/home`

   - Online seminar series "Communications in Numerical Linear Algebra", weekly seminar series, Mondays at 3:00 p.m. CET (Central European Time). `https://sites.google.com/view/commnla/home`

   - SIAM Conference on Applied Linear Algebra (LA21), held virtually, originally scheduled in New Orleans, LA, USA, May 17–21, 2021. ILAS 2021 conference was embedded in this conference. Bryan Shader was the ILAS-*ELA* 25$^{\text{th}}$ Anniversary Lecturer, Paola Boito was an ILAS Lecturer, Raf Vandebril was the ILAS Taussky-Todd Prize lecturer, and Lek-Heng Lim was a Hans Schneider Prize Lecturer. `https://www.siam.org/conferences/cm/conference/la21`

   - Western Canada Linear Algebra Meeting, held virtually. Brandon University, Manitoba, Canada, May 29–30, 2021. `https://www.brandonu.ca/wclam`

   - Applied Matrix Positivity, held virtually. International Centre for Mathematical Science, Edinburgh, Scotland, July 19–23, 2021. `https://www.icms.org.uk/events/workshops/amp`

   - International Workshop on Operator Theory and its Applications (IWOTA 2021), held virtually. Lancaster, England, UK, August 16–20, 2021. Vern Paulsen was an Israel Gohberg ILAS-IWOTA Lecturer. `https://www.lancaster.ac.uk/maths/iwotauk2021`

   - Matrix Equations and Tensor Techniques Workshop, Perugia, Italy, September 9–10, 2021, held in a hybrid format. `https://indico.cs.dm.unipi.it/event/7`

   - International Conference on Linear Algebra and its Applications (ICLAA 2021) CARAMS, MAHE, Manipal, India, December 15–17, 2021, held virtually. Stephen Kirkland was a Hans Schneider ILAS Lecturer. `https://carams.in/events/iclaa2021`

   - Latest trends and insights into matrix theory, iterative methods, and preconditioning. Conference celebrating Daniel Szyld's 65$^{\text{th}}$ birthday. Temple University, Philadelphia, PA, USA, March 24–26, 2022. `https://www.maths.tcd.ie/~ksoodha/szyld2022`

6. ILAS endorsed the following conferences of interest to ILAS members that will take place in the next months:

   - A joint meeting of ALAMA and "Due giorni di algebra lineare numerica" (ALAMA 2022-ALN2gg). Universidad de Alcalá (University of Alcalá, UAH), Alcalá de Henares, Spain, June 1–3, 2022. Françoise Tisseur will be an ILAS Lecturer. `https://congresosalcala.fgua.es/alama2020`

   - The 16$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii (WONRA 2022). Hotel Termas da Curia, Portugal, June 12–15, 2022. `http://www.mat.uc.pt/~wonra2022`

   - XXI Householder Symposium on Numerical Linear Algebra. Selva di Fasano, Italy, June 12–17, 2022. `https://users.ba.cnr.it//iac/irmanm21/HHXXI/index.html`

   - 9$^{\text{th}}$ International Conference on Matrix Analysis and Applications (ICMAA 2022). University of Aveiro, Aveiro, Portugal, June 15–17, 2022. `https://sites.google.com/view/icmaa-2022`

   - International Workshop on Operator Theory and its Applications (IWOTA 2022). Kraków, Poland, September 6–10, 2022. Hugo J. Woerdeman will be a Hans Schneider ILAS Lecturer. `https://iwota2022.urk.edu.pl`

7. The following ILAS conferences are scheduled:

   - The 24th ILAS Conference is scheduled to be held at National University of Ireland, Galway, June 20–24, 2022. The co-chairs of the organizing committee are Rachel Quinlan and Helena Šmigoc. Ilse Ipsen was chosen to deliver the *LAA* Lecture, Shmuel Friedland the *LAMA* Lecture, Misha Kilmer the SIAG/LA Lecture, Pauline van den Driessche the Hans Schneider Prize Lecture, and Paul van Dooren the Israel Gohberg ILAS-IWOTA Lecture. `http://ilas2020.ie`

   - The 25th ILAS Conference is scheduled to be held at Escuela Técnica Superior de Ingenieros de Montes, Forestal y del Medio Natural (The School of Forest Engineering and Natural Resources) of the Polytechnic University of Madrid, Madrid, Spain, June 12–16, 2023. The chair of the organizing committee is Fernando De Terán.

8. The Electronic Journal of Linear Algebra (*ELA*) is now in its 38th volume. *ELA*'s URL is `https://journals.uwyo.edu/index.php/ela`.

   Volume 37 was published in 2021 and contains 46 papers. *ELA* receives over 210 papers every year. Its acceptance rate is less than 25%. In 2021, 24,193 downloads and 66,177 abstract views of *ELA* papers occurred.

   Froilán M. Dopico (Universidad Carlos III de Madrid) is the Editor-in-Chief.

   On February 28, 2022, Michael Tsatsomeros (Washington State University) completed two consecutive terms as Editor-in-Chief of *ELA*, for a total of six years as Editor-in-Chief, which is the maximum period currently allowed by ILAS Bylaws. Michael will continue serving on the *ELA* Editorial Board as Advisory Editor. We sincerely thank Michael very much for his many years of great service to *ELA*, which started in 2003 as Associate Managing Editor.

9. *IMAGE* is the semi-annual bulletin for ILAS, available online at `https://ilasic.org/image`. The Editor-in-Chief is Louis Deaett (Quinnipiac University). In 2021, the website of *IMAGE* received 759 visits.

10. ILAS-NET is a moderated electronic newsletter for mathematicians worldwide, with a focus on linear algebra; until December 2021 it was managed by Sarah Carnochan Naqvi (University of Regina). Currently it is managed by Pietro Paparella (University of Washington, Bothell). We sincerely thank Sarah for her fourteen years of excellent service to our society.

    An archive of ILAS-NET messages is available at `https://ilasic.org/ilas-net`. To send a message to ILAS-NET, please send the message (preferably in text format) in an email to `pietrop@uw.edu` indicating that you would like it to be posted on ILAS-NET. If the message is approved, it will be posted soon afterwards. To subscribe to ILAS-NET, please go to `https://www.ilasic.org/ilas-net`

    By April 11, 2022, there were 1093 contacts in the ILAS-NET "audience", of which 965 were subscribers. In 2021, the website of ILAS-NET received 182 visits.

11. ILAS's website is located at `https://ilasic.org` and highlights the main activities of ILAS: the Electronic Journal of Linear Algebra (*ELA*), the conferences, *IMAGE*, ILAS-NET, and other activities. In addition, the website provides general information about ILAS (e.g., ILAS officers, bylaws, special lecturers, ILAS prizes) as well as links to pages of interest to the ILAS community. Currently it is managed by Dominique Guillot (University of Delaware). In 2021, the website of ILAS received 9,971 pageviews, from users from 96 different countries. The front page received 4,480 of these pageviews, the conference page 865, and the *IMAGE* page 759.

Finally, we want to express our great gratitude to all the officers of ILAS who all show wonderful dedication to the Society, as well as to all the individual members of ILAS and our corporate sponsors. Without any of them ILAS would not be what it is today.

Respectfully submitted,

Daniel B. Szyld, ILAS President (`szyld@temple.edu`); and
and Froilán M. Dopico, ILAS Vice President (`dopico@math.uc3m.es`).

ILAS 2021–2022 Treasurer's Report

April 1, 2021 – March 31, 2022

by Minerva Catral, ILAS Secretary/Treasurer

Net Account Balance on March 31, 2021			
Checking Account - Great Western		$ 39,710.39	
Certificate of Deposit 1		$ 60,962.22	
Certificate of Deposit 2		$ 100,183.01	
Accounts Payable		$ −738.00	
Accounts Receivable		$ 10,399.00	
			$ 210,516.62
General Fund		$ 128,887.43	
Israel Gohberg ILAS-IWOTA Lecture		$ 3,920.00	
Conference Fund		$ 9,352.29	
Olga Taussky-Todd/John Todd Fund		$ 11,121.68	
Hans Schneider Lecture Fund		$ 10,735.89	
Frank Uhlig Education Fund		$ 5,367.31	
Hans Schneider Prize Fund		$ 24,948.37	
ELA Fund		$ 592.54	
ILAS/*LAA* Fund		$ 15,591.11	
			$ 210,516.62
INCOME:			
Dues	$ 9,560.00		
Israel Gohberg ILAS-IWOTA Lecture Fund Donations	$ 1,510.00		
General Fund Donations	$ 3,100.00		
Conference Fund Donations	$ 35.00		
Taussky-Todd Fund Donations	$ 40.00		
Hans Schneider Lecture Fund Donations	$ 30.00		
Uhlig Education Fund Donations	$ 50.00		
Hans Schneider Prize Fund Donations	$ 70.00		
ELA Fund Donations	$ 910.00		
Richard Brualdi Early Career Prize Fund Donations	$ 40.00		
Corporate Dues Income	$ 550.00		
Interest – Great Western	$ 6.08		
Interest on Great Western Certificates of Deposit	$ 335.41		
Elsevier flow-through			
Misc Income	$ 338.51		
Total Income		$ 16,575.00	
EXPENSES:			
ILAS Conference Expenses			
ELA	$ 4,922.46		
IMAGE			
IWOTA			
PayPal/Credit Card Processing & Bank Fees	$ 410.01		
Non-ILAS Conferences			
Hans Schneider Lecture			
Hans Schneider Prize			
Taussky-Todd Lecture			
Business License	$ 61.25		
Election Costs	$ 290.33		
Web Hosting & Online Membership Forms	$ 431.87		
Elsevier flow-through			
Misc Expenses	$ 557.12		
Total Expenses		$ 6,673.04	
Net Account Balance on March 31, 2022			
Checking Account – Great Western		$ 155,858.96	
Certificate of Deposit 1		$ 61,114.62	
Accounts Receivable		$ 3,445.00	
			$ 220,418.58
General Fund		$ 135,104.39	
Israel Gohberg ILAS-IWOTA Lecture Fund		$ 5,430.00	
Conference Fund		$ 9,387.29	
Olga Taussky-Todd/John Todd Fund		$ 11,161.68	
Hans Schneider Lecture Fund		$ 10,765.89	
Frank Uhlig Education Fund		$ 5,417.31	
Hans Schneider Prize Fund		$ 25,018.37	
ELA Fund		$ 1,502.54	
Richard Brualdi Early Career Prize Fund		$ 1,040.00	
ILAS/*LAA* Fund		$ 15,591.11	
			$ 220,418.58

ILAS NEWS

ILAS Election Results

Froilán Dopico has been elected to a three-year term as ILAS Vice President, beginning March 1, 2022.

Paola Boito and Lek-Heng Lim were elected to three-year terms as members of the ILAS Board of Directors. They will start their terms on March 1, 2022.

Michael Tait Appointed ILAS Assistant Secretary/Treasurer

Michael Tait (Villanova University) has been appointed to the position of Assistant Secretary/Treasurer. He will work with current ILAS Secretary/Treasurer Minerva Catral on all aspects of ILAS membership and financial matters.

Renewal of ILAS Leadership

Contributed announcement from Daniel Szyld, ILAS President

Every March 1st there is some rotation and renewal of the ILAS leadership. This year, Hugo Woerdeman completed two terms as ILAS Vice President, while Leslie Hogben completed a year as past-Secretary/Treasurer.

Valeria Simoncini and Michael Tsatsomeros completed their terms on the ILAS Board of Directors.

Michael Tsatsomeros ended his second term as co-Editor-in-Chief of *ELA*, the *Electronic Journal of Linear Algebra*. He will continue as an Advisory Editor. Froilán Dopico is now the sole Editor-in-Chief of *ELA*.

Lek-Heng Lim Receives Guggenheim Fellowship

ILAS member (and recent recipient of the Hans Schneider Prize) Lek-Heng Lim of the University of Chicago has been awarded a 2022 Guggenheim Fellowship. Professor Lim is this year's sole recipient in the area of applied mathematics. (The two recipients in pure mathematics are Fields Medalist Manjul Bhargava of Princeton University, and Lauren K. Williams of Harvard University.) For details, see:

`https://news.uchicago.edu/story/two-university-chicago-scientists-named-2022-guggenheim-fellows`

ILAS Members Selected as 2022 SIAM Fellows

The Society for Industrial and Applied Mathematics (SIAM) recently announced its 2022 SIAM Fellows. Among those named are no fewer than four ILAS members.

- Zlatko Drmač of the University of Zagreb was recognized for contributions to algorithms with high relative accuracy in numerical linear algebra, model reduction, and system identification.
- Chen Greif of the University of British Columbia was recognized for contributions to scientific computing, especially in numerical linear algebra and its applications.
- Daniel Kressner of the École Polytechnique Fédérale de Lausanne was recognized for contributions in numerical linear and multilinear algebra and scientific computing.
- Lek-Heng Lim of the University of Chicago was recognized for pioneering contributions to numerical multilinear algebra, and for introducing high-level algebra, geometry, and topology to applied mathematics.

For details, see `https://sinews.siam.org/Details-Page/siam-announces-class-of-2022-fellows`.

ILAS Member Selected as 2022 AWM Fellow

The Association for Women in Mathematics (AWM) has selected ILAS member Jennifer "Jenny" Quinn among their 2022 class of Fellows. Jenny is recognized for "her outstanding achievements as a teacher, mentor, leader, expositor, and editor; for her pioneering service as AWM executive director; and for continued service as AWM volunteer and supporter."

For details, see https://awm-math.org/awards/awm-fellows/2022-fellows.

ILAS Members Featured in *Quanta Magazine*

Contributed announcement from Daniel Szyld, ILAS President

ILAS members Richard Brualdi, Leslie Hogben, and Carolyn Reinhart were featured in an article in *Quanta Magazine*: https://www.quantamagazine.org/in-new-math-proofs-artificial-intelligence-plays-to-win-20220307.

Anne Greenbaum Named AWM-SIAM Sonia Kovalevsky Lecturer

The Association for Women in Mathematics (AWM) and the Society for Industrial and Applied Mathematics (SIAM) have selected Anne Greenbaum as the 2022 Sonia Kovalevsky Lecturer. Her lecture "Two of My Favorite Problems" will be delivered at the hybrid SIAM Annual Meeting in Pittsburgh, Pennsylvania, July 11–15, 2022.

Professor Greenbaum was selected in recognition of her "long-lasting and significant impact on many aspects of numerical linear algebra." For the full press release from the AWM, visit:

https://awm-math.org/wp-content/uploads/2022/02/PR-Kovalevsky-2022-Greenbaum.pdf

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ILAS Receives Major Gift from Taylor & Francis

Contributed announcement from Daniel Szyld, ILAS President

Taylor & Francis, publishers of *Linear and Multilinear Algebra (LAMA)*, have made a major gift to ILAS of $60K, which will endow the *LAMA* lecture at each of the ILAS conferences (and possibly at other ILAS activities or programs).

We are grateful to Taylor & Francis for this donation, and for their continuous support of ILAS. Thanks are also due to ILAS member Stephen Kirkland, a co-Editor-in-Chief of *LAMA*, who helped make this donation possible.

ILAS Launches Program to Aid Mathematicians Affected by Conflicts

Contributed announcement from Daniel Szyld, ILAS President

The ILAS Board of Directors has launched a program to offer financial help to researchers in mathematics with connections to linear algebra around the world who are affected by wars, oppression, or persecution.

The program is named "ILAS Grant Program in Support of Mathematicians Working in Linear Algebra Affected by Conflicts" and calls for applications of affected persons for a grant of up to US$1,000.

Guidelines and more details can be found at https://ilasic.org/grants-conflict.

Ilse Ipsen to Deliver Olga Tausky-Todd Lecture at ICIAM 2023

ILAS member Ilse Ipsen from North Carolina State University was chosen to deliver the Olga Tausky-Todd Lecture at ICIAM 2023, the International Conference of Industrial and Applied Mathematics, to be held in Tokyo in August 2023.

See the announcement at https://iciam2023.org/1368.

Nominations Sought for Richard A. Brualdi Early Career Prize

The ILAS Richard A. Brualdi Early Career Prize committee is currently soliciting nominations for this prize.

The prize is awarded to an outstanding early-career researcher in the field of linear algebra who is within seven (7) years of receiving the Ph.D. (or equivalent degree) as of October 1 of the year before the award, for distinguished contributions to the field. The prize committee may consider exceptions to the seven-year rule in the case of career interruptions.

The deadline for submissions is October 1st, 2022.

The prize will be awarded at the ILAS meeting in Madrid, Spain, June 12–16, 2023. The complete guidelines are available at `https://ilasic.org/brualdi-guidelines`.

The members of the Prize Committee are: Marina Arav, Shmuel Friedland, Volker Mehrmann, and Daniel Szyld

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OBITUARY NOTICES

Thomas L. Markham, 1939–2021

Submitted by Tin-Yau Tam

Professor Thomas "Tom" L. Markham passed away on Monday, December 20, 2021, at the age of 82. Tom received his doctorate degree from Auburn University in 1967 under the supervision of the late Professor Emilie V. Haynsworth (1916–1985). Tom was a professor of mathematics at the University of South Carolina (USC) from 1968 to 1999, and served as Undergraduate Director of Mathematics at USC from 1996 to 1999. He retired as Distinguished Emeritus Professor in 1999. Tom was an active member of the community, and continued to publish a good number of papers in linear algebra after his retirement.

A more detailed obituary can be found at:

`https://www.legacy.com/us/obituaries/newsobserver/name/thomas-markham-obituary?id=31962370`.

He will be greatly missed by his family, students, friends, and colleagues.

Richard S. Varga, 1928–2022

Submitted by Daniel B. Szyld

Long-time ILAS member and Hans Schneider prize recipient Richard S. Varga died peacefully in the early hours of February 25, 2022. He was 93.

Richard made his mark in the profession in multiple ways across several areas, including matrix analysis, numerical methods, and approximation theory.

His first book, *Iterative Matrix Analysis*, completed when he was 32 years old, became a classic. He published a total of seven books and over 230 research papers, directed 25 Ph.D. dissertations, and mentored many younger colleagues.

In addition to the Hans Schneider Prize, Richard received many honors and awards, including a senior von Humboldt Prize and two honorary degrees (one from the University of Karlsruhe and one from the University of Lille). He also was selected as a SIAM Fellow and as an AMS Fellow.

But such numbers and accolades tell only part of the story. Richard was a gentleman with a sense of humor, and a generous man. He served the community in many ways. He was Editor-in-Chief of *Numerische Mathematik* for fourteen years, co-founding editor of *ETNA*, the *Electronic Transactions of Numerical Analysis*, serving as co-Editor-in-Chief for sixteen years, and editor of many important journals in linear algebra, numerical analysis, and approximation theory.

Many of us have been inspired by his work and his service ethic. We honor his career and his life.

CONFERENCE REPORTS

ILAS Partner Activities at the Joint Mathematics Meetings (held online) April 6–9, 2022

Report by Hugo Woerdeman

After initially being planned as an in-person meeting in Seattle in January, the 2022 Joint Mathematics Meetings took place virtually April 6–9. It was the first time ILAS was a partner, and we made our presence known by contributing an ILAS Special Lecture and three ILAS Special Sessions. ILAS lecturer Pauline van den Driessche (University of Victoria) did a wonderful job delivering the special lecture "Sign patterns meet dynamical systems", and the three special sessions listed below were also well attended:

- The Inverse Eigenvalue Problem for a Graph, Zero Forcing, Throttling and Related Topics, organized by Mary Flagg and Hein Van der Holst
- Matrix Analysis and Applications, organized by Luyining Gan, Tin-Yau Tam and Mohsen Aliabadi
- The Interplay of Matrix Analysis and Operator Theory, organized by Kelly Bickel, Hugo J. Woerdeman, Ryan K. Tully-Doyle, and Meredith Sargent

There were also two AMS Special Sessions organized by ILAS members, which drew many participants as well:

- Finding Needles in Haystacks: Approaches to Inverse Problems Using Combinatorics and Linear Algebra (an AMS Mathematics Research Communities session), organized by Shahla Nasserasr, Sam Spiro, and Emily J. Olson.
- Innovative and Effective Ways to Teach Linear Algebra, organized by David M. Strong, Sepideh Stewart, Megan Wawro, and Gil Strang.

We also anticipate a robust slate of ILAS activities at the 2023 Joint Mathematics Meetings, which will hopefully meet in person in Boston, Massachusetts, January 4–7, 2023. We are delighted that Apoorva Khare (Indian Institute of Science, Bangalore) will then deliver the ILAS Special Lecture.

Send News for *IMAGE* Issue 69

IMAGE seeks to publish all news of interest to the linear algebra community. Issue 69 of *IMAGE* is due to appear online on December 1, 2022. Send your news for this issue to the appropriate editor by October 15, 2022. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

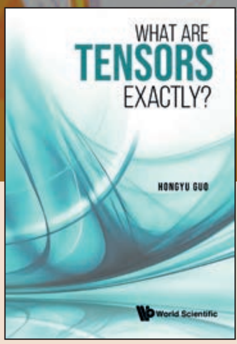
- feature articles to Sebastian Cioabă (`cioaba@udel.edu`)
- interviews of senior linear algebraists to Adam Berliner (`berliner@stolaf.edu`)
- problems and solutions to Rajesh Pereira (`pereirar@uoguelph.ca`)
- linear algebra education news to Anthony Cronin (`anthony.cronin@ucd.ie`)
- advertisements to Amy Wehe (`awehe@fitchburgstate.edu`)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (`jephianlin@gmail.com`)
- book reviews to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)

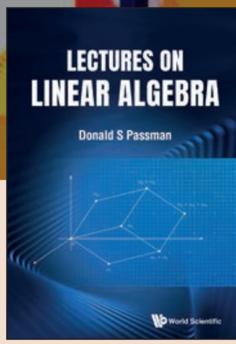
Send all other correspondence to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`).

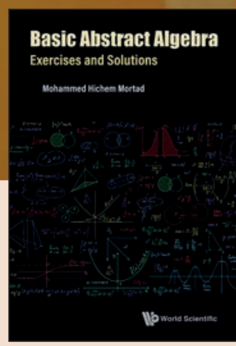
For past issues of *IMAGE*, please visit `http://www.ilasic.org/IMAGE`.

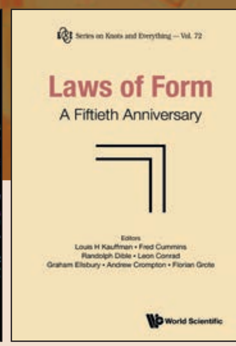
New Releases in ALGEBRA and Related Titles

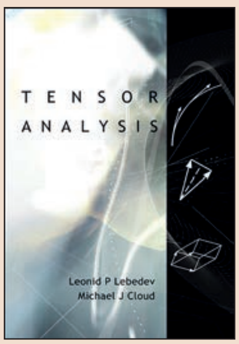
See All Titles in Algebra and Related Titles

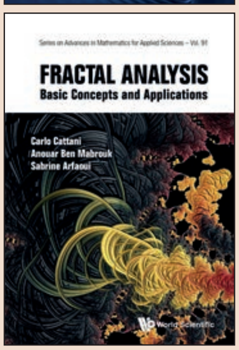


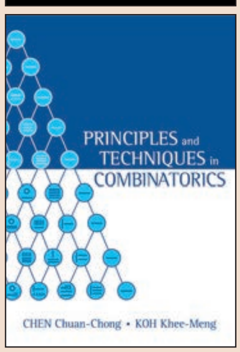


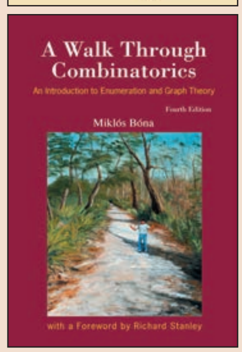












What Are Tensors Exactly?

by **Hongyu Guo** *(University of Houston-Victoria, USA)*

Readership: Researchers, professionals, academics, graduate students and undergraduate students in physics, AI, machine learning, and relativity and gravitation.

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UPCOMING CONFERENCES AND WORKSHOPS

The 9th International Conference on Matrix Analysis and Applications (ICMAA) Aveiro, Portugal, June 15–17, 2022

The 9th International Conference on Matrix Analysis and Applications, ICMAA 2022, will be held at the University of Aveiro, Aveiro, Portugal, June 15–17, 2022.

This meeting aims to stimulate the research and interactions of mathematicians in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and their applications, providing an opportunity to bring together researchers to exchange ideas and developments on these subjects.

The previous conferences were held in China (Beijing, Hangzhou), United States (Nova Southeastern University), Turkey (Selųk University, Konya), Vietnam (Duy Tan University, Da Nang), Japan (Shinshu University, Nagano Prefecture) and United States (University of Nevada, Reno). Former keynote speakers are Roger Horn, Richard Brualdi, Chi-Kwong Li, Steve Kirkland, Alexander A. Klyachko (ILAS guest speaker), Shmuel Friedland, Man-Duen Choi, Tsuyoshi Ando, Fumio Hiai and Lek-Heng Lim.

The keynote speaker of ICMAA 2022 is:

- Peter Šemrl, University of Ljubljana, Slovenia
  On Wigner's theorem

And the two invited speakers are:

- Natália Bebiano, University of Coimbra, Portugal
  An inverse eigenvalue problem for pseudo-Jacobi matrices
- Chi-Kwong Li, College of William and Mary, USA
  Linear maps preserving parallel pairs

The conference organizers are:

- Enide Andrade (Organizing Committee Chair), University of Aveiro, Aveiro, Portugal; Rute Lemos, University of Aveiro, Aveiro, Portugal; Tin-Yau Tam (Organizing Committee co-Chair), University of Nevada, Reno, USA; Qing-Wen Wang, Shanghai University, Shanghai, China; Fuzhen Zhang, Nova Southeastern University, Florida, USA.

The workshop is endorsed and sponsored by:

- The International Linear Algebra Society (ILAS)
- The Center for Research and Development in Mathematics and Applications (CIDMA)
- The Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA, the Center for Research and Development in Mathematics and Applications, within project UIDB/MAT/04106/2020.
- The Department of Mathematics, University of Aveiro, Portugal.

For detailed information and updates contact Enide Andrade (`enide@ua.pt`) or Rute Lemos (`rute@ua.pt`) if you have any questions, and for updates please visit the conference website at `http://sites.google.com/view/icmaa-2022`

Sponsors of ICMAA 2022:

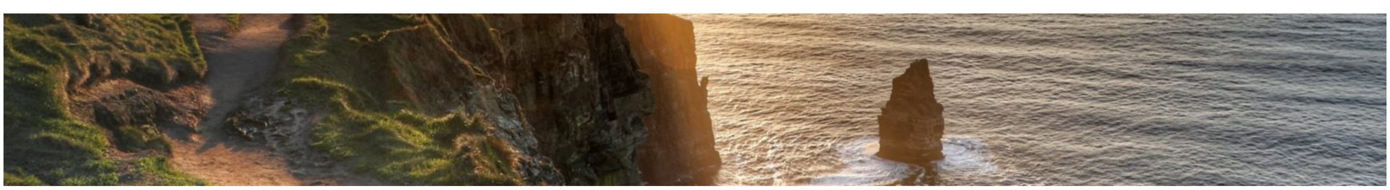
The 24th ILAS Conference: Classical Connections
Galway, Ireland, June 20–24, 2022

The 24th Conference of the International Linear Algebra Society will take place in Galway, Ireland, June 20–24, 2022. The venue will be the beautiful riverside campus of the National University of Ireland, Galway.

The conference theme is "Classical Connections." This will be reflected in the plenary programme and minisymposia, and all participants are encouraged to think about relating their themes to their historical roots.

Foundations of an attractive scientific programme are already lined up, and will feature the following plenary talks.

- Shmuel Friedland, University of Illinois Chicago (LAMA Lecture)
  Rank of a tensor and quantum entanglement
- Patrick Farrell, Oxford University
  Reynolds-robust preconditioners for the stationary incompressible Navier–Stokes and MHD equations
- Nicolas Gillis, Université de Mons
  Historical tour on the nonnegative rank
- Misha Kilmer, Tufts University (SIAG/LA Lecture)
- Monique Laurent, Centrum Wiskunde & Informatica
  Graphs, copositive matrices, and sums of squares of polynomials
- Clément de Seguins Pazzis, Lycée Sainte-Geneviève
  Decomposing matrices into quadratic ones
- Christiane Tretter, University of Bern
- Vilmar Trevisan, Universidade Federal do Rio Grande do Sul
  Eigenvalue Location of Symmetric Matrices
- Pauline van den Driessche, University of Victoria (Hans Schneider Prize Lecture)
  Linear Algebra is Everywhere: a Duo of Examples from Mathematical Biology
- Paul van Dooren, Université catholique de Louvain (Israel Gohberg ILAS-IWOTA Lecture)
  Strongly minimal self-conjugate linearizations for polynomial and rational matrices



In addition, the conference will feature 22 invited minisymposia (`http://ilas2020.ie/programme/minisymposia`):

1. *Graph spectra* – Domingos Cardoso, Claudia Justel and Renata del Vecchio
2. *Spectral properties of non-negative matrices* – Carlos Marijuán and Pietro Paparella
3. *Copositive and completely positive matrices and related topics* – Avi Berman, Mirjam Dür and Naomi Shaked-Monderer
4. *Mathematics of quantum information* – Rupert Levene and Ivan Todorov
5. *Combinatorial matrix theory* – Jane Breen and Roberto Canogar
6. *The inverse eigenvalue problem for graphs* – Jephian Lin and Polona Oblak
7. *General preservers* – Lajos Molnár

8. *Distance matrices of graphs* – Projesh Nath Choudhury and Apoorva Khare
9. *Linear algebra education* – Anthony Cronin and Sepideh Stewart
10. *Numerical linear algebra for PDEs* – Niall Madden
11. *The research and legacy of Richard A. Brualdi* – Adam Berliner, Louis Deaett and Seth Meyer
12. *Matrix positivity: theory and applications* – Alexander Belton and Dominique Guillot
13. *Rigidity and matrix completion* – James Cruickshank and Derek Kitson
14. *History of linear algebra* – Kirk Soodhalter and Jörg Liesen
15. *Companion matrix forms* – Fernando de Terán and Kevin Vander Meulen
16. *Riordan arrays and related topics* – Paul Barry, Gi-Sang Cheon and Tian-Xiao He
17. *Linear algebra for designs and codes* – Ronan Egan, Ilias Kotsireas, Padraig Ó Catháin and Eric Swartz
18. *Kemeny's constant on networks and its application* – Ángeles Carmona, Maria Jose Jimenez and Margarida Mitjana.
19. *Generalized inverses, operator matrices and tensor equations* – Dragana Cvetkovic Ilic, Yimin Wei and Qing Wen Wang
20. *Special matrices* – Natália Bebiano, Susana Furtado and Mikail Tyaglov
21. *Tensors for signals and systems* – Kim Batselier, Philippe Dreesen and Bori Hunyadi
22. *Coding theory and linear algebra over finite fields* – Eimear Byrne, Alberto Ravagnani and John Sheekey

The scientific organising committee consists of: Nair Abreu, Peter Cameron, Mirjam Dür, Ernesto Estrada, Vyacheslav Futorny, Stephen Kirkland, Yongdo Lim, Rachel Quinlan, Peter Šemrl, Helena Šmigoc, Françoise Tisseur, and Paul Van Dooren.

The local organizing committee consists of: Paul Barry, Jane Breen, Anthony Cronin, Richard Ellard, Kevin Jennings, Thomas Laffey, Niall Madden, Oliver Mason, Collette McLoughlin, Rachel Quinlan, Helena Šmigoc, and Kirk Soodhalter.

Registration for the conference is open at `http://ilas2020.ie/registration`, and will continue until June 10, 2022 at midnight IST (UTC+1:00).

Up-to-date information (including on accommodation and other practicalities) can be found on the conference website: `http://ilas2020.ie`.

E-mail queries may be addressed to `galway@ilas2020.ie`.

We are looking forward to gathering the ILAS community in Galway!

The 6$^{\text{th}}$ Workshop on Algebraic Designs, Hadamard Matrices & Quanta Kraków, Poland, June 27 – July 1, 2022

The 6$^{\text{th}}$ Workshop on Algebraic Designs, Hadamard Matrices & Quanta will be held at Jagiellonian University, as well as at the Institute of Mathematics, in Kraków, Poland.

Conference organizers:

- Karol Wojciech Życzkowski (chair)
- Wojciech Bruzda
- Wojciech Słomczyński
- Anna Karolina Szymusiak
- Anna Szczepanek

The list of confirmed invited speakers includes:

- Ingemar Bengtsson (Stockholm, Sweden)
- Robert Craigen (Winnipeg, Canada)
- Dane Flannery (Galway, Ireland)
- Shmuel Friedland (Chicago, USA)
- Dardo Goyeneche (Antofagasta, Chile)
- Markus Grassl (Gdańsk, Poland)
- Hadi Kharaghani (Lethbridge, Canada)
- Ilias Kotsireas (Waterloo, Canada)
- Máté Matolcsi (Budapest, Hungary)
- Koji Momihara (Kumamoto, Japan)
- Akihiro Munemasa (Tōhoku, Japan)
- Ion Nechita (Toulouse, France)
- Padraig Ó Catháin (Worcester, USA)
- Eric Swartz (William & Mary, USA)
- Behruz Tayfeh-Rezaie (Tehran, Iran)
- Mihály Weiner (Budapest, Hungary)
- Qing Xiang (Newark, USA)
- Danylo Yakymenko (Kyiv, Ukraine).

Registration for the conference has ended, though there may still be room for some additional (non-speaker) attendees.

For additional information or requests, visit the conference web site at `http://chaos.if.uj.edu.pl/hadamard2020` or send e-mail to `hadamard2020@uj.edu.pl`.

The 25th ILAS Conference
Madrid, Spain, June 12–16, 2023

The 25th conference of the International Linear Algebra Society, ILAS 2023, will be held June 12–16, 2023, in Madrid, Spain. The venue is the Escuela Técnica Superior de Ingeniería de Montes, Forestal y del Medio Natural, of the Universidad Politécnica de Madrid (Polytechnic University of Madrid).

The organizing committee consists of Manuel Alonso Morón, Roberto Canogar, Fernando De Terán (chair), Froilán M. Dopico, Ana María Luzón, Ana Marco, and José Javier Martínez.

The scientific committee will comprise: Raymond H. Chan (Hong-Kong), Fernando De Terán (Spain), Gianna M. Del Corso (Italy), Shaun Fallat (Canada), Heike Fassbender (Germany), Elias Jarlebring (Sweden), Linda Patton (USA), Jennifer Pestana (UK), João Queiro (Portugal), Naomi Shaked-Monderer (Israel), Daniel Szyld (ILAS President, USA), Raf Vandebril (ILAS Vice President for Conferences, Belgium) and Zdenek Strakos (Czech Republic).

More information about the conference, including plenary speakers, invited mini-symposia, and relevant dates, will be posted at the webpage of the conference: `http://www.ilas2023.es`

The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA)
Curia, Portugal, postponed to 2023

Due to the current international instability, the WONRA 2022 organizers have decided to postpone the workshop to be held in Portugal in June 2023, close to the ILAS meeting in Madrid. Additional information will be given on the conference web site: `http://www.mat.uc.pt/~wonra2022`

The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA) has as its purpose to stimulate the interchange of ideas on this subject and its applications. The previous meeting of this biennial workshop series celebrated the 100th anniversary of Toeplitz-Hausdorff Theorem. The high level of research activities on the topic after this fundamental result was established is due to the connections of the subject to many different branches of pure and applied mathematics, such as operator theory, functional analysis, C^*-algebras, Banach algebras, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics, etc. Moreover, a wide range of tools, including algebra, analysis, geometry, combinatorics and computer programming, are useful in its study. An informal workshop atmosphere will facilitate the exchange of ideas from different research areas.

The organizing committee consists of Natália Bebiano (CMUC, University of Coimbra, Portugal), Graça Soares (CMAT-UTAD, University of Trás-os-Montes e Alto Douro, Portugal), Rute Lemos (CIDMA, University of Aveiro, Portugal) and Ana Nata (CMUC, Polytechnic Institute of Tomar, Portugal).

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 67-2, 67-3, and 67-4. Solutions are invited to Problems 63-1 and 63-3; to all of the problems from issue 65; to parts (a) and (b) of Problem 66-2; to Problem 66-4; to Problem 67-1 and to all of the new problems from the present issue 68.

Problem 67-2: A Solution to Two Matrix Equations

Proposed by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, obaksalary@gmail.com
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, trenkler@statistik.tu-dortmund.de

Let A be an $m \times n$ complex matrix. Find all matrices X such that $AX = P_A$ and $XA = P_{A^*}$, where P_A and P_{A^*} denote the orthogonal projectors onto the column spaces (ranges) of A and A^*, respectively.

Solution 67-2.1 by Nebojša Č. Dinčić, *University of Niš, Niš, Serbia*, ndincic@hotmail.com

This solution is given in the more general setting of bounded linear operators on arbitrary Hilbert spaces. It is well-known fact that any closed-range operator $A \in L(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are arbitrary Hilbert spaces, has the operator matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{Ran}(A^*) \\ \mathrm{Null}(A) \end{bmatrix} \to \begin{bmatrix} \mathrm{Ran}(A) \\ \mathrm{Null}(A^*) \end{bmatrix}$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathrm{Ran}(A^*) \oplus \mathrm{Null}(A)$ and $\mathcal{K} = \mathrm{Ran}(A) \oplus \mathrm{Null}(A^*)$. Note that $A_1 \in L(\mathrm{Ran}(A^*), \mathrm{Ran}(A))$ is invertible.

Suppose that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{Ran}(A) \\ \mathrm{Null}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{Ran}(A^*) \\ \mathrm{Null}(A) \end{bmatrix}$$

for some X_1, X_2, X_3, X_4.

Letting $A^\dagger$ denote the Moore-Penrose inverse of A, we have $AA^\dagger = P_A$. Hence, $AX = P_A$ gives

$$AX = \begin{bmatrix} A_1X_1 & A_1X_2 \\ 0 & 0 \end{bmatrix} = AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $X_1 = A_1^{-1}$ and $X_2 = 0$. From $XA = A^\dagger A$ we have

$$\begin{bmatrix} X_1A_1 & 0 \\ X_3A_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $X_3 = 0$.

Therefore,

$$X = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & X_4 \end{bmatrix},$$

where $X_4 \in L(\mathrm{Null}(A^*), \mathrm{Null}(A))$ is arbitrary.

Solution 67-2.2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Let r denote the rank of A. We use the SVD $A = V\Sigma W^*$, where $V \in M_m(\mathbb{C})$ and $W \in M_n(\mathbb{C})$ are unitary, and $\Sigma = \left[\begin{smallmatrix} D & O \\ O & O \end{smallmatrix}\right]$, where D is an $r \times r$ diagonal matrix whose entries are all positive. With $Y = W^*XV$, the equations $AX = P_A$ and $XA = P_{A^*}$ become $\Sigma Y = V^*P_AV$ and $Y\Sigma = W^*P_{A^*}W$. The products V^*P_AV and $W^*P_{A^*}W$ are orthogonal projections whose ranges are the range of Σ and the range of Σ^*, respectively. Thus, writing $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right]$ to be compatible with the block structure of Σ, we have

$$\begin{bmatrix} D & O \\ O & O \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} D & O \\ O & O \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

These equations both hold if and only if $Y_{11} = D^{-1}$ and $Y_{12} = O$ and $Y_{21} = O$. Therefore the general solution to the equations $AX = P_A$ and $XA = P_{A^*}$ is

$$X = W \begin{bmatrix} D^{-1} & O \\ O & Z \end{bmatrix} V^*,$$

where Z is any $(n-r) \times (m-r)$ matrix.

Solution 67-2.3 by Johanns de Andrade Bezerra, *Federal University of Paraíba, João Pessoa, Brazil*, veganismo.direitoanimal@gmail.com

Note that $A^\dagger$ is a particular solution to the equations $AX = P_A$ and $XA = P_{A^*}$, since $P_A = AA^\dagger$ and $P_{A^*} = A^\dagger A$. According to [1, p. 41], the general solutions to the equations $AX = AA^\dagger$ and $XA = A^\dagger A$ are given, respectively, by

$$X = A^\dagger AA^\dagger + (I_n - A^\dagger A)W = A^\dagger + (I_n - A^\dagger A)W$$

and

$$X = A^\dagger AA^\dagger + Z(I_m - AA^\dagger) = A^\dagger + Z(I_m - AA^\dagger)$$

for any $n \times m$ complex matrices W and Z.

Thus, we have that

$$(I_n - A^\dagger A)W - Z(I_m - AA^\dagger) = 0 \tag{1}$$

and, again by [1, p. 41], the general solution of the equation (1) is given by

$$W = (I_n - A^\dagger A)V(I_m - AA^\dagger) + A^\dagger AM$$

and

$$Z = V - A^\dagger AV(I_m - AA^\dagger)$$

for any $n \times m$ complex matrices V and M. Therefore, by substituting for the Z in the second equation for X above and using the fact that $A^\dagger A$ and $I_n - A^\dagger A$ are complementary projections, we may conclude that any solution to the matrix system $AX = P_A$ and $XA = P_{A^*}$ is given by

$$X = A^\dagger + (V - A^\dagger AV(I_m - AA^\dagger))(I_m - AA^\dagger)$$

for any $n \times m$ complex matrix V. Or, equivalently, by substituting for W in the first equation for X above and using the fact that $AA^\dagger$ and $I_m - AA^\dagger$ are complementary projections, we may conclude that any solution to the matrix system $AX = P_A$ and $XA = P_{A^*}$ is given by

$$X = A^\dagger + (I_n - A^\dagger A)V(I_m - AA^\dagger)$$

for any $n \times m$ complex matrix V.

Reference

[1] J. K. Baksalary and R. Kala. The matrix equation $AX - YB = C$. *Linear Algebra Appl.*, 25:41–43, 1979.

Problem 67-3: A Numerical Range Area Problem

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let M be a 2×2 submatrix of a 3×3 unitary matrix. We show that the area of the numerical range of M is less than or equal to $\frac{\pi}{4}$.

Solution 67-3 by Pan Shun Lau, *University of Nevada, Reno, USA*, panlau@connect.hku.hk
and Chi-Kwong Li, *College of William and Mary, Williamsburg, Virginia, USA*, ckli@math.wm.edu

Denote by $W(A)$ the numerical range of a matrix A. For any 3×3 unitary matrix U, we have that $W(U)$ is the convex hull of the spectrum $\{a, b, c\}$ of U. If a, b and c do not lie on a straight line, then $W(U)$ is a non-degenerate triangle. In this case, a 2×2 matrix A is a principal submatrix of U if and only if $W(A)$ is an elliptical disk touching the three sides of $W(U)$; see [1]. In this situation, [2, Corollary 4.2] gives

$$\text{Area}(W(A)) \le \frac{\pi}{3\sqrt{3}}\text{Area}(W(U)),$$

where equality occurs when $W(A)$ touches the midpoint of each side of $W(U)$. It is not hard to show that a necessary condition for $W(U)$, and hence also for $W(A)$, to have maximum area is that $W(U)$ is an equilateral triangle as, for example, when U has spectrum $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. The area of this equilateral triangle is then $\frac{3\sqrt{3}}{4}$.

The above result allows for two easy generalizations. The first one answers Question 67-3.

- For any 2×2 submatrix A of a 3×3 unitary matrix U, $\text{Area}(W(A)) \le \frac{\pi}{3\sqrt{3}}\frac{3\sqrt{3}}{4} = \pi/4$.

  Proof. Note that by row and/or column permutations, every square submatrix of a unitary matrix can be regarded as a principle submatrix of some unitary matrix. □

- Let T be a 3×3 normal matrix. Then each 2×2 principal submatrix S of T satisfies

  $$\text{Area}(W(S)) \le \frac{\pi}{3\sqrt{3}}\text{Area}(W(T)).$$

  Moreover, there is a 3×3 unitary matrix with the same eigenvalues as T which has a 2×2 principal submatrix S with $W(S)$ touching the midpoint of each side of $W(T)$ and $\text{Area}(W(S)) = \frac{\pi}{3\sqrt{3}}\text{Area}(W(T))$.

  Proof. Assume that $W(T)$ is a non-degenerate triangle equal to the convex hull of its spectrum $\{\mu_1, \mu_2, \mu_3\}$. We may identify $\mathbb{C}$ with $\mathbb{R}^2$ and apply a real affine transform $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $x \mapsto xF + x_0$ for a 2×2 invertible matrix $F = (F_{ij})$ and $x_0 = (v_1, v_2) \in \mathbb{R}^2$, so that $\{f(\mu_1), f(\mu_2), f(\mu_3)\} = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. If $T = T_1 + iT_2$ for Hermitian matrices T_1 and T_2, then we can let $U = U_1 + iU_2$ with $U_1 = F_{11}T_1 + F_{21}T_2 + v_1 I_2$ and $U_2 = F_{12}T_1 + F_{22}T_2 + v_2 I_2$. Then we see that U is unitary with spectrum $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. A 2×2 principal submatrix S of T will be transformed to the corresponding principal submatrix A of U. Since an invertible affine transform will change the area by a fixed multiple, $\text{Area}(W(S)) = \frac{\pi}{3\sqrt{3}}\text{Area}(W(T))$ if and only if $\text{Area}(W(A)) = \frac{\pi}{3\sqrt{3}}\text{Area}(W(U))$. In fact, one can always construct such a submatrix A for U, and hence such a submatrix S for T. □

References

[1] J. P. Williams. On compressions of matrices. *J. London Math. Soc. (2)*, 3:526–530, 1971.

[2] D. Minda and S. Phelps. Triangles, ellipses, and cubic polynomials. *Amer. Math. Monthly*, 115(8):679–689, 2008.

Problem 67-4: The Schur Product of a Nilpotent Matrix and its Transpose

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let N be any nilpotent real matrix. Show that the Schur (entrywise) product of N with its transpose can never be a positive definite matrix.

Solution 67-4 by Lorenzo Ciardo, *Oxford University, Oxford, England*, lorenzo.ciardo@cs.ox.ac.uk

Suppose that N^k is the zero matrix, and let N have t rows and t columns. Then the kth powers of the eigenvalues of N are zero, which implies that all eigenvalues of N are zero. Therefore, $\text{Tr}(N^2)$ is zero, too, as it equals the sum of the squares of the eigenvalues of N. Letting $\circ$ denote the Schur product and e denote the vector all of whose entries are 1, we find

$$0 = \text{Tr}(N^2) = \sum_{i=1}^{t}(N^2)_{ii} = \sum_{i=1}^{t}\sum_{j=1}^{t} n_{ij}n_{ji} = \sum_{i=1}^{t}\sum_{j=1}^{t}(N \circ N^T)_{ij} = e^T(N \circ N^T)e,$$

so $N \circ N^T$ cannot be positive definite.

Also solved by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com.

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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce four new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 68-1: A Product of Projections

Proposed by Johanns DE ANDRADE BEZERRA, *Federal University of Paraíba, João Pessoa, Brazil*, veganismo.direitoanimal@gmail.com

A complex square matrix M which satisfies $M^2 = M$ is said to be a projection. Let A be an $n \times n$ Hermitian projection and B be a $n \times n$ projection. Show that if AB is a normal matrix, then AB must be a Hermitian projection.

Problem 68-2: The Expected Value of the Square of the Determinant of a Zero-One Matrix

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

A matrix is said to be a zero-one matrix if each entry of the matrix is either 0 or 1. What is the expected value of the square of the determinant of an $n \times n$ zero-one matrix? (Every element of the matrix is independently chosen and has equal probability of being 0 and being 1.)

Problem 68-3: Real Orthogonal Roots of Real Orthogonal Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

A real square matrix Q is said to be real orthogonal if $QQ^T = Q^TQ = I$. Let Q be an $n \times n$ real orthogonal matrix and let k be a positive integer. Show that the equation $X^k = Q$ has a solution where X is a real orthogonal matrix if and only if the equation $x^k = \det(Q)$ has a real solution.

Problem 68-4: A Circulant Limit

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

An $n \times n$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \pmod{n}$. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of complex numbers with only finitely many nonzero elements which satisfies $|x_1| > \sum_{k=2}^{\infty} |x_k|$. For each positive integer n, let C_n be the $n \times n$ circulant matrix whose first row is equal to the first n terms of the sequence $\{x_k\}_{k=1}^{\infty}$. Show that the limit $\lim_{n\to\infty} |\det(C_n)|^{\frac{1}{n}}$ exists.

Solutions to Problems 67-2, 67-3 and 67-4 are on pages 40–42.