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IMAGE

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FEATURE INTERVIEW

Using matrices to show results with little fuss

Rajendra Bhatia Interviewed by Apoorva Khare[1]

Rajendra Bhatia

A.K. - How did you get interested in linear algebra, and specifically in positive matrices?

R.B. - Positive matrices came much later. Getting into linear algebra was somewhat of a coincidence. I was doing my Ph.D. first at IIT Delhi (Indian Institute of Technology), then at ISI Delhi (Indian Statistical Institute). My Ph.D. advisor was K.R. Parthasarathy, and in the summer of 1976 he got interested in reading Tosio Kato's book [22] and ran a seminar on it. More or less casually, he posed the problem: "*Are the eigenvalues of matrices continuous?*"

Kato was motivated by physics, so most of the results given there were for Hermitian and unitary matrices. So we assumed that nothing was known for arbitrary matrices—which was a wrong assumption because numerical analysts had worked with them. Anyway, Parthasarathy gave me that problem. I discussed it with Kalyan K. Mukherjea, a topologist in Delhi, and eventually I answered it by giving bounds on the distance between the eigenvalues of two matrices. Only after doing this did we learn that Ostrowski and Henrici had already looked at the same question. But what I had done turned out to be different.

It was only later, when I went to the U.S., that I met people like Chandler Davis and learned more results in linear algebra and operator theory—majorization, unitarily invariant norms, and so on—which were needed for the problem I had tackled. But my interest in linear algebra started because of that seminar on Kato's book and the question that arose out of the seminar.

A.K. - And how about positive matrices, and your book on this topic?

R.B. - There was no real expert on matrix theory or matrix analysis in Delhi in those days. This was quite common at the time; we were a smaller community, and there were but a few research centers. At ISI Delhi, the faculty consisted of five or six statisticians and two mathematicians. So the areas of expertise were not too many. But I was always interested—and divided—between doing matrix analysis and working on other things, like C^*-algebras, mathematical physics... I was at Berkeley for one year as a postdoc; my advisor there was W.B. Arveson. So I learned quite a bit about C^*-algebras.

And, coming to positivity, I was myself working on it, and I found that many very good people in matrix analysis were not familiar with results in C^*-algebras. As a concrete example, I read papers (running into 20–30 pages) by electrical engineers-cum-mathematicians on the sensitivity of the Lyapunov equation. One of the things that took them a lot of work (and space) to compute was (say) the norm of the inverse of the Lyapunov map. Now of course, computing or estimating the norm of an arbitrary linear map (in finite or infinite dimensions) is a hard problem. However, in this case it was known to be a linear map which is positivity-preserving. And there was a fascinating C^*-algebra result by B. Russo and H.A. Dye (in [28]) that every such map attains its norm at the identity. Had the authors of those "Lyapunov papers" known some of these results, they wouldn't have spent so much time and effort to get norm bounds.

Then, in 2000, José Dias da Silva, who had visited ISI Delhi, invited me to deliver a lecture series in Lisbon. After some discussion, we settled on the following: I would present theorems about positivity-preserving maps, but only in matrix

[1]Department of Mathematics, Indian Institute of Science, Bangalore, Karnataka, India; khare@iisc.ac.in

language. This was because we felt that a matrix theory person would be unlikely to read papers on C^*-algebras, which are essentially dealing with infinite dimensions. There are lots of technicalities that one must be careful about. But for matrices one can show those results with less "fuss."

So I wanted to make broadly known—and moreover with accessible proofs—several C^*-algebraic results that experts in matrix theory may not know about. Such were my goals; and I gave a course in Lisbon 2001—returning to speak again in 2002 and 2003—in which theorems on C^*-algebras were developed only in the finite-dimensional setting. It was very good for me, because the audience consisted of some of the best linear algebraists around. So I had a mature, interested audience; and I could pitch my course at a high level.

Rajendra Bhatia at Berkeley in 1979

Each time I lectured in Lisbon, I added to the lecture notes on positivity I had written up the first time. Later, I also gave a course at ISI Delhi. That is how my interest in positivity was sustained—and also how my book [9] on positive definite matrices came about.

The book was started in 2001 with my lecture series, and finished by 2006. (It came out in 2007.) The final chapter in the book happened because I started, in the meantime, to work on the geometric mean of several positive definite matrices, which had been a longstanding problem. I was able to prove some results with John Holbrook in 2004. This went on to become a chapter in my book, and then I finalized the book. In fact, this chapter is what seems to have drawn attention to the book, of people who were working on brain-computer interfaces, radar imaging... Such people found the geometric mean to be very useful. In fact, if you look at the citations of the book, maybe more than half of them come from the engineering literature. The earlier parts of the book also gained in importance because of the explosion of work on quantum information.

A.K. - Of course, this is one of several books you have authored. Can you tell us how your first book ended up getting written?

R.B. - As I mentioned, for my Ph.D. thesis I worked on the perturbation of eigenvalues. I kept working on this, and from Chandler Davis I came to know of the important unsolved problems in this area at the time. After that, I was at TIFR (Tata Institute of Fundamental Research) in Bombay, which had an excellent library. I read the original papers on the kind of matrix analysis that I was doing, by Hermann Weyl, Ky Fan, George Pólya, and others [29, 19, 20, 26, 21], and from these readings I made my own notes. Combining these notes with the ones I had created from my earlier reading and research on the perturbation of eigenvalues, I ended up with a significant amount of material.

Later, I was visiting Hokkaido University, Sapporo, where Tsuyoshi Ando was. I gave lectures there, and by 1985 I had finalized my notes—which then became my first book [4]. While I had originally contemplated stepping away from the area after publishing my book, it generated enough interest and activity that I could not really leave the topic!

A.K. - How about your influential book *Matrix Analysis*?

R.B. - So as I said, my first book was ready in 1985, and was submitted—but not yet published (it appeared in print in 1987). In 1986, Chandler Davis came to Delhi. There, he saw my notes and suggested jointly expanding them into something more comprehensive. I told him that those notes were already submitted, but that I liked the idea of writing another book together on the subject. Of course, as is well known, Chandler was always involved in dozens of activities. So the plan we made was that we would give a graduate course together in Toronto, and write a book coming out of that course. We gave such a course on matrix analysis twice, in 1991 and 1994—and covered much of the content in my book. But given how busy Chandler was, the actual execution by us—of sitting down and writing the book—didn't get done beyond maybe one or two chapters' worth of polished notes. So in 1995 he suggested that I go ahead and write the book by myself. But his contribution to the book is unmistakable.

By then I had learned various other facts from different sources. These were the days before the Internet, and so often one would learn things in conversation. I used to be asked questions on matrix analysis, and I collected such questions and results. These too got added to the book [6], which appeared in 1997. And, like my 2007 book, various things we had worked out purely for their intrinsic interest drew the attention of more applied researchers—in this case, in quantum information theory and quantum computing—after a few years.

A.K. - Moving from books to papers: Can you tell us about some of the papers/results you can recall having written, and maybe the stories around them?

R.B. - I can recall a few papers that were important for different reasons. Let me mention them in chronological order.

1. One of my best papers is with Chandler Davis and Alan McIntosh [10]. It involves deep ideas from Fourier analysis. Our work was carried out during 1981–83, and, as there was no Internet at that time, we had to rely on letters. Alan McIntosh was a prominent Australian analyst—he was at Macquarie University in Sydney and then at ANU Canberra, where you [A.K.] are currently visiting. At that time, for a year he was an Australian in Paris. And Chandler Davis—a Canadian—was also in Paris, in and out. I was in Bombay and Delhi. So we had to rely on letters, and each letter took three weeks to reach the other person. Our progress was slow; I made a count of the number of letters exchanged during work on this paper, and it was 41! So the paper developed over 41 letters and four continents. And I think it was something that was technically difficult, and had very good ideas. Davis was a pioneer in this area. He had written a series of papers [16, 17, 18] called "The rotation of eigenvectors by a perturbation," and our paper took that body of work forward.

2. The second paper I can recall is in matrix inequalities. At the end of the 1980s, there were very few people working in this area. In 1988, Fuad Kittaneh and I wrote a paper [14] on a matrix/operator-valued arithmetic mean-geometric mean inequality. It seems to have stimulated a lot of activity in this area in the following years.

3. The third I can recall is the paper [15] with K.R. Parthasarathy—it is our only joint work. We did the work in 1997, and it appeared a few years later. In a sense, it carries forward the work on AM-GM inequalities—but it also brought in good ideas from Fourier and harmonic analysis. In particular, we made use of Bochner's theorem in proving positivity, which opened up new ways of thinking about such inequalities. Almost simultaneously, H. Kosaki got into it, and wrote several papers, Springer lecture notes, and so on. Some of that work is with F. Hiai, and they took it forward a lot, applying very good analysis to these matrix inequalities. So I remember that paper of ours. And to get just a bit technical: The key idea was that we needed to prove the positivity of a matrix with entries of the form $f(\lambda_i, \lambda_j)$, with λ_i real scalars. We first converted the matrix into a congruent matrix with entries of the form $g(\lambda_i - \lambda_j)$. Now the positivity of this matrix could be verified using Bochner's theorem for positive definite functions on $\mathbb{R}$.

4. The next one I recall is on the geometric mean, with John Holbrook [12]. This has an interesting story. I was the chairman of the Library Committee at ISI Delhi for quite a few years. As such, various books used to come to me from time to time. One such book was Serge Lang's *Fundamentals of Differential Geometry* [23], published by Springer in 1999 in their "Graduate Texts in Mathematics" series. I was flipping through its pages, and discovered an interesting section title. Geometers don't usually use the phrase "positive definite matrix"—they talk instead about the "Riemannian symmetric space $\mathrm{GL}_n/\mathrm{U}_n$", which might not appeal to matrix theorists instantly. But in Lang's book, one of the chapters opened with a section titled "$\mathrm{Pos}_n(\mathbb{R})\ldots$" This drew my attention. And then I read that Lang proved what he called the exponential metric increasing (EMI) property, following an elegant approach due to Dan Mostow.

   At that time, I was working on positive definite matrices. So I became curious about this inequality which Lang was describing, and which I didn't know. It turned out that in this chapter, Lang shows the EMI property—that leads to the fact that positive definite (real symmetric) matrices form a manifold of non-positive curvature. And as I read that chapter, I realized that if one knew a certain kind of linear algebra, then what Lang proved in that one chapter could be done in one page.

Rajendra Bhatia delivering the Hans Schneider Prize lecture "Another metric, another mean" at the 2017 ILAS meeting

I couldn't believe it. But I happened to be going to Yale that summer, so I looked for Lang while there. I couldn't find him, so I left a note in his mailbox, mentioning the proof, and asking if it looks correct, might it be useful, etc. Two months later, I got a warm, effusive note from Lang: "This is wonderful; I am visiting Berkeley right now, and I have given it to a colleague who is currently teaching a course in differential geometry; when I revise the book, I will use this proof; etc."

So I felt that one could further look into this. I published my proof of the EMI property—and its strengthenings—in a short note [7]. Shortly after, John Holbrook was visiting ISI Delhi and we started to discuss this circle of ideas, going in another direction as follows. The geometric mean of two matrices A and B was known, with a somewhat mysterious formula due to Pusz and Woronowicz, in a paper [27] on C^*-algebras, motivated by mathematical physics. With the realization that this formula

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

gives precisely the midpoint of the geodesic (in the aforementioned manifold) joining A and B, the next step becomes obvious: If for two matrices their geometric mean is the midpoint of the geodesic joining them, then for three matrices A, B, C it should be the barycentre of the "hyperbolic triangle" with vertices A, B, C. So that is what Holbrook and I developed in our paper. Unknown to us at that time, just a little earlier Maher Moakher had proposed the same definition of the geometric mean, derived several of its properties, and pointed out applications in elasticity and diffusion tensor imaging [24].

Coincidentally, as we were working on this, Ando, Li, and Mathias had a paper [1] in which they proposed a different candidate for the geometric mean of k positive definite matrices. They listed ten properties of that mean, and showed that their candidate satisfies those properties. So it became sort of incumbent on us to do the same for our "barycentric" mean. For some time, it wasn't clear that this held for the barycentric mean—there are properties, including operator monotonicity and concavity, that do not arise from geometric considerations, but from elsewhere. That all of these properties do indeed hold took some time to prove, in our work and that of others (principally Jimmie Lawson, Yongdo Lim, and Miklos Palfia).

Meanwhile, people working in applications—radar, data processing, etc.—also started to use this barycentric mean, and they found it useful. (They didn't care about operator monotonicity or concavity...) So, over the next 5–10 years, this mean became "the" geometric mean: It satisfies all the properties listed by Ando–Li–Mathias, plus more. But our getting into it was a coincidence, thanks to Lang's book coming to me as the chairman of the Library Committee.

5. Finally, let me mention recent work with Tanvi Jain and Yongdo Lim, on a different metric that is also useful in optimal transport, statistics, and so on: the Bures–Wasserstein distance. Because the preceding body of work on the barycentric/geometric mean was so useful to people in applications, and given its theoretical properties, we tried to develop similar results for the Bures–Wasserstein metric: barycenters of points under this distance, and their properties. Many people have worked on this program, usually proceeding via optimal transport and differential geometry. Some of the geometric mean results were thus known in this setting; others were not. But, using our experience from the geometric mean work, and proceeding via matrix theory—a different approach—we

were able to prove similar results to those for the geometric mean. This appeared a few years ago [13]. Bures was motivated by quantum information theory, while Wasserstein came from optimal transport. So this work should be useful in these areas, in addition to the theoretical side.

A.K. - You have founded—and still manage—an important series of mathematics books in India: the Texts and Readings in Mathematics (TRIM) series, which has published books not just by Indian mathematicians but also by prominent mathematicians from other countries. How did that happen?

R.B. - When I was a college student (first of physics and then of mathematics), between 1968–73, the textbooks that were prescribed were primarily by American publishers. For instance, for analysis we would have Rudin; for algebra, Herstein; for topology, Simmons; and so on. Virtually every textbook we studied was American. And the Indian books we saw were usually of the "answer key" flavor—for instance, someone published a book which had worked out most of the exercises in Herstein's book. So Indian books were looked down upon by good students—but they still did use them for the difficult problems! And I was unhappy with this then-prevalent idea that Western authors write books and Indian authors mostly write keys to these books.

This unhappiness lingered with me over the years. Meanwhile, in India people did, from time to time, bring out mathematics book series with original content. But these series did not last long.

Then, in the 1980s, when I was teaching (at the University of Bombay), someone asked me if I could prescribe an Indian book. And there wasn't any; which bothered me anew. So I kept thinking if this could be changed. Then, in 1990—when I was at ISI Delhi—I circulated among students and colleagues my notes on Fourier series. Some of my contemporaries said that I should publish those notes. I pointed out that mathematics publications in India at that time were rather sporadic. They suggested that we could all pitch in with a book each, and I believed them. I did ask that the first book not be mine, so the first book was *Linear Algebra and Linear Models* by Ravindra B. Bapat [2, 3], the second was my (notes on) *Fourier Series* [5, 8], and the third book was *Representations of Finite Groups* by Chitikila Musili [25].

We then began looking for publishers. The best-known publishers in India were subsidiaries of American publishers. And it became clear, when I talked to them, that their interest was in "buying cheap in India and selling expensive abroad." They would pay no royalties, print the text on cheap quality paper within India... Luckily, I found an Indian publisher, the Hindustan Book Agency, who was not so well-tested at the time, but in the long run has turned out to be very good. And, happily, it also coincided with the development and ever-increasing use of LaTeX. So mathematics books became easier to publish in India.

And so the TRIM Series started. My aim was to have mathematics books written mostly by Indians, published in India, and which could compete with series like Graduate Texts in Mathematics by Springer, or Graduate Studies in Mathematics by the AMS. It wasn't clear that it would succeed, but it did, and we are now 80+ books into the series. And we do have contributions by leading international authors—including Armand Borel, Alain Connes and Matilde Marcolli, David Mumford and Tadao Oda, and Terence Tao, among others. We have also reprinted books by John W. Milnor, Carl Ludwig Siegel, and Hermann Weyl. That said, most books are indeed written by Indian authors. And I am very happy that the TRIM Series has succeeded so far.

In fact, there has been a "reverse brain drain," as I would call it—earlier we used to buy rights from Western publishers, but now many books have gone in the other direction. Of course, in these 30+ years, the nature of publishing itself has changed. So now it doesn't seem that important to publish, because students can download books from the internet, and so on. So book publishing is no longer what it was when the TRIM Series first started. But at that time, I think, it was very important.

A.K. - Speaking of changes and the Internet—you have seen times change from using punch cards, to text-based computers, to modern-day laptops, and now, post-Covid, to the world of Zoom and Skype. Having been mathematically active during all these technological eras, can you share your thoughts about the ease and pace of doing research through the years?

R.B. - Well, I must confess I am not very tech-savvy. Of course, we are talking right now online via video-chat, sitting on different continents. But I have not used too much software in my work, just occasionally. But let me recall one example. In 1983, some version of MATLAB was available, and Chandler Davis had it. It wasn't on a "usual" type of desktop, but on some computer in the mathematics department. We used to test some of our conjectures using that software. And from his grant, he had employed one of the graduate students from the Computer Science department to help us, from time to time. So whenever we got stuck and didn't know how to use the machine, we would pick up the phone and call that person over to help us.

John Holbrook was into computers very early, and would do many calculations and tests on his computer. My first paper with John [11] was written in 1983 and typed by him with pre-LaTeX software. Colleagues were as impressed by the typesetting as by the results. Later, while working on the geometric mean in the early 2000s, one of our concerns was whether the Ando–Li–Mathias mean and our barycentric mean were in fact the same. John constructed a computer example to show they were not. Earlier, in 1989, John had constructed a counterexample to a long-standing conjecture on spectral variation of normal matrices on which several prominent linear algebraists had worked.

So I have seen my coworkers use computers from early days. But I go only occasionally to MATLAB to test some of my guesses and conjectures. I am not entirely satisfied with this state of affairs—namely, that some of my work has been interesting to computer scientists, but I use computers as little as I do!

Of course, the typing up of papers, exchanging drafts via e-mails, etc., has become much easier and faster over time, thanks to computers. In fact, I remember that in 1979–80, Donald Knuth—who was very happy to have developed TeX—gave a talk in Berkeley. He did a bit of a show—he turned around while giving his talk, and people could see "TeX" written on the back of his t-shirt. But nobody believed what he said, that we would soon be typesetting our own papers. In 1980, it seemed very far away.

So of course my working has changed with technology. And during the pandemic, the Internet really helped me keep in touch with colleagues and with ongoing research. On the other hand, browsing "freestyle" has stopped. Earlier, you would go to the library to search for a specific book or paper in some volume of a journal. And you would end up discovering an interesting book or paper, adjacent to the book or paper you meant to find; or maybe just pick up a book or volume in the library and learn new things "by accident." (I mentioned earlier the example of my paper with Holbrook on geometric means, which came out of my browsing through Lang's book.) But this has more or less stopped now.

A.K. - Finally, when did you join ILAS and can you share some of your thoughts on ILAS over the years?

R.B. - In fact, ILAS was preceded by the "International Matrix Group" (IMG), if I recall correctly. Hans Schneider was the force behind starting the IMG; I was told about it in 1988 by Ludwig Elsner. The way they functioned at the time was to circulate periodically a newsletter, together with some correspondence between the members. They asked me to be the representative for India for this group.

Rajendra Bhatia receiving the Hans Schneider Prize at the 2017 ILAS meeting

Soon after this, ILAS was formed. In August 1989 we had the inaugural ILAS conference in Provo, Utah—at Brigham Young University. It was a very good conference. (And I think it is also the only conference I have attended where no coffee was served!)

Over the years, I have seen ILAS grow and do excellent work for linear algebra, not only to keep its members updated about the latest research in the field, but also to advance the subject, attract more people, and to underline the importance of linear algebra and matrix theory in many different areas (both theoretical and applied) to the broader mathematical community. I think now many more people realize the importance of this field, especially with the recent thrust on quantum computing and other areas of engineering, applications to data science and beyond, and theoretical advances as well.

But yes, I started out in the "mother of ILAS"—the International Matrix Group—and have been with ILAS all along.

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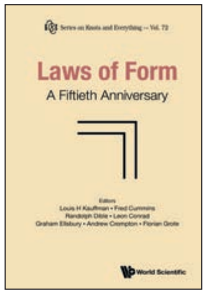

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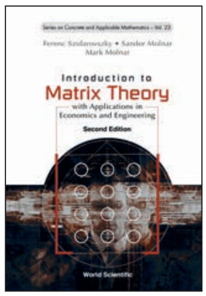

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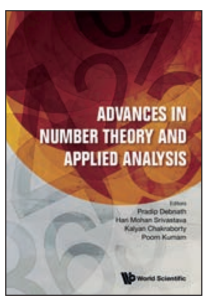

edited by **Pradip Debnath** *(Assam University, India)*, **Hari Mohan Srivastava** *(University of Victoria, Canada)*, **Kalyan Chakraborty** *(Kerala School of Mathematics, India)* & **Poom Kumam** *(King Mongkut's University of Technology Thonburi (KMUTT), Thailand)*

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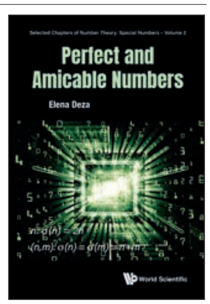

by **Elena Deza** *(Moscow State Pedagogical University, Russia)*

This book contains a detailed presentation on the theory of two classes of special numbers, perfect numbers, and amicable numbers, as well as some of their generalizations. It also gives a large list of their properties, facts and theorems with full proofs. This book contains a list of open problems and numerous questions related to generalizations of the classical case, which provides a broad perspective on the theory of these two classes of special numbers. Perfect and Amicable Numbers can be useful and interesting to both professional and general audiences.

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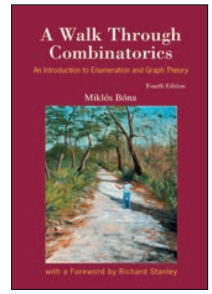

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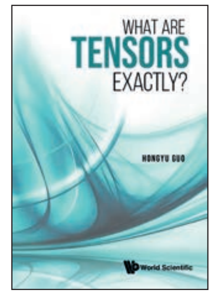

by **Hongyu Guo** *(University of Houston-Victoria, USA)*

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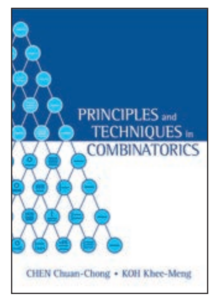

by **Chen Chuan-Chong** *(NUS, Singapore)* & **Koh Khee-Meng** *(NUS, Singapore)*

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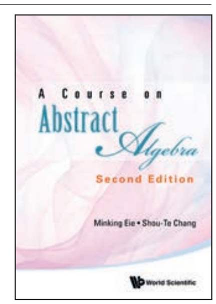

by **Minking Eie** & **Shou-Te Chang** *(National Chung Cheng University, Taiwan)*

It begins with a description of the algebraic structures of the ring of integers and the field of rational numbers. Abstract groups are then introduced. Technical results such as Lagrange's theorem and Sylow's theorems follow as applications of group theory. The theory of rings and ideals forms the second part of this textbook, with the ring of integers, the polynomial rings and matrix rings as basic examples. Emphasis will be on factorization in a factorial domain. The final part of the book focuses on field extensions and Galois theory to illustrate the correspondence between Galois groups and splitting fields of separable polynomials.

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LINEAR ALGEBRA EDUCATION

From Beginner to Expert: Using Python Labs to Increase Linear Algebra Fluency

Emily J. Evans, Brigham Young University, Provo, UT, USA, ejevans@mathematics.byu.edu

1. Introduction. More often than we would like as linear algebra instructors, a first course in linear algebra can be unappealing to students. If the focus of a first course is on the rich and deep mathematics of linear algebra, non-majors—and occasionally majors—find the class too challenging and not practical. If the focus is instead on more practical aspects of linear algebra, majors and non-majors alike frequently find the class boring, algorithmic, and unmotivated. Introducing applications to the curriculum helps, but not every faculty member is equally able to include applications in a natural way.

With these challenges in mind, at Brigham Young University (BYU) we have created a linear algebra curriculum that takes advantage of Python labs to provide motivation and applications, and to introduce students to real-world tools for solving linear algebra problems. Python was chosen specifically because it is freely available, relatively easy to use, rapidly becoming the language of data science, and accessible via Google Colaboratory through a web interface. Somewhat unusually, when we designed our linear algebra curriculum to use Python, we started with our second linear algebra course, instead of the beginning linear algebra course. The reason for this was two-fold; first, in a second linear algebra course, the majority of students are mathematics majors, allowing us a friendlier audience with which to determine what worked best and, second, this linear algebra course was designed for our applied mathematics majors, who typically are interested in getting their hands dirty with programming.

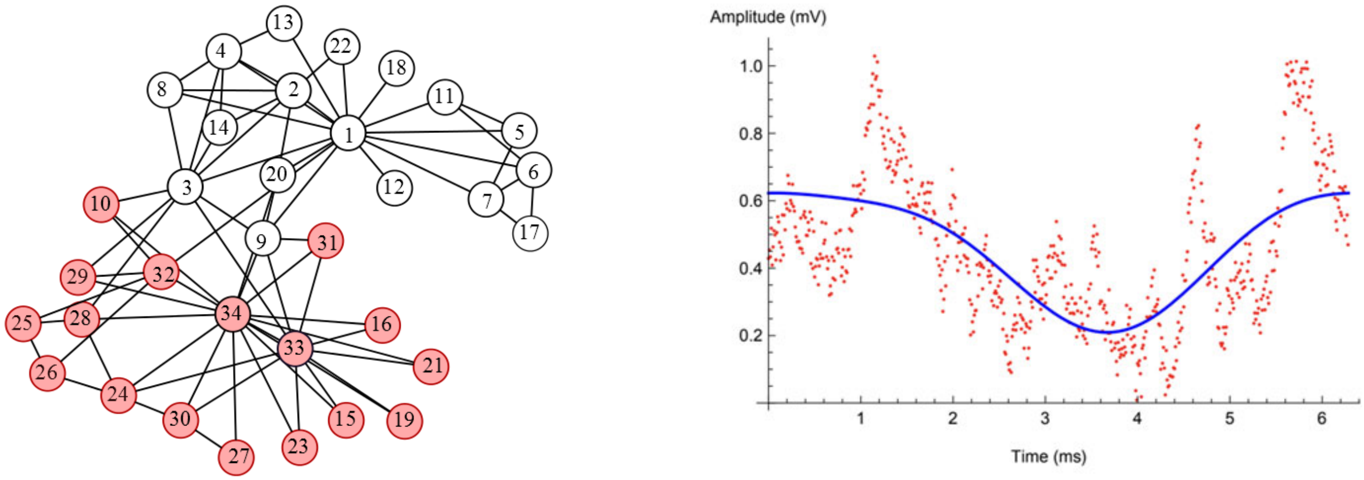
2. Advanced linear algebra. For our advanced linear algebra course, students are required to take the accompanying lab course, and these labs directly complement the classroom material. All of the labs, except for a few introductory ones, follow the pattern of: learn about a topic in class, implement that topic in the lab, learn to utilize relevant Python packages, and, finally, solve a real-world problem.

To illustrate this model, we consider one topic taught in class, the Drazin inverse, a generalized inverse that preserves certain spectral properties of a matrix. In class, the students are taught the properties of the Drazin inverse and why one would choose this inverse over other generalized inverses such as the Moore-Penrose pseudoinverse. For homework, students verify the computation of a Drazin inverse and then prove a variety of properties of the Drazin inverse. In this lab, the students create code to do the following:

1. Given a matrix and a candidate Drazin inverse, verify that the candidate inverse is the Drazin inverse of the given matrix. This reinforces the various properties of the Drazin inverse that the students have proved in class.
2. Given a matrix, calculate the Drazin inverse using the Schur decomposition of the matrix. This breaks our model somewhat, because no Python package exists to calculate the Drazin inverse. So instead we have students use a Python package to calculate the Schur decomposition, which they then use to determine the Drazin inverse.
3. Calculate the effective resistance between nodes in a given network using the Drazin inverse, and use the results to solve several real-world link prediction problems; see Figure 1(a) for a sample network for which the students predict the most probable link.

The response of students to the labs has overwhelmingly been positive, and students have reported that doing the labs both reinforces their understanding of linear algebra and makes them stronger programmers. The advanced linear algebra labs were developed as part of our applied and computational mathematics program (`https://acme.byu.edu`) and include labs that cover topics ranging from linear transformations to iterative methods. In addition to linear algebra labs, we also developed labs for a wide variety of applied mathematics topics, including optimization, data science, and numerical methods for differential equations.

3. Introductory linear algebra. Based on the success of our upper-division linear algebra course and the accompanying labs, the math department at BYU completely redesigned the curriculum for our introductory linear algebra course. Perhaps surprisingly, we reduced the number of topics taught in the classroom portion, allowing us to reduce the credit count from 3 credits to 2 credits. Of the topics that were removed, most were moved to the lab course, with the exception of a few proofs. This initially was met with pushback from the faculty, so we also created a new upper-division linear algebra course for our majors that allows students to dive deeper into the beauty of linear algebra. With this reduction in credits, we designed a new lab course which, although optional, over 90% of linear algebra students enroll in. The goal of this lab course is to reinforce the topics covered in class and introduce exciting applications of linear algebra. Like



(a) A sample network.

(b) Fitting data to a curve.

Figure 1: On the left, the social network studied by students when predicting links in the Drazin inverse lab, adapted from data by Wayne W. Zachary [1]. On the right, sample data and the best-fit curve from the least squares lab.

the advanced linear algebra labs, these labs are written in Python. Unlike the advanced linear algebra labs, we designed these labs to be offered as an online asynchronous course, supported by a group of teaching assistants that can offer one-on-one and small-group help. We have even automated the grading of these labs, and allow students to submit each lab as many times as they wish before the deadline. For us, one advantage of using teaching assistants and an automated grading system is that it has allowed us to develop and then teach the labs despite only a small fraction of our faculty being fluent in Python.

A few of the classroom topics that are reinforced in the labs include Gaussian elimination, Gram-Schmidt orthogonalization, and least squares. Each of these topics shares the characteristic that both in-class examples and homework are typically limited to relatively small matrices. However, in a lab setting, with access to software, much larger examples can be explored because students are no longer calculating by hand. Another very important aspect of these topics is that they are algorithmic and lend themselves very well to the students' programming them. We have discovered that having the students "teach" the computer to perform standard linear algebra operations reinforces their understanding of the material. Each of these reinforcement labs (which are a subset of the labs offered) also includes an interesting application or problem to solve using Python packages. For the students, learning to use these packages may be necessary for their degrees, programs, or careers. An example of a reinforcement problem can be seen in Figure 1(b), where students use least squares to fit a sine-like curve to data.

In addition to labs designed to reinforce classroom topics, we also include labs that introduce exciting applications to reveal the wide applicability of linear algebra. One such example is a data science lab that allows students to identify potentially cancerous tissue. In this lab, students review projections and write code that projects from an n-dimensional space to a plane. We then briefly teach them how covariance matrices can identify directions of greatest variance, and finally have them perform principal component analysis on a cancer data set.

To conclude, we have developed a collection of linear algebra labs that support the linear algebra learner, from beginner to expert. Python was chosen because of its widespread use, because it is open source, and also because good web browser-based solutions exist that make it very accessible to students. We have chosen to keep our curriculum dynamic by creating new labs every year or two so that students can be exposed to relevant and timely topics. Students have reported that their experience in the lab courses helps enhance their understanding of the topics and their appreciation of the utility of linear algebra.

Development of the labs was supported under a National Science Foundation TUES Phase II grant DUE-1323785.

References.

[1] W. W. Zachary. An information flow model for conflict and fission in small groups. *Journal of Anthropological Research*, 33(4):452–473, 1977.

JOURNAL ANNOUNCEMENTS

Special Issue of *Special Matrices* Dedicated to Frank J. Hall

Contributed announcement from Carlos Fonseca, *Special Matrices* Editor-in-Chief

The journal *Special Matrices* will publish a special issue dedicated to Frank J. Hall, in celebration of his contributions to matrix theory over the last four decades.

All contributions in the areas of matrix theory in which Professor Hall has made a significant impact, such as generalized inverses, combinatorial matrix theory, integer matrices, sign pattern matrices, nonnegative matrices, consimilarity, spectral graph theory, and Gershgorin theory, are welcome.

The invited editors for this special issue are:

- Zhongshan Li, Georgia State University, Atlanta, GA, USA
- Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL, USA
- Tin-Yau Tam, University of Nevada, Reno, NV, USA
- Bryan Shader, University of Wyoming, Laramie, WY, USA
- Marina Arav, Georgia State University, Atlanta, GA, USA
- Hein van der Holst, Georgia State University, Atlanta, GA, USA

All manuscripts will undergo the standard peer review process (single-blind, with at least two independent reviewers) employed by the journal. The deadline for submissions is December 20th, 2023.

Special Matrices is indexed by SCOPUS, Web of Science, MathSciNet, zbMATH, and other major databases, and has earned the DOAJ Seal, awarded to journals that demonstrate best practice in open access publishing

The *Special Matrices* home page is `https://www.degruyter.com/journal/key/spma/html`; details on the special issue are at `https://www.degruyter.com/publication/journal_key/SPMA/downloadAsset/SPMA_CFP_Frank_Hall.pdf`.

Send News for *IMAGE* Issue 71

IMAGE seeks to publish all news of interest to the linear algebra community. Issue 71 of *IMAGE* is due to appear online on December 1, 2023. Send your news for this issue to the appropriate editor by October 15, 2023. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- interviews of senior linear algebraists to Adam Berliner (`berliner@stolaf.edu`)
- problems and solutions to Rajesh Pereira (`pereirar@uoguelph.ca`)
- linear algebra education news and articles to Anthony Cronin (`anthony.cronin@ucd.ie`)
- advertisements to Amy Wehe (`awehe@fitchburgstate.edu`)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (`jephianlin@gmail.com`)
- proposals for book reviews and other articles to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)

Send all other correspondence to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`).

For past issues of *IMAGE*, please visit `https://www.ilasic.org/IMAGE`.

OBITUARY NOTICE

Ludwig Elsner, 1939–2023

Submitted by Volker Mehrmann

With deep sadness we announce that Ludwig Elsner has passed away. He died on February 25th, 2023, in Halle (Westfalia), Germany, at the age of 84. Ludwig, professor emeritus of mathematics at the University of Bielefeld, has been well known for his excellent work in linear and multi-linear algebra, in numerical linear algebra, and in particular for his inclusion theorems for eigenvalues, with their concise and elegant proofs.

His family, as well as his many friends, colleagues and students all remember him as an attentive, kind, calm, caring, competent and appreciative person. A more extensive obituary will be posted soon.

ILAS NEWS

Childcare Grants for ILAS Conferences

Contributed announcement from Daniel Szyld, ILAS President

ILAS will provide grants to ILAS members who are parents of young children so as to make it easier to attend an ILAS Conference. The grants, up to the amount of $500, can be used to pay for childcare either at the conference city, or the home city of the ILAS member. In the latter case, it is for any additional childcare expenses incurred because the ILAS member is attending the conference. The grant can also be used to subsidize the travel of a person who would take care of the child during the conference.

Applications need to be sent at least two months in advance of the conference dates and must include the name of the ILAS member, the name(s) of the child(ren) and their age(s), as well as the expected budget and type of expense. Applications can be sent directly to the ILAS President, who will communicate the decision to each applicant no later than one month after the application is received.

Reimbursements will be sent to the applicant after the applicant has attended the conference and once appropriate receipts for up to $500 have been submitted to the ILAS Secretary/Treasurer.

Other initiatives that ILAS has put into place to accommodate childcare needs relating to conference participation:

- Aiming to have a break every two hours or so during ILAS conferences to allow parents see their children if necessary.
- Invited speakers with young children may stay in apartments and not necessarily in a hotel.
- If a caretaker/companion is traveling with the parent, ILAS will facilitate free registration to the conference if such caretaker wishes to attend a few lectures.
- The ILAS President will aim to put in contact parents of young children in case they want to coordinate activities.
- Children will be welcome at the coffee breaks, excursions, and other social activities.

Taussky-Todd Prize Awarded to Stefan Güttel

ILAS has awarded the 2023 Taussky-Todd Prize to Stefan Güttel (University of Manchester) for "deep and impactful work on rational Krylov methods for nonlinear eigenvalue problems and matrix functions, in all aspects: analysis, software development, and applications." He will deliver the prize lecture at the 25$^{\text{th}}$ ILAS Conference in Madrid.

The selection committee consisted of Nair Abreu, Ilse Ipsen (chair), Joseph Landsberg, Pauline van den Driessche, and Daniel Szyld.

ILAS Election Results

Daniel B. Szyld has been re-elected to a further three-year term as ILAS President, beginning March 1, 2023.

Chi-Kwong Li and Fernando De Terán were elected to three-year terms as members of the ILAS Board of Directors. They began their terms on March 1, 2023.

ILAS President/Vice President Annual Report: May 1, 2023

Respectfully submitted by Daniel B. Szyld, ILAS President, szyld@temple.edu
and Froilán Dopico, ILAS Vice President, dopico@math.uc3m.es

The past year has been another momentous year for ILAS. Among the highlights are that the ILAS Community met again in person after the hard times of the COVID-19 pandemic in the 24$^{\text{th}}$ ILAS Conference held in Galway, Ireland (June 20–24, 2022); the first ILAS Richard A. Brualdi Early Career Prize was awarded to Michael Tait; the second participation of ILAS as a partner in the Joint Mathematics Meetings (JMM) 2023, the first time as partners in person; and a new childcare grant program for ILAS members who are parents of young children so as to make it easier to attend an ILAS Conference.

1. Board-approved actions since the last report include:

    - The Board approved a new dues structure for the Society. It was decided to maintain the previous dues of \$40 per year for the "standard" membership. A new category of Early Career Member was established and defined as a person within five years of receiving a PhD degree. The dues for Early Career members are at 50% of the standard membership, that is, \$20 per year. As in the past, any member can request that their dues be waived for financial hardship. Current student members continue to enjoy free membership. This dues structure is reflected in the ILAS website at `https://ilasic.org/join-ilas`. Lifetime membership continues to be available. Paying for more than one year at a time is also possible.
    - The Board approved new reimbursement guidelines for the travel and accommodation expenses for ILAS Executive Officers, other officers, and editors in Conferences. These guidelines state clearly in one document the set of common practices followed so far by ILAS on this issue.
    - The Board approved on October 20, 2022, some revisions of the ILAS Bylaws with the main goal of avoiding the possibility of a conflict of interest in the appointment of the nominating committee members. The new Bylaws can be found at ILAS Website.
    - The Board approved that the 26$^{\text{th}}$ Conference of the International Linear Algebra Society will be held in Kaohsiung, Taiwan, June 23–27, 2025, with Jephian C.-H. Lin (National Sun Yat-sen University) and Matthew M. Lin (National Cheng Kung University) as main organizers.
    - The Board approved an increase of \$50 for the institutional membership rates. This means an increase from \$200 to \$250 for the fees of silver institutional members and from \$350 to \$400 for the fees of gold institutional members.
    - The Board approved a grant childcare program for ILAS members who are parents of young children to make it easier to attend an ILAS Conference. The grants up to the amount of \$500 can be used to pay for childcare either at the conference city, or the home city of the ILAS member. In the latter case, it is for any additional childcare expenses incurred because the ILAS member is attending the conference. The grant can also be used to subsidize travel of a person who would take care of the child during the conference. Complete details can be found at `https://ilasic.org/childcare-grants`

2. Other news:

    - The inaugural ILAS Richard A. Brualdi Early Career Prize was awarded to Michael Tait (Villanova University, USA) "for outstanding contributions to spectral theory of graphs, developing new techniques, and settling several long-standing open problems in that area." Michael Tait will present the prize lecture at the 25$^{\text{th}}$ ILAS Conference in June 2023 in Madrid (Spain). The selection committee consisted of: Marina Arav, Volker Mehrmann, Shmuel Friedland (chair), and Daniel B. Szyld (ILAS President, ex officio).
    - The 2023 ILAS Taussky-Todd Prize was awarded to Stefan Güttel (University of Manchester, UK) "for deep and impactful work on rational Krylov methods for nonlinear eigenvalue problems and matrix functions, in all aspects: analysis, software development, and applications." Stefan Güttel will present the prize lecture at the 25$^{\text{th}}$ ILAS Conference in June 2023 in Madrid (Spain). The selection committee consisted of Nair Abreu, Ilse Ipsen (chair), Joseph Landsberg, Pauline van den Driessche, and Daniel B. Szyld (ILAS President, ex officio).
    - The Joint Mathematics Meetings 2023 (JMM 2023, an ILAS partner conference) was held in Boston, January 4–7, 2023. The ILAS address was given by Apoorva Khare (Indian Institute of Science, Bangalore) with title "Analysis and applications of Schur polynomials". There were four ILAS Special Sessions: (1) Matrix Analysis and Applications (organized by Hugo Woerdeman and Edward Poon); (2) Matrices and Operators (organized by Mohsen Aliabadi, Tin-Yau Tam, and Pan-Shun Lau); (3) The Inverse Eigenvalue Problem for a Graph and Zero Forcing (jointly presented with AIM (American Institute of Mathematics) and organized by Mary Flagg

and Bryan A. Curtis); and (4) Innovative and Effective Ways to Teach Linear Algebra (organized by David M. Strong, Gilbert Strang, Sepideh Stewart, and Megan Wawro).

- The Executive Board accepted the recommendation of the JMM Committee that Stephan Ramon Garcia from Pomona College (California) be the ILAS Lecturer at the Joint Mathematics Meetings 2024, to be held in San Francisco, January 3–6, 2024. In addition to this ILAS Lecture, there will be five ILAS Special Sessions scheduled at JMM 2024. The JMM committee consisted of Nair Abreu, Ilse Ipsen (chair), Joseph Landsberg, Pauline van den Driessche, and Daniel B. Szyld (ILAS President, ex officio).
- The Israel Gohberg ILAS-IWOTA Lecture Selection Committee selected the ILAS Member Mark Embree as Israel Gohberg Lecturer in the International Workshop on Operator Theory and its Applications (IWOTA) to be held at the University of Kent, UK, August 12–16, 2024. The selection committee consisted of Jussi Behrndt (IWOTA), Kelly Bickel (ILAS), Micheal Dritschel (IWOTA) and Ilya Spitkovsky (ILAS). More information on this lecture series can be found at `https://ilasic.org/special-lectures-support/`
- The 24$^{\text{th}}$ ILAS Conference at Galway (Ireland, June 20–24, 2022) was held in person. It was originally scheduled for 2020 but it had to be postponed due to the COVID-19 pandemic. Despite the uncertainties caused by the pandemic, the conference was very successful and included 260 registered participants, a total of 256 talks and posters (6 posters and 250 talks). There were 10 plenary talks and 40 contributed talks, the rest of the talks were in 22 minisymposia. Some highlights of the conference were the Hans Schneider Prize Lecture delivered by Pauline van den Driessche and the first Israel Gohberg ILAS-IWOTA Lecture in an ILAS Conference delivered by Paul Van Dooren.

3. ILAS elections ran November 1–30, 2022, and proceeded via electronic voting. The following were elected to offices with three-year terms that began on March 1, 2023:

   - President: Daniel B. Szyld
   - Board of Directors: Fernando De Terán and Chi-Kwong Li

   The following continue in the ILAS offices which they currently hold:

   - Vice President: Froilán M. Dopico (term ends February 28, 2025)
   - Secretary/Treasurer: Minerva Catral (term ends February 29, 2024)
   - Second Vice President (for ILAS conferences): Raf Vandebril (term ends February 28, 2026)
   - Assistant Secretary/Treasurer: Michael Tait (term ends February 28, 2026)
   - Board of Directors: Paola Boito (term ends February 28, 2025), Melina Freitag (term ends February 29, 2024), Apoorva Khare (term ends February 29, 2024) and Lek-Heng Lim (term ends February 28, 2025)

   On February 28, 2023, Hugo Woerdeman completed his one-year term on the Executive Board and on the Board of Directors as Past Vice President. We extend sincere thanks to Hugo for his very long and dedicated service to the Society. Among many other great contributions and initiatives, Hugo's work as ILAS Vice President was instrumental in launching and maintaining the new ILAS Website, as well as the ILAS Wikipedia page.

   Sebastian Cioabă and Dragana Cvetković Ilić completed their terms on the ILAS Board of Directors on February 28, 2023. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated.

   We also thank the members of the Nominating Committee – Raymond Chan, Steve Kirkland (Chair), Beatrice Meini, Bryan Shader and Helena Šmigoc – for their efforts on behalf of ILAS, and all of the nominees for their participation in the elections.

4. New appointments and reappointments:

   Second Vice President (for ILAS Conferences):

   Raf Vandebril

   Assistant Secretary/Treasurer:

   Michael Tait

   ILAS-NET Manager:

   Pietro Paparella

   ILAS Website Manager:

   Dominique Guillot

   ILAS Resident Registered Agent in the State of Florida:

   Fuzhen Zhang

Advisory Committee:

André Ran
Ilya Spitkovsky (chair)
Françoise Tisseur
Vilmar Trevisan

Education Committee:

Anthony Cronin
Judith McDonald
Rachel Quinlan
Sepideh Stewart (chair)
David Strong

Investment Oversight Committee:

Minerva Catral
Daniel Szyld
Michael Tait
Michael Tsatsomeros

Outreach and Membership Committee:

Minerva Catral
Andrii Dmytryshyn
Shahla Nasserasr (chair)
Federico Poloni

ILAS Richard A. Brualdi Early Career Prize Committee:

Marina Arav
Shmuel Friedland (Chair)
Volker Mehrmann

Israel Gohberg ILAS-IWOTA Lecture Committee:

Jussi Behrndt (IWOTA)
Kelly Bickel (ILAS)
Michael Dritschel (IWOTA)
Ilya Spitkovsky (ILAS)

5. ILAS endorsed the following conferences of interest to ILAS members that have taken place since the last President/Vice President annual report:

   - A joint meeting of ALAMA and "Due giorni di algebra lineare numerica" (ALAMA 2022-ALN2gg). Universidad de Alcalá, Alcalá de Henares, Spain, June 1–3, 2022. Françoise Tisseur was an ILAS Lecturer. `https://congresosalcala.fgua.es/alama2020/`
   - The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2022), Hotel Termas da Curia, Portugal, June 12–15, 2022. `http://www.mat.uc.pt/~wonra2022/`
   - XXI Householder Symposium on Numerical Linear Algebra, Selva di Fasano, Italy, June 12–17, 2022. Dario Bini was an ILAS Lecturer. `https://users.ba.cnr.it//iac/irmanm21/HHXXI/index.html`
   - 9th International Conference on Matrix Analysis and Applications (ICMAA 2022), University of Aveiro, Aveiro, Portugal, June 15–17, 2022. `https://sites.google.com/view/icmaa-2022`
   - International Workshop on Operator Theory and its Applications (IWOTA 2022), Kraków, Poland, September 6–10, 2022. Hugo J. Woerdeman was a Hans Schneider ILAS Lecturer. `https://iwota2022.urk.edu.pl`

6. ILAS endorsed the following conferences of interest to ILAS members that will take place in the next months:

   - Thematic Conference of the Spanish ALAMA Network on Laplacians and M-matrices in graphs, Universitat Politècnica de Catalunya, Barcelona, Spain, May 11–12, 2023. `https://mapthe.upc.edu/en/congresses/jornadas`
   - Western Canada Linear Algebra Meeting (WCLAM 2023), University of Regina, Saskatchewan, Canada, May 27–28, 2023. `https://uregina.ca/~sfallat/wclam`
   - The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2023), University of Coimbra, Coimbra, Portugal, June 7–9, 2023. `https://sites.google.com/view/wonra-2023`
   - 34th International Workshop on Operator Theory and its Applications (IWOTA 2023), University of Helsinki, Helsinki, Finland, July 31 – August 4, 2023. Stephan Ramon Garcia will be a Hans Schneider ILAS Lecturer. `https://www.helsinki.fi/en/conferences/iwota2023`
   - International Conference on Linear Algebra and its Applications (ICLAA 2023), Centre for Advanced Research in Applied Mathematics and Statistics (CARAMS), MAHE, Manipal, India, December 18–21, 2023. `https://carams.in/events/iclaa2023/`

7. The following ILAS conferences are scheduled:

   - The 25th ILAS Conference is scheduled to be held at Escuela Técnica Superior de Ingenieros de Montes, Forestales y del Medio Natural (The School of Forest Engineering and Natural Resources) of the Polytechnic University of Madrid, Madrid, Spain, June 12–16, 2023. The chair of the organizing committee is Fernando De Terán. The plenary speakers will be Carlos Beltrán (Universidad de Cantabria, Spain), Erin C. Carson (Charles University, Czech Republic), Stefan Güttel (University of Manchester, UK, ILAS Taussky-Todd Prize Lecture), Nicholas J. Higham (University of Manchester, UK, Hans Schneider Prize Lecture), Elias Jarlebring (KHT Stockholm, Sweden, SIAG-LA Lecture), Shahla Nasserasr (Rochester Institute of Technology, USA), Vanni Noferini (Aalto University, Finland, *LAMA* Lecture), Rachel Quinlan (University of Galway, Ireland), Michael Tait (Villanova University, USA, ILAS Richard A. Brualdi Early Career Prize Lecture), Cynthia Vinzant (University of Washington, USA).
   - The 26th ILAS Conference will be held in Kaohsiung, Taiwan, June 23–27, 2025, with Jephian C.-H. Lin (National Sun Yat-sen University) and Matthew M. Lin (National Cheng Kung University) as main organizers.

8. The following ILAS partner conferences are scheduled:

   - The 2024 Joint Mathematics Meetings (JMM) will be held in San Francisco, USA, January 3–6, 2024. Stephan Ramon Garcia from Pomona College (California) will be the ILAS Lecturer and five ILAS Special Sessions will be scheduled.
   - The SIAM Conference on Applied Linear Algebra will be held in Paris, France, May 12–16, 2024. Daniel Kressner and Laura Grigori are the chairs of the organizing committee. Andrii Dmytryshyn and Beatrice Meini will be the ILAS plenary speakers in the conference.

9. The Electronic Journal of Linear Algebra (*ELA*) is now in its 39th volume. *ELA*'s URL is `https://journals.uwyo.edu/index.php/ela`. The volume 38 was published in 2022 and contains 56 papers. *ELA* received 262 new submissions in 2022. The current acceptance rate is less than 23%. In 2022, 69774 downloads and 74673 abstract views of *ELA* papers occurred.

   Froilán M. Dopico (Universidad Carlos III de Madrid) is the Editor-in-Chief.

   ILAS members are strongly encouraged to submit their work to *ELA*, the flagship research journal of our society.

10. *IMAGE* is the semi-annual bulletin for ILAS, available online at `https://ilasic.org/image/`. The Editor-in-Chief is Louis Deaett (Quinnipiac University). In 2022, the website of *IMAGE* received 861 visits.

11. ILAS-NET is a moderated electronic newsletter for mathematicians worldwide, with a focus on linear algebra. It is managed by Pietro Paparella (University of Washington, Bothell).

    An archive of ILAS-NET messages is available at `https://ilasic.org/ilas-net/`. To send a message to ILAS-NET, please send the message (preferably in text format) in an email to `pietrop@uw.edu` indicating that you would like it to be posted on ILAS-NET. If the message is approved, it will be posted soon afterwards. To subscribe to ILAS-NET, please go to `https://www.ilasic.org/ilas-net/`

    By April 21, 2023, there were 1165 contacts in the ILAS-NET "audience," of which 1015 were subscribers.

12. ILAS's website is located at `https://ilasic.org/` and highlights the main activities of ILAS: the Electronic Journal of Linear Algebra (*ELA*), the conferences, *IMAGE*, ILAS-NET, and other activities. In addition, the website provides general information about ILAS (e.g., ILAS officers, bylaws, special lecturers, ILAS prizes, grant programs) as well as links to pages of interest to the ILAS community. Currently it is managed by Dominique Guillot (University of Delaware). In 2022, the website of ILAS received 11365 pageviews, from users from 93 different countries. The front page received 5096 of these pageviews, the conference page 912, and the *IMAGE* page 861.

Finally, we want to express our great gratitude to all the officers of ILAS who all show wonderful dedication to the society, as well as to all the individual members of ILAS and our corporate sponsors. Without any of them ILAS would not be what it is today.

Respectfully submitted,

Daniel B. Szyld, ILAS President (`szyld@temple.edu`); and
Froilán M. Dopico, ILAS Vice President (`dopico@math.uc3m.es`).

ILAS 2022–2023 Treasurer's Report

April 1, 2022 – March 31, 2023

by Minerva Catral, ILAS Secretary/Treasurer

Net Account Balance on March 31, 2022			
Checking Account - Great Western		$ 155,858.96	
Certificate of Deposit 1		$ 61,114.62	
Accounts Receivable		$ 3,445.00	
			$ 220,418.58
General Fund		$ 135,104.39	
Israel Gohberg ILAS-IWOTA Lecture Fund		$ 5,430.00	
Conference Fund		$ 9,387.29	
Taussky-Todd Fund		$ 11,161.68	
Hans Schneider Lecture Fund		$ 10,765.89	
Frank Uhlig Education Fund		$ 5,417.31	
Hans Schneider Prize Fund		$ 25,018.37	
Richard Brualdi Early Career Prize Fund		$ 1,040.00	
ELA Fund		$ 1,502.54	
ILAS/*LAA* Fund		$ 15,591.11	
			$ 220,418.58
INCOME:			
Dues	$ 13,360.00		
Israel Gohberg ILAS-IWOTA Lecture Fund Donations	$ 1,550.00		
General Fund Donations	$ 320.00		
Conference Fund Donations	$ 10.00		
Taussky-Todd Fund Donations	$ 10.00		
Hans Schneider Lecture Fund Donations	$ 10.00		
Uhlig Education Fund Donations	$ 20.00		
Hans Schneider Prize Fund Donations	$ 160.00		
ELA Fund Donations	$ 1,140.00		
Richard Brualdi Early Career Prize Fund Donations	$ 5,120.00		
Corporate Dues Income	$ 700.00		
Interest - First Interstate Bank	$ 24.96		
Interest on FIB Certificate of Deposit	$ 152.79		
Vanguard unrealized capital gains	$ 3,255.06		
Taylor and Francis Donation	$ 59,985.00		
Elsevier	$ 10,000.00		
Total Income		$ 95,817.81	
EXPENSES:			
ILAS Conference Expenses	$ 4,896.99		
ELA	$ 5,654.94		
IMAGE	$ 899.87		
IWOTA	$ 43.97		
JMM Expenses	$ 6,273.55		
PayPal/Credit Card Processing & Bank Fees	$ 700.92		
Non-ILAS Conferences	$ 2,632.65		
Hans Schneider Lecture	$ 1,500.00		
Hans Schneider Prize	$ 1,733.84		
LAMA Lecture	$ 1,614.77		
Taussky-Todd Lecture	$ 37.77		
Business License	$ 61.25		
Election Costs	$ 331.80		
Web Hosting, MailChimp & Online Membership Forms	$ 800.76		
Accounts Payable (ELA)	$ 1,047.96		
Misc Expenses	$ 103.88		
Total Expenses		$ 28,334.92	
Net Account Balance on March 31, 2023			
Checking Account - First Interstate Bank		$ 61,629.19	
Certificate of Deposit 1		$ 61,267.41	
PayPal		$ 12,797.77	
Vanguard		$ 153,255.06	
Accounts Payable		−$ 1,047.96	
			$ 287,901.47
General Fund		$ 155,750.00	
Israel Gohberg ILAS-IWOTA Lecture Fund		$ 7,002.63	
Conference Fund		$ 9,512.29	
Taussky-Todd Fund		$ 11,270.53	
Hans Schneider Lecture Fund		$ 9,407.71	
Frank Uhlig Education Fund		$ 5,503.56	
Hans Schneider Prize Fund		$ 23,750.74	
Richard Brualdi Early Career Prize Fund		$ 6,172.70	
ELA Fund		$ 1,161.08	
LAMA Fund		$ 58,370.23	
			$ 287,901.47

CONFERENCE REPORTS

ILAS at the Joint Mathematics Meetings
Boston, Massachusetts, USA, January 4–7, 2023

Report by Daniel B. Szyld and Hugo J. Woerdeman

The 2023 Joint Mathematics Meetings (JMM) took place in Boston, January 4–7, 2023. It was the second time that ILAS was a partner of the JMM, and the first in-person JMM meeting since the beginning of the ILAS-JMM partnership. It was great to see so many ILAS members at the JMM, enjoying the ILAS address, given by Apoorva Khare (Indian Institute of Science), and in the multiple ILAS Special Sessions.

Apoorva Khare's talk was entitled "Analysis applications of Schur polynomials," in which he presented several results involving Schur polynomials, viewed as functions on the positive orthant. The lecture was a true tour de force, involving material studied since the 18th century and by many others from more recent times. He was publicly recognized for his lecture at the JMM Prize Ceremony and awarded a framed certificate. The talk is available via the AMS YouTube channel and definitely worth watching. It can be viewed online at `https://www.youtube.com/watch?v=gHYnlpcuOLo`.

Apoorva Khare delivers the ILAS Invited Address at the 2023 JMM

The four special sessions were lively and well-attended. They were:

- Matrix Analysis and Applications (organized by Hugo J. Woerdeman and Edward Poon);
- Matrices and Operators (organized by Mohsen Aliabadi, Tin-Yau Tam, and Pan-Shun Lau);
- The Inverse Eigenvalue Problem for a Graph and Zero Forcing (jointly presented with AIM and organized by Mary Flagg and Bryan A. Curtis); and
- Innovative and Effective Ways to Teach Linear Algebra (organized by David M. Strong, Gil Strang, Sepideh Stewart, and Megan Wawro).

We hope to see many ILAS members at the next JMM, to be held in San Francisco, California, January 3–6, 2024. The 2024 ILAS address at JMM will be given by Stephan Ramon Garcia (Pomona College).

Among the over 5,100 attendees at the 2023 Joint Mathematics Meetings, many could be seen wearing an "ILAS" button, including ILAS President Daniel Szyld (left); Gilbert Strang fields questions at the ILAS Special Session on Innovative and Effective Ways to Teach Linear Algebra

New books from SIAM

A Journey through the History of Numerical Linear Algebra
Claude Brezinski, Gérard Meurant, and Michela Redivo-Zaglia

This expansive volume describes the history of numerical methods proposed for solving linear algebra problems, from antiquity to the present day. The authors focus on methods for linear systems of equations and eigenvalue problems and describe the interplay between numerical methods and the computing tools available at the time. The second part of the book consists of 78 biographies of important contributors to the field.

2022 • xx + 792 pages • Hardcover • 9781611977226 • List $135.00 • SIAM Member $94.50 • OT183

The Zen of Exotic Computing
Peter M. Kogge

The Zen of Exotic Computing is intended for computer science students interested in understanding alternative models of computing. It will also be of interest to researchers and practitioners interested in emerging technology such as quantum computing, machine learning, and AI.

2022 • xxxii + 386 pages • Softcover • 9781611977288 • List $89.00 • SIAM Member $62.30 • OT184

Advanced Reduced Order Methods and Applications in Computational Fluid Dynamics
Gianluigi Rozza, Giovanni Stabile, and Francesco Ballarin

Reduced order modeling is an important, growing field in computational science and engineering, and this is the first book to address the subject in relation to computational fluid dynamics. It focuses on complex parametrization of shapes for their optimization and includes recent developments in advanced topics such as turbulence, stability of flows, inverse problems, optimization, and flow control, as well as applications.

2022 • xxxviii + 462 pages • Softcover • 9781611977240 • List $99.00 • SIAM Member $69.30 • CS27

Mathematical Foundations of Finite Elements and Iterative Solvers
Paolo Gatto

This textbook describes the mathematical principles of the finite element method, a technique that turns a (linear) partial differential equation into a discrete linear system, often amenable to fast linear algebra. It examines the crucial interplay between analysis, discretization, and computations in modern numerical analysis; recounts historical developments leading to current state-of-the-art techniques; and, while self-contained, provides a clear and in-depth discussion of several topics, including elliptic problems, continuous Galerkin methods, iterative solvers, advection-diffusion problems, and saddle point problems.

2022 • x + 176 pages • Softcover • 9781611977080 • List $69.00 • SIAM Member $48.30 • CS26

The Less Is More Linear Algebra of Vector Spaces and Matrices
Daniela Calvetti and Erkki Somersalo

Designed for a proof-based course on linear algebra, this rigorous and concise textbook intentionally introduces vector spaces, inner products, and vector and matrix norms before Gaussian elimination and eigenvalues so students can quickly discover the singular value decomposition (SVD)—arguably the most enlightening and useful of all matrix factorizations. Gaussian elimination is then introduced after the SVD and the four fundamental subspaces and is presented in the context of vector spaces rather than as a computational recipe. This allows the authors to use linear independence, spanning sets and bases, and the four fundamental subspaces to explain and exploit Gaussian elimination and the LU factorization, as well as the solution of overdetermined linear systems in the least squares sense and eigenvalues and eigenvectors.

2022 • xii + 168 pages • Softcover • 9781611977394 • List $64.00 • SIAM Member $44.80 • OT187

Introduction to Linear Algebra, Sixth Edition
Gilbert Strang

The new 6th edition of Gilbert Strang's textbook *Introduction to Linear Algebra* combines serious purpose and a gentle touch, qualities also found in Strang's video lectures. The lectures for Math 18.06 at MIT are on OpenCourseWare ocw.mit.edu/courses. Two of the chapters - the first and the last - deserve special mention.

2022 • x + 430 pages • Hardcover • 9781733146678 • List $87.50 • SIAM Member $61.25 • WC20

An Applied Mathematician's Apology
Lloyd N. Trefethen

In 1940 G. H. Hardy published *A Mathematician's Apology*, a meditation on mathematics by a leading pure mathematician. Eighty-two years later, *An Applied Mathematician's Apology* is a meditation and also a personal memoir by a philosophically inclined numerical analyst, one who has found great joy in his work but is puzzled by its relationship to the rest of mathematics.

2022 • x +79 pages • Softcover • 9781611977189 • List $36.00 • SIAM Member $25.20 • OT182

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Outside North and South America order from Eurospan (*www.eurospanbookstore.com/siam*)

UPCOMING CONFERENCES AND WORKSHOPS

The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2023) Coimbra, Portugal, June 7–9, 2023

The 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2023) will be held at the Department of Mathematics, University of Coimbra, Coimbra, Portugal, on June 7–9, 2023, close to the 25th ILAS Conference in Madrid, Spain. The purpose of this workshop is to stimulate research and foster interaction between researchers interested in the subject of numerical ranges and numerical radii. A high level of research activity on the topic has resulted from connections between the subject and many different branches of pure and applied mathematics, such as operator theory, functional analysis, C^*-algebras, Banach algebras, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics, etc. Moreover, a wide range of tools, including algebra, analysis, geometry, combinatorics and computer programming, are useful in its study. An informal workshop atmosphere will facilitate the exchange of ideas from different scientific areas and, hopefully, the participants will be informed on the latest developments and new ideas. Contributions from both theoretical and applied perspectives are welcome. For further details, please visit `https://sites.google.com/view/wonra-2023`.

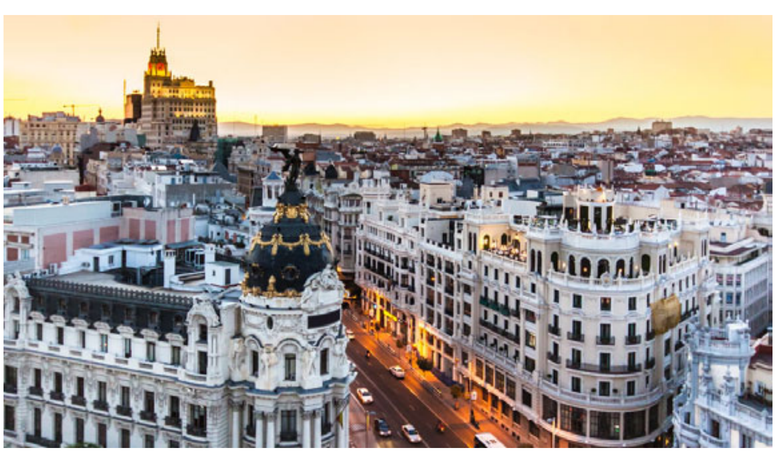
The Organizing Committee consists of Natália Bebiano (CMUC, University of Coimbra, `bebiano@mat.uc.pt`); Graça Soares (CMAT-UTAD, University of Trás-os-Montes e Alto Douro, `gsoares@utad.pt`); Rute Lemos (CIDMA, University of Aveiro, `rute@ua.pt`); Ana Nata (CMUC, Polytechnic Institute of Tomar, `anata@ipt.pt`).

The 25th ILAS Conference Madrid, Spain, June 12–16, 2023

The 25th conference of the International Linear Algebra Society, ILAS 2023, will be held June 12–16, 2023, in Madrid, Spain. The venue is the Escuela Técnica Superior de Ingenierí a de Montes, Forestal y del Medio Natural, of the Universidad Politécnica de Madrid (Polytechnic University of Madrid).

The conference will feature plenary talks from:

- Carlos Beltrán (Universidad de Cantabria)
- Erin C. Carson (Charles University)
- Stefan Güttel (University of Manchester, Taussky-Todd Prize)
- Nicholas J. Higham (University of Manchester, Hans Schneider Prize)
- Elias Jarlebring (KHT Stockholm, SIAG-LA speaker)
- Shahla Nasserasr (Rochester Institute of Technology)
- Vanni Noferini (Aalto University, LAMA speaker)
- Rachel Quinlan (NU Galway)
- Michael Tait (Villanova University, Richard A. Brualdi Early Career Prize)
- Cynthia Vinzant (University of Washington)



The organizing committee consists of:

- Roberto Canogar
- Fernando De Terán (chair)
- Froilán M. Dopico
- Ana María Luzón
- Ana Marco
- and José Javier Martínez
- Manuel Alonso Morón
- Raquel Viaña

The scientific committee consists of:

- Raymond H. Chan (Hong-Kong)
- Fernando De Terán (Spain)
- Gianna M. Del Corso (Italy)
- Shaun Fallat (Canada)
- Heike Fassbender (Germany)
- Elias Jarlebring (Sweden)
- Linda Patton (USA)
- Jennifer Pestana (UK)
- João Queiro (Portugal)
- Naomi Shaked-Monderer (Israel)
- Daniel Szyld (ILAS President, USA)
- Raf Vandebril (ILAS Vice President for Conferences, Belgium)
- Zdenek Strakos (Czech Republic).

More information about the conference, including talks and abstracts, invited mini-symposia, and relevant dates, is available at `http://www.ilas2023.es`.

The 34th International Workshop on Operator Theory and its Applications (IWOTA) Helsinki, Finland, July 31–August 4, 2023

The IWOTA conference series is the largest and most important annual event in operator theory and its applications, and it plays a leading role in the advancement of operator theory through the conference activities and proceedings. It contributes to future growth of the field through ample opportunities and partial financial support for early-career researchers and underrepresented groups to present their research and broaden their knowledge.

IWOTA 2023 will contribute to cross-fertilization with other fields of mathematics, such as complex analysis, functional analysis, harmonic analysis, linear algebra, mathematical physics, and inverse problems, as well as their various applications, and provides a medium for an intense exchange of new results, information and opinions, and for international collaboration in operator theory and its applications worldwide. It further sets directions for future research and also makes plans for future IWOTA conferences.

Deadlines:

June 15th, 2023 – Early registration
June 16th – July 10th, 2023 – Late registration

Invited Plenary Speakers:

- Stephan Ramon Garcia, Pomona College, USA (Hans Schneider ILAS Lecturer)
- Tuomas Hytönen, University of Helsinki, Finland
- Alexander Its, IUPUI, Indianapolis, USA
- Svetlana Jitomirskaya, University of California Irvine, USA
- Stephanie Petermichl, University of Würzburg, Germany
- Gideon Schechtman, Weizmann Institute, Israel
- Gunther Uhlmann, University of Washington, Seattle, USA
- Alexander Volberg, Michigan State University, USA

Organizing committee: Jani Virtanen (chair), University of Helsinki and University of Reading; Hans-Olav Tylli, University of Helsinki; Kari Astala, University of Helsinki. In collaboration with the IWOTA executive steering committee members: J. William Helton (chair), University of California San Diego; Igor Klep, University of Ljubljana, Slovenia; Hugo J. Woerdeman, Drexel University, Pennsylvania.

For further information, visit `https://www.helsinki.fi/en/conferences/iwota2023`.

Algebraic Statistics and Our Changing World: New Methods for New Challenges
Chicago, USA, September 18–December 15, 2023

The Institute for Mathematical and Statistical Innovation (IMSI), located in Chicago, will host a long program on Algebraic Statistics and Our Changing World: New Methods for New Challenges. The program will take place from September 18th to December 15th, 2023 and will include five workshops.

Applications for long-term visitors and workshop participants are now open at:

`https://www.imsi.institute/activities/algebraic-statistics-and-our-changing-world`

The focus will be mathematical and statistical challenges arising in applications central to our changing world, such as:

- modeling environmental and ecological systems so that we can better understand the effects of climate change on these systems,
- reimagining urban development and economic systems to address persistent inequity in daily living activities, and
- providing theoretical underpinnings for modern statistical learning techniques to understand the implications of their widespread use and for their easy adaptation to novel applications.

These are all hard challenges, but by bringing together biologists, social scientists, economists, statisticians, and mathematicians, and viewing the challenges collaboratively through the lens of algebraic statistics and, more generally, nonlinear algebra, we can utilize this new perspective to address these challenges side-by-side.

Combinatorics, algebra, and geometry underlie many of the statistical challenges present in the three focus applications of the program. For example, modeling and inferring biological and social networks rely on statistical models that are fundamentally algebraic, and in many cases estimation is done using combinatorial walks, while understanding modern statistical learning techniques relies on understanding principles rooted in algebraic geometry. Due to such underlying mathematical structures, algebraic and geometric methods have had a long history in statistics, from which the field of algebraic statistics has grown. By taking specific problems and identifying the underlying geometry and algebra, this program will pair domain-specific expertise and recent developments in algebraic statistics to develop interdisciplinary connections aimed at addressing new applications. The goal of the program is not only to see where we can make progress on these applications, but also to identify the mathematical and computational tools that we will need in the future, that we should start developing today.

Workshop on Spectral Graph Theory
Niterói, Brazil, October 24–27, 2023

The Brazilian community in spectral graph theory is very pleased to announce the 2023 Workshop on Spectral Graph Theory, in honor of Nair Abreu for her (70+3)rd birthday, that will be held at Universidade Federal Fluminense in Niterói, Rio de Janeiro, Brazil, on October 24–27, 2023. The aim of this workshop is to bring together both young and experienced researchers in subjects related to spectral graph theory and its applications. Plenary speakers include:

- Richard Brualdi, University of Wisconsin (USA)
- Steve Kirkland, University of Manitoba (Canada)
- Vladimir Nikiforov, University of Memphis (USA)
- Domingos Cardoso, Universidade de Aveiro (Portugal)
- Lilian Markenzon, Universidade Federal do Rio de Janeiro (Brazil)
- Chris Godsil, University of Waterloo (Canada)

The scientific organizing committee consists of: Claudia Justel, Instituto Militar de Engenharia (Brazil); Daniel Jaume, Universidad Nacional de San Luis (Argentina); Enide Andrade, Universidade de Aveiro (Portugal); Oscar Rojo, Universidad Católica del Norte (Chile); Renata Del-Vecchio, Universidade Federal Fluminense (Brazil); Sebastian Cioabă, University of Delaware (USA); Vilmar Trevisan, Universidade Federal do Rio Grande do Sul (Brazil).

Those interested in presenting their work should submit an abstract; the abstract submission deadline is June 30th. More information can be found at `http://spectralgraphtheory.org`.

International Conference on Linear Algebra and its Applications (ICLAA 2023) Manipal, India, December 18–21, 2023

The International Conference on Linear Algebra and its Applications (ICLAA 2023) will be organized by the Center for Advanced Research in Applied Mathematics & Statistics (CARAMS), Manipal Academy of Higher Education (MAHE), Manipal, India, and held December 18–21, 2023. The Scientific Advisory Committee consists of Ravindra B. Bapat (chairman), Manjunatha Prasad Karantha (convener), Steve Kirkland, and Simo Puntanen. The conference is endorsed by ILAS.

This conference is in sequel to the conferences CMTGIM 2012, ICLAA 2014, ICLAA 2017, and ICLAA 2020(21) held in Manipal during January 2012, December 2014, December 2017, and December 2021, respectively. The present conference shall provide an avenue for leading mathematicians, statisticians, and applied scientists who are working around the globe in the theme area to get together in physical space, interact with each other, discuss research issues, and introduce new innovations. The conference consists of special sessions on (i) Special Matrices, (ii) Matrix Methods in Statistics, and (iii) Matrices and Graphs, topics chosen in honor of leading mathematicians having a long-time assocation with CARAMS. Besides arranging invited talks from eminent speakers, the organizers invite participants to present their research in the sessions of contributed talks.

Publications. Research papers presented at the conference will have the opportunity for publication in one of the special issues dedicated to the conference in the following journals, subject to acceptance after the standard peer review process.

1. *Indian Journal of Pure and Applied Mathematics*, Springer
2. *AKCE International Journal of Graphs and Combinatorics*, Taylor & Francis
3. *Palestine Journal of Mathematics*, Palestine Polytechnic University

For the relevant timelines and other details, please visit the conference page at `https://carams.in/events/iclaa2023`.

Best paper award. To encourage research and motivate young researchers, the Best Paper Award competition will be conducted under the following three themes:

1. Matrix Theory, Special Matrices, and their Applications
2. Matrix Methods in Statistics, Linear Statistical Inference, and Applications
3. Matrix Methods in Graphs, Generalized Inverses, and Matrices over Different Algebraic Structures

Eligibility: Full research articles (which need not be original to the conference) on any of the above-mentioned themes are invited from those researchers (presenting authors) belonging to any of the following categories:

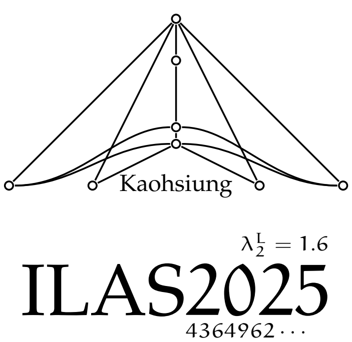
1. UG/PG Students and Ph.D. scholars within 1 year of Ph.D. registration
2. Any researcher of age less than 35 years (born on/after December 21, 1988)

All original papers selected for presentations will be forwarded to the review process in the appropriate conference publications. For more details, visit `https://carams.in/events/iclaa2023`.


Pre-conference workshop. The conference will be held following a pre-conference workshop, the "International Workshop on Special Matrices, Graphs, and Applications" (IWSMGA 2023) for which the resource persons are Abraham Berman, Ravindra B. Bapat, S. K. Neogy, Sivaramakrishnan Sivasubramanian, and others. For more details on the workshop, visit `https://carams.in/events/iwsmga2023`.

Registration is now open. For timelines, registration fees, and more details, visit `https://carams.in`. You may also write to the organizing secretary, K. Manjunatha Prasad, at `iclaa2023.mahe@gmail.com` or `kmprasad63@gmail.com`.

The 26^{th} ILAS Conference
Kaohsiung, Taiwan, June 23–27, 2025



We are delighted to announce that the 26^{th} Conference of the International Linear Algebra Society will be held in the vibrant city of Kaohsiung, Taiwan, June 23–27, 2025. This prestigious event will bring together leading experts, researchers, and enthusiasts from around the world to share their knowledge and explore the latest advancements in the field of linear algebra. For further details, including registration information and program updates, please visit the conference website:

`https://ilas2025.tw`

We look forward to welcoming you to this exciting event in 2025!

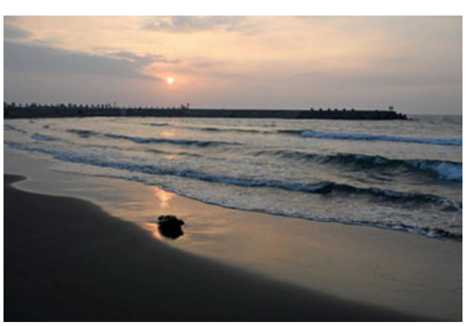


IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 63-1 and 69-1, as well as a second solution to Problem 63-3. Solutions are invited to all of the problems from issue 65; to parts (a) and (b) of Problem 66-2; to Problems 66-4, 68-2, 68-4, 69-2, 69-3 and to all of the new problems from the present issue 70.

Problem 63-1: Special Cases of a 100-Euro Conjecture

Proposed by Siegfried M. Rump, *Hamburg University of Technology, Hamburg, Germany*, `rump@tuhh.de`

When $v = (v_1, v_2, \ldots, v_n)^T$ and $w = (w_1, w_2, \ldots, w_n)^T$ are real n-vectors, we write $|v| \geq |w|$ to signify that $|v_k| \geq |w_k|$ for all k such that $1 \leq k \leq n$. If $A = [a_{ij}]$ is an $n \times n$ real matrix, we let $|A|$ denote the $n \times n$ matrix $[|a_{ij}|]$. The author has offered (see `https://www.tuhh.de/ti3/rump/100EuroProblem.pdf`) a 100-euro prize for the first proof or counterexample for the following conjecture: If A is an $n \times n$ real matrix such that all row sums of $|A|$ are equal to n, then there exists a nonzero $x \in \mathbb{R}^n$ such that $|Ax| \geq |x|$.

(a) Show that the conjecture is true if A is symmetric, $\operatorname{rank}(A) = 1$, or $\operatorname{rank}(|A|) \leq 2$.

(b) More generally, show that the conjecture holds if $|A|$ has all of its eigenvalues real.

Solution 63-1 by the proposer

(a) If $\operatorname{rank}(|A|) \leq 2$, then the matrix $|A|$ has the eigenvalue n and at most one more nonzero real eigenvalue λ. Similarly, if A is symmetric (resp., of rank 1) then $|A|$ is symmetric (resp., of rank 1) and in all of these cases $|A|$ has all of its eigenvalues real, so these are all special cases of part (b).

(b) One of the eigenvalues of $|A|$ is n; let the other eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{R}$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} a_{ji}| = \operatorname{Tr}(|A|^2) = n^2 + \sum_{i=1}^{n-1} \lambda_i^2 \geq n^2.$$

Hence, there exist i and j for which $|a_{ij} a_{ji}| \geq 1$. If $i = j$, then $|Ae_i| \geq |e_i|$, so we may henceforth assume that $i \neq j$. Observe that if D is a diagonal matrix whose entries are ± 1, then $|Ax| = |(AD)(Dx)|$ and $|Dx| = |x|$, so that, in particular, $|Ax| \geq |x|$ if and only if $|(AD)(Dx)| \geq |Dx|$. Hence, by right-multiplying by a suitable such D, we may assume that both a_{ij} and a_{ji} are nonnegative and hence that $a_{ij} a_{ji} \geq 1$. Now let $B = \left(\begin{smallmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{smallmatrix}\right)$ and let μ_1 and μ_2 be the eigenvalues of B. Then

$$(\mu_1 - \mu_2)^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = (\operatorname{Tr}(B))^2 - 4\det(B) = (a_{ii} + a_{jj})^2 - 4(a_{ii}a_{jj} - a_{ij}a_{ji}) = (a_{ii} - a_{jj})^2 + 4a_{ij}a_{ji} \geq 4.$$

Since $(\mu_1 - \mu_2)^2 > 0$, the eigenvalues μ_1 and μ_2 must be real (as opposed to a complex conjugate pair) and since $(\mu_1 - \mu_2)^2 \geq 4$, at least one must have modulus greater than or equal to 1. Let $(c, d)^T$ be an eigenvector of B corresponding to an eigenvalue of modulus greater than or equal to 1. Then, taking $x = ce_i + de_j$, we have $|Ax| \geq |x|$.

Problem 63-3: Products of Rectangular Circulant Matrices

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, `pereirar@uoguelph.ca`

An $n \times n$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \mod n$. Let m, n and q be positive integers. We can define a rectangular circulant matrix as follows: An $m \times n$ matrix C is said to be a rectangular circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \mod \gcd(m, n)$. As an example, a 6×9 rectangular circulant looks like

$$\begin{bmatrix} a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \\ a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \end{bmatrix}.$$

Let A be an $m \times n$ rectangular circulant and B be an $n \times q$ rectangular circulant.

(a) Show that AB is also a rectangular circulant.

(b) What is the maximum rank that AB can have?

(c) Show that the Moore-Penrose inverse of a rectangular circulant matrix is a rectangular circulant matrix.

Editor's note: Issue 69 of *IMAGE* featured a complete solution to this problem by M.J. de la Puente, *Universidad Complutense UCM, Madrid, Spain*, `mpuente@ucm.es`. After the publication of that issue, we received a solution of parts (a) and (c) from Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, `jeffrey.stuart@plu.edu`. This solution is significantly different from the one already published, and so we are including it here.

Solution 63-3.1 by M.J. de la Puente, *Universidad Complutense UCM, Madrid, Spain*, `mpuente@ucm.es`

Solution to all three parts presented in issue 69.

Solution 63-3.2 by Jeffrey Stuart, *Pacific Lutheran University, Tacoma, Washington, USA*, `jeffrey.stuart@plu.edu`

This solution is to parts (a) and (c) only. All matrices discussed in this solution are over a common field, either $\mathbb{R}$ or $\mathbb{C}$.

For each positive integer n, let P_n denote the $n \times n$ permutation matrix such that $p_{ij} = 1$ exactly when $j = i + 1 \bmod n$. Note that $(P_n)^{-1} = (P_n)^T$.

The five statements of the following theorem are immediately equivalent.

Theorem 1. *Let m and n be positive integers. Let C be an $m \times n$ real or complex matrix. The following are equivalent:*

1. *C is an $m \times n$ rectangular circulant matrix.*
2. *$P_m C = C P_n$.*
3. *$P_m C P_n^T = C$.*
4. *$C P_n^T = P_m^T C$.*
5. *C^T is an $n \times m$ rectangular circulant matrix.*

Proposition 2 (Problem 63 part (a))**.** *Let m, n and q be positive integers. If A is an $m \times n$ rectangular circulant matrix and B is an $n \times q$ rectangular circulant matrix, then AB is an $m \times q$ rectangular circulant matrix.*

Proof. By Theorem 1, we have $P_m A = A P_n$ and $P_n B = B P_q$, and hence

$$P_m (AB) = (P_m A) B = (A P_n) B = A (P_n B) = A (B P_q) = (AB) P_q.$$

By Theorem 1 again, the desired conclusion follows. □

Proposition 3 (Problem 63 part (c))**.** *Let m and n be positive integers, and let A be an $m \times n$ rectangular circulant matrix. Then the Moore-Penrose inverse A^+ of A is an $n \times m$ rectangular circulant matrix.*

Proof. For any $m \times n$ matrix M, the Moore-Penrose inverse of M^+ is the unique $n \times m$ matrix H that satisfies the following four properties:

$$MHM = M \tag{1a}$$
$$HMH = H \tag{1b}$$
$$(MH)^T = MH \tag{1c}$$
$$(HM)^T = HM \tag{1d}$$

Using (1a) with A and A^+, and using $P_m A P_n^T = A$,

$$P_m A P_n^T = A = AA^+A = \left(P_m A P_n^T\right) A^+ \left(P_m A P_n^T\right) = P_m A \left(P_n^T A^+ P_m\right) A P_n^T,$$

so

$$A = A\left(P_n^T A^+ P_m\right) A. \tag{2a}$$

Using (1b) with A and A^+, and using $P_m A P_n^T = A$,

$$P_n^T A^+ P_m = P_n^T \left(A^+ A A^+\right) P_m = P_n^T \left(A^+ \left(P_m A P_n^T\right) A^+\right) P_m = \left(P_n^T A^+ P_m\right) A \left(P_n^T A^+ P_m\right),$$

so

$$P_n^T A^+ P_m = \left(P_n^T A^+ P_m\right) A \left(P_n^T A^+ P_m\right). \tag{2b}$$

Using (1c) with A and A^+, using $P_m A P_n^T = A$, and using (4) of Theorem 1, we have

$$\begin{aligned}\left(A\left(P_n^T A^+ P_m\right)\right)^T &= P_m^T \left(A^+\right)^T P_n A^T = P_m^T \left(A^+\right)^T A^T P_m = P_m^T \left(AA^+\right)^T P_m \\ &= P_m^T \left(AA^+\right) P_m = \left(P_m^T A\right) A^+ P_m = \left(A P_n^T\right) A^+ P_m = A\left(P_n^T A^+ P_m\right),\end{aligned}$$

so

$$\left(A\left(P_n^T A^+ P_m\right)\right)^T = A\left(P_n^T A^+ P_m\right). \tag{2c}$$

Similarly, using (1d),

$$\left(\left(P_n^T A^+ P_m\right) A\right)^T = \left(P_n^T A^+ P_m\right) A. \tag{2d}$$

By (2a)–(2d), and the uniqueness of the Moore-Penrose inverse, $P_n^T A^+ P_m = A^+$, which implies that $A^+ P_m = P_n A^+$. By Theorem 1, then, A^+ is a rectangular circulant. □

Problem 69-1: Block Triangular k-Potent Matrices

Proposed by Jeffrey STUART, *Pacific Lutheran University, Tacoma, Washington, USA*, `jeffrey.stuart@plu.edu`

A complex matrix N is called *k-potent* if there exists a positive integer k such that $N^{k+1} = N$. Let A, B and C be complex matrices with A and B square. Let M be the complex matrix

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

(a) Suppose M is k-potent for some positive integer k. Show that A and B must be k-potent for the same k.

(b) Suppose A and B are k-potent matrices that have no eigenvalue in common. Show that M is k-potent for all choices of the matrix C.

(c) Suppose A and B are k-potent matrices that have at least one eigenvalue in common. Show that there exists a matrix C for which M is not k-potent.

Solution 69-1.1 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, `eaherman@gmail.com`

Each eigenvalue λ of a k-potent matrix N satisfies $\lambda^{k+1} = \lambda$, and so $\lambda = 0$ or λ is a kth root of unity. Let J be a Jordan normal form for N and assume for the sake of contradiction that J has a Jordan block $J(\lambda)$ that is not just a scalar. Since $N^{k+1} = N$, we have $J^{k+1} = J$ and hence $J(\lambda)^{k+1} = J(\lambda)$. However, the $(1, 2)$-entry of $J(\lambda)$ is 1 while the corresponding entry of $J(\lambda)^{k+1}$ is $(k+1)\lambda^k$. Thus, $\lambda^k = 1/(k+1)$. Regardless of whether $\lambda = 0$ or $\lambda^k = 1$, this yields a contradiction. Therefore, N is diagonalizable.

(a) Given the form of M, a straightforward induction argument shows that

$$M^{k+1} = \begin{bmatrix} A^{k+1} & C_k \\ O & B^{k+1} \end{bmatrix}, \quad \text{where} \quad C_k = \sum_{j=0}^{k} A^j C B^{k-j}.$$

When M is k-potent, we have $M^{k+1} = M$, and hence $A^{k+1} = A$ and $B^{k+1} = B$.

(b) Suppose P and Q are nonsingular matrices such that $P^{-1}AP$ and $Q^{-1}BQ$ are diagonal. Then

$$\begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix} M \begin{bmatrix} P & O \\ O & Q \end{bmatrix} = \begin{bmatrix} P^{-1}AP & P^{-1}CQ \\ O & Q^{-1}BQ \end{bmatrix}.$$

So we may as well assume that A and B are themselves diagonal. It suffices to show that $C = C_k$ holds for any choice of C. Since $C_k = \sum_{j=0}^{k} A^j C B^{k-j}$, the (i, j)-entry of C_k is $\sum_{j=0}^{k} \lambda^j c \mu^{k-j}$, where c is the (i, j)-entry of C, λ is ith diagonal entry of A, and μ is the jth diagonal entry of B. Thus, it suffices to show that $\sum_{j=0}^{k} \lambda^j \mu^{k-j} = 1$ when $\lambda \neq \mu$ and each of these eigenvalues is either 0 or a kth root of unity. If $\lambda = 0$, this equation simplifies to $\mu^k = 1$, which is true since $\mu \neq \lambda$. The equation holds similarly if $\mu = 0$. Otherwise, both eigenvalues are kth roots of unity and hence so is their quotient $r = \lambda/\mu \neq 1$. Dividing the equation by μ^k changes it to $\sum_{j=0}^{k} r^j = 1$, which simplifies to $\sum_{j=1}^{k} r^{j-1} = 0$. This last equation is true since $r^k = 1$, $r \neq 1$, and $0 = r^k - 1 = (r-1)\sum_{j=1}^{k} r^{j-1}$.

(c) As in (b), we may assume that A and B are diagonal. Suppose $\lambda = \mu$, where λ is the ith diagonal entry of A and μ is the jth diagonal entry of B. Let C be any matrix whose (i, j)-entry is not zero; say this value is $c \neq 0$. The (i, j)-entry of C_k is then $\sum_{j=0}^{k} \lambda^j c \mu^{k-j} = c \sum_{j=0}^{k} \lambda^k = c(k+1)\lambda^k$ (noting that in the case of $\lambda = \mu = 0$, we have $\lambda^0 \mu^k = \lambda^k \mu^0 = 0$). Because $\lambda = \mu$ is either 0 or a kth root of unity, $c(k+1)\lambda^k \neq c$, i.e., the matrices C_k and C differ in the (i, j)-entry.

Solution 69-1.2 by Shrinath HADIMANI, *Indian Institutes of Science Education and Research - Trivandrum, Kerala, India*, srinathsh3320@iisertvm.ac.in

The solutions to parts (a) and (c) are similar to Solution 69-1.1, so we publish here only the solution to part (b).

To prove part (b) we use the following theorem due to Sylvester (see [1, Theorem 2.4.4.1]).

Theorem 1. *Let $A \in M_n$, $B \in M_m$ and $C \in M_{n,m}$. The equation $AX - XB = C$ has a unique solution $X \in M_{n,m}$ if and only if A and B have no eigenvalues in common.*

Let C be any $n \times m$ matrix. Since A and B have no eigenvalues in common, by Sylvester's theorem the equation $AX - XB = -C$ has a unique solution X. Let $T = \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}$, where I_k denotes the $k \times k$ identity matrix. Note that

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} T = \begin{bmatrix} A & AX + C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & XB \\ 0 & B \end{bmatrix} = T \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Hence,

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = T \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} T^{-1}. \tag{2}$$

Therefore,

$$M^{k+1} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{k+1} = T \begin{bmatrix} A^{k+1} & 0 \\ 0 & B^{k+1} \end{bmatrix} T^{-1} = T \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} T^{-1} = M,$$

which means that M is k-potent.

Reference

[1] R.A. Horn and C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

Also solved by Shivani GOEL, *Indian Institute of Science, Bangalore, India*, shivani.goel.maths@gmail.com


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IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce three new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 70-1: A Decomposition for Nonzero Integer Matrices

Proposed by Gérald BOURGEOIS, *Université de la Polynésie française, FAA'A, Tahiti, Polynésie française*, bourgeois.gerald@gmail.com

Let $A \in M_n(\mathbb{Z})$ be a nonzero matrix. Show that there exist $G, N \in M_n(\mathbb{Z})$ satisfying $A = G + N$ where N is nilpotent of index n (i.e., $N^{n-1} \neq 0$ but $N^n = 0$) and all the eigenvalues of G (as a matrix over $\mathbb{C}$) are nonzero integers.

Problem 70-2: A Norm Inequality for Products of Triangular Matrices

Proposed by Lázsló LAJOS, *Eötvös Loránd University, Budapest, Hungary*, Laszlo47Lajos@gmail.com

Let $A_1, A_2, \ldots, A_k$ be real upper triangular 2×2 matrices, where $k \geq 3$. Then

$$\|A_1 A_2 \cdots A_{k-1} A_k \ - \ A_k A_{k-1} \cdots A_2 A_1\|_F \leq \prod_{i=1}^{k} \|A_i\|_F \, .$$

Problem 70-3: The Diagonal Entries of l^1-Normaloid Matrices

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

The l^1-norm on $\mathbb{R}^n$ is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$. The l^1 operator norm of an $n \times n$ matrix A is $\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1}$ and its spectral radius is defined by $r(A) = \max_{1 \leq i \leq n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. We say that A is l^1-*normaloid* if $r(A) = \|A\|_1$.

Let A be an l^1-normaloid real matrix with all entries nonzero. Show that the main diagonal entries of A are either all positive or all negative.

Solutions to 63-1 and 69-1, as well as a second solution to 63-3 are presented on pages 31–34.