The Bulletin of the International Linear Algebra Society

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Issue Number 72, pp. 1–41, Spring 2024

Contents continued on page 2.

Ongoing Online Seminars

IMAGE Problem Corner: Old Problems with Solutions

IMAGE Problem Corner: New Problems

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ISSN 1553-8991. Produced by the International Linear Algebra Society (ILAS). Two issues are published each year, on June 1 and December 1. The editors reserve the right to select and edit material submitted. All issues of *IMAGE* may be viewed online at `http://www.ilasic.org/IMAGE`. *IMAGE* is typeset using LaTeX. Photographs for this issue were obtained from websites referenced in articles, university department websites, conference websites, or from contributors directly, unless indicated otherwise here:

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With this issue, *IMAGE* welcomes Rachel Quinlan as Managing Editor, and bids farewell to Problem Corner editor Rajesh Pereira, whose generous tenure of service to *IMAGE* (and ILAS) in this capacity stretches all the way back to issue 57. Our tremendous thanks to Rajesh!

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FEATURE ARTICLE

Matrix Derivatives: Why and Where Did It Go Wrong?

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1. Introduction. The modern theory of matrix calculus rests on two pillars: a correct definition of the matrix derivative and the use of differentials. The necessity of a correct mathematical definition is obvious, but the use of differentials is also essential, because a differential does not alter the dimension of the matrix on which it operates. In this note I will briefly touch on differentials, but I will concentrate on describing the historical path towards finding the correct definition of the matrix derivative.

If we have a vector function $y = Ax$, where y has dimension $m \times 1$ and x has dimension $n \times 1$, then the derivative (Jacobian matrix) is an $m \times n$ matrix denoted by $Dy(x)$ or $(\partial y/\partial x)^T$ such that column i contains the partial derivatives of the components of y with respect to x_i and row s contains the partial derivatives of y_s with respect to the components of x. Since we are working with vectors y and x, a natural one-dimensional ordering of their components exists:

$$y = (y_1, y_2, \ldots, y_m)^T \qquad \text{and} \qquad x = (x_1, x_2, \ldots, x_n)^T.$$

There is no controversy about this definition.

The difficulty arises when we move from vectors to matrices. Now we have an $m \times p$ matrix function $Y = Y(X)$, where each element y_{st} of Y depends on an $n \times q$ matrix $X = (x_{ij})$. This produces $mnpq$ partial derivatives, and the question arises as to how these should be ordered. The matrices Y and X have a natural *two*-dimensional ordering of their elements, but no natural *one*-dimensional ordering. A one-dimensional ordering needs to be constructed, and this can be done in many ways, of which $\operatorname{vec} Y$ and $\operatorname{vec} X$ have become standard. (The vector $\operatorname{vec} X$ contains the columns of X, one underneath the other, starting with the first column.)

It is understandable that one wishes to maintain the matrix structure of Y and X when ordering the $mnpq$ partial derivatives. Such an ordering is possible, but it does not lead to the correct definition of the matrix derivative. The correct definition depends on a *one*-dimensional ordering of the matrices Y and X, that is, on $\operatorname{vec} Y$ and $\operatorname{vec} X$, because it is essential that each column of the derivative matrix contains the partial derivatives of all elements of Y with respect to a specific element of X and that each row contains the partial derivatives of a specific element of Y with respect to all elements of X. This mathematical necessity is in conflict with the intuitive desire to maintain the matrix structures of Y and X, and I think that this explains why it has taken so long to arrive at the correct definition and why there still exists so much resistance against it.

2. The pioneers. Until around 1970, the usual way of tackling matrix calculus problems was to resort to scalar differential calculus. A function expressed compactly in terms of matrices would be expanded in scalar form, differentiated by scalar methods, and then reassembled in the desired matrix form. Thus, to obtain the derivative of the $n \times n$ matrix function $Y = X^{-1}$, one would evaluate the n^4 elements $\partial y_{st}/\partial x_{ij}$ and reassemble in some way. Nowadays, we would write $dX^{-1} = -X^{-1}(dX)X^{-1}$, then vectorize to obtain

$$d \operatorname{vec} Y = d \operatorname{vec} X^{-1} = -((X^T)^{-1} \otimes X^{-1}) d \operatorname{vec} X,$$

and conclude that $DY(X) = -((X^T)^{-1} \otimes X^{-1})$ is the derivative. (The symbol $\otimes$ denotes the Kronecker product.)

Perhaps it was Herbert Turnbull who undertook the first attempt to define a matrix derivative. For any square $n \times n$ matrix function Y of a square $n \times n$ matrix X, Turnbull in his 1928 paper [18] introduces the operator Ω such that

$$(\Omega Y)_{ij} = \sum_{s=1}^{n} \frac{\partial y_{sj}}{\partial x_{si}},$$

and obtains some results based on this definition. This is obviously a very special case and thus does not provide a general theory of matrix calculus.

The real pioneers were Paul Dwyer and Jack Macphail. In their 1948 paper [5], they consider two special cases, namely the case where X is a scalar ($n = q = 1$) and the case where Y is a scalar ($m = p = 1$). If X is a scalar, say x, and we have a matrix function $Y = Y(x)$, then we are understandably tempted to define its "derivative" as $\dot{Y}$ (produced in LaTeX by the command `\dot{Y}`) such that $\dot{Y}$ has the same dimension as Y, for example

$$Y = \begin{pmatrix} x & 2x^3 & 3x^{-4} \\ e^x & \sin x & \log x \end{pmatrix} \quad \Longrightarrow \quad \dot{Y} = \begin{pmatrix} 1 & 6x^2 & -12x^{-5} \\ e^x & \cos x & x^{-1} \end{pmatrix},$$

where the example is taken from [5], but the notation $\dot{Y}$ is my own, emphasizing the fact that this matrix contains all the partial derivatives but is *not* the derivative $DY(x)$. More generally, for any $m \times p$ matrix function $Y = Y(x)$ of a scalar x, we have

$$\dot{Y}(x) = \sum_{s=1}^{m} \sum_{t=1}^{p} \frac{dy_{st}}{dx} E_{st}, \tag{1}$$

where E_{st} is a matrix (of appropriate order) containing a 1 in its (s,t)-entry and 0s elsewhere.

Alternatively, if Y is a scalar function, say y, of a matrix $X = (x_{ij})$, then we may be tempted to define its "derivative" as $\breve{y}$ (produced by the command `\breve{y}`) such that, for example,

$$y = x_{11}x_{23} - x_{13}x_{21} \quad \Longrightarrow \quad \breve{y} = \begin{pmatrix} x_{23} & 0 & -x_{21} \\ -x_{13} & 0 & x_{11} \end{pmatrix},$$

where $\breve{y}$ has the same dimension as X. Generalizing, we have, for any scalar function y of an $n \times q$ matrix X,

$$\breve{y}(X) = \sum_{i=1}^{n} \sum_{j=1}^{q} \frac{\partial y}{\partial x_{ij}} E_{ij}. \tag{2}$$

The matrices $\dot{Y}$ and $\breve{y}$ thus defined contain all the partial derivatives, *but they are not derivatives*. The derivatives of Y and y are, respectively,

$$DY(x) = \frac{d \operatorname{vec} Y}{dx} = \operatorname{vec} \dot{Y}$$

and

$$Dy(X) = \frac{\partial y}{\partial (\operatorname{vec} X)^T} = (\operatorname{vec} \breve{y})^T.$$

Dwyer and Macphail [5] study $\dot{Y}$ and $\breve{y}$, which they call "symbolic" matrix derivatives—perhaps to emphasize that these are not real derivatives but just manipulations—and provide some rules for operations with these symbolic derivatives.

As an example, they find the partial derivatives of $Y = AXBX^TC$. That is, they work out $\dot{Y}(x_{ij})$ and $\breve{y}_{st}(X)$, which is far from easy using their algebra. In modern notation, we simply take the differential

$$dY = A(dX)BX^TC + AXB(dX)^TC,$$

and then vectorize to obtain

$$\begin{aligned} d \operatorname{vec} Y &= (C^TXB^T \otimes A) d \operatorname{vec} X + (C^T \otimes AXB) d \operatorname{vec} X^T \\ &= \left(C^TXB^T \otimes A + (C^T \otimes AXB)K\right) d \operatorname{vec} X, \end{aligned}$$

using the commutation matrix K (of appropriate order)[1]. The derivative is then given by

$$DY(X) = \frac{\partial \operatorname{vec} Y}{\partial (\operatorname{vec} X)^T} = C^TXB^T \otimes A + (C^T \otimes AXB)K.$$

[1]The commutation matrix K is defined by the property that $K \operatorname{vec} A = \operatorname{vec} A^T$.

Although the work of Dwyer and Macphail was innovative and important, it suffered from two weaknesses. Not only are their "derivatives" not derivatives, but also their notation does not distinguish between their two definitions. They use $\partial Y/\partial X$ for both cases, specializing to $\partial Y/\partial x_{ij}$ (what we call $\dot{Y}(x_{ij})$) and $\partial y_{st}/\partial X$ (what we call $\breve{y}_{st}(X)$), respectively. Unfortunately, many authors followed their notation, which added to the confusion. The approach of Dwyer and Macphail was practical, not mathematical. Their purpose was to establish a system of principles and rules that would assist in the practical task of obtaining and organizing the partial derivatives. They were not interested in the underlying mathematical theory, and made no reference to multivariable (vector) calculus or the theory of multilinear algebra. This was unfortunate and has caused much misery.

Almost twenty years passed without much happening.[2] Then Paul Dwyer extended and simplified the results in [5], based in part on results obtained in William Wrobleski's Ph.D. thesis, defended in 1963 under Dwyer's supervision. Dwyer, in his 1967 paper [4], considers again only scalar functions of a matrix and matrix functions of a scalar, obtains some form of a chain rule, and applies his results to Jacobian determinants, some of them quite intricate (even involving symmetric matrices, where we would now use the duplication matrix). No attempt is made to define a general matrix derivative or to connect the results to the established literature on vector calculus.

An important and crucial step forward was the idea to work via differentials, moving from differentials to derivatives only at the last step. This idea was introduced by Heinz Neudecker in 1967 [12] and applied to the "derivative" $\breve{\phi}$, even managing a preliminary version of the first and second identification theorems [10, Theorems 5.11 and 6.6]. No full treatment was provided in this first attempt, since only scalar functions of a matrix (e.g., trace, determinant) were considered.

3. Why bother? Before we move to the general matrix derivative, why bother? Why is the ordering of the partial derivatives so important? Surely it does not matter how we order the partial derivatives as long as we have all of them collected in one matrix. But it *does* matter, as I tried to explain in 2010 in [8]. The derivative is not just a collection of partial derivatives; it is a mathematical object with a meaning, just like the inverse. Consider the following three matrices:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} -5 & 2 \\ -1 & 3 \end{pmatrix}.$$

Then $B = A^{-1}$, but the matrix C contains the same elements as B and has the additional property that the components of $\operatorname{vec} C$ are in increasing order. So, why not use C as the inverse of A? This is clearly preposterous, because the inverse is more than a collection of its elements—it is a mathematical entity with a meaning.

Exactly the same is true for the derivative. The derivative is not just a collection of partial derivatives. In particular, we want to be able to use a chain rule, we want to interpret the rank of the derivative, and we want to use its determinant in transformation theorems. This is only possible with a correct definition of the matrix derivative.

4. Matrix derivatives. Until the late 1960s, nobody had succeeded in developing a viable and mathematically correct calculus for matrices. The derivative of a vector is not a vector but a matrix. But what is the derivative of a matrix? A supermatrix? But we have no algebra for supermatrices.[3] Intuitively, we wish to keep the matrix structure of Y and X intact when we define the derivative. And this misguided intuition is precisely the reason why it has taken so long to arrive at the correct definition and why there is still so much resistance. Because we can't. We have to impose a one-dimensional ordering on the matrices Y and X, and this means we have to map these matrices into vectors, usually $\operatorname{vec} Y$ and $\operatorname{vec} X$. Since the two matrices are now transformed into vectors, the step from vector calculus to matrix calculus is easy.

Two papers, appearing almost simultaneously in the *Journal of the American Statistical Association* in 1969, define three matrix "derivatives"—matrices containing all partial derivatives of a matrix function Y with respect to an argument matrix X. Heinz Neudecker's paper [13] was the first to appear, followed shortly by the paper [17] of Derrick Tracy and Paul Dwyer. These three matrices are defined in (3), (4), and (5) below.

[2] Some papers and books on matrix calculus, including my own *Matrix Differential Calculus* with Neudecker, mention Bodewig's 1959 book, *Matrix Calculus* [3]. This, however, is a book about computational aspects of matrix *algebra*. Matrix *calculus* is not discussed apart from one comment in the preface.

[3] There exists, however, an algebra for tensor products of which the Kronecker product is a special case; see Pollock [15], who defines tensors in the context of multilinear algebra.

First, based on the expression for $\dot{Y}(x_{ij})$ from (1), define $\dot{Y}(X)$ by assembling the matrices $\dot{Y}(x_{ij})$ in a block structure,

$$\dot{Y}(X) = \sum_{i=1}^{n}\sum_{j=1}^{q} E_{ij} \otimes \dot{Y}(x_{ij}) = \sum_{ij}\sum_{st} \frac{\partial y_{st}}{\partial x_{ij}}(E_{ij} \otimes E_{st}), \tag{3}$$

which is a partitioned matrix of order $mn \times pq$ where each block is of order $m \times p$ (like Y). Second, based on the equation (2) for $\breve{y}_{st}(X)$, define $\breve{Y}(X)$ as

$$\breve{Y}(X) = \sum_{s=1}^{m}\sum_{t=1}^{p} E_{st} \otimes \breve{y}_{st}(X) = \sum_{st}\sum_{ij} \frac{\partial y_{st}}{\partial x_{ij}}(E_{st} \otimes E_{ij}), \tag{4}$$

also of order $mn \times pq$, where each block is of order $n \times q$ (like X). Given the properties of the commutation matrix, we have

$$K_{mn}\dot{Y}K_{qp} = \breve{Y},$$

providing a one-to-one correspondence between the two matrices. For example, if $Y = AXB$, then

$$\dot{Y} = (\operatorname{vec} A)(\operatorname{vec} B^T)^T \qquad \text{and} \qquad \breve{Y} = (\operatorname{vec} A^T)(\operatorname{vec} B)^T,$$

neither of which is the derivative of Y. The derivative is $DY(X) = B^T \otimes A$, and in general

$$\begin{aligned} DY(X) = \frac{\partial \operatorname{vec} Y}{\partial (\operatorname{vec} X)^T} &= \sum_{ij}\sum_{st} \frac{\partial y_{st}}{\partial x_{ij}}(\operatorname{vec} E_{st})(\operatorname{vec} E_{ij})^T \\ &= \sum_{tj}\sum_{si} \frac{\partial y_{st}}{\partial x_{ij}}(E_{tj} \otimes E_{si}), \end{aligned} \tag{5}$$

which is of order $mp \times nq$, where each block is of order $m \times n$ and contains the partial derivative of one column of Y with respect to one column of X.

Neither Neudecker nor Tracy and Dwyer seems to have realized that there is an essential difference between $\dot{Y}$ and $\breve{Y}$ on the one hand, and DY on the other hand. Both papers recognize that the algebra through DY is easier, but they continue to think that the ordering of the partial derivatives is up to the author and does not really matter. This view persisted until at least 1985 and is echoed is Daan Nel's 1980 survey paper [11], where he writes:

> "Although it does not look so at first glance, the vector rearrangement method [that is, the arrangement through (5)] proved to be a very powerful method for writing the matrix derivative as a simple and recognizable statement in terms of the original matrices."

A third element of confusion was added by the fact that Neudecker in 1969 used the wrong definition for the Jacobian from vector analysis. If y is an $m \times 1$ vector function of an $n \times 1$ vector x, then the derivative $Dy(x)$ is an $m \times n$ matrix, but Neudecker defines it as the transpose, that is as an $n \times m$ matrix, which one might call the "gradient" matrix. Hence, if $Y = AX$, he finds $DY(X) = I \otimes A^T$ rather than $I \otimes A$, and if X happens to be a function of Z, say $X = BZ$, then the chain rule (using the wrong definition) gives $DY(Z) = I \otimes B^T A^T$ rather than the correct expression $DY(Z) = I \otimes AB$. Remarkably, Tracy and Dwyer, in their 1969 paper, made the same mistake as Neudecker; maybe they copied Neudecker's definition (or vice versa). Since both papers appeared in a high-prestige journal, the mistake persisted in the literature for at least ten years.

In the 1970s, a large literature followed, mostly concentrating on further properties of $\dot{Y}$ and $\breve{Y}$. Matrix calculus was required in control theory [19], econometrics [1], psychology [2], statistics [11, 16], and in many other fields, but a mathematically correct and easy-to-apply theory of matrix calculus was still lacking. In 1978, Peter Bentler and Sik-Yum Lee wrote [2]:

> "Although multivariable calculus has become a standard mathematical topic in recent years [...], the differential calculus of matrix functions is not yet a mature enough field to have developed general matrix calculus rules that reduce to the standard pragmatic rules of scalar calculus when the matrices involved are of order one. It is by no means an easy task to move from scalar calculus to matrix calculus."

Their paper is an important step forward, because they use the correct concept of matrix derivative, although they don't use differentials. In 1979, Harold Henderson and Shayle Searle [6] also use the correct concept, and they do use differentials. Unfortunately, both Bentler–Lee and Henderson–Searle confuse gradient and derivative, so that the chain rule

$$DY(Z) = DY(X)\, DX(Z)$$

reads $DY(Z) = DX(Z)\, DY(X)$ in their notation.

The confusion between gradient and derivative, which started with the papers [13, 17] by Neudecker and Tracy–Dwyer in 1969, was finally resolved by Stephen Pollock in his 1979 book [14], where he defines (correctly) the derivative of an $m \times 1$ vector function y with respect to an $n \times 1$ vector x as an $m \times n$ matrix, remarking that "this definition is at variance with a common convention which arrays the partial derivatives in an $n \times m$ matrix" [14, p. 75]. Pollock also employs the correct generalization from vector to matrix calculus.

Heinz Neudecker and I decided in 1981 to write *Matrix Differential Calculus* [10] and it took us seven years to complete. Three years before the book came out, we felt that a statement was required concerning the correct definition of the matrix derivative and the beauty and strength of the use of differentials. Hence, in our 1985 paper [9], we attempted to convince the reader that there is only one correct definition, and we argued strongly against the use of $\dot{Y}$ and $\check{Y}$ on the grounds of mathematical correctness and computational efficiency.

Maybe the phrase "only one correct definition" is too strong. Essential is that we organize both Y and X into vectors, for which we typically employ $\operatorname{vec} Y$ and $\operatorname{vec} X$, the column-by-column vectorization. But we could also use the row-by-row vectorization or indeed any other vectorization. The essence is not what type of vectorization is used, but that we employ a vectorization. The column-by-column vectorization orders the components of X as $(x_{11}, x_{21}, \dots)$, while the row-by-row vectorization (advocated by Pollock in 1985 [15]) orders the components of X lexicographically as $(x_{11}, x_{12}, \dots)$, which may be more elegant and logical. If $Y = AXB$ then $\operatorname{vec} Y = (B^T \otimes A) \operatorname{vec} X$, and hence

$$\operatorname{vec} Y^T = K(B^T \otimes A) \operatorname{vec} X = (A \otimes B^T) K \operatorname{vec} X = (A \otimes B^T) \operatorname{vec} X^T.$$

Realizing that $\operatorname{vec} X^T$ is precisely the row-by-row vectorization, the derivative based on the row-by-row vectorization is $A \otimes B^T$ rather than $B^T \otimes A$, which is perhaps more intuitive. The choice between these alternative orderings of the components is a matter of taste, and $\operatorname{vec} X$ (column-by-column) is now standard. But the fact that the phrase "only one correct definition" is too strong does not imply that $\dot{Y}$ and $\check{Y}$ are correct definitions of the derivative. They are not.

Our 1985 paper did not end the discussion, and this was the reason I wrote another critical paper [8] twenty-five years later where essentially the same message is delivered. The current note has a different flavor: rather than trying to convince the reader, I have tried to understand and explain the confusion that has clouded the field of matrix calculus for so many decades.

5. Patterned matrices. The correct definition of derivative and the subsequent chain rule also make it easy to deal with patterned matrices. A matrix is "patterned" or obeys a "linear structure" [7] if its elements are subject to a linear restriction, such as symmetry, lower triangularity, or diagonality. In a patterned matrix A, there exists a subset of $\operatorname{vec} A$ containing the essential elements of A. If $\psi(A)$ denotes this subvector of $\operatorname{vec} A$, then there exists a unique matrix Δ such that $\operatorname{vec} A = \Delta \psi(A)$. In the case of symmetry, $\psi(A) = \operatorname{vech}(A)$ and Δ is the duplication matrix D.[4]

From the differential

$$d \operatorname{vec} Y = DY(X)\, d \operatorname{vec} X$$

we would conclude that the derivative is $DY(X)$, but if X turns out to be patterned, we immediately have, by the chain rule,

$$d \operatorname{vec} X = \Delta\, d\psi(X),$$

[4] The vector $\operatorname{vech}(A)$ is the "half-vector" obtained from $\operatorname{vec} A$ by deleting all elements of A above the diagonal, and the duplication matrix D is defined by the property that $D \operatorname{vech}(A) = \operatorname{vec} A$ for any symmetric matrix A.

and hence

$$d\,\mathrm{vec}\,Y = DY(X)\,\Delta\,d\psi(X),$$

leading to the derivative

$$\frac{\partial\,\mathrm{vec}\,Y}{\partial(\psi(X))^T} = DY(X)\,\Delta.$$

Acknowledgements. I am grateful to Karim Abadir and Stephen Pollock for discussions on notation, definitions, and many other things that helped clarify my mind.

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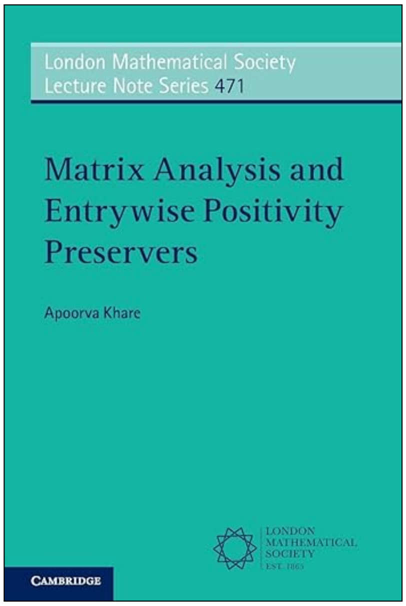
BOOK REVIEWS

Matrix Analysis and Entrywise Positivity Preservers, by Apoorva Khare

LMS Lecture Note Series 471, Cambridge, March 2022, ISBN 978-1-1088-6712-2, xxii+292 pages
Reviewed by Chi-Kwong Li, College of William and Mary, ckli@math.wm.edu



This monograph delves into the study of operations preserving various notions of positivity in matrices, including positive semidefinite matrices, totally nonnegative matrices, and more. Beyond matrices, the exploration extends to positivity notions within function spaces, metric spaces, and local compact groups, enriching the scope of matrix analysis.

The monograph is structured into three comprehensive parts, the first of which focuses on positive semidefinite and Hankel totally nonnegative matrices, exploring functions acting entrywise on such matrices to preserve positivity. Similar problems are considered for matrices with prescribed zero patterns, with special attention given to power functions acting on matrices of a fixed size. Entrywise power preservers of monotonicity and superadditivity are also studied, culminating in a discourse on Loewner convexity and singular matrix encoders, adding depth to our understanding of positivity preservation.

The second part delves into foundational results on matrix positivity preservers, discussing entrywise functions preserving either positive semidefinite matrices, or totally nonnegative Hankel matrices, of all sizes. In addition to covering classical techniques and proof methodology introduced by authors such as Fekete, Horn, Loewner, Pólya, Rudin, Schoenberg, and Vasudeva, the author also includes new techniques in the discussion. Preservers of Loewner positivity, monotonicity, and convexity on kernels over infinite domains are considered, alongside additional results on Euclidean distance matrices, functions acting outside forbidden diagonal blocks of matrices, and real-valued functions with positive divided differences.

The third part tackles the challenging problem of characterizing entrywise functions on matrices preserving positivity, focusing on polynomial preservers. On this topic, the discussion includes some recent work of the author and his collaborators. By a result of Schoenberg and Rudin, the only entrywise polynomial preservers of positive matrices for all dimensions are the ones with nonnegative coefficients. The problem for matrices of a fixed size has been open for a long time. Here the author presents his own recent work with collaborators classifying the sign patterns of entrywise polynomial preservers of $N \times N$ positive semidefinite matrices. The result is also extended to power series that are countable sums of real powers. Characterizations are also given of entrywise polynomial preservers of totally nonnegative matrices and totally nonnegative Hankel matrices. In particular, it is shown that there are polynomials with a negative leading coefficient such that the entrywise polynomial map will preserve positive semidefinite matrices. The extreme (negative) value of the leading coefficient for such polynomials is determined.

One might think that the monograph focuses on a special topic in matrix analysis of limited interest. However, this is not the case. In fact, the study of positive definite matrices, totally nonnegative matrices, kernels, and sequences has been central to research related to many pure and applied topics. The author of this monograph has done a wonderful job of presenting the subject in a clear and systematic fashion. He provides a nice description of the history of the development of the subject and indicates connections to different areas. Not only does the author organize and present the results in a lucid manner, but the proofs are described and arranged so that readers can appreciate the interesting connections between different ideas and techniques from a wide spectrum of subjects including matrix theory, operator theory, real and complex analysis, optimization, and algebraic combinatorics. The historical notes and questions in the monograph naturally lead to future research directions.

This book is a wonderful resource for researchers and will be an excellent text for graduate study.

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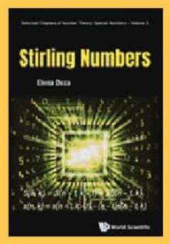
New Releases

Stirling Numbers

by **Elena Deza** (Moscow Pedagogical State University, Russia)

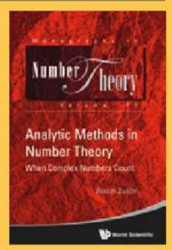


This book collects much of the scattered material on the two subclasses of Stirling numbers to provide a holistic overview of the topic. From the combinatorial point of view, Stirling numbers of the second kind, S(n, k), count the number of ways to partition a set of n different objects (i.e., a given n-set) into k non-empty subsets.

468pp | Pub date: Jan 2024
Hardcover 978-981-127-809-9 | **US$158 / £145**

Analytic Methods in Number Theory

When Complex Numbers Count
by **Wadim Zudili**n (Radboud University Nijmegen, The Netherlands)

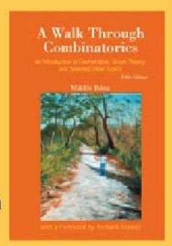


The present book takes a semi-systematic review of analytic achievements in number theory ranging from classical themes about primes, continued fractions, transcendence of π and resolution of Hilbert's seventh problem to some recent developments on the irrationality of the values of Riemann's zeta function, sizes of non-cyclotomic algebraic integers and applications of hypergeometric functions to integer congruences.

192pp | Pub date: Sep 2023
Hardcover 978-981-127-931-7 | **US$78 / £70**

A Walk Through Combinatorics

An Introduction to Enumeration, Graph Theory, and Selected Other Topics
5th Edition
by **Mikló s Bó na** (University of Florida, USA)

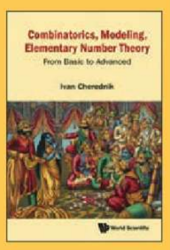


"This is still one of the best introductions to combinatorics."
Mathematical Association of America

636pp | Pub. date: Aug 2023
Hardcover 978-981-127-784-9 | **US$128 / £120**
eBook for Individual 978-981-127-786-3 | **US$51 / £45**

Combinatorics, Modeling, Elementary Number Theory

From Basic to Advanced
by **Ivan Cherednik** (The University of North Carolina at Chapel Hill, USA)



This book is mostly based on the author's 25 years of teaching combinatorics to two distinct sets of students: first-year students and seniors from all backgrounds.

392pp | Pub date: Jun 2023
Hardcover 978-981-126-539-6 | **US$118 / £110**

All-Time Bestseller

Principles and Techniques in Combinatorics

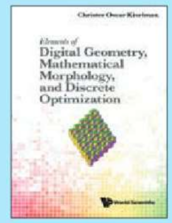
Solutions Manual
by **Kean Pew Foo, Mingyan, Simon Lin**
(University of Illinois at Urbana-Champaign, USA)

Solutions are written in a relatively self-contained manner, with little undergraduate mathematics assumed. This caters to a wide range of readers, and makes it easy for the reader to understand the thought process behind the solutions to each problem.

440pp | Pub. date: Oct 2018
Softcover 978-981-3238-84-8 | **US$45 / £40**

Elements of Digital Geometry, Mathematical Morphology, and Discrete Optimization

by **Christer Oscar Kiselman** (Uppsala University, Sweden)



"The book offers an attractive option for a course dealing with set theory, general topology. Analysis, and group theory, and their application to digital setting, mathematical morphology and discrete optimization."

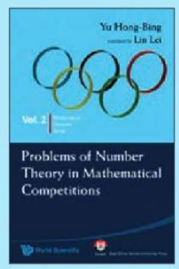
Mathematical Reviews Clippings
American Mathematical Society

488pp | Pub. date: Feb 2022
Hardcover 978-981-124-829-0 | **US$158 / £145**

Mathematical Olympiad Series - Vol 2

Problems of Number Theory in Mathematical Competitions

by **Hong-Bing Yu** (Suzhou University, China)
Translated by: Lei Lin (East China Normal University, China)

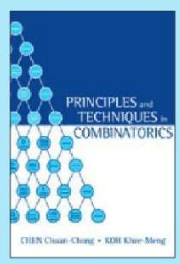


In this book, the author introduces some basic concepts and methods in elementary number theory via problems in mathematical competitions.

116pp | Pub. date: Sep 2009
Softcover 978-981-4271-14-1 | **US$34 / £28**

Principles and Techniques in Combinatorics

by **Chen Chuan-Chong** (NUS, Singapore), Koh Khee-Meng (NUS, Singapore)



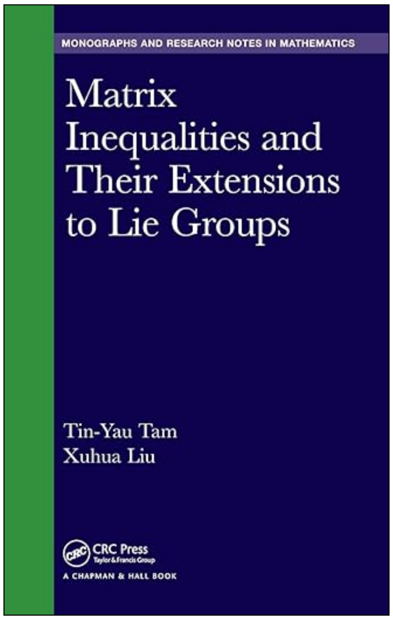
"This book should be a must for all mathematicians who are involved in the training of Mathematical Olympiad teams, but it will also be a valuable source of problems for university courses."
Mathematical Reviews

312pp | Pub. date: Jul 1992
Softcover 978-981-02-1139-4 | **US$39 / £32**

S/EP/03/24/04/RM

Matrix Inequalities and Their Extensions to Lie Groups, by Tin-Yau Tam and Xuhua Liu

Chapman & Hall/CRC, March 2018, ISBN 978-1-4987-9616-3, x+148 pages
Reviewed by M. S. Moslehian, Ferdowsi University of Mashhad, `moslehian@um.ac.ir`



This research monograph focuses on some of the most important and fundamental matrix inequalities and explores methods to systematically extend them to Lie groups. The text is well organized, presenting proofs in a clear and accessible manner. An overview of the book's contents is as follows.

The first chapter delves into fundamental aspects of matrix theory, encompassing matrix decompositions like polar decomposition, singular value decomposition, and QR decomposition; concepts of majorization and Löwner order; unitarily invariant norms; compound matrices; and various matrix inequalities.

The second chapter elucidates the analytic and algebraic structures inherent to semisimple Lie groups and Lie algebras. It delves into the Cartan decomposition, Iwasawa decomposition, and complete multiplicative Jordan decomposition, laying the foundational groundwork for subsequent chapters.

In the third chapter, the authors utilize Lie group decompositions in order to broaden matrix inequalities to connected noncompact real semisimple Lie groups. In this context, they introduce several renowned inequalities, such as those of Golden–Thompson, Araki–Lieb–Thirring, and Bernstein.

Chapter 4 is devoted to various matrix inequalities, including the inequalities of Kato, Nakamoto, and Horn, as well as McIntosh (Cauchy–Schwartz type) inequalities related to the spectral norm

$$\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = s_1(A),$$

along with their extensions to Lie groups.

Chapter 5 deals with matrix inequalities such as the Audenaert and Simon inequalities related to unitarily invariant norms, along with their extensions to Lie groups.

In Chapter 6, the focus shifts to the inequalities of Ando–Hiai and Hiai–Petz regarding the t-geometric mean

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$, for $n \times n$ positive definite complex matrices A and B. These concepts are further extended to Lie groups in terms of Kostant's preorder. In addition, t-geometric means find a definition in symmetric spaces of noncompact type. Such spaces have nonpositive curvature, with an intriguing convexity property of geodesic triangles.

In Chapter 7, the Kostant linear and nonlinear convexity theorems are investigated. Inspired by the well-known theorem of Schur–Horn on the eigenvalues and diagonal entries of Hermitian matrices, Mirsky's query in the 1960 publication [1] sought a relationship between the singular values and diagonal entries of a complex matrix. A theorem by Thompson and Sing completely answered this inquiry. The authors show that the Schur–Horn theorem and the Thompson–Sing theorem are special cases of Kostant's linear convexity theorem.

A foundational understanding of linear algebra, matrix analysis, and Lie groups is essential for grasping the content of this book. The reviewer anticipates the incorporation of more essential matrix inequalities in subsequent editions. Nonetheless, this book is highly recommended for postgraduate students and mathematicians intrigued by the topic.

References.

[1] L. Mirsky. Symmetric gauge functions and unitarily invariant norms. *Quart. J. Math. Oxford*, 11:50–59, 1960.

OBITUARY NOTICES

David H. Carlson, 1936–2024

Submitted by Biswa Datta

Our beloved friend and colleague David H. Carlson passed away peacefully on the evening of April 28th, 2024 at his residence in the city of Encinitas in southern California, which had been his home for the last several years. He is survived by his wife, Joyce Carlson, and several children and grandchildren.

Dave Carlson was one of the earliest graduate students of the late Hans Schneider. He himself made some notable contributions to the theory and educational aspects of linear algebra and collaborated with several well-known researchers in the field. He was one of the long-time editors of *Linear Algebra and its Applications.* He also served as Vice Chair of the SIAM Activity Group on Linear Algebra for several years.

He will be missed by us forever.

Nicholas J. Higham, 1961–2024

Submitted by Stefan Güttel and Daniel Kressner

It is with deep sadness that we announce the passing of Nicholas J. Higham, a leading figure in numerical analysis and numerical linear algebra, on January 20, 2024, at the age of 62. Nick spent most of his career at the University of Manchester, where he held the title of Royal Society Research Professor and Richardson Professor of Applied Mathematics. He was renowned for his work in numerical analysis and linear algebra, which spanned a wide range of topics, including rounding error analysis, linear systems, eigenvalue problems, and matrix functions. His scientific contributions have shaped entire fields, defining the way a whole generation of scientists understand and pursue linear algebra and numerical computing.

Nick authored highly regarded monographs that have set new standards in clarity of exposition, scholarship, and attention to detail. His 700-page research monograph *Accuracy and Stability of Numerical Algorithms*, published by SIAM in 1996 and 2002, has been highly acclaimed by both mathematicians and users of numerical methods. Alan Edelman described it as "the next 'Bible' of accuracy and stability," alluding to the classic 1965 book *The Algebraic Eigenvalue Problem* by renowned numerical analyst J. H. Wilkinson. Indeed, Nick will be regarded by many as Wilkinson's successor. His research was in the same spirit as Wilkinson's, representing a blend of mathematical theory and practical issues. Nick's monograph *Functions of Matrices: Theory and Computation*, published by SIAM in 2006, has solidified the theoretical and algorithmic foundations of matrix functions and has firmly established them as the third pillar of (numerical) linear algebra, next to linear systems and eigenvalue problems. In early 2024 he completed a new monograph entitled *Numerical Linear Algebra and Matrix Analysis*, which will be published shortly.

The great care and ingenuity Nick put into his books and mathematical writing leave him as a humbling role model for authors of mathematical books to come. He authored the popular *Handbook of Writing for the Mathematical Sciences* (already in its 3rd edition, SIAM 2020), which is widely used in writing courses and is given to all incoming graduate students in some mathematics departments. Nick also co-authored *How to Be Creative: A Practical Guide for the Mathematical Sciences* (SIAM 2022), the *MATLAB Guide* (3rd edition, SIAM 2017), and edited the 1000-pages-strong *Princeton Companion to Applied Mathematics* (Princeton University Press 2015), contributing about 100 pages himself.

Nick resolved several important questions on the accuracy and stability of linear algebra computations, such as proving the numerical stability of Cholesky factorization for semidefinite matrices (earning him the Leslie Fox Prize for Numerical Analysis in 1988), demonstrating the stability of Strassen's fast matrix multiplication method (dispelling a prevalent myth from the 1980s), and proving the stability of the Bunch–Kaufman pivoting strategy for symmetric indefinite linear systems (which had been unclear despite its wide use since the 1970s). The influence of all this work on algorithm and software design has been profound. In particular, the choice of methods and error bounds during the development of the widely used LAPACK library has been strongly influenced by Nick's analyses. Nick also made seminal contributions to polynomial eigenvalue problems, mixed-precision algorithms, probabilistic rounding error analysis, and matrix functions. His highly cited work on the scaling-and-squaring method is the foundation of MATLAB's implementation of the matrix exponential, one of the most important matrix functions. Indeed, many of his algorithms are implemented in MATLAB, the NAG Library, and in various open-source packages.

Nick's research has been recognized through numerous prizes and honors, including the ILAS Hans Schneider Prize in Linear Algebra in 2022, which he received "for fundamental contributions in the analysis of a wide range of numerical linear algebra problems and matrix functions," and the SIAM George Pólya Prize for Mathematical Exposition in 2021. Nick has tirelessly served our community, for example through his role as President of SIAM from 2017–2018 and on the ILAS Board of Directors. He served on the editorial boards of many journals and was Editor-in-Chief of SIAM's book series on Fundamentals of Algorithms.

Nick is survived by his wife, Françoise Tisseur, and their sons, Thomas and Freddie. He will be greatly missed by family, friends, and colleagues worldwide. His legacy will live on through his monumental contributions to numerical analysis and the countless lives he touched through his work and kindness.

Send News for *IMAGE* Issue 73

IMAGE seeks to publish all news of interest to the linear algebra community. Issue 73 of *IMAGE* is due to appear online on December 1, 2024. Send your news for this issue to the appropriate editor by October 15, 2024. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- book reviews to Mohsen Aliabadi (`maliabadisr@ucsd.edu`)
- linear algebra education news and articles to Anthony Cronin (`anthony.cronin@ucd.ie`)
- interviews of senior linear algebraists to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)
- problems and solutions to Rajesh Pereira (`pereirar@uoguelph.ca`)
- advertisements to Amy Wehe (`awehe@fitchburgstate.edu`)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (`jephianlin@gmail.com`)
- other articles and proposals to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)

Send all other correspondence to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`).

For past issues of *IMAGE*, please visit `https://www.ilasic.org/IMAGE`.

ILAS NEWS

David Bindel Selected as 2024 SIAM Fellow

The Society for Industrial and Applied Mathematics (SIAM) recently announced its 2024 SIAM Fellows. Among those selected for this honor was ILAS member David Bindel of Cornell University.

David was recognized for "contributions in numerical linear algebra and its innovative use in broad areas of computational science and engineering."

For details, see `https://sinews.siam.org/Details-Page/announcing-the-2024-class-of-siam-fellows`.

ILAS Election Results

Minerva Catral has been re-elected to a further three-year term as ILAS Secretary/Treasurer, starting March 1st, 2024.

Stefan Güttel and Naomi Shaked-Monderer were elected to three-year terms as members of the ILAS Board of Directors. They began their terms on March 1st, 2024.

Special thanks are due to the Board members who rotated off after three years of service, Melina Freitag and Apoorva Khare.

ILAS President/Vice President Annual Report: May 1, 2024

Respectfully submitted by Daniel B. Szyld, ILAS President, `szyld@temple.edu`
and Froilán M. Dopico, ILAS Vice President, `dopico@math.uc3m.es`

The past year has been another momentous year for ILAS. Among the highlights are the 25th ILAS Conference held in Madrid, Spain (June 12–16, 2023) with 480 participants; the third participation of ILAS as a partner in the Joint Mathematics Meetings (JMM) 2024; and new funds for students approved by the ILAS Board of Directors.

1. Board-approved actions since the last report include:

   - The Board moved on June 13, 2023 to remove the list of ILAS members from the ILAS website. This decision was made in part to comply with the European Union regulation about protection of personal data.

   - The Board approved on June 16, 2023 the renewal of the agreement between the AMS and ILAS concerning the conduct of the Joint Mathematics Meetings (JMM). With this agreement, ILAS will be a partner in the annual JMM. The agreement covered JMM2024 and JMM2025. Immediately after JMM2024, this agreement was automatically extended to cover JMM2026. This rolling forward to cover two years at a time will continue into the future. The agreement may be terminated by ILAS or the AMS upon two-year advance notice in writing.

   - The Board moved on June 16, 2023 to change the name of the "Taussky-Todd Prize" to the current "ILAS Olga Taussky and John Todd Prize".

   - The Board moved on September 22, 2023 to renew the agreement between Springer, IWOTA, and ILAS. With this agreement IWOTA committed to publish at least three proceedings with Springer for the IWOTA Conferences held in the years 2025–2030. In return Springer will give a donation of US$2000 to the Israel Gohberg ILAS-IWOTA Fund in each of the years 2026, 2028 and 2030 for organizing three Israel Gohberg ILAS Lectures at the conferences IWOTA 2026, ILAS 2028 and IWOTA 2030.

   - The Board moved on October 16, 2023 to create a "Prize Canvassing Committee". The purpose of this committee will be to identify ILAS members who might qualify for non-ILAS awards, e.g., AMS Fellow, SIAM Fellow, and possibly others. Then the committee will identify a person or persons who will prepare the nominations.

   - The Board moved on October 16, 2023 to increase the funding for the program "ILAS Support for non-ILAS Conferences" (`https://ilasic.org/non-ilas-guidelines/`). Now ILAS Lectureships at non-ILAS conferences may be supported by up to US$1250 for expenses and general support of non-ILAS conferences may be up to US$1000. Previously these quantities were US$1000 and US$750, respectively.

   - The Board approved on October 16, 2023 two new initiatives with the goal of helping Ph.D. students with fewer funding opportunities to attend ILAS Conferences. There will be ten grants of US$700 each for the travel expenses of Ph.D. students attending an ILAS Conference. ILAS will set aside funds to cover the registration of any deserving Ph.D. student who requests a registration fee waiver. This will be in place for the next two ILAS Conferences, and afterwards the Board will reassess the program.

   - The Board moved on April 18, 2024 to continue without changes the "ILAS Grant Program in support of Mathematicians working in Linear Algebra affected by conflicts." (`https://ilasic.org/grants-conflict/`).

2. Other news:

   - The 25th ILAS Conference was held June 12–16, 2023, at the "Escuela Técnica Superior de Ingenieros de Montes, Forestales y del Medio Natural" of the Universidad Politécnica de Madrid (Spain). The conference attracted 480 participants, mostly coming from Europe, but also from America and Asia. The program included 10 plenary lectures, 5 invited minisymposia, 26 contributed minisymposia, and 123 contributed talks, resulting in a total of 419 talks. Some highlights of the conference were the Hans Schneider Prize Lecture delivered by Nicholas J. Higham; the ILAS Olga Taussky and John Todd Prize Lecture delivered by Stefan Güttel; the first ILAS Richard A. Brualdi Early Career Prize Lecture delivered by Michael Tait; a minisymposium on the occasion of the 60th birthday of Steve Kirkland (organized by Hermie Monterde, Sarah Plosker, Jane Breen, and Sooyeong Kim) and another minisymposium on the occasion of the 70th birthday of Michael Overton (organized by Sara Grundel, Tim Mitchell, and Julio Moro).

- The Joint Mathematics Meetings 2024 (JMM 2024, an ILAS partner conference) was held in San Francisco, January 3–6, 2024. The ILAS Invited Address was given by the ILAS member Stephan Ramon Garcia (Pomona College, USA) with the title "Fast food for thought: what can chicken nuggets tell us about linear algebra?". There were six ILAS Special Sessions: (1) Graphs and Matrices (organized by Jane Breen and Stephen Kirkland); (2) Generalized Numerical Ranges and Related Topics (organized by Tin-Yau Tam and Pan-Shun Lau); (3) Linear Algebra, Matrix Theory, and its Applications (organized by Stephan Ramon Garcia and Konrad Aguilar); (4) Sign-pattern Matrices and their Applications (organized by Bryan L. Shader and Minerva Catral); (5) Spectral and Combinatorial Problems for Nonnegative Matrices and their Generalizations (organized by Pietro Paparella and Michael J. Tsatsomeros); and (6) Innovative and Effective Ways to Teach Linear Algebra (organized by David M. Strong, Sepideh Stewart, Gilbert Strang and Megan Wawro).

- The Executive Board accepted the recommendation of the JMM Committee that Anne Greenbaum from the University of Washington (USA) will give the ILAS Invited Address at the Joint Mathematics Meetings 2025, to be held in Seattle, January 8–11, 2025. In addition to this ILAS Lecture, there will be five ILAS Special Sessions scheduled at JMM 2025. The JMM committee consisted of Orly Alter, Ángeles Carmona Mejía, Mark Embree, Apoorva Khare (chair), and Daniel Szyld (ILAS President, ex officio).

- The ILAS Board approved on October 16, 2023 the request of the organizers of the 35$^{\text{th}}$ International Workshop on Operator Theory and its Applications (IWOTA), to be held at the University of Kent, UK, August 12–16, 2024, for an ILAS Lectureship for the ILAS member Melina Freitag.

- The *LAA* Lecture Committee selected Daniel Kressner as *LAA* lecturer for the ILAS 2025 Conference to be held at Kaohsiung, Taiwan, June 23–27, 2025.

3. ILAS elections ran November 1–30, 2023, and proceeded via electronic voting. The following were elected to offices with three-year terms that began on March 1, 2024:

   - Secretary/Treasurer: Minerva Catral
   - Board of Directors: Stefan Güttel and Naomi Shaked-Monderer

   The following continue in the ILAS offices which they currently hold:

   - President: Daniel Szyld (term ends February 28, 2026)
   - Vice President: Froilán M. Dopico (term ends February 28, 2025)
   - Second Vice President (for ILAS conferences): Raf Vandebril (term ends February 28, 2026)
   - Assistant Secretary/Treasurer: Michael Tait (term ends February 28, 2026)
   - Board of Directors: Paola Boito (term ends February 28, 2025), Fernando De Terán (term ends February 28, 2026), Chi-Kwong Li (term ends February 28, 2026) and Lek-Heng Lim (term ends February 28, 2025).

   Melina Freitag and Apoorva Khare completed their terms on the ILAS Board of Directors on February 29, 2024. We thank them for their valuable contributions as Board members; their service to ILAS is most appreciated.

   We also thank the members of the Nominating Committee – Leslie Hogben, Beatrice Meini (Chair), Helena Šmigoc, Tin-Yau Tam, and Hugo Woerdeman – for their efforts on behalf of ILAS, and all of the nominees for their participation in the elections.

4. New appointments and reappointments:

   ILAS-NET Manager:

   Leonardo Robol

   Prize Canvassing Committee:

   Richard Brualdi
   Chen Greif (chair)
   Ilse Ipsen
   Pauline van den Driessche
   Daniel Szyld (ILAS President, ex officio)

JMM Committee for proposing ILAS-JMM Lecturers for 2025, 2026, and 2027:

Orly Alter
Ángeles Carmona Mejía
Mark Embree
Apoorva Khare (chair)
Daniel Szyld (ILAS President, ex officio)

Hans Schneider Prize Committee for 2025:

Leslie Hogben (chair)
Steve Kirkland
Misha Kilmer
Volker Mehrmann
Beatrice Meini
Daniel Szyld (ILAS President, ex officio)

LAA Lecture Committee for the ILAS 2025 Conference:

Dario Bini
Richard Brualdi (chair)
Dragana Cvetkovic
Daniel Szyld (ILAS President, ex officio)

Israel Gohberg ILAS-IWOTA Lecture Selection Committee for the conferences IWOTA 2026, ILAS 2028 and IWOTA 2030:

Jussi Behrndt
Kelly Bickel
Michael Dritschel
Ilya Spitkovsky (chair)

- ILAS endorsed the following conferences of interest to ILAS members that have taken place since the last President/Vice President annual report:
  - Thematic Conference of the Spanish ALAMA Network on Laplacians and M-matrices in graphs, Universitat Politècnica de Catalunya, Barcelona, Spain, May 11–12, 2023. `https://mapthe.upc.edu/en/congresses/jornadas`
  - Western Canadian Linear Algebra Meeting (WCLAM 2023), University of Regina, Saskatchewan, Canada, May 27–28, 2023. `https://uregina.ca/~sfallat/wclam`
  - 16th Workshop on Numerical Ranges and Numerical Radii (WONRA 2023), University of Coimbra, Coimbra, Portugal, June 7–9, 2023. `https://sites.google.com/view/wonra-2023`
  - 34th International Workshop on Operator Theory and its Applications (IWOTA 2023), University of Helsinki, Helsinki, Finland, July 31 – August 4, 2023. Stephan Ramon Garcia was the Hans Schneider ILAS Lecturer. `https://www.helsinki.fi/en/conferences/iwota2023`
  - International Conference on Linear Algebra and its Applications (ICLAA 2023), Centre for Advanced Research in Applied Mathematics and Statistics (CARAMS), MAHE, Manipal, India, December 18–21, 2023. `https://carams.in/events/iclaa2023/`
- ILAS endorsed the following conferences of interest to ILAS members that will take place in the next months:
  - SIAM Conference on Applied Linear Algebra (LA24), Sorbonne Université, Paris, France, May 13–17, 2024. Daniel Kressner and Laura Grigori are the chairs of the organizing committee. The ILAS members Andrii Dmytryshyn and Beatrice Meini will be the ILAS plenary speakers at the conference. `https://www.siam.org/conferences/cm/conference/la24`
  - ALAMA 2024, the 8th meeting of the Spanish Thematic Network of Linear Algebra, Matrix Analysis, and Applications (ALAMA), Universidad de Oviedo, Gijón campus, Spain, June 12–14, 2024. `https://www.unioviedo.es/alama2024/`

  - 35$^{\text{th}}$ International Workshop on Operator Theory and its Applications (IWOTA 2024), University of Kent, Canterbury, UK, August 12–16, 2024. The ILAS member Mark Embree will be the Israel Gohberg ILAS-IWOTA Lecturer, and the ILAS member Melina Freitag will be an ILAS supported lecturer. `https://blogs.kent.ac.uk/iwota2024/`

- The following ILAS conference is scheduled:
  - The 26$^{\text{th}}$ ILAS Conference will be held in Kaohsiung, Taiwan, June 23–27, 2025, with Jephian C.-H. Lin (National Sun Yat-sen University) as Chair of the Organizing Committee. The plenary speakers will be Haim Avron (Tel Aviv University, Israel, SIAG/LA Lecturer), Fan Chung Graham (University of California, San Diego, USA), Fumio Hiai (Tohoku University, Japan), Daniel Kressner (EPFL, Switzerland, *LAA* Lecturer), Ren-Cang Li (University of Texas at Arlington, USA), Karen Meagher (University of Regina, Canada), Polona Oblak (University of Ljubljana, Slovenia), Fernando De Terán (Universidad Carlos III de Madrid, Spain), Karol Życzkowski (Jagiellonian University, Poland), and the Hans Schneider Prize winner (to be announced).

- The following ILAS partner conference is scheduled:
  - The Joint Mathematics Meetings 2025 will be held in Seattle, January 8–11, 2025. The ILAS member Anne Greenbaum from the University of Washington (USA) will give the ILAS Invited Address and six ILAS Special Sessions are scheduled.

- The Electronic Journal of Linear Algebra (*ELA*) is now in its 40$^{\text{th}}$ volume. *ELA*'s URL is `https://journals.uwyo.edu/index.php/ela`. The Editorial Board was renewed in depth on January 1, 2024 (`https://journals.uwyo.edu/index.php/ela/about/editorialTeam`). Volume 39 was published in 2023 and contains 43 papers. *ELA* received 256 new submissions in 2023. The current acceptance rate is less than 23%. In 2023, 79404 downloads and 73582 abstract views of *ELA*papers occurred.

  Froilán M. Dopico (Universidad Carlos III de Madrid) is the Editor-in-Chief.

  ILAS members are strongly encouraged to submit their work to *ELA*, the flagship research journal of our society.

- *IMAGE* is the semi-annual bulletin for ILAS, available online at `https://ilasic.org/image/`. The Editor-in-Chief is Louis Deaett (Quinnipiac University, USA). In 2023, the website of *IMAGE* received 667 visits.

- ILAS-NET is a moderated electronic newsletter for mathematicians worldwide, with a focus on linear algebra. It is currently managed by Leonardo Robol (Università di Pisa, Italy), as of January 1st, 2024. It was previously managed by Pietro Paparella (University of Washington, Bothell, USA).

  An archive of ILAS-NET messages is available at `https://ilasic.org/ilas-net/`. To send a message to ILAS-NET, please send the message (preferably in text format) in an email to `leonardo.robol@unipi.it` indicating that you would like it to be posted on ILAS-NET. If the message is approved, it will be posted soon afterwards. To subscribe to ILAS-NET, please go to `https://www.ilasic.org/ilas-net/`

  On April 22, 2024, there were 1071 active subscribers to the ILAS-NET newsletter.

- ILAS's website is located at `https://ilasic.org/` and highlights the main activities of ILAS: the Electronic Journal of Linear Algebra (*ELA*), the conferences, *IMAGE*, ILAS-NET and other activities. In addition, the website provides general information about ILAS (e.g., ILAS officers, bylaws, special lecturers, ILAS prizes, grant programs) as well as links to pages of interest to the ILAS community. Currently it is managed by Dominique Guillot (University of Delaware, USA). In 2023, the website of ILAS received 11529 pageviews from users from 112 countries. The front page received 4579 of these pageviews, the conference page 2043, and the *IMAGE* page 667.

Finally, we want to express our great gratitude to all the officers of ILAS who all show wonderful dedication to the society, as well as to all the individual members of ILAS and our corporate sponsors. Without any of them ILAS would not be what it is today.

Respectfully submitted,

Daniel B. Szyld, ILAS President (`szyld@temple.edu`); and
Froilán M. Dopico, ILAS Vice President (`dopico@math.uc3m.es`).

ILAS 2023–2024 Treasurer's Report

April 1, 2023 – March 31, 2024

by Minerva Catral, ILAS Secretary/Treasurer

Net Account Balance on March 31, 2023

Checking Account - First Interstate Bank		$ 61,629.19	
Certificate of Deposit 1		$ 61,267.41	
PayPal		$ 12,797.77	
Vanguard		$ 153,255.06	
Accounts Payable		−$ 1,047.96	
			$ 287,901.47
General Fund		$ 155,750.00	
Israel Gohberg ILAS-IWOTA Lecture Fund		$ 7,002.63	
Conference Fund		$ 9,512.29	
ILAS Olga Taussky and John Todd Prize Fund		$ 11,270.53	
Hans Schneider Lecture Fund		$ 9,407.71	
Frank Uhlig Education Fund		$ 5,503.56	
Hans Schneider Prize Fund		$ 23,750.74	
ILAS Richard A. Brualdi Early Career Prize Fund		$ 6,172.70	
ELA Fund		$ 1,161.08	
LAMA Fund		$ 58,370.23	
			$ 287,901.47
INCOME:			
Dues	$ 8,610.00		
Israel Gohberg ILAS-IWOTA Lecture Fund Donations	$ 854.21		
General Fund Donations	$ 3,600.00		
Conference Fund Donations	$ 10.00		
ILAS Olga Taussky and John Todd Prize Fund Donations	$ 10.00		
Hans Schneider Lecture Fund Donations	$ 0.00		
Uhlig Education Fund Donations	$ 20.00		
Hans Schneider Prize Fund Donations	$ 100.00		
ELA Fund Donations	$ 190.00		
ILAS Richard A. Brualdi Early Career Prize Fund Donations	$ 30.00		
Corporate Dues Income	$ 2,150.00		
Interest - First Interstate Bank	$ 19.31		
Interest on FIB Certificate of Deposit	$ 154.01		
Vanguard Unrealized Capital Gains	$ 31,796.54		
Accounts Receivable	$ 100.00		
Miscellaneous	$ 6.15		
Total Income		$ 47,650.22	
EXPENSES:			
ILAS Conference Expenses	$ 11,270.66		
ELA	$ 8,988.86		
IMAGE	$ 844.20		
IWOTA	$ 56.22		
JMM Expenses	$ 1,082.17		
PayPal/Credit Card Processing & Bank Fees	$ 523.40		
Non-ILAS Conferences	$ 1,376.16		
Hans Schneider ILAS Lecture	$ 1,000.00		
Hans Schneider Prize	$ 749.40		
LAMA Lecture	$ 1,284.62		
ILAS Olga Taussky and John Todd Prize	$ 1,337.57		
ILAS Richard A. Brualdi Early Career Prize	$ 1,663.05		
Business License	$ 70.00		
Election Costs	$ 450.33		
Web Hosting, MailChimp & Online Membership Forms	$ 663.05		
Child Care Grant	$ 533.66		
Total Expenses		$ 31,893.35	

Net Account Balance on March 31, 2024

Checking Account - First Interstate Bank		$ 20,564.32	
Certificate of Deposit		$ 50,000.00	
PayPal		$ 7,942.42	
Vanguard		$ 225,051.60	
Accounts Receivable		$ 100.00	
			$ 303,658.34
General Fund		$ 159,829.53	
Israel Gohberg ILAS-IWOTA Lecture Fund		$ 8,376.08	
Conference Fund		$ 10,494.17	
ILAS Olga Taussky and John Todd Prize Fund		$ 11,049.13	
Hans Schneider Lecture Fund		$ 9,558.62	
Frank Uhlig Education Fund		$ 6,083.03	
Hans Schneider Prize Fund		$ 25,690.90	
ILAS Richard A. Brualdi Early Career Prize Fund		$ 5,294.14	
ELA Fund		$ 1,507.73	
LAMA Fund		$ 65,775.01	
			$ 303,658.34

CONFERENCE REPORTS

Events organized by Red ALAMA
Vallclara and Pamplona, Spain, 2023–2024

Report by Ángeles Carmona

The Spanish Network of Linear Algebra takes great delight in informing you about various events that have taken place in late 2023 and early 2024. It seems to be the year of linear algebra in Spain—a year unusually rich with related happenings. Here we report on two recent events in which the ALAMA network has participated.

1. **The 2nd MAPTHE Workshop** held from the 1st to the 5th of December, 2023, in Vallclara, Spain (`https://mapthe.upc.edu/en/mapthe_2/2ndWorkshop`). This workshop serves as a meeting point for members of MAPTHE, a research group from UPC-BarcelonaTech, and ALAMA (Linear Algebra, Matrix Analysis, and Applications) network members. At these meetings, we share the latest advances and define future directions in line with the objectives of the funded projects of which MAPTHE is a part. Additionally, we foster collaboration with other researchers attending the meetings. The focus this time was primarily on random walks in networks and the study of associated parameters, as well as on advances in inverse problems. We also reviewed outcomes from the 6th ALAMA Conference, which took place in Barcelona in May 2023 [see *IMAGE* issue 71, p. 16].

   This workshop was funded by the MICINN project (Ministry of Science, Innovation, and Universities: Project MTM2017-90682-REDT), the Universitat Politècnica de Catalunya (UPC), the UPC Department of Mathematics, and the ALAMA Network.

Participants and relatives at the 2nd MAPTHE Workshop (Tarragona, Spain)

2. **A special session titled "New trends in Linear Algebra and Matrix Analysis"** organized by Á. Carmona, C. Marijúan, and A. Roca on behalf of the ALAMA Network at the biennial meeting of the RSME (Real Sociedad Matemática Española).

   This session showcased a broad spectrum of current work and emerging developments in the fields of linear algebra and matrix analysis, and their applications. Topics covered in this session included: companion forms; inverse spectral problems; structured matrices; solving large systems (iterative methods, preconditioning, etc.); generalized inverses; control theory; rational matrices; polynomial matrices; matrix equations; spectral theory of matrices; pseudospectra; discrete potential theory; and software development. We believe this session was incredibly beneficial for fostering synergies within the Network itself. Moreover, Pamplona, the host city for the conference, proved a welcoming and pleasant city for such a gathering.

Participants at the biennial meeting RSME 2024

ILAS at the Joint Mathematics Meetings San Francisco, USA, January 3–6, 2024

Report by Minerva Catral and Daniel B. Szyld

The 2024 Joint Mathematics Meetings (JMM) was held in San Francisco, California on January 3–6, 2024. ILAS had a strong presence at the meeting, with five ILAS Special Sessions and the ILAS Invited Address delivered by Stephan Ramon Garcia. ILAS has been a JMM partner since 2022.

The ILAS Invited Address by Stephan Ramon Garcia was entitled "Fast Food for Thought: What Can Chicken Nuggets Tell Us About Linear Algebra?" and drew a large audience. The talk, which centered around a simple chicken nuggets question, connected everything from analysis and combinatorics to probability theory, and of course linear algebra! Stephan was publicly recognized for his lecture at the JMM Prize Ceremony and awarded a framed certificate. The talk is available on YouTube at `https://www.youtube.com/watch?v=EEwBiS6v768`.

The titles and organizers of the ILAS special sessions were:

- Graphs and Matrices (Jane Breen, Stephen Kirkland)
- Generalized Numerical Ranges and Related Topics (Tin-Yau Tam, Pan-Shun Lau)
- Linear Algebra, Matrix theory, and its Applications (Stephan Ramon Garcia, Konrad Aguilar)
- Sign-pattern Matrices and Their Applications (Bryan Shader, Minnie Catral)
- Spectral and combinatorial problems for nonnegative matrices and their generalizations (Pietro Paparella, Michael Tsatsomeros)
- Innovative and Effective Ways to Teach Linear Algebra (David Strong, Sepideh Stewart, Gilbert Strang, Megan Wawro)

Also, an AIM Special Session on Graphs and Matrices was organized by ILAS members Mary Flagg and Bryan Curtis.

The next JMM will take place in Seattle on January 8–11, 2025, where the ILAS Invited Address will be given by Anne Greenbaum.

(top-bottom, left-right): Stephan Ramon Garcia answers a question after giving the ILAS Invited Address; Stephan Ramon Garcia and Daniel Szyld at the JMM Prize Ceremony; Daniel Szyld and Gilbert Strang; Speakers at the ILAS Special Session on Innovative and Effective Ways to Teach Linear Algebra (left-right): Peyam Ryan Tabrizian, Frank Uhlig, Sheldon Axler, Sepideh Stewart, Gilbert Strang, Jeffrey Meyer, Mike Michailidis

International Conference on Linear Algebra and its Applications (ICLAA) Manipal, India, December 18–21, 2023

Report by Umashankara Kelathaya

The International Conference on Linear Algebra and its Applications (ICLAA 2023), organized by the Center for Advanced Research in Applied Mathematics & Statistics (CARAMS), Manipal Academy of Higher Education (MAHE), was held in Manipal, India, December 18–21, 2023. The conference, endorsed by ILAS, was the fifth in its sequence, following CMTGIM 2012, ICLAA 2014, ICLAA 2017, and ICLAA 2021.

The inauguration of ICLAA 2023

Like the preceding conferences, ICLAA 2023 focused on classical matrix theory and linear algebra, nonnegative matrices and special matrices, matrices and graphs, matrix methods in statistics, linear statistical inference, etc. About 200 delegates registered for ICLAA 2023, including 28 invited speakers and 67 contributing speakers from different parts of the world. The list of invited speakers included Rafikul Alam, Ravindra B. Bapat, Sudeep R. Bapat, Abraham Berman, N. Eagambaram, Shaun M. Fallat, Stephen J. Haslett, Surender Kumar Jain, Debajit Kalita, Rajesh Kannan, Apoorva Khare, Stephen J. Kirkland, Mika Mattila, David Raj Micheal, Bojan Mohar, Samir K. Neogy, Sukanta Pati, Simo Puntanen, T. E. S. Raghavan, Balaji Ramamurthy, T. S. S. R. K. Rao, Dietrich von Rosen, Arumugam S, Peter Šemrl, Ajit Iqbal Singh, K. C. Sivakumar, Sivaramakrishnan Sivasubramanian, and Murali K. Srinivasan. Despite a few dropouts due to unavoidable constraints, it is evident that the community of scholars within the focus areas of linear algebra, matrix methods in statistics, and matrices and graphs is not only consolidating, but also growing as a society with a strong bond. ICLAA 2023 provided a platform for renowned mathematicians and statisticians to come together and discuss research problems. It provided ample time for young scholars to present their contributions before eminent scholars. Every contributing speaker got not less than twenty minutes to present their results. Also, ICLAA 2023 included three special lectures, from senior scientists, aimed at encouraging young scholars.

The sponsors of ICLAA 2023 are the National Board for Higher Mathematics (NBHM), the Science and Engineering Research Board (SERB), and the Indian National Science Academy (INSA). About 14 senior scientists (invited speakers) including Profs. R. B. Bapat, S. K. Jain, T. E. S. Raghavan, S. K. Neogy, T. S. S. R. K. Rao, Apoorva Khare, etc., and 13 young scientists among the participants benefited from this financial support for their travel to and from Manipal.

The ILCAA 2023 welcome address

ICLAA 2023 opened with a formal inaugural function on December 18th. Dr. K. Manjunatha Prasad (Organizing Secretary) provided the opening remarks and described the background of the ICLAA series and CARAMS. An overview of ICLAA 2023 was provided by Prof. R. B. Bapat (Chairman, Scientific Committee). Dr. Narayana Sabhahit, Pro VC - Tech. & Sci., MAHE, Manipal presided over the inaugural function, and Profs. Peter Šemrl and Stephen J. Haslett were the chief guests.

Plenary and invited talks were organized in 14 different sessions, and contributed talks in 12 sessions, including parallel sessions. The best paper award contest was held in two categories and three themes. The conference included three special sessions, "Special Matrices", "Matrix Methods in Statistics", and "Matrices, Graphs, and Applications". These sessions were dedicated to the great linear algebraists – Abraham Berman, Simo Puntanen, Ravindra B. Bapat, and Stephen J. Kirkland – who have been associated with the ICLAA series for the last several years.

Besides the busy scientific schedule, a joint excursion to Kota Shivarama Karantha Memorial Theme Park was arranged on the evening of December 19th, featuring the presentation of a cultural program consisting of Yakshagana, a form of traditional Indian theater developed in coastal Karnataka. This was followed by a musical fountain show.

The participants of ICLAA 2023

The third day of the conference, December 20th, was dedicated to Prof. R. B. Bapat on the special occasion of his birthday. CARAMS, MAHE, along with all the participants of ICLAA 2023, celebrated with Prof. Bapat's family.

ICLAA 2023 concluded on December 21, 2023 with the valedictory session. The chief guests at this function were Prof. T. E. S. Raghavan and Prof. Surender Kumar Jain. Prof. Simo Puntanen joined the program virtually. After the distribution of best paper awards and addresses by the chief guests, the valedictory session closed with a vote of thanks by Dr. K. Manjunatha Prasad.

This conference followed the preconference workshop IWSMGA 2023, which focused on Special Matrices, Graphs, and their Applications. Eminent personalities in the field, like Profs. Abraham Berman, S. K. Neogy, and Sivaramakrishnan Sivasubramanian, etc., delivered lectures and discussed problems in the tutorial sessions. Nearly 60 participants from different parts of India benefited from this workshop.


(left-right): Cultural program, Yakshagana; Prof. Peter Šemrl addressing the gathering

The following four publications dedicated to ICLAA 2023 are planned. Two are journal publications, namely in:

1. *Indian Journal of Pure and Applied Mathematics*, Springer
2. *AKCE International Journal of Graphs and Combinatorics*, Taylor & Francis

and the other two are book volumes with invited and contributed chapters:

3. *Special Matrices and Applications*, published by Springer
4. *Linear Algebra, Matrices and their Applications*, published by AMS Contemporary Mathematics.

The last date for submission of articles is August 31, 2024. For more details about the publications dedicated to ICLAA 2023, please visit `https://carams.in/events/publications-iclaa-2023`.

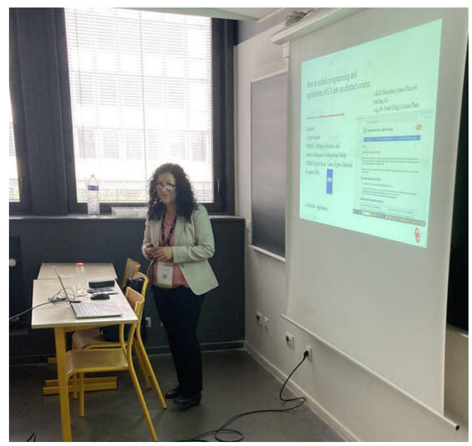
SIAM Conference on Applied Linear Algebra Paris, France, May 13–17, 2024

Report by Daniel B. Szyld

ILAS has an ongoing collaboration with the SIAM Activity Group on Linear Algebra (SIAG-LA), and we always have a significant presence at their conferences, while many members of the SIAG-LA come to our conferences.

This year, the SIAM Conference on Applied Linear Algebra took place at the Sorbonne Université, Paris, during the week of May 13–17, 2024, and while the ILAS presence was less abundant than in other years, it was still important. About half of the members of the ILAS Board of Directors were present.

The conference attendance was an all-time record of 650 participants. The program was very full, starting every day at 8AM and ending most days at 7:20PM. There were 130 minisymposia sessions and 16 contributed sessions. In total there were 8 parallel sessions at all times. Two such sessions were organized by the ILAS Education Committee, and they were very well attended with well-received talks. There were also three minisymposia sessions honoring Steve Mackey for his 70th birthday. They were of the highest scientific level and it was a very fitting homage to Steve's career up to this point.



Sepideh Stewart presents in the Linear Algebra Education minisymposium

Two plenary speakers represented ILAS and gave excellent talks: "Eigenstructures of Low-rank Matrix Polynomials" by Andrii Dmytryshyn (Örebro University, Sweden) and "A Journey in the Kemeny Constant: Centrality Measures, Stochastic Complementation and Infinite-dimensional Matrices" by Beatrice Meini (University of Pisa, Italy).

There was also a special session as a memorial appreciation of Nick Higham's impact on the field of Numerical Linear Algebra and contributions to the community. It was a very emotional event.

During the conference dinner, SIAM Vice President for Programs Jim Nagy exalted the cooperation between the two societies. As always, in the next two years our ILAS Conferences will take place, followed in 2027 by the next SIAM Conference on Applied Linear Algebra.

(left-right): Plenary lectures by Andrii Dmytryshn and Beatrice Meini

UPCOMING CONFERENCES AND WORKSHOPS

The 8th Linear Algebra, Matrix Analysis and Applications Meeting (ALAMA) Gijón, Spain, June 12–14, 2024

The thematic network of Linear Algebra, Matrix Analysis and Applications (ALAMA, https://www.red-alama.es) has organized its eighth meeting for June 2024 in Gijón, Asturias, Spain. This edition continues the biennial series following the editions in Vitoria-Gasteiz (2008), Valencia (2010), Leganés (2012), Barcelona (2014), León (2016), Alicante (2018) and Alcalá de Henares (2022).

The main objective of the meeting is to gather researchers whose work is related to linear algebra, matrix analysis, and their applications, so that they can exchange results, experiences and ideas.

The members of the Scientific Committee are Pedro Alonso, Luca Bergamaschi, Fernando de Terán, Silvia Marcaida, Juan Manuel Peña, María Luisa Serrano and María Teresa Trobajo. The Local Committee is made up of Pedro Alonso, Jorge Jiménez, Mariano Mateos, Antonio Palacios, Set Pérez, María Luisa Serrano and Jesús Suárez.

The meeting will include three invited talks. In addition, there will be five minisymposia, comprising a total of 47 communications, and 34 further individual communications, all of which will be presented in parallel sessions. About 90 participants are expected, including from Belgium, China, Colombia, France, India, Italy, Morocco, Portugal, and Spain.

The plenary speakers are:

- Yuji Nakatsukasa (University of Oxford)
- S. Irene Díaz (University of Oviedo)
- Plamen Koev (San Jose State University)

Abstracts of the papers presented will be included in a book of conference proceedings, as is customary for this type of meeting.

Thanks are due to the sponsors of the event: University of Oviedo (UO), the Fundación Cajastur, the Mathematics Department of UO, the Spanish Society of Applied Mathematics (SEMA) and Ayuntamiento de Gijón (Asturias).

For more information, please visit the conference web site at https://www.unioviedo.es/alama2024.

Workshop on Matrices and Operators Reno, USA, June 15–16, 2024

The University of Nevada, Reno (Reno, Nevada, USA) will host the Workshop on Matrices and Operators on Saturday, June 15th and Sunday, June 16th, 2024. The workshop aims to promote research and collaboration among researchers with interests in matrix theory, operator theory, and related topics. Participants are encouraged to present their latest findings and discoveries, and share their experiences and research problems, fostering an atmosphere of collaboration and mutual learning. Through dialogue and the exchange of knowledge, the workshop will enhance the collective growth and development of the research community involved in matrix theory, operator theory, and related fields.

Confirmed Speakers:

- Dominique Guillot, University of Delaware, USA
- Jimmie Lawson, Louisiana State University, USA
- Avleen Kaur, University of British Columbia, Canada
- Chi-Kwong Li, College of William and Mary, USA
- Jephian Chin-Hung Lin, National Sun Yat-sen University, Taiwan
- Qing-Wen Wang, Shanghai University, China
- Xiang Xiang Wang, University of Nevada, Reno, USA
- Xuzhou Zhan, Beijing Normal University, Zhuhai, China
- Xiaodong Zhang, Shanghai Jiao-Tong University, China

The organizers of the workshop are Valentin Deaconu, University of Nevada, Reno, USA, vdeaconu@unr.edu; Alex Kumjian, University of Nevada, Reno, USA, alex@unr.edu; Pan Shun Lau, University of Nevada, Reno, USA, plau@unr.edu; Chi-Kwong Li, College of William and Mary, USA, ckli@math.wm.edu; and Tin-Yau Tam (Chair), University of Nevada, Reno, USA, ttam@unr.edu.

The conference will be hosted at the Davidson Math and Science Center, University of Nevada, Reno, USA. The conference banquet will be held at 6PM on Saturday, June 15th, 2024.

For more details on the conference, see:

`https://sites.google.com/view/tinyautam/homepage/mao-2024`

Special session on Linear Algebra, Matrix Analysis and Applications *at the* XXVIII Congress of Differential Equations and Applications/XVIII Congress of Applied Mathematics (CEDYA/CMA) Bilbao, Spain, June 24–28, 2024

This special session is part of the program of the biennial congress of the Spanish Society of Applied Mathematics (SEMA). Since its first edition in 1978, this meeting has served as a benchmark for both national and international researchers in fields such as differential equations, dynamical systems, optimization, control, numerical analysis, and scientific computing. The ALAMA Network has organized a special session with a focus on the Network's young researchers.

Invited Speakers:

- Ana Mancho (Instituto de Ciencias Matemáticas ICMAT, Madrid, Spain)
- Esmeralda Mainar Maza (Universidad de Zaragoza, Spain)
- Katharina Schratz (Sorbonne Université, Paris, France)
- Adelia Sequeira (Department of Mathematics at the IST, University of Lisbon, Portugal)
- Malgorzata Peszynska (Department of Mathematics, Oregon State University, USA)
- Mária Lukácová-Medvidová (Johannes Gutenberg-Universität, Mainz, Germany)
- Giancarlo Sangalli (Dipartimento di Matematica, Università di Pavia, Italy)
- David Pardo (Instituto BCAM y UPV/EHU, Bilbao, Spain)
- Lia Bronsard (McMaster University, Canada)

Scientific committee:

- Begoña Cano Urdiales (University of Valladolid, Spain)
- Elena Gaburro (INRIA and University of Verona, Italy)
- Carlos García Cervera (University of California, Santa Barbara, USA)
- David Lannes (CNRS et Université de Bordeaux, France)
- María Luisa Rapún Banzo (Polytechnic University of Madrid, Spain)
- Carmen Rodrigo Cardiel (IUMA and University of Zaragoza, Spain)
- Luis Vega González (BCAM and University of the Basque Country, Spain)

For further information, visit:

`https://www.sema.org.es/es/cedya2024`

Special sessions on Linear Algebra, Matrix Analysis and Applications *and on* Random walks, Kemeny's constant and applications *at* Mathematical Modelling in Engineering & Human Behaviour (MM&HB) 2024 Valencia, Spain, July 10–12, 2024

The objective of the Mathematical Modelling in Engineering and Human Behaviour conference is to develop an interdisciplinary forum in areas like medicine, sociology, business and engineering, where the latest mathematical techniques can be discussed in a common and understandable language with experts in cross-disciplinary areas. The conference is aimed at gathering researchers who need mathematics for the formulation and analysis of models.

Two special sessions of particular interest to the linear algebra community will be included:

1. Special session **"Linear Algebra, Matrix Analysis and Applications"** organized by José Mas, on topics in matrix analysis such as total positivity, sign regular matrices, combined matrices and inverse problems.

   Invited Speakers:

   - Begoña Cantó, Universitat Politècnica de València
   - Andrés M. Encinas, Universitat Politècnica de Catalunya
   - Ana Marco, Universidad de Alcalá
   - José Martínez de las Heras, Universidad de Alcalá
   - Juan Manuel Peña, Universidad de Zaragoza
   - Álvaro Samperio, Universidad de Valladolid

2. Special session **"Random walks, Kemeny's constant and applications"** organized by Alicia Roca. The goal of this minisymposium is to present the concept of Kemeny's constant, problems related to its calculation, and some applications.

   Invited Speakers:

   - Aida Abiad, TU Eindhoven
   - María José Jiménez, Universitat Politècnica de Catalunya
   - Robert E. Koij (plenary lecture), University of Technology Delft
   - Àlvar Martín, Universitat Politècnica de Catalunya

Organizing committee of MME&HB 2024: Begoña Cantó, Juan Carlos Cortés, Antonio Hervás, Damián Ginestar, Dolors Roselló, Francisco Chicharro, Alicia Roca, Antoni Vidal, Elena López.

For further information, visit `https://imm.webs.upv.es/jornadas/2024/home.html`.

Special sessions on Combinatorial Matrix Theory *and on* Spectral Graph Theory *at the* Women in Combinatorics Virtual Conference 2024 (to be held online) July 15–16, 2024

The inaugural iteration of the Women in Combinatorics Virtual Conference 2024 will be held virtually July 15–16, 2024, meeting 9AM – 4PM Eastern Time (UTC−04:00) each day.

Included are a special session on Combinatorial Matrix Theory (organized by Jane Breen and Hermie Monterde) and one on Spectral Graph Theory (organized by Sasmita Barik and Hermie Monterde).

Registration for the conference is free and open to everyone. To register, complete the form at

`https://forms.gle/QouqsKgN7DkqeFo47`

no later than 11:59PM on July 12th, 2024. Zoom links will be sent to registered participants. For more information about this conference, please visit the conference website:

`https://www.womenincombinatorics.com/virtual-conference-2024`

Special session on Linear Algebra and Matrix Analysis *at the* European Congress of Mathematics Sevilla, Spain, July 15–19, 2024

The European Congress of Mathematics (ECM) is held by the European Mathematical Society (EMS) every four years and stands as the second-largest mathematical event globally. The inaugural ECM took place in Paris in 1992, and since then it has been held quadrennially in various European cities: Budapest in 1996, Barcelona in 2000, Stockholm in 2004, Amsterdam in 2008, Krakow in 2012, Berlin in 2016, and Portorož in 2021 (originally scheduled for 2020 but postponed due to the coronavirus pandemic).

The 2024 ECM will include a special session entitled "Linear Algebra and Matrix Analysis" that has been organized by Fernando De Terán (Universidad Carlos III de Madrid), Alicia Roca (Universitat Politècnica de Valencia), and Ángeles Carmona (Universitat Politècnica de Catalunya, Barcelona, Spain). This minisymposium will be devoted to presentations of recent results in the areas of applied linear algebra, numerical linear algebra, matrix analysis, and their applications, by researchers from different European countries. The aim of the minisymposium is twofold:

1. to bring together current researchers in the field in order to share results, new problems or proposals, possible new applications, and ideas or techniques that could be involved in the study of those problems; and
2. to foster the relationship between researchers from different European countries who are currently working in the areas of applied and numerical linear algebra and matrix analysis.

The minisymposium intends to support the recently created Topical Activity Group E-ANLAMA within the framework of the European Mathematical Society.

Invited Speakers:

- Aida Abiad, Technische Universität Eindhoven, the Netherlands
- Andrii Dmytryshyn, Örebro University, Sweden
- Andrés M. Encinas, Universitat Politècnica de Catalunya, Spain
- Bruno Lannazzo, Università di Perugia, Italy
- Volker Mehrmann, Technische Universität Berlin, Germany
- Julio Moro, Universidad Carlos III de Madrid, Spain
- João Queiró, Universidade de Coimbra, Portugal
- Fernando De Terán, Universidad Carlos III de Madrid, Spain

The deadlines for abstract submission and for early registration have passed, but late registration remains possible. For further information, visit:

`https://www.ecm2024sevilla.com/index.php`

The 35th International Workshop on Operator Theory and its Applications (IWOTA) Canterbury, UK, August 12–16, 2024

The IWOTA conference series is the largest and most important annual event in operator theory and its applications, bringing together leading international experts from pure mathematics and application areas to trace the future development of operator theory and related areas such as complex analysis, harmonic analysis, linear algebra, random matrix theory, and mathematical physics, as well as their applications, including control theory, signal processing, and AI.

IWOTA 2024 will take place August 12–16, 2024 at the University of Kent in Canterbury, UK. It will provide a medium for an intense exchange of new results, information and opinions, and for international collaboration in operator theory and its applications worldwide. It will further set directions for future research through the conference activities and proceedings, and also serve as the setting for the planning of future IWOTA conferences.

A substantial part of IWOTA consists of special sessions whose organisers have been selected to ensure a coherent, diverse and attractive agenda of research activity and talks. Special sessions provide opportunities for all participants to present their results and interact with other researchers with similar interests.

Invited Plenary Speakers:

- Mark Embree, Virginia Tech (Israel Gohberg ILAS-IWOTA Lecturer)
- Rupert Frank, LMU Munich
- Melina Freitag, Potsdam University
- Stéphane Gaubert, École Polytechnique, Paris
- Mariana Haragus, University of Franche-Comté
- Svetlana Jitomirskaya, University of California, Irvine and Georgia Tech
- Barbara Kaltenbacher, University of Klagenfurt
- Gitta Kutyniok, LMU Munich
- Malabika Pramanik, University of British Columbia

The IWOTA 2024 Organizing Committee consists of Bas Lemmens, Ana Loureiro and Ian Wood (University of Kent), Marina Iliopoulou (National and Kapodistrian University of Athens) and Marco Marletta (Cardiff University) in collaboration with the IWOTA executive steering committee members: J. William Helton (chair, University of California, San Diego), Sanne ter Horst (North-West University, South Africa), Igor Klep (University of Ljubljana, Slovenia), Irene Sabadini (Polytecnico di Milano, Italy), and Hugo J. Woerdeman (Drexel University, Pennsylvania).

Early registration for the conference has now passed; from June 1st, registration fees are £300 (£180 for early-career researchers). For further information, including a list of special sessions and a link to the registration page, see:

https://blogs.kent.ac.uk/iwota2024

Applied Matrix Positivity II
Edinburgh, UK, November 4–8, 2024

A workshop entitled "Applied Matrix Positivity II" will take place at the International Centre for Mathematical Sciences (ICMS), Edinburgh, UK during the week of November 4–8, 2024.

This workshop is a sequel to the ICMS workshop "Applied Matrix Positivity", which was held online in July 2021.

Participants will explore connections between the following three themes.

- Positivity transforms
- Total positivity and time-frequency analysis
- Spatial statistics and machine learning

These have the overarching concept of a positive-definite kernel as a common golden thread. Our aim is to foster new exchanges and collaborations between theoretical and applied groups working on these topics of great current interest.

The week-long workshop will consist of a preliminary day of talks devoted to foundational aspects of each theme, followed by three thematic days and concluding with the final day devoted to open discussion and the formulation of problems and plans for future research.

Further details regarding the workshop and participation will appear on the ICMS website here:

https://www.icms.org.uk/workshops/2024/applied-matrix-positivity-ii

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The 9$^{\text{th}}$ Linear Algebra Workshop (LAW'25)
Portorož, Slovenia, June 2–6, 2025

http://www.law05.si/law25

Having brought researchers together once every three years for almost 30 years, after a recent "pandemic delay" the LAW'xx meetings are back again, hosted by the University of Primorska, and co-organized by the University of Ljubljana and the Institute of Mathematics, Physics, and Mechanics (IMFM). The aim of this workshop is twofold: to mark the round anniversary of the founder of these gatherings, Heydar Radjavi, and to restart them again.

Confirmed Invited Speakers:

- Jane Breen, Ontario Tech University, Canada
- Doug Farenick, University of Regina, Canada
- João Gouveia, University of Coimbra, Portugal
- Laurent Marcoux, University of Waterloo, Canada
- Lajos Molnár, University of Szeged and University of Budapest, Hungary
- Clément de Seguins Pazzis, Université de Versailles Saint-Quentin-en-Yvelines, France
- Helena Šmigoc, University College Dublin, Ireland
- Ryotaro Tanaka, Tokyo University of Science, Japan

The Scientific Committee of LAW'25 consists of Dijana Ilišević (University of Zagreb, Croatia); Raphael Loewy (Technion - Israel Institute of Technology, Israel); Mitja Mastnak (Chair, Saint Mary's University, Canada); Martin Mathieu (Queen's University Belfast, United Kingdom); João Filipe Queiró (University of Coimbra, Portugal); and Konrad Schmüdgen (University of Leipzig, Germany).

The Organizing Committee of LAW'25 consists of Ljiljana Arambašić (University of Zagreb, Croatia); Bojan Kuzma (Local Organizer, University of Primorska, Slovenia); Matjaž Omladič (Chair, University of Ljubljana, Slovenia); Nik Stopar (University of Ljubljana, Slovenia); and Aljaž Zalar (University of Ljubljana, Slovenia).

Young researcher sections. Short presentations by Ph.D. students or postdocs are welcome, as usual at LAW'xx meetings.

Working groups. Much of math research is done through discussing open problems. We always do that, but not at conferences, at least not officially. At LAW'xx gatherings, we do; here are two prearranged working groups for LAW'25:

- **Local to global properties of collections of matrices** (lead by Mitja Mastnak and Heydar Radjavi). We are interested in the following question about a collection of n-by-n matrices: if it satisfies some interesting local property, does this imply some significant global property? For example: if for all A, B in an irreducible semigroup we have that the spectrum of $AB - BA$ is real, then the semigroup is simultaneously similar to a semigroup of real matrices. The aim of this working group is to study some open problems in the area and perhaps also come up with new open problems.
- **Moment problems and positive polynomials** (lead by Aljaž Zalar). Given a (finite or infinite) sequence of numbers indexed by monomials and a closed set X, the moment problem (MP) asks to characterize criteria for the existence of a Borel measure on X such that its moments coincide with the given sequence. Duality with positive polynomials has led to many new results in MPs. The aim of this working group is to discuss open problems in this area.

Deadlines: Extended abstracts of contributions (whether general or "young") should be submitted before November 30th, 2024 in order to be reviewed on time.

Geography: As past LAW'xx meetings have visited changing locations in Slovenia, a seaside area was chosen this time, a former Venetian salt-harvesting "colony", where people are still Slovene-Italian bilingual. It includes saltpans at Sečovlje (Sicciole), the touristy place Portorož (Portorose) in the middle, and an ancient salt exporting port-fortress, Piran (Pirano).

XXII Householder Symposium on Numerical Linear Algebra
Ithaca, USA, June 8–13, 2025

The next Householder Symposium will be held June 8–13, 2025 at Cornell University in Ithaca, NY.

The Householder Symposia originated in a series of meetings organized by Alston Householder, Director of the Mathematics Division of Oak Ridge National Laboratory and Ford Professor at the University of Tennessee. These international meetings were devoted to matrix computations and linear algebra and were held in Gatlinburg, Tennessee. They had a profound influence on the subject. The last "Gatlinburg" conference held in Gatlinburg was in 1969 on the occasion of Householder's retirement. At the time, it was decided to continue the meetings but vary the place. Since then, these meetings have been held at three-year intervals in a variety of venues and the series has been renamed in honor of Alston Householder.

This meeting is the twenty-second in the series. As envisioned by Householder, the meeting is informal, emphasizing an intermingling of young and established researchers. Topics include numerical linear and multilinear algebra, matrix theory, including probabilistic algorithms, and related areas such as optimization, differential equations, signal and image processing, network analysis, data analytics, and systems and control. The seventeenth Householder Prize for the best Ph.D. thesis in numerical linear algebra since January 1st, 2022 will also be presented at the meeting.

Attendance at the meeting is by invitation only. Applications will open later this year and are due on October 31st, 2024. More information about the meeting and application process can be found at `https://householder-symposium.org`.

The Householder committee includes Zhaojun Bai (University of California, Davis, USA); David Bindel (Cornell University, USA); Julianne Chung (Emory University, USA); James Demmel (chair, University of California, Berkeley, USA); Froilán Dopico (Universidad Carlos III de Madrid, Spain); Zlatko Drmač (University of Zagreb, Croatia); Heike Fassbender (Technical University of Braunschweig, Germany); Sherry Li (Lawrence Berkeley National Laboratory, USA); James Nagy (Emory University, USA); and Alison Ramage (University of Strathclyde, UK).

The local organizing committee consists of David Bindel, Anil Damle (chair), and Alex Townsend, all of Cornell University.

The 17$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii (WONRA) Taichung, Taiwan, June 19–21, 2025

The 17$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii (WONRA) will be held at the National Chung Hsing University, Taichung, Taiwan, June 19 (Thu) – 21 (Sat), 2025. The purpose of the workshop is to stimulate research and foster interactions between researchers interested in the study of numerical ranges and numerical radii. This subject has a long and distinguished history, with connections and applications to different branches of pure and applied science such as operator theory, functional analysis, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, and quantum information science. In fact, the 2025 WONRA will have a special session on Quantum Information Theory, in connection to the International Quantum Year 2025.

For additional information, please visit the workshop website:

`https://sites.google.com/email.nchu.edu.tw/wonra2025`

The 2025 WONRA organizing committee consists of:

- Ray-Kuang Lee, National Tsing Hua University, Taiwan (`rklee@ee.nthu.edu.tw`)
- Chi-Kwong Li, College of William & Mary, USA (`ckli@math.wm.edu`)
- Raymond Nung-Sing Sze, Hong Kong Polytechnic University, Hong Kong (`raymond.sze@polyu.edu.hk`)
- Ming-Cheng Tsai, National Taipei University of Technology, Taiwan (`mctsai2@ntut.edu.tw`)
- Ya-Shu Wang, National Chung Hsin University, Taiwan (`yashu@dragon.nchu.edu.tw`)
- Ngai-Ching Wong, National Sun Yat-sen University, Taiwan (`wong@math.nsysu.edu.tw`)

The 26$^{\text{th}}$ ILAS Conference Kaohsiung, Taiwan, June 23–27, 2025

Kaohsiung

$\lambda_2^L = 1.6$

ILAS2025

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The 26$^{\text{th}}$ Conference of the International Linear Algebra Society will be held in the vibrant city of Kaohsiung, Taiwan, June 23–27, 2025. This prestigious event will bring together leading experts, researchers, and enthusiasts from around the world to share their knowledge and explore the latest advancements in the field of linear algebra.

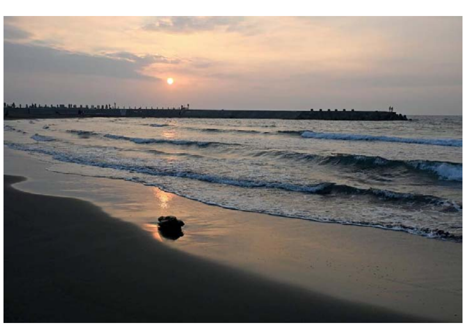
Plenary speakers:

- Haim Avron (Tel Aviv University) **SIAG/LA Lecture**
- Fan Chung Graham (University of California, San Diego)
- Fumio Hiai (Tohoku University)
- Daniel Kressner (EPFL) ***LAA* Lecture**
- Ren-Cang Li (University of Texas at Arlington)
- Karen Meagher (University of Regina)
- Polona Oblak (University of Ljubljana)
- Fernando De Terán Vergara (Universidad Carlos III de Madrid)
- Karol Życzkowski (Jagiellonian University)
- TBA **Hans Schneider Prize Lecture**

For further details, including registration information and program updates, please visit the conference website:

`https://ilas2025.tw`

We look forward to welcoming you to this exciting event in 2025!



ONGOING ONLINE SEMINARS

Algebraic Graph Theory Seminar

https://math.uwaterloo.ca/~agtheory

Host: University of Waterloo

Schedule: weekly on Mondays

Time: 11:30AM, Waterloo (Ontario, Canada) time

Most recent talk:

Symmetric Nonnegative Trifactorization Rank

Helena Šmigoc (University College Dublin, Ireland)

Next talk:

June 3rd, 2024

Which graphs are determined by their total numbers of walks?

Wei Wang (Xi'an Jiaotong University, China)

Contact: Sabrina Lato (smlato@uwaterloo.ca)

Matrix Seminar

https://docs.google.com/document/d/1MswSd16JqsZE294kYCXujLio4cnAiuYv6QKRc6BxvI0/edit

Host: University of Nevada, Reno

Schedule: biweekly on Fridays

Time: 4:15PM, Reno (Nevada, USA) time

Most recent talk:

Extreme Points of Infinite-dimensional Symmetric Doubly Stochastic Matrices

Selcuk Koyuncu (University of North Georgia, USA)

Next talk: Fall 2024

Contact: Pan Shun Lau (plau@unr.edu)

05C50 Online

https://sites.google.com/view/05c50online/home

Host: University of Manitoba

Schedule: biweekly on Fridays

Time: 10:00AM, Winnipeg (Manitoba, Canada) time

Most recent talk:

Spectral properties of a structured sign pattern related to a system of second order ODEs

Minerva Catral (Xavier University, USA)

Next talk: Fall 2024

Contact: Hermie Monterde (monterdh@myumanitoba.ca)

IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 71-1 and 71-2. Solutions are invited to Problems 68-2, 68-4, 69-2, 69-3, 70-2, 71-3, and to all of the new problems from the present issue 72.

Problem 71-1: A Property of 4×4 Hadamard Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

Let n be an even number. An $n \times n$ matrix H is called a Hadamard matrix if its entries lie in the set $\{-1, 1\}$ and it satisfies $HH^T = nI_n$. A *monomial matrix* is a square matrix that has at most one nonzero entry in each row and in each column.

(a) Show that, for any 4×4 Hadamard matrix H, there exists a unitary monomial matrix U and a rank-one matrix R such that $H = 2U + R$.

(b) Show that no $n \times n$ Hadamard matrix is the sum of a monomial matrix and a rank-one matrix when $n > 4$.

Solution 71-1 by Ferdinand BARBARINO, *University of Mons, Mons, Belgium*, giovanni.barbarino@gmail.com

(a) Notice first that a monomial matrix U is the product of a diagonal matrix D and a permutation matrix P, where U is unitary if and only if D is unitary. Let R denote a rank-one matrix.

Now let H be any Hadamard 4×4 matrix and let h_1 be its first column. Then, for any matrices D, P and R, with DP unitary monomial and R of rank one, $H = 2DP + R$ if and only if $\mathrm{diag}(h_1)H = 2\,\mathrm{diag}(h_1)DP + \mathrm{diag}(h_1)R$. Moreover, in this case, $\mathrm{diag}(h_1)H$ is still Hadamard, $2\,\mathrm{diag}(h_1)DP$ is still unitary monomial, and $\mathrm{diag}(h_1)R$ still has rank one. As a consequence, we can assume without loss of generality that every entry in the first column of H is 1. Since the columns of H are orthogonal, each of the other columns must have exactly 2 entries equal to -1 and 2 entries equal to 1. Multiplying by a permutation matrix (either on the left or on the right) preserves the Hadamard, monomial and rank-one properties, so we can permute the rows of H such that the second column is $[-1\ -1\ 1\ 1]^T$.

Applying further permutations in a similar way to adjust the final two columns, we find that the only Hadamard matrix we need to consider is

$$H = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} = 2 \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

(b) Now let n be even with $n > 4$. If H is Hadamard and $H = DP + R$, where DP is monomial and R is of rank one, then multiplying on the right by P^T preserves all properties, so we have $H = D + R$.

Moreover, if h_1 is the first column of H, then multiplying on the left by $\mathrm{diag}(h_1)$ still preserves the Hadamard, diagonal, and rank-one properties, so we may again suppose that the first column of H is composed of all 1s. As a consequence, the first column of R is similarly composed of all 1s, except maybe for the first entry, since R and H are equal outside the diagonal.

As in the previous point, the columns of H are orthogonal, so all the columns except the first must have exactly $\frac{n}{2}$ entries equal to -1 and $\frac{n}{2}$ entries equal to 1. As a consequence, all corresponding columns of R will have at least $\frac{n}{2} - 1 \geq 2$ entries equal to -1 and $\frac{n}{2} - 1 \geq 2$ entries equal to 1. At the same time, those columns are scalar multiples of the first one (since R has rank one) and so, within each of them, all but one entry must be equal, which is impossible.

Also solved by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

Problem 71-2: Cauchy-Schwarz and the Characteristic Polynomial

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let $\mathbb{F}$ be either the field of real numbers or the field of complex numbers. Suppose N is a nonzero nilpotent $n \times n$ matrix with entries in $\mathbb{F}$, that $k \in (0, 1)$, and that $p(x)$ is a monic polynomial of degree n with all of its coefficients in $\mathbb{F}$. Show that there exists an $n \times n$ matrix M with entries in $\mathbb{F}$ having characteristic polynomial $p(x)$ and satisfying $|\mathrm{Tr}(MN)|^2 > k\,\mathrm{Tr}(M^*M)\,\mathrm{Tr}(N^*N)$.

Solution 71-2 by Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, `eaherman@gmail.com`

Since the only eigenvalue of N is zero, N is similar to a block diagonal matrix in which each block is a Jordan block with 1s along the superdiagonal and 0s elsewhere, except that one block could be all 0s. If there is such a zero block, we may assume it is placed last. Since N is nonzero, there is at least one nonzero block. If N is real, then, since all its eigenvalues are also real, the similarity transformation matrix may be chosen to be real. We begin by proving the special case where N equals its Jordan canonical form, in the form we just described.

For any real or complex matrix A, the trace of A^*A is the sum of the squares of the absolute values of its entries. Thus, $\mathrm{Tr}(N^*N)$ is the number of 1s in N, which we will denote by s_N; that is, $s_N = \sum_{i=1}^{n-1} N(i,i+1)$. We define the matrix $M = M_n$ as follows. The first row of M is denoted $a_1, \dots, a_n$, and its first subdiagonal is defined, for some fixed value $\alpha \neq 0$, by $M(i,i-1) = 1 + (\alpha - 1)N(i-1,i)$, for $i = 2, \dots, n$; that is, a subdiagonal entry of M equals α when the corresponding superdiagonal entry of N is 1 and equals 1 when that superdiagonal entry is 0. The values of $a_1, \dots, a_n$ and α are yet to be determined, and all remaining entries of M are 0. Hence, the other two traces with which we are concerned are given by

$$\mathrm{Tr}(M^*M) = \sum_{i=1}^{n} |a_i|^2 + s_N\alpha^2 + (n-1-s_N)1^2$$

and

$$\mathrm{Tr}(MN) = s_N\alpha.$$

Expanding along the last column of the matrix, we find that, for $n = 3, \dots, n$, a recursive formula for the characteristic polynomial $\det(\lambda I_n - M)$ is given by

$$\begin{aligned}\det(\lambda I_n - M_n) &= \lambda \det(\lambda I_{n-1} - M_{n-1}) \\ &\quad + (-1)^{n-1}(-a_n)\det\begin{bmatrix} -M(2,1) & \lambda & 0 & \cdots & 0 \\ 0 & -M(3,2) & \lambda & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -M(n-1,n-2) & \lambda \\ 0 & 0 & \cdots & 0 & -M(n,n-1) \end{bmatrix} \\ &= \lambda \det(\lambda I_{n-1} - M_{n-1}) - a_n q_{n-1},\end{aligned}$$

where we let q_{n-1} be the product of the first $n-1$ subdiagonal entries of M. In particular, each q_{n-1} is equal to α^k for some $k > 0$, where $k \neq 0$ is a consequence of $N \neq 0$. Therefore,

$$\det(\lambda I_n - M_n) = -a_n q_{n-1} - a_{n-1}q_{n-2}\lambda - \cdots - a_3 q_2 \lambda^{n-3} + \lambda^{n-2}\det(\lambda I_2 - M_2). \tag{1}$$

Due to the fact that $N \neq 0$, together with our assumption on the order of the Jordan blocks, we have $M_2 = \left[\begin{smallmatrix} a_1 & a_2 \\ \alpha & 0 \end{smallmatrix}\right]$. Therefore, $\det(\lambda I_2 - M_2) = \det\left[\begin{smallmatrix} \lambda - a_1 & -a_2 \\ -\alpha & \lambda \end{smallmatrix}\right] = -a_2\alpha - a_1\lambda + \lambda^2$, which we can combine with (1) to obtain

$$\det(\lambda I_n - M_n) = -\sum_{j=0}^{n-2} a_{n-j}q_{n-1-j}\lambda^j - a_1\lambda^{n-1} + \lambda^n,$$

where we define $q_1 = \alpha$. Further, for convenience of notation, let $q_0 = 1$. Then, if $p(x) = \sum_{j=0}^{n-1} c_j x^j + x^n$, it suffices to let $a_i = -\frac{c_{n-i}}{q_{i-1}}$ for $i = 1, 2, \dots, n$. Then the desired inequality can be written as

$$s_N^2\alpha^2 > k\left(\sum_{i=1}^{n}\left|\frac{c_{n-i}}{q_{i-1}}\right|^2 + s_N\alpha^2 + (n-1-s_N)\right)s_N$$

or, after dividing through by the left side,

$$1 > \frac{k}{s_N\alpha^2}\left(\sum_{i=1}^{n}\left|\frac{c_{n-i}}{q_{i-1}}\right|^2 + (n-1-s_N)\right) + k.$$

Since the only occurrences of α within the outermost parentheses are in the denominator q_{i-1}, the first term on the right tends to 0 as α gets large. Therefore, since $0 < k < 1$, the above inequality can be made true by selecting a sufficiently large value for α.

We now prove this result for the general case where $N = SJS^{-1}$ is an arbitrary nonzero nilpotent matrix, J is its Jordan form and S is an invertible matrix. For all $m \in \mathbb{Z}^+$, there exists a matrix A_m with characteristic polynomial $p(x)$ such that

$$|\operatorname{Tr}(A_m J)|^2 > \frac{m-1}{m} \operatorname{Tr}(A_m^* A_m) \operatorname{Tr}(J^* J).$$

Let $\|\cdot\|$ be any fixed matrix norm and let $\hat{A}_m = \dfrac{A_m}{\|A_m\|}$. By the Bolzano–Weierstrass Theorem, $\{\hat{A}_m\}_{m=1}^{\infty}$ has a convergent subsequence $\{\hat{A}_{m_j}\}_{j=1}^{\infty}$. Let A be the limit of this sequence. For $n = 1, 2, 3, \ldots$, we have

$$\frac{m-1}{m} < \frac{\left|\operatorname{Tr}(\hat{A}_m J)\right|^2}{\operatorname{Tr}(\hat{A}_m^* \hat{A}_m) \operatorname{Tr}(J^* J)} \leq 1,$$

where the right-hand inequality follows from the Cauchy–Schwarz Inequality. Hence, in the limit we obtain

$$\frac{|\operatorname{Tr}(AJ)|^2}{\operatorname{Tr}(A^* A) \operatorname{Tr}(J^* J)} = 1.$$

By the equality condition of the Cauchy–Schwarz Inequality, the above shows that A must be a scalar multiple of J. Hence, the sequence $\{S\hat{A}_{m_j} S^{-1}\}_{j=1}^{\infty}$ converges to a scalar multiple of $SJS^{-1} = N$ and thus

$$\lim_{j \to \infty} \frac{\left|\operatorname{Tr}(SA_{m_j} S^{-1} N)\right|^2}{\operatorname{Tr}((SA_{m_j} S^{-1})^* SA_{m_j} S^{-1}) \operatorname{Tr}(N^* N)} = 1,$$

from which the result follows.

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce three new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 72-1: Pascal Matrices

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

The $n \times n$ Pascal matrix P_n is the matrix whose (i,j)th entry is $\binom{i+j-2}{i-1}$. Show that if n is a power of 2, then every entry of the matrix $(P_n)^3 + I$ is an even number.

Problem 72-2: Diagonal Scaling and Matrix Powers

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let M be an $n \times n$ matrix and let $\|\cdot\|$ be the operator norm. Suppose that, for all integers k with $1 \leq k \leq n$, the matrix M^k has all main diagonal entries equal to 0. Show that for any $\epsilon > 0$, there exists a positive definite diagonal matrix D such that $\|D^{-1}MD\| < \epsilon$.

Problem 72-3: Numerical Range of a 2×2 matrix

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

It is well known that the numerical range of a 2×2 matrix is an ellipse. Suppose M is a 2×2 matrix whose numerical range $W(M)$ is a non-degenerate ellipse. Show that M is unitarily equivalent to a matrix A where $2\sqrt{|a_{12}a_{21}|}$ is the distance between the two foci of $W(M)$.

Solutions to Problems 71-1 and 71-2 are on pages 38–40.